Discrete Applied Mathematics

Zombie number of the Cartesian product of graphs

--Manuscript Draft--

Manuscript Number:	DA11466R2
Article Type:	Note
Keywords:	Cartesian product of graphs; Zombie Number; NP-hard
Corresponding Author:	Behnam Bahrak University of Tehran Tehran, Tehran IRAN, ISLAMIC REPUBLIC OF
First Author:	Ali Keramatipour
Order of Authors:	Ali Keramatipour
	Behnam Bahrak
Abstract:	Zombies and Survivors is a variant of the pursuit-evasion game Cops and Robbers , with the difference that zombies must always move closer to one of their closest survivors. The game is played on a simple graph by two players. The goal of the zombies is to catch the survivors while survivors' objective is to avoid being captured. The zombie number of G, denoted as $\zn(G)$, is the minimum number of zombies required to capture a single survivor on G, no matter what moves survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al.\cite{Fitz16} about the zombie number of the Cartesian product of two graphs. This result provides a new proof for $\zn(Q_n) = \ceil*{\frac{2n}{3}}$. We also introduce a new problem regarding capture time in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph G, with and without a limited capture time .

Dear reviewers,

Thank you for your time to review our paper. We truly appreciate all your helpful comments, suggestions, and corrections for improving this paper. In response to your suggestions, we have outlined the changes and provided the necessary explanations.

In response to the comments of Reviewer #1:

Abstract

-- P1 L19: I am not sure why you use italics in various spots in the abstract

→ **In this revision, we only use italics on the newly defined terms and phrases like "G-zombie" or "G-move".**

-- P1 L24 and throughout the paper: I would use “the survivor” instead of “survivor”

→ **We made this change for most of “survivor”s throughout the paper.**

Introduction

-- P1 L47: “cops should not” should be “cops need not”

-- P1 L48: “They can either hold...” should be its own sentence.

→ **We applied these changes in this revision.**

-- P2 L15: There should not be a comma after “Capture time in a game”

-- P2 L17: “exists” should be “exist”

→ **We corrected these in this revision and moved capture time related definitions from section 3 to here.**

-- P2 L26 and throughout the paper: Notation for Cartesian product looks strange. Is there a better way to typeset?

→ **We shortened the square notation.**

-- P2 L41, L45, L47: Instead of 3, 4, 5, write “Section 3”, “Section 4”, “Section 5”

→ **We have changed these in this revision.**

Zombie number of the Cartesian product of two graphs

-- P3 L6: “notations” should be “notation”

-- P3 L8: It would be clearer to write “For 1 ≤ i ≤ m, define Gi to be the subgraph induced by..”

-- P3 L9: It would be clearer to write “Similarly, for 1 ≤ j ≤ n, define Hj to be the subgraph induced by..”

-- P3 L9: It is redundant to define (j, i) as the common vertex Gi and Hj . You could remind the reader of the notation.

-- P3 L20: It would be clearer to write ”Define distI (j, k) to be the distance between vertices j and k on a graph I. Define the length of a path P to be len(P).”

-- P3 L23: It would be clearer to write “For a vertex (u,v) ∈ G□H, define its G-equivalent vertex to be u ∈ V (G). You should also make the definition of G-equivalent graph more formal then “where we put”.

→ **We have changed these in this revision.**

-- P3 L52: It would seem clearer to talk about G- and H-edges rather than moves.

-- P4 L6: I think this proof should be more formal. Instead of writing some moves are happening, you could specify that the path of the same length with G-edges before H-edges has all G-edges in Gy and all H edges in Hu.

→ **Since Reviewer #3 suggested to remove the proof of this lemma and only state it as a simple fact, we have done so and this proof is not included in the new revision of the paper.**

-- P4 L30: I think the sentence “If a = y and the survivor makes an H-move, ...” is not necessary (the zombie strategy does depends only on the position when it is their turn).

→ **We have removed this sentence.**

-- P4 L39: I think the sentence “Now for each G-move...” is not clear. Is the point you’re trying to make that whenever the survivor makes an H-move, that the zombies will all be on Hx?

→ **Yes. Instead of H-equivalent we should have used H_x. We corrected this in the new revision.**

-- P4 L49, 53: It would be clearer to write “Theorem 2” rather than “theorem proved above” and “proved conjecture”

→ **We have changed these in the new revision.**

Capture time in Cartesian product of graphs

-- P5 L23 and throughout: I think in Theorem 5 you could replace diameter of the graph with radius (just start the cops at the vertices that minimize the length of the longest path). Also you could subtract 1 from the capture time since it is only necessary for the G or H-zombies to catch the survivor, not both.

→ **We applied these comments in this revision.**

-- P5 L26: Theorem 2 should be captitalized.

-- P5 L35: “total number” should be “the total number”

→ **We have changed these in this revision.**

Limited capture time zombie number problem is NP-Hard

-- P5 L42: Quotation mark is backward

-- P5 L46: You should include a reference that dominating set is NP-Hard.

-- P5 L54: “Also NG[u] represents...” should be “Let NG[u] represent...” and you should specify whether this set includes u itself (is it the closed or open neighborhood)?

-- P6 L6: First sentence should be “The LCZk problem...”

-- P6 L7: The instance should be: “Let G = (V,E) be a simple undirected graph and let zc and k be positive integers.”

-- P6 L9: The second sentence of the question is not needed (or should be placed outside the formal definition).

-- P6 L13: The instance should be “Let G = (V,E) be a simple undirected graph and let d be a positive integer.”

→ **We applied all these corrections in this revision.**

-- P6 L30: You should explicitly state adjacencies for (v, i)’s.

→ **We stated the exact path in the new version. We think it is much clearer now and adjacencies are stated as well.**

-- P6 L43: Why are the k’s capitalized in the figure?

→ **It was a mistake. We used lowercase letters in this revision.**

-- P6 L48: Also note the number of edges in Gk is O(n).

→ **Reviewer #3 noted that this line is too obvious and it is not needed to be mentioned. In this revision we just stated that Gk can be built in polynomial time.**

-- P6 L50: It would be clearer to say: “Let S be the set of vertices v ∈ V (G) such that v ∈ S′ or (v, i) ∈ S`.”

→ **We changed this in the new revision.**

-- P7 L17: I am not convinced by the proof of Lemma 7. It is certainly true that if the survivor moves n + 1 times, they will be on the same vertex twice. But that does not imply that the zombies will be in the same position both times (even if their total distance is the same). Without this fact I don’t think you can argue that the survivor could repeat the same moves.

→ **The proof of Lemma 7 was wrong and it is removed from this revision. Since it was not used anywhere else in the paper, other theorems still hold.**

Zombie number problem is NP-Hard

-- P7 L45: The instance should be: “Let G = (V, E) be a simple undirected graph and let zc be a positive integer.”

-- P8: Proof of Theorem 8 is correct, but could be a bit clearer.

→ **We changed these in the new revision and rewrote the proof of Theorem 8 (Theorem 7 in this revision) to make it clearer.**

In response to the comments of Reviewer #2:

-- P1. Line 29. The final G in the Abstract needs to be in math mode.

-- P1. Lines 43-45. Since the zombie player starts first, why present the survivor's details before the zombie's details?

-- P2. Lines 15-16. The attempt to define capture time has too many grammar mistakes to be understood. Please revise this.

-- P2. Lines 39-47. Don't say "2", "3", etc. Say "Section 2", etc.

-- P3. Line 16. H-Move should be H-move

→ **We applied all these corrections in this revision.**

-- P3-4. In the proof of Lemma 1 there are several mentions of moves. However, the lemma does not involve any players, and so there are no entities that would be in motion. I believe that several of the references to "moves" in this proof should instead refer to "edges".

→ **Since Reviewer #3 suggested to remove the proof of this lemma and only state it as a simple observation, we have done so and this proof is not included in the new revision of the paper.**

-- P5. L26. Theorem 2 (not theorem 2)

-- P5. L42. The quotes are in the wrong direction. This also happens elsewhere in the paper, such as in the References.

→ **We have changed these in this revision.**

-- P5. L44. I believe that the class that has been described is P, not NP. The authors should ensure that NP is correctly defined.

→ **This part was removed in the new version.**

-- P5. L53. Do the authors truly want the closed neighborhood of u, and not the open neighborhood? If yes, then they should say so. Alternatively, if they want the open neighborhood, then they should not use the notation for closed neighborhood.

→ **We refer to this as closed neighborhood in the new version.**

-- P6. L52-53. Surely vertex $(u,k-1)$ is adjacent to an other vertex, and so the statement that "there is no vertex at distance k or less" is false.

→ **We meant there is no zombie at distance k or less. We changed it in this revision.**

-- P7. L22. I believe "each two rounds" is meant to be "each round"

-- P7. L28-30. I agree with the pigeonhole principle statement. But I am not yet convinced that the final sentence of the paragraph is correct. Couldn't the positions of the players have now changed from the previous time when the survivor was on the previously visited vertex, and if so, might there now be an opportunity for distances to be reduced in the coming moves (which will not be the same moves as before)?

→ **The proof of Lemma 7 was wrong and it is removed from this revision. Since it was not used anywhere else in the paper, other theorems still hold.**

In response to the comments of Reviewer #3:

2. Technical Details

→ **1: We changed the definition and now refer to this game as "Zombies and Survivor" instead of "Zombies and Survivors"**

→ **2: We changed this in the new version. Now the zombies player is given "z" zombies instead of choosing "z" himself.**

→ **3: We changed this in the new version and made a customized notation "z" for z_c. By c in z_c we meant "count" but it was definitely not clear!**

→ **4: A new sentence is added to this revision to emphasize the rule.**

→ **5: We moved the definition of capture time from page 5 to section 1 and also added the reference you mentioned to clarify this parameter.**

→ **6: We replaced this with "capt" function in this revision.**

→ **7: "G-equivalent graph" is a new copy of graph "G", where we put each survivor and zombies, on their "G-equivalent" position. Since it is a graph, we believe that "G-equivalent graph" suits it more than "G-equivalent game position".**

→ **8: As you suggested, we removed the proof and just stated the lemma.**

→ **9:**

(a) Above figure 2, we stated that survivor is located at (x,y). In this revision we have added a new sentence to remind the reader.

(b) p_H refers to a single object which is a path in graph H. Every G-zombie will move along this path in its H-subgraph. Also choosing a single path is key to this proof, since if we use different paths for our zombies, they might end up in different H-subgraphs which makes us unable to discuss the game on a single H.

(c) We applied your suggestion in this revision

(d) We made some changes to make the proof shorter and easier to understand. Please let us know if it needs to be clearer.

→ **10: We removed the appendix and added it to the supplementary materials of the paper.**

→ **11: We changed this in the new version.**

→ **12: We added the reference to this revision.**

→ **13: We corrected the definition and also moved it to above figure 1.**

→ **14: We replaced diameter with radius and wrote "Let $rad(G)$ represent the G's radius".**

→ **15: These lines are removed in this revision.**

→ **16: We changed this to closed neighborhood.**

→ 17: k is not an input to the problem and is just a parameter. We applied your suggestion and changed the phrase to "INSTANCE of LCZ_k problem".

→ 18: We changed this in the new version.

→ 19: We have applied these changes in this revision.

→ 20: We included this reference and also mentioned the cops and robbers version of the problem at page 2.

→ 21: The proof of Lemma 7 was wrong and it is removed from this revision. Since it was not used anywhere else in the paper, other theorems still hold.

→ 22: The reference is included in this revision.

→ 23: Thanks for your comment, we applied your suggestion to this revision.

→ 24: In this revision we use B_i instead of K_i to make the sentence clearer.

→ 25: We applied your comment in this revision.

→ 26: We now state that the game is played on a "simple connected graph"

→ 27: These parts of proof were modified using Reviewer #1's comments.

Zombie number of the Cartesian product of graphs

Ali Keramatipour, Behnam Bahrak*

School of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran, Iran

Abstract

Zombies and Survivor is a variant of the pursuit-evasion game *Cops and Robber(s)*, with the difference that zombies must always move closer to the survivor. The game is played on a simple connected graph by two players. The goal of the zombies is to catch the survivor while the survivor's objective is to avoid being captured. The *zombie number* of G, denoted as $z(G)$, is the minimum number of zombies required to capture a single survivor on G, no matter what moves the survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al.[1] about the zombie number of the Cartesian product of two graphs. This result provides a new proof for $z(Q_n) = \lceil \frac{2n}{3} \rceil$. We also introduce a new problem regarding *capture time* in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph G, with and without a limited *capture time*.

Key words: Cartesian Product of Graphs, Zombie Number, NP-Hard

1. Introduction

The *Zombies and Survivor* game is played on a simple connected graph by two players. The deterministic version of this game [1] is played as follows. Initially, the zombie player is given z zombies and places them on the graph's vertices. This initial arrangement that zombie player chooses is essential for him to win. Then the survivor player chooses one single vertex which is the survivor's initial position. Starting with the zombie player, on each player's turn, the zombie player must move each zombie to one of its adjacent vertices so that they get closer to the survivor, while the survivor player either moves to an adjacent vertex or stays at his current location. Here lies the difference between *Zombies and Survivor* and *Cops and Robber(s)* games, as in *Cops and Robber(s)* cops need not necessarily get closer to the robber(s). They can either hold their current position, get closer, or further away from the robber(s). Although zombies are not as intelligent as cops, they can still choose their path intelligently between the shortest paths. If any

*Corresponding author
Email addresses: alikeramatipour@ut.ac.ir (Ali Keramatipour), bahrak@ut.ac.ir (Behnam Bahrak)

zombie and the survivor ever occupy the same vertex, the survivor is captured and the zombie player wins. The zombie number of a graph G, denoted as $z(G)$, is the minimum number of zombies required so that the zombie player can always capture the survivor, no matter how survivor moves.

The Cartesian product $G \square H$ of two graphs G and H, is a graph with vertex set of $V(G) \times V(H)$, where vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $\{u_2, v_2\} \in E_H$, or $u_2 = v_2$ and $\{u_1, v_1\} \in E_G$ [3]. Figure 1 shows an example of the Cartesian product of two graphs.

Capture time of a game [5], noted as $capt(G, z)$, represents the maximum number of moves that the survivor can avoid being caught on a graph G, while the zombie player tries to catch the survivor as quickly as possible using z zombies. If the survivor can win, $capt(G, z)$ goes to infinity.

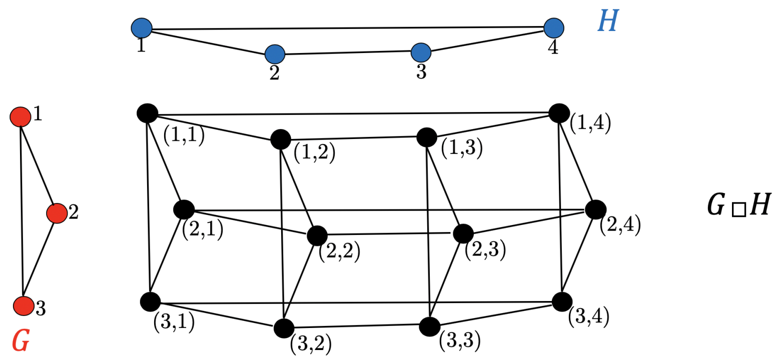


Figure 1: $C_3 \square C_4$ an example of the Cartesian product

Our contributions can be summarized as follows:

1) In [1], Fitzpatrick et al. conjectured that $z(G \square H) \leq z(G) + z(H)$. In section 2, we prove this conjecture and use it to show that $z(Q_n) = \lceil \frac{2n}{3} \rceil$.

2) In section 3, we provide a new bound on *capture time* of *Zombies and Survivor* game played on the Cartesian product of two graphs.

3) In section 4, we introduce a variation of *Zombies and Survivor* game, in which the zombie player is restricted to win in a limited number of moves, and prove it belongs to NP-Hard class of problems.

4) The complexity of the *Cops and Robber(s)* game is known [7]. In section 5, we prove that the original *Zombies and Survivor* game belongs to NP-Hard class of problems.

2. Zombie number of the Cartesian product of two graphs

To prove $z(G \Box H) \leq z(G) + z(H)$, we show that $z(G) + z(H)$ zombies are enough for the zombie player to capture the survivor on $G \Box H$.

To explain the proof we first need to define some notation. Assume H and G have m and n vertices, respectively.

For $1 \leq i \leq m$, define G_i to be the subgraph induced by vertices $(u, v = i)$. Similarly, for $1 \leq j \leq n$, define H_j to be the subgraph induced by vertices $(u = j, v)$.

In the Cartesian product of G and H, each G_i $(1 \leq i \leq m)$ is isomorphic to G, and each H_j $(1 \leq j \leq n)$ is isomorphic to H. Also the common vertex between G_i and H_j is (j, i), and (x, y) is the vertex where the survivor is located. Figure 2 illustrates these definitions.

A *G-edge* is an edge in one of the G_is and an *H-edge* is an edge in one of the H_js. A *G-move* is a move made on a *G-edge*. Similarly, an *H-move* is a move made on an *H-edge*. If the survivor decides to remain in its current vertex, this move is considered both a *G-move* and an *H-move*.

Define $dist_I(j, k)$ to be the distance between vertices j and k on a graph I. Define the length of a path P to be $len(P)$.

For vertex (u, v), define its G-equivalent vertex to be vertex $u \in G$, and its H-equivalent vertex, to be vertex $v \in H$. G-equivalent graph is a graph G where each zombie and the survivor is put on its G-equivalent vertex. H-equivalent graph is defined in the same way.

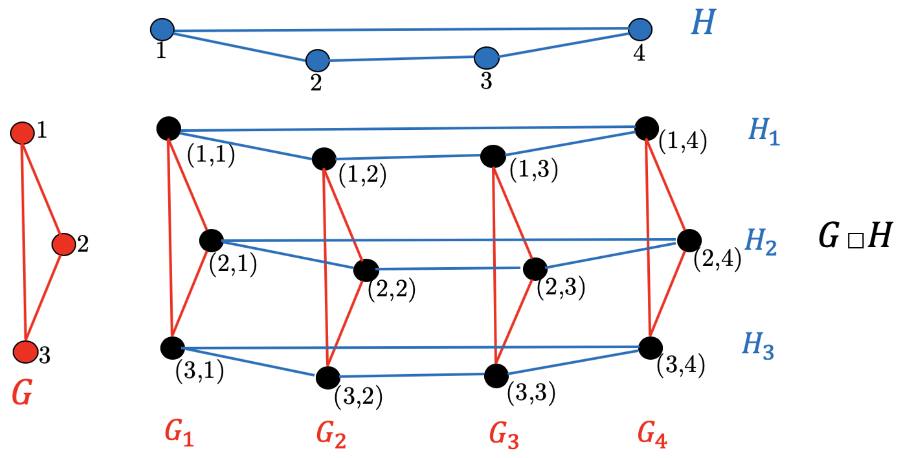


Figure 2: $G \Box H$, G_is, and H_is.

We start off by stating a simple lemma regarding distance in Cartesian product:

Lemma 1. $dist_{G \Box H}((x, y), (u, v)) = dist_G(x, u) + dist_H(y, v)$.

Theorem 2. $z(G\square H) \le z(G) + z(H)$.

Proof. We provide a winning strategy for the Cartesian product of G and H using $z(G)+z(H)$ zombies. First, we place $z(G)$ zombies, that have a winning strategy on a single G, $G_{a=1}$ and call them *G-zombie*s. We do the same for $H_{b=1}$ and call them *H-zombie*s. Also recall that (x, y) is the survivor's vertex.

Consider one of the shortest paths between vertices a (the index of G-subgraph shared by *G-zombie*s) and y in H and call it p_H. We also define p_G in the same manner between b and x.

On each zombie turn, if $a \neq y$, each *G-zombie* will move along the p_H path in its corresponding H subgraph. According to lemma 1, since zombies' and the survivor's equivalents on H are getting closer, thus their actual vertices on $G\square H$ are getting closer as well and this move is possible. Since they are all moving along similar paths (in their corresponding H-subgraphs) they will still share the same G-subgraph. Now consider when $a = y$, *G-zombie*s will play their winning strategy (that they had on a single G) in this case. This move is also possible since in G's strategy, zombies would get closer to the survivor on each turn. This means for those turns that $a = y$ holds, if we consider the G-equivalent graph between *G-zombie*s and the survivor, it is just like a simple game played on a single G. *H-zombie*s will follow the same strategy but in their corresponding environment.

Suppose using this strategy the survivor wins, therefore the survivor must be able to do either infinite *G-move*s or infinite *H-move*s. Without loss of generality, suppose the survivor makes infinite *H-move*s. After *len*(p_G) number of *H-move*s, *H-zombie*s will get to H_x. Now for each *G-move* made by the survivor and having zombies mimicking it, they will still be on H_x. Since the survivor can do infinite *H-move*s and prevent being caught, it means that the survivor could also avoid being caught on a single H which contradicts our assumption.

□

An example for further understanding can be found in supplementary materials of this paper.

Corollary 3. $z(Q_n) \le \left\lceil \frac{2n}{3} \right\rceil$

Proof. We prove this by using both induction and Theorem 2. First note that the Cartesian product of hypercube graphs Q_m and Q_n is equal to Q_{m+n}. It is easy to see $z(Q_3) = 2$, $z(Q_2) = 2$, and $z(Q_1) = 1$. For $n > 3$, we consider Q_n as the Cartesian product of Q_3 and Q_{n-3}. Using the induction base, we know that $z(Q_{n-3}) \le \left\lceil \frac{2n-6}{3} \right\rceil$. According to Theorem 2, $z(Q_n) \le z(Q_{n-3}) + z(Q_3)$ and $z(Q_{n-3}) \le \left\lceil \frac{2n-6}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil - 2$, we can see that $z(Q_n) \le \left\lceil \frac{2n}{3} \right\rceil$. □

It is already proved that at least $\left\lceil \frac{2n}{3} \right\rceil$ zombies are needed to capture one survivor on graph Q_n (Theorem 16 of [1]):

Theorem 4. *For each integer $n \ge 1$, $z(Q_n) \ge \left\lceil \frac{2n}{3} \right\rceil$.*

Combining Corollary 3 and Theorem 4 we can conclude that $z(Q_n) = \lceil \frac{2n}{3} \rceil$. This proves Conjecture 18 from [1] which is already proved in [2] with a different method.

3. Capture time in Cartesian product of graphs

Let $rad(G)$ represent the G's radius.

Theorem 5. $capt(G \square H, z_G + z_H) < rad(G) + rad(H) + capt(G, z_G) + capt(H, z_H)$

Proof. By using z_G zombies as *G-zombies* and z_H zombies as *H-zombies*, and having them follow the same set of moves provided in Theorem 2, with the difference that initially a and b are centers of graphs H and G. We show that the survivor's *G-moves* cannot exceed $rad(H) + capt(G, z_G)$. With the same conclusion, it can be shown that *H-moves* cannot exceed $rad(G) + capt(H, z_H)$ as well.

According to the definition of the radius of a graph, after at most first $radius(H)$ *G-moves* that the survivor makes, $a = y$ holds. Now since for each *G-move* made from now on by the survivor, *G-zombies* can follow their strategy on a G graph, and after at most $capt(G, z_G)$ *G-moves*, the survivor will be captured.

Since each of the survivor's moves is either a *G-move* or an *H-move* or both, after less than $rad(G)+rad(H)+capt(G, z_G)+capt(H, z_H)$ moves the survivor will be captured. □

4. Limited capture time zombie number problem is NP-Hard

A well known example of an NP-hard problem is the dominating-set problem in graph theory[4]. A dominating-set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating-set for G.

We define $Lcz(G, k)$ (*limited capture time zombie number*) as the minimum number of zombies needed so that zombie player is able to capture the survivor in at most k moves on graph G.

Let $N_G[u]$ represent the closed neighborhood of u in graph G.

The LCZ_k problem is defined as below:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer z.
QUESTION: Is $Lcz(G, k) \leq z$?

The dominating-set problem is defined below:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer d.
QUESTION: Is $\gamma(G) \leq d$?

Theorem 6. $LCZ_k \in$ *NP-Hard*

Proof. We prove this by reducing the dominating-set problem to LCZ_k in polynomial time.

Consider k to be an arbitrary positive integer. We construct a new graph G'_k from G. Suppose G has n vertices. For each $v \in V(G)$, we add $k-1$ new vertices $(v, 1 \leq i < k)$ making a new path, $(v, (v,1), (v,2), ..., (v, k-1))$ (as shown in figure 3). Creating G'_k can be done in polynomial time.

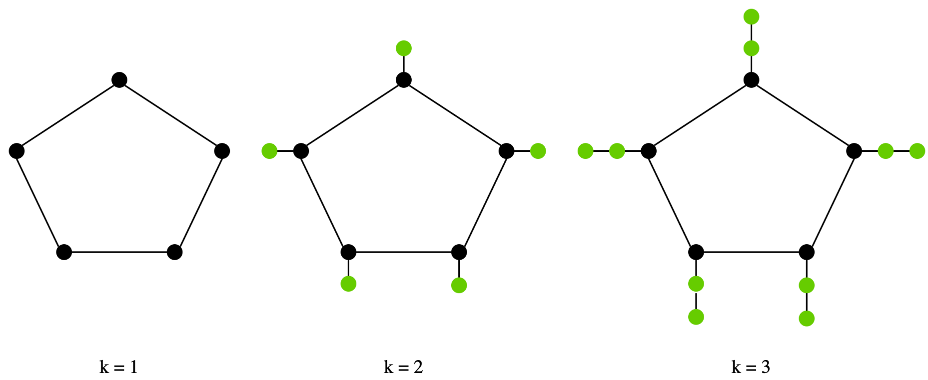


Figure 3: G'_k obtained from $G = C_5$ where for $k = 1, 2, 3$

We prove this theorem by showing $Lcz(G'_k, k) = \gamma(G)$. Consider set S' as zombies' initial vertices on G'_k, so that zombies are able to capture the survivor within k moves. Let S be the set of vertices $v \in V(G)$ such that $v \in S'$ or $(v, i) \in S'$. If there is a vertex $u \in V(G)$ not dominated by S, then there is no vertex v such that $v \in S'$ and $v \in N_{G'_k}[u] \cup \{(u, 1 \leq i < k)\}$. By having the survivor in vertex $(u, k-1)$, there is no zombie at distance k or less from him, which means he will not be captured. Thus $\gamma(G) \leq Lcz(G'_k, k)$ holds.

Now for each vertex v in one of G's smallest dominating-sets, place a zombie on vertex v of G'_k. These zombies can capture the survivor in at most k moves. To show this, consider the survivor's initial vertex, if it is not a newly added vertex, he can be captured in one move. Now suppose the survivor is initially on (u, i). Since u is dominated by a zombie, after zombies' first move, the survivor will be trapped inside the u's path, and would be captured in at most k moves. Therefore, $Lcz(G'_k, k) \leq \gamma(G)$.

By combining these results, $Lcz(G'_k, k) = \gamma(G)$. Therefore the dominating-set problem is reduced to LCZ_k.

□

5. Zombie Number Problem is NP-Hard

Now define zombie number (Z) problem:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer z.
QUESTION: Is $z(G) \leq z$?

Theorem 7. *$Z \in$ NP-Hard class.*

Proof. We reduce dominating-set problem to Z.

To do this, we add $n = |V(G)|$ copies of $K_{n,n}$ to G. We call the newly obtained graph H, and call the G-subgraph simply as G, and the i-th $(1 \leq i \leq n)$ bipartite subgraph as B_i. (i, j, b) represents the j-th vertex in B_i's part b $(b = 1, 2)$ and labels $1, 2, .., n$ represent G's vertices. For each vertex (i, j, b) we connect it to vertices in $N_G[j]$ (See figure 4). Since we are adding $2n^2$ new vertices, building H can be done in polynomial time.

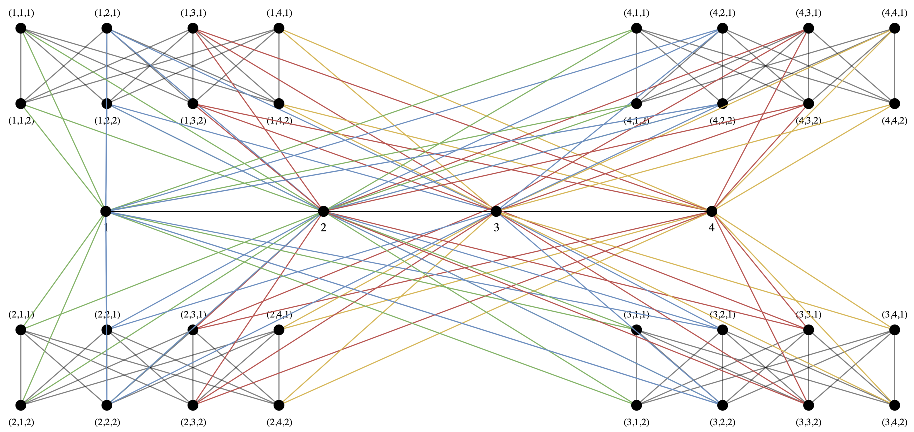


Figure 4: H obtained from P_4, zombie player needs at least $\gamma(P_4) = 2$ zombies to win

By having zombies on each vertex of G's dominating-set, the survivor will be captured on the first move and zombie player wins. Thus, $z(H) \leq \gamma(G)$. Now suppose we have zombies less than the domination number of graph: $z < \gamma(G)$. We prove the survivor can avoid being captured.

Since there are n bipartite subgraphs, one of them is initially free of zombies. Further, since zombies are not dominating G, there will be a vertex $w \in V(G)$ with no zombies as its neighbors. Thus the survivor can safely start at vertex $(k, v = w, b = 1)$ (let (k, v, b) denote the survivor's vertex).

Zombies in G (e.g. at vertex u) are at distance 2 from the survivor $(u \to (k, u, 3-b) \to (k, v, b))$, which means after their move all of them should be at one of (k, v, b)'s neighbors, that is, $N_G[v]$ or, $(k, 1 \leq i \leq n, 3-b)$. Therefore, each zombie joining B_k does not share the same partition of B_k as the survivor, since it has to be on one of its neighbors.

On each survivor turn, assume that the survivor is at (k, v, b) and zombies are either in G, or in $B_{(i \neq k)}$, or in B_k at a vertex of the form $(k, j, 3-b)$ (not in the same partition as the survivor). Since there is a vertex $w \in G$ not being dominated by zombies, the survivor should move to vertex $(k, w, 3-b)$. Since the survivor is now sharing the same partition as all zombies in B_k, and $N_G[w]$ is empty of zombies, none of his neighbors is occupied by a zombie and the survivor will be safe by following this strategy indefinitely.

It is now proved that $z(H) = \gamma(G)$, thus the dominating-set problem is reduced to Z.

□

References

[1] Fitzpatrick, Shannon L., J. Howell, Margaret-Ellen Messinger, and David A. Pike. “A deterministic version of the game of zombies and survivors on graphs.” Discrete Applied Mathematics 213 (2016): 1-12.

[2] Offner, David, and Kerry Ojakian. “Comparing the power of cops to zombies in pursuit-evasion games.” Discrete Applied Mathematics (2019).

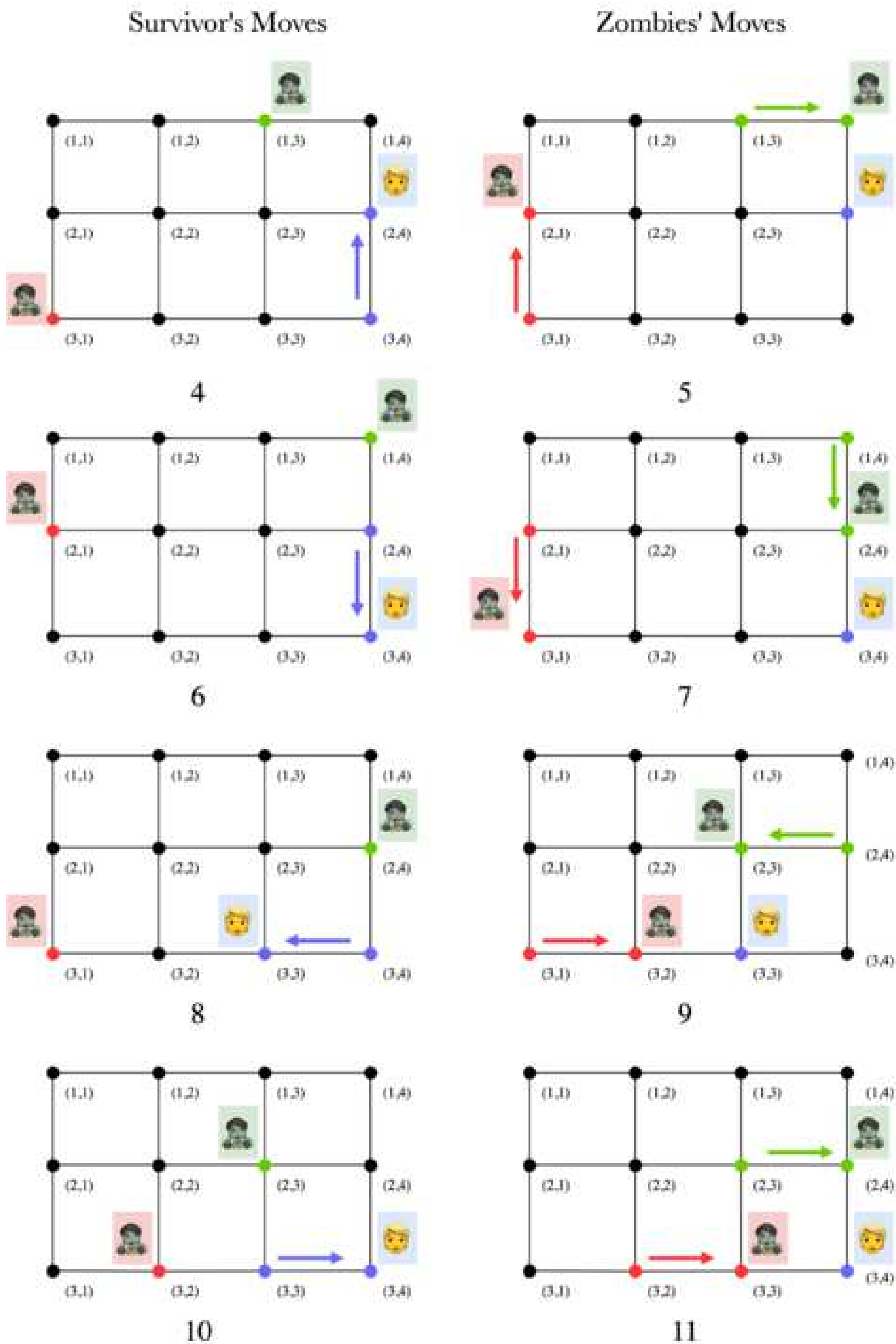
[3] West, Douglas B. “Introduction to Graph Theory.” Prentice hall, (1996).

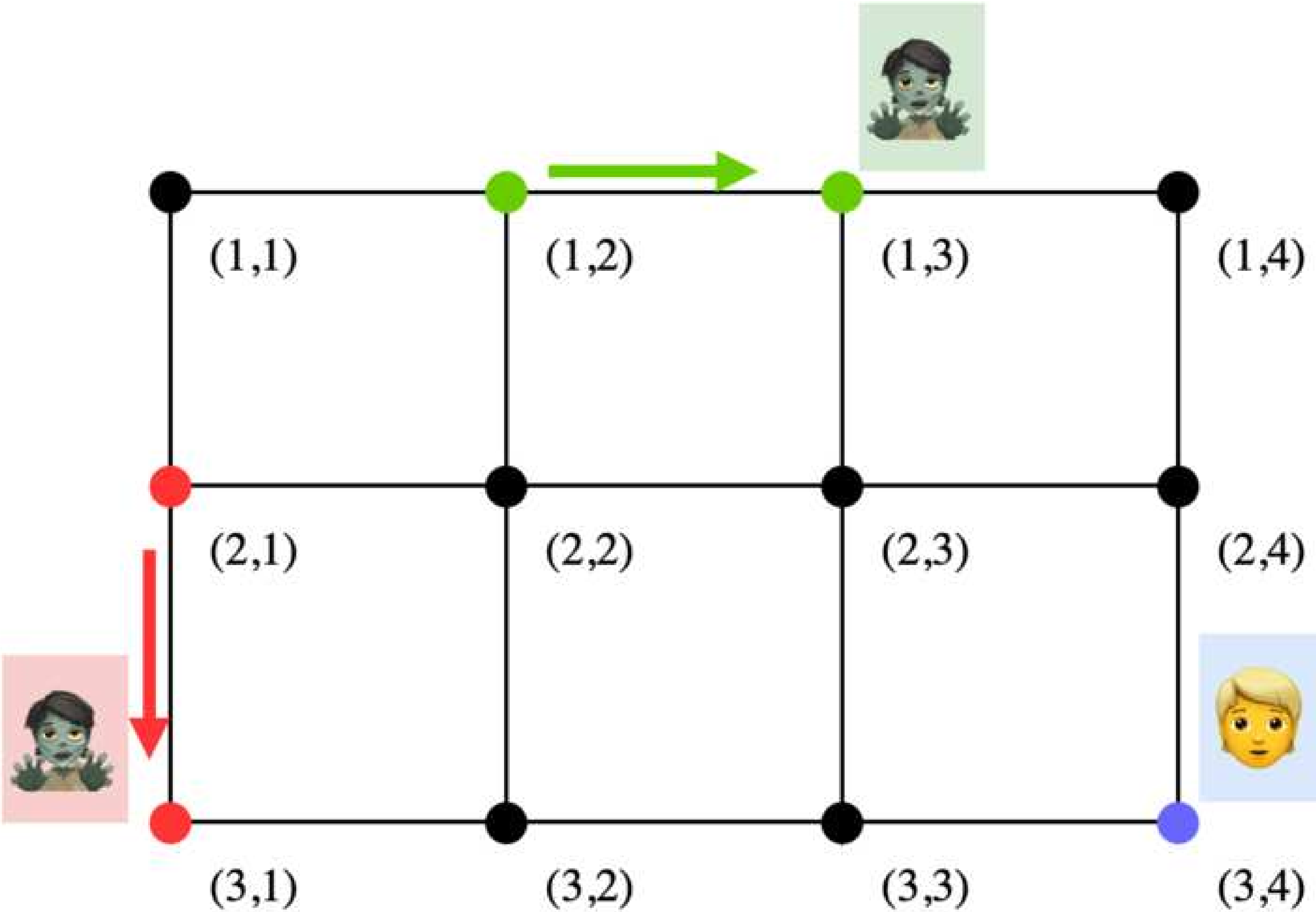
[4] Hopcroft, John E., Rajeev Motwani, and Jeffrey D. Ullman, “Introduction to Automata Theory, Languages, and Computation.” Pearson Addison-Wesley, (2007).

[5] Bonato, Anthony, Petr Golovach, Gena Hahn, and Jan Kratochvíl, “The capture time of a graph”, Discrete Mathematics 309 (2009), 5588–5595.

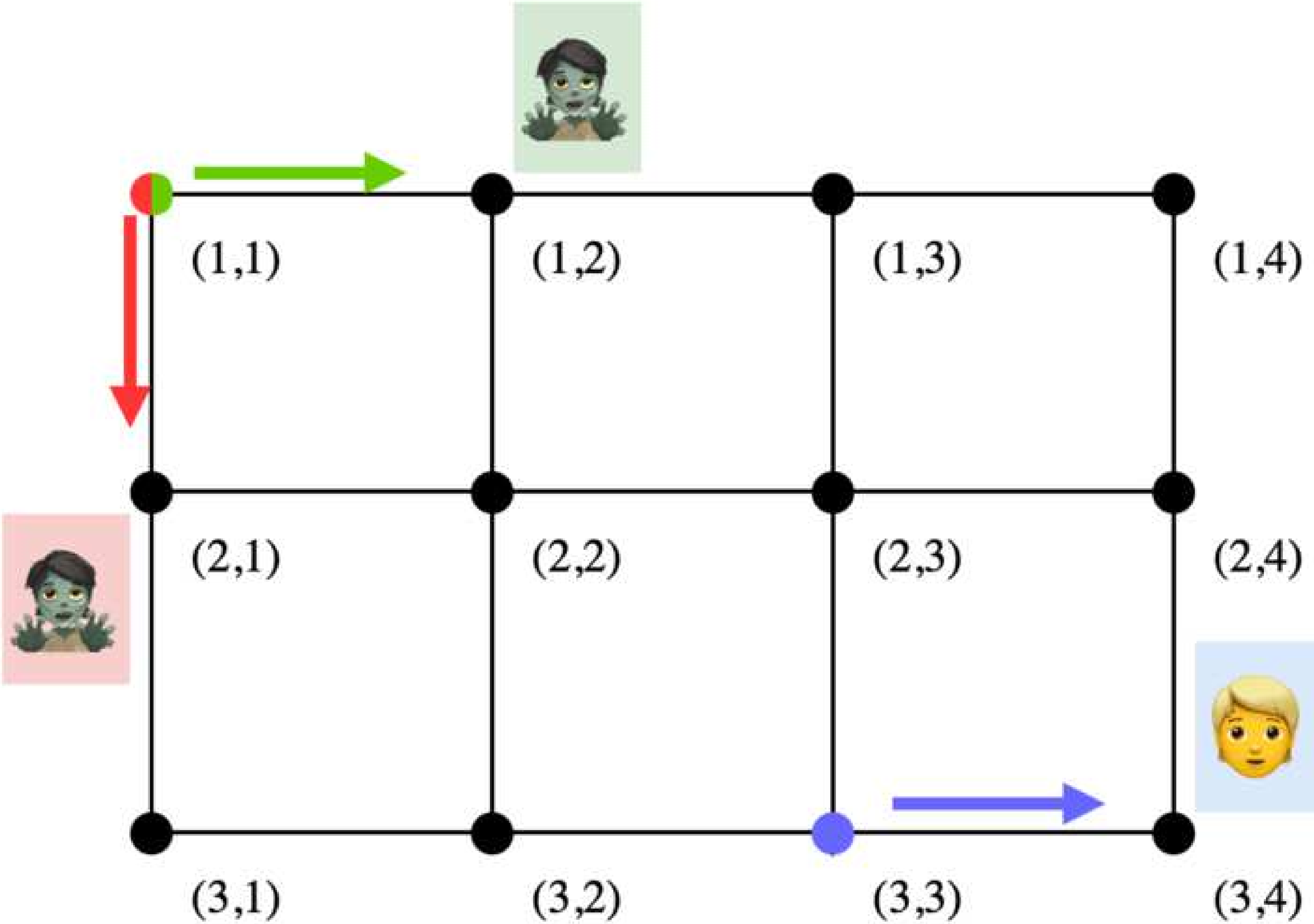
[6] Fomin, Fedor V., Petr A. Golovach, Jan Kratochvíl, Nicolas Nisse, and Karol Suchan, “Pursuing a fast robber on a graph”, Theoretical Computer Science, 411 (2010) 1167 - 1181

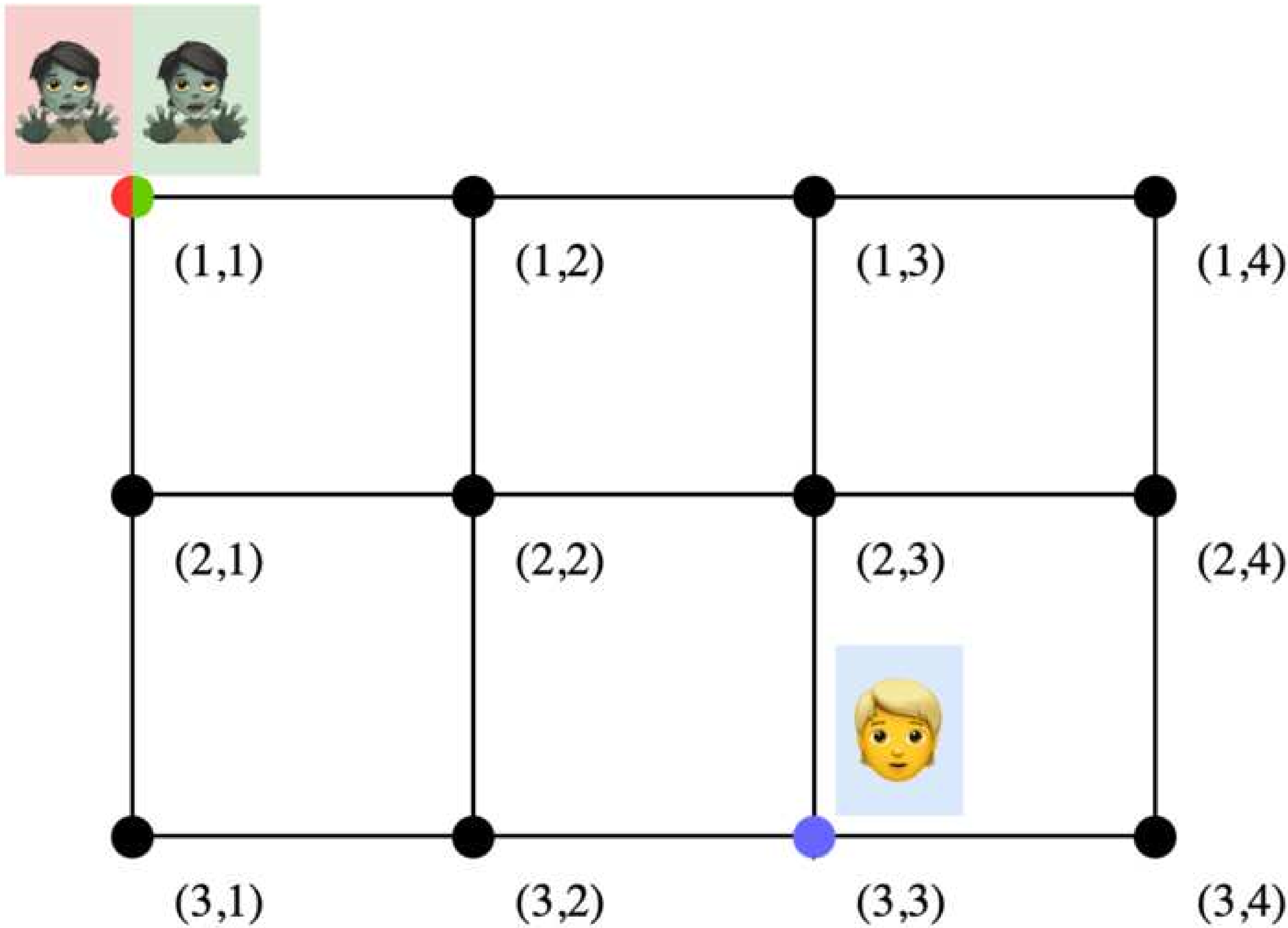
[7] W. B. Kinnersley, “Cops and Robbers is EXPTIME-complete”, Journal of Combinatorial Theory Series B, 111 (2015), 201–220.


Survivor's Moves
Zombies' Moves
4
5
6
7
8
9
10
11



(1,1)
(1,2)
(1,3)
(1,4)
(2,1)
(2,2)
(2,3)
(2,4)
(3,1)
(3,2)
(3,3)
(3,4)


(1,1)
(1,2)
(1,3)
(1,4)
(2,1)
(2,2)
(2,3)
(2,4)
(3,1)
(3,2)
(3,3)
(3,4)



(1,1)
(1,2)
(1,3)
(1,4)
(2,1)
(2,2)
(2,3)
(2,4)
(3,1)
(3,2)
(3,3)
(3,4)

A supplementary example for theorem 2

Click here to access/download
Supplementary Interactive Plot Data (CSV)
example.tex

Ali Keramatipour: Conceptualization, Formal analysis, Investigation; Methodology, Validation, Visualization, Writing – original draft

Behnam Bahrak: Project administration; Supervision; Writing – review & editing