Dear reviewers,

Thank you for taking the time to review our paper and special thanks for your invaluable comments and suggestions for improving this paper. We have carefully studied all your comments and made corrections and modification to the paper to address all of them. In response to your suggestions, we have outlined the changes and provided the necessary explanations.

In response to the comments of Reviewer 1:

1- "The proof of (*) is correct, but it's the only result in the paper (short note) and I do not think it merits publication on its own."

We added several new results to the paper on capture time and complexity of the zombies and survivor game.

2- "p.1, line 39: "as in cop-version cops should not" [this doesn't make sense]"

In “cops and rubbers” (as explained in the revised version of the paper), cops can choose any edge and move along it or keep their current positions, while in “zombies and survivor”, zombies must always move and get closer to the survivor.

3- "p.1, line 43: "player always has a winning strategy" [do the zombies exactly have a 'strategy'? They follow a simple algorithm, so it's unclear to the reader; it's also unclear what move the zombies make when faced with multiple shortest paths]:"

By having a winning strategy, we meant zombie player can always win no matter how the survivor moves. We changed the statement in this revision to clarify the sentence. When facing with multiple shortest paths, the zombies simply choose one randomly. We mentioned this in the revised version.

4- "p.1, line 45 and line 48 The word "the" should precede "Cartesian product""

We corrected this in the revised version.

5- "p.2, Figure 1: the figure is a little confusing given the numbers of the vertices in H (I assume it was copied from an example illustrating a colouring or independent set?, but given the preceding paragraph referring to indexed vertices, it could be confusing.)"

We redraw figure 1 to clarify the indices we use in the proof.

6- "p.3, line 31: the word "the" should precede "zombie number""

We corrected this in the revised version.

7- "p.4, line 17: subscript on Z(Q_2)"

We corrected this in the revised version.

8- "p.4, line 18: the word "Cartesian" is capitalized elsewhere in the proof."

We corrected this in the revised version.

In response to the comments of Reviewer 2:

Our proof was unclear and there existed some typos regarding the indices used for vertices of Cartesian product. We modified the proof in the revised version, used better notations, and provided an example in the appendix.

1- "I think that I have found some flaws in the proof given in the current manuscript. Most importantly, the premise stated on P3 L39 that "Since G-Zombies are initially on the same G_i after each move they would still be on the same G_i" is, I believe, false. To describe a situation that illustrates that this statement is false, suppose that there are 3 G-Zombies, located on three distinct vertices, say u, v, w of some G_i. By virtue of being G-Zombies, these three zombies will move within G_i before choosing to move along an H-edge. However, suppose that the survivor is located in the same copy of H as vertex v. Then the zombie at v will now move along an H-edge, whereas the zombies at u and w will move along G-edges. u,v,w will now fail to be in the same copy of G, contrary to what the authors have stated in their proof."

G-zombies will move along H-edges (HG-path in the previous version) if possible. They prioritize H-moves over G-moves. Our strategy makes them share the same G-subgraph. Now if one can do an H-move, they will all do the same.

2- "P1 L49. $G \Box H$ is defined on vertex set $V(G) \times V(H)$. Hence by fixing u in the pair (u,v) it is a copy of H, not a copy of G, that is obtained. Likewise, fixing v yields a copy of G, not H"

We corrected this in the revised version.

3- "Figure 1 shows $G \Box H$ with the copies of G taking vertical form. But in Figure 2 they are shown horizontally. I recommend using a consistent portrayal."

We corrected this in the revised version.

4- "P3 L53. Although the distance between the two stated vertices is at most n, the survivor is not guaranteed to be stationary. So it is unclear that H_y can be reached within n H-moves."

Our proof was unclear here. We changed it and also added an example in appendix to address this issue. H_y should be replaced with H_x since the survivor's H-subgraph would be H_x (we corrected this in the revised version). When survivor does an H-move, its H subgraph won't change. This means, H-zombies will follow their path and get to the survivor's H-subgraph (H_x). So after at most n H-moves by the survivor, H-zombies will be in H_x. From now on, for each G-move made by survivor, they will mimic it so they would still be in H_x. For each H-move made by survivor, H-zombies will play the game just like they would on a single H graph. Since they can win on H, survivor cannot do unlimited H-moves.

5- "P3 L54. The authors state that by making H-moves the H-Zombies will reach H_y. But if they are making H-moves, then these zombies are staying within their current copy of H."

This statement was also unclear which is corrected in this revision. We meant that after at most n H-moves done by the survivor, and not the H-zombies.

In response to the comments of Reviewer 3:

On writing suggestions:

1- "P1 L25: "We also use the proved conjecture to provide a new proof for $Z(Q_n) = \lceil 2n/3 \rceil$ " could be written more efficiently as something like "This result provides a new proof for $Z(Q_n) = \lceil 2n/3 \rceil$ " In general there are many places where the writing could be tighter, and such changes would result in a more readable manuscript. I suggest the authors go through the paper looking for such places (Phrases like "We can now see that" can be replaced by "thus" for example.)"

Thank you for your suggestions. We went through the paper and looked for such statements. We believe it is much more readable now.

2- "P1 L17: "The Zombies and Survivors" should be "Zombies and Survivors""

We corrected this in the revised version.

3- "P1 L33: "The Zombies and Survivor game" should be "The Zombies and Survivors game""

Since in this paper we are dealing with only one survivor, we used "the zombies and survivor game".

4- "P1 L36: “Starting by” should be “Starting with””

We corrected this in the revised version.

5- "P1 L45: “Cartesian product” should be “The Cartesian product””

We corrected this in the revised version.

6- "P1 L48: “Cartesian product” should be “the Cartesian product””

We corrected this in the revised version.

7- "P4 L17: Z (Q 2) should be Z(Q 2) (parenthesis should not be subscript).”

We corrected this in the revised version.

8- "P4 L19: “proved conjecture” should be “Theorem 2””

We corrected this in the revised version.

9- "P1 L40: “cop-version” should be “Cops and Robbers” (I would also advise referring to the Cop game as “Cops and Robbers” throughout, rather than sometimes as “Cops and Robber”)”

Since we are considering the game with one robber, we referred to this game as "Cops and Robber". In the new version we changed it to "Cops and Robber(s)".

Mathematical suggestions:

10- " P2 L18: You could add to the caption that this is the Cartesian product G \Box H = C_3 \Box C_4 of G = C_3 and H = C_4 . The vertex labels do not seem to have a purpose and should be removed (or made consistent with the notation used in the paper).”

We corrected this in the revised version.

11- "P2 L33: You already defined G i and H j on Page 1 so you should not repeat the definition here.”

We corrected this in the revised version.

12- "P2 L35: I would suggest adopting the convention that the vertices of G and H are numbered and just referring to the vertex (i, j) rather than V i,j (the V doesn't add any information)."

We made this change in the revised version.

13- "P3 L25: In Figure 2, H and G do not necessarily have the same number of vertices, so there should not be n copies of each."

We corrected this in the revised version.

14- "P3 L29: "Conjecture 2" should be "Theorem 2""

We corrected this in the revised version.

15- "P3 L30: The sentence before the proof does not add anything and should be removed."

We corrected this in the revised version.

16- "P2 L22: I think it is more standard to use z for zombie number rather than Z."

We used a new notation for the zombie number in this revision. Also "Z" is now used to define the zombie number problem.

17- "P2 L41: Lemma 1 is not used in the rest of the paper and should be deleted. IF it turns out to be useful, my comments about it are below."

We removed the lemma completely. We added another simple lemma which we believe is needed for understanding the proof and to show that the moves made by zombies are possible.

18- "P3 L34: There are a number of issues with the proof. The main one is that the zombie strategy is not precisely defined. Given a survivor vertex and a set of zombie vertices, the exact move made by each zombie must be clearly specified. Below I propose a way to make your idea a precise strategy. Since the strategy is not precisely defined, it also makes the proof of correctness difficult. But once you specify the strategy precisely, you need to verify that every move that the zombies make is legal, and brings them closer to their goal. One way to do this would be to introduce notation for the current G i for the G-zombies and the survivor (E.g. G z and G s , and similar for H-zombies), and note that until either the G-zombies are in the same G i as the survivor or the H zombies are in the same H j as the survivor that the distance in H edges from G s to G z plus the distance in G edges from H s to H z strictly decreases after each move. Then once G s = G z (or H s = H z) you can use a more precise argument that the robber will only be able to make a finite number of moves on G-edges (or H-edges). Once G s = G z and H s = H z the robber can pnly make a finite a number of moves of any kind before being caught.

Here is a proposal for one way to make your strategy precise. First you would define what it means for zombies to start in a winning position (can catch the survivor no matter where they are), or to be in a winning position relative to the survivor's current position, and specify that the G-zombies begin in a winning position on one of the G i 's and the H-zombies begin in a winning position for one of the H j 's

Suppose the survivor is at (x, y). Then on their turn, if the G-zombies are in G i , and i =/y, all G-zombies should move to the same G j , where j is one step closer to y in H than i. If i = y, then the G-zombies should play their winning strategy on G i (Note in the proof of correctness, you need to argue why following this strategy keeps the G-zombies in a Gwinning position–this is a subtle point, and the reason is because they do not start playing their strategy on G until they are in the same G i as the survivor, at which point they are in a winning position (because their first coordinates have not changed). From that point on, every time the survivor makes a G move, the G-zombies counter with their winning strategy on G, thus maintaining their winning position.

You can specify the strategy (and justification) for H the same way."

We really appreciate your comments for improving the proof. In this revision, we used a fixed path (p_G and p_H) for the zombies to follow. The statement "G-Zombies will move on an HG-path" is not correct, since the H-part of the HG-path needs to be unique. Otherwise G-zombies might lose their arrangement and will not be able to follow their strategy when they reach G_y. But by using a unique path, this will not be a problem.

19- Suggestions for extension:

Thanks for your suggestions. By considering the capture time and bounding it in our strategy, we could formulate a new variation of the zombies and survivors game in this revision. This new problem is similar to the original problem with the condition that zombie player must win by limited number of moves. We also proved this new problem and the original version of the game belongs to NP-Hard class of problems.

In response to the comments of Reviewer 4:

"This review concerns the paper "Zombie number of the Cartesian product of graphs" written by Ali Keramatipour and Behnam Bahrak. The paper focuses on the graph-theoretic game of Zombies and Survivors, introduced by the 2016 paper of Fitzpatrick, Howell, Messinger, and Pike. The main point of this paper is to prove a conjecture from that paper which states: z(G 2 H) ≤ z(G) + z(H), where z(G) is the zombie number of the graph G, i.e. the fewest number of zombies need to capture the survivor. Once they have proven that conjecture, they derive another conjecture as a corollary.

First, I make some small technical points. Lemma 1 is not necessary. It seems much clearer to omit the obvious Lemma 1 and simply state in the proof of Conjecture 2, something like this:

if the survivor is not in the same copy of G as the G-zombies, then the G-zombies all move into a copy of G which is closer to the survivor, while if the G-zombies are in the same copy of G as the survivor, then they all move closer to the survivor following their strategey to win in G.

While the work appears accurate, there are two serious issues with the paper. First, the writing is filled with errors of usage, grammar, etc. and the proofs should be expressed more clearly. It is beyond the demands of a reviewer of a math article to point out the many errors. I would recommend the authors attempt to get outside help proof-reading the language and presentation. Second is that the proof is rather easy, just a small modification of the corresponding proof for the cop number that $c(G\ 2\ H) \leq c(G) + c(H)$, where $c(G)$ is the cop number of graph G. If this paper was not answering a conjecture from another paper, I would recommend rejecting this paper, since the result is not that significant. However, the simple result of this paper does answer a conjecture from the 2016 paper of Fitzpatrick, Howell, Messinger, and Pike. And furthermore, armed with this conjecture, a number of results from the 2016 paper, some with involved proofs, follow immediately. It seems like an oversight of the 2016 paper to give this as a conjecture, rather than actually proving it. Thus, despite the simplicity of the result, it would certainly be worth having this result published.

Thus, I do not accept or reject this paper, but leave it up to the editors to decide if a sufficiently edited and cleaned up version of this paper is worthy of publication, or if more results would need to be added to it. If nothing is added to this paper, but it is just cleaned up, I can not really see this as a “paper”, but more of a “note” which could be useful to have in a published form.”

1- “Lemma 1 is not necessary.”

We removed the lemma completely. We added another simple lemma which we believe is needed for understanding the proof and to show that the moves made by zombies are possible

2- "First, the writing is filled with errors of usage, grammar, etc. and the proofs should be expressed more clearly.”

Thank you for your suggestions. We went through the paper and looked for such statements. We believe it is much more readable now.

3- “If nothing is added to this paper, but it is just cleaned up, I can not really see this as a “paper”, but more of a “note” which could be useful to have in a published form.”

Three new results are added to this revision. Also we submitted this work as a “note” instead of “paper”, since it was less than 10 pages and according to the journal’s policy we couldn’t submit it as a full length article.

Best Regards,

Ali Keramatipour

Behnam Bahrak