Zombie number of the Cartesian product of graphs

Ali Keramatipour, Behnam Bahrak*

School of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran, Iran

Abstract

Zombies and Survivor is a variant of the pursuit-evasion game *Cops and Robber(s)*, with the difference that zombies must always move closer to the survivor. The game is played on a simple graph by two players. The goal of the zombies is to catch the survivor while the survivor's objective is to avoid being captured. The *zombie number* of G, denoted as $z(G)$, is the minimum number of zombies required to capture a single survivor on G, no matter what moves the survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al.[1] about the zombie number of the Cartesian product of two graphs. This result provides a new proof for $z(Q_n) = \left\lceil \frac{2n}{3} \right\rceil$. We also introduce a new problem regarding *capture time* in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph G, with and without a limited *capture time*.

Keywords: Cartesian Product of Graphs, Zombie Number, NP-Hard

1. Introduction

The *Zombies and Survivor* game is played on a simple graph by two players (we only discuss connected graphs, proofs can be easily modified for disconnected graphs). The deterministic version of this game [1] is played as follows. Initially, the zombie player is given z zombies and places them on the graph's vertices. This initial arrangement that zombie player chooses is essential for him to win. Then the survivor player chooses one single vertex which is the survivor's initial position. Starting with the zombie player, on each player's turn, the zombie player must move each zombie to one of its adjacent vertices so that they get closer to the survivor, while the survivor player either moves to an adjacent vertex or stays at his current location. Here lies the difference between *Zombies and Survivor* and *Cops and Robber(s)* games, as in *Cops and Robber(s)* cops need not necessarily get closer to the robber(s). They can either hold their current position, get closer, or further away from the robber(s). Although zombies are not as intelligent

as cops, they can still choose their path intelligently between the shortest paths. If any zombie and the survivor ever occupy the same vertex, the survivor is captured and the zombie player wins. The zombie number of a graph G, denoted as $z(G)$, is the minimum number of zombies required so that the zombie player can always capture the survivor, no matter how survivor moves.

The Cartesian product $G\square H$ of two graphs G and H, is a graph with vertex set of $V(G) \times V(H)$, where vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $\{u_2, v_2\} \in E_H$, or $u_2 = v_2$ and $\{u_1, v_1\} \in E_G$ [3]. Figure 1 shows an example of the Cartesian product of two graphs.

Capture time of a game [5], noted as $capt(G, z)$, represents the maximum number of moves that the survivor can avoid being caught on a graph G, while the zombie player tries to catch the survivor as quickly as possible using z zombies. If the survivor can win, $capt(G, z)$ goes to infinity.
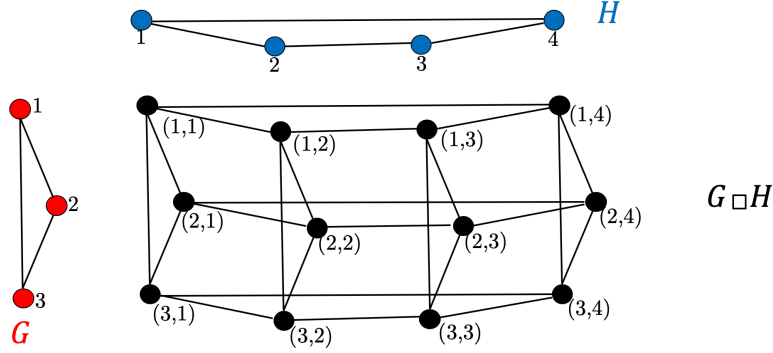

Figure 1: $C_3 \square C_4$ an example of the Cartesian product

Our contributions can be summarized as follows:

1) In [1], Fitzpatrick et al. conjectured that $z(G\square H) \leq z(G) + z(H)$. In section 2, we prove this conjecture and use it to show that $z(Q_n) = \lceil \frac{2n}{3} \rceil$.

2) In section 3, we provide a new bound on *capture time* of *Zombies and Survivor* game played on the Cartesian product of two graphs.

3) In section 4, we introduce a variation of *Zombies and Survivor* game, in which the zombie player is restricted to win in a limited number of moves, and prove it belongs to NP-Hard class of problems.

4) The complexity of the *Cops and Robber(s)* game is known [7]. In section 5, we prove that the original *Zombies and Survivor* game belongs to NP-Hard class of problems.

2. Zombie number of the Cartesian product of two graphs

To prove $z(G \square H) \leq z(G) + z(H)$, we show that $z(G) + z(H)$ zombies are enough for the zombie player to capture the survivor on $G \square H$.

To explain the proof we first need to define some notation. Assume H and G have m and n vertices, respectively.

For $1 \leq i \leq m$, define G_i to be the subgraph induced by vertices $(u, v = i)$. Similarly, for $1 \leq j \leq n$, define H_j to be the subgraph induced by vertices $(u = j, v)$.

In the Cartesian product of G and H, each G_i $(1 \leq i \leq m)$ is isomorphic to G, and each H_j $(1 \leq j \leq n)$ is isomorphic to H. Also the common vertex between G_i and H_j is (j, i), and (x, y) is the vertex where the survivor is located. Figure 2 illustrates these definitions.

A *G-edge* is an edge in one of the G_is and an *H-edge* is an edge in one of the H_js. A *G-move* is a move made on a *G-edge*. Similarly, an *H-move* is a move made on an *H-edge*. If the survivor decides to remain in its current vertex, this move is considered both a *G-move* and an *H-move*.

Define $dist_I(j, k)$ to be the distance between vertices j and k on a graph I. Define the length of a path P to be $len(P)$.

For vertex (u, v), define its *G*-equivalent vertex to be vertex $u \in G$, and its *H*-equivalent vertex, to be vertex $v \in H$. *G*-equivalent graph is a graph G where each zombie and the survivor is put on its *G*-equivalent vertex. *H*-equivalent graph is defined in the same way.
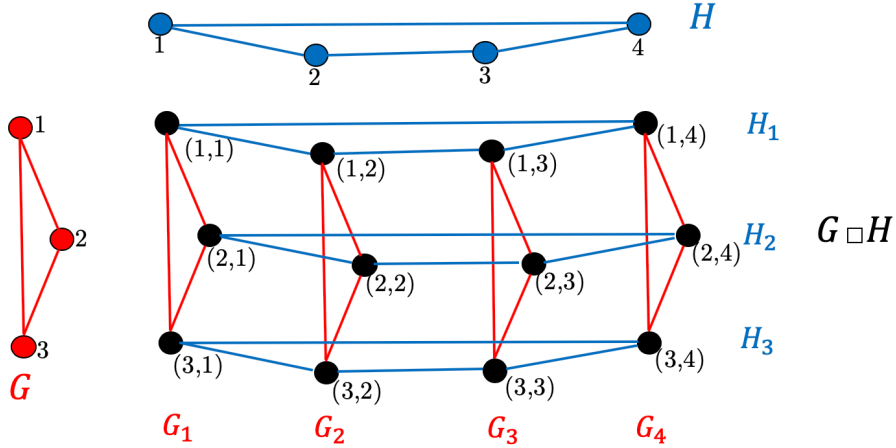

Figure 2: $G \square H$, G_is, and H_is.

We start off by stating a simple lemma regarding distance in Cartesian product:

Lemma 1. $dist_{G \square H}((x, y), (u, v)) = dist_G(x, u) + dist_H(y, v)$.

Theorem 2. $z(G \square H) \leq z(G) + z(H)$.

Proof. We provide a winning strategy for the Cartesian product of G and H using $z(G) + z(H)$ zombies. First, we place $z(G)$ zombies, that have a winning strategy on a single G, $G_{a=1}$ and call them *G-zombie*s. We do the same for $H_{b=1}$ and call them *H-zombie*s. Also recall that (x, y) is the survivor's vertex.

Consider one of the shortest paths between vertices a (the index of G-subgraph shared by *G-zombie*s) and y in H and call it p_H. We also define p_G in the same manner between b and x.

On each zombie turn, if $a \neq y$, each *G-zombie* will move along the p_H path in its corresponding H subgraph. According to lemma 1, since zombies' and the survivor's equivalents on H are getting closer, thus their actual vertices on $G \square H$ are getting closer as well and this move is possible. Since they are all moving along similar paths (in their corresponding H-subgraphs) they will still share the same G-subgraph. Now consider when $a = y$, *G-zombie*s will play their winning strategy (that they had on a single G) in this case. This move is also possible since in G's strategy, zombies would get closer to the survivor on each turn. This means for those turns that $a = y$ holds, if we consider the G-equivalent graph between *G-zombie*s and the survivor, it is just like a simple game played on a single G. *H-zombie*s will follow the same strategy but in their corresponding environment.

Suppose using this strategy the survivor wins, therefore the survivor must be able to do either infinite *G-move*s or infinite *H-move*s. Without loss of generality, suppose the survivor makes infinite *H-move*s. After $len(p_G)$ number of *H-move*s, *H-zombie*s will get to H_x. Now for each *G-move* made by the survivor and having zombies mimicking it, they will still be on H_x. Since the survivor can do infinite *H-move*s and prevent being caught, it means that the survivor could also avoid being caught on a single H which contradicts our assumption.

$\square$

An example for further understanding can be found in supplementary materials of this paper.

Corollary 3. $z(Q_n) \leq \left\lceil \frac{2n}{3} \right\rceil$

Proof. We prove this by using both induction and Theorem 2. First note that the Cartesian product of hypercube graphs Q_m and Q_n is equal to Q_{m+n}. It is easy to see $z(Q_3) = 2$, $z(Q_2) = 2$, and $z(Q_1) = 1$. For $n > 3$, we consider Q_n as the Cartesian product of Q_3 and Q_{n-3}. Using the induction base, we know that $z(Q_{n-3}) \leq \left\lceil \frac{2n-6}{3} \right\rceil$. According to Theorem 2, $z(Q_n) \leq z(Q_{n-3}) + z(Q_3)$ and $z(Q_{n-3}) \leq \left\lceil \frac{2n-6}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil - 2$, we can see that $z(Q_n) \leq \left\lceil \frac{2n}{3} \right\rceil$. $\square$

It is already proved that at least $\left\lceil \frac{2n}{3} \right\rceil$ zombies are needed to capture one survivor on graph Q_n (Theorem 16 of [1]):

Theorem 4. *For each integer $n \geq 1$, $z(Q_n) \geq \left\lceil \frac{2n}{3} \right\rceil$.*

Combining Corollary 3 and Theorem 4 we can conclude that $z(Q_n) = \left\lceil \frac{2n}{3} \right\rceil$. This proves Conjecture 18 from [1] which is already proved in [2] with a different method.

3. Capture time in Cartesian product of graphs

Let $rad(G)$ represent the G's radius.

Theorem 5. $capt(G \square H, z_G + z_H) < rad(G) + rad(H) + capt(G, z_G) + capt(H, z_H)$

Proof. By using z_G zombies as *G-zombie*s and z_H zombies as *H-zombie*s, and having them follow the same set of moves provided in Theorem 2, with the difference that initially a and b are centers of graphs H and G. We show that the survivor's *G-move*s cannot exceed $rad(H) + capt(G, z_G)$. With the same conclusion, it can be shown that *H-move*s cannot exceed $rad(G) + capt(H, z_H)$ as well.

According to the definition of the radius of a graph, after at most first $radius(H)$ *G-move*s that the survivor makes, $a = y$ holds. Now since for each *G-move* made from now on by the survivor, *G-zombie*s can follow their strategy on a G graph, and after at most $capt(G, z_G)$ *G-move*s, the survivor will be captured.

Since each of the survivor's moves is either a *G-move* or an *H-move* or both, after less than $rad(G) + rad(H) + capt(G, z_G) + capt(H, z_H)$ moves the survivor will be captured. $\square$

4. Limited capture time zombie number problem is NP-Hard

A well known example of an NP-hard problem is the dominating-set problem in graph theory[4]. A dominating-set for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating-set for G.

We define $Lcz(G, k)$ (*limited capture time zombie number*) as the minimum number of zombies needed so that zombie player is able to capture the survivor in at most k moves on graph G.

Let $N_G[u]$ represent the closed neighborhood of u in graph G.

The LCZ_k problem is defined as below:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer z.
QUESTION: Is $Lcz(G, k) \leq z$?

The dominating-set problem is defined below:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer d.
QUESTION: Is $\gamma(G) \leq d$?

Theorem 6. $LCZ_k \in$ *NP-Hard*

Proof. We prove this by reducing the dominating-set problem to LCZ_k in polynomial time.

Consider k to be an arbitrary positive integer. We construct a new graph G'_k from G. Suppose G has n vertices. For each $v \in V(G)$, we add $k-1$ new vertices $(v, 1 \leq i < k)$ making a new path, $(v, (v, 1), (v, 2), ..., (v, k-1))$ (as shown in figure 3). Creating G'_k can be done in polynomial time.
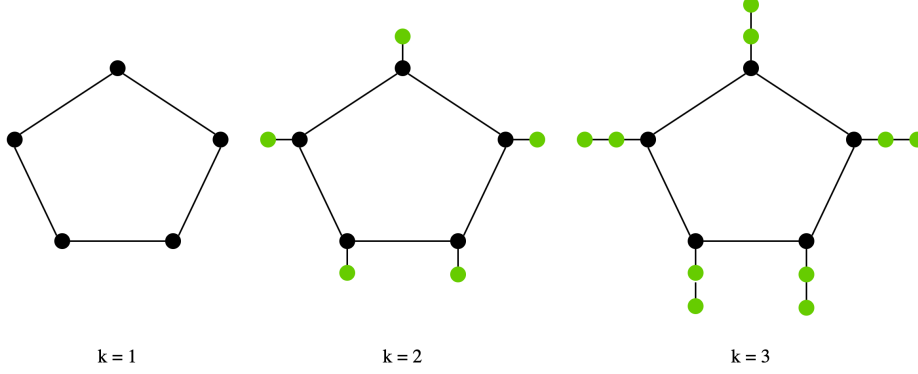

Figure 3: G'_k obtained from $G = C_5$ where for $k = 1, 2, 3$

We prove this theorem by showing $Lcz(G'_k, k) = \gamma(G)$. Consider set S' as zombies' initial vertices on G'_k, so that zombies are able to capture the survivor within k moves. Let S be the set of vertices $v \in V(G)$ such that $v \in S'$ or $(v, i) \in S'$. If there is a vertex $u \in V(G)$ not dominated by S, then there is no vertex v such that $v \in S'$ and $v \in N_{G'_k}[u] \cup \{(u, 1 \leq i < k)\}$. By having the survivor in vertex $(u, k-1)$, there is no zombie at distance k or less from him, which means he will not be captured. Thus $\gamma(G) \leq Lcz(G'_k, k)$ holds.

Now for each vertex v in one of G's smallest dominating-sets, place a zombie on vertex v of G'_k. These zombies can capture the survivor in at most k moves. To show this, consider the survivor's initial vertex, if it is not a newly added vertex, he can be captured in one move. Now suppose the survivor is initially on (u, i). Since u is dominated by a zombie, after zombies' first move, the survivor will be trapped inside the u's path, and would be captured in at most k moves. Therefore, $Lcz(G'_k, k) \leq \gamma(G)$.

By combining these results, $Lcz(G'_k, k) = \gamma(G)$. Therefore the dominating-set problem is reduced to LCZ_k.

$\square$

5. Zombie Number Problem is NP-Hard

Now define zombie number (Z) problem:
INSTANCE: A simple undirected graph $G = (V, E)$ and a positive integer z.
QUESTION: Is $z(G) \leq z$?

Theorem 7. $Z \in$ *NP-Hard class.*

Proof. We reduce dominating-set problem to Z.

To do this, we add $n = |V(G)|$ copies of $K_{n,n}$ to G. We call the newly obtained graph H, and call the G-subgraph simply as G, and the i-th ($1 \leq i \leq n$) bipartite subgraph as B_i. (i, j, b) represents the j-th vertex in B_i's part b ($b = 1, 2$) and labels $1, 2, .., n$ represent G's vertices. For each vertex (i, j, b) we connect it to vertices in $N_G[j]$ (See figure 4). Since we are adding $2n^2$ new vertices, building H can be done in polynomial time.
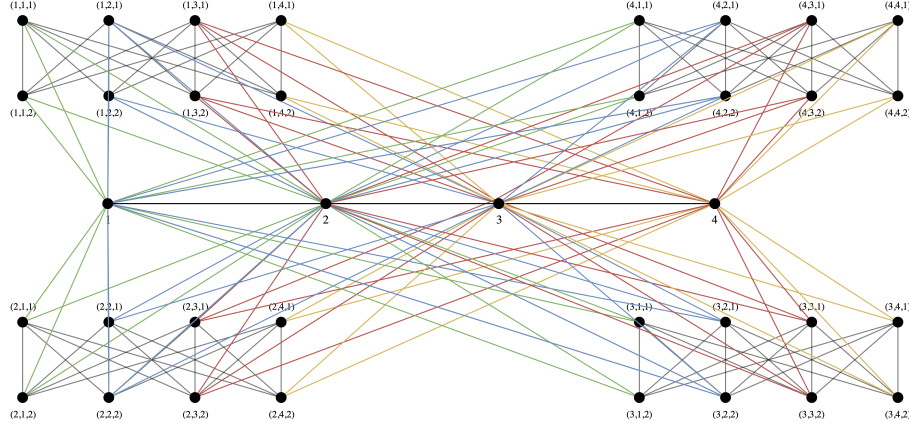

Figure 4: H obtained from P_4, zombie player needs at least $\gamma(P_4) = 2$ zombies to win

By having zombies on each vertex of G's dominating-set, the survivor will be captured on the first move and zombie player wins. Thus, $z(H) \leq \gamma(G)$. Now suppose we have zombies less than the domination number of graph: $z < \gamma(G)$. We prove the survivor can avoid being captured.

Since there are n bipartite subgraphs, one of them is initially free of zombies. Further, since zombies are not dominating G, there will be a vertex $w \in V(G)$ with no zombies as its neighbors. Thus the survivor can safely start at vertex $(k, v = w, b = 1)$ (let (k, v, b) denote the survivor's vertex).

Zombies in G (e.g. at vertex u) are at distance 2 from the survivor ($u \rightarrow (k, u, 3-b) \rightarrow (k, v, b)$), which means after their move all of them should be at one of (k, v, b)'s neighbors, that is, $N_G[v]$ or, $(k, 1 \leq i \leq n, 3 - b)$. Therefore, each zombie joining B_k does not share the same partition of B_k as the survivor, since it has to be on one of its neighbors.

On each survivor turn, assume that the survivor is at (k, v, b) and zombies are either in G, or in $B_{(i \neq k)}$, or in B_k at a vertex of the form $(k, j, 3-b)$ (not in the same partition as the survivor). Since there is a vertex $w \in G$ not being dominated by zombies, the survivor should move to vertex $(k, w, 3 - b)$. Since the survivor is now sharing the same partition as all zombies in B_k, and $N_G[w]$ is empty of zombies, none of his neighbors is occupied by a zombie and the survivor will be safe by following this strategy indefinitely.

It is now proved that $z(H) = \gamma(G)$, thus the dominating-set problem is reduced to Z.

$\square$

References

[1] Fitzpatrick, Shannon L., J. Howell, Margaret-Ellen Messinger, and David A. Pike. "A deterministic version of the game of zombies and survivors on graphs." Discrete Applied Mathematics 213 (2016): 1-12.

[2] Offner, David, and Kerry Ojakian. "Comparing the power of cops to zombies in pursuit-evasion games." Discrete Applied Mathematics (2019).

[3] West, Douglas B. "Introduction to Graph Theory." Prentice hall, (1996).

[4] J. E. Hopcroft, R. Motwani, and J. D. Ullman. "Introduction to Automata Theory, Languages, and Computation." Pearson Addison-Wesley, (2007).

[5] A. Bonato, P. Golovach, G. Hahn, and J. Kratochvıl, "The capture time of a graph", Discrete Mathematics 309 (2009), 5588–5595.

[6] Fomin, Golovach, Kratochvil, Nisse, "Pursing fast robber in graphs", Theoretical Computer Science, 411 (2010) 1167 - 1181

[7] W. B. Kinnersley, "Cops and Robbers is EXPTIME-complete", Journal of Combinatorial Theory Series B, 111 (2015), 201–220.