

# Discrete Applied Mathematics

## Zombie number of the Cartesian product of graphs

--Manuscript Draft--

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Abstract:	<p>Zombies and Survivors is a variant of the pursuit-evasion game Cops and Robbers , with the difference that zombies must always move closer to one of their closest survivors. The game is played on a simple graph by two players. The goal of the zombies is to catch the survivors while survivors' objective is to avoid being captured. The zombie number of <math>G</math>, denoted as <math>zn(G)</math>, is the minimum number of zombies required to capture a single survivor on <math>G</math>, no matter what moves survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al. \cite{Fitz16} about the zombie number of the Cartesian product of two graphs. This result provides a new proof for <math>zn(Q_n) = \lceil \frac{2n}{3} \rceil</math>. We also introduce a new problem regarding capture time in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph <math>G</math>, with and without a limited capture time .</p>

Dear reviewers,

Thank you for taking the time to review our paper and special thanks for your invaluable comments and suggestions for improving this paper. We have carefully studied all your comments and made corrections and modification to the paper to address all of them. In response to your suggestions, we have outlined the changes and provided the necessary explanations.

**In response to the comments of Reviewer 1:**

1- "The proof of (\*) is correct, but it's the only result in the paper (short note) and I do not think it merits publication on its own."

We added several new results to the paper on capture time and complexity of the zombies and survivor game.

2- "p.1, line 39: "as in cop-version cops should not" [this doesn't make sense]"

In "cops and robbers" (as explained in the revised version of the paper), cops can choose any edge and move along it or keep their current positions, while in "zombies and survivor", zombies must always move and get closer to the survivor.

3- "p.1, line 43: "player always has a winning strategy" [do the zombies exactly have a 'strategy'? They follow a simple algorithm, so it's unclear to the reader; it's also unclear what move the zombies make when faced with multiple shortest paths]:"

By having a winning strategy, we meant zombie player can always win no matter how the survivor moves. We changed the statement in this revision to clarify the sentence. When facing with multiple shortest paths, the zombies simply choose one randomly. We mentioned this in the revised version.

4- "p.1, line 45 and line 48 The word "the" should precede "Cartesian product""

We corrected this in the revised version.

5- "p.2, Figure 1: the figure is a little confusing given the numbers of the vertices in H (I assume it was copied from an example illustrating a colouring or independent set?, but given the preceding paragraph referring to indexed vertices, it could be confusing.)"

We redraw figure 1 to clarify the indices we use in the proof.

6- "p.3, line 31: the word "the" should precede "zombie number""

We corrected this in the revised version.

7- "p.4, line 17: subscript on  $Z(Q_2)$ "

We corrected this in the revised version.

8- "p.4, line 18: the word "Cartesian" is capitalized elsewhere in the proof."

We corrected this in the revised version.

#### **In response to the comments of Reviewer 2:**

Our proof was unclear and there existed some typos regarding the indices used for vertices of Cartesian product. We modified the proof in the revised version, used better notations, and provided an example in the appendix.

1- "I think that I have found some flaws in the proof given in the current manuscript. Most importantly, the premise stated on P3 L39 that "Since  $G_i$ -Zombies are initially on the same  $G_i$  after each move they would still be on the same  $G_i$ " is, I believe, false. To describe a situation that illustrates that this statement is false, suppose that there are 3  $G$ -Zombies, located on three distinct vertices, say  $u, v, w$  of some  $G_i$ . By virtue of being  $G$ -Zombies, these three zombies will move within  $G_i$  before choosing to move along an  $H$ -edge. However, suppose that the survivor is located in the same copy of  $H$  as vertex  $v$ . Then the zombie at  $v$  will now move along an  $H$ -edge, whereas the zombies at  $u$  and  $w$  will move along  $G$ -edges.  $u, v, w$  will now fail to be in the same copy of  $G$ , contrary to what the authors have stated in their proof."

$G$ -zombies will move along  $H$ -edges ( $HG$ -path in the previous version) if possible. They prioritize  $H$ -moves over  $G$ -moves. Our strategy makes them share the same  $G$ -subgraph. Now if one can do an  $H$ -move, they will all do the same.

2- "P1 L49.  $G \times H$  is defined on vertex set  $V(G) \times V(H)$ . Hence by fixing  $u$  in the pair  $(u, v)$  it is a copy of  $H$ , not a copy of  $G$ , that is obtained. Likewise, fixing  $v$  yields a copy of  $G$ , not  $H$ "

We corrected this in the revised version.

3- "Figure 1 shows  $G \times H$  with the copies of  $G$  taking vertical form. But in Figure 2 they are shown horizontally. I recommend using a consistent portrayal."

We corrected this in the revised version.

4- "P3 L53. Although the distance between the two stated vertices is at most  $\$n\$$ , the survivor is not guaranteed to be stationary. So it is unclear that  $\$H\_y\$$  can be reached within  $\$n\$$   $\$H\$$ -moves."

Our proof was unclear here. We changed it and also added an example in appendix to address this issue.  $\$H\_y\$$  should be replaced with  $\$H\_x\$$  since the survivor's H-subgraph would be  $\$H\_x\$$  (we corrected this in the revised version). When survivor does an H-move, its H subgraph won't change. This means, H-zombies will follow their path and get to the survivor's H-subgraph ( $\$H\_x\$$ ). So after at most  $\$n\$$  H-moves by the survivor, H-zombies will be in  $\$H\_x\$$ . From now on, for each G-move made by survivor, they will mimic it so they would still be in  $\$H\_x\$$ . For each H-move made by survivor, H-zombies will play the game just like they would on a single H graph. Since they can win on H, survivor cannot do unlimited H-moves.

5- "P3 L54. The authors state that by making  $\$H\$$ -moves the  $\$H\$$ -Zombies will reach  $\$H\_y\$$ . But if they are making  $\$H\$$ -moves, then these zombies are staying within their current copy of  $\$H\$$ ."

This statement was also unclear which is corrected in this revision. We meant that after at most  $\$n\$$  H-moves done by the survivor, and not the H-zombies.

### **In response to the comments of Reviewer 3:**

On writing suggestions:

1- "P1 L25: "We also use the proved conjecture to provide a new proof for  $Z(Q\ n) = \lceil 2n/3 \rceil$  " could be written more efficiently as something like "This result provides a new proof for  $Z(Q\ n) = \lceil 2n/3 \rceil$  " In general there are many places where the writing could be tighter, and such changes would result in a more readable manuscript. I suggest the authors go through the paper looking for such places (Phrases like "We can now see that" can be replaced by "thus" for example.)"

Thank you for your suggestions. We went through the paper and looked for such statements. We believe it is much more readable now.

2- "P1 L17: "The Zombies and Survivors" should be "Zombies and Survivors""

We corrected this in the revised version.

3- "P1 L33: "The Zombies and Survivor game" should be "The Zombies and Survivors game""

Since in this paper we are dealing with only one survivor, we used "the zombies and survivor game".

4- "P1 L36: "Starting by" should be "Starting with""

We corrected this in the revised version.

5- "P1 L45: "Cartesian product" should be "The Cartesian product""

We corrected this in the revised version.

6- "P1 L48: "Cartesian product" should be "the Cartesian product""

We corrected this in the revised version.

7- "P4 L17:  $Z(Q^2)$  should be  $Z(Q^2)$  (parenthesis should not be subscript)."

We corrected this in the revised version.

8- "P4 L19: "proved conjecture" should be "Theorem 2""

We corrected this in the revised version.

9- "P1 L40: "cop-version" should be "Cops and Robbers" ( I would also advise referring to the Cop game as "Cops and Robbers" throughout, rather than sometimes as "Cops and Robber")"

Since we are considering the game with one robber, we referred to this game as "Cops and Robber". In the new version we changed it to "Cops and Robber(s)".

Mathematical suggestions:

10- " P2 L18: You could add to the caption that this is the Cartesian product  $G \boxtimes H = C_3 \boxtimes C_4$  of  $G = C_3$  and  $H = C_4$  . The vertex labels do not seem to have a purpose and should be removed (or made consistent with the notation used in the paper)."

We corrected this in the revised version.

11- "P2 L33: You already defined  $G_i$  and  $H_j$  on Page 1 so you should not repeat the definition here."

We corrected this in the revised version.

12- "P2 L35: I would suggest adopting the convention that the vertices of G and H are numbered and just referring to the vertex (i, j) rather than  $V_{i,j}$  (the V doesn't add any information)."

We made this change in the revised version.

13- "P3 L25: In Figure 2, H and G do not necessarily have the same number of vertices, so there should not be n copies of each."

We corrected this in the revised version.

14- "P3 L29: "Conjecture 2" should be "Theorem 2"'"

We corrected this in the revised version.

15- "P3 L30: The sentence before the proof does not add anything and should be removed."

We corrected this in the revised version.

16- "P2 L22: I think it is more standard to use z for zombie number rather than Z."

We used a new notation for the zombie number in this revision. Also "Z" is now used to define the zombie number problem.

17- "P2 L41: Lemma 1 is not used in the rest of the paper and should be deleted. IF it turns out to be useful, my comments about it are below."

We removed the lemma completely. We added another simple lemma which we believe is needed for understanding the proof and to show that the moves made by zombies are possible.

18- "P3 L34: There are a number of issues with the proof. The main one is that the zombie strategy is not precisely defined. Given a survivor vertex and a set of zombie vertices, the exact move made by each zombie must be clearly specified. Below I propose a way to make your idea a precise strategy. Since the strategy is not precisely defined, it also makes the proof of correctness difficult. But once you specify the strategy precisely, you need to verify that every move that the zombies make is legal, and brings them closer to their goal. One way to do this would be to introduce notation for the current  $G_i$  for the G-zombies and the survivor (E.g.  $G_z$  and  $G_s$ , and similar for H-zombies), and note that until either the G-zombies are in the same  $G_i$  as the survivor or the H zombies are in the same  $H_j$  as the survivor that the distance in H edges from  $G_s$  to  $G_z$  plus the distance in G edges from  $H_s$  to  $H_z$  strictly decreases after each move. Then once  $G_s = G_z$  (or  $H_s = H_z$ ) you can use a more precise argument that

the robber will only be able to make a finite number of moves on G-edges (or H-edges). Once  $G_s = G_z$  and  $H_s = H_z$  the robber can only make a finite number of moves of any kind before being caught.

Here is a proposal for one way to make your strategy precise. First you would define what it means for zombies to start in a winning position (can catch the survivor no matter where they are), or to be in a winning position relative to the survivor's current position, and specify that the G-zombies begin in a winning position on one of the  $G_i$ 's and the H-zombies begin in a winning position for one of the  $H_j$ 's

Suppose the survivor is at  $(x, y)$ . Then on their turn, if the G-zombies are in  $G_i$ , and  $i \neq y$ , all G-zombies should move to the same  $G_j$ , where  $j$  is one step closer to  $y$  in  $H$  than  $i$ . If  $i = y$ , then the G-zombies should play their winning strategy on  $G_i$  (Note in the proof of correctness, you need to argue why following this strategy keeps the G-zombies in a winning position—this is a subtle point, and the reason is because they do not start playing their strategy on  $G$  until they are in the same  $G_i$  as the survivor, at which point they are in a winning position (because their first coordinates have not changed). From that point on, every time the survivor makes a  $G$  move, the G-zombies counter with their winning strategy on  $G$ , thus maintaining their winning position.

You can specify the strategy (and justification) for  $H$  the same way."

We really appreciate your comments for improving the proof. In this revision, we used a fixed path ( $p_G$  and  $p_H$ ) for the zombies to follow. The statement "G-Zombies will move on an HG-path" is not correct, since the H-part of the HG-path needs to be unique. Otherwise G-zombies might lose their arrangement and will not be able to follow their strategy when they reach  $p_y$ . But by using a unique path, this will not be a problem.

19- Suggestions for extension:

Thanks for your suggestions. By considering the capture time and bounding it in our strategy, we could formulate a new variation of the zombies and survivors game in this revision. This new problem is similar to the original problem with the condition that zombie player must win by limited number of moves. We also proved this new problem and the original version of the game belongs to NP-Hard class of problems.

#### **In response to the comments of Reviewer 4:**

"This review concerns the paper "Zombie number of the Cartesian product of graphs" written by Ali Keramatipour and Behnam Bahrak. The paper focuses on the graph-theoretic game of Zombies and Survivors, introduced by the 2016 paper of Fitzpatrick, Howell, Messinger, and Pike. The main point of this paper is to prove a conjecture from that paper which states:  $z(G \times H) \leq z(G) + z(H)$ , where  $z(G)$  is the zombie number of the graph  $G$ , i.e. the fewest number of zombies need to capture the survivor. Once they have proven that conjecture, they derive another conjecture as a corollary.

First, I make some small technical points. Lemma 1 is not necessary. It seems much clearer to omit the obvious Lemma 1 and simply state in the proof of Conjecture 2, something like this:

if the survivor is not in the same copy of  $G$  as the G-zombies, then the G-zombies all move into a copy of  $G$  which is closer to the survivor, while if the G-zombies are in the same copy of  $G$  as the survivor, then they all move closer to the survivor following their strategy to win in  $G$ .

While the work appears accurate, there are two serious issues with the paper. First, the writing is filled with errors of usage, grammar, etc. and the proofs should be expressed more clearly. It is beyond the demands of a reviewer of a math article to point out the many errors. I would recommend the authors attempt to get outside help proof-reading the language and presentation. Second is that the proof is rather easy, just a small modification of the corresponding proof for the cop number that  $c(G \cup H) \leq c(G) + c(H)$ , where  $c(G)$  is the cop number of graph  $G$ . If this paper was not answering a conjecture from another paper, I would recommend rejecting this paper, since the result is not that significant. However, the simple result of this paper does answer a conjecture from the 2016 paper of Fitzpatrick, Howell, Messenger, and Pike. And furthermore, armed with this conjecture, a number of results from the 2016 paper, some with involved proofs, follow immediately. It seems like an oversight of the 2016 paper to give this as a conjecture, rather than actually proving it. Thus, despite the simplicity of the result, it would certainly be worth having this result published.

Thus, I do not accept or reject this paper, but leave it up to the editors to decide if a sufficiently edited and cleaned up version of this paper is worthy of publication, or if more results would need to be added to it. If nothing is added to this paper, but it is just cleaned up, I can not really see this as a “paper”, but more of a “note” which could be useful to have in a published form.”

1- “Lemma 1 is not necessary.”

We removed the lemma completely. We added another simple lemma which we believe is needed for understanding the proof and to show that the moves made by zombies are possible

2- “First, the writing is filled with errors of usage, grammar, etc. and the proofs should be expressed more clearly.”

Thank you for your suggestions. We went through the paper and looked for such statements. We believe it is much more readable now.

3- “If nothing is added to this paper, but it is just cleaned up, I can not really see this as a “paper”, but more of a “note” which could be useful to have in a published form.”

Three new results are added to this revision. Also we submitted this work as a “note” instead of “paper”, since it was less than 10 pages and according to the journal’s policy we couldn’t submit it as a full length article.

Best Regards,

Ali Keramatipour

Behnam Bahrak



# Zombie number of the Cartesian product of graphs

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## Abstract

*Zombies and Survivors* is a variant of the pursuit-evasion game *Cops and Robbers*, with the difference that zombies must always move closer to one of their closest survivors. The game is played on a simple graph by two players. The goal of the zombies is to catch the survivors while survivors' objective is to avoid being captured. The *zombie number* of  $G$ , denoted as  $z(G)$ , is the minimum number of zombies required to capture a single survivor on  $G$ , no matter what moves survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al.[1] about the zombie number of the *Cartesian product* of two graphs. This result provides a new proof for  $z(Q_n) = \lceil \frac{2n}{3} \rceil$ . We also introduce a new problem regarding *capture time* in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph  $G$ , with and without a limited *capture time*.

*Key words:* Cartesian Product of Graphs, Zombie Number, NP-Hard

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## 1. Introduction

The *Zombies and Survivors* game is played on a simple graph by two players. The deterministic version of this game [1] is played as follows (note that we only consider the game with a single survivor). Initially, the zombie player chooses a number  $z_c$  and places  $z_c$  zombies on the graph vertices. Then the survivor player chooses one single vertex which is the survivor's initial position. Starting with the zombie player, on each player's turn, the survivor player either moves to an adjacent vertex or stays at his current location, while zombie player must move each zombie to one of its adjacent vertices so that they get closer to the survivor. Here lies the difference between *Zombies and Survivor* and *Cops and Robber(s)* games, as in *Cops and Robber(s)* cops should not necessarily get closer to the robber(s), they can either hold their current position, get closer, or further away from the robber(s). Although zombies are not as intelligent as cops, they can still choose their path intelligently between the shortest paths. If any

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zombie and the survivor ever occupy the same vertex, the survivor is captured and the zombie player wins. The zombie number of a graph  $G$ , denoted as  $z(G)$ , is the minimum number of zombies required so that the zombie player can always capture the survivor, no matter how survivor moves.

The Cartesian product  $G \square H$  of two graphs  $G$  and  $H$ , is a graph with vertex set of  $V(G) \times V(H)$ , where vertices  $(u_1, u_2)$  and  $(v_1, v_2)$  are adjacent if and only if  $u_1 = v_1$  and  $\{u_2, v_2\} \in E_H$ , or  $u_2 = v_2$  and  $\{u_1, v_1\} \in E_G$  [3]. Figure 1 shows an example of the Cartesian product of two graphs.

*Capture time* in a game, is the maximum number of moves survivor can avoid being captured, while zombie player has  $z_c$  zombies. If this goes to infinity, a survivor-win play exist.

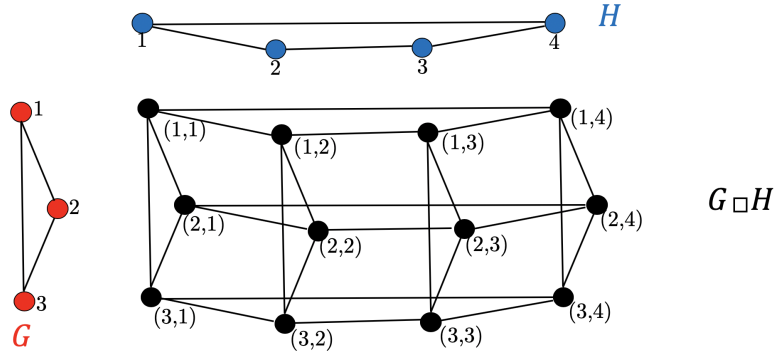


Figure 1:  $C_3 \square C_4$  an example of the Cartesian product

Our contributions can be summarized as follows:

1) In [1], Fitzpatrick et al. conjectured that  $z(G \square H) \leq z(G) + z(H)$ . In 2 We prove this conjecture and use it to show that  $z(Q_n) = \lceil \frac{2n}{3} \rceil$ .

2) In 3, we provide a new bound on *capture time of zombies and survivor* game played on the Cartesian product of two graphs.

3) We introduce a variation of *zombies and survivor* game, in which the zombie player is restricted to winning in a limited number of moves in 4, and prove it belongs to *NP-Hard* class of problems.

4) In 5, we prove that the original *zombies and survivor* game belongs to *NP-Hard* class of problems.

## 2. Zombie number of the Cartesian product of two graphs

To prove  $z(G \square H) \leq z(G) + z(H)$ , we show that  $z(G) + z(H)$  zombies are enough for the zombie player to capture the survivor on  $G \square H$ .

To explain the proof we first need to define some notations. Assume  $H$  and  $G$  have  $m$  and  $n$  vertices, respectively.

We define  $G_i$  as the induced subgraph by all vertices  $(u, v)$  in  $G \square H$ , where  $v = i$ . Similarly  $H_j$  is defined as the induced subgraph by vertices  $(u, v)$  in  $G \square H$ , where  $u = j$ .

In the Cartesian product of  $G$  and  $H$ , each  $G_i$  ( $1 \leq i \leq m$ ) is isomorphic to  $G$ , and each  $H_j$  ( $1 \leq j \leq n$ ) is isomorphic to  $H$ . We name the common vertex between  $G_i$  and  $H_j$ ,  $(j, i)$ . Also  $(x, y)$  is the vertex where the survivor is located. Figure 2 illustrates these definitions.

A  $G$ -edge is an edge in one of the  $G_i$ s and an  $H$ -edge is an edge in one of the  $H_j$ s. A  $G$ -move is a move made on a  $G$ -edge. Similarly, an  $H$ -Move is a move made on an  $H$ -edge. If the survivor decides to remain in its current vertex, this move is considered both a  $G$ -move and an  $H$ -move.

$dist_I(j, k)$  is the distance between  $j$  and  $k$  vertices on a graph  $I$ . Length of a path  $P$  is shown by  $len(P)$ .

For vertex  $(u, v)$ ,  $G$ -equivalent vertex, is vertex  $u$  on graph  $G$ , and  $H$ -equivalent vertex, is vertex  $v$  on graph  $H$ .  $G$ -equivalent graph is a graph where we put each zombie and the survivor on its  $G$ -equivalent vertex.  $H$ -equivalent graph is defined in the same way.

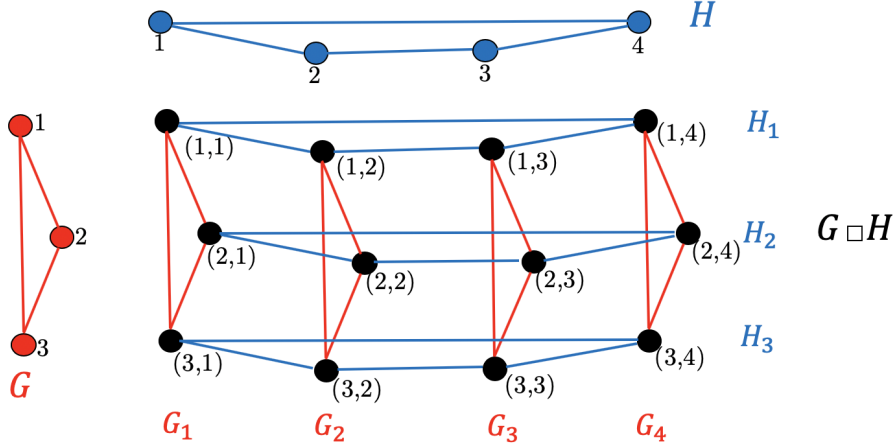


Figure 2:  $G \square H$ ,  $G_i$ s, and  $H_i$ s.

**Lemma 1.**  $dist_{G \square H}((x, y), (u, v)) = dist_G(x, u) + dist_H(y, v)$ .

*Proof.* Since there is a path from  $(x, y)$  to  $(x, v)$  with length  $dist_H(y, v)$  and a path from  $(x, v)$  to  $(u, v)$  of length  $dist_G(x, u)$ , we only need to prove there can be no path with length less than  $dist_G(x, u) + dist_H(y, v)$ . Suppose not, this path uses some  $G$ -moves and some  $H$ -moves. If a  $G$ -move is followed by an  $H$ -move (or vice-versa), then we can swap these moves and still end up in the same vertex. For example if  $(u_1, v_1) \rightarrow (u_2, v_1) \rightarrow$

( $u_2, v_2$ ) is happening, we can do  $(u_1, v_1) \rightarrow (u_1, v_2) \rightarrow (u_2, v_2)$ . Since we can swap each two moves of different type, suppose all  $G$ -moves happen before all  $H$ -moves in the shortest path. Since this path has a length less than  $\text{dist}_G(x, u) + \text{dist}_H(y, v)$ , we have found a path in either  $G$  between  $x$  and  $u$ , with length less than  $\text{dist}_G(x, u)$ , or in  $H$  between  $y$  and  $v$ , with length less than  $\text{dist}_H(y, v)$ , which is a contradiction. Thus the statement holds.  $\square$

**Theorem 2.**  $z(G \square H) \leq z(G) + z(H)$ .

*Proof.* We provide a winning strategy for the Cartesian product of  $G$  and  $H$  using  $z(G) + z(H)$  zombies. First, we place  $z(G)$  zombies, that have a winning strategy on a single  $G$ ,  $G_{a=1}$  and call them  $G$ -zombies. We do the same for  $H_{b=1}$  and call them  $H$ -zombies.

Consider one of the shortest paths between vertices  $a$  (the index of  $G$ -subgraph shared by  $G$ -zombies) and  $y$  in  $H$  and call it  $p_H$ . We also define  $p_G$  in the same manner between  $b$  and  $x$ .

On each zombie turn, if  $a \neq y$ , each  $G$ -zombie will move along the  $p_H$  path in its corresponding  $H$  subgraph. According to Lemma 1, since zombies' and survivor's equivalents on  $H$  are getting closer, thus their actual vertices on  $G \square H$  are getting closer as well and this move is possible. Since they are all moving along similar paths (in their corresponding  $H$ -subgraphs) they will still share the same  $G$ -subgraph. Now consider when  $a = y$ ,  $G$ -zombies will play their winning strategy (that they had on a single  $G$ ) in this case. This move is also possible since in  $G$ 's strategy, zombies would get closer to survivor on each turn. If  $a = y$  and survivor makes an  $H$ -move,  $G$ -zombies will maintain their positioning by mimicking the exact same move on their corresponding  $H$ . This means for those turns that  $a = y$  holds, if we consider the  $G$ -equivalent graph between  $G$ -zombies and the survivor, it is just like a simple game played on a single  $G$ .  $H$ -zombies will follow the same strategy but in their corresponding environment.

Suppose using this strategy  $G \square H$  is *survivor-win*, then the survivor must either do infinite  $G$ -moves or infinite  $H$ -moves. Without loss of generality, suppose the survivor makes infinite  $H$ -moves, we prove that this is not possible. After  $\text{len}(p_G)$  number of  $H$ -moves,  $H$ -zombies will get to  $H_x$ . Now for each  $G$ -move made by survivor and having zombies chasing him, nothing changes in their  $H$ -equivalent graph. Since the survivor can do infinite  $H$ -moves and prevent being caught, it means that the survivor could also avoid being caught on a single  $H$  which contradicts our assumption.  $\square$

An example for further understanding can be found at Appendix A.

**Corollary 3.**  $z(Q_n) \leq \lceil \frac{2n}{3} \rceil$

*Proof.* We prove this by using both induction and the theorem proved above. First note that the Cartesian product of hypercube graphs  $Q_m$  and  $Q_n$  is equal to  $Q_{m+n}$ . It is easy to see  $z(Q_3) = 2$ ,  $z(Q_2) = 2$ , and  $z(Q_1) = 1$ . For  $n > 3$ , we consider  $Q_n$  as the Cartesian product of  $Q_3$  and  $Q_{n-3}$ . Using the induction base, we know that  $z(Q_{n-3}) \leq \lceil \frac{2n-6}{3} \rceil$ . According to the proved conjecture  $z(Q_n) \leq z(Q_{n-3}) + z(Q_3)$  and  $z(Q_{n-3}) \leq \lceil \frac{2n-6}{3} \rceil = \lceil \frac{2n}{3} \rceil - 2$ , we can see that  $z(Q_n) \leq \lceil \frac{2n}{3} \rceil$ .  $\square$

It is already proved that at least  $\lceil \frac{2n}{3} \rceil$  zombies are needed to capture one survivor on graph  $Q_n$  (Theorem 16 of [1]):

**Theorem 4.** *For each integer  $n \geq 1$ ,  $z(Q_n) \geq \lceil \frac{2n}{3} \rceil$ .*

Combining *Corollary 3* and *Theorem 4* we can conclude that  $z(Q_n) = \lceil \frac{2n}{3} \rceil$ . This proves Conjecture 18 from [1] which is already proved in [2] with a different method.

### 3. Capture time in Cartesian product of graphs

We define a new parameter, *capture time* of the game, noted as  $\text{capt}(G, z_c)$ , which represents the maximum number of moves that survivor can avoid being caught on a graph  $G$ , when the zombie player uses  $z_c$  zombies. Zombie player tries to make  $\text{capt}(G, z_c)$  as least as possible, while survivor tries to maximize it. Also  $\text{diam}(G)$  is the length of  $G$ 's diameter.

**Theorem 5.**  $\text{capt}(G \square H, z_G + z_H) \leq \text{diam}(G) + \text{diam}(H) + \text{capt}(G, z_G) + \text{capt}(H, z_H)$

*Proof.* By using  $z_G$  zombies as  $G$ -zombies and  $z_H$  zombies as  $H$ -zombies, and having them follow the same set of moves provided in theorem 2, we show that survivor's  $G$ -moves cannot exceed  $\text{diam}(H) + \text{capt}(G, z_G)$ . With the same conclusion, it can be shown that  $H$ -moves cannot exceed  $\text{diam}(G) + \text{capt}(H, z_H)$  as well.

According to the definition of the diameter of a graph, after at most first  $\text{diam}(H)$   $G$ -moves that survivor makes,  $a = y$  holds. Now since for each  $G$ -move made from now on by survivor,  $G$ -zombies can follow their strategy on a  $G$  graph, and after at most  $\text{capt}(G, z_G)$   $G$ -moves, survivor will be captured.

Since each of survivor's moves is either a  $G$ -move or an  $H$ -move or both, total number of moves cannot exceed  $\text{diam}(G) + \text{diam}(H) + \text{capt}(G, z_G) + \text{capt}(H, z_H)$ .  $\square$

### 4. Limited capture time zombie number problem is NP-Hard

*NP-hardness* (non-deterministic polynomial-time hardness) is, in computational complexity theory, a class of problems that are informally "at least as hard as the hardest problems in NP class". A problem is assigned to the NP (non-deterministic polynomial time) class if it is solvable in polynomial time by a non-deterministic Turing machine.

A well known example of an NP-hard problem is the *dominating-set* problem in graph theory. A *dominating-set* for a graph  $G$  is a subset  $D$  of  $V(G)$  such that every vertex not in  $D$  is adjacent to at least one member of  $D$ . The domination number  $\gamma(G)$  is the number of vertices in the smallest dominating-set for  $G$ .

We define  $\text{Lcz}(G, k)$  (*limited capture time zombie number*) as the minimum number of zombies needed so that zombie player is able to capture the survivor in at most  $k$  moves on graph  $G$ .

Also  $N_G[u]$  represents the set of neighbors of  $u$  in graph  $G$ .

$LCZ_k$  problem is defined as below:

INSTANCE: Let  $G = (V, E)$  be a simple undirected graph. Given a graph  $G$  and two positive integers  $z_c$  and  $k$ .

QUESTION: Is  $Lcz(G, k) \leq z_c$  ? In other words, can the zombie player capture the survivor after at most  $k$  moves using  $z_c$  zombies in graph  $G$  ?

The dominating-set problem is defined below:

INSTANCE: Let  $G = (V, E)$  be a simple undirected graph. Given a graph  $G$  and a positive integer  $d$ .

QUESTION: Is  $\gamma(G) \leq d$  ?

**Theorem 6.**  $LCZ_k \in NP\text{-}Hard$

*Proof.* We prove this by reducing the dominating-set problem to  $LCZ_k$  in polynomial time.

To have a better understanding, consider the case where  $k = 1$ . For zombie player being able to capture the survivor in one move, every vertex not occupied by a zombie, should have a zombie neighbor. This is exactly the definition of a dominating-set. This simply shows that  $LCZ_1 \in NP\text{-}Hard$  class.

Now consider  $k$  to be an arbitrary integer bigger than 1. We construct a new graph  $G'_k$  from  $G$ . Suppose  $G$  has  $n$  vertices. For each vertex  $v \in V(G)$ , we add a *new* path with  $k$  vertices ending in  $v$  (as shown in figure 3), and name each *new* vertex  $(v, i)$  for  $1 \leq i < k$ .

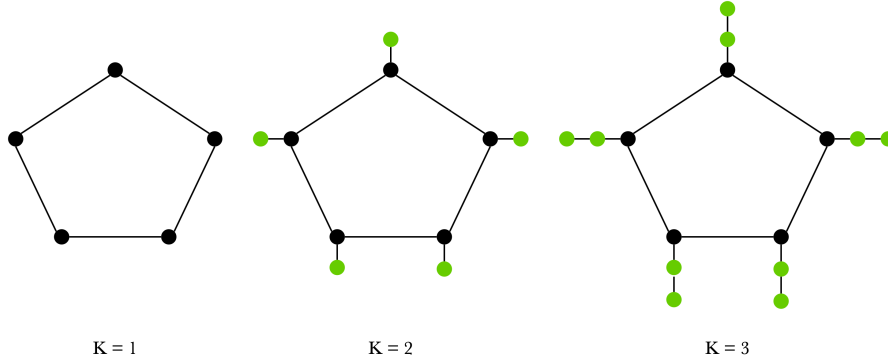


Figure 3:  $G'_k$  obtained from  $G = C_5$  where for  $k = 1, 2, 3$

$|V(G'_k)| = nk$  is of  $O(n)$ . Thus creating  $G'_k$  can be done in polynomial time. Consider set  $S'$  as zombies' initial vertices on  $G'_k$ , so that zombies are able to capture the survivor within  $k$  moves. For each vertex  $v \in S'$  or  $(v, i) \in S'$  add  $v$  to a new set of  $G$ 's vertices,  $S$ . If there is a vertex  $u \in G$  not dominated by  $S$ , then there is no vertex  $v$  such that  $v \in S'$  and  $v \in N_{G'_k}[u] \cup \{u, (u, 1 \leq i < k)\}$ . By having the survivor in vertex  $(u, k - 1)$ , there is no vertex at distance  $k$  or less from him, which means he will not be captured. Thus  $\gamma(G) \leq Lcz(G'_k, k)$  holds.

Now for each vertex  $v$  in one of  $G$ 's smallest dominating-sets, place a zombie on vertex  $v$  of  $G'_k$ . These zombies can capture the survivor in at most  $k$  moves. To show this, consider survivor's initial vertex, if it is not a *newly* added vertex, he can be captured in one move. Now suppose survivor is initially on  $(u, i)$ . Since  $u$  is dominated by a zombie, after zombies' first move, survivor will be trapped inside the  $u$ 's path, and would be captured in at most  $k$  moves. Therefore,  $Lcz(G'_k, k) \leq \gamma(G)$ .

By combining these results,  $Lcz(G'_k, k) = \gamma(G)$ . Therefore the dominating-set problem is reduced to  $LCZ_k$ . □

**Lemma 7.** *Let  $n$  be the number of vertices of graph  $G$ . If survivor can avoid being captured on  $G$  after  $(n + 1) \times n \times \text{diam}(G)$  moves, he can forever avoid being captured on  $G$ .*

*Proof.* Define *zombieDist* as the sum of the distances between each zombie and the survivor. It is not hard to see that after each two rounds of play, that is each player has played once, *zombieDist* will not increase, since zombies are always getting closer to the survivor. Now we show that if *zombieDist* does not strictly decrease after each player takes turn for  $n + 1$  times, survivor can avoid being captured forever. Consider the sequence of vertices occupied by survivor in  $n + 1$  consecutive moves. By pigeonhole principle, one vertex has been seen by survivor at least twice. If survivor keeps repeating those moves, he will maintain his distance from zombies and will avoid being captured forever.

Therefore, for a graph  $G$  which zombie-player can win, after each  $(n + 1)$  moves, *zombieDist* should strictly decrease. *zombieDist* is at most  $n \times \text{diam}(G)$ , since there is not more than  $n$  zombies and each zombie is at distance at most  $\text{diam}(G)$  from survivor (this bound can be easily improved). Thus, after  $(n + 1) \times n \times \text{diam}(G)$  moves, zombies would capture the survivor. □

By using this lemma, we can see  $LCZ_k$  problem for  $k > (n + 1) \times n \times \text{diam}(G)$  and a graph with  $n$  vertices are the same as the problem  $LCZ_{(n+1) \times n \times \text{diam}(G)}$ , and by solving  $LCZ_{(n+1) \times n \times \text{diam}(G)}$  for  $G$ , we get  $z(G)$  as well.

## 5. Zombie Number Problem is NP-Hard

Now define zombie number ( $Z$ ) problem:  
 INSTANCE: Let  $G = (V, E)$  be a simple undirected graph. Given graph  $G$  and an integer  $z_c$ .  
 QUESTION: Is  $z(G) \leq z_c$ ?

**Theorem 8.**  $Z \in \text{NP-Hard class}$ .

*Proof.* We reduce dominating-set problem to  $Z$ .

To do this, we add  $n$  new complete bipartite graphs,  $K_{n,n}$ , to  $G$ . We call the newly obtained graph  $H$ , and call the  $G$ -subgraph simply as  $G$ , and the  $i$ -th ( $1 \leq i \leq n$ )

bipartite subgraph as  $K_i$ .  $(i, j, b)$  represents the  $j$ -th vertex in  $K_i$ 's part  $b$  ( $b = 1, 2$ ). For each vertex  $(i, j, b)$  we connect it to vertices  $j$  and  $N_G[j]$  (See figure 4). Since we are adding  $2n^2$  new vertices, building  $H$  can be done in polynomial time.

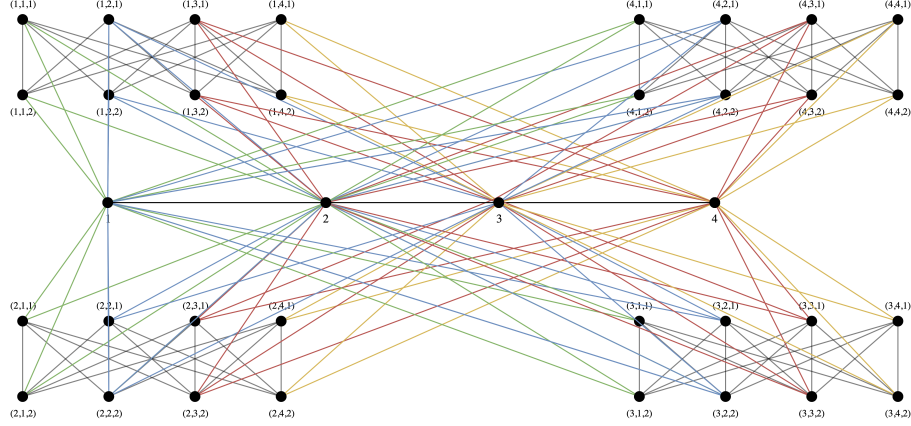


Figure 4:  $H$  obtained from  $P_4$ , zombie player needs at least  $\gamma(P_4) = 2$  zombies to win

By having zombies on each vertex of  $G$ 's dominating-set, survivor will be captured on the first move and zombie player wins. Thus,  $z(H) \leq \gamma(G)$ . Now suppose we have zombies less than the domination number of graph:  $z_c < \gamma(G)$ . We prove survivor can avoid being captured.

Suppose zombie player has placed his zombies. Since we have  $n$  bipartite subgraphs and  $z_c < n$ , there is a bipartite subgraph, without any zombies in it ( $k$ -th bipartite graph). Since the vertices corresponding to these zombies are not dominating  $G$ , there is a vertex  $v$ , not dominated by them. We place the survivor on vertex  $(k, v, b = 1)$  and therefore, survivor has no neighbors occupied by a zombie. On each survivor turn, there is a vertex  $v$  in  $G$  not dominated by a zombie, therefore survivor will move to vertex  $(k, v, 3 - b)$ . We prove by following this move, he will survive. On zombies' turn, zombie player has zombies on either  $G$  or some  $K_i$  ( $i \neq k$ ) or  $K_k$ . Initially there is no zombie in  $K_k$ , this means whenever a zombie wants to join  $K_k$ , it has to be in  $G$  in order to reach  $K_k$ , as there is no connection between  $K_i$ -subgraphs. Zombies in  $G$  (e.g. at vertex  $u$ ) are at distance 2 from survivor ( $u \rightarrow (k, u, 3 - b) \rightarrow (k, v, b)$ ), which means after their move all of them should be at one of  $(k, v, b)$ 's neighbors, that is,  $v, N_G[v]$  or,  $(k, 1 \leq i \leq n, 3 - b)$ . Therefore, each zombie joining  $K_k$  does not share the same partition as survivor's. Survivor now moves to  $(k, v, 3 - b)$  and will be sharing the same partition as zombies' in  $K_k$ . As survivor does not have any neighbors in  $G$  or  $K_k$  on each zombie player's turn, he will not be captured. Therefore,  $\gamma(G) \leq z(H)$ .

It is now proved that  $z(H) = \gamma(G)$ , thus the dominating-set problem is reduced to  $Z$ .

□



## References

- [1] Fitzpatrick, Shannon L., J. Howell, Margaret-Ellen Messinger, and David A. Pike. "A deterministic version of the game of zombies and survivors on graphs." *Discrete Applied Mathematics* 213 (2016): 1-12.
- [2] Offner, David, and Kerry Ojakian. "Comparing the power of cops to zombies in pursuit-evasion games." *Discrete Applied Mathematics* (2019).
- [3] West, Douglas B. "Introduction to Graph Theory." Prentice hall, (1996).

## Appendix A. An example for zombie number of Cartesian product of two graphs

**Example 9.**  $z(P_3 \square P_4) = 2$

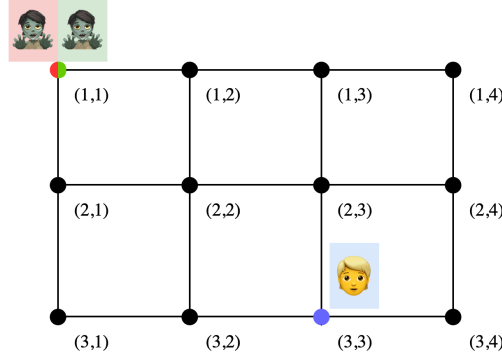


Figure A.5:  $P_3 \square P_4$  and initial vertices

It is easy to show that  $z(P_3) = z(P_4) = 1$ . On each of these path graphs, zombie's initial position could be any vertex of the graph. For this example, we put the  $G$ -zombie and  $H$ -zombie ( $G = P_3$  and  $H = P_4$ ) both on vertex  $(1,1)$ . We show the survivor with blue color,  $H$ -zombie with red, and  $G$ -zombie with green.  $G$ -zombie will try to get to the same  $G_i$  as survivor's which is  $G_3$  using an  $H$ -edge.  $H$ -zombie will try to get to  $H_3$  (See figure A.6).

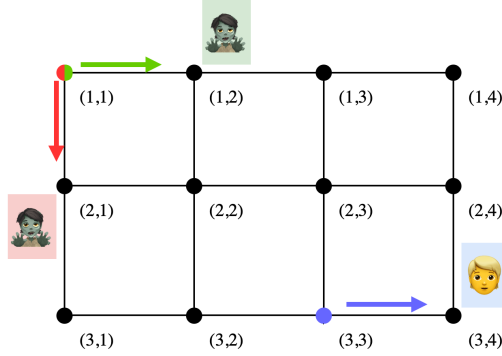


Figure A.6: First move of players

After zombies' move the survivor must move. No matter what move he makes, either  $G$ -zombie has made itself closer to  $H_x$  or  $H$ -zombie has made itself closer to  $G_y$ . In this case,  $H$ -zombie got closer to  $H_x$ . Since neither  $H$  or  $G$ -zombies share  $H_x$  or  $G_y$  with the survivor, they will still try to achieve that (See figure A.7).

Now  $H$ -zombie shares the same copy of  $H$  as survivor and it is survivor's turn. If survivor moves to another  $H_i$ ,  $H$ -zombie will mimic the move. If survivor makes an  $H$ -move,  $H$ -zombie will do whatever it did on a single  $H$  for capturing the survivor. This

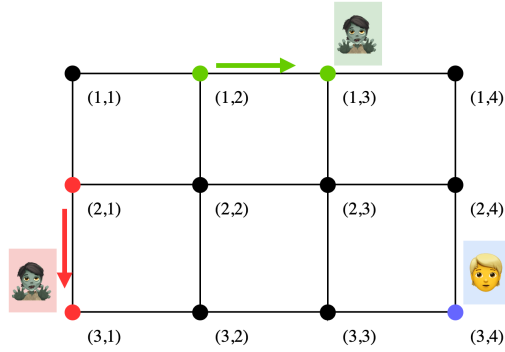
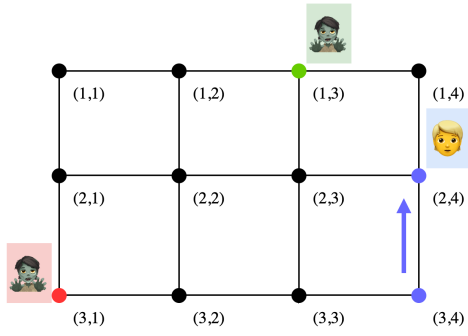


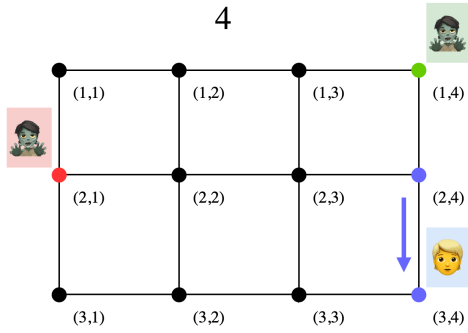
Figure A.7: Second move made by zombies, third in total

means survivor cannot do infinite *H-moves*. Thus for him being able to survive he has to do infinite *G-moves*, which again leads to *G-zombie* capturing him. For other moves, you can see figure A.8.

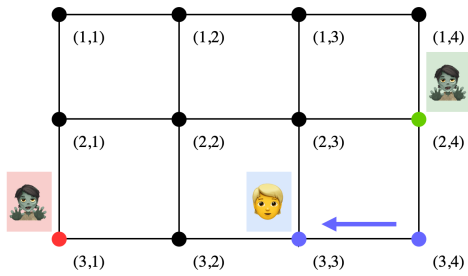
## Survivor's Moves



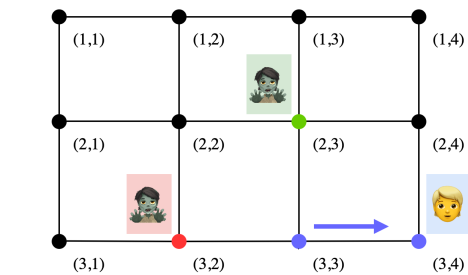
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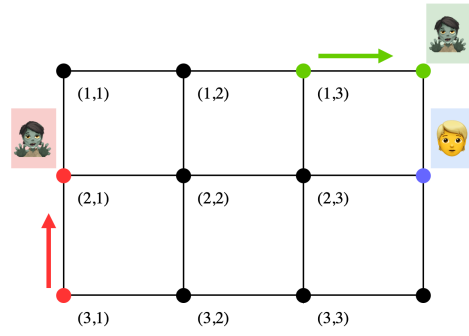


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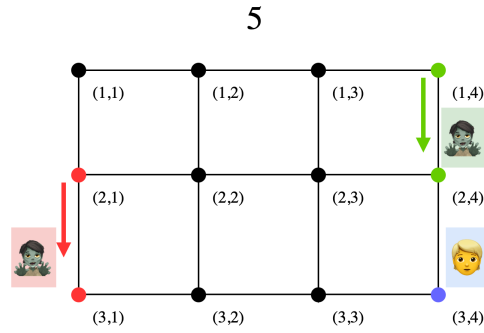


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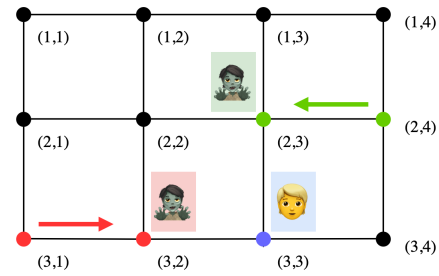
## Zombies' Moves



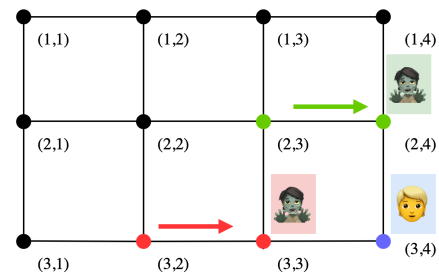
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Figure A.8: Other moves made by players

**Ali Keramatipour:** Conceptualization, Formal analysis, Investigation; Methodology, Validation, Visualization, Writing – original draft

**Behnam Bahrak:** Project administration; Supervision; Writing – review & editing