Zombie number of the Cartesian product of graphs

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Abstract

Zombies and Survivors is a variant of the pursuit-evasion game *Cops and Robbers*, with the difference that zombies must always move closer to one of their closest survivors. The game is played on a simple graph by two players. The goal of the zombies is to catch the survivors while survivors' objective is to avoid being captured. The *zombie number* of G, denoted as $z(G)$, is the minimum number of zombies required to capture a single survivor on G, no matter what moves survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al.[1] about the zombie number of the *Cartesian product* of two graphs. This result provides a new proof for $z(Q_n) = \lceil \frac{2n}{3} \rceil$. We also introduce a new problem regarding *capture time* in the Cartesian product of two graphs. At last, we study computational complexity of finding the zombie number of a graph G, with and without a limited *capture time*.

Keywords: Cartesian Product of Graphs, Zombie Number, NP-Hard

1. Introduction

The *Zombies and Survivors* game is played on a simple graph by two players. The deterministic version of this game [1] is played as follows (note that we only consider the game with a single survivor). Initially, the zombie player chooses a number z_c and places z_c zombies on the graph vertices. Then the survivor player chooses one single vertex which is the survivor's initial position. Starting with the zombie player, on each player's turn, the survivor player either moves to an adjacent vertex or stays at his current location, while zombie player must move each zombie to one of its adjacent vertices so that they get closer to the survivor. Here lies the difference between *Zombies and Survivor* and *Cops and Robber(s)* games, as in *Cops and Robber(s)* cops should not necessarily get closer to the robber(s), they can either hold their current position, get closer, or further away from the robber(s). Although zombies are not as intelligent as cops, they can still choose their path intelligently between the shortest paths. If any

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zombie and the survivor ever occupy the same vertex, the survivor is captured and the zombie player wins. The zombie number of a graph G, denoted as $z(G)$, is the minimum number of zombies required so that the zombie player can always capture the survivor, no matter how survivor moves.

The Cartesian product $G\Box H$ of two graphs G and H, is a graph with vertex set of $V(G) \times V(H)$, where vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $\{u_2, v_2\} \in E_H$, or $u_2 = v_2$ and $\{u_1, v_1\} \in E_G$ [3]. Figure 1 shows an example of the Cartesian product of two graphs.

Capture time in a game, is the maximum number of moves survivor can avoid being captured, while zombie player has z_c zombies. If this goes to infinity, a survivor-win play exist.

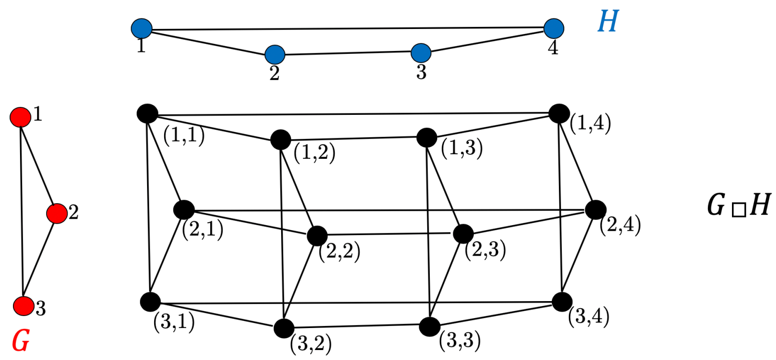

Figure 1: $C_3\Box C_4$ an example of the Cartesian product

Our contributions can be summarized as follows:

1) In [1], Fitzpatrick et al. conjectured that $z(G\Box H) \leq z(G) + z(H)$. In 2 We prove this conjecture and use it to show that $z(Q_n) = \lceil \frac{2n}{3} \rceil$.

2) In 3, we provide a new bound on *capture time* of *zombies and survivor* game played on the Cartesian product of two graphs.

3) We introduce a variation of *zombies and survivor* game, in which the zombie player is restricted to winning in a limited number of moves in 4, and prove it belongs to *NP-Hard* class of problems.

4) In 5, we prove that the original *zombies and survivor* game belongs to *NP-Hard* class of problems.

2. Zombie number of the Cartesian product of two graphs

To prove $z(G\Box H) \leq z(G) + z(H)$, we show that $z(G) + z(H)$ zombies are enough for the zombie player to capture the survivor on $G\Box H$.

To explain the proof we first need to define some notations. Assume H and G have m and n vertices, respectively.

We define G_i as the induced subgraph by all vertices (u, v) in $G \square H$, where $v = i$. Similarly H_j is defined as the induced subgraph by vertices (u, v) in $G \square H$, where $u = j$.

In the Cartesian product of G and H, each G_i $(1 \leq i \leq m)$ is isomorphic to G, and each H_j $(1 \leq j \leq n)$ is isomorphic to H. We name the common vertex between G_i and H_j, (j, i). Also (x, y) is the vertex where the survivor is located. Figure 2 illustrates these definitions.

A *G-edge* is an edge in one of the G_is and an *H-edge* is an edge in one of the H_js. A *G-move* is a move made on a *G-edge*. Similarly, an *H-Move* is a move made on an *H-edge*. If the survivor decides to remain in its current vertex, this move is considered both a *G-move* and an *H-move*.

$dist_I(j, k)$ is the distance between j and k vertices on a graph I. Length of a path P is shown by $len(P)$.

For vertex (u, v), G-equivalent vertex, is vertex u on graph G, and H-equivalent vertex, is vertex v on graph H. G-equivalent graph is a graph where we put each zombie and the survivor on its G-equivalent vertex. H-equivalent graph is defined in the same way.

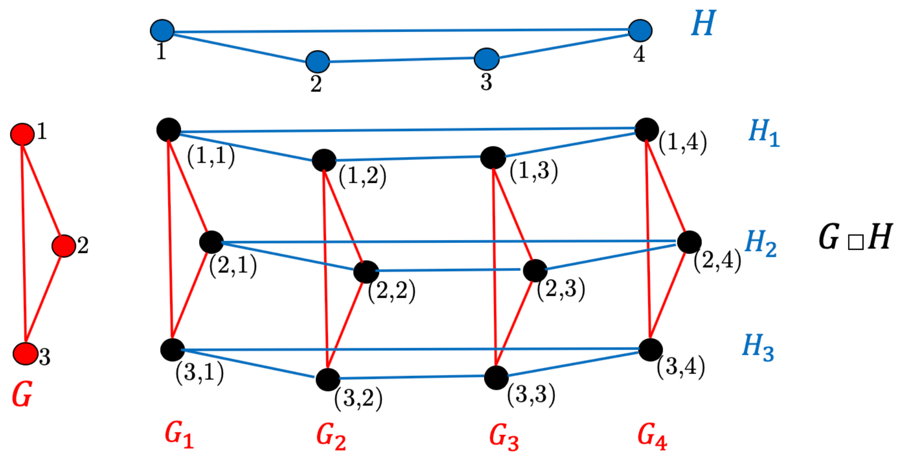

Figure 2: $G \square H$, G_is, and H_is.

Lemma 1. $dist_{G \square H}((x, y), (u, v)) = dist_G(x, u) + dist_H(y, v)$.

Proof. Since there is a path from (x, y) to (x, v) with length $dist_H(y, v)$ and a path from (x, v) to (u, v) of length $dist_G(x, u)$, we only need to prove there can be no path with length less than $dist_G(x, u) + dist_H(y, v)$. Suppose not, this path uses some *G-moves* and some *H-moves*. If a *G-move* is followed by an *H-move* (or vice-versa), then we can swap these moves and still end up in the same vertex. For example if $(u_1, v_1) \rightarrow (u_2, v_1) \rightarrow$

(u_2, v_2) is happening, we can do $(u_1, v_1) \to (u_1, v_2) \to (u_2, v_2)$. Since we can swap each two moves of different type, suppose all *G-move*s happen before all *H-move*s in the shortest path. Since this path has a length less than $dist_G(x, u) + dist_H(y, v)$, we have found a path in either G between x and u, with length less than $dist_G(x, u)$, or in H between y and v, with length less than $dist_H(y, v)$, which is a contradiction. Thus the statement holds. $\square$

Theorem 2. *$z(G \square H) \leq z(G) + z(H)$.*

Proof. We provide a winning strategy for the Cartesian product of G and H using $z(G) + z(H)$ zombies. First, we place $z(G)$ zombies, that have a winning strategy on a single G, $G_{a=1}$ and call them *G-zombie*s. We do the same for $H_{b=1}$ and call them *H-zombie*s.

Consider one of the shortest paths between vertices a (the index of G-subgraph shared by *G-zombie*s) and y in H and call it p_H. We also define p_G in the same manner between b and x.

On each zombie turn, if $a \neq y$, each *G-zombie* will move along the p_H path in its corresponding H subgraph. According to Lemma 1, since zombies' and survivor's equivalents on H are getting closer, thus their actual vertices on $G \square H$ are getting closer as well and this move is possible. Since they are all moving along similar paths (in their corresponding H-subgraphs) they will still share the same G-subgraph. Now consider when $a = y$, *G-zombie*s will play their winning strategy (that they had on a single G) in this case. This move is also possible since in G's strategy, zombies would get closer to survivor on each turn. If $a = y$ and survivor makes an *H-move*, *G-zombie*s will maintain their positioning by mimicking the exact same move on their corresponding H. This means for those turns that $a = y$ holds, if we consider the G-equivalent graph between *G-zombie*s and the survivor, it is just like a simple game played on a single G. *H-zombie*s will follow the same strategy but in their corresponding environment.

Suppose using this strategy $G \square H$ is *survivor-win*, then the survivor must either do infinite *G-move*s or infinite *H-move*s. Without loss of generality, suppose the survivor makes infinite *H-move*s, we prove that this is not possible. After $len(p_G)$ number of *H-move*s, *H-zombie*s will get to H_x. Now for each *G-move* made by survivor and having zombies chasing him, nothing changes in their H-equivalent graph. Since the survivor can do infinite *H-move*s and prevent being caught, it means that the survivor could also avoid being caught on a single H which contradicts our assumption.

$\square$

An example for further understanding can be found at Appendix A.

Corollary 3. *$z(Q_n) \leq \lceil \frac{2n}{3} \rceil$*

Proof. We prove this by using both induction and the theorem proved above. First note that the Cartesian product of hypercube graphs Q_m and Q_n is equal to Q_{m+n}. It is easy to see $z(Q_3) = 2$, $z(Q_2) = 2$, and $z(Q_1) = 1$. For $n > 3$, we consider Q_n as the Cartesian product of Q_3 and Q_{n-3}. Using the induction base, we know that $z(Q_{n-3}) \leq \lceil \frac{2n-6}{3} \rceil$. According to the proved conjecture $z(Q_n) \leq z(Q_{n-3}) + z(Q_3)$ and $z(Q_{n-3}) \leq \lceil \frac{2n-6}{3} \rceil = \lceil \frac{2n}{3} \rceil - 2$, we can see that $z(Q_n) \leq \lceil \frac{2n}{3} \rceil$. $\square$

It is already proved that at least $\lceil \frac{2n}{3} \rceil$ zombies are needed to capture one survivor on graph Q_n (Theorem 16 of [1]):

Theorem 4. *For each integer $n \geq 1$, $z(Q_n) \geq \lceil \frac{2n}{3} \rceil$.*

Combining *Corollary 3* and *Theorem 4* we can conclude that $z(Q_n) = \lceil \frac{2n}{3} \rceil$. This proves Conjecture 18 from [1] which is already proved in [2] with a different method.

3. Capture time in Cartesian product of graphs

We define a new parameter, *capture time* of the game, noted as $capt(G, z_c)$, which represents the maximum number of moves that survivor can avoid being caught on a graph G, when the zombie player uses z_c zombies. Zombie player tries to make $capt(G, z_c)$ as least as possible, while survivor tries to maximize it. Also $diam(G)$ is the length of G's diameter.

Theorem 5. $capt(G\square H, z_G + z_H) \leq diam(G) + diam(H) + capt(G, z_G) + capt(H, z_H)$

Proof. By using z_G zombies as *G-zombie*s and z_H zombies as *H-zombie*s, and having them follow the same set of moves provided in theorem 2, we show that survivor's *G-move*s cannot exceed $diam(H) + capt(G, z_G)$. With the same conclusion, it can be shown that *H-move*s cannot exceed $diam(G) + capt(H, z_H)$ as well.

According to the definition of the diameter of a graph, after at most first $diam(H)$ *G-move*s that survivor makes, $a = y$ holds. Now since for each *G-move* made from now on by survivor, *G-zombie*s can follow their strategy on a G graph, and after at most $capt(G, z_G)$ *G-move*s, survivor will be captured.

Since each of survivor's moves is either a *G-move* or an *H-move* or both, total number of moves cannot exceed $diam(G) + diam(H) + capt(G, z_G) + capt(H, z_H)$. $\square$

4. Limited capture time zombie number problem is NP-Hard

NP-hardness (non-deterministic polynomial-time hardness) is, in computational complexity theory, a class of problems that are informally "at least as hard as the hardest problems in NP class". A problem is assigned to the NP (non-deterministic polynomial time) class if it is solvable in polynomial time by a non-deterministic Turing machine.

A well known example of an NP-hard problem is the *dominating-set* problem in graph theory. A *dominating-set* for a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating-set for G.

We define $Lcz(G, k)$ (*limited capture time zombie number*) as the minimum number of zombies needed so that zombie player is able to capture the survivor in at most k moves on graph G.

Also $N_G[u]$ represents the set of neighbors of u in graph G.

LCZ_k problem is defined as below:
INSTANCE: Let $G = (V, E)$ be a simple undirected graph. Given a graph G and two positive integers z_c and k.
QUESTION: Is $Lcz(G, k) \leq z_c$? In other words, can the zombie player capture the survivor after at most k moves using z_c zombies in graph G ?

The dominating-set problem is defined below:
INSTANCE: Let $G = (V, E)$ be a simple undirected graph. Given a graph G and a positive integer d.
QUESTION: Is $\gamma(G) \leq d$?

Theorem 6. $LCZ_k \in$ *NP-Hard*

Proof. We prove this by reducing the dominating-set problem to LCZ_k in polynomial time.

To have a better understanding, consider the case where $k = 1$. For zombie player being able to capture the survivor in one move, every vertex not occupied by a zombie, should have a zombie neighbor. This is exactly the definition of a dominating-set. This simply shows that $LCZ_1 \in$ NP-Hard class.

Now consider k to be an arbitrary integer bigger than 1. We construct a new graph G'_k from G. Suppose G has n vertices. For each vertex $v \in V(G)$, we add a *new* path with k vertices ending in v (as shown in figure 3), and name each *new* vertex (v, i) for $1 \leq i < k$.

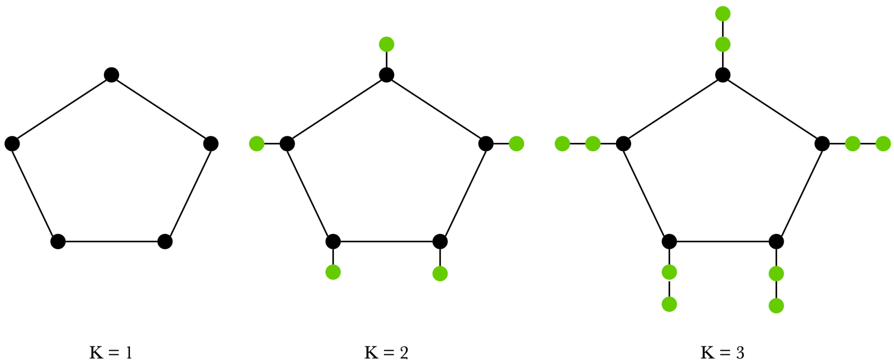

Figure 3: G'_k obtained from $G = C_5$ where for $k = 1, 2, 3$

$|V(G'_k)| = nk$ is of $O(n)$. Thus creating G'_k can be done in polynomial time. Consider set S' as zombies' initial vertices on G'_k, so that zombies are able to capture the survivor within k moves. For each vertex $v \in S'$ or $(v, i) \in S'$ add v to a new set of G's vertices, S. If there is a vertex $u \in G$ not dominated by S, then there is no vertex v such that $v \in S'$ and $v \in N_{G'_k}[u] \cup \{u, (u, 1 \leq i < k)\}$. By having the survivor in vertex $(u, k-1)$, there is no vertex at distance k or less from him, which means he will not be captured. Thus $\gamma(G) \leq Lcz(G'_k, k)$ holds.

Now for each vertex v in one of G's smallest dominating-sets, place a zombie on vertex v of G'_k. These zombies can capture the survivor in at most k moves. To show this, consider survivor's initial vertex, if it is not a *new*ly added vertex, he can be captured in one move. Now suppose survivor is initially on (u, i). Since u is dominated by a zombie, after zombies' first move, survivor will be trapped inside the u's path, and would be captured in at most k moves. Therefore, $Lcz(G'_k, k) \leq \gamma(G)$.

By combining these results, $Lcz(G'_k, k) = \gamma(G)$. Therefore the dominating-set problem is reduced to LCZ_k.

□

Lemma 7. *Let n be the number of vertices of graph G. If survivor can avoid being captured on G after $(n+1) \times n \times diam(G)$ moves, he can forever avoid being captured on G.*

Proof. Define *zombieDist* as the sum of the distances between each zombie and the survivor. It is not hard to see that after each two rounds of play, that is each player has played once, *zombieDist* will not increase, since zombies are always getting closer to the survivor. Now we show that if *zombieDist* does not strictly decrease after each player takes turn for $n+1$ times, survivor can avoid being captured forever. Consider the sequence of vertices occupied by survivor in $n+1$ consecutive moves. By pigeonhole principle, one vertex has been seen by survivor at least twice. If survivor keeps repeating those moves, he will maintain his distance from zombies and will avoid being captured forever.

Therefore, for a graph G which zombie-player can win, after each $(n+1)$ moves, *zombieDist* should strictly decrease. *zombieDist* is at most $n \times diam(G)$, since there is not more than n zombies and each zombie is at distance at most $diam(G)$ from survivor (this bound can be easily improved). Thus, after $(n+1) \times n \times diam(G)$ moves, zombies would capture the survivor. □

By using this lemma, we can see LCZ_k problem for $k > (n+1) \times n \times diam(G)$ and a graph with n vertices are the same as the problem $LCZ_{(n+1)\times n\times diam(G)}$, and by solving $LCZ_{(n+1)\times n\times diam(G)}$ for G, we get $z(G)$ as well.

5. Zombie Number Problem is NP-Hard

Now define zombie number (Z) problem:
INSTANCE: Let $G = (V, E)$ be a simple undirected graph. Given graph G and an integer z_c.
QUESTION: Is $z(G) \leq z_c$?

Theorem 8. *$Z \in$ NP-Hard class.*

Proof. We reduce dominating-set problem to Z.

To do this, we add n new complete bipartite graphs, $K_{n,n}$, to G. We call the newly obtained graph H, and call the G-subgraph simply as G, and the i-th $(1 \leq i \leq n)$

bipartite subgraph as K_i. (i, j, b) represents the j-th vertex in K_i's part b $(b = 1, 2)$. For each vertex (i, j, b) we connect it to vertices j and $N_G[j]$ (See figure 4). Since we are adding $2n^2$ new vertices, building H can be done in polynomial time.

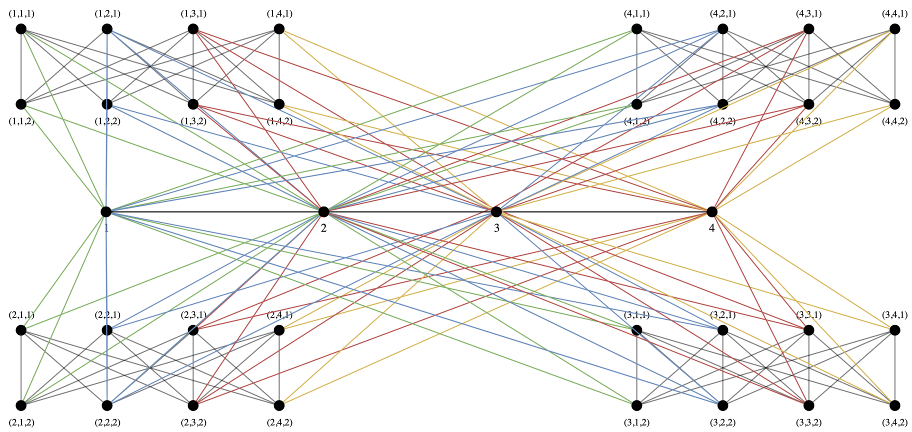

Figure 4: H obtained from P_4, zombie player needs at least $\gamma(P_4) = 2$ zombies to win

By having zombies on each vertex of G's dominating-set, survivor will be captured on the first move and zombie player wins. Thus, $z(H) \leq \gamma(G)$. Now suppose we have zombies less than the domination number of graph: $z_c < \gamma(G)$. We prove survivor can avoid being captured.

Suppose zombie player has placed his zombies. Since we have n bipartite subgraphs and $z_c < n$, there is a bipartite subgraph, without any zombies in it (k-th bipartite graph). Since the vertices corresponding to these zombies are not dominating G, there is a vertex v, not dominated by them. We place the survivor on vertex $(k, v, b = 1)$ and therefore, survivor has no neighbors occupied by a zombie. On each survivor turn, there is a vertex v in G not dominated by a zombie, therefore survivor will move to vertex $(k, v, 3 - b)$. We prove by following this move, he will survive. On zombies' turn, zombie player has zombies on either G or some K_i $(i \neq k)$ or K_k. Initially there is no zombie in K_k, this means whenever a zombie wants to join K_k, it has to be in G in order to reach K_k, as there is no connection between K_i-subgraphs. Zombies in G (e.g. at vertex u) are at distance 2 from survivor $(u \to (k, u, 3 - b) \to (k, v, b))$, which means after their move all of them should be at one of (k, v, b)'s neighbors, that is, $v, N_G[v]$ or, $(k, 1 \leq i \leq n, 3 - b)$. Therefore, each zombie joining K_k does not share the same partition as survivor's. Survivor now moves to $(k, v, 3 - b)$ and will be sharing the same partition as zombies' in K_k. As survivor does not have any neighbors in G or K_k on each zombie player's turn, he will not be captured. Therefore, $\gamma(G) \leq z(H)$.

It is now proved that $z(H) = \gamma(G)$, thus the dominating-set problem is reduced to Z.

□

References

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[3] West, Douglas B. ”Introduction to Graph Theory.” Prentice hall, (1996).

Appendix A. An example for zombie number of Cartesian product of two graphs

Example 9. $z(P_3 \square P_4) = 2$

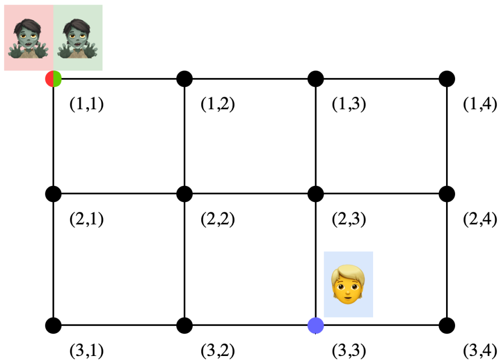

Figure A.5: $P_3 \square P_4$ and initial vertices

It is easy to show that $z(P_3) = z(P_4) = 1$. On each of these path graphs, zombie's initial position could be any vertex of the graph. For this example, we put the *G-zombie* and *H-zombie* ($G = P_3$ and $H = P_4$) both on vertex $(1,1)$. We show the survivor with blue color, *H-zombie* with red, and *G-zombie* with green. *G-zombie* will try to get to the same G_i as survivor's which is G_3 using an *H-edge*. *H-zombie* will try to get to H_3 (See figure A.6).

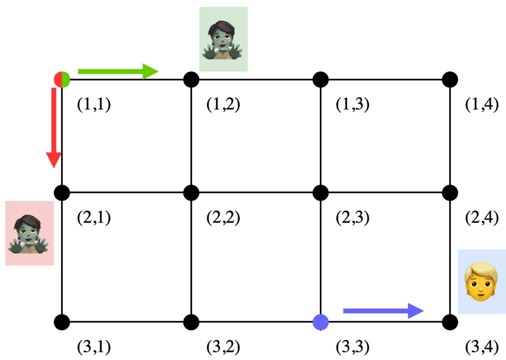

Figure A.6: First move of players

After zombies' move the survivor must move. No matter what move he makes, either *G-zombie* has made itself closer to H_x or *H-zombie* has made itself closer to G_y. In this case, *H-zombie* got closer to H_x. Since neither *H or G-zombies* share H_x or G_y with the survivor, they will still try to achieve that (See figure A.7).

Now *H-zombie* shares the same copy of H as survivor and it is survivor's turn. If survivor moves to another H_i, *H-zombie* will mimic the move. If survivor makes an *H-move*, *H-zombie* will do whatever it did on a single H for capturing the survivor. This

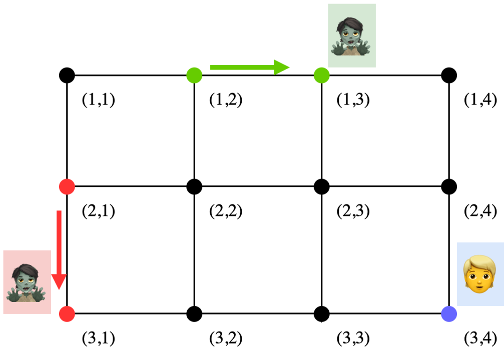

Figure A.7: Second move made by zombies, third in total

means survivor cannot do infinite *H-move*s. Thus for him being able to survive he has to do infinite *G-move*s, which again leads to *G-zombie* capturing him. For other moves, you can see figure A.8.

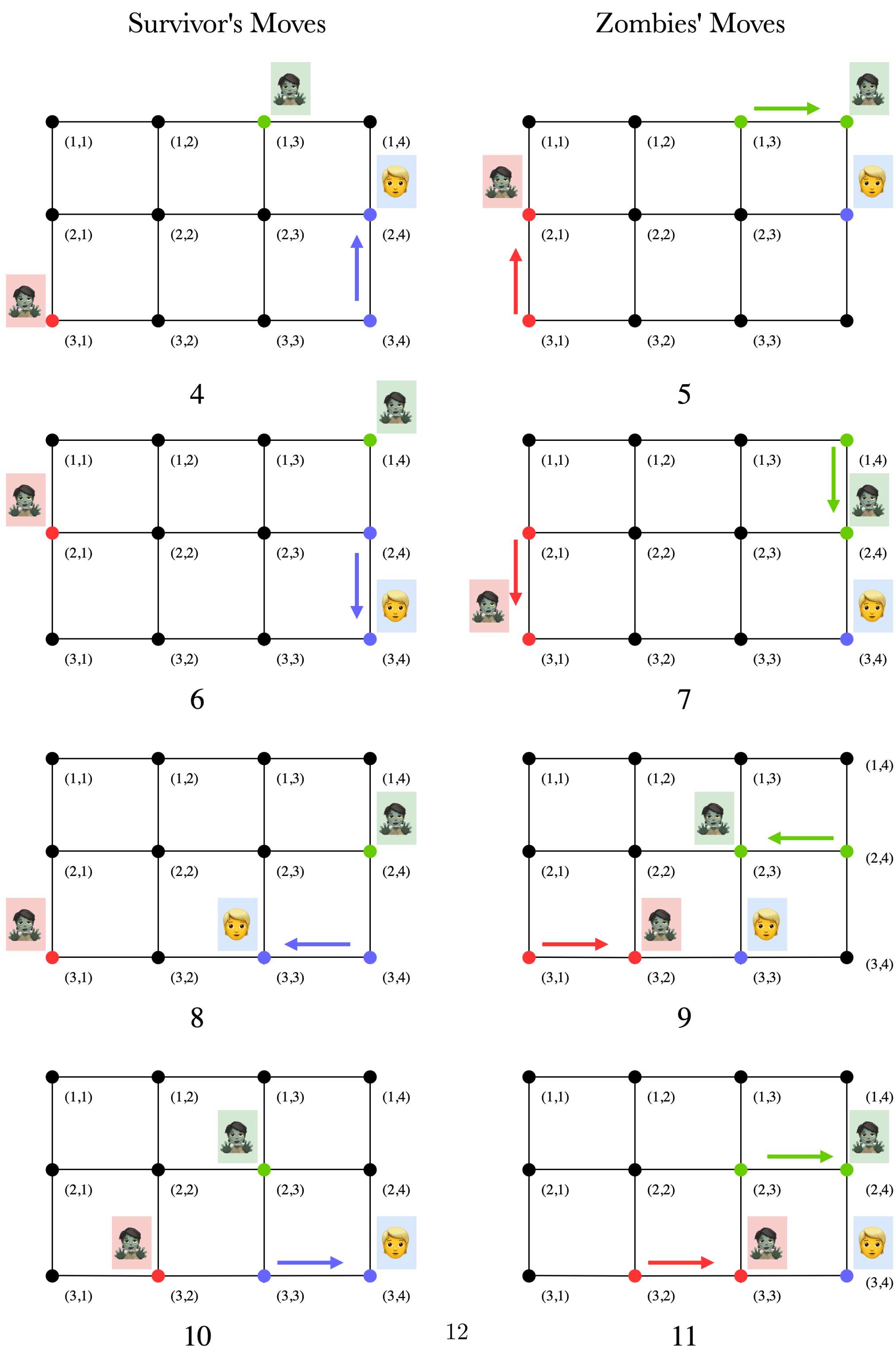

Figure A.8: Other moves made by players