%&pdflatex

\documentclass[1p]{elsarticle}

\usepackage{lineno,hyperref}

\usepackage{amsmath, amssymb, amscd, amsthm, amsfonts}

\usepackage{mathtools}

\modulolinenumbers[5]

\DeclarePairedDelimiter\ceil{\lceil}{\rceil} \DeclarePairedDelimiter\floor{\lfloor}{\rfloor}

\newtheorem{theorem}{Theorem}

\newtheorem{lemma}[theorem]{Lemma}

\newtheorem{conjecture}[theorem]{Conjecture}

\newtheorem{corollary}[theorem]{Corollary}

\newtheorem{example}[theorem]{Example}

\journal{Discrete Applied Mathematics}

\bibliographystyle{elsarticle-num}

\begin{document}

\begin{frontmatter}

\title{Zombie number of the Cartesian product of graphs}

%% or include affiliations in footnotes:

\author{Ali Keramatipour}

\ead{alikeramatipour@ut.ac.ir}

\author{Behnam Bahrak\corref{correspondingauthor}}

\cortext[correspondingauthor]{Corresponding author}

\ead{bahrak@ut.ac.ir}

\address{School of Electrical and Computer Engineering, College of Engineering, University of Tehran, Tehran, Iran}

\begin{abstract}

{\it Zombies and Survivors} is a variant of the pursuit-evasion game {\it Cops and Robbers}, with the difference

that zombies must always move closer to the survivor. The game is played on a simple graph by two players. The

goal of the zombies is to catch the survivor while survivor's objective is to avoid being captured. The zombie

number of $G$, denoted as $z(G)$, is the minimum number of zombies required to capture the survivor, no matter

what moves survivor makes. In this paper, we prove a conjecture by Fitzpatrick et al. \cite{Fitz16} about the

zombie number of the Cartesian product of two graphs. This result provides a new proof for $z(Q\_n) =

\ceil\*{\frac{2n}{3}}$. We also introduce a new problem regarding {\it capture time}

in Cartesian product of two graphs. At last in section \ref{np-capturetime}, we prove another result, we define

{\it capture time} problem more precisely and prove it belongs to {\it NP-Hard} set of problems.

<<Rewrite the highlighted part. It’s vague. We cannot define capture time and then make it more precise. In mathematics, definitions must be solid and precise in the first place. Also you shouldn’t use section numbers in the abstract.>>

\end{abstract}

\begin{keyword}

Cartesian Product of Graphs\sep Zombie Number\sep NP-Hard

\end{keyword}

\end{frontmatter}

\section{Introduction}\label{section-introduction}

The {\it Zombies and Survivors} game is played on a simple graph by two players. The deterministic version of this game

\cite{Fitz16} is played as follows (note that we only consider the game with a single survivor in this paper).

Initially, the zombie player chooses a number $z$ and places $z$ zombies on the graph vertices. Then the survivor player

chooses one single vertex which is the survivor's initial position. Starting with the zombie player, on each turn,

survivor player either moves to an adjacent vertex or stays at his current location, while zombie player must move each

zombie to one of its adjacent vertices so that they get closer to the survivor. Here lies the difference between {\it

Zombies and Survivor} and {\it Cops and Robber} games, as in {\it Cops and Robber}, cops should not necessarily get

closer to the {\it Robber}, they can either hold their current position, get closer, or further away from the robber. If

any zombie and the survivor ever occupy the same vertex, the survivor is captured and the zombie player wins. The zombie

number of a graph $G$, denoted as $z(G)$, is the minimum number of zombies required so that the zombie player always has

a winning play on graph $G$. If a winning play exist, zombies choose a set of initial vertices, and a move on their turn

when they are faced with multiple, so that the survivor will be captured no matter how he moves.

<<Vague. Rewrite.>>

The Cartesian product $G \square H$ of two graphs $G$ and $H$, is a graph with vertex set of $V(G)\times V(H)$, where

vertices $(u\_1 , u\_2)$ and $(v\_1 , v\_2)$ are adjacent if and only if $u\_1 = v\_1$ and $ \{ u\_2 , v\_2 \} \in E\_{H} $, or

$u\_2 = v\_2$ and $ \{u\_1 , v\_1 \} \in E\_{G}$ \cite{West02}.

<<Did the reviewers ask to remove $\times$?>>

Figure \ref{fig:p2} shows an example of the Cartesian product

of two graphs. We define $G\_{i}$ as the induced graph by all vertices $(u,v)$ in $G \square H$, where $v=i$. Similarly

$H\_{j}$ is defined as the induced graph by vertices $(u,v)$ in $G \square H$, where $u=j$.

<<This part shouldn’t be in the introduction. Please move notation definition to the next section.>>

Capture time of a game, is the number of moves survivor can avoid being captured, if this can go to infinity, a survivor-win play exist.

<<Shouldn’t this be the “maximum” number?>>

NP-hardness (non-deterministic polynomial-time hardness) is, in computational complexity theory, the defining property

of a class of problems that are informally "at least as hard as the hardest problems in NP". A simple example of an

NP-hard problem is the dominating set problem in graph theory. A problem is assigned to the NP (nondeterministic polynomial time) class if

it is solvable in polynomial time by a nondeterministic Turing machine.

<<This doesn’t belong to introduction section. Move it to section 5>>

\begin{figure}[h!]

\centering

\includegraphics[width=0.9\linewidth]{fig/CpWest.png}

\caption{$ C\_3 \square C\_4$: an example of a Cartesian product}

\label{fig:p2}

\end{figure}

In \cite{Fitz16}, Fitzpatrick et al. conjectured that $z(G \square H) \leq z(G) + z(H)$, and showed the correctness of

this inequality for specific cases of $G$ and $H$. In this paper we prove the conjecture for all graphs $G$ and $H$, and

use it to show that $z(Q\_n) = \ceil\*{\frac{2n}{3}}$.

<<Now that you have more contribution, please summarize them in this section. You can write:>>

Our contributions can be summarized as follows:

1. We prove the Fitzpatrick conjecture for all graphs $G$ and $H$, and use it to show that $z(Q\_n) = \ceil\*{\frac{2n}{3}}$.
2. We define …
3. We prove NP-hardness of …

\section{Conjecture and Proof}\label{conj-proof}

To prove $z(G \square H) \leq z(G) + z(H)$, we show that $z(G) + z(H)$ zombies are enough for the zombie player to

capture the survivor on $G \square H$.

To explain the proof we first need to define some notations. Assume $H$ and $G$ have $m$ and $n$ vertices, respectively.

We define $G\_{i}$ as the induced graph by all vertices $(u,v)$ in $G \square H$, where $v=i$. Similarly

$H\_{j}$ is defined as the induced graph by vertices $(u,v)$ in $G \square H$, where $u=j$.

In the Cartesian product of $G$ and $H$, each $G\_{i}$ $(1 \leq i \leq m)$ is isomorphic to $G$, and each $H\_{j}$ $(1

\leq j \leq n)$ is isomorphic to $H$. Figure \ref{fig:p1} illustrates these definitions. We name the common vertex between $G\_{i}$

and $H\_{j}$, $(i,j)$. Also $(x,y)$ is the vertex where the survivor is located. A {\it G-move} is a move made on one of

the $G\_{i}$'s edges. Similarly, an {\it H-Move} is a move made on one of $H\_{j}$'s edges. If the survivor decide to remain in

its current position, this move is considered both a {\it G-move} and an {\it H-move}. <<Why? Shouldn’t this be neither?>> A {\it G-edge} is an edge in one of

the $G\_{i}$s and an {\it H-edge} is an edge in one of the $H\_{j}$s. A {\it winning state} is a state that zombie player

can catch survivor no matter what set of moves survivor makes, from that particular state. A $G$-equivalent graph, is

defined as a graph where each {\it G-zombie} in, say in vertex $(z\_x,z\_y)$, is placed on $z\_x$ vertex and survivor is placed on

$x$ vertex. $H$-equivalent is also defined in the same way.

<<What is a G-zombie? I think you have defined it in the proof of the next theorem. BTW Isn’t the notation unnecessarily complex?>>

$(u,v)$'s equivalent vertex on $G$ and $H$ is $u$ and $v$ respectively.

$dist\_I(j,k)$ is the distance between $j$ and $k$ vertices on a graph $I$.

\begin{figure}[h!]

\centering

\includegraphics[width=0.9\textwidth]{fig/cp3.png}

%\includegraphics[width=200pt]{cp.png}

\caption{$G \square H$, $G\_i$s, and $H\_i$s .}

\label{fig:p1}

\end{figure}

\begin{lemma} \label{shortestpathlemma}

$dist\_{G \square H}((x,y),(u,v)) = dist\_G(x,u) + dist\_H(y,v)$.

\end{lemma}

\begin{proof}

Since there is a path from $(x,y)$ to $(x,v)$ with length $dist\_H(y,v)$ and a path from $(x,v)$ to $(u,v)$ of length

$dist\_G(x,u)$, we only need to prove there can be no path with length less than $dist\_G(x,u) + dist\_H(y,v)$.

Suppose not, this path uses some {\it G-moves} and some {\it H-moves}. If a {\it G-move} is followed by

an {\it H-move} (or vice-versa), then we can swap these moves and still end up in the same vertex. For example if $(u\_1,v\_1)

\rightarrow (u\_2,v\_1) \rightarrow (u\_2,v\_2)$ is happening, we can do $(u\_1,v\_1) \rightarrow (u\_1,v\_2) \rightarrow

(u\_2,v\_2)$. Since we can swap each two moves of different types, suppose all {\it G-move}s happen before all {\it

H-move}s in the shortest path. Since this path has a length less than $dist\_G(x,u) + dist\_H(y,v)$, we have found a

path in either $G$ between $x$ and $u$ with length less than $dist\_G(x,u)$, or in $H$ between $y$ and $v$ with

length less than $dist\_H(y,v)$, which is a contradiction. Thus the statement holds.

\end{proof}

\begin{theorem}

$z(G \square H) \leq z(G) + z(H)$.

\end{theorem}

\begin{proof}

We provide a winning strategy for the Cartesian product of $G$ and $H$ using $z(G)+z(H)$ zombies. First, we place

$z(G)$ zombies, that have a winning strategy on a single $G$, say $G\_{1}$ and call them {\it G-zombies}. We do the

same for $H\_{1}$ and call them {\it H-zombies}. Initially all {\it G-zombies} share the same $G\_{i}$. We name the

subgraph $G\_{i}$ shared by them $G\_{z}$. We define $H\_{s}$ in the same manner. As stated before, initially $z=s=1$.

<<Can’t we simply say: initially all the G-zombies are on G\_1 and all the H-zombies on H\_1?>>

Consider one of the shortest paths between

vertices $z$ and $y$ in an $H$ and call it $p\_H$. We also define $p\_G$ in the same manner between $s$ and $x$. We

name these paths' length, $d\_H$ and $d\_G$, respectively.

<<Shouldn’t we specify which H graph? For example H\_j?>>

On each zombie turn, if $z \neq y$, each {\it G-zombie} will move along the $d\_H$ path in its corresponding $H$

subgraph. According to Lemma \ref{shortestpathlemma}, since zombies' and survivor's equivalents on $H$ are getting closer, thus their actual vertices on $G

\square H$ are getting closer as well. Now consider when $z = y$. The {\it

G-zombies} will only play their winning strategy (that they had on a single $G$ ) in this case. <<Isn’t G\_i better?>> This move is also

possible since in $G$'s strategy, zombies would get closer to survivor on each turn . If $z = y$ and survivor makes

an {\it H-move}, {\it G-zombies} will only try to maintain their positioning by mimicking the exact same move on

their corresponding $H$. This means for those turns that $z=y$ holds, if we consider the $G$-equivalent graph

between {\it G-zombies} and survivor, it is just like a simple game played on a single $G$. The {\it H-zombies} will

follow the same strategy but in their corresponding environment.

Suppose using this strategy $G \square H$ is {\it survivor-win}, then our survivor must do infinite moves in at

least one direction. <<I think the word “direction” is misleading>> Without loss of generality, suppose the survivor makes infinite {\it H-moves}. We prove that

this is not possible. After $d\_G$ number of {\it H-moves}, the {\it H-zombies} will get to $H\_x$. Now for each {\it

G-move} made by survivor and having zombies chasing him, nothing changes in their $H$-equivalent graph. Since the

survivor can do infinite {\it H-moves} and prevent being caught, it means that survivor could also avoid being

caught on a single $H$ which contradicts our assumption.

\end{proof}

We now provide an example for further understanding:

<<I think we should move the example to an appendix because it is not part of the contribution of this paper>>

\begin{example} $z(P\_3 \square P\_4 ) = 2$

\end{example}

\begin{figure}[h!]

\centering

\includegraphics[width=0.5\linewidth]{fig/p34m1.png}

\caption{$P\_3 \square P\_4$ and initial vertices}

\label{fig:p3}

\end{figure}

It is easy to show that $z(P\_3) = z(P\_4) = 1$. On each of these path graphs, the zombie's initial position could be any vertex of

the graph. For this example, we put the {\it G-zombie} and {\it H-zombie} ($G = P\_3$ and $H = P\_4$) both on vertex

$(1,1)$. We show the survivor with blue color, {\it H-zombie} with red, and {\it G-zombie} with green. {\it G-zombie}

will try to get to the same $G\_{i}$ as survivor which is $G\_3$ using an {\it H-edge}. {\it

H-zombie} will try to get to $H\_3$. See Figure \ref{fig:p4}.

\begin{figure}[h!]

\centering

\includegraphics[width=0.5\linewidth]{fig/p34m2.png}

\caption{First move of players}

\label{fig:p4}

\end{figure}

After zombie's move the survivor must move. No matter what move he makes, either {\it G-zombie} has made itself closer

to $H\_x$ or {\it H-zombie} has made itself closer to $G\_y$. In this case, {\it H-zombie} <<Or G-zombie?>> got closer to $H\_x$. Since

neither {\it H or G-zombies} share $H\_x$ or $G\_y$ with the survivor, they will still try to achieve that (See Figure \ref{fig:p5}).

\begin{figure}[h!]

\centering

\includegraphics[width=0.5\linewidth]{fig/p34m3.png}

\caption{Second move made by zombies, third in total}

\label{fig:p5}

\end{figure}

Now {\it H-zombie} shares the same copy of $H$ as survivor and now it is survivor's turn. If survivor moves to another $H\_i$,

{\it H-zombie} will mimic the move. If survivor makes an {\it H-move}, {\it H-zombie} will do whatever it did on a

single $H$. This means survivor cannot do infinite {\it H-moves}. Thus for him being able to survive he has to do

infinite {\it G-moves}, which again leads to {\it G-zombie} capturing him. For other moves, you can see Figure

\ref{fig:p6}. We cannot discuss each possible survivor move since they are a lot, but you can still easily apply the

strategy provided above.

\begin{figure}[h!]

\centering

\includegraphics[width=0.6\linewidth]{fig/p34m6.png}

\caption{Other moves made by players}

\label{fig:p6}

\end{figure}

\begin{corollary}

\label{C3}

$z(Q\_{n}) \leq \ceil\*{\frac{2n}{3}}$

\end{corollary}

\begin{proof}

We prove this by using both induction and the theorem proved above. First note that the Cartesian product of

hypercube graphs $Q\_{m}$ and $Q\_{n}$ is equal to $Q\_{m+n}$. It is easy to see $z(Q\_3) = 2$, $z(Q\_2) = 2$, and

$z(Q\_1) = 1$. For $n > 3$, we consider $Q\_n$ as the Cartesian product of $Q\_3$ and $Q\_{n-3}$. Using the induction

base, we know that $z(Q\_{n-3}) \leq \ceil\*{\frac{2n - 6}{3}}$. According to the proved conjecture $z(Q\_n) \leq

z(Q\_{n-3}) + z(Q\_3)$ and $z(Q\_{n-3}) \leq \ceil\*{\frac{2n - 6}{3}} = \ceil\*{\frac{2n}{3}} - 2$, we can see that

$z(Q\_n) \leq \ceil\*{\frac{2n}{3}}$ .

\end{proof}

It is already proved that at least $\ceil\*{\frac{2n}{3}}$ zombies are needed to capture one survivor on graph $Q\_n$

(Theorem 16 of \cite{Fitz16}):

\begin{theorem}

\label{T4}

For each integer $n \geq 1$, $z(Q\_n) \geq \ceil\*{\frac{2n}{3}} $.

\end{theorem}

Combining {\it Corollary \ref{C3}} and {\it Theorem \ref{T4}} we can conclude that $z(Q\_n) = \ceil\*{\frac{2n}{3}}$.

This proves Conjecture 18 from \cite{Fitz16} which is already proved in \cite{Offner19} with a different method.

\section{Capture time and Cartesian product}\label{capturetime}

We define two new parameters, $CT(G,z)$ (capture time) and $ZCT(G,k)$ (zombie capture time), where $G$ is a graph,

and $z$ and $k$ are two integers.

$CT(G,z)$ is the number of turns that survivor can avoid being caught, assuming that both players choose their best

strategies. Zombie player will play with $z$ zombies and tries to make this number as least as possible, while

survivor tries to maximize this.

$ZCT(G,k)$ is the minimum number of {\it zombies} needed so that zombie player is able to capture the survivor in at

most $k$ turns.

Also $diam(G)$ is the length of $G$'s diamater.

\begin{theorem}

\label{T5}

$CT( G \square H, Z\_G + Z\_H ) \leq diam(G) + diam(H) + CT(G, Z\_G) + CT(H, Z\_H)$

\end{theorem}

\begin{proof}

We prove this by showing that survivor's {\it G-moves} cannot exceed $diam(H) + CT(G, Z\_G)$. With the same

conclusion, it can be shown that {\it H-moves} cannot exceed $diam(G) + CT(H, Z\_H)$ as well.

After first $diam(H)$ {\it G-moves} that survivor makes, $z$ = $y$ holds. Now since for each {\it G-move} made

from now on by survivor, {\it G-zombies} can follow their strategy on a $G$ graph, after at most $CT(G,Z\_G)$

{\it G-moves}, survivor will be captured.

Since each of survivor's moves is either {\it G-move} or an {\it H-move} or even both (check definition above),

total number of moves cannot exceed $diam(G) + diam(H) + CT(G, Z\_G) + CT(H, Z\_H)$.

\end{proof}

\section{Zombie Capture time Problem is NP-Hard}\label{np-capturetime}

We define zombie capture time ($ZCT\_k$) problem as below:

INSTANCE: Let $G = (V,E)$ be a simple undirected graph. Given graph $G$ and a positive integer $z$.

QUESTION: Is $ZCT(G,k) \leq z$ ? In other words, can we capture the survivor after at most $k$ turns using $z$ zombies in graph $G$ ?

The {\it dominating set} problem is a well-known NP-Hard problem and is defined below:

INSTANCE : Given a graph G and an integer z.

QUESTION : Does G have a dominating set of size at most z ?

\begin{theorem}

$ZCT\_k$ $\in$ NP-Hard

\end{theorem}

\begin{proof}

We prove this by reducing a well-known NP-Hard problem (dominating set problem) to $ZCT\_k$ in polynomial time.

To have a better understanding, consider the case where $k=1$. For zombie player being able to capture survivor in

one move, every vertex not occupied by a zombie, should have a zombie neighbor. This is exactly the definition

of a dominating set. Dominating set is a subset $D$ of $V(G)$ such that every vertex not in $D$ is adjacent to

at least one vertex of $D$. This simply shows that $ZCT\_1 \in$ NP-Hard

Now consider $k$ to be an arbitrary number bigger than 1. We construct a new graph $G'$ from $G$. Suppose $G$

has $n$ vertices. For each vertex $v \in V(G)$, we add a {\it new} path with $k$ vertices ending in $v$.

\ref{fig:p7} We name each {\it new} vertex $(v,i)$ for $1 \leq i \leq k - 1$. Also a vertex $v$ is dominated by

vertex $u$ if $v = u$ or $v \in N[u]$. A vertex $v$ is dominated by a set $S$, if there is a $u \in S$ which

dominates $v$.

\begin{figure}[h!]

\centering

\includegraphics[width=0.9\linewidth]{fig/ZCT.png}

\caption{$G'$ obtained from $G = C\_5$ where for $k = 1,2,3$}

\label{fig:p7}

\end{figure}

$|V(G')|$ is of $O(nk)$. Thus creating $G'$ can be done in polynomial time. Now we solve $ZCT\_k$ on graph $G'$

and get a number $z$, and a set $S$ of vertices as our zombies' intial positions. We build a set $DS$ on $G$

from set $S$ on $G'$. For each vertex $v \in S$ or $(v,i) \in S$ add $v$ to $DS$. Now suppose $DS$ is not a

dominating set, thus there is a vertex $u$ not dominated by $DS$. This means there was no zombie on vertices $u$

and $(u,i)$ on graph $G'$. By having our survivor on $(u,k-1)$, there is no zombie with distance of $k$ or less

from him which means that he will not be captured if he doesn't move. Thus, $DS$ must be a dominating set and $

|minimumDominatingSet(G)| \leq |DS| \leq |S| \leq z$.

Now consider minimum dominating set of $G$, $MDS$. For each vertex $v \in MDS$ place a zombie on vertex $v$

of $G'$. These zombies can capture the survivor in at most $k$ moves. Consider survivor's intial vertex. It

should not be on {\it old} vertices of $G'$ since they are all dominated by our zombies and survivor would be

captured in one move. Suppose survivor is initially on $(u,i)$. If $u$ is occupied by a zombie already, that

zombie will move along the $u-path$ and since our survivor is trapped, he will be captured within $k$ moves. If

there is no zombie on $u$ there is a vertex $u'$ occupied zombie which $ \{u',u\} \in E(G)$. On its move, zombie

will move to $u$. Thus our survivor is trapped again and will be captured withing $k$ moves. Therefore, $z \leq

|minimumDominatingSet(G)|$.

By combining results, $z = |minimumDominatingSet(G)|$. Therefore dominating set problem is reduced to $ZCT\_k$.

In the following lemma \ref{limit-moves}, we prove this assumption changes nothing.

\end{proof}

\begin{lemma}

\label{limit-moves}

If survivor can avoid being captured after $(n + 1) \* n^2$ moves, he can avoid being captured forever.

\end{lemma}

\begin{proof}

Define $zombieDist$ as sum of the distances between each zombie and survivor. It is not hard to see that after 2

rounds of play, that is each player has played once, $zombieDist$ won't increase, since zombies are always

getting closer. Now we show that if $zombieDist$ does not strictly decrease after each player takes turn for $n

+ 1$ times, survivor can avoid being captured forever. Consider the sequence of vertices occupied by survivor in

last $n + 1$ moves. By pigenhole principle, one vertex has been seen by survivor atleast twice. If survivor keeps

repeating those moves, he will maintain his distance from zombies and will avoid being captured forever.

Therefore, for a graph $G$ which zombie has a strategy to win, after each $(n + 1)$ move, $zombieDist$ should

strictly decrease. $zombieDist$ is at most $n^2$, since there is not more than $n$ zombies and each zombie is at

distance at most $n$ from survivor (this bound can be easily improved). Thus, after $(n + 1) \* n^2$ moves,

zombies would capture the survivor.

\end{proof}

By using this lemma, we can see $ZCK\_k$ problem for $k > (n + 1) \* n^2$ and a graph with $n$ vertices are as same as

the problem for $ZCK\_{(n + 1) \* n^2}$, and by solving $ZCK\_{(n + 1) \* n^2}$ for $G$, we get $z(G)$ as well.

\section{Zombie Number Problem is NP-Hard}\label{np-zombienumber}

Now we define zombie number ($Z$) problem:

INSTANCE: Let $G = (V,E)$ be a simple undirected graph. Given graph $G$.

QUESTION: Is $Z(G) \leq z$ ?

\begin{theorem}

$Z \in$ NP-Hard

\end{theorem}

\begin{proof}

We reduce dominating-set problem to $Z$.

To do this, we add $n$ new complete bipartite graphs $K\_{n,n}$ to $G$. We call the new graph obtained $H$ and

recall the $G$-subgraph simply as $G$ and $i$-th $(1 \leq i \leq n)$ bipartite subrgraph as $K\_i$ . We name the

$j$-th $(1 \leq i \leq n)$ vertex in $K\_i$ in $b$-th $(b = 1,2)$ part $(i,j,b)$. For each

vertex $(i,j,b)$ we connect it to vertices $j$ and $N\_G[j]$ (neighbours of vertex $j$) in $G$. A vertex is We solve

the $Z$ problem on this new graph.

By having zombies on each vertex of $G$'s dominating-set, survivor will be captured on the first move and zombie

player wins. Thus, $z(H) \leq |DS(G)|$. Now suppose we have zombies less than $z < |DS(G)|$. We prove survivor can

avoid being captured.

Suppose zombie player has placed his zombies. Since we have $n$ bipartite subgraphs and $z < n$, there is a

bipartite subgraph, without any zombies in it ($k$-th bipartite graph). Since zombies are not dominating $G$,

there is a vertex $v$, not dominated by them. We place the survivor on vertex $(k,v,b)$ $b = 1$. Survivor has no

neighbours occupied by a zombie. On each survivor turn, there is a vertex $v$ in $G$ not dominated by a zombie,

survivor will move to vertex $(k,v,3 - b)$. We prove by following this strategy, he will survive. On zombie

turn, zombie player has zombies on either $G$ or some $K\_i$ $(i \neq k)$ or $K\_k$. Initially there is no zombie

in $K\_k$. Now whenever a zombie wants to join $K\_k$, first it has to be in $G$ in order to reach $K\_k$, since

there is no connection between $K$-subgraphs. Zombies in $G$ (e.g. at vertex $u$) are at distance 2 from

survivor $(u \rightarrow (k,u,3 - b) \rightarrow (k,v,b))$, which means at the end of their turn all of them should be at one of

$(k,v,b)$'s neighbours, $v , N[v] $ or, $ (k,1 \leq i \leq n,3 - b)$ on their next move. Therefore, each zombie

joining $K\_k$ does not share the same partition as survivor, and on the next survivor move, he goes to vertex

$(k,v,3-b)$ and shares the same partition with all zombies in $K\_k$, and since this vertex has no neighbour in

$G$ or other $K$-subgraphs, survivor will not be captured.

Now we have proved with zombies less than $|DS(G)|$, survivor will not be captured and with $|DS(G)|$ number of

zombies, he will be. This means by being able to solve $Z$ problem, we can obtain $DS(G)$, therefore dominating

set problem is reduced to $Z$.

\end{proof}

\begin{thebibliography}{999}

\bibitem{Fitz16}

Fitzpatrick, Shannon L., J. Howell, Margaret-Ellen Messinger, and David A. Pike. "A deterministic version of the

game of zombies and survivors on graphs." Discrete Applied Mathematics 213 (2016): 1-12.

\bibitem{Offner19}

Offner, David, and Kerry Ojakian. "Comparing the power of cops to zombies in pursuit-evasion games." Discrete

Applied Mathematics (2019).

\bibitem{West02}

West, Douglas B. "Introduction to Graph Theory." Prentice hall, (1996).

\bibitem{reviewer}

One of our reviewer's notes, Discrete Applied Mathematics, (2020).

\end{thebibliography}

\end{document}