Practical Work: Numerical simulation of dynamical systems

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1 Context

This practical work is to learn how to solve an ODE(Ordinary Differential Equation) numerically with the Euler's method. Afterwards, I am going implement another numerical method called Heun's method which is a bit improved version of Euler's. These underlying methods are implemented to solve Lotka-Volterra and Lorenz equations specifically in this practical work. Afterwards, some of the solutions are visually analyzed to observe the beauty behind them.

2 Implementation and results

In the practical work, it has been asked to implement 2 methods called **Euler's** and **Heun's** method. However, let me answer the following questions first:

1. **Q:** What do the terms $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent in the Lotka-Volterra systems?

   A: They represent the growth rates of preys and predators respectively.

2. **Q:** What do the parameters α, β, γ and δ represent?

   A: Relationship between preys and predators(i.e. rate of birth/morality).

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = -\gamma y + \delta xy$$

Figure 1: Lotka-Volterra equations

2.1 Euler's method

Principle behind the Euler's method. To solve an ODE numerically, we should use the definition of derivative:

$$f'(x) = \lim_{h\to 0} \frac{df(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

From the definition, $f(x+h)$ can be deduced easily:

$$\lim_{h\to 0} f(x+h) = f(x) + \lim_{h\to 0} f'(x)h$$

Since h is to be assigned to a constant [relatively small] value bigger than 0 in the actual implementation, limit part of the equations can be removed bringing a trade-off with accuracy drop in the final result:

$$f(x+h) \approx f(x) + f'(x)h$$

Results. Here is the not well approximated visualization of Lotka-Volterra system.

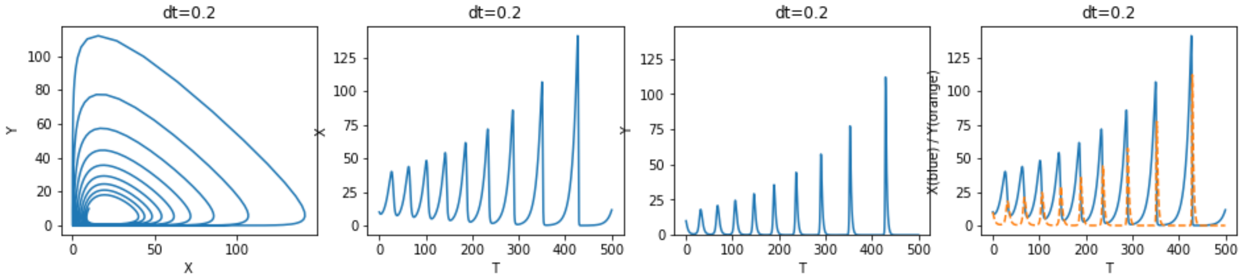


Figure 2: Lotka Volterra ($x = 10, y = 10, \alpha = 0.1, \beta = 0.02, \gamma = 0.4, \delta = 0.02, \Delta t = 0.2$)

X - predators, Y - preys, T - time. Here it is obvious that the populations of both predators and preys grow by the time passes.

The images below illustrates behaviour of the system according to the different Δt values(time differences).

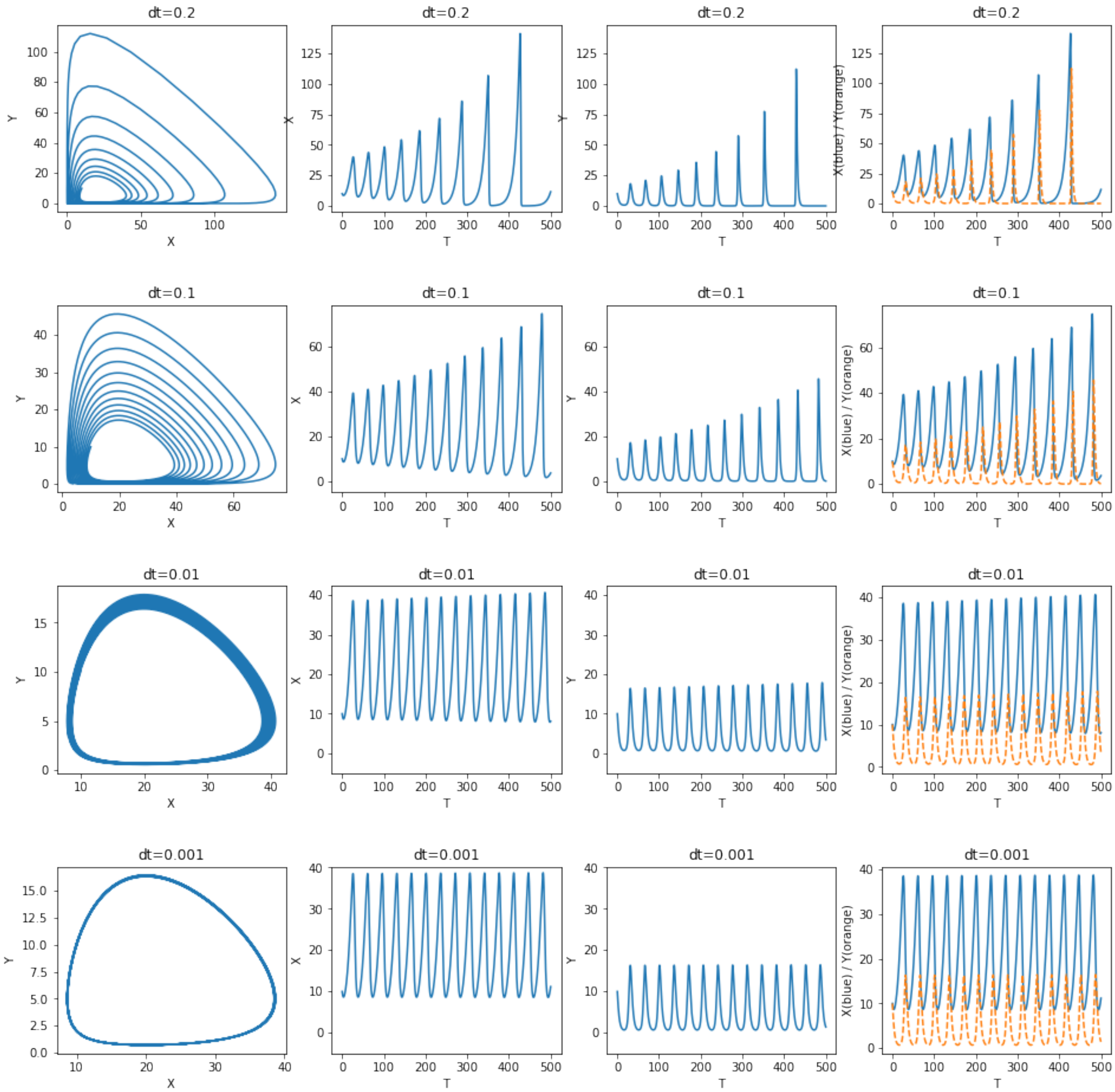


Figure 3: Lotka Volterra ($x = 10, y = 10, \alpha = 0.1, \beta = 0.02, \gamma = 0.4, \delta = 0.02$)

In these images, we can clearly see that how different time differences make difference in the calculations. When $\Delta t = 10^{-3}$, the populations just oscillate with a stable amplitude. This means that although the populations were growing when $\Delta t = 0.2$ (predators' faster than preys'), the population of predators fluctuates between [approximately] 10 and 40 and the population of preys fluctuates between [approximately] 0 and 18 when $\Delta t = 10^{-3}$.

2.2 Heun's method

This method is used to correct results obtained by the Euler's method by using trapezoidal rule to find the area under the curve. The basic principle of this rule is as follows:

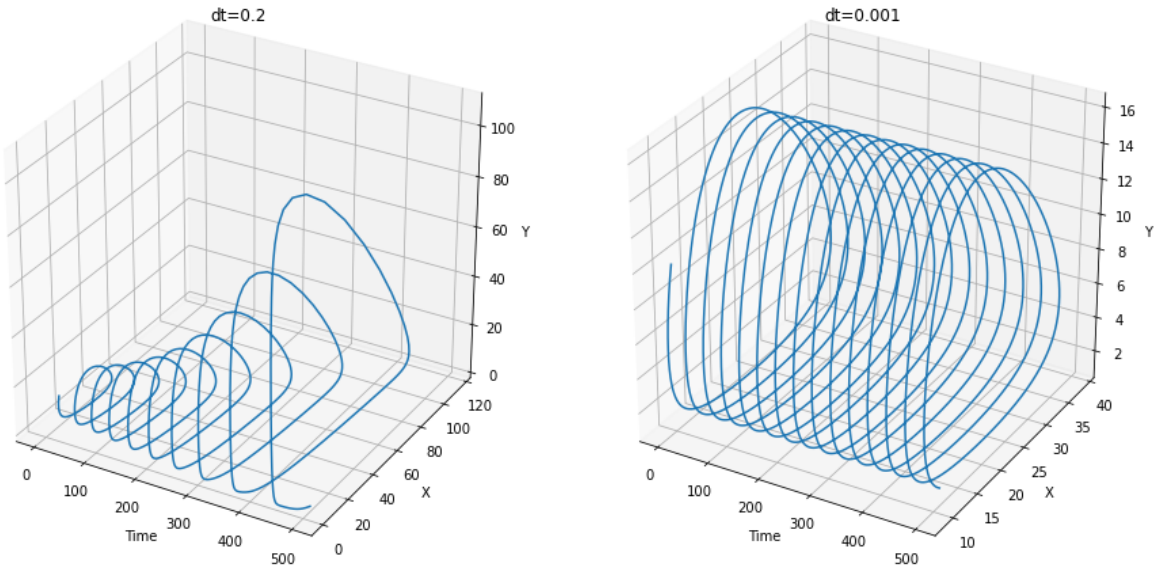
$$lim_{h\to 0}\frac{df(x)}{h} = f'(x) \implies \int_{x_0}^{x_0+h} f'(x)h = f(x_0+h) - f(x_0) \tag{2}$$

To find the area under the curve of $f'(x)$ in the interval of (x_0, x_0+h), trapezoidal rule is used:

$$\int_{x_0}^{x_0+h} f'(x)h = \frac{1}{2}(f'(x_0) + f'(x_0+h))h$$

Therefore, the equation 2 can be rewritten as follows:

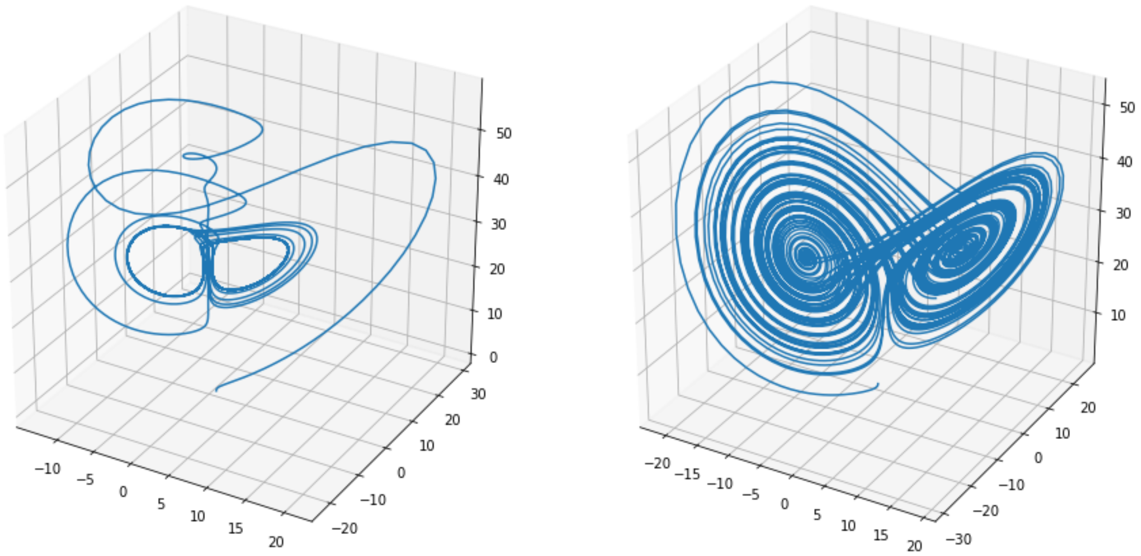
$$f(x_0+h) = f(x_0) + \frac{h}{2}(f'(x_0) + f'(x_0+h))$$



(a) $\Delta t = 0.2$ (b) $\Delta t = 10^{-3}$

Figure 4: 3D visualization of Lotka-Volterra system

2.3 Solving Lorenz ODEs



(a) $x = -1, y = 0, z = 1$

(b) $x = -1, y = 0.001, z = 1$

Figure 5: 3D visualization of Lorenz system ($\sigma = 10, \rho = 28, \beta = 8/3$)

3 Conclusions

We can conclude that these systems are deterministic chaotic which means that we sometimes cannot predict the outcomes of these systems even if there are well-defined formulas without any use of randomness. This is due to the small measurement errors(i.e. we can assume that $\Delta y = 10^{-3}$ was a measurement error in Lorenz system) that we make. As we observed in Lotka-Volterra system even if $\Delta t = 0.2$ was relatively small, it was not small enough to predict the system's behaviour right. Since we cannot use the number defined as $lim_{h \to 0} h$ (because this is an ideology), we always have to replace it with some constant which is small enough for the system to be predicted "right" and the problem is, how do we know what number is good enough for the system to predict its approximate outcome?

Another conclusion: The life of insects could be more complex than ours BUT since we are dealing with both our and their lives, ours is more complex :)