Comparative Analysis of the Existing Solutions for Bertrand's Paradox

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***Abstract*—Bertrand's Paradox, a cornerstone of probability theory, questions the fundamental principles of randomness and uniformity, arising from an apparently innocent experiment involving randomly chosen chords within a circle. This experiment exposes fundamental flaws in how probability theory handles randomness. Despite its historical significance and ongoing relevance, a discernible gap exists in the literature regarding a unified understanding of Bertrand's Paradox. This paper aims to bridge this gap by exploring its origins, implications, and the prominent solutions proposed in the literature. Edwin Jaynes' "Well-Posed Problem," a novel perspective reinterpreting Bertrand's challenge while maintaining its intrinsic randomness, serves as a central topic of discussion. Additionally, the Principle of Meta-indifference forms the foundation for Marinoff's Solution, and Jinchang Wang and Rodger Jackson's insightful analysis offers additional perspectives for dissecting the conundrum. The heart of the discussion lies in a critical analysis of these solutions, aiming to provide a reasoned argument for the most convincing resolution. This study attempts to demystify Bertrand's Paradox through a thorough investigation, giving readers a tangible comprehension of the complexities involved. The work challenges existing concepts of randomness and contributes to the ongoing discussion on probability theory by providing a reasoned argument for the most appealing resolution.**

I. Introduction

In many cases, solving mathematical problems requires one to come up with unique ideas or methods. This is because our initial understanding or intuition about the problem might not be easily translated into mathematical language with the tools we usually have at hand. So, what might seem like a tough part of a problem is often just one trying to convert these initial understandings into solid mathematical terms that can help one find a solution.

Joseph Louis François Bertrand was a French mathematician who worked in the field of number theory, differential geometry, probability theory, economics and thermodynamics. In 1945, he conjectured that there is at least one prime between n and $2n-2$ for every $n > 3$ [1], now named as the Bertrand's Postulate. In 1849, he was the first to define real numbers using what is now known as a Dedekind Cut [1] [2]. He is also famous for proposing a paradox in the field of probability theory, now known as the Bertrand's Paradox [3], which has since stood as a cornerstone example in the study of probability theory. The paradox itself stems from a very simple experiment involving the random selection of chords within a circle. Despite its seemingly simple nature, this experiment leads to varying and contradictory probabilities based on the method chosen for chord selection. His contributions to mathematics, especially his work on probability has been a significant force in further research and advance in this field.

Introduced by Bertrand in his book "Culcul des Probabilités" (1889) [3], this paradox presents a perplexing scenrio in probability theory. At its core, Bertrand's Paradox asks a seemingly innocent and straightforward question, *"What is the probability that a randomly chosen chord in a circle is longer than the side of the inscribed equilateral triangle?"* [3] The paradox arises because the term "randomly chosen" is not well-defined in this context, leading to multiple interpretations, and, consequently, different probabilities. This fundamental inconsistency challenges the very foundations of probability theory, and our understanding of randomness. The paradox has significant implications for our understanding of randomness and uniformity, two fundamental concepts in probability theory. It challenges the notion that a problem can have a single, definitive answer, instead suggesting that the answer is open to the way one perceives or interprets the problem at hand.

Central to the understanding of Bertrand's Paradox is the Principle of Indifference, which states that in the absence of any other information, one should assume that all possible outcomes are equally likely [4]. This principle is often used to assign probabilities to events, particularly in situations where the domain is finite. However, its applications in scenarios with potentially infinite outcomes such as the Bertrand's Paradox, for which it is the perfect example, shows that this principle may not necessarily result in well-defined results for probabilities. Bertrand's Paradox, therefore, challenges the very foundations of probability theory, and our understanding of randomness. Randomness, in probability theory, refers to phenomena for events that are not completely predictable in advance. That is, it is a measure of the unpredictability of an outcome. A random variable is an assignment of a numerical value to every possible outcome of an event space. The Bertrand's Paradox shows that if probabilities are not well-defined, then the mechanism that generates random variables will also not be well defined.

Through our paper, we aim to provide a clear, comprehensive overview of Bertand's Paradox, and how it forces us to reconsider our understanding of randomness and uniformity, two fundamental concepts in probability theory. We begin with a detailed explanation of the paradox, followed by a review of prominent solutions proposed in literature. The heart of this paper lies in the critical analysis and comparison of those solutions, culminating in a reasoned argument for the most convincing resolution.

II. Explanation

A. Principle of Indifference

The *Principle of Indifference*, also known as the *Principle of Insufficient Reason*, is a fundamental concept in probability theory and decision making. It asserts that in the absence of any distinguishing information, one should assign equal probabilities to all possible outcomes. Originating from the works of Jacob Bernoulli and Laplace, this principle is especially relevant in contexts where there is a clear lack of bias or preference among outcomes, such as in games of chance [5].

Mathematically, the principle can be stated as If there are n mutually exclusive and collectively exhaustive outcomes, and there is no basis to consider any one of these outcomes more likely than the others, then each outcome should be assigned a probability of $\frac{1}{n}$. This approach is intuitively appealing and simplifies the assignment of probabilities in situations where little or no empirical data is available.

However, the application of this principle must be approached with caution. For example, consider a simple scenario involving the toss of a fair coin. Intuitively, we assign a probability of 0.5 to each outcome (heads or tails) due to the symmetry of the situation. This symmetry provides a 'sufficient reason' for the application of the principle. Yet, when the principle is applied to more complex situations, such as predicting weather conditions or assessing risk in financial markets, its utility becomes less clear. In these cases, the outcomes do not possess inherent symmetries, and the state space can be partitioned in numerous ways, leading to different probability assignments.

The [5] further elaborates on the challenges of applying this principle in continuous settings. For instance, when dealing with a continuous interval, such as $[0, 1]$, assigning a uniform distribution (a continuous analog of the principle of indifference) becomes problematic. The continuous state space often contains an infinite number of possible outcomes, making it difficult to justify the assignment of equal probabilities.

While the Principle of Indifference serves as a useful heuristic in certain situations, especially where outcomes are symmetric and information is scarce, its application in more complex and continuous domains is fraught with challenges. It often results in arbitrary and potentially misleading probability assignments, necessitating a more nuanced approach in these contexts. The principle remains a topic of debate and refinement within the fields of probability theory and decision analysis.

B. Bertrand's Paradox

Bertrand's Paradox is a captivating conundrum that challenges our intuition about probability and randomness. At its heart, this paradox revolves around a simple geometric scenario: a circle and a chord drawn at random within it. The central question it raises, as elegantly put by Bertrand himself, asks about the chances of this randomly drawn chord being longer than the side of an equilateral triangle inscribed within the same circle. This seemingly straightforward question, first posed in Bertrand's seminal work in 1907 [3], leads us into a maze of probabilistic reasoning.

The circle in this puzzle is defined by a radius, let's call it r, and our attention is drawn to the length of a randomly chosen chord, which we'll denote as l. The comparison is made against the side of an equilateral triangle that perfectly fits inside the circle, with each side measuring $\sqrt{3}r$ The mathematical challenge here is to determine the probability that l exceeds $\sqrt{3}r$ a question that symbolizes $\mathbb{P}\left[l > \sqrt{3}r\right]$.

What makes Bertrand's Paradox particularly intriguing is the approach to its solution. Bertrand himself introduced three different methods, each grounded in the principle of indifference, which is a foundational concept in probability theory suggesting that in the absence of any distinct preferences, all outcomes should be considered equally likely. However, the paradox lies in the fact that each of these methods, though seemingly reasonable and valid, leads to a different answer. This discrepancy invites us to delve deeper into the nature of probability and challenges our understanding of what it means to make a 'random' choice.

As we examine these methods and their implications, we are not just solving a mathematical puzzle; we are also exploring the philosophical underpinnings of probability theory. Bertrand's Paradox doesn't just ask us to calculate a probability; it invites us to contemplate the very principles that guide our reasoning in situations of uncertainty and randomness. Each approach to the problem illuminates a different aspect of this complex interplay between geometry, probability, and philosophy.

1) The Method of Random Endpoints: This method is predicated on the selection of two random points situated upon the circle's perimeter and involves the construction of the chord that links these two points. To accurately determine the probability in question, one can envisage the triangle being rotated in such a manner that one of its vertices coincides with one extremity of the chord. Upon observation, it becomes evident that if the chord's opposing extremity falls upon the arc that extends between the endpoints of the triangle's side that is opposite to the initial point, the chord's length will invariably surpass that of the triangle's corresponding side. Notably, this particular arc represents exactly one-third of the circle's entire circumference. Consequently, the probability of a chord, drawn at random, having a length that is greater than that of the triangle's side, is correspondingly deduced to be precisely $\frac{1}{3}$. This conclusion is drawn from the geometric properties inherent in the circle and the triangle, and the specific positional relationship between the chord and the triangle within the circle.

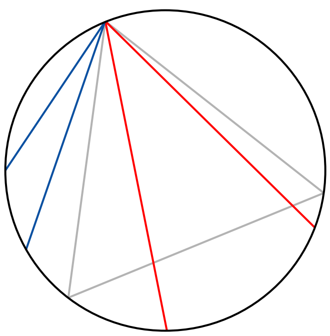

Figure 1: Illustration of random chords using the first selection method; red = longer than triangle side, blue = shorter - reference

2) The Method of Random Radial Point: In this particular methodology, the process commences with the selection of a radius within the circle. Following this, a point is chosen along the chosen radius. A chord is then constructed, which not only passes through this designated point but also is perpendicular to the radius initially selected. To calculate the probability in focus, one can imagine the inscribed triangle being rotated in such a way that one of its sides becomes perpendicular to the radius. The situation where the chord's length surpasses that of the triangle's corresponding side arises when the point selected on the radius is closer to the center of the circle than the point where the triangle's side intersects with the radius. Considering that the side of the triangle effectively bisects the radius, it can be inferred that the probability of a randomly constructed chord being longer than the side of the triangle is estimated to be $\frac{1}{2}$. This estimation is based on the geometrical relationship between the circle, the radius, the chord, and the inscribed triangle, particularly focusing on the relative positions of the chord and the triangle concerning the circle's radius.

3) The Method of Random Midpoint: This technique entails the selection of a random point within the confines of the circle, followed by the construction of a chord that identifies this point as its midpoint. The length of the chord

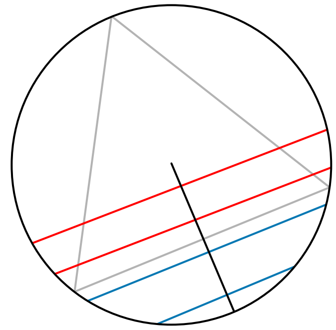

Figure 2: Random chords using the second selection method - reference

will surpass that of the triangle's side if the chosen point resides within the bounds of a concentric circle, which possesses a radius that is half the size of the larger circle's radius. Given that the area of this inner circle constitutes one-fourth of the total area of the larger circle, the likelihood of a randomly generated chord having a length greater than that of the side of the triangle is thereby calculated to be $\frac{1}{4}$. This conclusion is derived from understanding the spatial relationship and proportional areas between the two concentric circles and examining how this interplay affects the length of the chord about the side of the triangle.

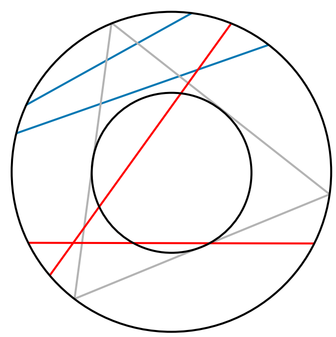

Figure 3: Random chords using the third selection method - reference

III. Proposed Solutions

A. The Well-Posed Solution

In 1973, Edwin Jaynes proposed a solution to Bertrand's paradox in his paper "The Well-Posed Problem" [6]. In this paper he redefined Bertrand's problem in such a manner that he does "no violence" to the problem it remains just as random as Bertrand originally defined it to be. He defines the problem as such,

"A large straw is tossed randomly into the circle, if it falls such that it intersects the circle, what will be the probability that the chord formed due to this intersection is longer than a side of the equilateral triangle inscribed within the circle?"

He explains that there are several ways of describing the equally possible solutions for this and all of them lead to different results, thereby pointing out the logical inconsistencies in Laplace's Principle of Indifference. Three of the possible solutions that he then refers to in his paper are to assign uniform probability density to:

- **Solution A:** The linear distance between the center of the chord and the circle ($p_A = 1/2$)
- **Solution B:** Angles of intersection of the chords on the circumference ($p_B = 1/3$)
- **Solution C:** The center of the chord over the interior area of the circle ($p_C = 1/4$)

These solutions are the redefined versions of the three possible solutions that Bertrand suggested himself. Before exploring the work of Jaynes it is important to note that many mathematicians and philosophers including Bertrand himself have called this problem an "ill-posed" problem due to vague definition of "randomness" in this context.

This problem has been approached in probability theory through the Frequentist approach as the Principle of Indifference is rejected unequivocally. This implies that conducting an experiment is the only way to assign probabilities and therefore answer Bertrand's question.

Jaynes also talks about the invariance argument given by Poincare for this paradox in which he defines the problem as straight lines being drawn at random on the xy plane, located using the parameters u and v such that the equation for the line is given as $ux + vy = 1$. Poincare then asks, which probability density $p(u, v)dudv$ has the property of invariance under the Euclidean rotations and translations. The solution to this is

$$p(u, v) = \left(u^2 + v^2\right)^{-\frac{3}{2}}$$

However as convincing as this argument might look like, this has been rejected and Bertrand's take on this problem has been preferred over Poincare's as a random shower of straws cannot be expected to hold the property that Poincare suggested for the lines to have.

In his proposed solution, Jaynes considers two additional symmetries that must exist for this problem to have a definite solution. These two relevant symmetries are not apparent due to the slightly obscure definition of the problem. As Jaynes pointed out, neither the original formulation of the problem nor its redefinition (by Jaynes himself) specifies the exact location or size of the circle. For the problem to exhibit the property of randomness and to possess a definite and universal solution, it must be 'indifferent' to both of these conditions. From this point Jaynes begins his rigorous solution to this problem to prove the invariance of the problem with respect to the size and the location of the circle in the following manner:

1) Rotational Invariance: In this setting for a circle of radius R, the chord drawn on it has the position (r, θ) (given in polar coordinates). As the distribution of chord lengths is solely determined by the radial distribution, it follows that the probability density over the internal area is actually independent of θ.

As the problem does not specify the direction in which the observer is facing, a general and definite solution should stand irrespective of the direction in which the observer is facing. If two observers watching the experiment have an angular difference of α between their lines of sight. If both of them use the describe the experiment from their own reference frame and assign the probability density $f(r, \theta)$ and $g(r, \theta_\alpha)$, given that both of them are describing the same experiment, it follows that

$$f(r, \theta) = g(r, \theta - \alpha)$$

This property should regardless of the existence of rotational symmetry. However, due to rotational symmetry the situation appears as the same to both the observers, hence both the functions $f(r, \theta)$ and $g(r, \theta)$ are the same. As the property holds true for $0 \leq \alpha \leq 2\pi$ it can be concluded that

$$f(r, \theta) = f(r) \tag{1}$$

2) Scale Invariance: Using the Rotational Invariance principle for such a problem it is reduced to determining the distribution function $f(r)$ which can be normalized as,

$$\int_0^{2\pi} \int_0^R f(r) r dr d\theta = 1 \tag{2}$$

Now, if another circle is considered to exist within the main circle with radius aR where $0 < a \leq 1$. For the smaller circle there is a $h(r) r dr d\theta$ probability that the center of the chord lies in the area $dA = r dr d\theta$.

Now, if there is a straw that intersects that the smaller circle, it must also draw a chord on the large circle. Hence, $f(r)$ must be proportional to $h(r)$ and according to conditional probability this relation can be formalized as

$$f(r) = 2\pi h(r) \int_0^{aR} f(r) r dr \tag{3}$$

For $0 < a \leq 1$ and $0 \leq r \leq aR$. Again, eq. 3 should hold irrespective of scale invariance. Considering the case of two observers with different sized eyeballs. To them the problem would appear the same for both the circles, hence for a general solution it must be shown that both the problems are scaled versions of each other. The only way to do so is to compare the area elements for both the circles.

$$\begin{aligned} h(ar)(ar)d(ar)d\theta &= f(r) r dr d\theta \\ h(ar)(ar)^2 &= f(r) r^2 \\ a^2 h(ar) &= f(r) \end{aligned} \tag{4}$$

Combining eq. 3 and eq. 4 gives,

$$a^2 f(ar) = 2\pi f(r) \int_0^{aR} f(u) u du \tag{5}$$

For $0 < a \leq 1$ and $0 \leq r \leq aR$. Differentiating eq. 5 w.r.t a, substituting $a = 1$ and solving the resulting differential equation gives

$$f(r) = \frac{q r^{q-2}}{2\pi R^q} \tag{6}$$

Where q is a constant and $0 < q < \infty$, and cannot be determined using scale invariance. If $q = 1$ then **Solution A** is compatible with scale invariance and if $q = 2$ then **Solution C** becomes compatible with scale invariance, however **Solution B** cannot be compatible with scale invariance as it corresponds to $f(r) = (R^2 - r^2)^{-\frac{1}{2}}$, which does not align with eq. 6. This implies that at most, the probability assignment for **Solution B** will be compatible with circle of only one size and not any other circle inscribed within that.

3) Translational Invariance: There can be a situation where the straw tossed at random intersects two circles C, C', both of radius R with a relative displacement of b between them. Fig. 4 shows that the midpoint of the chord w.r.t circle C is P with co-ordinates (r, θ) (the prime quantities are mentioned accordingly for the other circle). Through simple geometrical analysis,

$$r' = |r - b\cos(\theta)|$$

$$\theta' = \begin{cases} \theta & r > b\cos(\theta) \\ \theta + \pi & r < b\cos(\theta) \end{cases}$$

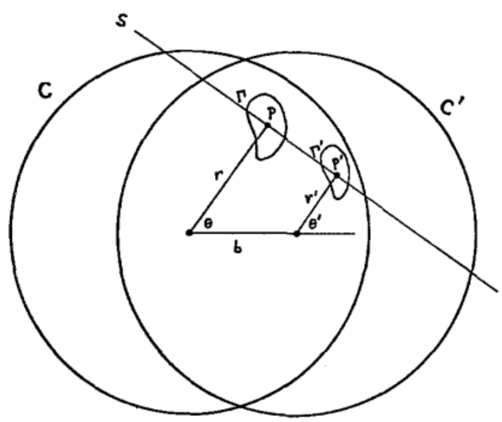

Figure 4: A straw S intersecting two circles displaced by b [6]

When P varies over the region Γ, P' varies over the region Γ', and a one-to-one mapping can be drawn from Γ onto Γ'.

Yet again, as there was no information given about the location of the circle, for two observers displaced by some distance, the problems for both the circles would appear the same. The necessity for consistency mandates here that the probability densities for both the circles such that it takes the form of equation 6 for the same values of q as discussed earlier. This further leads to the assignment of the same probability density to both the circles as not only is there a correspondence between the between the intersection of the circles but essentially these are the same events.

The probability of a Chord intersecting C having its midpoint in Γ is

$$\int_{\Gamma} f(r) r dr d\theta = \left(\frac{q}{2\pi R^q}\right) \int_{\Gamma} r^{q-1} dr d\theta \tag{7}$$

Similarly for C' and Γ',

$$\frac{q}{2\pi R^q} \int_{\Gamma}^{\prime} (r')^{q-1} dr' d\theta' = \left(\frac{q}{2\pi R^q}\right) \int_{\Gamma} |r - b\cos(\theta)|^{q-1} dr d\theta \tag{8}$$

Eq. 7 and Eq. 8 will be equal for any Γ iff $q = 1$. Thereby uniquely determining the distribution $f(r)$. Hence, the proposed **Solution C** is also eliminated due to lack of translational invariance.

B. Marinoff's Solution and Principle of Meta-indifference

Marinoff proposed his solution in 1994, which is one of the most accepted solution for the Bertrand's Paradox. He claimed that the Bertrand's problem is vaguely posed therefore using different approaches we get different but self consistent results. According to Marinoff, the principle of indifference is consistently applicable to infinite sets as long as problems can be formulated unambiguously. The principle of indifference suggests that in the absence of any relevant information to differentiate between possible outcomes, one should assign equal probabilities to each outcome. He suggest that using the distinction strategy, we can resolve a paradox whose identity is indeterminate into several distinct determinate problems which are not ill-posed.

The Bertrand's paradox is vaguely posed but we know that using the prescribed triangle to find the probability of the chord being longer is not ambiguous. The ambiguity or vagueness lies in the random process for generating chords. It can also be seen in above explanation that one clearly faces methodological alternatives while generating random chords. Thus it can be realized that Bertrand's 3 answers are constructed as replies to 3 different questions regarding the probability of the chord being larger than the side of inscribed circle when it is generated [7]

- by a procedure on the circumference of the circle?
- by a procedure outside the circle?
- by a procedure inside the circle?

Marinoff's argues that Bertrand's question confounds distinct problems. He quotes Keynes's statement in his paper, a re-owned British economist, who emphasizes about using alternative approaches carefully such that the principle of indifference can be applied unambiguously. Conclusively, it will not confuse distinct problems and will lead to such results in geometrical probability which are unambiguously valid. Van Frassen, a famous Dutch-American philosopher, said that to solve Bertrand's paradox it should be known or told what is random process, which events are equiprobable and which parameter should be assumed to be uniformly distributed [8]. However, when we are given a uniform distribution of a parameter, we eventually reject the principle of indifference. According to him, a generic problem such as Bertrand's paradox has multiple if not infinite solutions, and it depends on how the problem was interpreted and what geometric entities were assumed to be uniformly distributed.

Marinoff's argument is that many versions of Bertrand's paradox can be resolved by using distinction strategy in which there is a generic singular question. All the solutions depend upon the consistent application of principle of indifference to infinite sets leading to non contradicting results. Whenever we have ignorance or uncertainty, we can imply principle of indifference. Extending this principle of meta-indifference was proposed which implies a level of indifference not just among the outcomes but also among the various ways we might measure or assign probabilities to those outcomes.

Principle of meta-indifference When dealing with a set of possible outcomes X, a collection of measurable events σ-algebra, $\sum$, on X, and a set of probability measures M on those events, and assuming there's no reason to prefer one measure over another in M, then all measures in M are treated as equally probable. The probability of any specific event X is then calculated by averaging the probabilities assigned to $X$by each measure in M.
For all x in $\sum$, P(x) = the mean over all μ in M of μ(x). [8]

C. Jinchang Wang and Rodger Jackson's solution

1) Bertrand's Paradox can only have one solution: Jinchang Wang and Rodger Jackson in 2011, came up with an intriguing solution to Bertrand's paradox. It is an added-on version of the Marinoff and Jaynes proposed solutions. To attempt to solve a paradox, a person should attempt to solve the following three aims:

- The paradox should have only a single solution.
- Prove the correct solution through proper mathematical rigor along with disproving the other incorrect solutions.
- Point out the hurdles that were faced by the people while interpreting this paradox and explain what was lacking in the original explanation.

Now, keeping the above pointers in mind. Let us move on and now delve into establishing the fact that this paradox can only have one solution.

The authors mention that there is a basic question that has been skipped by almost all of the readers while skimming through this paradox and that is:"everyone agrees on what Bertrand-chords are like – they are homogeneously or uniformly distributed over the circle." [9]
To get a further in-depth meaning of the idea, refer to the following figures:

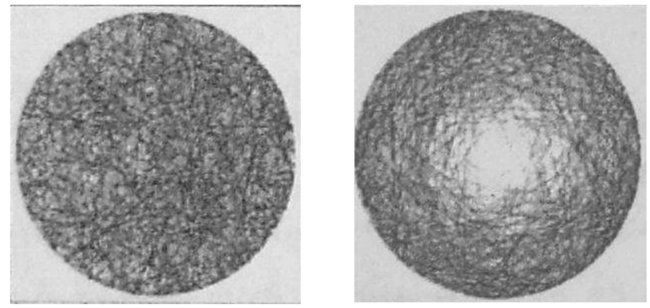

Figure 5: Left: Fig A ; Right: Fig B

Upon being inquired about which distribution of chords are Bertrand-chords, everyone would be inclined to choose Fig A since it is homogenously distributed whereas Fig B contains some hollow space in between, therefore, it is not homogenous. From this, we can also create a relation that if a distribution is random then, it is meant to be homogenous. The terms are interchangeable since both of the terms mean that the chords are distributed uniformly across the circle.

To recall some basic probability knowledge, let us choose a uniform distribution from [0,1]. What would be the probability of the pointers in the range to be greater than 0.5? The answer is straightforward i.e. 0.5 (50%) which signifies that it will have only a single solution. Therefore, as Bertrand chords are also uniformly distributed (homogenous), there should only be one correct solution out of the three. It is interesting to note that Fig B can also be one of the possible distributions but that would be counted as non-homogenous which would lead to multiple solutions. However, it is straightforward to decline since the basis of the Bertrand chords is that they are random or homogenous for this particular solution.

2) Bertrand-Chords formalized definition: We are fixing ourselves to two dimensions (XY-plane), and the chord direction in this context refers to the angle from the chord to the positive fixed x-axis in the clockwise direction. The chords are formed in a circle which is centered at the origin (O) and of radius (r). The range for the angle (α) of the chord direction is in the range from $[0°, 180°)$ with the distribution being continuous. As the data is continuous, therefore, the probability of having a certain α is zero. Adding to that, let $D_{\alpha-normal}$ be the line passing through the diameter of the circle such that it crosses the chords perpendicularly. The uniform points formed due to the intersection of the chords and the $D_{\alpha-normal}$ can be stated as follows:
Definition A: "Chords in C are Bertrand-chords or homogeneously distributed chords if and only if their chord-directions are uniformly distributed over range $[0°, 180°)$, and for any α between 0 and 180, the intersecting points of the chords in C_α with diameter $D_{\alpha-normal}$ are uniformly distributed along $D_{\alpha-normal}$." [9], where C_α is a chord at α angle with the x-axis.

3) Disprove of incorrect solutions and prove of the correct solution: **Solution** $P = \frac{1}{4}$
This solution is only acceptable when the points on the in-

tersection of the diameter and the chords are uniformly distributed. For visualization, refer to Figure 3. They counter-argue this solution as follows: [9]

I The set of all points which are the midpoints of the intersection on the Bertrand-chords are homogenously distributed.

II If all of the chords in the circle are homogenously distributed, then their midpoints are also homogenously distributed.

III If a random midpoint (M) is marked on the circle and the corresponding chord is generated while keeping M as the midpoint of the chord would lead to the set of chords being non-homogenously distributed in the circle.

IV If all of the chords (drawn randomly) in the circle are homogenously distributed, then their midpoints are non-homogenously distributed.

In this evaluation, arguments (I) and (II) are equivalent which are conveying the same idea. The differentiating or the disprove starts from (III) which simply says to mark a random point (M) in the circle and then draw a chord that meets the diameter perpendicularly at the center of itself. If you repeat this process for high iterations, you would observe a pattern like Fig. B which would lead to the distribution of the chords to be non-homogenous.

To disprove it, argument (IV) is introduced which is simply the inverse of argument (II). Therefore, establishing the fact that (II) is false. As (II) is false and (I) is the equivalent version of the argument (II) makes it to be false too. Thus, this proves that this is not the solution to this paradox. The proof of the argument (III) has been shown by Jinchang Wang and Rodger Jackson in their work. [9]

Solution $P = \frac{1}{3}$

The authors introduces the new term here, "Chord Angle β", which is the angle between the one point of the chord and the radius passing through the same point iin the circle. For visualization, refer to the following figure: [9]

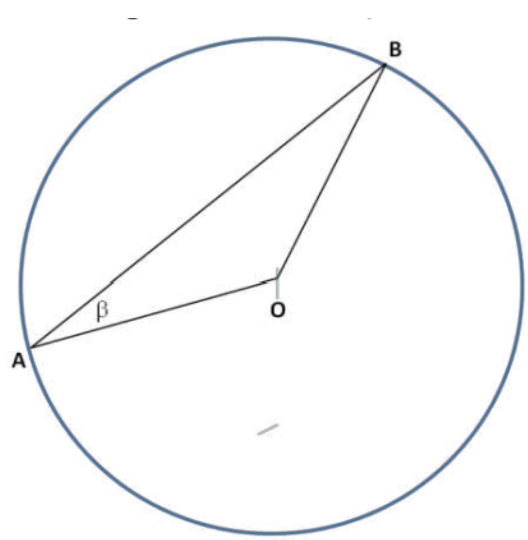

Figure 6

Let us define the above figure mathematically. The figure contains a circle centered at origin (O) and radius (r) with a chord (AB) and radiuses (OA and OB) with (OA) touching chord point A. Therefore, chord angle (β) is the angle between the chord and the radius at point A. It can be visualized from the diagram that it is an isosceles triangle with angle ($\angle ABO = \beta$). The range for β is $0° - 90°$.

It is important to notice that this is different from chord-direction which is in-respect to the fixed x-axis of the circle placed in a XY-plane. For this solution to be true, the chord-angles (β) needs to be uniformly distributed from $0° - 90°$. If they are, the, this solution is true and should accepted.

They counter-argue this solution as follows: [9]

I If the chords present in the circle are all Bertrand-chords, then the angles are uniformly distributed across the circle between $0° - 90°$.

II If the chords are randomly distributed (homogenously distributed) throughout the circle, then again, the set of angles are uniformly distributed between $0° - 90°$.

III If a set of chord's angles for a set of chords in the circle are uniformly distributed from $0° - 90°$, then, this would lead to the non-homogeneity of the chords in the circle.

IV If a set of homogenous set of chords of the circle are chosen, then the angles of the chords are not uniformly distributed from $0° - 90°$.

We can see the similarity of the disprove from the last solution. It starts with some base cases which are then falsified by a mathematical rigorous proof by the authors. [9] The range is from $0° - 90°$ and the allowed angle for the length to be greater than the triangle's side length is from $30°$ leading to a ratio of $\frac{1}{3}$.

The author states in (I) and (II) that if the chords are distributed homogenously or either the angles (β) are distributed homogenously, the paradox would hold true but they are not true according to the authors. They utilize (III) and (IV) to counter the solution and makes sure that it is not being accepted in the future by any one who tries to understand Bertrand's paradox.

The author utilizes the fact of placing the chords at the intersection of the radius with randomly generated chord-angle (β). However, this randomization leads to the chords spread to be non-homogenous across the circle. To generate a similar argument as (II) to counter the solution, (IV) has been generated which uses (III) and presents a similar version of itself which says that if the chords-set are homogenous i.e. Bertrand-chords exists, then, the chord-angle range is not uniformly distributed.

It can be concluded using same lines of reasoning as of the last disprove of the solution. (IV) is just a reversed version of (II) with chord-angles coming out to be non-homogenous which is a contradiction. This means that (II) is false and if (II) is false, this leads to the interchangeable argument (I) to be false too. Therefore, leading to the founding arguments to crumble and fail down. This would lead us to the next solution which is the only correct version with the solution of this paradox according to the authors, Jinchang Wang and Rodger Jackson.

Solution $P = \frac{1}{2}$

The authors have accepted this as the only correct solution for the paradox. For visualization, refer to Figure 2. From the figure, it can be observed that it is integral for the chords to be perpendicular every time they cross the radius line of the circle. This integral condition makes sure that there is half length of the radius as compared to the original length (R), leading to $\frac{1}{2}$. This could be re-stated as follows: [9]

I If there are C Bertrand-chords in the circle, then, the midpoints due to the intersection of the radius and the chord at any given radius would result in a uniform distribution.

Now, let us recall the definition A mentioned at the start of the discussion of this solution which conveys the same idea for the uniform distribution of the Bertrand-chords in which chords (C_α) were intersecting $D_{\alpha-normal}$ orthogonally. Since it matches with the definition of the homogenous or random distribution meaning that this solution is valid. The probability argument of $\frac{1}{2}$ is also valid using probability rules and the (I) also matches with the **Definition A**, thus, this solution is sound enough to be accepted as the only solution to the paradox.

IV. Discussion & Analysis

Marinoff's solution to Bertrand's Paradox assumes the premise that the question posed by the Paradox is generic in nature and not general. However, there does not seem to be a strong case for rejecting it if it's the latter. Van Frassen also highlighted this that if we are known what is the process being randomization and how different chords are generated, there is no ignorance hence no need of principle of indifference. As there is distinction strategy would fail here, therefore the solution is also flawed. Moreover, even if we ignore this and use the principle of metaindifference to find the answer of the statistically generalized generic question, it might result in consistent numerical probabilities or not. If we get the prior result then it suggests that the principle of indifference is applicable here as the problem is well posed. However, if we get the latter results, it means that no such probability exist therefore referring to the failure of principle of metaindifference.

Jaynes is credited with making a substantial contribution by suggesting $\frac{1}{2}$ as the right answer, offering rigorous and convincing proofs, and offering theoretically sound techniques for producing Bertrand-chords. Although Jaynes has been criticized for failing to refute competing answers and imposing restrictions to drive the problem from the realm of ill-posed problems to the realm of well-posed problems, his analytical rigor strengthens the validity of his solution. His solution holds immense significance due to its relevance to the problems in Physics and engineering like predicting the viscosity of gases given the total gas energy and the average particle density. Moreover, what Jaynes did with his solution, was not only to not base his solution based on pure thought and devised experiments to match the probabilities laid down by Bertrand. Jaynes argues throughout his paper that frequent trials matter the most for assigning probabilities in problems like these. He also points out the whole at the heart of Principle of indifference, which becomes apparent when there is lack of knowledge or known details regarding certain issues. His way of working around different invariances to make sure that the problem is not conditioned on restriction which stops from being generalized problem, works as an inspiration for others in terms of critical analysis of such simple yet entangled problem. Although his solution lead to the answer that only **Solution A** can be accepted as the correct solution, as the other two were ruled out with his rigorously formulated argument. This however, did not go unchallenged!

The book challenges Marinoff's claim that the dilemma is ill-posed because a "random chord" is ambiguous. Bertrand-chords, according to the text, have a consistent and well-defined idea that lays the groundwork for a unique solution.

The work presents Jaynes' answer as analytically persuasive, displaying a better comprehension of the issue and aiding in the resolution of Bertrand's dilemma by disproving competing theories and clearing up misconceptions. This assertion is refuted in the text, which contends that Bertrand-chords are homogeneously distributed and have a distinct and consistent idea. The analysis states that just $\frac{1}{2}$ is the answer to Bertrand's dilemma, resolving the paradox, and finds that Jaynes and Marinoff's views were incorrect, leading to useless debates and various solutions.

V. Conclusion

In conclusion, the Bertrand's paradox reveals a rich landscape of different solutions, each with its own unique insights into the intricacies of probability theory. For example, Marinoff's geometric approach offers an elegant solution by taking into account the spatial relationship between chords and lines. Jaynes's Bayesian lens calls into question the principle of indifference, emphasizing the need for a well-defined problem. His solution shows the power of using prior information systematically to improve probability assignments. Jinchang Wang's and Rodger Jackson's solution adds another layer of complexity to the discourse. Their geometrically-motivated approach attempts to reconcile different ways of chord selection. A comparison of these solutions highlights the significance of methodological choices and underlying assumptions when determining probabilities. The Bertrand paradox itself reminds us of the nuances and difficulties in probability theory. The ongoing discourse around the subject encourages a deeper comprehension of fundamental principles in probability, and a nuanced appreciation of the diverse perspectives that contribute to the resolution of the paradox.

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