CS/Math 113 - Problem Set 5

Dead TAs Society
Habib University - Spring 2023

Week 06

Problems

Problem 1. [Chapter 2.1, Question 10] Determine whether these statements are true or false.

(a) $\phi \in \{\phi\}$

(b) $\phi \in \{\phi, \{\phi\}\}$

(c) $\{\phi\} \in \{\phi\}$

(d) $\{\phi\} \in \{\{\phi\}\}$

(e) $\{\phi\} \subset \{\phi, \{\phi\}\}$

(f) $\{\{\phi\}\} \subset \{\phi, \{\phi\}\}$

(g) $\{\{\phi\}\} \subset \{\{\phi\}, \{\phi\}\}$

Solution:

(a) True

(b) True

(c) False

(d) True

(e) True

(f) False

(g) False

Problem 2. [Chapter 2.1, Question 21] Find the power set of these sets where a and b are distinct elements.

(a) $\{a\}$

(b) $\{a, b\}$

(c) $\{\phi, \{\phi\}\}$

Solution:

(a) $\mathcal{P} = \{\phi, \{a\}\}$

(b) $\mathcal{P} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

(c) $\mathcal{P} = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Problem 3. [Chapter 2.1, Question 22] Can you conclude that $A = B$ if A and B are two sets with the same power set ?

Solution:
The Power Set $\mathcal{P}(\mathbb{A})$ contains the all the possible subsets of the set $\mathbb{A}$. Therefore, the Union of all elements in the Power Set would result in the original set of $\mathbb{A}$.
Therefore, if A and B are two sets such that $\mathcal{P}(A) = \mathcal{P}(B)$, then the Union of the elements of $\mathcal{P}(A) =$ the Union of the elements of $\mathcal{P}(B)$. Hence, it can be concluded that $A = B$.

Problem 4. [Chapter 2.1, Question 23] How many elements does each of these sets have where a and b are distinct elements ?

(a) $\mathcal{P}(\{a, b, \{a, b\}\})$

(b) $\mathcal{P}(\{\phi, a, \{a\}, \{\{a\}\}\})$

(c) $\mathcal{P}(\mathcal{P}(\phi))$

Solution:

(a) We have 3 elements, therefore there are $2^3 = 8$ distinct elements.

(b) We have 4 elements, therfore there are $2^4 = 16$ distinct elements.

(c) The Power Set of null set has only 1 element [$\mathcal{P}(\phi) = \{\phi\}$]. Then the power set of the power set has $2^1 = 2$ elements $\implies \mathcal{P}(\mathcal{P}(\phi)) = \{\phi, \{\phi\}\}$

Problem 5. [Chapter 2.1, Question 25] Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$

Solution:

Since we have **if and only if**, then we will have to prove that the statement holds both sides.

Case 1: If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$

Consider any arbitrary element a such that $a \in A$. Then $\{a\} \subseteq A$. Therefore, $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, and $\{a\} \in \mathcal{P}(A)$, it follows (by transitivity) that $\{a\} \in \mathcal{P}(B)$. Since $\{a\}$ exists within the power set of B, then that implies that $\{a\} \subseteq B \implies a \in B$. Hence proved if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Case 2: If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Consider any arbitrary element a such that $a \in A$. Since $A \subseteq B$, then $a \in B$. Then we know that $\{a\} \subseteq A \implies \{a\} \in \mathcal{P}(A)$. Since $a \in B$, then $\{a\} \subseteq B \implies \{a\} in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Hence proved.

Problem 6. [Chapter 2.1, Question 26] Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Solution:

For two sets A and B, $A \times B$ is defined as a pair (a, b) where $a \in A,\ b \in B$. We know that $A \subseteq C$, and $B \subseteq D$, then for any arbitrary elements a and b, $a \in A,\ b \in B \implies a \in C,\ b \in D$. Therefore, $(a, b) \in C \times D$. Since $(a, b) \in A \times B$, therefore $A \times B \subseteq C \times D$. Hence shown.

Problem 7. [Chapter 2.1, Question 27] Let $A = \{a, b, c, d\}$ and $B = \{x, y\}$. Find

(a) $A \times B$

(b) $B \times A$

Solution:

(a) $A \times B = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y), (d, x), (d, y)\}$

(b) $B \times A = \{(x, a), (x, b), (x, c), (x, d), (y, a), (y, b), (y, c), (y, d)\}$

Problem 8. [Chapter 2.1, Question 38] Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$

Solution:

Consider that $A \neq B$. Then there exists some arbitrary element a in A such that a does not exists in B. $\exists a \in A$ such that $a \notin B$. And we know that $A \neq \phi$ and $B \neq \phi$[A and B are not empty]. Then for any arbitrary element $b \in B$, $(a, b) \in A \times B$ however, $(a, b) \notin B \times A$. The same can be said for any arbitrary element $b \in B$ where $b \notin A$. Therefore, $A \times B \neq B \times A$ for two non empty sets A and B if $A \neq B$ which implies it is only the case when $A = B$. Hence proved.

Problem 9. [Chapter 2.1, Question 44] Prove or disprove that if A, B, and C are nonempty sets, and $A \times B = B \times C$, then $B = C$

Solution:

$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
$B \times C = \{(b, c) \mid b \in B \wedge c \in C\}$
If $A \times B = B \times C$, then for any pair $(a, b) \in A \times B$, we can conclude that $(a, b) \in B \times C \implies a \in B,\ b \in C$. Therefore, $\forall a \in A \rightarrow a \in B$. Similarly, $\forall b \in B \rightarrow b \in C \equiv A \subseteq B \subseteq C$ and that without loss of generality $C \subseteq B \subseteq A$. Therefore $B = C$. Hence proved.

Problem 10. [Chapter 2.2, Question 5] Prove the complementation law in Table 1 by showing that $\bar{\bar{A}} = A$

Solution:

By definition of complement, $\forall a \in A \rightarrow a \notin \bar{A}$. Similarly by the definition of the complement, $\forall a' \in \bar{A} \rightarrow a' \notin A$.
Then $\forall a \bar{\bar{A}} \rightarrow a \notin \bar{A}$. Since a does not exist in $\bar{A}$, then by the definition, a exists in A. Hence we can conclude that $\bar{\bar{A}} = A$.

This proof can be written in another way:
$\bar{\bar{A}} = \{x | \neg x \in \bar{A}\} = \{x | \neg\neg x \in A\} = \{x | x \in A\} = A$

Hence proved.

Problem 11. [Chapter 2.2, Question 11] Let A and B sets. Prove the commutative laws from Table 1 by showing that

(a) $A \cup B = B \cup A$

(b) $A \cap B = B \cap A$

Solution:

(a) By the definition, $A \cup B = \{x|x \in A \vee x \in B\}$
Then from the definition,
$A \cup B = \{x|x \in A \vee x \in B\} = \{x|x \in B \vee x \in A\} = B \cup A$.
Hence proved.

(b) By the definition, $A \cap B = \{x|x \in A \wedge x \in B\}$
Then from the definition,
$A \cap B = \{x|x \in A \wedge x \in B\} = \{x|x \in B \wedge x \in A\} = B \cap A$
Hence proved

Problem 12. [Chapter 2.2, Question 19] Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

(a) by showing each side is a subset of the other side

(b) using a membership table.

Solution:

(a) By definition, $\overline{A \cap B \cap C} = \{x|x \notin A \vee x \notin B \vee x \notin C\} = \{x|x \in \bar{A} \vee x \in \bar{B} \vee x \in \bar{C}\} = \bar{A} \cup \bar{B} \cup \bar{C}$. Hence shown that $\overline{A \cap B \cap C} \subseteq \bar{A} \cup \bar{B} \cup \bar{C}$.
Conversely, by definition $\bar{A} \cup \bar{B} \cup \bar{C} = \{x|x \in \bar{A} \vee x \in \bar{B} \vee x \in \bar{C}\} = \{x|x \notin A \vee x \notin B \vee x \notin C\} = \overline{A \cap B \cap C}$. Hence proved that $\bar{A} \cup \bar{B} \cup \bar{C} \subseteq \overline{A \cap B \cap C}$

(b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	0	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0

Hence shown that their truth values are same, hence they are logically equivalent.

Problem 12. Prove or disprove that for all sets A, B,and C we have

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution:

(a) Suppose that $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in (B \cup C) \implies y \in B \vee y \in C$. Since y exists in either B or C, and x exists in A, then $(x, y) \in (A \times B) \vee (x, y) \in (A \times C) \implies (x, y) \in (A \times B) \cup (A \times C)$. Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Similarly, suppose that $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in (A \times B) \vee (x, y) \in (A \times C) \implies x \in A$, and $y \in B \vee y \in C \implies y \in (B \cup C)$. So, $(x, y) \in A \times (B \cup C)$. Therefore $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Hence proved that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) Suppose that $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C \implies y \in B,\ y \in C$. Since y exists in both B and C, and x exists in A, then $(x, y) \in A \times B$ and $(x, y) \in A \times C \implies (x, y) \in (A \times B) \cap (A \times C)$. Therefore $A \times (B \cap C) \subseteq (A \times B) \cap (B \times C)$.

Similarly, suppose $(x, y) \in (A \times B) \cap (B \times C)$. Then we know that $(x, y) \in A \times B$, and $(x, y) \in A \times C$. Therefore, $x \in A,\ y \in B,\ y \in C$. Hence $y \in B \cap C \implies (x, y) \in A \times (B \cap C)$. Therefore $(A \times B) \cap (B \times C) \subseteq A \times (B \cap C)$. Hence proved that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Problem 13. [Chapter 2.2, Question 44] Show that if A and B are finite sets, then $A \cup B$ is a finite set.

Solution:

For any finite sets A and B, suppose that A has n elements and B has m elements where n and m are natural numebers. Then $A \cup B$ at most has $n + m$ elements. Since n and m are natural numbers, $n + m$ is also a natural number. Therefore, $A \cup B$ is finite.

Problem 14. [Chapter 2.2, Question 45] Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

Solution:

We know that A is an infinite set. Then consider that $A \cup B$ is a finite set. Then $A \cup B$ has a total number of n elements, where n is any natural number. However, we know that

A is infinite, therefore, A has more than n elements as A is infinite. Hence we have a contradiction. So $A \cup B$ cannot have n elements, but will have all elements of A, which implies $A \cup B$ is also infinite.