CS/Math 113 - Problem Set 1

Habib University - Spring 2023

Week 01

1 Definition

Definition 1. (Integer) An integer is a number with no decimal or fractional part and it includes negative and positive numbers, including zero. $[n \in \mathbb{Z}]$

Definition 2. (Even Integer) An integer is even if it can be written as $2k$ where k is an integer. $[n = 2k \mid k \in \mathbb{Z}]$

Definition 3. (Odd Integer) An integer is even if it can be written as $2k+1$ where k is an integer. $[n = 2k + 1 \mid k \in \mathbb{Z}]$

Definition 4. (Parity) The parity of an integer is its property of being even or odd.

Definition 5. (Natural Numbers) Natural numbers are a set of positive numbers from 1 to ∞ $[n \in \mathbb{N}]$

Definition 6. (Rational Numbers) Rational numbers are any numbers that can be expressed in the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$ $[n \in \mathbb{Q}]$

Definition 7. (Divisiblity) A nonzero integer m divides an integer n provided that there is an integer q such that $n = mq$. We say that m is a divisor of n and that m is a factor of n and use the notation $m|n$

2 Problems

Using the definitions above, solve the following problems.

Problem 1. Prove that the sum of two odd integers is even.

Problem 2. Prove that the product of two even integers is even.

Problem 3. Prove that the product of any two rational numbers is also a rational number.

Problem 4. Prove that the square of any natural number is also a natural number.

Problem 5. Prove that the square of any rational number is also a rational number.

Problem 6. In each case either prove the statement or find a counterexample.

(a) The sum of any three consecutive integers (positive or negative) is divisible by 3.

(b) The product any two even integers is divisible by 4.

(c) The product of any four consecutive integers (positive or negative) is divisible by 8.

(d) If $a - b$ has remainder 0 when divided by m, then a and b have remainders 0 when divided by m.

(e) If n is an odd integer, then $3n + 3$ is divisible by 6

Problem 7. Prove that the product of five consecutive integers is divisible by 120.

Problem 8. Prove that the sum of two postive integers of the same parity (odd/even) is even.

Problem 9. Prove or disprove that if $a + b$ is an odd integer, then both $a + x$ and $b + x$ are odd integers, where a, b, and x are integers.

Solution:

Problem 1. Odd + Odd is even.
Then the sum can be represented as $2k + 1 + 2k + 1 = 4k + 2 = 2(2k + 1)$.
Let $\alpha = 2k + 1$. Then $2(2k + 1) = 2\alpha$ which is of the form $2k$ from the definition. Hence is even.

Problem 2. Even x Even = Even.
$2k \times 2k = 4k = 2(2k)$.
Let $\alpha = 2k$. Then $2(2k) = 2\alpha$ which is of the form $2k$ from the definition. Hence is even.

Problem 3. Rational x Rational = Rational
Let x and y be rational numbers. Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$.
Then $x \times y = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.
Then ac and bd are also some integers, and it follows that the product must be rational from the definition.

Problem 4. Let n be a natural number. Then the square of n is $n \times n = n^2$ which is just the same multiple of the number. Hence it must exist in $\mathbb{N}$.

Problem 5. Rational squared = Rational.
Let x be a rational number. Then $x = \frac{a}{b}$.
$x^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$.
Since a and b are integers, their square must also be an integer.
Then, let $p = a^2$ and $q = b^2$.
So $x^2 = \frac{p}{q}$ which is a rational number from the definition.

Problem 6.
(a) Let $n_1 = n, n_2 = n + 1, n_3 = n + 2$ be consecutive integers.
Then $n_1 + n_2 + n_3 = n + n + 1 + n + 2 = 3n + 3 = 3(n + 1)$.
Since the sum is a multiple of 3, the sum is divisible by 3.
(b) Let $x = 2k$ and $y = 2l$.
Then $x \times y = 2k \times 2l = 4kl$.
Since the product is a multiple of 4, it must be divisible by 4.
(c) In 4 consecutive integers, there have to be 2 even numbers. Let the first number be n. Suppose n is even. Then $n = 2k \mid k \in \mathbb{Z}$. $n + 2 = 2k + 2 = 2(k + 1)$ will also be divisible by 2. Since both n and $n + 2$ are even and consecutive even integers, one of them must be divisible by 4. Therefore, the product of those two integers will be divisible by 2 and 4 both, so it will be divisible by 8. If n is odd, then $n+1$ must be even, and $n+3$ must also be even. Then $n + 1 = 2k$ and $n + 3 = 2k + 2$. One of these must be divisible by 4 so the product will be divisible by 2 and 4 both, hence it will be divisible by 8.
(d) Let $a = 3, b = 1$, and $m = 2$.
Then $a - b = 2$ and has remainder 0 when divided by m. However, neither a nor b gives a remainder of 0 when divided by m.
(e) Let $n = 2k + 1$.
Then $3n + 3 = 3(2k + 1) + 3$
$= 6k + 3 + 3$
$= 6k + 6$
$= 6(k + 1)$ which is divisible by 6. Hence proved.

Problem 7. In 5 consecutive integers $n, n + 1, n + 2, n + 3, n + 4$, exactly one integer will be divisible by 5, there must be at least 2 consecutive even integers out of which one must be divisible by 4 as they are consecutive, and at least one must be divisible by 3. Then at least one number is divisible by 2, one is divisible by 3, one is divisible by 4 and one is divisible by 5. Then the product of all these numbers must also be divisible by $2 \times 3 \times 4 \times 5$. The product comes out to be 120. Hence the number must be a multiple of 120, thus has to be divisible by 120.

Problem 8. Let $x = 2k + 1$ and $y = 2l + 1$.
Then $x + y = 2k + 1 + 2l + 1 = 2(k + l) + 2 = 2(k + l + 1)$.
Let $t = k + l + 1$.
Then $x + y = 2t$ which is even.

Similarly, let $x = 2k$ and $y = 2l$.
Then $x + y = 2k + 2k = 4k = 2(2k)$ which is even.

Problem 9. If $a + b$ is odd, then either a is odd, or b is odd. As odd + odd results in an even number, and even + even is even.
If a is odd, then b is even. However, x can either be even or odd. If x is even, then $a + x$ is odd, but $b + x$ is even as b was even. If x is odd, then $a + x$ is even as odd + odd is even, and $b + x$ is odd.
The same argument holds if b is odd and a is even. Hence disproved.
A simple counterexample can be shown. Suppose $a = 3$ and $b = 2$. Then $a + b = 3 + 2 = 5$ which is odd. Let $x = 1$. Then $a + x = 3 + 1$ which is even. Hence disproved.