Weekly Challenge 06: Proofs
CS/MATH 113 Discrete Mathematics

Ali Muhammad Asad

Habib University — Spring 2023

1. Perfect Universe

A *perfect universe* is a universe where the combination of any two elements of the universe yields a unique element of the universe (we write the 'combination' of element a with element b as the element ab), such that the following holds in the universe (henceforth referred to as U):

- Associativity: For all elements a, b, c in U, $(ab)c = a(bc)$.
- Existence of an *identity* element: There exists an element e in U such that when combined with any element a in U, it does not change a i.e. $\exists e \in U \ni \forall a \in U, ea = ae = a$. If e is such an element of U we call e the *identity* of U.
- Existence of *enemies*: For each element a in U, there exists an element b in U such that a combined with b produces the identity element e of U i.e. $\forall a \in U, \exists b \in U \ni ab = ba = e$. If b is such an element for a, we call b the *enemy* of a.

Note that a perfect universe need not be commutative, i.e., it is not necessary for all elements $a, b \in U$ to have the property that $ab = ba$.

(a) Prove that *in a perfect universe, there is only one identity element.*

Solution: From the definition, we know there must be some element e in U such that it does not change any other element a of the universe U. $\exists e \in U \ni \forall a \in U, ea = ae = a$.

To prove that there exists only one identity element means that the identity element must be unique. That there can only exist one such element.

We can show that if we consider two identity elements, they turn out to be one and the same.
Consider an identity element e_1. Then by the definition, $e_1 a = a e_1 = a$.

Similarly, consider an identity element e_2. Then by the definition, $e_2 a = a e_2 = a$.

Then on premultiplying either e_1 or e_2 by a, we get; $e_1 a = e_2 a$.

We know from the properties of the universe that there is an existence of enemies, that is $\forall a \in U, \exists b \in U \ni ab = ba = e$.
From the above property, we know there must be some such element b such that we get both the identity elements e_1 and e_2.
Then $e_1 ab = e_2 ab \implies e_1 = e_2$. Therefore we get that $e_1 = e_2$.

Further, on post multiplying, $ae_1 = ae_2$
$bae_1 = bae_2 \implies e_1 = e_2$.

Henceforth, we can see that either on post multiplying, or premultiplying, we get back the identity elements, furthermore, that both different identity elements that we considered are equal to each other. Therefore, both are one and the same.

Hence there must be only one identity element in the universe.

(b) Prove that *in a perfect universe, every element has a unique enemy.*

Solution:
From the definition of enemies, we know there must be some element b in U such that we get back the identity element e. $\forall a \in U, \exists b \in U \ni ab = ba = e$. Then $ab = e$ and $ba = e$ (We already know that there must be some unique identity element).

Consider an element a, with two different enemeies b_1 and b_2. Then $b_1 a = ab_1 = e$ and $b_2 a = ab_2 = e$.

To show that there is a unique enemey, we must show that b_1 and b_2 are the same.

Since b_1 and b_2 are both the enemeies of a, then we can say that $b_1 ab_2 = eb_2$.
We know that $ab_2 = e \implies b_1 e = eb_2$. From the definition, and uniqueness of the identity element, we know that $b_1 e = b_1$ and $eb_2 = b_2$. Therefore, $b_1 = b_2$.

Further, since b_1 is also the enemy of a, we can say that $b_1 ab_2 = b_1 e$. Then $b_1 a = e \implies eb_2 = b_1 e$. Similarly, by the identity element, $b_2 = b_1$.

Hence shown that both the enemies are equal to each other, therefore they are the same. Hence, for any arbitrary element a, a must have some unique enemy.

(c) Prove that *for any elements a and b of a perfect universe, the enemy of ab is the same as the enemy of b combined with the enemy of a.*

Solution: *Can be easily done through the definitions, use Associativity, the existence of identity elements[further that the identity element is unique for ab] and the unique existence of enemies. This result follows from the above properties, definitions, and uniqueness proofs.*