Practice Set 1

TA Solution

January 2022

Questions

Question 1: There are 2000 students on campus who own Team Fortress 2, Plants vs Zombies, or Kerbal Space Program. If 500 students own all three games, 200 own only Team Fortress 2, 350 own only Plants vs. Zombies, and 150 own only Kerbal Space Program, how many of these games in total are owned by Florida Tech students?

Question 2:

- Explain what it means for two sets to be equal.
- Describe as many of the ways as you can to show that two sets are equal.
- Show in at least two different ways that the sets $A - (B \cap C)$ and $(A - B) \cup (A - C)$ are equal.

Question 3: Which of the following set descriptions gives the set $\{2, 8, 14, 20, 26, 32\}$?

a) $\{n \in \mathbb{N} \mid n = 2x + 6 \text{ for some integer } x \text{ such that } 1 < x < 6\}$

b) $\{n \in \mathbb{N} \mid n = 6x + 2 \text{ for some integer } x \text{ such that } 1 < x < 6\}$

c) $\{n \in \mathbb{N} \mid n = 6x + 2 \text{ for some integer } x \text{ such that } 0 < x < 6\}$

d) None of the above

Question 4: Let $B = \{2, 3, 6, 9, 11\}$ and $C = \{1, 4, 6, 11, 15\}$. Which of the following sets are not any of $B \cup C$, $B \cap C$, and $B - C$?

a) $\{1, 6, 9, 15\}$

b) $\{6, 11\}$

c) $\{2, 3, 9\}$

d) None of the above

Question 5: Describe each of the following sets in set builder notation.

- $B = \{1, 2, 5, 10, 17, 26, 37, 50, \dots\}$
- $C = \{1, 5, 9, 13, 17, 21, \dots\}$
- $\mathrm{E} = \{\text{lemon}, \text{lime}, 1, 3, 5, 7, \dots\}$

Question 6: The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [Hint: First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

Question 7: Can you conclude that $A = B$ if A and B are two sets with the same power set?

Question 8: Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Question 9: Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Question 10: Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$ by

- showing each side is a subset of the other side.
- using a membership table.

Question 11: Prove the first distributive law by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solutions

Answer 1:

$$A = \textbf{People who own Team Fortress 2}$$
$$B = \textbf{People who own Plants vs Zombies}$$
$$C = \textbf{People who own Kerbal Space Program}$$
$$|A \cup B \cup C| = 2000$$
$$|A \cap B \cap C| = 500$$
$$|A \cap \overline{B} \cap \overline{C}| = 200$$
$$|\overline{A} \cap B \cap \overline{C}| = 350$$
$$|\overline{A} \cap \overline{B} \cap C| = 150$$

We know how many own all 3 or just 1 and everyone else owns 2 games. Therefore the total number of games owned is

$$500 \times 3 + 200 + 350 + 150 + (2000 - 500 - 200 - 350 - 150) \times 2 = 1500 + 700 + 1600 = 3800$$

Answer 2: For two sets to be equal, it means that all elements in one set are in the other and vice versa (each set is a subset of the other).
We can show it by a membership table (see whether for each case, the element is in a set if and only if it is in the other that is supposedly equal to it). We can also use set identities or show that the sets are subset of each other.
Way 1: Show $A - (B \cap C) = (A - B) \cup (A - C)$ using set identities

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{(B \cap C)} && \text{Definition of Set complement} \\ &= A \cap (\overline{B} \cup \overline{C}) && \text{DeMorgan's Law} \\ &= (A \cap \overline{B}) \cup (A \cap \overline{C}) && \text{Distributive Law} \\ &= (A - B) \cup (A - C) && \text{Definition of set complement} \end{aligned}$$

Way 2: Show $A - (B \cap C) = (A - B) \cup (A - C)$ showing each is a subset of the other
Show $A - (B \cap C) \subseteq (A - B) \cup (A - C)$
Let $x \in A - (B \cap C)$
then $x \in A \cap \overline{B \cap C}$
that is $x \in A \wedge \neg(x \in B \wedge x \in C)$
then $x \in A \wedge (x \notin B \vee x \notin C)$
that is $x \in A \wedge (x \in \overline{B} \vee x \in \overline{C})$
then $(x \in A \wedge x \in \overline{B}) \vee (x \in A \wedge x \in \overline{C})$
that is $(x \in A \cup \overline{B}) \vee (x \in A \cup \overline{C})$
then using definition of set complement,$(x \in A - B) \vee (x \in (A - C))$
then $x \in (A - B) \cup (A - C)$

Then show that $(A - B) \cup (A - C) \subseteq A - (B \cap C)$
Let $x \in (A - B) \cup (A - C)$
then $x \in (A - B) \vee x \in (A - C)$
that is using definition of set complement, $x \in (A \cup \overline{B}) \vee x \in (A \cup \overline{C})$
then $(x \in A \wedge x \in \overline{B}) \vee (x \in A \wedge x \in \overline{C})$
that is $x \in A \wedge (x \in \overline{B} \vee x \in \overline{C})$
then $x \in A \wedge (x \notin B \vee x \notin C)$
that is $x \in A \wedge \neg(x \in B \wedge x \in C)$
then $x \in A \cap \overline{B \cap C}$
that is $x \in A - (B \cap C)$
Hence shown

Answer 3: Difference is of 6 in each and starting is 2, therefore c

Answer 4:

$$B \cup C = \{1, 2, 3, 4, 6, 9, 11, 15\}$$

.

$$B \cup C = \{6, 11\}$$

$$B - C = \{2, 3, 9\}$$

Hence **a**

Answer 5:

$$B = \{x^2 + 1 |\ x \in \mathbb{N}\}$$

$$C = \{4x + 1 |\ x \in \mathbb{N}\}$$

$$C = \{x |\ x \in \{\text{lemon,lime}\} \vee (x \in \mathbb{N} \wedge x \bmod 2 = 1)\}$$

Answer 6: Firstly show that if $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ then $a = c$ and $b = d$. If $a \neq b$, then $\{\{a\}, \{a, b\}\}$ contains two elements, both sets and exactly one contains one element, and exactly one contains 2 elements, therefore $\{\{c\}, \{c, d\}\}$ must have the same property, hence $c \neq d$, so $\{c\}$ is the set containing one element and hence $\{a\} = \{c\}$ so $a = c$. The two element sets must also be equal and since $b \neq a = c$, $b = d$. The other possibility is that $a = b$ then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$ and hence $\{\{c\}, \{c, d\}\} = \{\{c\}\}$ and that can only happen when $c = d$. Since the set is $\{c\}$, therefore $a = c$ and hence $b = d$.
The next part we need to prove is that if $a = b, c = d$ then $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, This is immediate as we can see by substitution. Therefore the statement is true and proof is complete.

Answer 7: Yes. Yes we can. $A \in P(A) \subseteq P(B)$ hence $A \subseteq B$ similarly $B \in P(B) \subseteq P(A)$ hence $B \subseteq A$. Since each set is a subset of each other, they are the same.

Answer 8: Proof by counter example: Let $A = \{1\}$, then $(A \times A) \times A = ((1, 1), 1) \neq (1, 1, 1)) = A \times A \times A$.

Answer 9: We want to prove all elements (y, z) in $A \times B$ are in $C \times D$.

$$(y, z) \in A \times B \implies y \in A \wedge z \in B$$

Since $A \subseteq C$ and $B \subseteq D$, therefore

$$y \in C \wedge z \in D$$

and this implies that $(y, z) \in C \times D$
Answer 10: First show that $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$
Let $x \in \overline{A \cap B \cap C}$
then $x \notin A \cap B \cap C$
that means $x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}$
therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$. Then show that $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$
Let $x \in \overline{A} \cup \overline{B} \cup \overline{C}$
that means $x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}$
therefore $x \notin A \vee x \notin B \vee x \notin C$
that means x can not be in intersection of A, B, C therefore
$x \notin A \cap B \cap C$
and we can conclude that $x\overline{A \cap B \cap C}$

Using membership table

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

Table 1: Membership Table

Answer 11: If x is in $A \cup (B \cap C)$ then x is either in A or in $(B \cup C)$. Therefore, we have to consider two cases:

1. If $x \in A$, then $x \in (A \cup B)$ as well as in $A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.
2. If $x \in B \cap C$, then $x \in A \cup B$ because $x \in B$, and $x \in (A \cup C)$ because $x \in C$. Hence, again $x \in (A \cup B) \cap (A \cup C)$.

This shows that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
To finish the proof, we also have to prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Then x is either in $(A \cup B)$ or $(A \cup C)$. Again, two cases.

1. x is in the above set and $x \in A$. In this case, it must also be in $A \cup (B \cap C)$
2. x is in the above set and $x \in B$. In this case, it must also be in C (Similarly if x was in the above set and $x \in C$, then it would have to be in B, therefore these two cases are the same) . Hence $x \in (B \cap C)$ therefore it is in $A \cup (B \cap C)$.

This proves the subset inequality. Since both sets are subset of each other, therefore this law is true and you can do your Discrete Math assignments.