Homework 3: Inference and Relations

upper-bound

CS/MATH 113 Discrete Mathematics
Habib University, Spring 2022

1 Inference

1. Prove the validity of the following arguments using inference rules. Mention the rule(s) that you use at each step.

   (a) 5 points

P1	If it is Sunday today, then we play cricket or basketball.
P2	If the basketball field is occupied, we don't play basketball.
P3	It is Sunday today, and the basketball field is occupied.
C	We play cricket or volleyball.

Solution: Let us define our propositions.
S: It's Sunday
C: We play cricket
B: We play basketball
O: The basketball field is occupied
V: We play volleyball

The given argument can then be expressed as follows.

P1	$S \implies (C \vee B)$
P2	$O \implies \neg B$
P3	$S \wedge O$
C	$C \vee V$

The proof proceeds below.

$$S \quad \text{Simplification P3} \tag{1}$$
$$O \quad \text{Simplification P3} \tag{2}$$
$$C \vee B \quad \text{Modus Ponens P1, (1)} \tag{3}$$
$$\neg B \quad \text{Modus Ponens P2, (2)} \tag{4}$$
$$C \quad \text{Disjunctive Syllogism (3), (4)} \tag{5}$$
$$C \vee V \quad \text{Addition (5)} \tag{6}$$

Eq. (6) is the same as the conclusion. Hence, the given argument is valid. □

   (b) 10 points

P1	Ahmed failed the course, but attended every lecture.
P2	Everyone who did the homework every week passed the course.
P3	If a student passed the course, then they did some of the homework.
C	Not every student did every homework assignment.

Solution: Let us define our propositional functions and domains.
$F(x)$: x failed the course, where $x \in$ set of students.
$H(x, y)$: x did y, where $x \in$ set of students and $y \in$ set of homework assignments.
$A(x)$: x attended all lectures, where $x \in$ set of students.
The given argument can then be expressed as follows.

P1	$F(Ahmed) \wedge A(Ahmed)$
P2	$\forall x(\forall y H(x, y) \rightarrow \neg F(x))$
P3	$\forall x(\neg F(x) \rightarrow \exists y H(x, y))$
C	$\neg\forall x \forall y H(x, y)$

The proof proceeds below.

$$F(Ahmed) \quad \text{Simplification P1} \tag{7}$$
$$\forall y H(Ahmed, y) \implies \neg F(Ahmed) \quad \text{Universal Instantiation P2} \tag{8}$$
$$\neg\forall y H(Ahmed, y) \quad \text{Modus Tollens (7), (8)} \tag{9}$$
$$\exists y \neg H(Ahmed, y) \quad \text{De Morgan's Law (9)} \tag{10}$$
$$\exists x \exists y \neg H(x, y) \quad \text{Existential Generalization (10)} \tag{11}$$
$$\neg\forall x \forall y H(x, y) \quad \text{2 times De Morgan's Law (11)} \tag{12}$$

Eq. (12) is the same as the conclusion. Hence, the given argument is valid. □

2. Consider the statement: The remainder of the square of any odd number when divided by 4 is 1.
   (a) $\boxed{\text{5 points}}$ Write the above statement using predicate logic notation and prove it.

Solution: Let us define our propositional functions and domains.
$O(x)$: x is odd, where $x \in \mathbb{Z}$
$R(x, d, r)$: The remainder of x when divided by d is r, where $x, d, r \in \mathbb{Z}$
Then the statement can be written as:

$$\forall x(O(x) \implies R(x^2, 4, 1)).$$

Let us attempt a direct proof.
First, we establish how the predicates can be written mathematically.
$O(x)$: $\exists k\,(x = 2k + 1), k \in \mathbb{Z}$
$R(x^2, 4, 1)$: $\exists m\,(x^2 = 4m + 1), m \in \mathbb{Z}$
Now,

$$\begin{aligned} x &= 2k + 1 \\ \implies x^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4(k^2 + k) + 1 \\ &= 4m + 1 && \text{where } m = k^2 + k \end{aligned}$$

□

   (b) $\boxed{\text{5 points}}$ Write the statement from above using a bi-conditional instead of a conditional. Prove whether the new statement holds.

Solution: The bi-conditional would be

$$\forall x\ (O(x) \iff R(x^2, 4, 1)).$$

We have already proved one of the implications. It remains to prove that

$$\forall x\ (R(x^2, 4, 1) \implies O(x))$$

Let us attempt a proof by contraposition, i.e. to prove:

$$\begin{aligned} &\forall x\ (\neg O(x) \implies \neg R(x^2, 4, 1)) \\ &\forall x\ (E(x) \implies \neg R(x^2, 4, 1)) \qquad (13) \end{aligned}$$

where $E(x) : x$ is even.
Let us attempt a direct proof of (13).

$$\begin{aligned} & x = 2k \\ \implies\ & x^2 = (2k)^2 \\ & = 4k^2 \\ & = 4m && \text{where } m = k^2 \\ \implies\ & R(x, 4, 0) \end{aligned}$$

□

3. $\boxed{\text{5 points}}$ Show that these statements about the real number x are equivalent: (i) x is irrational, (ii) $\frac{x}{2}$ is irrational. Which proof method did you use?

Solution: Consider the predicates
$R(x)$: x is rational, i.e. $x \in \mathbb{Q}$
$I(x) = \neg R(x)$: x is irrational, i.e. $x \notin \mathbb{Q}$

Then, to prove: $I(x) \iff I(\frac{x}{2})$.
That is, to prove: $I(x) \implies I(\frac{x}{2})$ and $I(\frac{x}{2}) \implies I(x)$.

<u>Case 1</u>: $I(x) \implies I(\frac{x}{2})$
Let us prove this by contraposition, i.e. to prove:

$$\neg I\left(\frac{x}{2}\right) \implies \neg I(x)$$

$$\text{i.e. } R\left(\frac{x}{2}\right) \implies R(x)$$

$$\exists p, q \in \mathbb{Z}, q \neq 0\ \frac{x}{2} = \frac{p}{q} \implies \exists (m, n) \in \mathbb{Z}^2, n \neq 0\ x = \frac{m}{n}$$

Now, assume

$$\begin{aligned}\frac{x}{2} &= \frac{p}{q}\\ \implies x &= \frac{2p}{q}\\ &= \frac{m}{n} && \text{where } m = 2p, n = q\end{aligned}$$

Case 1 proved.

Case 2: $I(\frac{x}{2}) \implies I(x)$
Let us also prove this by contraposition, i.e. to prove:

$$\begin{aligned}\neg I(x) &\implies \neg I(\frac{x}{2})\\ R(x) &\implies R(\frac{x}{2})\\ \exists(p,q) \in \mathbb{Z}^2, q \neq 0\ x = \frac{p}{q} &\implies \exists(m,n) \in \mathbb{Z}^2, n \neq 0\ \frac{x}{2} = \frac{m}{n}\end{aligned}$$

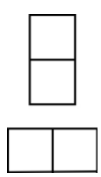
Now, assume

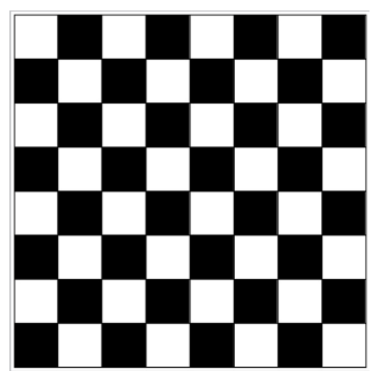
$$\begin{aligned}x &= \frac{p}{q}\\ \implies \frac{x}{2} &= \frac{p}{2q}\\ &= \frac{m}{n} && \text{where } m = p, n = 2q\end{aligned}$$

Case 2 proved. □

4. This question refers to the *tiling* or *tessellation* operation.

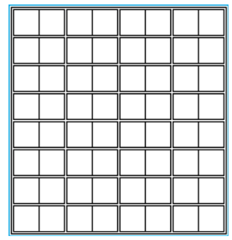
Given a standard checkerboard and dominoes, answer the following questions. Explain your answer for each question.





(a) $\boxed{\text{5 points}}$ Can we tile the standard checkerboard using dominoes?

Solution: Rephrased this means: Does such a tiling for the checkerboard exist. Yes! One example provides a constructive existence proof.



(b) $\boxed{\text{5 points}}$ Can we tile a checkerboard obtained by removing one of the four corner squares of a standard checkerboard?

Solution: Proof that we cannot tile such a checkerboard (= there does not exist a tiling for such a checkerboard) by contradiction:
Assuming there exists such a tiling for our checkerboard. We can notice, that our checkerboard has 64 - 1 = 63 squares.
Since each domino has two squares, a board with a tiling must have an even number of squares. The number 63 is not even. We have a contradiction, hence our first assumption that we cannot tile such a checkerboard is true. $\square$

(c) $\boxed{\text{5 points}}$ Can we tile a board obtained by removing both the upper left and the lower right squares of a standard checkerboard?

Solution: Proof that we cannot tile such a checkerboard (= there does not exist a tiling for such a checkerboard) by contradiction:
Assuming there exists such a tiling for our checkerboard. Then we can notice, that there are 62 squares in this board. Hence, to tile it we need 31 dominos.
Key fact: Each domino covers one black and one white square. Therefore the tiling covers 31 black squares and 31 white squares. Our board has either 30 black squares and 32 white squares or 32 black squares and 30 white squares. We have a contradiction, hence our first assumption that we cannot tile such a checkerboard is true. $\square$

5. $\boxed{\text{10 points}}$ The following is a murder case solved by Sherlock Holmes, in "A Study in Scarlet" :
"And now we come to the great question as to the reason why. Robbery has not been the object of the murder, for nothing was taken. Was it politics, then, or was it a woman? That is the question which confronted me. I was inclined from the first to the latter supposition. Political assassins are only too glad to do their work and fly. This murder had, on the contrary, been done most deliberately, and the perpetrator has left his tracks all over the room, showing he had been there all the time."
From these, Sherlock Holmes concluded: "It was a woman". Translate the above argument to statements in predicate logic and prove its validity.

Solution: Defining our propositional variables:
R = It was robbery
T = Something was taken
P = It was political
W = It was a woman
L = Assassin left immediately
M = Assassin left marks all over the place

(1)	$\neg T \to \neg R$	(Premise)
(2)	$\neg T$	(Premise)
(3)	$\neg R$	(Modus ponens of (1) and (2))
(4)	$\neg R \to P \vee V$	(Premise)
(5)	$P \vee V$	(Modus ponens of (3) and (4))
(6)	$P \to L$	(Premise)
(7)	$M \to \neg L$	(Premise)
(8)	M	(Premise)
(9)	$\neg L$	(Modus ponens of (7) and (8))
(10)	$\neg P$	(Modus tolens of (6) and (9))
$\therefore$	W	(Conclusion: Disjunctive Syllogism from (10) and (5))

Thus Sherlock Holmes' argument is valid. □

2 Equivalence Relation

6. Prove or disprove whether each of the relations represented below is an equivalence relation.
   (a) $\boxed{\text{5 points}}$ R on $\mathbb{R} = \{(x, y) \mid xy \geq 0\}$
   (b) $\boxed{\text{5 points}}$ R on $\mathbb{R} = \{(x, y) \mid x = 1\}$
   (c) $\boxed{\text{5 points}}$ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
   (d) $\boxed{\text{5 points}}$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
   (e) $\boxed{\text{5 points}}$ $R_1 \cap R_2$ where R_1 and R_2 are equivalence relations on a set, S.

Solution:

(a) R is not an equivalence relation because it is not transitive.
For example, $(-3, 0) \in R$ and $(0, 4) \in R$, but $(-3, 4) \notin R$. □

(b) R is not an equivalence relation because it is not reflexive.
For example, $(2, 2) \notin R$. □

(c) R is an equivalence relation because it satisfies the 3 properties.

Reflexive: All entries on the main diagonal are 1.

Symmetric: The matrix is symmetric.

Transitive: This can be verified from the matrix. □

(d) R is an equivalence relation because it satisfies the 3 properties.

Reflexive: All entries on the main diagonal are 1.

Symmetric: The matrix is symmetric.

Transitive: This can be verified from the matrix. □

(e) $R_1 \cap R_2$ is an equivalence relation because it satisfies the 3 properties.

Reflexive: R_1 and R_2 are equivalence relations on S.
$\forall a \in S\ (a, a) \in R_1$ and $\forall a \in S\ (a, a) \in R_2$.
$\therefore \forall a \in S\ (a, a) \in (R_1 \cap R_2)$.

Symmetric: Consider $(a, b) \in (R_1 \cap R_2)$.
Then $(a, b) \in R_1$ and $(a, b) \in R_2$.
Then $(b, a) \in R_1$ and $(b, a) \in R_2$.
Therefore, $(b, a) \in (R_1 \cap R_2)$.

Transitive: Consider $(a, b) \in (R_1 \cap R_2)$ and $(b, c) \in (R_1 \cap R_2)$.
Then $(a, b) \in R_1$ and $(b, c) \in R_1$.
Then $(a, c) \in R_1$.
Similarly $(a, c) \in R_2$.
Therefore $(a, c) \in (R_1 \cap R_2)$. □

Let R be a relation from a set A to a set B. The *inverse relation* from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The *complementary relation* $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$. A relation R on the set A is *irreflexive* if $\forall a \in A\colon (a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

7. (a) $\boxed{\text{5 points}}$ Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$.

Solution:
Case 1: R is symmetric $\implies R = R^{-1}$.
If $R = \emptyset$, then the claim is trivially true.
Otherwise there are 2 cases.
Case 1.1: R is symmetric $\implies R \subseteq R^{-1}$.
Consider $(a, b) \in R$.
Then $(b, a) \in R$ (R is symmetric) and $(a, b) \in R^{-1}$ (definition of R^{-1}).
Case 1.2: R is symmetric $\implies R^{-1} \subseteq R$.
Consider $(a, b) \in R^{-1}$.
Then $(b, a) \in R$ (definition of R^{-1}) and $(a, b) \in R$ (R is symmetric).

Case 2: $R = R^{-1} \implies R$ is symmetric.
If $R = \emptyset$, then the claim is trivially true.
Otherwise, consider $(a, b) \in R$.
Then $(a, b) \in R^{-1}$ ($R = R^{-1}$) and $(b, a) \in R$ (definition of R^{-1}). □

(b) $\boxed{\text{5 points}}$ Show that the relation R on a set A is reflexive if and only if $\overline{R}$ is irreflexive.

Solution:
Case 1: R is reflexive $\implies \overline{R}$ is irreflexive.
Consider $(a, b) \in \overline{R}$ for $a, b \in A$.
Then $(a, a) \in R$ and $(a, a) \notin \overline{R}$ (R is reflexive, and definition of $\overline{R}$).
Similarly, $(b, b) \notin \overline{R}$.
That is, $a \neq b$.

Case 2: $\overline{R}$ is irreflexive $\implies R$ is reflexive.
Consider $a \in A$.
Then $(a, a) \notin \overline{R}$ ($\overline{R}$ is irreflexive) and $(a, a) \in A$ (definition of $\overline{R}$). □

(c) 5 points Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

Solution:
Case 1: R is reflexive and transitive $\implies R^n \subseteq R$
Consider $(a, b) \in R^n$.
Then $(a, x_1) \in R, (x_1, x_2) \in R, (x_2, x_3) \in R, \ldots, (x_{n-1}, b) \in R$ (definition of composition).
Then $(a, b) \in R$ (R is transitive).
Case 2: R is reflexive and transitive $\implies R \subseteq R^n$
Consider $(a, b) \in R$.
Then $(a, a) \in R$ (R is reflexive) and $(a, a) \in R^{n-1}$ (definition of composition).
Then $(a, b) \in R^n$ (definition of composition). □

(d) 5 points Show that the relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Solution:
Case 1: R is transitive $\implies R^n \subseteq R$
Proved in Case 1 above.
Case 2: $R^n \subseteq R \implies R$ is transitive.
Consider $(a, b) \in R$ and $(b, c) \in R$.
Then $(a, c) \in R^2$ (definition of composition).
Then $(a, c) \in R$ ($R^2 \subseteq R$). □

8. Given the matrix, M_R, for a relation, R, on a finite set, A, explain how to obtain the following matrices?
   (a) 5 points $M_{R^{-1}}$
   (b) 5 points $M_{\overline{R}}$

Solution:
(a) The matrix is the transpose of M_R, i.e. $M_{R^{-1}}(a, b) = M_R(b, a)$.
(b) The matrix is obtained as $M_{R^{-1}}(a, b) = 1 - M_R(a, b)$.

9. The following questions refer to the relations, R and S, involving the set $X = \{a, b, c\}$. Specifically, R and S are relations on 2^X, the power set of X. For the definitions of R and S given below, prove whether each is an equivalence relation.
   (a) 5 points $R = \{(A, B) \mid |A| = |B|\}$.
   (b) 5 points $S = \{(A, B) \mid |A| < |B|\}$.

Solution:
(a) R is an equivalence relation because it satisfies the 3 properties.

Reflexive: $|A| = |A|$ for any set, A.
Therefore $\forall A \in 2^X \; (A, A) \in R$.

Symmetric: For any sets, A and B, $|A| = |B| \implies |B| = |A|$.
Therefore, $(A, B) \in R \implies (B, A) \in R$.

Transitive: For any sets, A, B and C, $|A| = |B| \wedge |B| = |C| \implies |A| = |C|$.
Therefore, $(A, B) \in R \wedge (B, C) \in R \implies (A, C) \in R$. □

(b) S is not an equivalence relation because it is not reflexive.
For example, $(\{a\}, \{a\}) \notin R$. □

10. 5 points A partition P_1 is called a *refinement* of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2. Given equivalence relations, R_1 and R_2, on a set, A, and the corresponding partitions, P_1 and P_2, show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

Solution: A partition contains non-empty subsets. If P_1 and P_2 are the partitions of A corresponding to R_1 and R_2, then R_1 and R_2 are not empty.

Case 1: $R_1 \subseteq R_2 \implies P_1$ is a refinement of P_2.
Consider a set, $P_{1,a} \in P_1$ corresponding to $[a]_{R_1}$ for some $a \in A$.
Then $a \in P_{1,a}, (a, a) \in R_1$ and $(a, a) \in R_2$.
Then there exists a set $[a]_{R_2} = P_{2,a} \in P_2$ and $a \in P_{2,a}$.
Thus, for any set in P_1, there exists a set in P_2 that is a superset.
Furthermore, as the sets in P_2 are disjoint, the superset is unique.

Case 2: P_1 is a refinement of $P_2 \implies R_1 \subseteq R_2$.
Consider $(a, b) \in R_1$. Then a belongs to some set $P_{1,a} \in P_1$ such that $P_{1,a} = [a]_{R_1}$.
Similarly, $b \in P_{1,a}$.
Then there exists a set $P_{2,a} \in P_2$ such that $a \in P_{2,a}$ and $b \in P_{2,a}$.
That is, $(a, b) \in R_2$. □

3 Ordering

11. Prove or disprove whether each of the relations represented below is a partial order.

(a) 5 points $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) 5 points $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Solution:

(a) This relation is not a partial order because it is not anti-symmetric, as illustrated by the highlighted entries in the matrix. $\begin{bmatrix} 1 & \color{red}{1} & 1 \\ \color{red}{1} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ □

(b) This relation is not a partial order because it is not anti-symmetric, as illustrated by the highlighted entries in the matrix. $\begin{bmatrix} 1 & \color{red}{1} & 1 & 0 \\ \color{red}{1} & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ □

12. $\boxed{\text{5 points}}$ Given a poset (S, R), its *dual* is (S, R^{-1}). Show that the dual is also a poset.

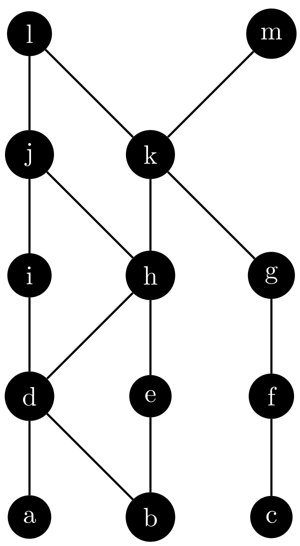
Solution: We have to prove that if R is reflexive, anti-symmetric, and transitive, then so is R^{-1}.

Reflexive: R is reflexive so $\forall a \in S\ (a, a) \in R$.
Therefore $\forall a \in S\ (a, a) \in R^{-1}$.

Anti-symmetric: Consider $(a, b) \in R^{-1}$ and $(b, a) \in R^{-1}$.
Then $(b, a) \in R$ and $(a, b) \in R$, and $a = b$.

Transitive: Consider $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$.
Then $(b, a) \in R$ and $(c, b) \in R$.
Then $(c, a) \in R$ and $(a, c) \in R^{-1}$. □

13. Answer the following questions for the partial order represented by the given Hasse diagram.



(a) $\boxed{\text{2 points}}$ Find the maximal elements.

(b) $\boxed{\text{2 points}}$ Find the minimal elements.

(c) $\boxed{\text{1 point}}$ Is there a greatest element? If so, what is it?

(d) $\boxed{\text{1 point}}$ Is there a least element? If so, what is it?

(e) $\boxed{\text{2 points}}$ Find all upper bounds of $\{a, b, c\}$.

(f) $\boxed{\text{1 point}}$ Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) $\boxed{\text{2 points}}$ Find all lower bounds of $\{f, g, h\}$.

(h) $\boxed{\text{1 point}}$ Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) maximal elements: $\{l, m\}$.

(b) minimal elements: $\{a, b, c\}$.

(c) There is no greatest element.

(d) There is no least element.

(e) Upper bounds of $\{a, b, c\}$: k, l, m.

(f) Least upper bound of $\{a, b, c\}$: k.

(g) Lower bounds of $\{f, g, h\}$: none.

(h) Greatest lower bound of $\{f, g, h\}$: none.