Midterm Exam

CS/MATH 113 Discrete Mathematics

Habib University, Spring 2022

Total Marks: 30 Date: Monday, 28 February, 2022.

1. $\boxed{\text{5 points}}$ Show that $(P \implies (Q \vee R)) \equiv ((P \wedge \neg Q) \implies R)$.

Solution:

$$\begin{array}{lrr} & P \implies (Q \vee R) & \text{LHS} \\ \iff & \neg P \vee (Q \vee R) & \text{definition of implication} \\ \iff & (\neg P \vee Q) \vee R & \text{associativity} \\ \iff & \neg(P \wedge \neg Q) \vee R & \text{De Morgans} \\ \iff & (P \wedge \neg Q) \implies (R) & \text{definition of implication} \end{array}$$

$\square$

2. $\boxed{\text{5 points}}$ Given the sets, $A, B,$ and C, where C is non-empty, show that

$$((A \times C) = (B \times C)) \implies (A = B).$$

Solution: Given $(A \times C) = (B \times C)$ and $C \neq \emptyset$
Case 1: Let $A = \emptyset$

$$\begin{array}{lrr} \implies & (A \times C) = \emptyset & \text{Cartesian product with an empty set} \\ \implies & (B \times C) = \emptyset & \text{premise } (A \times C) = (B \times C) \\ \implies & B = \emptyset & \text{since } C \neq \emptyset \\ \implies & A = B & \text{desired conclusion} \end{array}$$

Similarly, $(B = \emptyset) \implies (A = \emptyset)$
Case 2: Let A, B and C be non-empty. Then

$$\begin{array}{rrr} \forall a \in A, c \in C & (a,c) \in (A \times C) & \text{Definition of the Cartesian product} \\ & (A \times C) \subset (B \times C) & \text{premise} \\ \implies & (a,c) \in (B \times C) & \text{subset definition} \\ \implies & a \in B & \forall a \in A \\ \implies & A \subset B & \end{array}$$

Similarly, we prove $B \subset A$. Therefore, $A = B$. $\square$

3. $\boxed{\text{10 points}}$ Let $x, y \in \mathbb{R}$ and $Q(x, y)$ be the propositional function, $x + y = x - y$. Determine the truth value of each of the following statements, also providing an explanation or a counterexample, as applicable, with each.

(a) $\forall x \forall y\, Q(x, y)$

Solution: The statement reads
For all x, for all y, $x + y = x - y$
This is FASLE as we have a counterexample, when $x = 1, y = 1$, then $x + y = 2 \neq x - y = 0$.

(b) $\forall x \exists y\, Q(x, y)$

Solution: The statement reads
For all x, there exists a y such that $x + y = x - y$
This is TRUE as indeed, when $y = 0$ then $x + y = x - y = x, \forall x$.

(c) $\exists x \forall y\, Q(x, y)$

Solution: The statement reads
There exists x, such that for all y $x + y = x - y$
This is FALSE.
Proof by contradiction
Suppose the statement was true i.e. there exists x, such that for all y $x + y = x - y$
Then for all y, $y = -y$, but that is not the case if $y \neq 0$.
Note that considering any one value of x is not sufficient here to disprove the statement, as there might have been some other value of x making the statement true. So we need a general result rather than a counterexample.

(d) $\exists y\, Q(1, y)$

Solution: The statement reads
When $x = 1$, there exists a y such that $1 + y = 1 - y$
This is TRUE as indeed, when $y = 0$ then $1 + y = 1 - y = 1$.
Note that this is just an instance of the universal true statement given in part b and therefore has to be true.

(e) $\exists x \exists y\, Q(x, y)$

Solution: The statement reads
There exists x, there exists a y such that $x + y = x - y$
This is TRUE, when for instance $x = 1, y = 0$.
Note again that since the universal statement in part b is true, our statement needs to be true aswell.

4. $\boxed{\text{5 points}}$ Show that $((A \cap B) = (A \cup B)) \implies (A = B)$.

Solution:

(1)	$A \subset (A \cup B)$	since $\forall a \in A, a \in A \cup B$
(2)	$(A \cup B) = (A \cap B)$	premise
(3)	$\implies A \subset (A \cap B)$	implied from (1) and (2)
(4)	$(A \cap B) \subset B$	since $\forall x \in A \cap B, x \in B$
(5)	$\implies A \subset B$	implied from (3) and (4)

Similarly, $B \subset A$.
Therefore, $A = B$ $\square$

5. $\boxed{\text{5 points}}$ In this problem we prove $\{12a + 25b | a, b \in \mathbb{Z}\} = \mathbb{Z}$.

(a) Let $S = \{12a + 25b | a, b \in \mathbb{Z}\}$. Show that $S \subset \mathbb{Z}$.

Solution: Since a and b are integers and since we know that product and sum of integers are integers again, $\forall a, b \in \mathbb{Z}, 12a + 25b \in \mathbb{Z}$. Thus the set $S = \{12a + 25b | a, b \in \mathbb{Z}\} \subset \mathbb{Z}$. $\square$

(b) Next to show $\mathbb{Z} \subset S$, we need to prove that $\forall x \in \mathbb{Z}, \exists a, b \in \mathbb{Z}$ such that $x = \underline{12a + 25b}$. Fill in the blank and then complete the proof by choosing appropriate values of a and b.

Solution: For any $x \in \mathbb{Z}$ keeping $a = -2x$ and $b = x$ produces $x = 12a + 25b$. $\square$