Recitation: Relations

The world is a hierarchy

April 1, 2022

Definitions/Important Concepts

Partition and Equivalence Classes

Equivalent: If R is an equivalence relation, a is equivalent to b if aRb.

Partition: A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union

Equivalence Class of a w.r.t R ($[a]_R$): The set of all elements of A that are equivalent to a.

Ordering

Poset: A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Comparable: The elements a and b of a poset $(S, \preceq)$ are called comparable if either $a \preceq b$ or $b \preceq a$.

Total Order: If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order.

Maximal Element: An element of a poset that is not less than any other element of the poset.

Alternate: a is maximal in the poset $(S,\preceq)$ if $\nexists b \in S : a \prec b$.

Minimal Element: An element of a poset that is not greater than any other element of the poset.

Alternate: a is minimal in the poset $(S,\preceq)$ if $\nexists b \in S : b \prec a$.

Greatest Element: An element of a poset greater than all other elements in this set.

Alternate: a is the greatest element if $\forall b \in A : b \preceq a$

Least Element: An element of a poset less than all other elements in this set.

Alternate:a is the least element if $\forall b \in A : a \preceq b$

Upper Bound of a set: An element in a poset greater than all other elements in the set.

Alternate: Assume $A \subseteq (S, \preceq).u \in S$ is an upper bound of A if $\forall a \in A : a \preceq u$

Lower Bound of a set: An element in a poset less than all other elements in the set.

Alternate: Assume $A \subseteq (S, \preceq).l \in S$ is a lower bound of A if $\forall a \in A : l \preceq u$

Least Upper Bound of a set: An upper bound of the set that is less than all other upper bounds.

Greatest Lower Bound of a set: A lower bound of the set that is greater than all other lower bounds.

Questions

1. Justify that for an equivalence relation R for an equivalence class $[\mathbf{c}]$ $\forall a, b \in [\mathbf{c}](a, b) \in R$, or every element in the equivalence class has relation with every element in the equivalence class

2. Determine the number of different equivalence relations on a set with three elements a, b, c by listing them.

3. Let us assume that F is a relation on the set $\mathbb{R}$ real numbers defined by xFy if and only if $x - y$ is an integer. Prove that F is an equivalence relation on $\mathbb{R}$.
   (a) What is the equivalence class of 1 for this equivalence relation?
   (b) What is the equivalence class of 1/2 for this equivalence relation?

4. Can you draw the Hasse diagram for the relation $\{(a, b) \mid a > b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$? If not, then why?

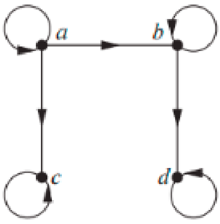
5. Which of these relations(R) on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.
   1. $\{(0, 0), (2, 2), (3, 3)\}$
   b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$
   c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$
   d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$
   e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

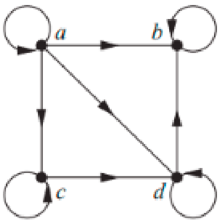
6. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if
   a) a is no shorter than b ?
   b) a weighs more than b ?
   c) $a = b$ or a is a descendant of b ?
   d) a and b do not have a common friend?

7. Which of these are posets?
   1. $(\mathbf{Z}, =)$
   2. $(\mathbf{Z}, \nmid)$

8. Determine whether the relations represented by these zero-one matrices are partial orders.
   a.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
   b.
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

9. Determine whether relations with the following directed graph is a partial order.

(a) 

(b) 

(c) 

(d) 

10. **EXAMPLE:** Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid \text{a divides b}\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

11. Prove that the greatest element is unique when it exists.

12. Prove the least element is unique when it exists.

13. Draw a Hasse Diagram for the following cases.
    (a) Divisibility on the set $\{1, 2, 3, 5, 7, 11, 13\}$
    (b) less than or equal to relation on $\{0, 2, 5, 10, 11, 15\}$.

14. Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.
    (a) Draw a Hasse Diagram representing this poset.
    (b) Find the maximal elements.
    (c) Find the minimal elements.
    (d) Is there a greatest element?
    (e) Is there a least element?
    (f) Find all upper bounds of $\{\{2\}, \{4\}\}$.
    (g) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.
    (h) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.
    (i) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists

15. Let **R** be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C1 , C2), where C1 and C2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C2 can be obtained from C1 either by rotating the checkerboard or by rotating it and then reflecting it.
    (a) Show that **R** is an equivalence relation.
    (b) What are the equivalence classes of **R**?

16. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.

    Definition: A partition P1 is called a refinement of the partition P2 if every set in P1 is a subset of one of the sets in P2.