Homework 4: Functions, Induction, Graphs

Sample Solution

CS/MATH 113 Discrete Mathematics
Habib University, Spring 2022

Functions

1. $\boxed{\text{5 points}}$ Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to every student,

   a) their mobile phone number.
   b) their student identification number.
   c) their final grade in the class.
   d) their home town.
   e) their Ehsas hour appointment.

Solution:

a) if every student has a unique phone number, i.e. no 2 students share a phone number. For this, there must be at least as many available phone numbers as the number of students.

b) if every student has a unique ID, i.e. no 2 students are assigned the same ID. For this, there must be at least as many available IDs as the number of students.

c) if every student has a unique grade, i.e. no 2 students earn the same grade. For this, there must be at least as many available grades as the number of students.

d) if every student has a unique home town, i.e. no 2 students come from the same town. For this, f must consider, i.e. must have in its codomain, at least as many towns as the number of students.

e) if every student has a unique Ehsas hour appointment, i.e. no 2 students get the same time slot. For this, there must be at least as many available time slots as the number of students.

2. $\boxed{\text{5 points}}$ If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

Solution: Yes, g is one-to-one. Proof follows.

Proof. We show that $g(a) = g(b) \implies a = b$.
Assume two elements a, b, such that $g(a) = g(b)$.
Applying the function f on each side of this equation yields: $f(g(a)) = f(g(b))$.
This can be rewritten as: $(f \circ g)(a) = (f \circ g)(b)$
As we know that $f \circ g$ is one-to-one, we can conclude $a = b$. □

3. $\boxed{\text{5 points}}$ Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one. Give an example of a decreasing function from $\mathbb{R}$ to itself that is not one-to-one.

Solution: We attempt a proof by contradiction.

Proof. Assume that a strictly decreasing function, f, is not one-to-one.
Then there exist 2 distinct real numbers, $a_1, a_2 \in \mathbb{R}$, such that

$$f(a_1) = f(a_2).$$

There are 2 cases to consider.
Case I: $a_1 < a_2$: As f is strictly decreasing, it must be that $f(a_1) > f(a_2)$. $\perp$
Case II: $a_1 > a_2$: As f is strictly decreasing, it must be that $f(a_1) < f(a_2)$. $\perp$ □

An example of a decreasing function, $f : \mathbb{R} \to \mathbb{R}$, that is not one-to-one, is

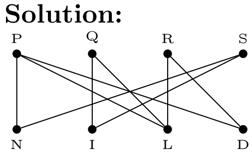
$$f(x) = -\text{floor}(x)$$

Graphs

4. $\boxed{\text{5 points}}$ To lift the spirits of the students on their return, Habib University has decided to build 4 new courtyards–Nature, Ice, Light, and Darkness. Designs are invited and bids are received for the courtyards from 4 architect firms as follows. Parveen Prime can design Nature, Light, and Darkness; Queen Quratulain can design Ice and Light; Reena Rani can design Light and Darkness; and Super Sonam can design Nature and Ice.

(a) $\boxed{\text{5 points}}$ Use a bipartite graph to model the four architects and the courtyards that they can design.

Solution:



(b) $\boxed{\text{5 points}}$ Use Hall's theorem to determine whether there is an assignment of architects to courtyards so that each architect is assigned one courtyard to design.

Solution: We will have to consider the neighborhood of every subset of architects.
As there are 4 architects, there are $2^4 = 16$ subsets.
One of these 16 subsets is the empty subset and can be ignored.
$N(\{P\}) = \{N, L, D\}$.
$N(\{Q\}) = \{I, L\}$.
$N(\{R\}) = \{L, D\}$.
$N(\{S\}) = \{N, I\}$.
$N(\{P, Q\}) = \{N, I, L, D\}$.
$N(\{P, R\}) = \{N, L, D\}$.
$N(\{P, S\}) = \{N, I, L, D\}$.
$N(\{Q, R\}) = \{I, L, D\}$.

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$N(\{P, Q, R, S\}) = \{N, I, L, D\}$.
In each case, the neighborhood is at least as large as the subset considered.
Thus, the condition in Hall's theorem is satisfied and a complete matching from architects to courtyards exists.

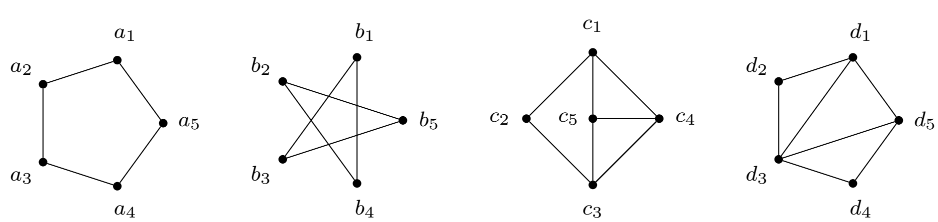
(c) $\boxed{\text{5 points}}$ Provide the assignment, if it exists, of architects to courtyards such that each architect is assigned to a courtyard that she can design.

Solution: One matching is: $\{(P, N), (Q, L), (R, D), (S, I)\}$

5. The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism.* Two simple graphs that are not isomorphic are called *non-isomorphic.*

In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between the vertices of the two graphs that preserves the adjacency relationship. Isomorphism of simple graphs is an equivalence relation.

Determine which of the following pairs of graphs are isomorphic. Provide an isomorphism (which vertex is mapped to which other vertex) or a rigorous argument that none exists.



Graph A Graph B Graph C Graph D

(a) $\boxed{\text{5 points}}$ Graph A and Graph B

Solution: The graphs are isomorphic.

Proof. Following is a bijection: $f(a_1) = b_1, f(a_2) = b_3, f(a_3) = b_5, f(a_4) = b_2, f(a_5) = b_4$. □

(b) $\boxed{\text{5 points}}$ Graph A and Graph C

Solution: The graphs are not isomorphic.

Proof. c_5 has a degree of 3 and all vertices in A have a degree of 2. c_5 cannot be the image of any vertex in A. Therefore, no bijection exists. □

(c) $\boxed{\text{5 points}}$ Graph A and Graph D

Solution: The graphs are not isomorphic.

Proof. d_3 has a degree of 4 and all vertices in A have a degree of 2. d_3 cannot be the image of any vertex in A. Therefore, no bijection exists. □

(d) $\boxed{\text{5 points}}$ Graph B and Graph C

Solution: The graphs are not isomorphic.

Proof. c_5 has a degree of 3 and all vertices in B have a degree of 2. c_5 cannot be the image of any vertex in B. Therefore, no bijection exists. □

(e) $\boxed{\text{5 points}}$ Graph B and Graph D

Solution: The graphs are not isomorphic.

Proof. d_3 has a degree of 4 and all vertices in B have a degree of 2. d_3 cannot be the image of any vertex in B. Therefore, no bijection exists. □

(f) $\boxed{\text{5 points}}$ Graph C and Graph D

Solution: The graphs are not isomorphic.

Proof. d_3 has a degree of 4 and no vertex in C has this degree. d_3 cannot be the image of any vertex in C. Therefore, no bijection exists. □

6. $\boxed{\text{5 points}}$ Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

Solution: We attempt a proof by contradiction.

Proof. Assume that no two vertices in the graph have the same degree. That is, every vertex has a distinct degree. Then, there are as many distinct degrees in the graph as there are vertices.

We note that in a simple graph with n vertices, the largest possible degree is $n-1$ (a vertex can at most be connected to all the other $n-1$ vertices). So the set of possible degrees is, $\{0, 1, 2, 3, \ldots, n-1\}$.

We further note that the graph cannot simultaneously have a vertex with degree 0, i.e. an isolated vertex, and a a vertex with degree $(n-1)$. i.e. a vertex connected with all the other vertices. Therefore, the set of possible degrees is, either $\{0, 1, 2, 3, 4, \ldots, n-2\}$ or $\{1, 2, 3, 4, \ldots, n-1\}$.

In both cases, the number of possible degrees is $(n-1)$.$\perp$ □

Note: this can also be argued using the pigeonhole principle (also Section 6.2 in the course textbook).

7. $\boxed{\text{5 points}}$ The *complementary graph* $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Given G with v vertices and e edges, how many edges are there in $\overline{G}$? Justify your answer.

Solution: If G has v vertices and e edges, then the number of edges in $\overline{G}$ is: $\dfrac{v(v-1)}{2} - e$.

Proof. G and $\overline{G}$ have the same v vertices.
From the definition, the edges in G and $\overline{G}$ form a partition of the edges of K_v.
K_v is known to have $\dfrac{v(v-1)}{2}$ edges and G has e edges.
Therefore, the number of edges in $\overline{G}$ is: $\dfrac{v(v-1)}{2} - e$. □

Induction

8. Prove the following using induction.

(a) $\boxed{\text{5 points}}$ Given a a relation R that is reflexive and transitive, $R^n = R$ for all positive integers n.

Solution: The statement is $P(n) : R$ is reflexive and transitive $\implies R^n = R$ where $n \in \mathbb{Z}^+$.
Basis Step: Check $P(n)$ for $n = 1$.
$P(n) = P(1) : R$ is reflexive and transitive $\implies R^1 = R$.
This is trivially true. Therefore, the base case, i.e. $P(1)$ is true.
Inductive Hypothesis: Assume $P(n)$ to be true for $n = k$.
That is, assume the following to be true. $P(k) : R$ is reflexive and transitive $\implies R^k = R$.
Inductive Step: Check $P(n)$ for $n = k + 1$.
That is, we investigate $P(k + 1) : R$ is reflexive and transitive $\implies R^{k+1} = R$.
This leads to 2 cases:
Case 1: R is reflexive and transitive $\implies R^{k+1} \subseteq R$.
Case 2: R is reflexive and transitive $\implies R \subseteq R^{k+1}$.

Case 1: R is reflexive and transitive $\implies R^{k+1} \subseteq R$.
Consider $(a, b) \in R^{k+1}$.
Then, from the definition of composition, $(a, c) \in R^k$ and $(c, b) \in R$.
Then, from the inductive hypothesis, $(a, c) \in R$ and $(c, b) \in R$.
Then, from the transitivity of R, $(a, b) \in R$.
Case 2: R is reflexive and transitive $\implies R \subseteq R^{k+1}$.
Consider $(a, b) \in R$.
From the reflexivity of R, $(a, a) \in R$.
Then, from the inductive hypothesis, $(a, a) \in R^k$.
Then, from the definition of composition, $(a, b) \in R^{k+1}$.
This completes the inductive step. □

(b) $\boxed{\text{5 points}}$ A relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Solution: *As proved in the course textbook for Theorem 1 in Section 9.1.* There are two cases:
Case 1: R is transitive $\implies R^n \subseteq R$.
Case 2: $R^n \subseteq R \implies R$ is transitive.
Case 2: $R^n \subseteq R \implies R$ is transitive.
A direct proof suffices.
Consider $(a, b) \in R$ and $(b, c) \in R$.

Then, from the definition of composition, $(a,c) \in R^2$.
Then, from the antecedent, $(a,c) \in R$.
Case 1: R is transitive $\implies R^n \subseteq R$.
The statement is $P(n): R$ is transitive $\implies R^n \subseteq R$ where $n \in \mathbb{Z}^+$.
Basis Step: Check $P(n)$ for $n = 1$.
$P(n) = P(1): R$ is transitive $\implies R^1 \subseteq R$.
This is trivially true. Therefore, the base case, i.e. $P(1)$ is true.
Inductive Hypothesis: Assume $P(n)$ to be true for $n = k$.
That is, assume the following to be true. $P(k): R$ is transitive $\implies R^k \subseteq R$.
Inductive Step: Check $P(n)$ for $n = k+1$.
That is, we investigate $P(k+1): R$ is transitive $\implies R^{k+1} \subseteq R$.
Consider $(a,b) \in R^{k+1}$.
Then, from the definition of composition, $(a,c) \in R^k$ and $(c,b) \in R$.
Then, from the inductive hypothesis, $(a,c) \in R$ and $(c,b) \in R$.
Then, from the transitivity of R, $(a,b) \in R$.

This completes the inductive step. □

9. $\boxed{\text{5 points}}$ Prove via induction that a complete graph with n vertices contains $\frac{n(n-1)}{2}$ edges.

Solution: We perform the induction over the number, n, of vertices in the complete graph, K_n.
Let the statement be $P(n): K_n$ contains $\frac{n(n-1)}{2}$ edges.
Basis Step: Check $P(n)$ for $n = 1$.
$P(n) = P(1): K_1$ contains $\frac{1(1-1)}{2} = 0$ edges.
This is true. Therefore, the base case, i.e. $P(1)$ is true.
Inductive Hypothesis: Assume $P(n)$ to be true for $n = k$.
That is, assume the following to be true. $P(k): K_k$ contains $\frac{k(k-1)}{2}$ edges.

Inductive Step: Check $P(n)$ for $n = k+1$.
That is, we investigate $P(k+1): K_{k+1}$ contains $\frac{k(k+1)}{2}$ edges.

Consider K_{k+1} and take away a vertex, v_{k+1}, and its incident edges.
The resulting graph is K_k which, by the inductive hypothesis, contains $\frac{k(k-1)}{2}$ edges.
Now reintroduce v_{k+1} to obtain K_{k+1}.
As it is a complete graph, v_{k+1} has an edge with each of the other k vertices.
That is, K_{k+1} adds k edges to K_k.
The total number of edges in $K_{k+1} = \frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$.
This is as claimed in $P(k+1)$.
This completes the inductive step. □