Solution to Homework I: Sets

CS/MATH 113 Course Team

Habib University – Spring 2022

1. $\boxed{\text{5 points}}$ Write down $\mathcal{P}(X)$ if $X = \{\emptyset, \{\alpha, \beta, \gamma\}, \gamma, \{\{\alpha, \beta\}\}\}$.

Solution:

$$
\begin{aligned}
\mathcal{P}(X) = \{&\emptyset,\\
&\{\emptyset\}, \{\{\alpha,\beta,\gamma\}\}, \{\gamma\}, \{\{\{\alpha,\beta\}\}\},\\
&\{\emptyset, \{\alpha,\beta,\gamma\}\}, \{\emptyset,\gamma\}, \{\emptyset, \{\{\alpha,\beta\}\}\}, \{\{\alpha,\beta,\gamma\},\gamma\}, \{\{\alpha,\beta,\gamma\}, \{\{\alpha,\beta\}\}\}, \{\gamma, \{\{\alpha,\beta\}\}\},\\
&\{\emptyset, \{\alpha,\beta,\gamma\}, \gamma\}, \{\emptyset, \{\alpha,\beta,\gamma\}, \{\{\alpha,\beta\}\}\}, \{\emptyset, \gamma, \{\{\alpha,\beta\}\}\}, \{\{\alpha,\beta,\gamma\}, \gamma, \{\{\alpha,\beta\}\}\},\\
&\{\emptyset, \{\alpha,\beta,\gamma\}, \gamma, \{\{\alpha,\beta\}\}\}\}
\end{aligned}
$$

There should be a total of $2^4 = 16$ elements.

2. (a) $\boxed{\text{5 points}}$ Assume that RO has asked for your help to generate a set that contains all the possible pairs of DSSE faculty and DSSE courses at Habib University. Describe the sets and set operations that you can use to provide RO the desired set.

Solution: Let us define the following sets.
A : the set of all DSSE courses
B : the set of all DSSE faculty members.
Then, the Cartesian product, $B \times A$, will contain all possible pairings of DSSE faculty with DSSE courses.

(b) $\boxed{\text{5 points}}$ Imagine that the the operation above is extended to include an additional set that contains all the time slots when a course can be scheduled. Explain the outcome of the obtained set.

Solution: Let C denote the additional set, i.e. the set that contains all the time slots for each course. This set can be used to figure out a scheduling system for the courses offered at Habib University and which faculty member(s) can teach a course. It will give all possible triples of faculty member, course and timing that the course can be offered. One can use it to help avoid clashes and determine which courses can be offered in a given semester.

3. The *symmetric difference* of two sets A and B is defined as $A \oplus B = (A - B) \cup (B - A)$. It is also known as the *disjunctive union* as it contains all those elements which are in either of those sets, but not in their intersection.

   (a) $\boxed{\text{5 points}}$ Prove that $A \oplus B = (A \cup B) - (A \cap B)$.

   Solution: The most direct way (not always the simplest) to prove that $A = B$ for sets, is to prove both $A \subseteq B$ and $B \subseteq A$.

   Proof. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   <u>Case I</u>: To prove: $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

   Let $x \in (A \cup B) - (A \cap B)$
   $\implies x \in (A \cup B), x \notin (A \cap B)$
   $\implies x \in A$ or $x \in B, x \notin (A \cap B)$
   Case 1: Suppose $x \in A, x \notin B$. $\therefore x \in (A - B)$. $\therefore x \in (A - B) \cup (B - A)$.
   Case 2: Suppose $x \in B, x \notin A$. $\therefore x \in (B - A)$. $\therefore x \in (A - B) \cup (B - A)$.
   Either way $x \in (A - B) \cup (B - A)$
   $\implies (A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

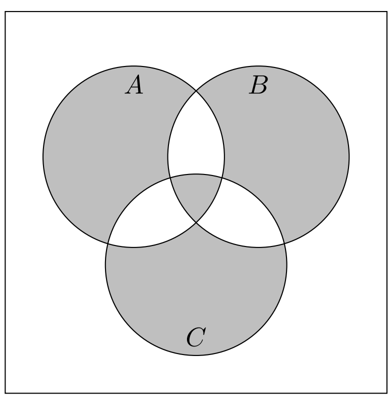
   <u>Case I</u>: To prove: $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

   Let $x \in (A - B) \cup (B - A)$
   $\implies x \in (A - B)$ or $x \in (B - A)$
   Case 1: Suppose $x \in (A - B)$. $\therefore x \in (A \cup B), x \notin (A \cap B)$. $\therefore x \in (A \cup B) - (A \cap B)$.
   Case 1: Suppose $x \in (B - A)$. $\therefore x \in (A \cup B), x \notin (A \cap B)$. $\therefore x \in (A \cup B) - (A \cap B)$.
   Either way $x \in (A \cup B) - (A \cap B)$.
   $\implies (A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

   □

   (b) $\boxed{\text{5 points}}$ For three sets A, B, and C we definine the symmetric difference as $A \oplus B \oplus C = (A \oplus B) \oplus C$, i.e. the two-set definition is applied twice. Draw the Venn diagram of this.

   Solution:



(c) $\boxed{\text{5 points}}$ Using the insights from above, express $A \oplus B \oplus C$ in the same manner as given in part a). That is, using the set operations: union, intersection, difference and/orcomplement. Show your working.

Solution: The expression would be $A \oplus B \oplus C = (A \cup B \cup C) - (A \cap B) - (A \cap C) - (B \cap C) \cup (A \cap B \cap C)$.

4. Let A be the set of all numbers that are divisible by 6 and B the set of all numbers that are divisible by 10.

(a) $\boxed{\text{5 points}}$ Write the sets A and B in set notation and describe $A \cap B$ as simply as possible.

Solution: There are multiple ways to write it down. One example:

$$\begin{aligned} A &= \{x \in \mathbb{N} \mid x \bmod 6 = 0\} \\ B &= \{x \in \mathbb{N} \mid x \bmod 10 = 0\} \\ A \cap B &= \{x \in \mathbb{N} \mid x \bmod 6 = 0 \text{ and } x \bmod 10 = 0\} \\ &= \{x \in \mathbb{N} \mid x \bmod 30 = 0\}. \end{aligned}$$

(b) $\boxed{\text{10 points}}$ Describe the set $A \oplus B$, i.e. the symmetric difference of A and B, using set notation. Provid a proof that the set you indicate is indeed the symmetric difference of A and B.

Solution:

$$\begin{aligned} A \oplus B = C = \quad & (A - B) \cup (B - A) \\ = \quad & \{x \in \mathbb{N} \mid x \bmod 6 = 0, x \bmod 10 \neq 0\} \\ & \cup \{x \in \mathbb{N} \mid x \bmod 10 = 0, x \bmod 6 \neq 0\} \\ = \quad & \{x \in \mathbb{N} \mid x \bmod 10 = 0 \text{ or } x \bmod 6 = 0, x \bmod 30 \neq 0, \} \end{aligned}$$

Proof. $C = (A \cup B) - (A \cap B)$
<u>Case I</u>: To prove: $C \subseteq (A \cup B) - (A \cap B)$

$$\begin{aligned} &\text{Let } x \in C \\ &\implies x \bmod 10 = 0 \text{ or } x \bmod 6 = 0, x \bmod 30 \neq 0 \\ &\implies x \in A \text{ or } x \in B, x \notin (A \cap B) \\ &\implies x \in (A \cup B), x \notin (A \cap B) \\ &\implies x \in (A \cup B) - (A \cap B) \end{aligned}$$

<u>Case II</u>: To prove: $(A \cup B) - (A \cap B) \subseteq C$

$$\begin{aligned} &\text{Let } x \in (A \cup B) - (A \cap B) \\ &\implies x \in (A \cup B), x \notin (A \cap B) \\ &\implies x \in A \text{ or } x \in B, x \notin (A \cap B) \\ &\implies x \bmod 10 = 0 \text{ or } x \bmod 6 = 0, x \bmod 30 \neq 0 \\ &\implies x \in C \end{aligned}$$

$\square$

(c) $\boxed{\text{5 points}}$ Given $U = \{x \in \mathbb{N} \mid x \leq 60\}$, list the elements of A, B, and $A \oplus B$.

Solution:

$$\begin{aligned} A =& \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60\} \\ B =& \{10, 20, 30, 40, 50, 60\} \\ A \oplus B =& \{6, 10, 12, 18, 20, 24, 30, 36, 40, 42, 48, 50, 54, 60\} - \{30, 60\} \\ =& \{6, 10, 12, 18, 20, 24, 36, 40, 42, 48, 50, 54\} \end{aligned}$$

5. Show that $\overline{A \cup \overline{B}} = \overline{A} \cap B$.

(a) $\boxed{\text{5 points}}$ by using set identities.

Solution:

Proof. $\overline{A \cup \overline{B}} = \overline{A} \cap B$
Applying De Morgan's law to the left hand side:

$$\begin{aligned} \overline{A \cup \overline{B}} &= \overline{A} \cap \overline{\overline{B}} \\ &= \overline{A} \cap B \end{aligned}$$

$\square$

(b) $\boxed{\text{5 points}}$ by proving that each set is a subset of the other.

Solution:

Proof. $\overline{A \cup \overline{B}} \subseteq \overline{A} \cap B$ and $\overline{A} \cap B \subseteq \overline{A \cup \overline{B}}$
<u>Case I</u>: <u>To prove</u>: $\overline{A \cup \overline{B}} \subseteq \overline{A} \cap B$
Let $x \in \overline{A \cup \overline{B}}$
$\implies x \notin (A \cup \overline{B})$
$\implies (x \notin A) \wedge (x \notin \overline{B})$
$\implies (x \in \overline{A}) \wedge (x \in B)$
$\implies x \in \overline{A} \cap B$
<u>Case II</u>: To prove: $\overline{A} \cap B \subseteq \overline{A \cup \overline{B}}$
Let $x \in \overline{A} \cap B$
$\implies (x \in \overline{A}) \wedge (x \in B)$
$\implies (x \notin A) \wedge (x \notin \overline{B})$
$\implies x \notin (A \cup \overline{B})$
$\implies x \in \overline{A \cup \overline{B}}$ $\square$