Practice Problems: Proofs
CS 113 Discrete Mathematics

kisi ka bharosa nahi

Habib University – Spring 2022

1. Show that $\sqrt{2}$ is irrational. In other words, $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$

Solution: Assume $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.
And $\frac{p}{q}$ is the lowest form it can be.

$$\left(\frac{p}{q}\right)^2 = 2$$

$$p^2 = 2q^2$$

This implies p is even which means $p = 2k$, for some $k \in \mathbb{Z}$

$$4k^2 = 2q^2$$

$$2k^2 = q^2$$

This implies q is even.
But p and q cant both be even as they are in the lowest form possible thus the 2 would be canceled.
Here we have a contradiction.
Thus $\sqrt{2}$ cannot be written in form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$
Thus $\sqrt{2}$ is irrational.

2. Explain what you must do to disprove the statement: $x^3 + 5x + 3$ has a root between $x = 0$ and $x = 1$

Solution: The statement in logical notation is

$$\exists x \text{ such that } (0 < x < 1 \wedge x^3 + 5x + 3 = 0)$$

Giving a counterexample is not enough. Saying that when $x = 0.5$ then $x^3 + 5x + 3 \neq 0$ is not sufficient.

To disprove this statement, we need to prove that the **negation is true** which is

$\neg \exists x$ such that $0 < x < 1 \wedge x^3 + 5x + 3 = 0 \equiv \forall x$ such that $\neg(0 < x < 1 \wedge x^3 + 5x + 3 = 0)$

Or in English

For all x, it is not the case that both x is between 0 and 1 and $x^3 + 5x + 3 = 0$

3. Prove that for any integer n the number $n^2 + 5n + 13$ is odd

Solution:

4. State the statement of Contradiction and verify that it is a valid argument.
Hint: In contradiction we are saying that A implies B is the same as saying that A and $\neg B$ happening together is false.

Solution:

5. Show through contraposition the following proposition is true: $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Solution:

6. Prove that "$(a + b)^2 = a^2 + b^2$" is **not** an algebraic identity where $a, b \in \mathbb{R}$

Solution:

7. Prove the following claim: There exists irrational numbers a and b such that a^b is rational.

Solution:

8. Show that $x^n + y^n = z^n$ has no solutions where $x, y, z \in \mathbb{Z}$ with and $x \neq 0, \quad y \neq 0, \quad z \neq 0$ whenever $n \in \mathbb{Z}$ and $n > 2$

Solution:

9. Prove that for m and n integers, if 2 divides m or 10 divides n, then 4 divides m^3n^2

Solution:

10. Show that there are infinitely many primes, in other words the set containing all prime numbers is infinite.
Definition: A prime number is a Natural number that is only divisible by 1 and itself, and has to be divisible by 2 different numbers.
Fundamental Theorem of arithmetic: Every integer $N > 1$ has a prime factorization, meaning either N is itself prime or can be written as a product of prime numbers.

Solution:

11. Show a direct proof that you are worthy of love

Solution: This statement is true by the axiom of humanity. The proof is trivial.

12. Give a counterexample to the statement

"If n is an integer and n^2 is divisible by 4, then n is divisible by 4"

Solution:

13. Show through contraposition the following proposition is true : If $x^2 - 6x + 5$ is even, then x is odd.

Solution:

14. In this question we will prove Euclid's Lemma that if p is a prime number that divides ab then p divides a or p divides b.
We shall prove this by proving a lemma and using a corollary from that lemma.
Well ordering principle: Every set of positive integers have a smallest element.
Division algorithm: if $a, b \in \mathbb{Z}$, where $b > 0$, then there exists unique $q, r \in \mathbb{Z}$, $a = bq + r$ where, $0 \leq r < b$

    (a) **Bezout's lemma:** for all integers a and b there exist integers s and t such that $gcd(a, b) = as + bt$

Solution:

(b) **Corollary of bezout's lemma:** If a and b are relatively prime then $as + bt = 1$

Solution:

(c) Using the above corollary prove Euclid's lemma.

Solution:

15. In solving coding questions, you would want to know whether your solution is valid or not in all cases. You can easily do that by giving a proof of correctness. Come up with a solution and give a proof of correctness for the following problem.

https://codeforces.com/problemset/problem/1635/C

Solution: