Midterm Exam

CS/MATH 113 Discrete Mathematics

Sections $\{L1, L2, \ldots, L5\} - \{L3\}$

Habib University, Spring 2022

Total Marks: 30	Date: Tuesday, 1 March, 2022.
Duration: 75 minutes	Time: 1130–1245h

Student ID: ______________________________

Student Name: ______________________________

DO NOT TURN OVER UNTIL INSTRUCTED.

Instructions:

1. Please deposit your bag and devices at the front of the classroom.
2. You are only allowed to have writing utensils, your cheat sheet, your HU ID card, and a water bottle with you during the exam.
3. Please display your HU ID card clearly on your desk next to you.
4. This exam pack consists of 6 sides. This is the first side. The next 4 contain a question on each along with space for solutions. And the last one provides additional space for solutions. Once you are allowed to turn over this side, make sure your exam pack has all the sides. Notify an instructor if it does not.
5. Fill your HU ID, e.g. xy01042, and name above. Once you are allowed to turn over this side, also fill in your ID and name in the provided space on top of *every* side.
6. Enter your solutions in the provided space below the questions on each page. The last page contains no questions and provides additional space in case your solution overruns the space provided on any of the previous pages.
7. Keep your answers short and direct.
8. Collaboration of any sort or possession of forbidden items, e.g. cell phone during the exam will result in immediate expulsion from the exam room and invocation of Habib University's Disciplinary Code afterward.

GOOD LUCK!

1. Prove the following logical equivalences.
   (a) $\boxed{\text{2 points}}$ $((p \implies q) \vee (p \implies r)) \equiv (p \implies (q \vee r))$
   (b) $\boxed{\text{3 points}}$ $((p \implies r) \wedge (q \implies r)) \equiv ((p \vee q) \implies r)$
   (c) $\boxed{\text{4 points}}$ $\neg(p \iff q) \equiv (p \iff \neg q)$

Solution:

(a) We argue by deriving RHS from LHS using logical equivalences.

1)	$((p \implies q) \vee (p \implies r))$	LHS
2)	$((\neg p \vee q) \vee (\neg p \vee r))$	implication
3)	$\neg p \vee \neg p \vee q \vee r$	commutativity
4)	$\neg p \vee q \vee r$	idempotence
5)	$p \implies (q \vee r)$	implication

5) is the same as RHS. □

(b) We argue by deriving RHS from LHS using logical equivalences.

1)	$((p \implies r) \wedge (q \implies r))$	LHS
2)	$((\neg p \vee r) \wedge (\neg q \vee r))$	implication
3)	$(\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (\neg q \wedge r) \vee (r \wedge r)$	distribution
4)	$(\neg p \wedge \neg q) \vee (\neg p \wedge r) \vee (\neg q \wedge r) \vee r$	idempotence
5)	$(\neg p \wedge \neg q) \vee r$	absorption
6)	$\neg(p \vee q) \vee r$	De Morgan's law
7)	$(p \vee q) \implies r$	implication

7) is the same as RHS. □

(c) We argue by deriving RHS from LHS using logical equivalences.

1)	$\neg(p \iff q)$	LHS
2)	$\neg((p \implies q) \wedge (q \implies p))$	biconditional
3)	$\neg(p \implies q) \vee \neg(q \implies p)$	De Morgan's law
4)	$\neg(\neg p \vee q) \vee \neg(\neg q \vee p)$	implication
5)	$(p \wedge \neg q) \vee (q \wedge \neg p)$	De Morgan's law
6)	$(p \vee q) \wedge (p \vee \neg p) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p)$	distribution
7)	$(p \vee q) \wedge T \wedge T \wedge (\neg q \vee \neg p)$	negation
8)	$(p \vee q) \wedge (\neg q \vee \neg p)$	domination
9)	$(\neg p \vee \neg q) \wedge (q \vee p)$	commutativity
10)	$(p \implies \neg q) \wedge (\neg q \implies p)$	implication
11)	$p \iff \neg q$	biconditional

11) is the same as RHS. □

2. Prove the validity of the following arguments where the propositions are separated by semi-colons.
   (a) $\boxed{\text{2 points}}$ $p \wedge q; p \implies (\neg q \vee r); r \implies s; \therefore s$
   (b) $\boxed{\text{3 points}}$ $p \implies q; \neg q \vee r; r \implies (t \vee s); \neg s \wedge p; \therefore t$
   (c) $\boxed{\text{4 points}}$ $(\neg p \wedge q) \implies (r \vee s); \neg p \implies (r \implies w); (s \implies t) \vee p; \neg p \wedge q; \therefore w \vee t$

Solution:

(a) We proceed by deducing the conclusion from the premises by applying rules of inference.

1)	p	simplification on P1
2)	$\neg q \vee r$	Modus Ponens on P2 and 1)
3)	q	simplification on P1
4)	r	disjunctive syllogism on 2) and 3)
5)	s	Modus Ponens on 4) and P3

5) is the same as the conclusion. □

(b) We proceed by deducing the conclusion from the premises by applying rules of inference.

1)	p	simplification on P4		
2)	q	Modus Ponens on P1 and 1)		
3)	r	disjunctive syllogism on 2) and P2)		
4)	$t \vee s$	Modus Ponens on 3) and P35)	$\neg s$	simplification on P4
6)	t	disjunctive syllogism on 4) and 5))		

6) is the same as the conclusion. □

(c) We proceed by deducing the conclusion from the premises by applying rules of inference.

1)	$\neg p$	simplification on P4
2)	$r \vee s$	Modus Ponens on P1 and P4
3)	$r \implies w$	Modus Ponens on 1) and P2
4)	$\neg r \vee w$	implication, 3)
5)	$\neg s \vee t$	implication, P3
6)	$s \vee w$	resolution on 2) and 4)
7)	$w \vee t$	resolution on 5) and 6)

7) is the same as the conclusion. □

3. Write each of the following statements in logical notation, taking care to identify, as applicable, the used propositions, predicates, and domains.

   (a) $\boxed{\text{2 points}}$ All prime numbers except 2 are odd.

   (b) $\boxed{\text{2 points}}$ The start and end dates of the semester cannot be the same.

   (c) $\boxed{\text{3 points}}$ The number of edges in a simple graph with n nodes is no more than $\frac{n(n-1)}{2}$.

Solution:

(a) Let the domain be numbers, and let us define the following predicates.
$P(x)$: x is prime
$O(x)$: x is odd
Then the statement can be written as: $\forall x\,((P(x) \wedge x \neq 2) \implies O(x))$.

(b) Let us define the following predicates,
$S(d, s)$: d is the start of s
$E(d, s)$: d is the end of s
where d is from the set of dates, and s is from the set of semesters.

Then the statement can be written as: $\forall d_1 \forall d_2 \forall s((S(d_1, s) \wedge E(d_2, s)) \implies d_1 \neq d_2)$.

(c) Let us define the following predicates,
$N(G, x)$: G has x nodes
$E(G, x)$: G has x edges
where G is from the set of simple graphs, and x is from the set of numbers.

Then the statement can be written as: $\forall G \forall n \forall e\left((N(G, n) \wedge E(G, e)) \implies e \leq \frac{n(n-1)}{2}\right)$.

4. $\boxed{\text{5 points}}$ A *partition* of a set, A, is a set of non-empty subsets, A_i, of A, such that every element a in A is in exactly one of these subsets (i.e. A is a disjoint union of the subsets). [Wikipedia]

   Prove the validity of the following argument where the propositions are separated by semi-colons.

   $$\{A_1, A_2\} \text{ is a partition of the set } A; \{A_1, A_2, A_3\} \text{ is a partition of the set } A \cup B; \therefore A_3 \subseteq B$$

   Solution: We proceed by reasoning about the given premises and inferring information about A_3 from them in order to arrive at the conclusion.

1)	$A_1 \cup A_2 = A$	definition of partition, P1
2)	$A_1 \cup A_2 \cup A_3 = A \cup B$	definition of partition, P2
3)	$A \cup A_3 = A \cup B$	substituting 1) in 2)
4)	$A_3 \subseteq A \cup B$	definition of union, 3)
5)	$A_3 \cap A = A_3 \cap (A_1 \cup A_2)$	using 1)
6)	$A_3 \cap A = (A_3 \cap A_1) \cup (A_3 \cap A_2)$	distribution, 5)
7)	$A_3 \cap A = \emptyset \cup \emptyset$	definition of partition, P2, 6)
8)	$A_3 \cap A = \emptyset$	definition of union, 7)
9)	$A_3 \subseteq B$	from 4) and 8)

   9) is the same as the conclusion. □

Continued Solutions: