Solution to Homework 2: Logic 1
CS 113 Discrete Mathematics
Habib University – Spring 2020

Don't Grade Me

Propositional Logic

1. Prove or disprove the following claims using truth tables. In each case, explicitly state your conclusion and how it is supported by the truth table.

   (a) $\boxed{\text{5 points}}$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

   Solution:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(\neg(p \vee q)) \iff (\neg p \wedge \neg q)$
F	F	F	T	T	T	T	T
F	T	T	F	T	F	F	T
T	F	T	F	F	T	F	T
T	T	T	F	F	F	F	T

   We see that $(\neg(p \vee q)) \iff (\neg p \wedge \neg q)$ is a tautology.
   $\implies (\neg(p \vee q)) \equiv (\neg p \wedge \neg q)$.
   Hence proved.

   (b) $\boxed{\text{5 points}}$ $(p \vee q) \implies \neg r \equiv (\neg p \wedge \neg q) \wedge \neg r$.

   Solution: For ease of notation, let

$$A : (p \vee q) \implies \neg r$$
$$B : (\neg p \wedge \neg q) \wedge \neg r$$

   So we have to prove that $A \equiv B$.

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \vee q$	$\neg p \wedge \neg q$	A	B	$A \iff B$
F	F	F	T	T	T	F	T	T	T	T
F	F	T	T	T	F	F	T	T	F	F
F	T	F	T	F	T	T	F	T	F	F
F	T	T	T	F	F	T	F	F	F	T
T	F	F	F	T	T	T	F	T	F	F
T	F	T	F	T	F	T	F	F	F	T
T	T	F	F	F	T	T	F	T	F	F
T	T	T	F	F	F	T	F	F	F	T

We see that $A \iff B$ is not a tautology.
$\implies A \not\equiv B$.

2. We want to write the statement, "A person is popular only if they are cool or funny", in propositional logic.

   (a) $\boxed{\text{5 points}}$ Identify three simple propositions, $p, q,$ and r, needed for the representation and write out the corresponding expression that uses them to represent the given sentence.

   Solution: The simple propositions are as follows.
   p : The person is popular.
   q : The person is cool.
   r : The person is funny.
   The expression is : $p \implies (q \vee r)$.

   (b) $\boxed{\text{5 points}}$ For your expression identified above, write the converse, contrapositive, and inverse in propositional logic as well as complete English sentences.

   Solution: The expression is : $p \implies (q \vee r)$.

	Logical Notation	English sentence
Converse	$(q \vee r) \implies p$	A person is cool or funny only if they are popular
Contrapositive	$(\neg q \wedge \neg r) \implies \neg p$	A person is not cool and not funny only if they are not popular
Inverse	$\neg p \implies (\neg q \wedge \neg r)$	A person is not popular only if they are neither funny nor cool

   Note that there are other correct ways to express the above.

3. $\boxed{\text{5 points}}$ A small company makes widgets in a variety of constituent materials (aluminum, copper, iron), colors (red, green, blue, grey), and finishes (matte, textured, coated). Although there are many combinations of widget features, the company markets only a subset of the possible combinations. The following sentences are constraints that characterize the possibilities.

   1. aluminum $\vee$ copper $\vee$ iron
   2. aluminum $\implies$ grey
   3. copper $\wedge \neg$ coated $\implies$ red
   4. coated $\wedge \neg$ copper $\implies$ green
   5. green $\vee$ blue $\iff \neg$ textured $\wedge \neg$ iron

   Suppose that a customer places an order for a copper widget that is both green and blue with a matte finish.

   (a) $\boxed{\text{5 points}}$ Using the propositions above, express the order as a compound proposition in logical notation.

Solution: The order can be represented as: copper $\wedge$ green $\wedge$ blue $\wedge$ matte.
That is, the truth value of each of the above is True. The truth value of all other propositions is False.

(b) $\boxed{\text{5 points}}$ Determine which constraints are satisfied and which are violated for the order, and provide an explanation.

Solution: In the table below, a value of 1 for a constraint indicates that the constraint is satisfied and a value of 0 indicates that it is violated.

Constraint	Satisfied	Explanation
aluminum $\vee$ copper $\vee$ iron	1	The constraint is satisfied because copper is T.
aluminum $\implies$ grey	1	The implication is satisfied because both the antecedent and consequent are F.
copper $\wedge \neg$ coated $\implies$ red	0	The implication is not satisfied because the antecedent is T but the consequent is F.
coated $\wedge \neg$ copper $\implies$ green	1	The implication is satisfied because both the antecedent is F and the consequent is T.
green $\vee$ blue $\iff \neg$ textured $\wedge \neg$ iron	1	The biconditional is satisfied because both sides have a truth value of T.

4. $\boxed{\text{5 points}}$ You are given four cards each of which has a number on one side and a letter on another. You place them on a table in front of you and the four cards read: A 5 2 J. Which cards would you turn over in order to test the following rule?

Cards with 5 on one side have J on the other side.

Explain your choice.

Solution: The rule can be written as $5 \implies J$, where the antecedent indicates the sign on one side and the consequent indicates the sign on the other side of the card.

In the table below, a value of 1 for a card indicates that I will turn it and a value of 0 indicates that I will not turn it.

Card	Turned	Explanation
A	1	The consequent of the implication is False. Whenever this is case, the implication's truth value depends on the value of the antecedent. If the antecedent is True, i.e. there is a 5 on the other side, the implication will be False. And if the antecedent is False, i.e. there is not a 5 on the other side, the implication will be True. Therefore this card needs to be turned to test the validity of the rule.
5	1	The antecedent of the implication is True. The truth value of the implication in this case is the same as that of the consequent. So the letter on the other side will impact the validity of the rule. If the letter on the other side is J, the implication will be True, otherwise the implication will be False. Therefore this card needs to be turned to test the validity of the rule.
2	0	The antecedent of the implication is False. Whenever this is case, the implication is True regardless of the truth value of the consequent. So, the letter on the other side of the card will make no difference on the validity of the rule.
J	0	The consequent of the implication is True. Whenever this is case, the implication is True regardless of the truth value of the antecedent. So, the number on the other side of the card will make no difference on the validity of the rule.

5. An argument is said to be *valid* if its *conclusion* can be inferred from its *premises.* An argument that is not valid is called an *invalid* argument, or a *fallacy.* For each of the arguments below, identify the simple propositions involved, write the premises and conclusion(s) in logical notation using the identified simple propositions, and decide whether it is valid. Justify your decision.

   (a) $\boxed{\text{5 points}}$ If I am wealthy, then I am happy. I am happy, therefore, I am wealthy.

Solution: The simple propositions are as follows.
p : I am wealthy.
q : I am happy.
The argument is

$$\begin{array}{l} p \implies q \\ q \\ \hline p \end{array}$$

This argument is invalid because p does not derive from the premises.

Given the implication, the consequent q can be inferred from the antecedent p, and as per the contrapositive, which is logically equivalent, $\neg p$ can be inferred from $\neg q$. There is no information about any inference from q and the conclusion p *cannot* be inferred from the premises. Therefore, this argument is invalid. This form of fallacy is common and is called, "affirming the consequent".

Another proof approach could be to test whether $((p \implies q) \wedge q) \implies p$ is a tautology.

(b) $\boxed{\text{5 points}}$ If Ahmed drives his car, he is at least 18 years old. Ahmed does not drive a car. Therefore, Ahmed is not yet 18 years old.

Solution: The simple propositions are as follows.
p : Ahmed drives his car.
q : Ahmed is at least 18 years old.
The argument is

$$\begin{array}{l} p \implies q \\ \neg p \\ \hline \neg q \end{array}$$

This argument is invalid because $\neg q$ does not derive from the premises.

Given the implication, the consequent q can be inferred from the antecedent p, and as per the contrapositive, which is logically equivalent, $\neg p$ can be inferred from $\neg q$. There is no information about any inference from $\neg p$ and the conclusion $\neg q$ *cannot* be inferred from the premises. Therefore, this argument is invalid. This form of fallacy is common and is called, "denying the antecedent".

Another proof approach could be to test whether $((p \implies q) \wedge \neg p) \implies \neg q$ is a tautology.

(c) $\boxed{\text{5 points}}$ If I study, then I will not fail CS 113. If I do not play cards too often, then I will study. I failed CS 113. Therefore, I played cards too often.

Solution: The simple propositions are as follows.
p : I study.
q : I will fail CS 113.
r : I play cards too often.
The argument is

$$\begin{array}{ll} p \implies \neg q & (1) \\ \neg r \implies p & (2) \\ q & (3) \\ \hline r & \end{array}$$

This argument is valid. The proof is given below.

Inference	Premises	
$\neg p$	(1) $\wedge$ (3)	(4)
r	(2) $\wedge$ (4)	

$\square$

6. $\boxed{\text{5 points}}$ One of your TA's has hidden a manual titled, "Sacred Secrets: How to Earn an A+ and Keep your Mind", somewhere on campus. As they could themselves not benefit from this

manual, the directions they have left for you to find the manual are as follows.

1. There is a hint at Learn Courtyard or at the Gym.
2. If your TA is sitting in Ehsas or they are absent, then there is a hint at Learn Courtyard.
3. If your TA is not sitting in Ehsaas, then there is a hint at the Gym.
4. If there are people in Learn Courtyard, then there is no hint at Learn Courtyard.
5. If there is a hint at Learn Courtyard, then the manual is at Zen Garden.
6. If there is hint at the Gym, then the manual is at Earth Courtyard.
7. If your TA is absent, then the manual is at Fire Courtyard.

You notice that there are people in Learn Courtyard. Where is the manual?

Identify the relevant simple propositions to model the above in propositional logic. Represent the above situation using propositional logic and describe the steps needed to infer the location of the manual.

Solution: The simple propositions are as follows.
p : There is a hint at Learn Courtyard.
q : There is a hint at the Gym.
r : Your TA is sitting in Ehsas.
s : Your TA is absent.
t : There are people in Learn Courtyard.
u : The manual is at Zen Garden.
v : The manual is at Earth Courtyard.
w : The manual is at Fire Courtyard.

We have the following information.

$$p \vee q \tag{1}$$
$$r \vee s \implies p \tag{2}$$
$$\neg r \implies q \tag{3}$$
$$t \implies \neg p \tag{4}$$
$$p \implies u \tag{5}$$
$$q \implies v \tag{6}$$
$$r \implies w \tag{7}$$
$$t \tag{8}$$

We can infer the following.

$$\neg p \qquad \text{Modus Ponens } (8), (4) \tag{9}$$
$$q \qquad \text{Disjunctive Syllogism } (9), (1) \tag{10}$$
$$v \qquad \text{Modus Ponens } (6)(10) \tag{11}$$

Therefore the manual is at Earth Courtyard.

7. $\boxed{\text{5 points}}$ A TV channel is reporting a terrorist attack on a shopping mega-mall. The mega-mall website claims that the mall closes only in case of an attack. It is known that a sale is on whenever the mega-mall is open, and that many people come when there is a sale. A crime expert explained that in case of an attack, neighbors end up hearing firing sounds and calls are made to the local police. Phone logs indicate no recent calls to the police.

   (a) $\boxed{\text{5 points}}$ We are not sure about the TV report, but we trust all the other sources. Is the mega-mall open?

   Solution: The simple propositions are as follows.
   p : The mega-mall is closed.
   q : The mega-mall is attacked.
   r : There is a sale at the mega-mall.
   s : There are many people at the mega-mall.
   t : Neighbors hear firing sounds.
   u : Calls are made to the local police.
   We have the following information.

$$\begin{aligned} p &\implies q && (12)\\ \neg p &\implies r && (13)\\ r &\implies s && (14)\\ q &\implies (t \wedge u) && (15)\\ &\neg u && (16) \end{aligned}$$

   We can infer the following.

$$\begin{aligned} &\neg u \vee \neg t && \text{Addition (16)} && (17)\\ &\neg t \vee \neg u && \text{Commutativity (17)} && (18)\\ &\neg(t \wedge u) && \text{DeMorgan's law (18)} && (19)\\ &\neg q && \text{Modus Tollens (15), (19)} && (20)\\ &\neg p && \text{Modus Tollens (12), (20)} && (21) \end{aligned}$$

   Therefore the mega-mall is open.

   (b) $\boxed{\text{5 points}}$ Is the TV report true?

   Solution: From (20), we have $\neg q$. That is, the mega-mall is not attacked. Therefore the TV report is false.

Predicate Logic

8. (a) $\boxed{\text{5 points}}$ There is a third quantifier often used in predicate logic called the *Uniqueness Quantifier*, $\exists! x\ P(x)$ which is read as, "$P(x)$ is true for one and only one x in the domain",

or "there is a *unique* x such that $P(x)$". Give an example of a propositional function $P(x)$ and a corresponding domain, such that $\exists! x\ P(x)$ is a true proposition.

Solution: Consider the domain $\mathbb{Z}$ of integers and the predicate $P(x) : x - 1 = 0$. Then the following is True:

$$\exists! x \in \mathbb{Z}\ (x - 1 = 0)$$

(b) $\boxed{\text{5 points}}$ The uniqueness quantifier can be expressed using the other two quantifiers but is still used on its own as it shortens the logical terms. In particular,

$$\exists! x\ P(x) \equiv \exists x\ (P(x) \wedge \forall y\ (P(y) \rightarrow y = x)) \tag{22}$$

Express the proposition on the right above in English and explain why it is equivalent to the left hand side, i.e. to the uniquely quantified propositional function. You may explain in words; a formal proof is not yet required.

Solution: The proposition on the right claims that there is an x in the domain for which $P(x)$ is True. And if $P(x)$ is True for any other value y from the domain, y must be the same as x. This is the same as saying that x is the only value in the domain for which $P(x)$ is True. In other words, there is only one x or a unique x in the domain which makes $P(x)$ True.

(c) $\boxed{\text{5 points}}$ Express $\neg\exists! x P\ (x)$ in a similar way as (22). Provide an expression in formal notation as well as in English. Also, give an example of a true proposition $\neg\exists! x\ P(x)$ by slightly changing the one you gave in part (a).

Solution: The solution involves the negation of an implication so let us work that out first.

$$\begin{aligned} & \neg(p \implies q) \equiv \neg(\neg p \vee q) \\ \implies\quad & \neg(p \implies q) \equiv p \wedge \neg q && \text{(deMorgan's Law and double negation)} \end{aligned} \tag{23}$$

Now let us negate Equivalence (22) by moving the negation inside.

$$\begin{aligned} & \exists! x\ P(x) \equiv \exists x\ (P(x) \wedge \forall y\ (P(y) \rightarrow y = x)) \\ \implies\quad & \neg\exists! x\ P(x) \equiv \neg\exists x\ (P(x) \wedge \forall y\ (P(y) \rightarrow y = x)) && \text{(negate both sides)} \\ & \equiv \forall x\ \neg(P(x) \wedge \forall y\ (P(y) \rightarrow y = x)) && \text{(de Morgan's law for quantifier)} \\ & \equiv \forall x\ (\neg P(x) \vee \neg\forall y\ (P(y) \rightarrow y = x)) && \text{(deMorgan"s law for connectives)} \\ & \equiv \forall x\ (\neg P(x) \vee \exists y\ \neg(P(y) \rightarrow y = x)) && \text{(de Morgan's law for quantifier)} \\ & \equiv \forall x\ (\neg P(x) \vee \exists y\ (P(y) \wedge y \neq x)) && \text{(Equivalence (23))} \\ & \equiv \forall x\ (P(x) \implies \exists y\ (P(y) \wedge y \neq x)) && (\neg p \vee q \equiv p \implies q) \end{aligned}$$

The expression on the right now says that if there is an x in the domain for which $P(x)$ is True, then there is at least one other distinct value, y, from the domain for which $P(y)$ is True.

Changing the domain of the earlier proposition to negative integers, $\mathbb{Z}^-$, makes the uniqueness quantifier false, i.e.

$$\neg\exists! x \in \mathbb{Z}^- \ (x - 1 = 0)$$

Alternately, we can modify $P(x)\colon x - 1 = 0$ to $Q(x)\colon x^2 - 1 = 0$ and apply it to the original domain $\mathbb{Z}$ of integers to obtain

$$\neg\exists! x \in \mathbb{Z} \ (x^2 - 1 = 0)$$

9. For each of the statements given below, perform the following.

   1. Express the statement in formal notation using quantifiers.
   2. Express the negation of the statement in formal notation such that no negation is left to the quantifier.
   3. Express the negated statement above as a statement in English.

   (a) $\boxed{\text{5 points}}$ No one can have Pakistani and Indian citizenship.

   Solution:

   1. Domain : All people
      $P(x)$: x has Pakistani citizenship.
      $I(x)$: x has Indian citizenship.
      Statement : $\neg\exists x \ (I(x) \land P(x)) \equiv \forall x \ \neg(I(x) \land P(x)) \equiv \forall x \ (\neg I(x) \lor \neg P(x))$
   2. $\neg\neg\exists x \ (I(x) \land P(x)) \equiv \exists x \ (I(x) \land P(x))$ (double negation)
   3. Someone has both Pakistani and Indian citizenship.

   (b) $\boxed{\text{5 points}}$ If everyone does their homework and goes to the recitations, no one will be badly prepared for the exams.

   Solution:

   1. Domain : All people
      $H(x)$: x does their homework.
      $R(x)$: x goes to the recitations.
      $P(x)$: x is badly prepared for the exam.
      Statement : $\forall x \ (H(x) \land R(x)) \implies \forall x \ \neg P(x)$
   2. $$\begin{aligned} \neg(\forall x \ (H(x) \land R(x)) \implies \forall x \ \neg P(x)) &\equiv \forall x \ (H(x) \land R(x)) \land \neg\forall x \ \neg P(x) \quad \text{(Equivalence (23))} \\ &\equiv \forall x \ (H(x) \land R(x)) \land \exists x \ P(x) \end{aligned}$$

3. Everyone did their homework and went to the recitations but someone is still badly prepared for the exams.

(c) 5 points No student has solved at least one exercise in every section of the book.

Solution:

1.

$Solve(st, ex)$	:	st has solved ex
$In(ex, sec)$	:	ex is in sec
Domain for st	:	all students
Domain for ex	:	all exercises
Domain for sec	:	all sections of the book
Statement	:	$\neg\exists st\ \forall sec\ \exists ex\ Solve(st, ex) \wedge In(ex, sec)$

2. $\neg\neg\exists st\ \forall sec\ \exists ex\ Solve(st, ex) \wedge In(ex, sec) \equiv \exists st\ \forall sec\ \exists ex\ Solve(st, ex) \wedge In(ex, sec)$
3. Some student has solved at least one exercise in every section of the book.

(d) 5 points No one has climbed every mountain in Pakistan.

Solution:

1.

$C(x, y))$	:	x has climbed y.
Domain for x	:	All people
Domain for y	:	Mountains in Pakistan
Statement	:	$\neg\exists x\forall y\ C(x, y) \equiv \forall x\exists y\ \neg C(x, y)$

2. $\neg\neg\exists x\forall y\ C(x, y) \equiv \exists x\forall y\ C(x, y)$ (double negation)
3. Someone has climbed every mountain in Pakistan.

10. Translate the specifications below into English using the given propositional functions.
$F(p)$: The printer p is out of service
$B(p)$: Printer p is busy
$L(j)$: Print job j is lost
$Q(j)$: Print job j is queued

(a) 5 points $\exists p\ (F(p) \wedge B(p)) \rightarrow \exists j\ L(j)$

Solution: If any printer is busy and out of service, then some print job is lost.

(b) 5 points $(\forall p\ B(p) \wedge \forall j\ Q(j)) \rightarrow \exists j\ L(j)$

Solution: If all printers are busy and all jobs are queued, then some print job is lost.

11. Express each of the system specifications below using suitable predicates, quantifiers, and logical connectives.

(a) $\boxed{\text{5 points}}$ At least one mail message can be saved if there is a disk with more than 10KB of free space.

Solution:

$Save(m)$	:	m can be saved
Domain for m	:	all mail messages
$Free(d, mem)$	:	d has more than mem KB free space
Domain for d	:	all disks
Domain for mem	:	units of storage
Statement	:	$\exists d\ Free(d, 10) \implies \exists m\ Save(m)$

(b) $\boxed{\text{5 points}}$ The system mailbox can be accessed by everyone in the group if the file system is locked.

Solution:

$Access(x, y)$	:	x can access y.
Domain for x	:	people in the group
Domain for y	:	systems
$State(y, z)$	:	y is in state z.
Domain for z	:	system states
Statement	:	$State(s.file, locked) \rightarrow \forall x\ Access(x, s.mailbox)$

12. Consider the propositions below for which the domain of all variables is $\mathbb{Z}$. For each proposition,

1. Express the proposition in English,
2. State its truth value and provide an explanation if it is true or a counterexample if it is false, and
3. Specify a domain for which the proposition has the other truth value.

(a) $\boxed{\text{5 points}}$ $\forall x \forall y\ (x^2 = y^2 \rightarrow x = y)$

Solution:

1. If the squares of two integers are equal, then the integers must have already been equal.
2. This proposition is False. A counterexample is $(x, y) : (1, -1)$. (One counterexample is sufficient to disprove universality.)
3. One domain for which the proposition is true is $\mathbb{Z}^+$.

(b) $\boxed{\text{5 points}}$ $\forall x \exists y\ (y^2 = x)$

Solution:

1. Every integer is the square of some integer.
2. This proposition is False. A counterexample is $x = 2$ for which no $y \in \mathbb{Z}$ can be found to satisfy the statement. (One counterexample is sufficient to disprove universality.)
3. One domain for which the proposition is true is $\mathbb{R}^+$.

(c) $\boxed{\text{5 points}}$ $\exists x \forall y\ (x \leq y^2)$

Solution:

1. There exists an integer which is less than or equal to the square of any other integer.
2. This proposition is True. It is possible to find a value, e.g. $x = 0$, that makes the proposition True, as $0 \leq y^2$ for all $y \in \mathbb{Z}$. (One example is sufficient to prove existence.)
3. One other domain for which the proposition is true is $\mathbb{R}$. But if we take $\mathbb{R}^+$, then such a number does not exist. In particular for every real number x, you can then find a number y, such that $y^2 < x$.

(d) $\boxed{\text{5 points}}$ $\forall x \forall y\ \exists z\ (x - z = y)$

Solution:

1. The difference of any 2 integers is an integer.
2. This proposition is True. The difference of 2 integers is again an integer.
3. One domain for which the proposition is False is $\mathbb{Z}^+$. A counterexample for that domain is $(x, y) : (5, 7)$. z would have to be -2 which is not in the domain.