Recitation: Relations

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Definitions

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively as follows:

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Questions

1. Find the error in the "proof" of the following "theorem".
   Theorem: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
   Proof: Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

2. Let

   - $R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}$, the "greater than" relation
   - $R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation
   - $R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}$, the "less than" relation

   Find

   (a) $R_1 \circ R_1$

   (b) $R_1 \circ R_2$

   (c) $R_1 \circ R_3$

3. The relation R on a set A is transitive if and only if $R \circ R \subseteq R$

4. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 1), (3, 4), (3, 5), (4, 2), (4, 5), (5, 1), (5, 2)$, and $(5, 4)$. Find R^2, R^3, R^4, R^5

5. Show that for a reflexive relation R, $R^{-1} \subseteq R \circ R^{-1}$

6. Represent each of these relations on $\{1, 2, 3\}$ with a matrix

   - $\{(1, 1), (1, 2), (1, 3)\}$
   - $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$

7. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices

1.
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

2.
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

8. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, ..., 100\}$ consisting of the first 100 positive integers have if R is

   1. $\{(a, b) \mid a > b\}$
   2. $\{(a, b) \mid a = 1\}$
   3. $\{(a, b) \mid a \neq b\}$

10. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?

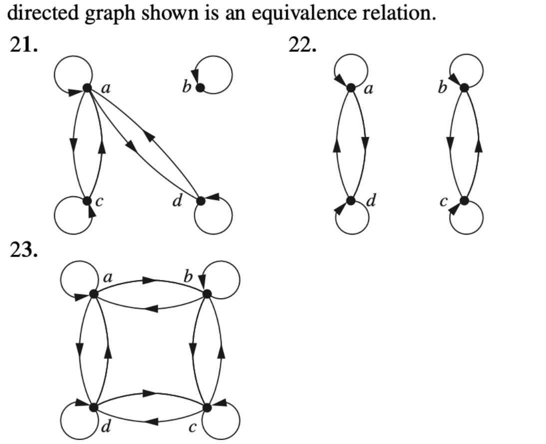
11. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1}. Note that $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

12. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

   1. $\{(a, b) \mid a$ and b are the same age$\}$
   2. $\{(a, b) \mid a$ and b have the same parents$\}$
   3. $\{(a, b) \mid a$ and b share a common parent$\}$
   4. $\{(a, b) \mid a$ and b have met$\}$
   5. $\{(a, b) \mid a$ and b speak a common language$\}$

13. Justify whether the given relations are equivalence relations or not.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.



14. Justify that for an equivalence relation R for an equivalence class $[\mathbf{c}]$ $\forall a, b \in [\mathbf{c}], (a, b)$, or every element in the equivalence class has relation with every element in the equivalence class

15. Let R be a relation that is reflexive and transitive. Prove that $R^2 = R$.

16. Show that $R \circ R = R$ for equivalence relations

17. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$