Recitation: Relation Proofs

Fix your relations

April 8, 2022

Questions

1. $\boxed{\text{5 points}}$ Given Theorem 1 and Definition 1 below, prove Theorem 2.

   Theorem 1. *Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent.*
   $$(i) aRb \quad (ii) [a] = [b] \quad (iii) [a] \cap [b] \neq \emptyset$$

   Definition 1. *A partition of a set, A, is a set of non-empty subsets, A_i, of A, such that every element a in A is in exactly one of these subsets (i.e. A is a disjoint union of the subsets). [Wikipedia]*

   Theorem 2. *Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.*

2. $\boxed{\text{5 points}}$ Prove the following theorem.

   Theorem 3. *Let R be an equivalence relation on a set A. The following statements for elements a and b of A are equivalent.*
   $$(i) aRb \quad (ii) [a] = [b] \quad (iii) [a] \cap [b] \neq \emptyset$$

3. $\boxed{\text{5 points}}$ Given non empty set S and a partition A_i of S. Prove that $R = \bigcup_{A_i} (A_i \times A_i)$ is an equivalence relation on S. Moreover the equivalence classes of R are exactly A_i.

4. $\boxed{\text{5 points}}$ Show that a finite nonempty poset has a maximal element.

5. $\boxed{\text{5 points}}$ The following is a summary of someone's attempt to prove that exists only one unique God. Find the error in the proof.

   Proof: Assume there is more than one unique God.
   Insert convincing argument to show that this leads to a contradiction.
   Since having more than one unique God results in a contradiction, there exists only one unique God.
   $\square$

6. $\boxed{\text{5 points}}$ "Prove the least element is unique when it exists.". State the error in the following proofs.

   Proof 1: Assume that Least element exists and it is not unique. Let a be a least element and g be another least element and $a \neq g$. Then by the definition of least element, $g \prec a$. But we assumed that a is the least element. Therefore, this is a contradiction and hence we can say that, The least element is unique when it exists.

   Proof 2: Let's assume that there is more than one unique least element, with a and b being two distinct least elements. For the relation R to be antisymmetric, a must be equal to be ($a = b$), given that $(a, b) \in R$ and $(b, a) \in R$. Since a and b are equal to each other, we can conclude that there is only one unique least element.

7. $\boxed{\text{5 points}}$ The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

8. $\boxed{\text{5 points}}$ Fix the problem in the proof. "Show that for a reflexive relation R, $R^{-1} \subseteq R \circ R^{-1}$"

   Proof: By reflexivity, $(a, a) \in R$, therefore $(b, a) \in R \circ R^{-1}$ $\square$

9. $\boxed{\text{5 points}}$ Show that a subset of an antisymmetric relation is also antisymmetric.

10. $\boxed{\text{5 points}}$ A relation R is called circular if aRb and bRc imply that cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.