Linear Algebra – Math 205
Lecture – 2, Exercise Set
(SPRING 2023)
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Exercise Set 1.3 Solution

Question 1

Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix.

(a) BA.
Solution: **In the case of multiplication, the number of columns in the first matrix should be equal to the number of rows in the second matrix**, The matrix B has four rows and five columns (4×5), whereas the matrix A has four rows and five columns (4×5). It means multiplication is not defined since the number of columns in B does not match the number of rows in A.

(b) $AC + D$.
Solution: The matrix A has four rows and five columns (4×5), and the matrix C has five rows and two columns (5×2). Multiplication is defined as the number of columns in A equaling the number of rows in C, resulting in matrix AC being 4×2. **In the case of addition or subtraction, matrices must be of equal size**, and AC and D have the same size (4×2) in this question. The final matrix $AC + D$ has a size of 4×2.

(c) $AE + B$.
Solution: The matrix A has four rows and five columns (4×5), and the matrix E has five rows and four columns (5×4). Multiplication is defined as the number of columns in A equaling the number of rows in E, resulting in matrix AE being 4×4. But adding matrices AE and B is not defined.

(d) $AB + B$.
Solution: The matrix A has four rows and five columns (4×5), while the matrix B has four rows and five columns (5×4). Hence, since multiplication is not defined, we cannot go any further.

(e) $E(A + B)$.
Solution: The matrix A has four rows and five columns (4×5), and the matrix B has four rows and five columns (4×5). The addition is defined as the size of A equals the size of B, resulting in matrix $(A + B)$ being 4×5. Now the matrix E has size 5×4, so multiplication is defined in $E(A + B)$ and has a size of 5×5.

(f) $E(AC)$.
Solution: The matrix A has a size of 4×5, and the matrix C has a size of

5×2; hence, multiplication is defined, and the resulting matrix (AC) is 4×2. For further multiplication, the matrix $E(AC)$ is defined since E has size 5×4 and (AC) has 4×2, so the resulting matrix has a size of 5×2.

(g) $E^T A$
Solution: The matrix E^T has size 4×5 and A has 4×5. Hence, multiplication is not defined since the number of columns in E^T does not match the number of rows in A.

(h) $(A^T + E)D$
Solution: The matrix A^T has size 5×4 and E has 5×4, hence addition is defined and the size of matrix $A^T + E$ is 5×4. The multiplication of $(AT + E)D$ is defined because the number of columns in $(AT + E)$ and the number of rows in D are both four. The resulting matrix, $(AT + ED)$, is 5×2 in size.

Question 7

Use the method of Example 7 to find

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

(a) the first row of AB, i.e. A_1B.

$$\begin{bmatrix} 3 & -2 & 7 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 18+0+49 & -6-2+49 & 12-6+35 \end{bmatrix} = \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$$

(b) the third row of AB, i.e. A_3B.

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 0+0+63 & 0+4+63 & 0+12+45 \end{bmatrix} = \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$$

(c) the second column of AB, i.e. AB_2.

$$\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -6-2+49 \\ -12+5+28 \\ 0+4+63 \end{bmatrix} = \begin{bmatrix} 41 \\ 35 \\ 67 \end{bmatrix}$$

(d) the first column of BA, i.e. B_1A.

$$\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 18-12+0 \\ 0+6+0 \\ 21+42+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$$

(e) the third row of AA, i.e. A_3A.

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 0+24+0 & 0+20+36 & 0+16+81 \end{bmatrix} = \begin{bmatrix} 24 & 56 & 97 \end{bmatrix}$$

(f) the third column of AA, i.e. AA_3.

$$\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 21-8+63 \\ 42+20+36 \\ 0+16+81 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

Question 8

Let A and B be the matrices in Exercise 7. Use the method of Example 9 to

(a) Express each column matrix AB of as a linear combination of the column matrices of A

$$AB = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$
$$\begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$$

(b) express each column matrix BA of as a linear combination of the column matrices of B

$$BA = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 70 \\ 6 & 17 & 31 \\ 63 & 41 & 122 \end{bmatrix}$$

The column matrices of BA can be expressed as linear combinations of the column matrices of B as follows:

$$\begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix} = 3 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} -6 \\ 17 \\ 41 \end{bmatrix} = -2 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} 70 \\ 31 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

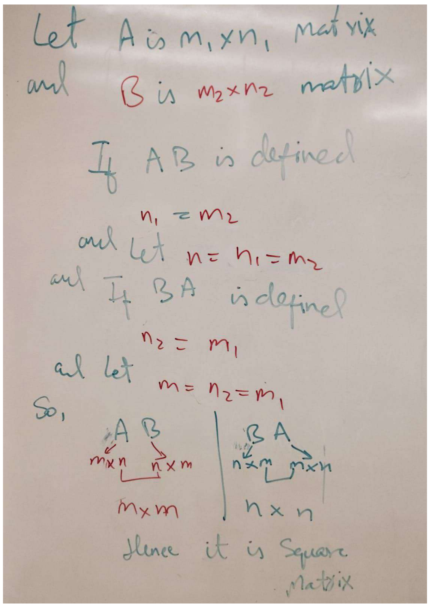
Question 12

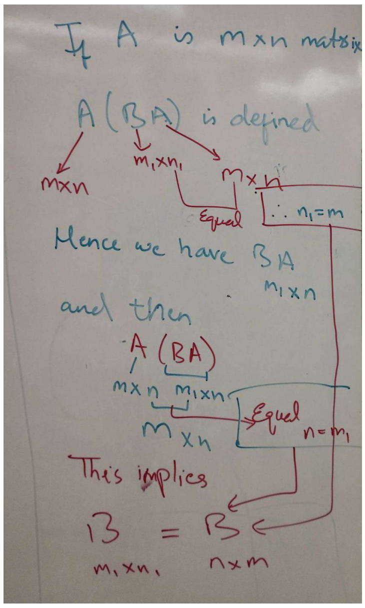
Use the method of Example 7 to find

(a) Show that if AB and BA are both defined, then AB and BA are square matrices.

Solution: If AB is defined, a column of A is equal to rows of B; if BA is defined, it means a column of B equals rows of A. This means that AB and BA have equal rows and columns and are thus square matrices.

Mathematically for both parts.





(b) Show that if A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

Solution: For defining $A(BA)$, one should define BA first, and if BA is defined, it means a column of B is equal to rows of A, and as A is $m \times n$, then BA must be $m_1 \times n$. Now for $A(BA)$, number of columns of A (n)should be equal to number of rows BA (m_1), which means $A(BA)$ must have resulting size $m \times n$. As a result, B is a $ntimesm$ matrix.

Question 13

In each part, find matrices A, x, and b that express the given system of linear equations as a single matrix equation Ax = b.

(a)

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 9 & -1 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

(b)

$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 \qquad + x_2 - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

$$\begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 0 & 1 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

Question 17

In each part, determine whether block multiplication can be used to compute AB from the given partitions. If so, compute the product by block multiplication.

(a)

$$A = \left[\begin{array}{rrr|r} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{rr|r} 2 & 1 & 4 \\ -3 & 5 & 2 \\ \hline 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

Solution: The partitioning of A and B makes them each effectively 2×2 matrices, so block multiplication might be possible. However, if

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \text{ and } B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}\right]$$

then the products $A_{11}B_{11}, A_{12}B_{21}, A_{11}B_{12}, A_{12}B_{22}, A_{21}B_{11}, A_{22}B_{21}, A_{21}B_{12}$, and $A_{22}B_{22}$ are all undefined. If even one of these is undefined, block multiplication is impossible.

(b)

$$A = \left[\begin{array}{rrrr} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right], \quad B = \left[\begin{array}{r|r|r} 2 & 1 & 4 \\ -3 & 5 & 2 \\ 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

Solution: The partitioning of A makes them each effectively $A_{11} = 2 \times 4$ and $A_{21} = 1 \times 4$ matrices and the partitioning of B makes $B_{11} = 4 \times 1$, $B_{12} = 4 \times 1$ and $B_{13} = 4 \times 1$ matrices.

$$A = \left[\begin{array}{c} A_{11} \\ A_{21} \end{array}\right] \text{ and } B = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \end{array}\right]$$

Hence the products $A_{11}B_{11}, A_{11}B_{12}, A_{11}B_{13}, A_{21}B_{11}, A_{21}B_{12}$, and $A_{21}B_{13}$ are defined.

$$A_{11}B_{11} = \left[\begin{array}{rrrr} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{array}\right] \left[\begin{array}{r} 2 \\ -3 \\ 7 \\ 0 \end{array}\right] = \left[\begin{array}{r} -1 \\ 37 \end{array}\right]$$

$$A_{11}B_{12} = \left[\begin{array}{rrrr} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{array}\right] \left[\begin{array}{r} 1 \\ 5 \\ -1 \\ 3 \end{array}\right] = \left[\begin{array}{r} 23 \\ -13 \end{array}\right]$$

$$A_{11}B_{12} = \left[\begin{array}{rrrr} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{array}\right] \left[\begin{array}{r} 4 \\ 2 \\ 5 \\ -3 \end{array}\right] = \left[\begin{array}{r} -10 \\ 8 \end{array}\right]$$

$$A_{21}B_{11} = \left[\begin{array}{cccc} 1 & 5 & 6 & 1 \end{array}\right] \left[\begin{array}{r} 2 \\ -3 \\ 7 \\ 0 \end{array}\right] = \left[\begin{array}{c} 29 \end{array}\right]$$

$$A_{21}B_{12} = \begin{bmatrix} 1 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 23 \end{bmatrix}$$

$$A_{21}B_{12} = \begin{bmatrix} 1 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 41 \end{bmatrix}$$

Question 18

(a) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

Proof. Let A has $m \times n$ and B has $n \times p$, since AB is defined. Assume that the entries of i-th row of A are all zeros. We claim that the i-th row of AB is a row of zeros.

To see this, pick an entry c_{ij} in i-th row of AB. By the definition of multiplication of AB, we have

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Since the i-th row of A is zero, we have $a_{i1} = a_{i2} = \cdots = a_{in} = 0$.

$$c_{ij} = 0b_{1j} + 0b_{2j} + \cdots + 0b_{nj} = \sum_{k=1}^{n} 0b_{kj} = 0.$$

Hence, the i-th row of AB is a row of zeros.

OR

In general, if $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ik}b_{kj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Let the i-th row of A is zero, so we have $a_{i1} = a_{i2} = \cdots = a_{in} = 0$.

$$c_{ij} = 0b_{1j} + 0b_{2j} + \cdots + 0b_{nj} = \sum_{k=1}^{n} 0b_{kj} = 0.$$

Hence, the i-th row of AB is a row of zeros.