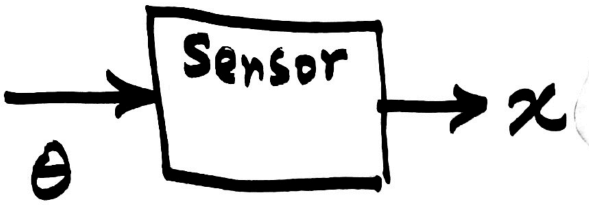
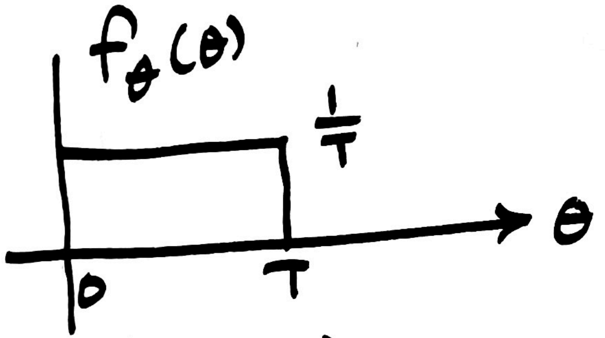
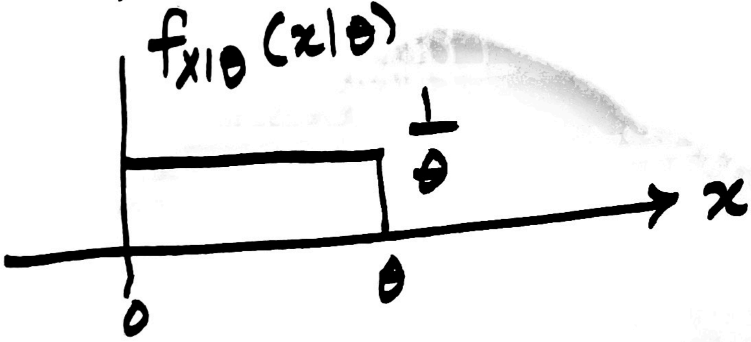


θ is uniformly distributed, $\theta \sim \mathcal{U}[0, T]$

Given θ, the observation x is also uniformly distributed $x|\theta \sim \mathcal{U}[0, \theta]$



$$f_\theta(\theta) = \begin{cases} \frac{1}{T} & 0 \leq \theta \leq T \\ 0 & \text{otherwise} \end{cases}$$



$$f_{X|\theta}(x|\theta) = \begin{cases} \frac{1}{\theta}, & \text{for } 0 \leq x \leq \theta \\ \text{zero,} & \text{otherwise} \end{cases}$$

We may obtain joint density

$$f_{X,\theta}(x, \theta) = f_{X|\theta}(x|\theta) f_\theta(\theta) = \begin{cases} \frac{1}{T\theta} & \text{for } 0 \leq x \leq \theta \leq T \\ \text{zero, otherwise} \end{cases}$$

$$f_X(x) = \int f_{X,\theta}(x,\theta)\, d\theta = \frac{1}{T} \int_x^T \frac{1}{\theta}\, d\theta = \frac{\ln(T) - \ln(x)}{T}$$

for $0 \leq x \leq T$.

$$f_{\theta|x}(\theta|x) = \frac{f_{x,\theta}(x,\theta)}{f_x(x)}$$

$$= \begin{cases} \dfrac{1}{\theta(\ln(T) - \ln(x))}, & 0 \leq x \leq \theta \leq T \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{\theta} = E[\theta|x] = \int \theta f_{\theta|x}(\theta|x)\, d\theta$$

$$= \int_x^T \frac{1}{\ln(T) - \ln(x)}\, d\theta$$

$$\boxed{\hat{\theta} = \frac{T - x}{\ln(T) - \ln(x)}}$$