Lecture 11

Normal Distribution and Error Functions (Chapter 2)
Limit Laws and Theorems (Chapter 5)
Distributions Derived from Normal Distribution (Chapter 6)
Confidence Interval (Chapter 7, page 217-219) + (Chapter 8, page 279-302)

Defining Standard Error in Point Estimation

The distribution of $\hat{\theta}_n$ is called the **sampling distribution**.

The standard deviation of $\hat{\theta}_n$ is called the **standard error**, denoted by se:

$$\sigma_{\hat{\theta}} = \text{se} = \sqrt{\mathbf{var}\left(\hat{\theta}_n\right)}$$

Often the standard error depends upon the true parameters of the unknown distribution. This is hard to obtain the true value of standard error.

Estimating Standard Error in Point Estimation

Often, the standard error depends on the unknown F.
In those cases, se is an unknown quantity but we usually can estimate it.

The estimated standard error is denoted by $\widehat{\text{se}}$ or $s_{\hat{\theta}}$

$$s_{\hat{\theta}} = \widehat{\text{se}} = \sqrt{\text{var}(\hat{\theta}_n)}\Big|_{\theta=\hat{\theta}_n}$$

Estimating Standard Error in Point Estimation

Example: Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$ and let

$\hat{p}_n = \dfrac{1}{n}\sum_i X_i.$

Then

$\mathbb{E}(\hat{p}_n) = \dfrac{1}{n}\sum_i \mathbb{E}(X_i) = p$

so $\hat{p}_n$ is unbiased.

The standard error is

$\text{se} = \sigma_{\hat{p}} = \sqrt{\textbf{var}(\hat{p}_n)} = \sqrt{p(1-p)/n}.$

The estimated standard error is

$\widehat{\text{se}} = s_{\hat{p}} = \sqrt{\hat{p}(1-\hat{p})/n}.$

Normal Distribution

The density function of the normal distribution depends on two parameters, μ and σ (where $-\infty < \mu < \infty, \sigma > 0$):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

The cdf of the Gaussian random variable is given by

$$F_X(x) = \mathrm{P}[X \le x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(\lambda-\mu)^2/2\sigma^2} d\lambda$$

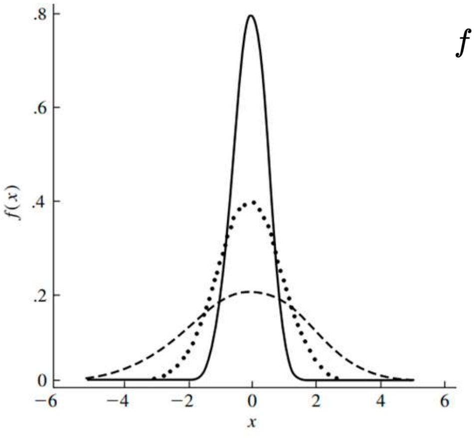


FIGURE 2.13 Normal densities, $\mu = 0$ and $\sigma = .5$ (solid), $\mu = 0$ and $\sigma = 1$ (dotted), and $\mu = 0$ and $\sigma = 2$ (dashed).

Proposition
If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$,
then $Y \sim N(a\mu + b, a^2\sigma^2)$.

Normal Distribution

The parameters μ and σ are called the mean and standard deviation of the normal density.

The CDF of normal distribution, denoted by Φ, cannot be evaluated in closed-form.

As shorthand for the statement "X follows a normal distribution with parameters μ and σ," it is convenient to use $X \sim N(\mu, \sigma^2)$.

From the form of the density function, we see that the density is symmetric about μ,

$$f_X(\mu - x) = f_X(\mu + x),$$

where it has a maximum, and that the rate at which it falls off is determined by σ.

The special case for which $\mu = 0$ and $\sigma = 1$ is called the standard normal density, denoted as $Z \sim N(0,1)$. The CDF of Z is expressed as:

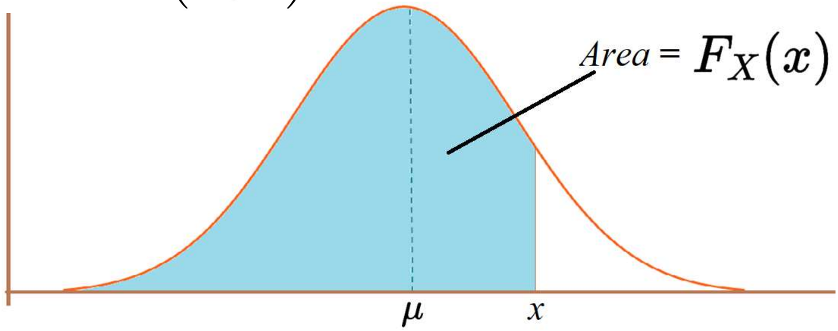
$$\Phi(z) = \mathrm{P}[Z \le z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}\lambda^2} d\lambda$$

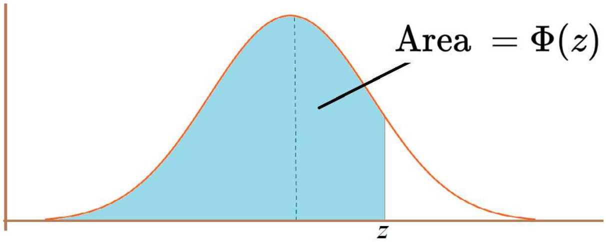
Normal Distribution

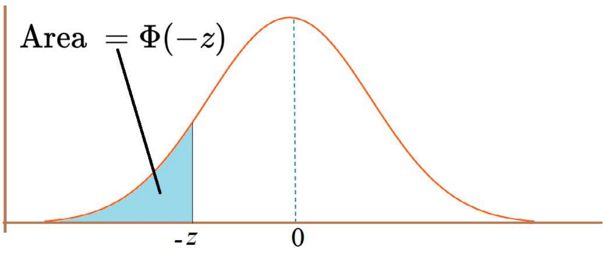
$$F_X(x) = \mathrm{P}[X \leq x] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(\lambda-\mu)^2/2\sigma^2} d\lambda$$

The change of variable $t = (\lambda - \mu)/\sigma$ results in

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-t^2/2} dt = \Phi\left(\frac{x-\mu}{\sigma}\right)$$







$$\Phi(-z) = 1 - \Phi(z)$$

Normal Distribution – *Error* function

The error function is defined as: $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$

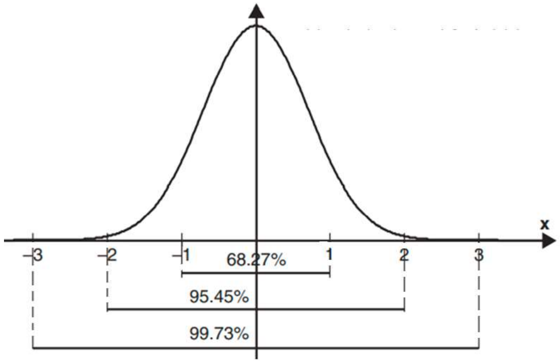
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\lambda^2} d\lambda = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{2}} e^{-t^2} dt$$
$$= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{0} e^{-t^2} dt + \int_{0}^{x/\sqrt{2}} e^{-t^2} dt \right)$$

By using the well-known integral $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$, it follows that

$$\Phi(x) = \tfrac{1}{2}\left(1 + \operatorname{erf}(x/\sqrt{2})\right)$$

$$P(x_0 < X < x_1) = F_X(x_1) - F_X(x_0) = \Phi\left(\frac{x_1 - \mu}{\sigma}\right) - \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$$

$$\begin{aligned} P(|X - \mu| \le \delta) &= P(-\delta \le X - \mu \le \delta) \\ &= P\left(-\frac{\delta}{\sigma} \le \frac{X - \mu}{\sigma} \le \frac{\delta}{\sigma}\right) \\ &= \Phi\left(\frac{\delta}{\sigma}\right) - \Phi\left(-\frac{\delta}{\sigma}\right) \\ &= 2\Phi\left(\frac{\delta}{\sigma}\right) - 1 \end{aligned}$$



$$P(-1 < Z < +1) = \int_{-1}^{+1} f(z)dz = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0.6827$$

$$P(-2 < Z < +2) = \int_{-2}^{+2} f(z)dz = \int_{-2}^{+2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0.9545$$

$$P(-3 < Z < +3) = \int_{-3}^{+3} f(z)dz = \int_{-3}^{+3} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0.9973$$

Some Important Theorems

For a family of independent identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^{\infty}$, the partial sum S_n is defined by

$$S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n$$

Associated with S_n is the sample mean $\widehat{\mu}_1 = \bar{X}_n$, as given by

$$\widehat{\mu}_1 = \bar{X}_n = \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{N} X_i$$

Let the true mean of X_i be $\mu = \mathbb{E}[X_i]$.

Strong Law of Large Number

For a family of i.i.d. random variables $\{X_i\}_{i=1}^{\infty}$, suppose that the mean $\mu = \mathbb{E}[X_i]$ exists. Then,

$$\lim_{n\to\infty} \widehat{\mu}_1 = \lim_{n\to\infty} \bar{X}_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu$$

with probability one.

The theorem can also be expressed as $\lim_{n\to\infty} \mathbb{P}\left[\frac{S_n}{n} = \mu\right] = 1.$

Weak Law of Large Number

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{S_n}{n} - \mu\right| > \epsilon\right] = 0$$

for every $\epsilon > 0$.

Proof: We first find $E(\bar{X}_n)$ and $\operatorname{Var}(\bar{X}_n)$:

$$E(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu$$

Since the X_i are independent,

$$\operatorname{Var}(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}(X_i) = \frac{\sigma^2}{n}$$

The desired result now follows immediately from Chebyshev's inequality, which gives

$$P(|\bar{X}_n - \mu| > \varepsilon) \leq \frac{\operatorname{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0, \quad \text{as } n \to \infty$$

Central Limit Theorem

For a family of i.i.d. random variables $\{X_i\}_{i=1}^{\infty}$ with mean μ and variance $\sigma^2 > 0$, let

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}}.$$

Then, for $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}[Z_n \leq x] = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\lambda^2/2} d\lambda$$

where $\Phi(\cdot)$ is the standard normal distribution function.

$$Z_n \sim \mathcal{N}(0,1)$$

An estimator is asymptotically normal if

$$\frac{\widehat{\theta}_n - \theta}{\sigma_{\widehat{\theta}_n}} \rightsquigarrow \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty$$

The estimation error in sample mean scaled by the standard error is thus asymptotically normal

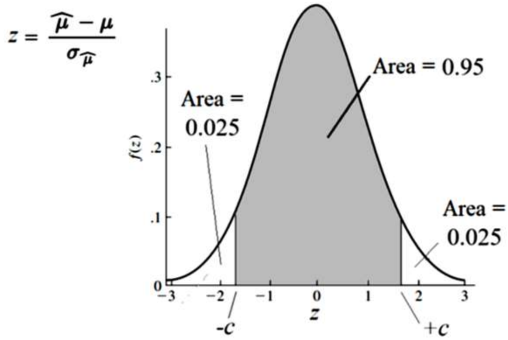
$$\frac{\widehat{\mu}_n - \mu}{\sigma_{\widehat{\mu}_n}} = \frac{\widetilde{\mu}_n}{\sigma_{\widehat{\mu}_n}} \rightsquigarrow \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty$$

Confidence Interval

A **confidence interval** for a population parameter, θ
(as calculated from the given samples of a random variable X),
is a random interval that contains θ with some specified probability.

For example, we computed a sample mean $\widehat{\mu}$ of some random variable X, let the standard error of the estimation be $\sigma_{\widehat{\mu}}$, a 95% confidence interval for the true μ is a random interval that contains the true μ with probability 0.95.

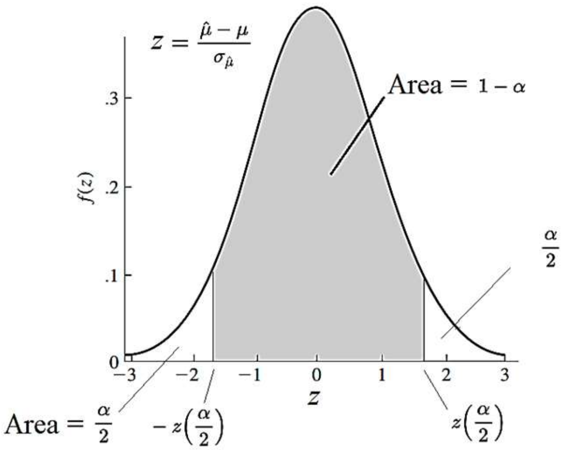
Find c such that $P\left(-c \leq \dfrac{\widehat{\mu} - \mu}{\sigma_{\widehat{\mu}}} \leq c\right) \approx 0.95$



$$P\left(-c \leq \frac{\hat{\mu} - \mu}{\sigma_{\hat{\mu}}} \leq c\right) \approx 1 - \alpha$$

Let $c := z\left(\dfrac{\alpha}{2}\right)$
denote the value of z
where the tail area is $\alpha/2$.

$$P\left(-z\left(\frac{\alpha}{2}\right) \leq \frac{\hat{\mu} - \mu}{\sigma_{\hat{\mu}}} \leq z\left(\frac{\alpha}{2}\right)\right) \approx 1 - \alpha$$



Elementary manipulation of the inequalities gives

$$P\left(\widehat{\mu} - z\left(\frac{\alpha}{2}\right)\sigma_{\widehat{\mu}} \leq \mu \leq \widehat{\mu} + z\left(\frac{\alpha}{2}\right)\sigma_{\widehat{\mu}}\right) \approx 1 - \alpha$$

That is, the probability that μ lies in the interval $\hat{\mu} \pm z(\alpha/2)\sigma_{\hat{\mu}}$ is approximately $1 - \alpha$.

The interval is thus called a $100(1 - \alpha)\%$ confidence interval.

It is important to understand that this interval is random, and that the preceding equation states that the probability that this random interval covers the true μ is $1 - \alpha$.

TABLE Critical values for standard Gaussian random variable.

α	$z(\alpha)$
0.1000	1.2816
0.0500	1.6449
0.0250	1.9600
0.0100	2.3263
0.0050	2.5758
0.0025	2.8070
0.0010	3.0903
0.0005	3.2906
0.0001	3.7191

Chi-Square Distribution

DEFINITION

If Z is a standard normal random variable, the distribution of $U = Z^2$ is called the chi-square distribution with 1 degree of freedom. ■

The chi-square distribution with 1 degree of freedom is denoted χ_1^2. It is useful to note that if $X \sim N(\mu, \sigma^2)$, then $(X - \mu)/\sigma \sim N(0, 1)$, and therefore $[(X - \mu)/\sigma]^2 \sim \chi_1^2$.

DEFINITION

If $U_1, U_2, \ldots, U_n$ are independent chi-square random variables with 1 degree of freedom, the distribution of $V = U_1 + U_2 + \cdots + U_n$ is called the *chi-square distribution with n degrees of freedom* and is denoted by χ_n^2. ■

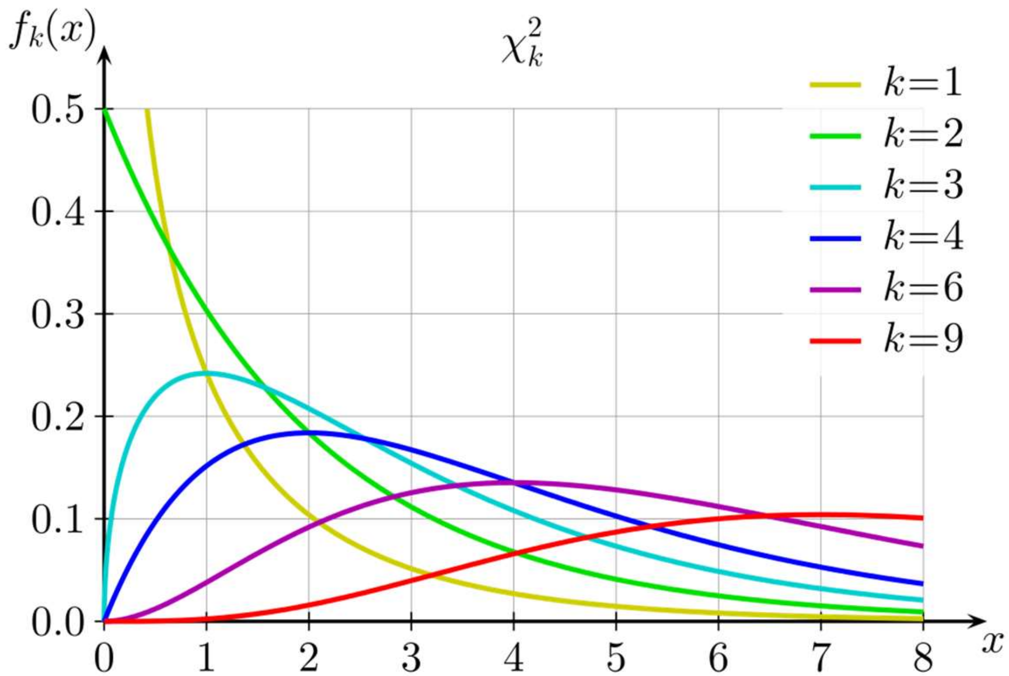
Its density is

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}, \qquad v \geq 0$$

Its moment-generating function is

$$M(t) = (1 - 2t)^{-n/2}$$

Also, $E(V) = n$ and $\text{Var}(V) = 2n$. To indicate that V follows a chi-square distribution with n degrees of freedom, we write $V \sim \chi_n^2$. A notable consequence of the definition of the chi-square distribution is that if U and V are independent and $U \sim \chi_n^2$ and $V \sim \chi_m^2$, then $U + V \sim \chi_{m+n}^2$.



t Distribution

DEFINITION

If $Z \sim N(0, 1)$ and $U \sim \chi^2_n$ and Z and U are independent, then the distribution of $Z/\sqrt{U/n}$ is called the ***t* distribution** with n degrees of freedom. ■

The density function of the t distribution with n degrees of freedom is

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2}$$

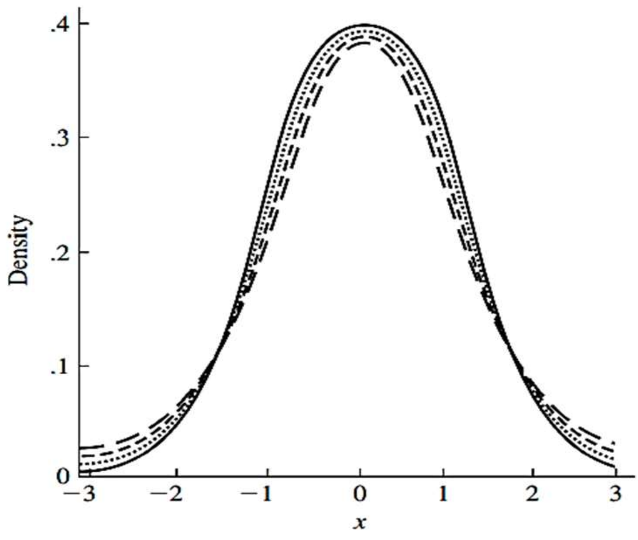


FIGURE 6.1 Three t densities with 5 (long dashes), 10 (short dashes), and 30 (dots) degrees of freedom and the standard normal density (solid line).

Sample mean and Sample variance

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

These are called the **sample mean** and the (**unbiased**) **sample variance**, respectively.

Sample mean and Sample variance

THEOREM **B**

The distribution of $(n-1)S^2/\sigma^2$ is the chi-square distribution with $n-1$ degrees of freedom.

COROLLARY **B**

Let $\overline{X}$ and S^2 be as given at the beginning of this section. Then

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

We have found earlier that the maximum likelihood estimates of μ and σ^2 from an i.i.d. normal sample are

$$\widehat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

A confidence interval for μ is based on the fact that

$$\frac{\bar{X}-\mu}{\sqrt{S^2/n}} \sim t_{n-1}$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

where t_{n-1} denotes the t distribution with $n-1$ degrees of freedom.

Let $t_{n-1}(\alpha/2)$ denote that point beyond which the t distribution with $n-1$ degrees of freedom has probability $\alpha/2$.

Since the t distribution is symmetric about 0 , the probability to the left of $-t_{n-1}(\alpha/2)$ is also $\alpha/2$. Then, by definition,

$$P\left(-t_{n-1}(\alpha/2) \le \frac{\sqrt{n}(\bar{X}-\mu)}{S} \le t_{n-1}(\alpha/2)\right) = 1-\alpha$$

The inequality can be manipulated to yield

$$P\left(\bar{X} - \frac{S}{\sqrt{n}}t_{n-1}(\alpha/2) \le \mu \le \bar{X} + \frac{S}{\sqrt{n}}t_{n-1}(\alpha/2)\right) = 1-\alpha$$

According to this equation, the probability that μ lies in the interval

$$\bar{X} \pm S t_{n-1}(\alpha/2)/\sqrt{n}$$

is $1-\alpha$.

Note that this interval is random: The center is at the random point $\bar{X}$

t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
two-tails	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Now let us turn to a confidence interval for σ^2. From earlier material

$$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$$

where χ^2_{n-1} denotes the chi-squared distribution with $n-1$ degrees of freedom.

Let $\chi^2_m(\alpha)$ denote the point beyond which the chi-square distribution with m degrees of freedom has probability α.

It then follows by definition that

$$P\left(\chi^2_{n-1}(1-\alpha/2) \leq \frac{n\hat{\sigma}^2}{\sigma^2} \leq \chi^2_{n-1}(\alpha/2)\right) = 1-\alpha$$

Manipulation of the inequalities yields

$$P\left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)} \le \sigma^2 \le \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right) = 1-\alpha$$

Therefore, a $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right)$$

Note that this interval is not symmetric about $\hat{\sigma}^2$.
It is not of the form $\hat{\sigma}^2 \pm c$.

Value of $\chi^2_{n;\alpha}$ such that $\text{Prob}[\chi^2_n > \chi^2_{n;\alpha}] = \alpha$

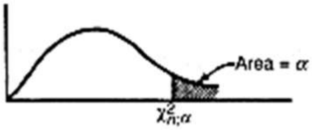


TABLE IV

Chi-Square (χ^2) Distribution

Degrees of Freedom	Area to the Right of Critical Value							
	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01
1	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	14.042	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278
29	14.257	16.047	17.708	19.768	39.087	42.557	45.722	49.588
30	14.954	16.791	18.493	20.599	40.256	43.773	46.979	50.892

A simulation illustrates these ideas: The following experiment was done on a computer 20 times.
A random sample of size $n = 11$ from normal distribution with $\mu = 10$ and $\sigma^2 = 9$ was generated.
From the samples, $\bar{X}$ and $\hat{\sigma}^2$ were calculated and 90% confidence intervals for μ and σ^2 were constructed.
Thus at the end there were 20 intervals for μ and 20 intervals for σ^2.
Horizontal lines are drawn at the true values $\mu = 10$ and $\sigma^2 = 9$.

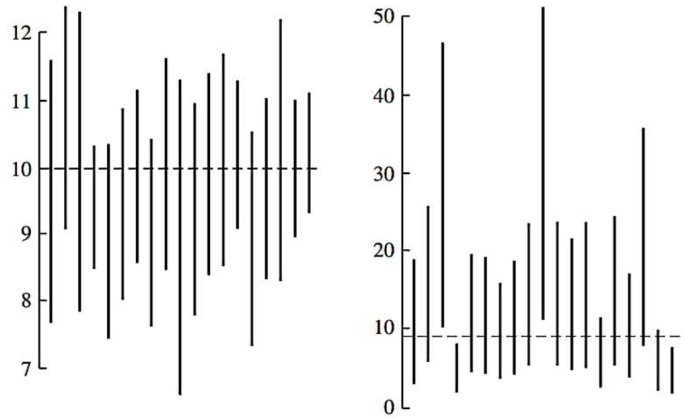


FIGURE 8.8 20 confidence intervals for μ (left panel) and for σ^2 (right panel) as described in Example A. Horizontal lines indicate the true values.