MARKING SCHEME

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LINEAR ALGEBRA, MIDTERM [Total Marks: 100]

(SPRING 2022)

Question 1: [10 Marks]

Prove that for an invertible matrix A, its inverse is unique. [6 MARKS]

Let B and C be two inverses of A.

$\because$ B is an inverse, $BA = I$ — ①

Multiplying both sides of ① on right by C:

$(BA)C = IC = C$ — ②

But $(BA)C = B(AC) = BI = B$ — ③ $(\because AC = I)$

From ② and ③, $B = C$

inverse of A is unique

For a matrix C, if we know that $\det(C) > 0$, then show that $(C^{-1})^T = (C^T)^{-1}$. [4 MARKS]

$$(C^{-1})^T C^T = (CC^{-1})^T$$
$$= I^T$$
$$= I$$

$$\therefore (C^{-1})^T = (C^T)^{-1}$$

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Question 2: [10 Marks]

Let Y and Z be $n \times n$ matrices.

(a) Prove that if $I_n = YZ$ then $Z = Y^{-1}$. [5 MARKS]

$\det(YZ) = \det(I) = 1$

$\Rightarrow \det(Y)\det(Z) = 1$

$\Rightarrow \det(Y) \neq 0$ and $\det(Z) \neq 0$ } Y^{-1}, Z^{-1} exist

3

$Y^{-1}YZ = Y^{-1}I$

$IZ = Y^{-1}$

$Z = Y^{-1}$

2

(b) Prove that if $I_n = ZY$ then $Z = Y^{-1}$. [5 MARKS]

Since Y^{-1} exists (proved in (a)):

$I = ZY$

$IY^{-1} = ZYY^{-1}$

$Y^{-1} = ZI$

$Y^{-1} = Z$ (Proved)

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<u>Question 3 [10 MARKS]</u>

If B is an $m \times m$ square matrix, and the homogenous system $B\underline{x} = \underline{0}$ has only the trivial solution, then show that the reduced row echelon form of B is the identity matrix I_m. [7 MARKS]

Th. 1.5.3 (b) $\Rightarrow$ (c)

Let $B\underline{x} = \underline{0}$ be* $m \times m$ system of eq. with only trivial sol. The sys. corresponding to RREF of the augmented matrix is:

$$① \quad \begin{matrix} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \vdots \\ & & x_m & = 0 \end{matrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & 0 \\ a_{21} & a_{22} & \cdots & a_{2m} & 0 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} & 0 \end{bmatrix} \quad ②$$

② can be reduced to the aug-matrix for ① in form $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ through EROs

Ignoring the last column of 0's in each of the matrices, we can conclude that RREF of B is identity matrix I_m.

Give two different row Echelon forms for the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$. [3 MARKS]

① $R_2 \to R_2 - 2R_1$ $\quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

② $R_1 \to R_1 - 3R_2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(other correct answers possible)

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Question 4 [10 MARKS]

Let $A\underline{x} = \underline{b}$ be any consistent system of linear equations and let $\underline{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\underline{x} = \underline{x}_1 + \underline{x}_0$ where $\underline{x}_0$ is a solution of the homogenous linear system $A\underline{x} = \underline{0}$. Show also that every matrix of this form is a solution.

Let $\underline{x}^*$ be any solution to $A\underline{x}=\underline{b}$. Then: $A\underline{x}^* = \underline{b}$. We also know that $A\underline{x}_1 = \underline{b}$

$\therefore A\underline{x}^* - A\underline{x}_1 = \underline{b} - \underline{b} = \underline{0} = A(\underline{x}^* - \underline{x}_1)$ [4]

$\therefore (\underline{x}^* - \underline{x}_1)$ is a solution to the homo. system $A\underline{x}=\underline{0} \Rightarrow \underline{x}^* = \underline{x}_1 + (\underline{x}^* - \underline{x}_1) \Rightarrow$ required form [3]

For a matrix of the form $\underline{x} = \underline{x}_1 + \underline{x}_0$,

$\Rightarrow A\underline{x} = A\underline{x}_1 + A\underline{x}_0 = \underline{b} + \underline{0} = \underline{b}$ [3]

$\therefore \underline{x}$ is a solution (with format $\underline{x}_1 + \underline{x}_0$)

Question 5 [10 MARKS]

If $\underline{u}$ and $\underline{v}$ are vectors in $\mathbb{R}^n$ then prove that $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$. [5 MARKS]

$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = \underline{u}\cdot\underline{u} + 2(\underline{u}\cdot\underline{v}) + \underline{v}\cdot\underline{v}$

$= \|\underline{u}\|^2 + 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2$

$\leq \|\underline{u}\|^2 + 2|\underline{u}\cdot\underline{v}| + \|\underline{v}\|^2$ (absolute value property)

$\leq \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2$ (CS inequality)

$= (\|\underline{u}\| + \|\underline{v}\|)^2$

Take $\sqrt{\ }$ throughout: $\|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$

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If $\underline{u}$, $\underline{v}$, and $\underline{w}$ are vectors in $\mathbb{R}^n$ then show that $d(\underline{u},\underline{v}) \leq d(\underline{u},\underline{w}) + d(\underline{w},\underline{v})$. [5 MARKS]

$d(\underline{u},\underline{v}) = \|\underline{u}-\underline{v}\|$
$= \|(\underline{u}-\underline{w})+(\underline{w}-\underline{v})\|$
$\leq \|\underline{u}-\underline{w}\|+\|\underline{w}-\underline{v}\|$ (using part a)
$= d(\underline{u},\underline{w})+d(\underline{w},\underline{v})$

Question 6 [10 MARKS]

Prove or disprove the following claim: "***It is possible for a vector space to contain only two distinct elements.***" [5 MARKS]

Not possible! Every vector space must contain the $\underline{0}$ vector. For any other non-zero vector, it must contain all its scalar multiples as well. So, FALSE [4]

[1]+ Distinct scalar multiples of non-zero vector must be unequal $\Rightarrow k_1 \neq k_2 \Rightarrow k_1-k_2 \neq 0 \Rightarrow (k_1-k_2)\underline{v} \neq 0 \Rightarrow k_1\underline{v} \neq k_2\underline{v}$ $\forall \underline{v} \neq 0$

Let S be the set of all real numbers in $(0,\infty)$. If S is a vector space under the operations

$\underline{u}+\underline{v}=\underline{u}\,\underline{v}$ AND $k\underline{u}=\underline{u}^k$, then identify the zero element of S and show that it truly is the zero element. [5 MARKS]

$\underline{0} = 1$ [2]

$\underline{u}+\underline{0} = \underline{0}+\underline{u} \Rightarrow$ for any real no. $\underline{u}=x$:

$x+1 = x(1) = x$

$\therefore \underline{0}=1$ fulfils conditions of zero vector

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Question 7 [10 MARKS]

Prove that for any vectors $\underline{a}, \underline{b}$, and $\underline{c}$, the vectors $(\underline{a} - \underline{b})$, $(\underline{b} - \underline{c})$ and $(\underline{c} - \underline{a})$ form a linearly dependent set. [5 MARKS]

For all real k, we have (show this with ANY $k \neq 0$)

$k(\cancel{\underline{a}} - \cancel{\underline{b}}) + k(\cancel{\underline{b}} - \cancel{\underline{c}}) + k(\cancel{\underline{c}} - \cancel{\underline{a}}) = \underline{0}$

$\therefore$ the vectors are linearly dependent
(k must be non-zero scalar)

Prove that if $\{\underline{a}, \underline{b}\}$ is a linearly independent set, and $\underline{c} \notin span\{\underline{a}, \underline{b}\}$, then $\{\underline{a}, \underline{b}, \underline{c}\}$ is also linearly independent. [5 MARKS]

Using +/- theorem, $\{\underline{a}, \underline{b}\} \cup \{\underline{c}\}$ is also linearly independent.
Hence, $\{\underline{a}, \underline{b}, \underline{c}\}$ is also linearly independent

OR

Proof by Contradiction: Use definitional equation of linear independence.
If $\underline{c}$ has non-zero coefficient k,
then $k_1 \underline{c} = k_2 \underline{a} + k_2 \underline{b}$ and hence,
$\underline{c} \in span\{\underline{a}, \underline{b}\} \Rightarrow$ not possible

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Question 8 [10 MARKS]

Given a set S with two or more vectors, prove that if S is linearly independent then no vector in S is expressible as a linear combination of other vectors in S. [5 MARKS]

Let $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$. Assume $\exists\, \underline{v}_i \in S$ s.t. it equals a linear combination of the others i.e. $\exists$ scalars $k_1, k_2 \ldots k_n$ s.t. $\underline{v}_i = k_1\underline{v}_1 + \ldots + k_n\underline{v}_n$

Then, $\underline{v}_i + k_1\underline{v}_1 + \ldots + k_n\underline{v}_n = \underline{0}$

This is clearly non-trivial, since the coefficient for $\underline{v}_i = 1 \neq 0 \Rightarrow$ Contradiction

So, such a vector $\underline{v}_i$ doesn't exist.

Given a set S with two or more vectors, prove that if no vector in S is expressible as a linear combination of other vectors in S, then S is linearly independent. [5 MARKS]

Suppose the set $S = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ is linearly dependent. Then $\exists$ some non-zero k_i s.t. $k_1\underline{v}_1 + k_2\underline{v}_2 + \ldots k_i\underline{v}_i + \ldots + k_n\underline{v}_n = \underline{0}$.

$\Rightarrow k_i\underline{v}_i = k_1\underline{v}_1 + \ldots + k_n\underline{v}_n$

This gives a contradiction, since now, $\underline{v}_i$ is a linear combination of the others.

So, S must be linearly independent.

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Question 9 [10 MARKS]

Let S be a finite set of vectors in a ***finite dimensional*** vector space V. Then prove that if $span(S) = V$ but S is linearly dependent, then S contains a proper subset B (i.e. $B \subset S$) such that B is a basis for V.

Theorem 5.4.6.(a)

$\because$ linearly dependent, $\exists$ $\underline{v}$ in S that is expressible as a linear combination of other vectors in S. Using +/- theorem, we can remove $\underline{v}$ from S, giving S' that still spans V. [4]

- If S' linearly indep, S' is a basis for V $\Rightarrow$ Proof Complete
- If S' linearly dependent, then remove some appropriate vector from S' to produce S'' that still spans V. Continue this till you arrive at set of vectors in S that's lin. indep. & spans V.

This subset B of S is a basis for V.

Question 10 [10 MARKS]

Show that the set of all polynomials in P_n that have a horizontal tangent at $x = 1$ is a subspace of P_n.

Horizontal tangent means derivative is zero. [2]

- For closure under vector addition: [4]

$$\frac{d(p_1+p_2)}{dx}(1) = \frac{dp_1}{dx}(1) + \frac{dp_2}{dx}(1)$$

- For closure under scalar multiplication [4]

$$\frac{d(kp)}{dx}(1) = k\frac{dp}{dx}(1)$$