Linear Algebra – Math 205
Exercise Set of Lect 9 & 10
(SPRING 2023)
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Exercise Set 2.1 Solution

Question 01

Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

(a) Find all the minors of A. (b) Find all the cofactors.

Solution:

(a)

$$\mathrm{M}_{11} = \text{minor of } a_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 7 * 4 - (-1) * 1 = 29, \mathrm{M}_{12} = 21, \mathrm{M}_{13} = 27, \mathrm{M}_{21} = -11, \mathrm{M}_{22} = 13,$$

$\mathrm{M}_{23} = -5, \mathrm{M}_{31} = -19, \mathrm{M}_{32} = -19, \mathrm{M}_{33} = 19$

(b)

$\mathrm{C}_{11} = i^{1+1}\mathrm{M}_{11} = 29, \mathrm{C}_{12} = -21, \mathrm{C}_{13} = 27, \mathrm{C}_{21} = 11, \mathrm{C}_{22} = 13, \mathrm{C}_{23} = 5, \mathrm{C}_{31} = -19, \mathrm{C}_{32} = 19, \mathrm{C}_{33} = 19$

Question 08

Evaluate $det(A)$ by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$$

Solution: Expand by first row: $|A| = (k+1) \cdot \begin{vmatrix} k-3 & 4 \\ k+1 & k \end{vmatrix} - (k-1) \cdot \begin{vmatrix} 2 & 4 \\ 5 & k \end{vmatrix} + 7 \cdot \begin{vmatrix} 2 & k-3 \\ 5 & k+1 \end{vmatrix}$

$|A| = (k+1) \cdot ((k-3) * k - 4 * (k+1)) - (k-1)(2 * k - 4 * 5) + 7 \cdot (2 * (k+1) - (k-3) * 5)$

$|A| = k^3 - 8k^2 - 10k + 95$

Question 22

Show that the matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is invertible for all values of θ; then find A^{-1} using Theorem 2.1.2.

Solution: Theorem, If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\det(A) = 1\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = (\cos\theta^2 + \sin\theta) = 1$$

$$\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 24(a)

$$\begin{aligned} 4x + y + z + w =& 6 \\ 3x + 7y - z + w =& 1 \\ 7x + 3y - 5z + 8w =& -3 \\ x + y + z + 2w =& 3 \end{aligned}$$

Solve $Ax = b$ by the Cramer's Rule.

Solution:

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 6 \\ 1 \\ -3 \\ 3 \end{pmatrix}; |A| = -424$$

$$A_1 = \begin{pmatrix} 6 & 1 & 1 & 1 \\ 1 & 7 & -1 & 1 \\ -3 & 3 & -5 & 8 \\ 3 & 1 & 1 & 2 \end{pmatrix}; |A_1| = -424; x = -424/-424 = 1$$

$$A_2 = \begin{pmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{pmatrix}; |A_2| = 0; y = 0/-424 = 0$$

$$A_3 = \begin{pmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{pmatrix}; |A_3| = -848; z = -848/-424 = 2$$

$$A_4 = \begin{pmatrix} 4 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & -3 \\ 1 & 1 & 1 & 3 \end{pmatrix}; |A_4| = 0; w = 0/-424 = 0$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Question 25

Prove that if $\det(A) = 1$ and all the entries in A are integers, then all the entries in A^{-1} are integers.

Solution: This follows from Theorem 2.1.2 and the fact that the cofactors of A are integers if A has only integer entries, since integers are closed under multiplication, addition and subtraction. $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \operatorname{adj}(A)$

Question 26

Let $A\mathbf{x} = \mathbf{b}$ be a system of n linear equations in n unknowns with integer coefficients and integer constants. Prove that if $\det(A) = 1$, the solution x has integer entries.

Solution: From the previous solution, the fact that the cofactors of A are integers if A has only integer entries. $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \operatorname{adj}(A)$ and multiplication, addition and subtraction of integers are closed. Hence x has also integer entries.

Exercise Set 2.2 Solution

Question 12

Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find

(a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

Solution: Here two EROs($R_1 \leftrightarrows R_2, R_2 \leftrightarrows R_3$) have been applied, hence determinant is $-6 * -1 * -1 = -6$.

(b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$

Solution: Here three EROs($3 * R_1, -1 * R_2, 4 * R_3$), hence determinant is $-6 * 3 * -1 * 4 = 72$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

Solution: Here one ERO($R_1 + R + 3$), hence determinant remains -6.

(d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

Solution: Here two EROs($-3 * R_1, R_3 + (-4) * R_2$) have been applied, hence determinant is $-6 * -3 = 18$.

Question 13

Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

Solution:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}\end{aligned}$$

Add $-a$ times Row 1 to Row 2; add $-a^2$ times Row 1 to Row 3. Since $b^2 - a^2 = (b-a)(b+a)$, we add $-(b+a)$ times Row 2 to Row 3 to obtain

$$\begin{aligned}\det(A) &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & \left(c^2-a^2\right)-(c-a)(b+a) \end{vmatrix} \\ &= (b-a)\left[\left(c^2-a^2\right)-(c-a)(b+a)\right] \\ &= (b-a)(c-a)[(c+a)-(b+a)] \\ &= (b-a)(c-a)(c-b)\end{aligned}$$

Question 14

Use an argument like that in the proof of Theorem 2.1.3 to show that

(a)

$$\det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -a_{13}a_{22}a_{31}$$

Solution: Expand by Column 1, we have $\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 - 0 - a_{31} \begin{vmatrix} 0 & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

(b)

$$\det \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$$

Solution: Expand by Column 1, we have $a_{41} \begin{vmatrix} 0 & 0 & a_{14} \\ 0 & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$. Now Expand again by Column 1, we have $a_{14}(0 - 0 - a_{32} \begin{vmatrix} 0 & a_{14} \\ a_{23} & a_{24} \end{vmatrix}) = a_{14}(-a_{32}(0 - a_{23}a_{14})) = a_{14}a_{23}a_{32}a_{41}$

Question 15

Prove the following special cases of Theorem 2.2.3.

(a) $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(b) $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Solution: In each case, d will denote the determinant on the left and, as usual, $\det(A) = \sum \pm a_{1j_1}a_{2j_2}a_{3j_3}$, where $\sum$ denotes the sum of all such elementary products.

(a) $\sum \pm a_{2j_1}a_{1j_2}a_{3j_3} = -\sum \pm a_{1j_2}a_{2j_1}a_{3j_3} = -\det(A)$

(b) $\sum \pm \left(a_{1j_1} + ka_{2j_2}\right) a_{2j_2}a_{3j_3} = \sum \pm a_{1j_1}a_{2j_2}a_{3j_3} = \det(A)$