Habib University
shaping futures

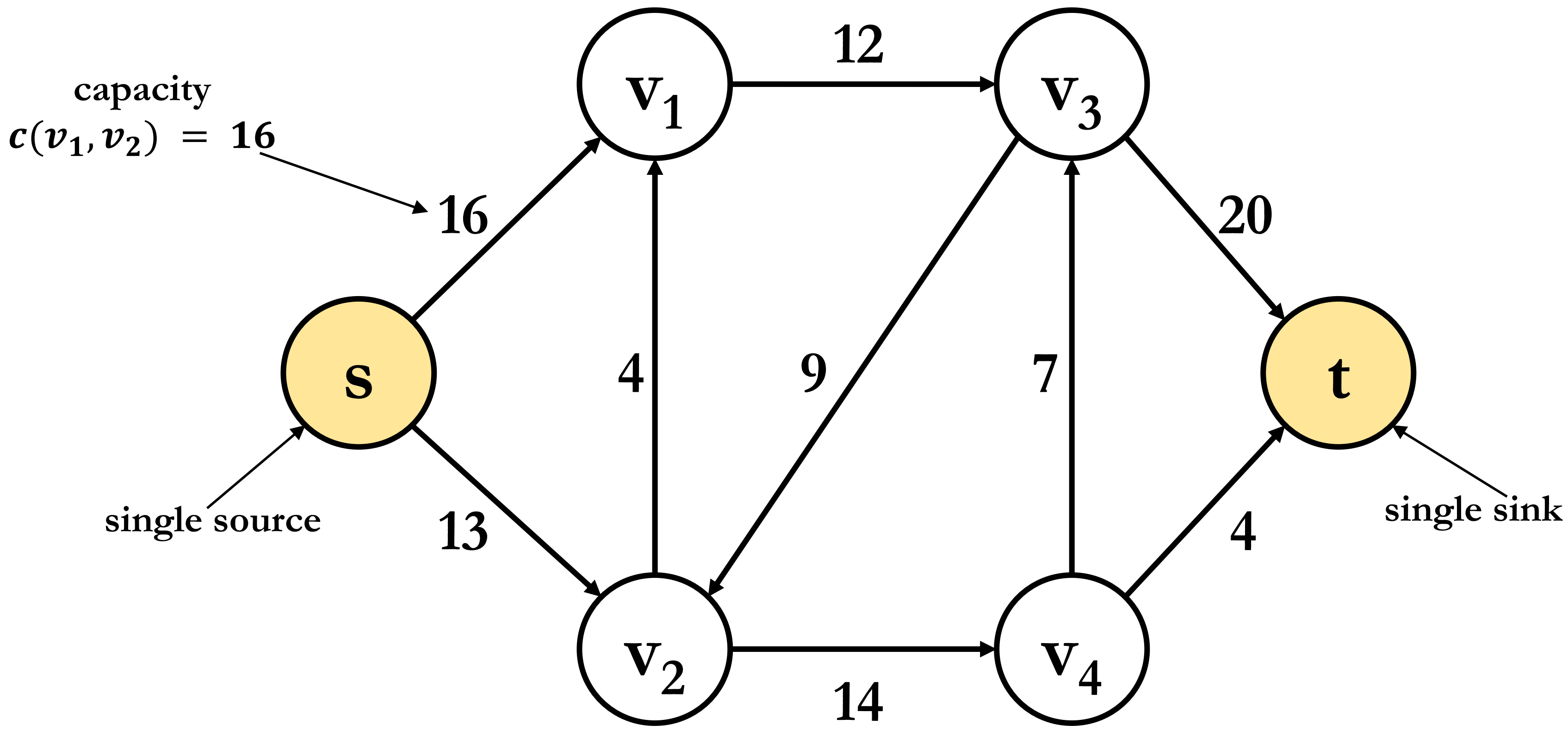
Maximum Flows

CS 412 – Week 08

Shah Jamal Alam

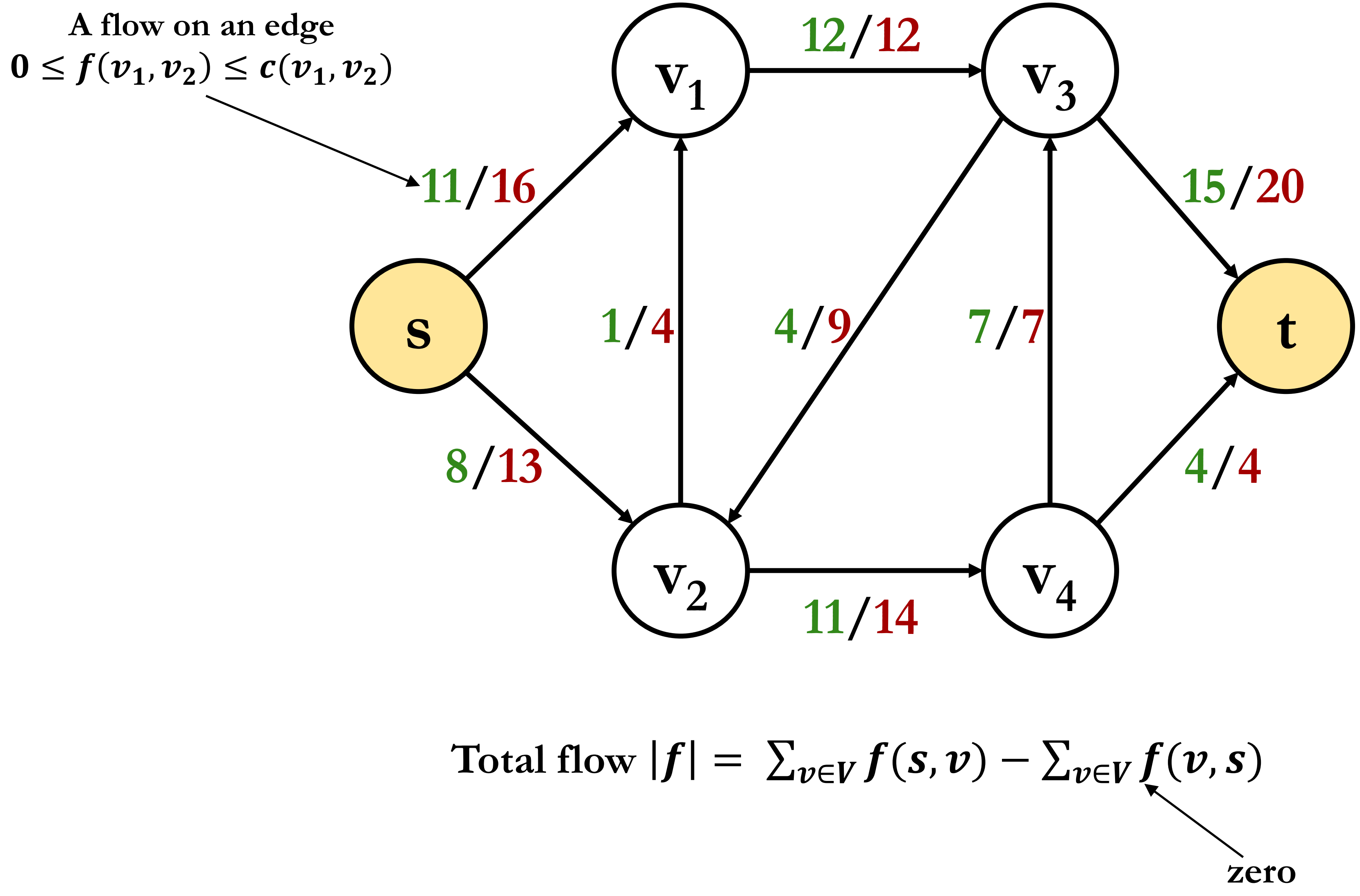
Source: *CLRS*

A flow graph $G = (V, E)$ with a *source* and a *sink*



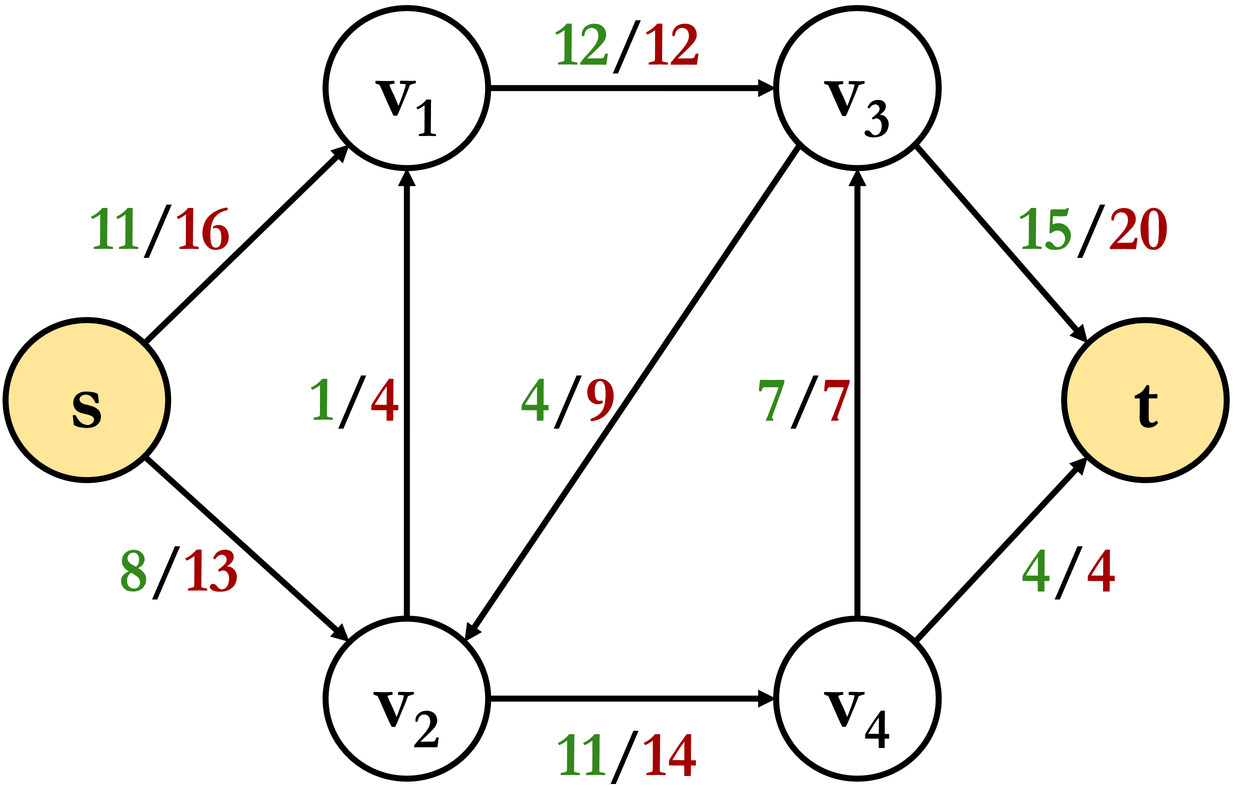
Every vertex $v \in V\{s, t, \}$ is on a path from $s \rightsquigarrow t$.

An example of a flow in G with a total flow $|f| = 19$.

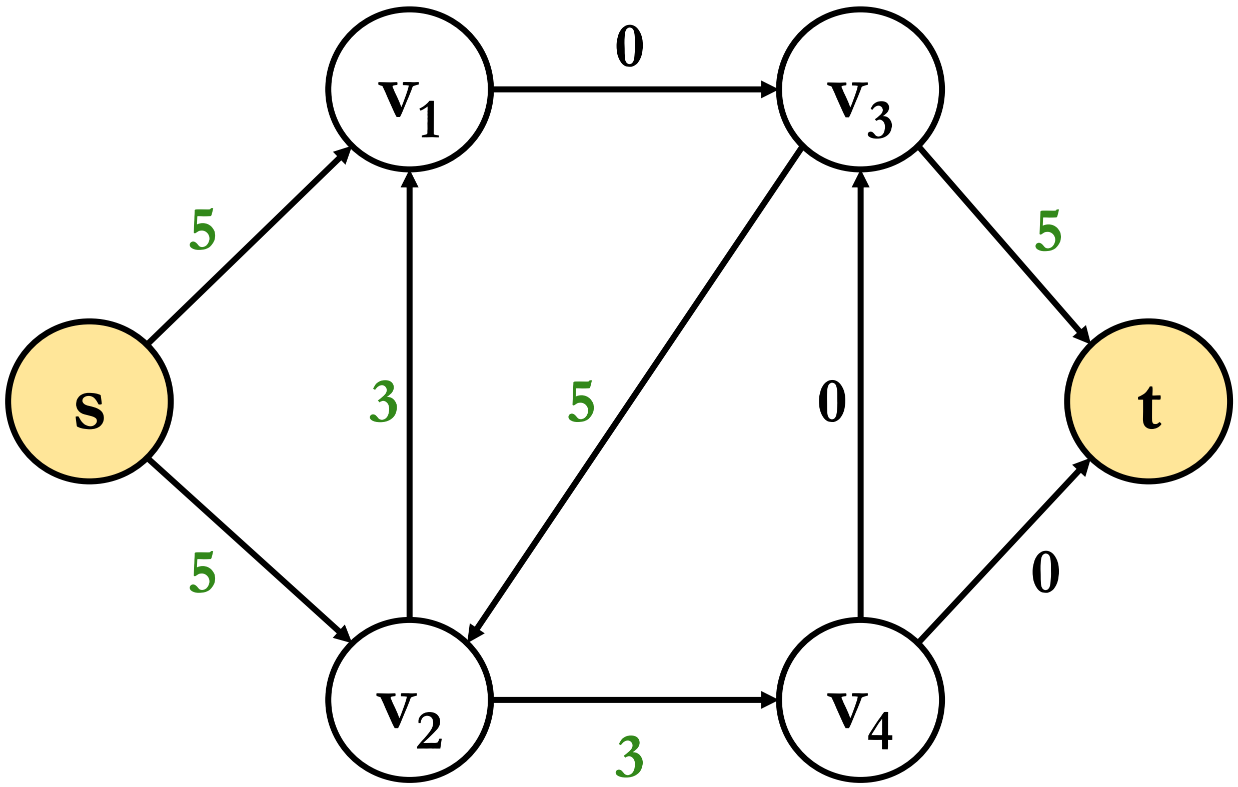


The Residual Network G_f of the example graph G – I

$$\textit{Residual capacity } c_f(u,v) = \begin{cases} c(u,v) - f(u,v), & \textit{if } (u,v) \in E \\ f(v,u), & \textit{if } (v,u) \in E \\ 0, & \textit{otherwise} \end{cases}$$



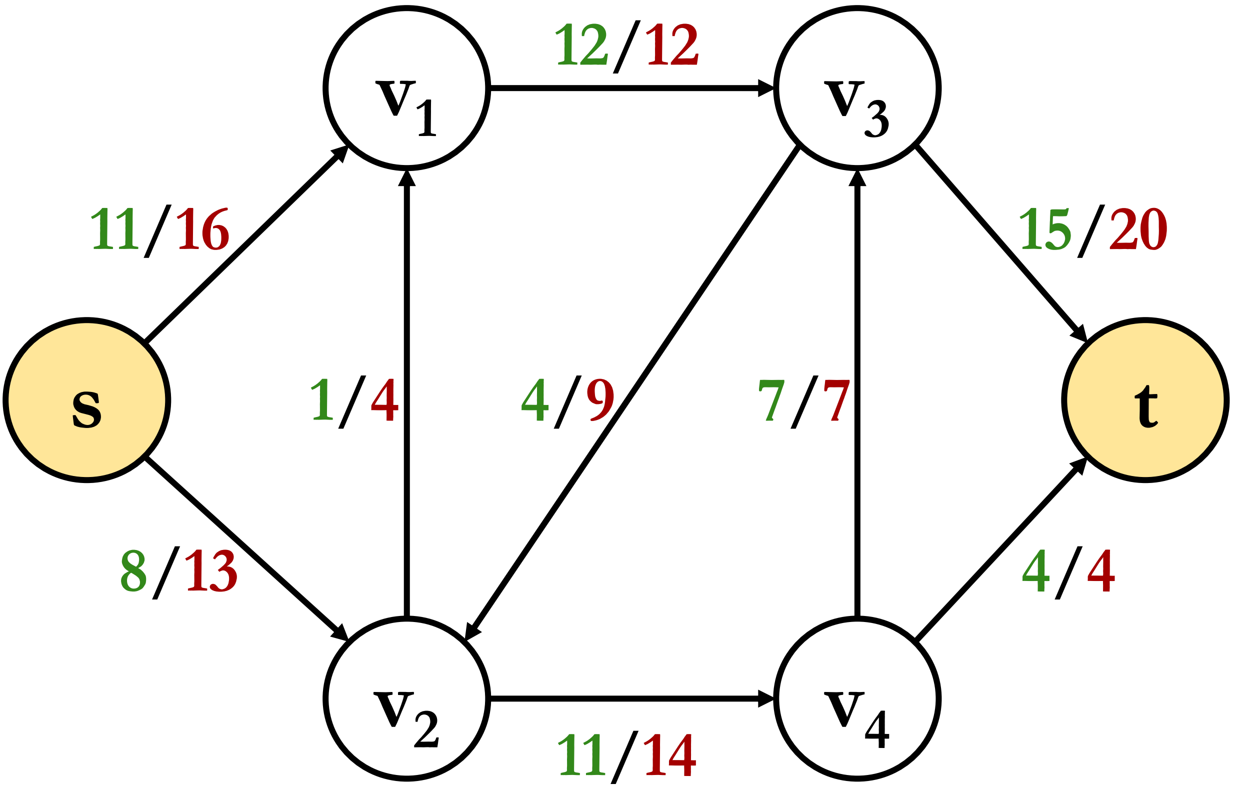
$G = (V, E)$



$G_f = (V, E_f)$

The Residual Network G_f of the example graph G – II

$$Residual\ capacity\ c_f(u,v) = \begin{cases} c(u,v) - f(u,v), & if\ (u,v) \in E \\ f(v,u), & if\ (v,u) \in E \\ 0, & otherwise \end{cases}$$



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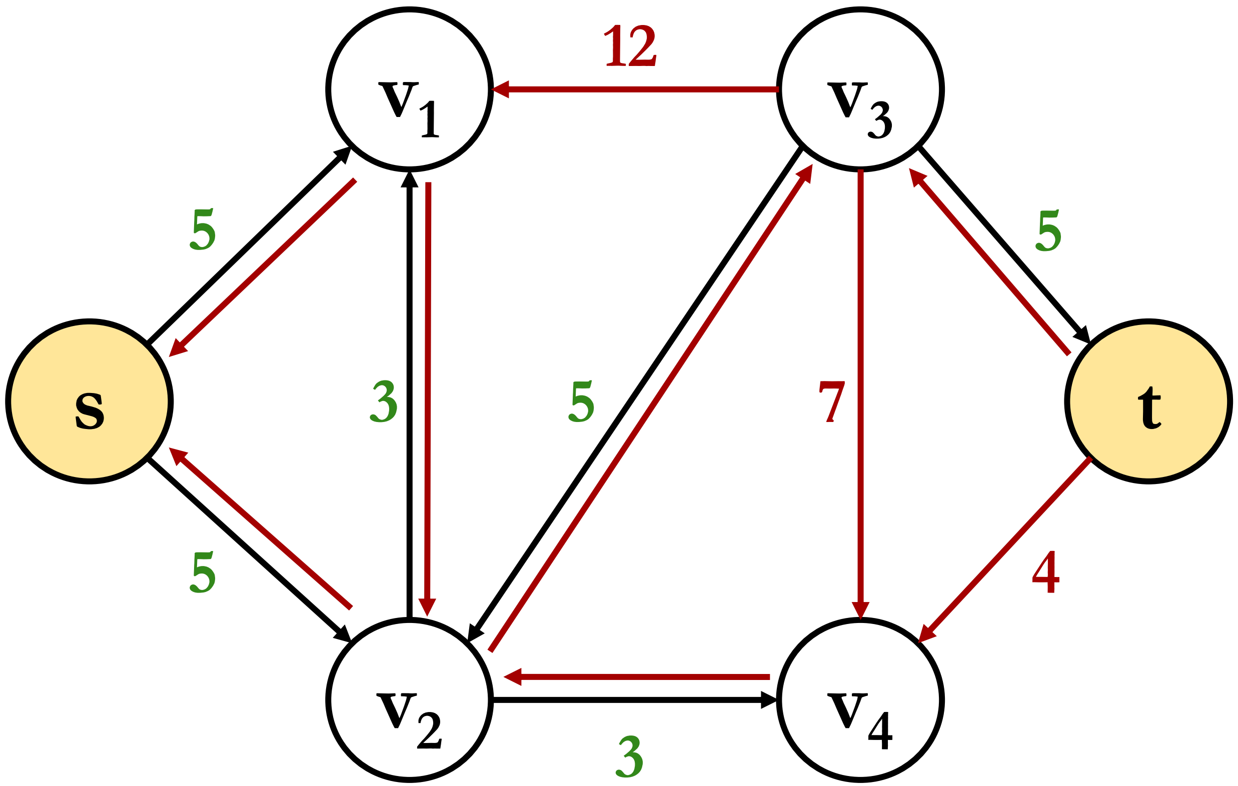
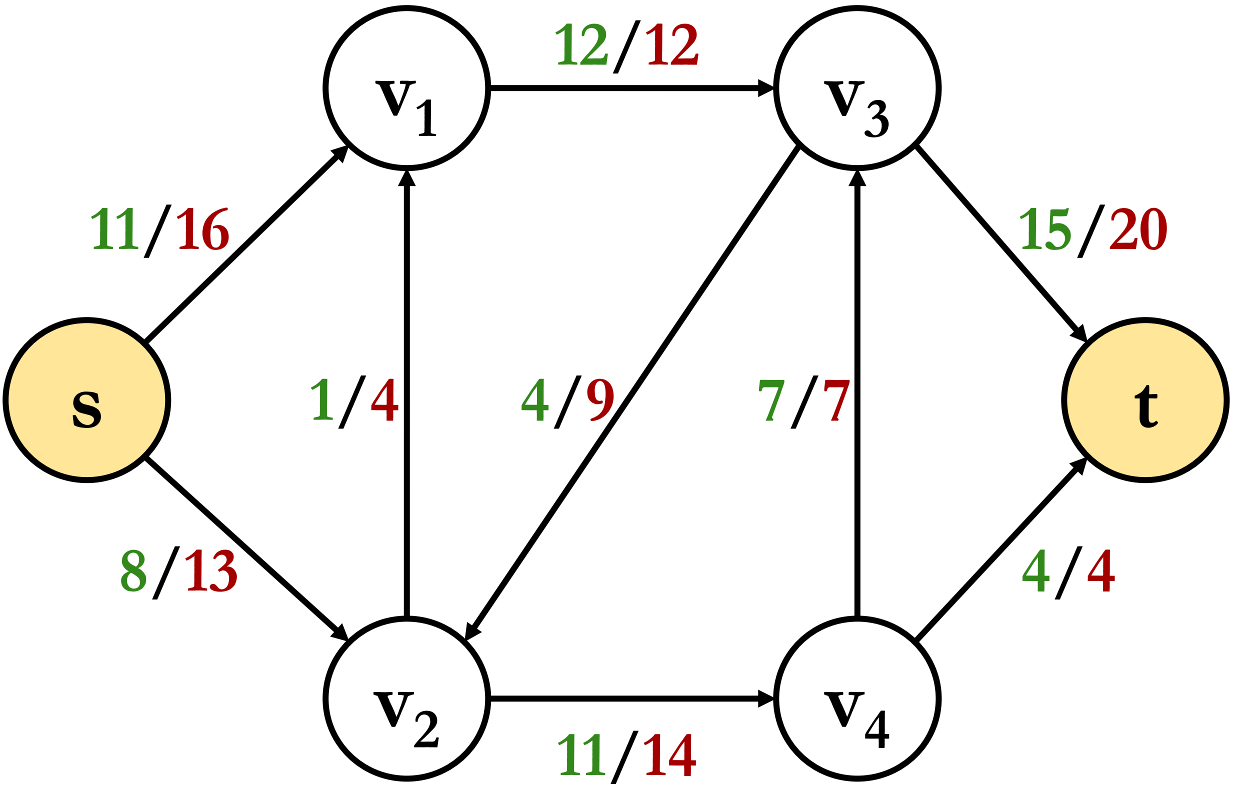


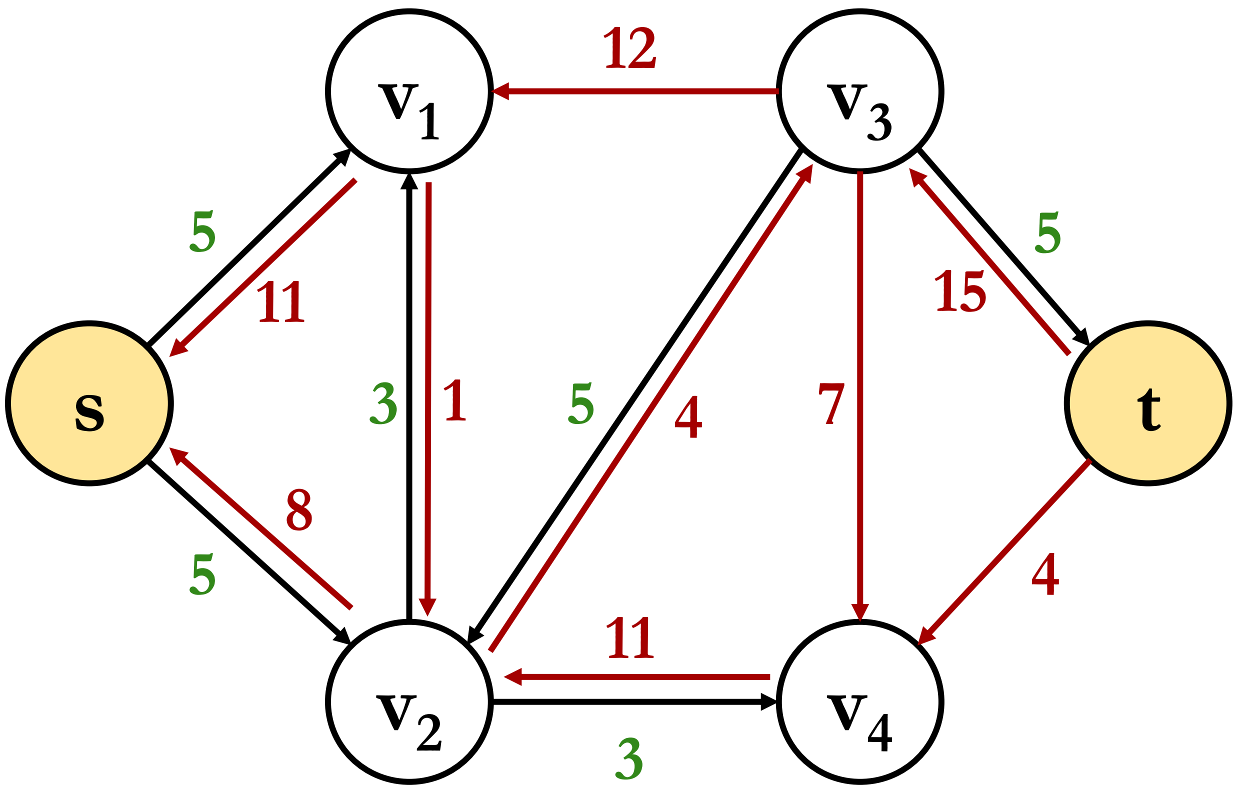
$G_f = (V, E_f)$

The Residual Network G_f of the example graph G – III

$$\textit{Residual capacity } c_f(u,v) = \begin{cases} c(u,v) - f(u,v), & \text{if } (u,v) \in E \\ f(v,u), & \text{if } (v,u) \in E \\ 0, & \text{otherwise} \end{cases}$$



$G = (V, E)$



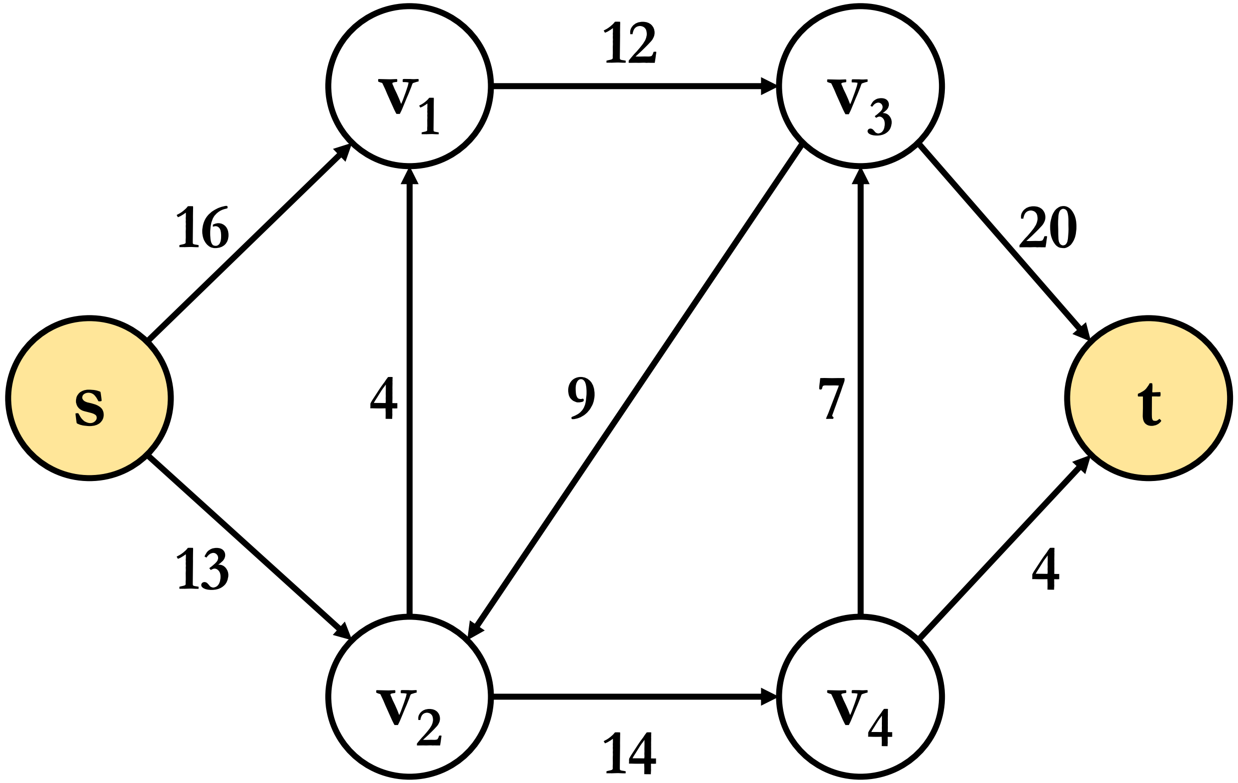
$G_f = (V, E_f)$

The Ford-Fulkerson method

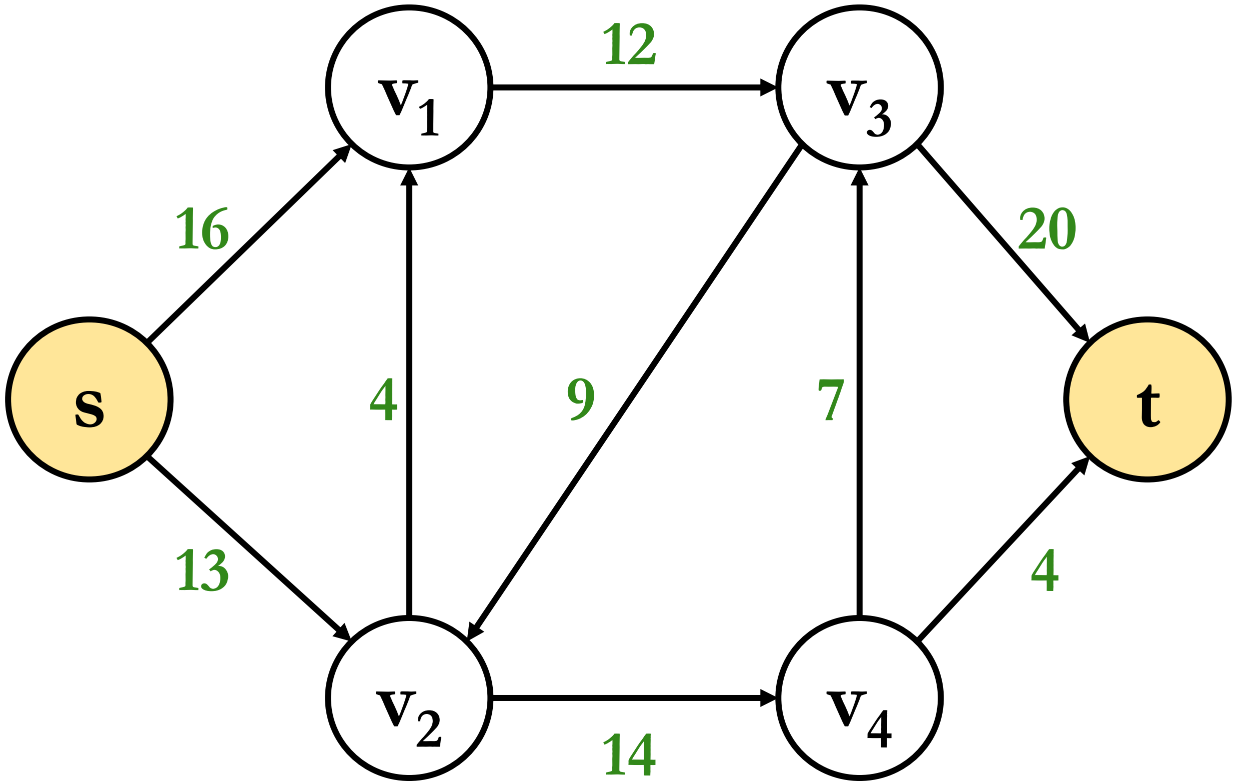
```
FORD-FULKERSON-METHOD(G, s, t)
1  initialize flow f to 0
2  while there exists an augmenting path p in the residual network G_f
3      augment flow f along p
4  return f
```

The Ford-Fulkerson method

$\boxed{|f| = 0}$

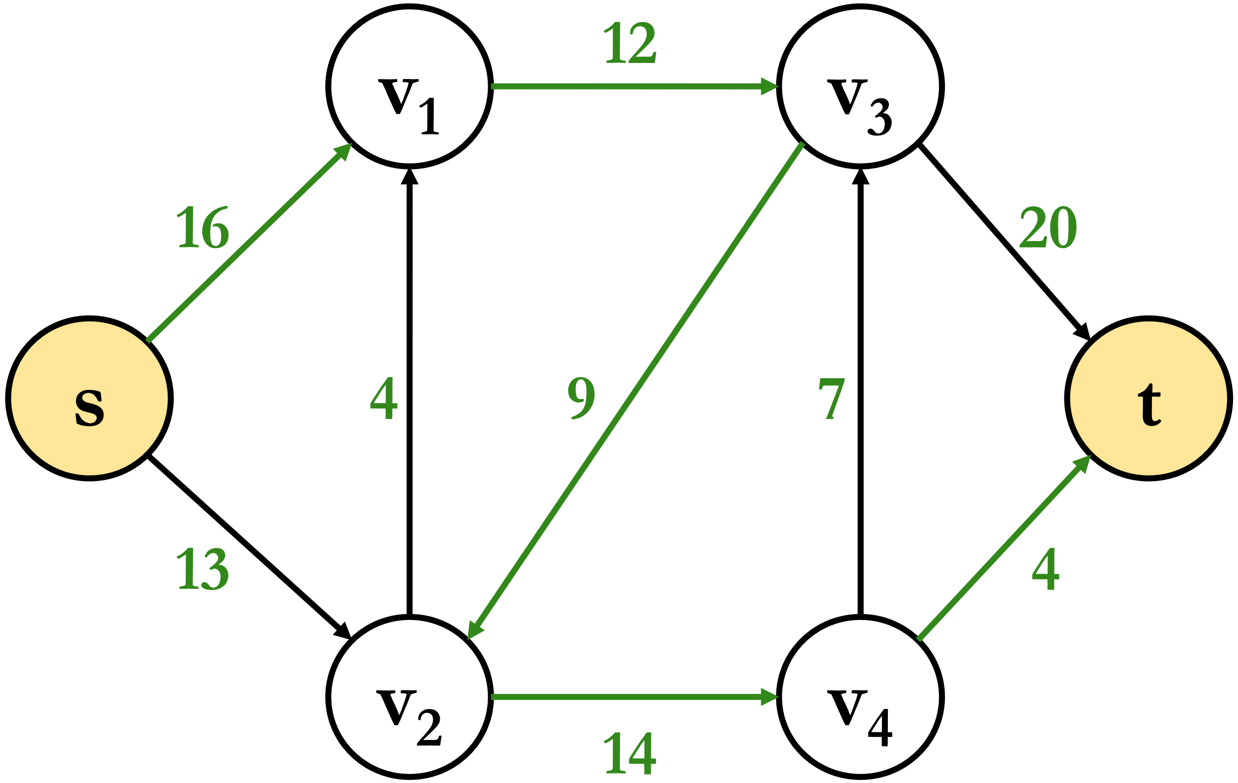


$$G = (V, E)$$



$$G_f = (V, E_f)$$

The Ford-Fulkerson method



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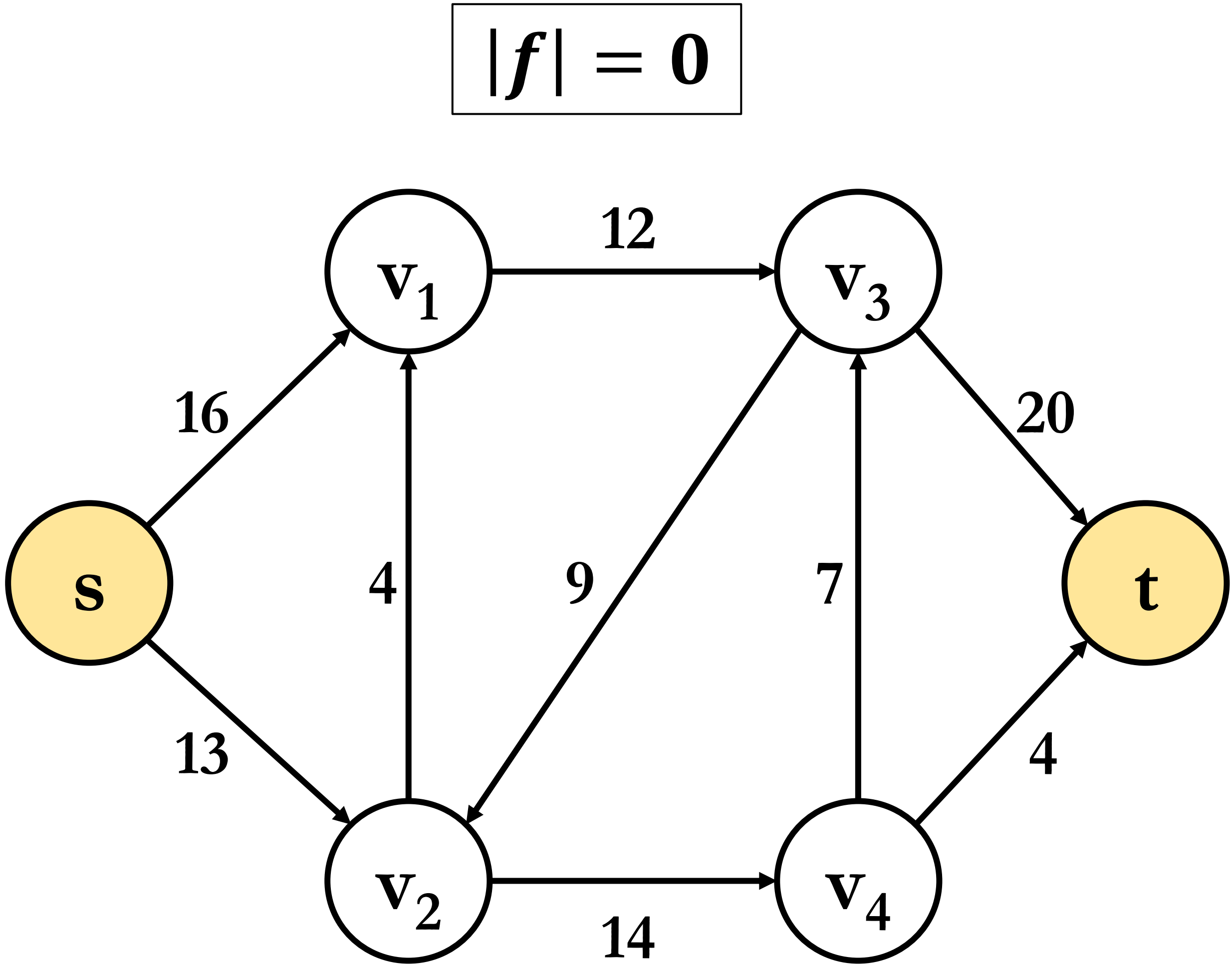
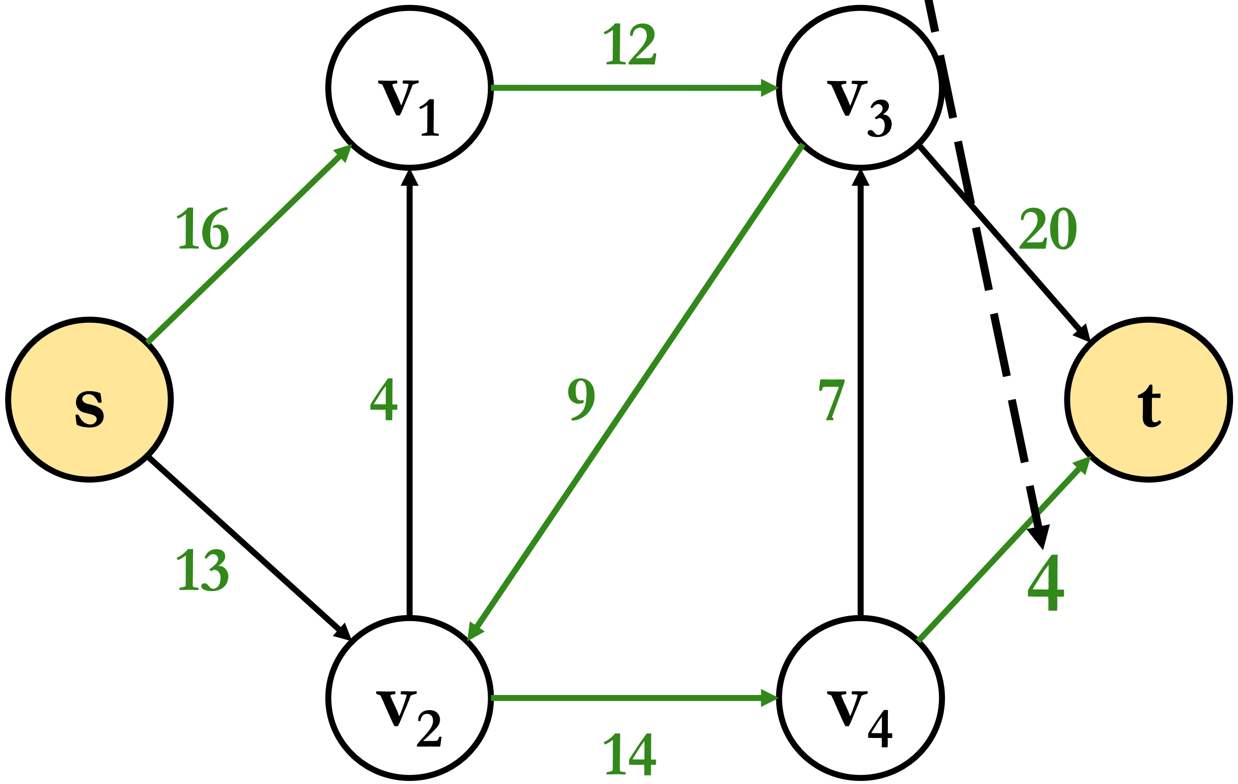


$G = (V, E)$

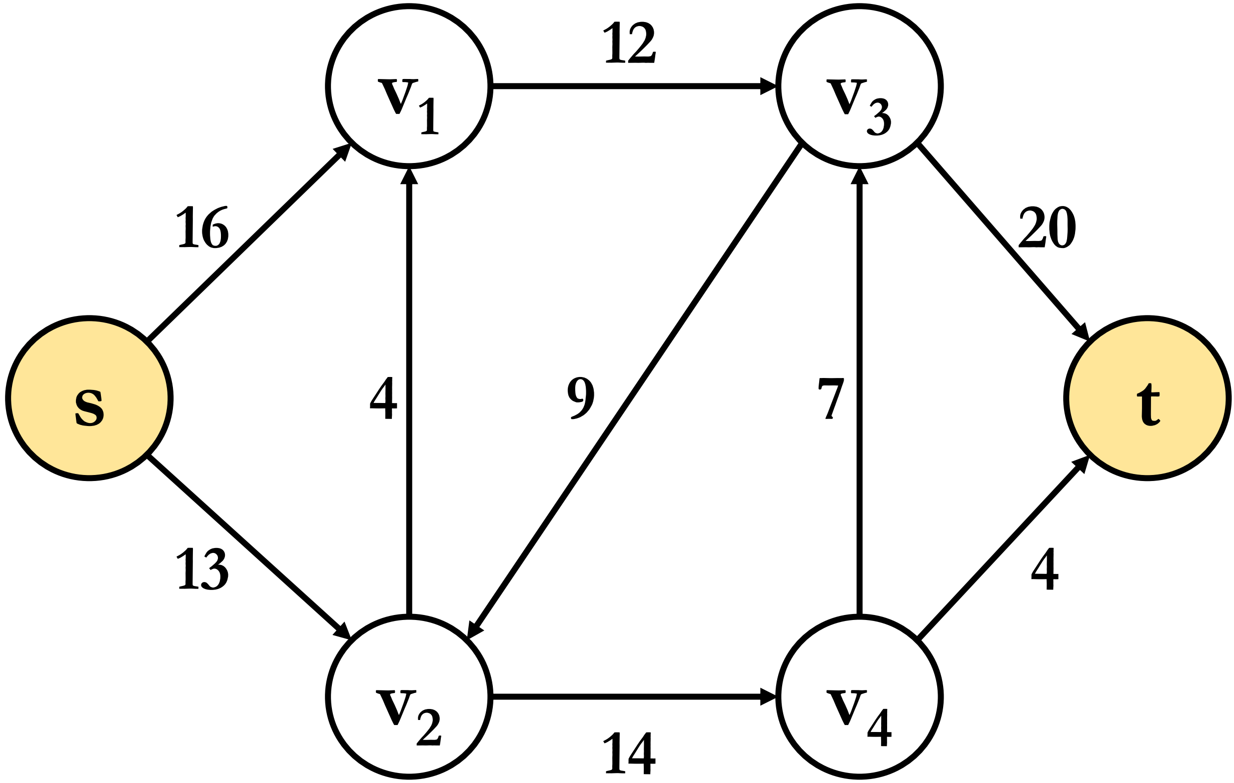
The Ford-Fulkerson method

$$c_f(p) = min\{c_f(u,v): (u,v) \text{ is on } p: s \leadsto t\}$$

$$|f| = 0$$

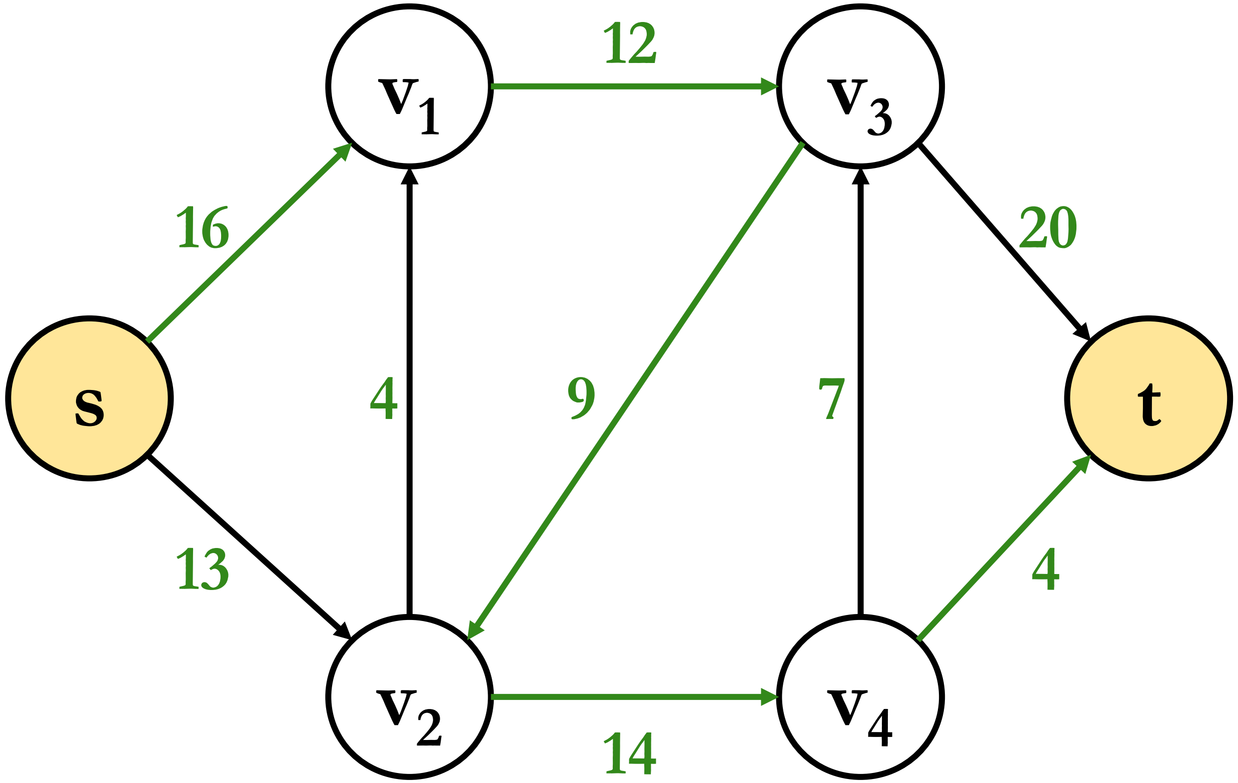


$$G_f = (V, E_f)$$



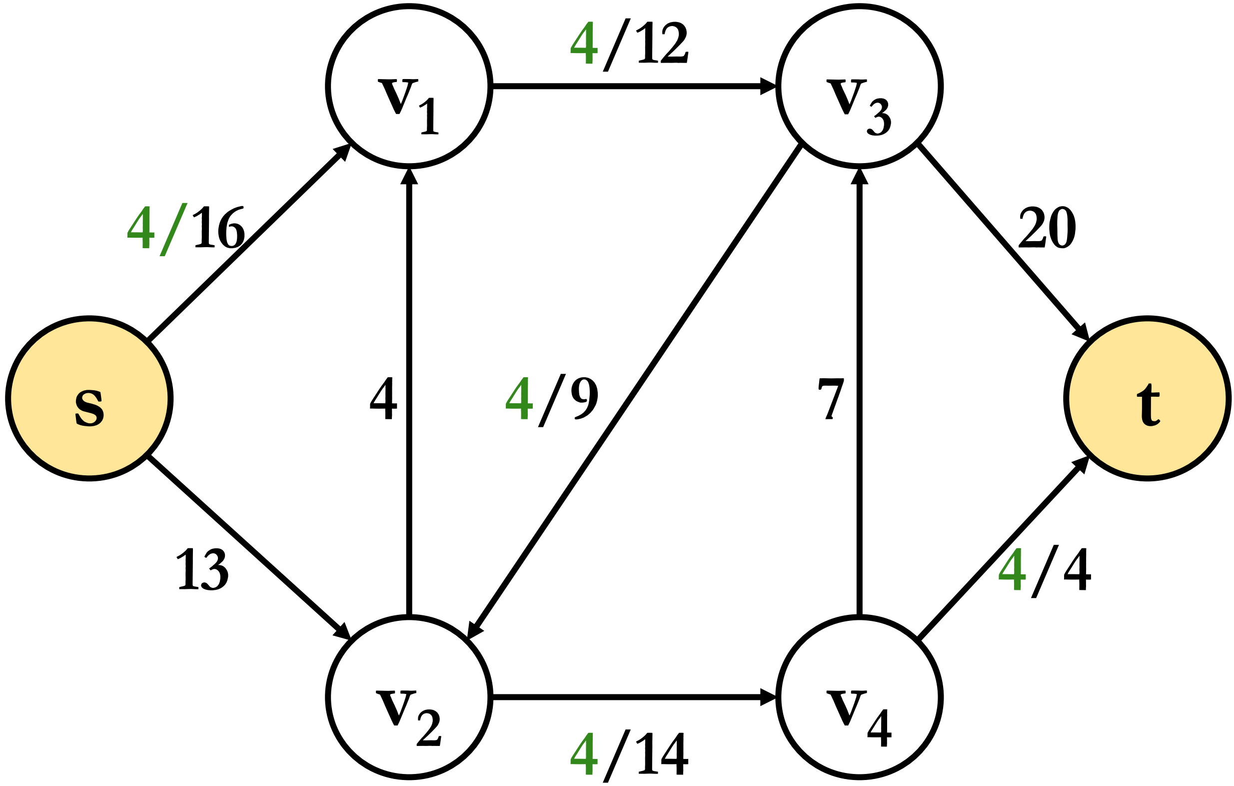
$$G = (V, E)$$

The Ford-Fulkerson method



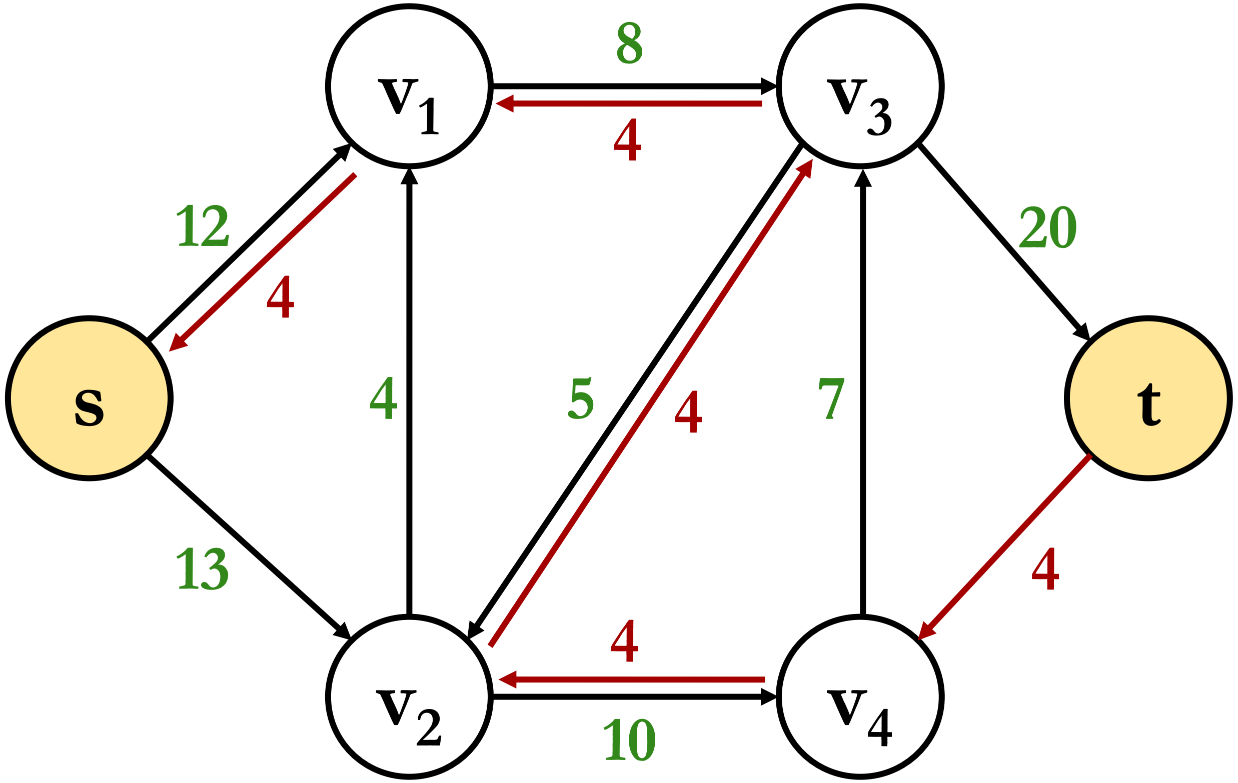
$G_f = (V, E_f)$

$|f| = 4$



$G = (V, E)$

The Ford-Fulkerson method



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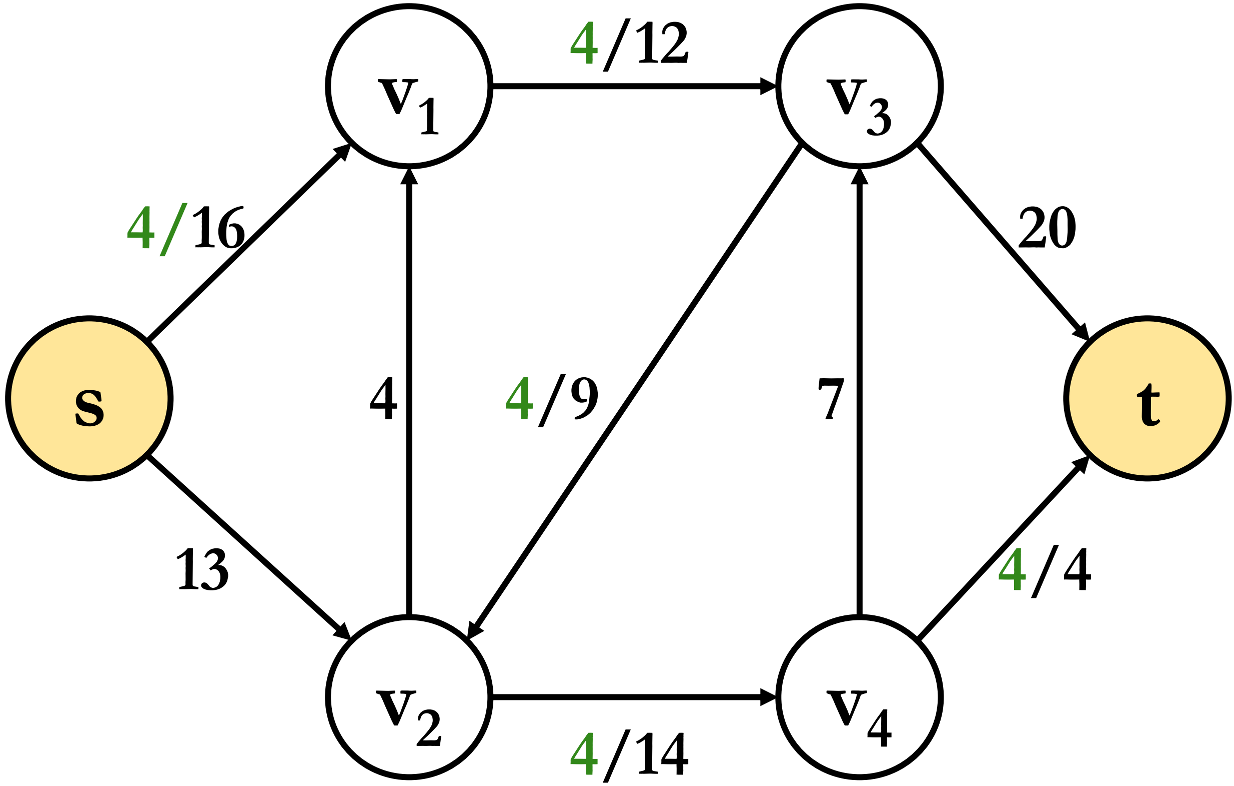
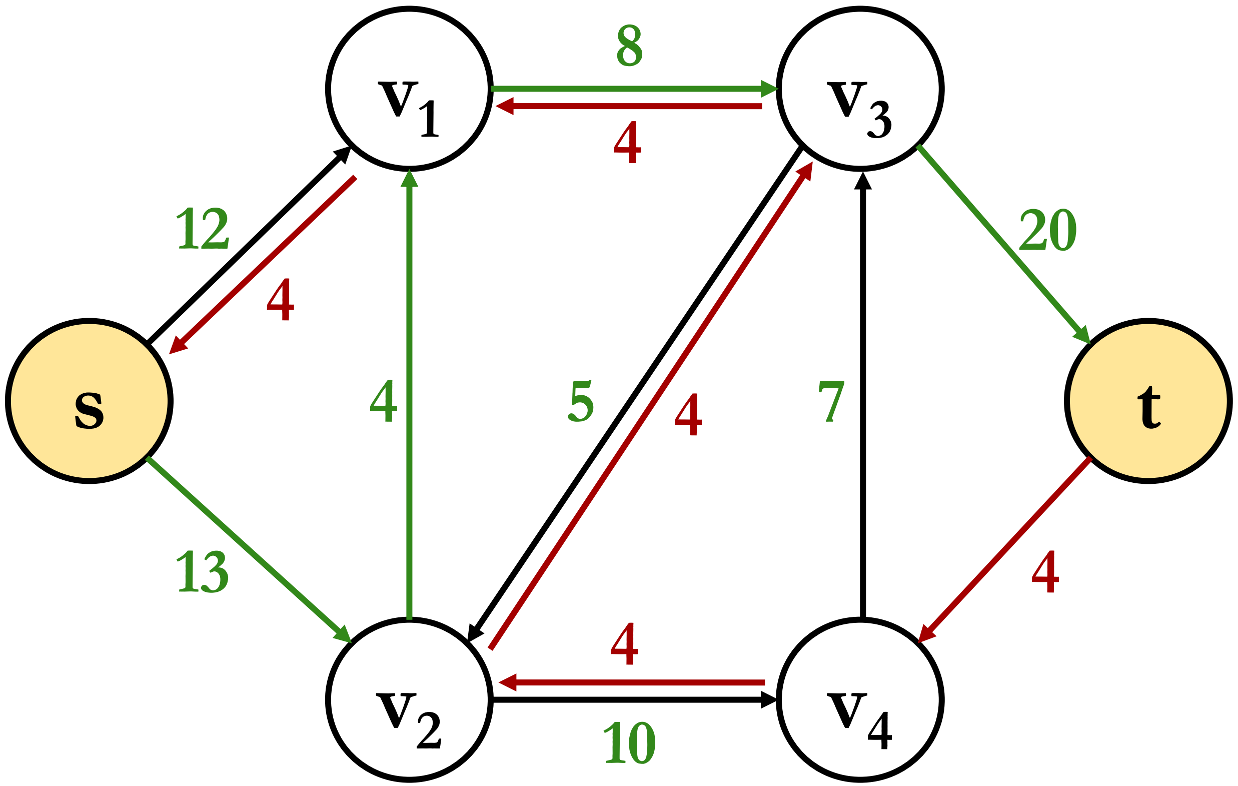


$G = (V, E)$

The Ford-Fulkerson method



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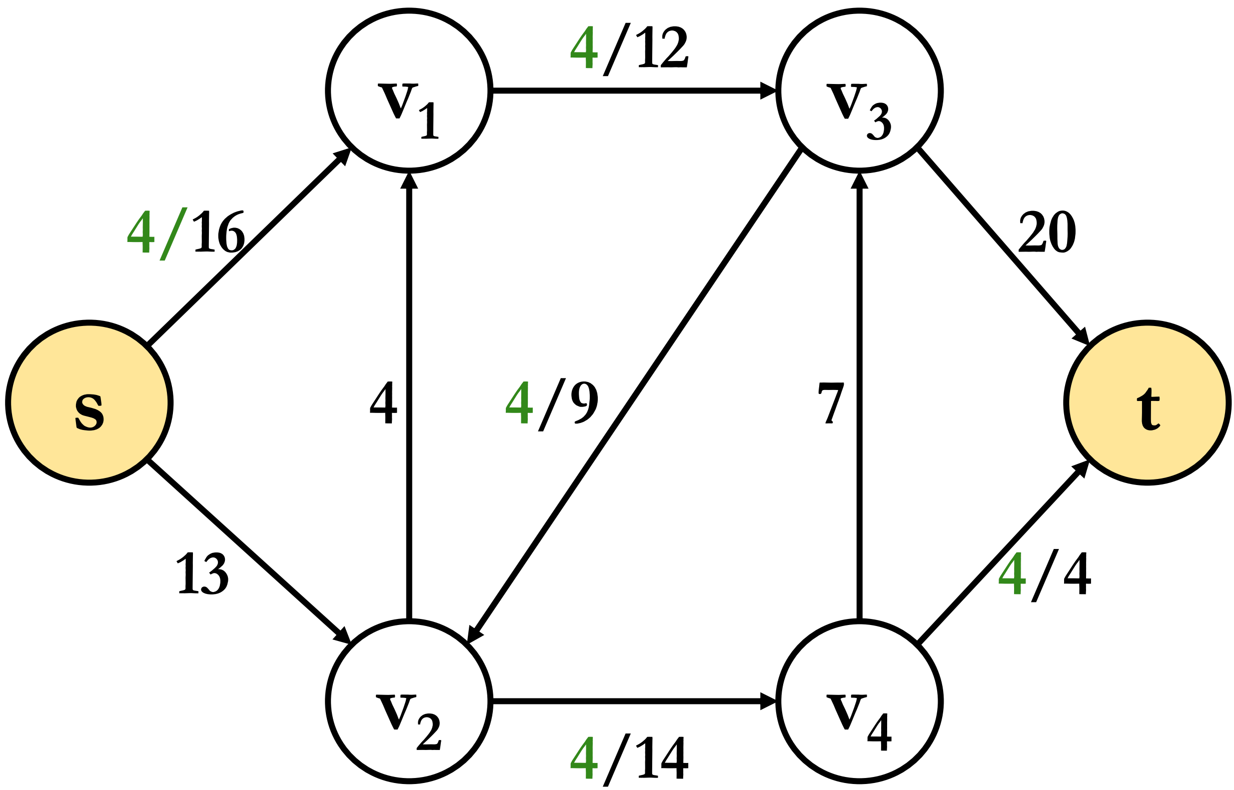
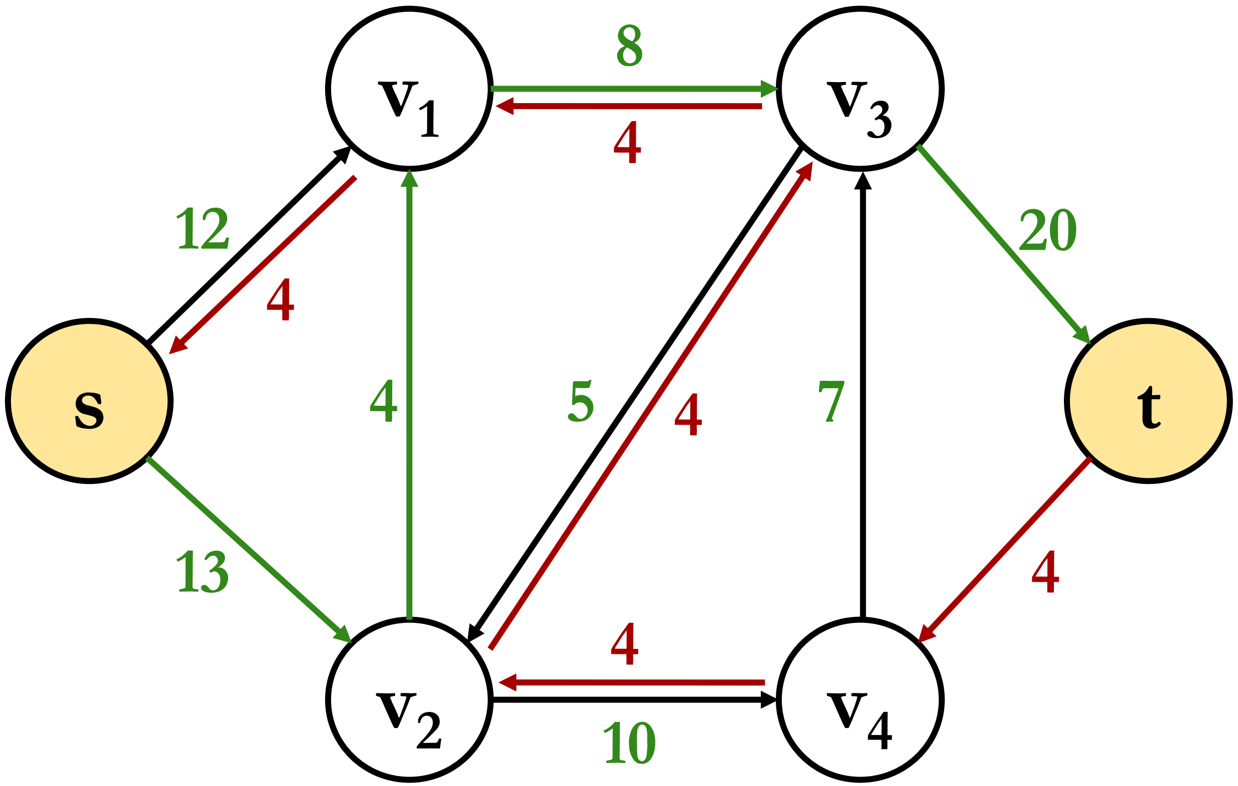


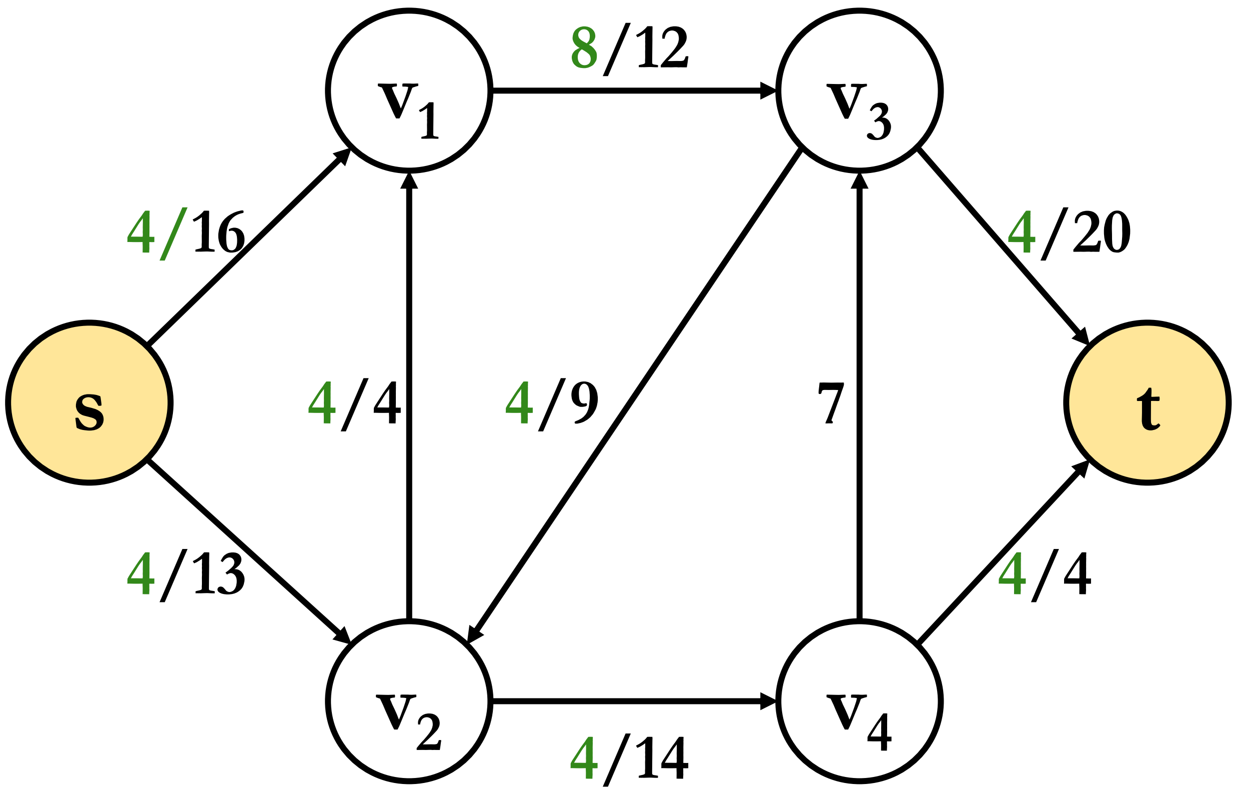
$G = (V, E)$

The Ford-Fulkerson method



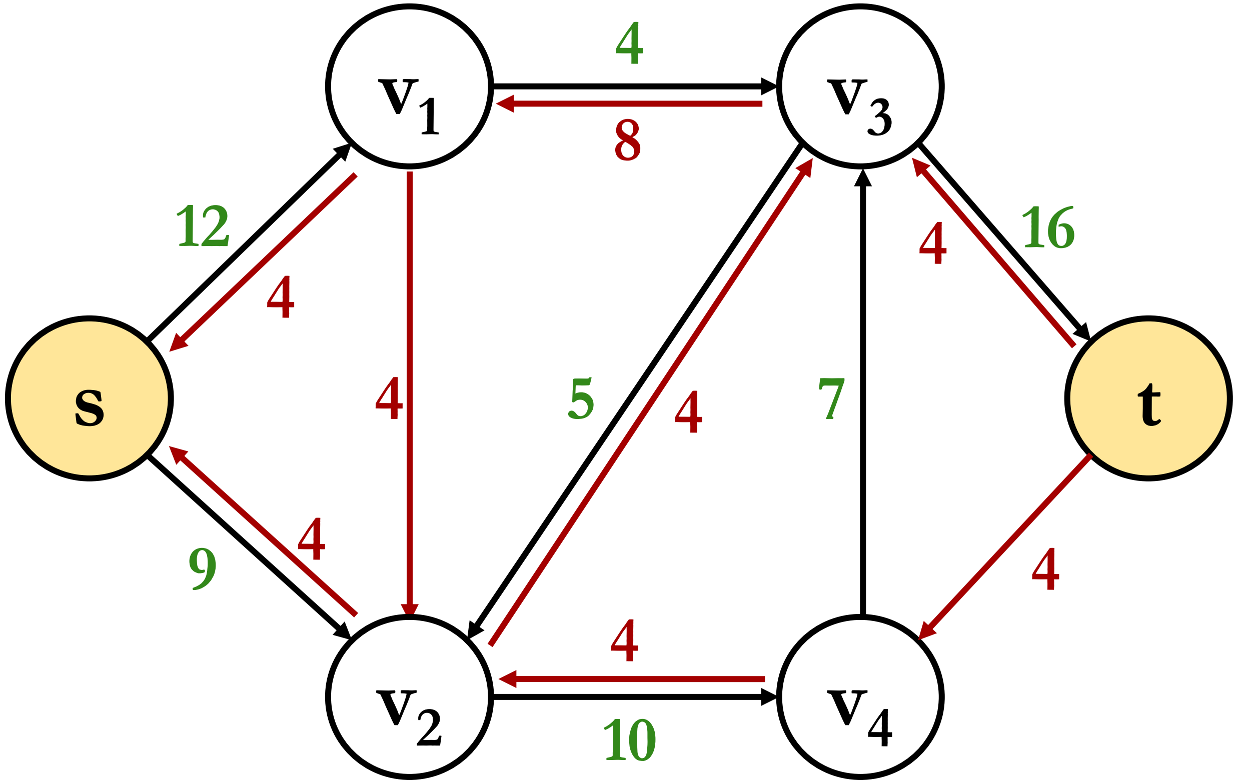
$G_f = (V, E_f)$

$|f| = 8$



$G = (V, E)$

The Ford-Fulkerson method



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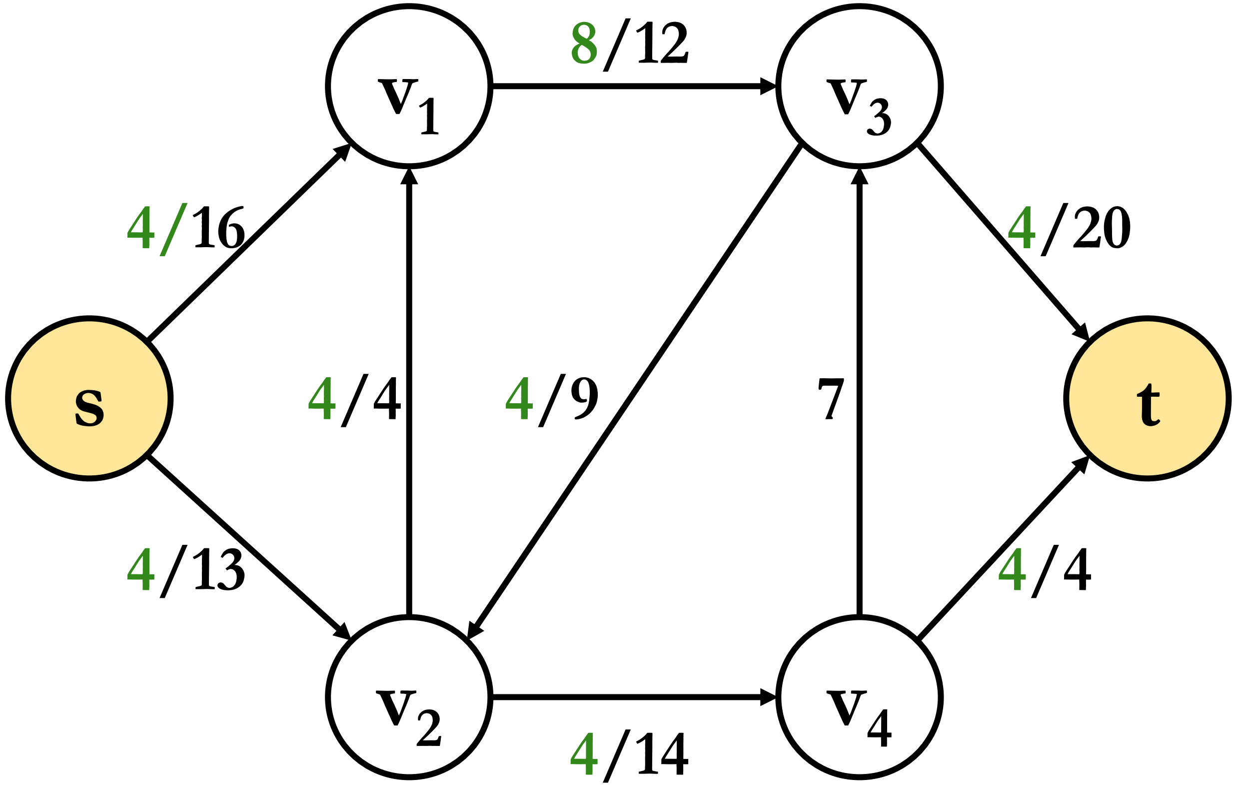
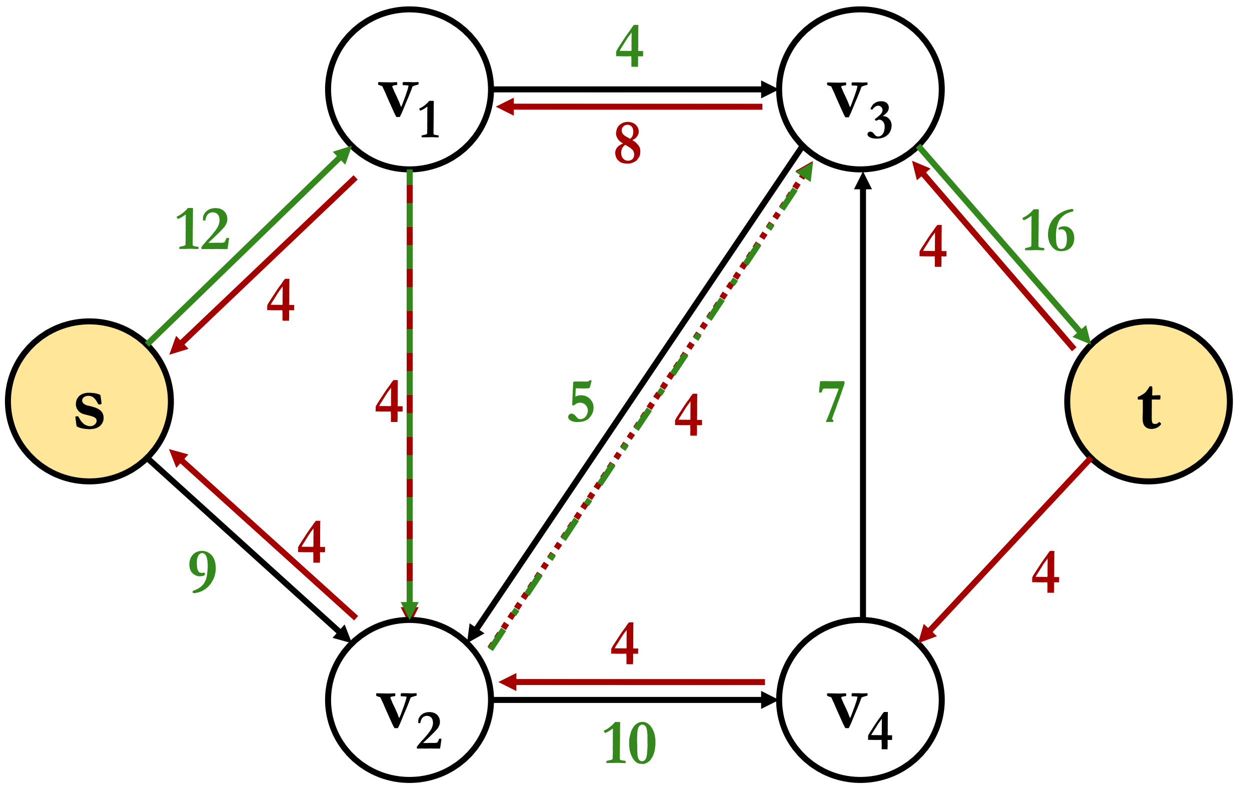


$G = (V, E)$

The Ford-Fulkerson method



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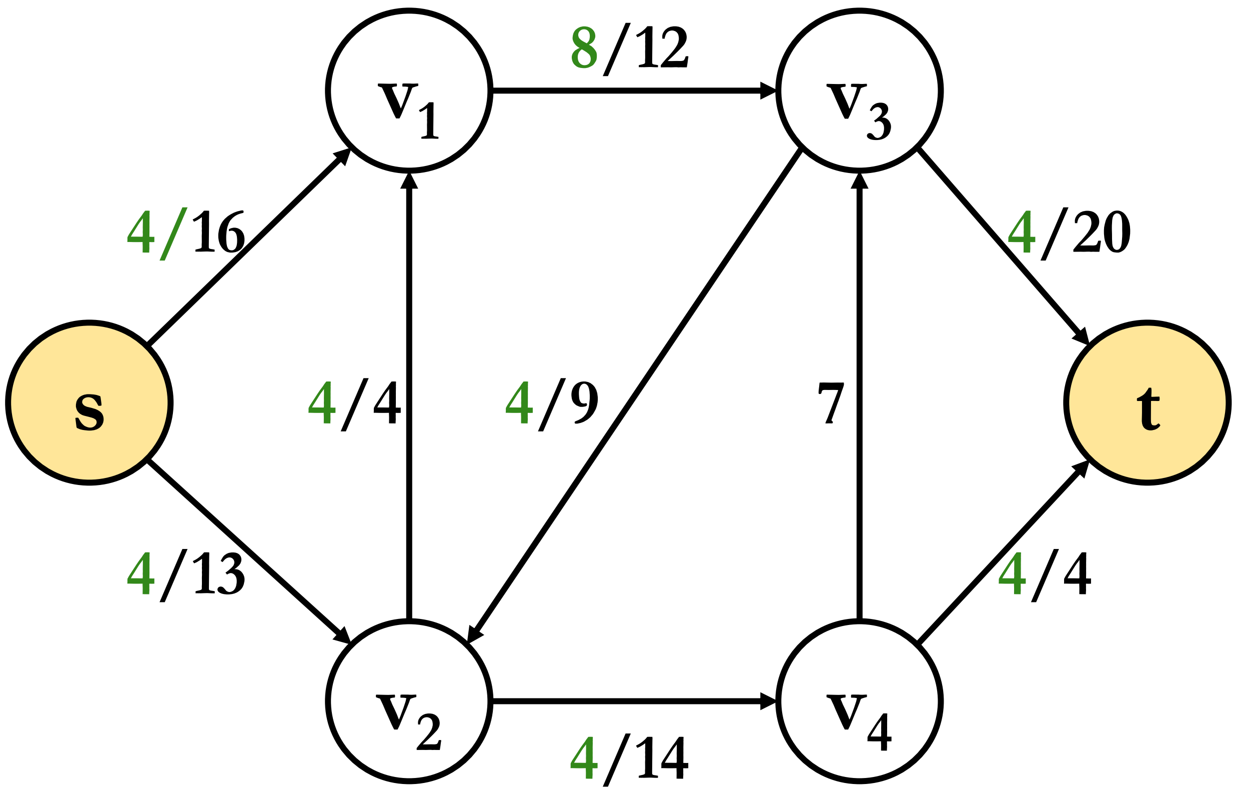
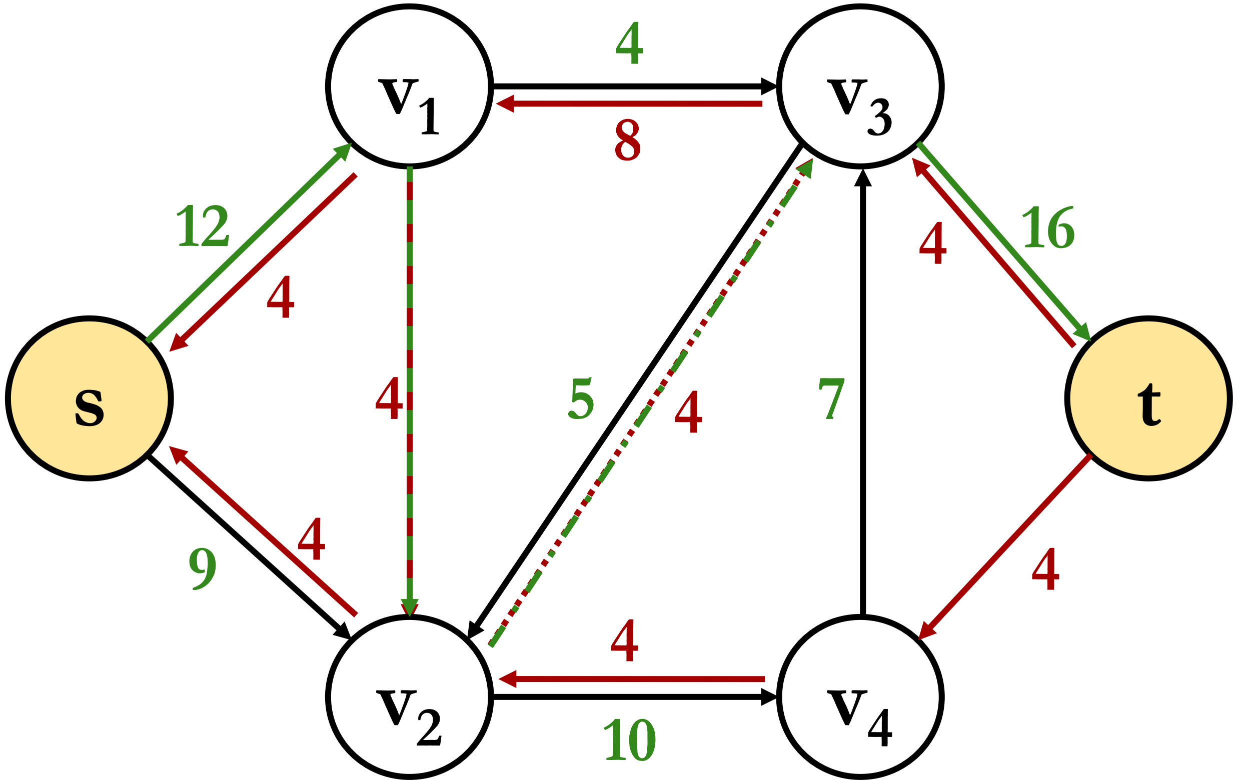


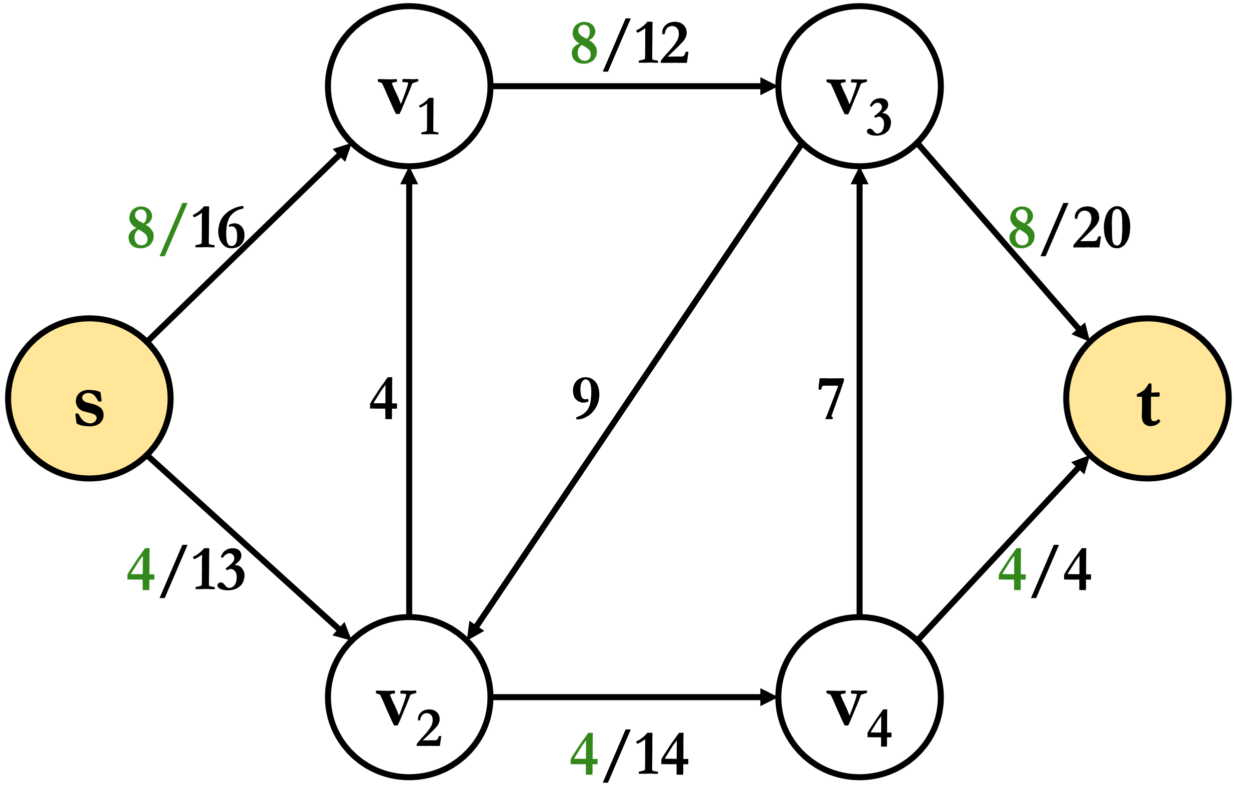
$G = (V, E)$

The Ford-Fulkerson method



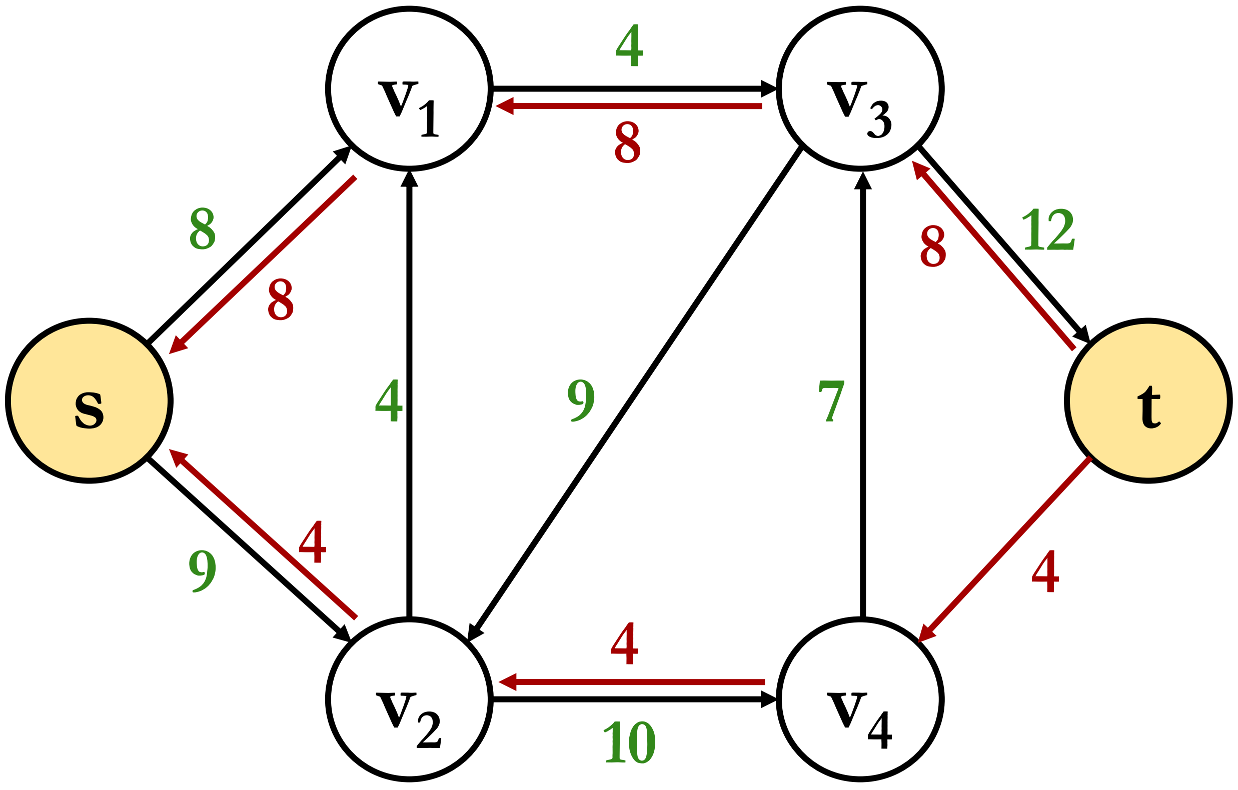
$G_f = (V, E_f)$

$|f| = 12$



$G = (V, E)$

The Ford-Fulkerson method



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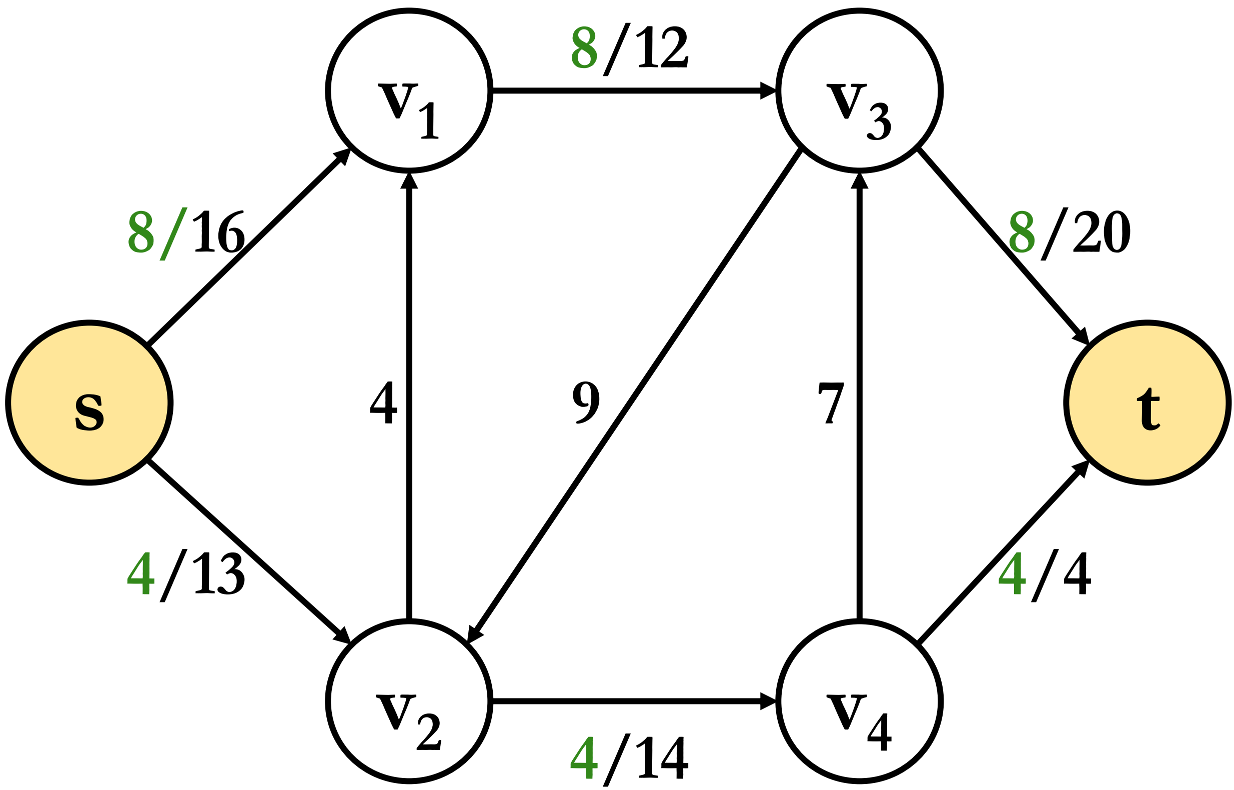

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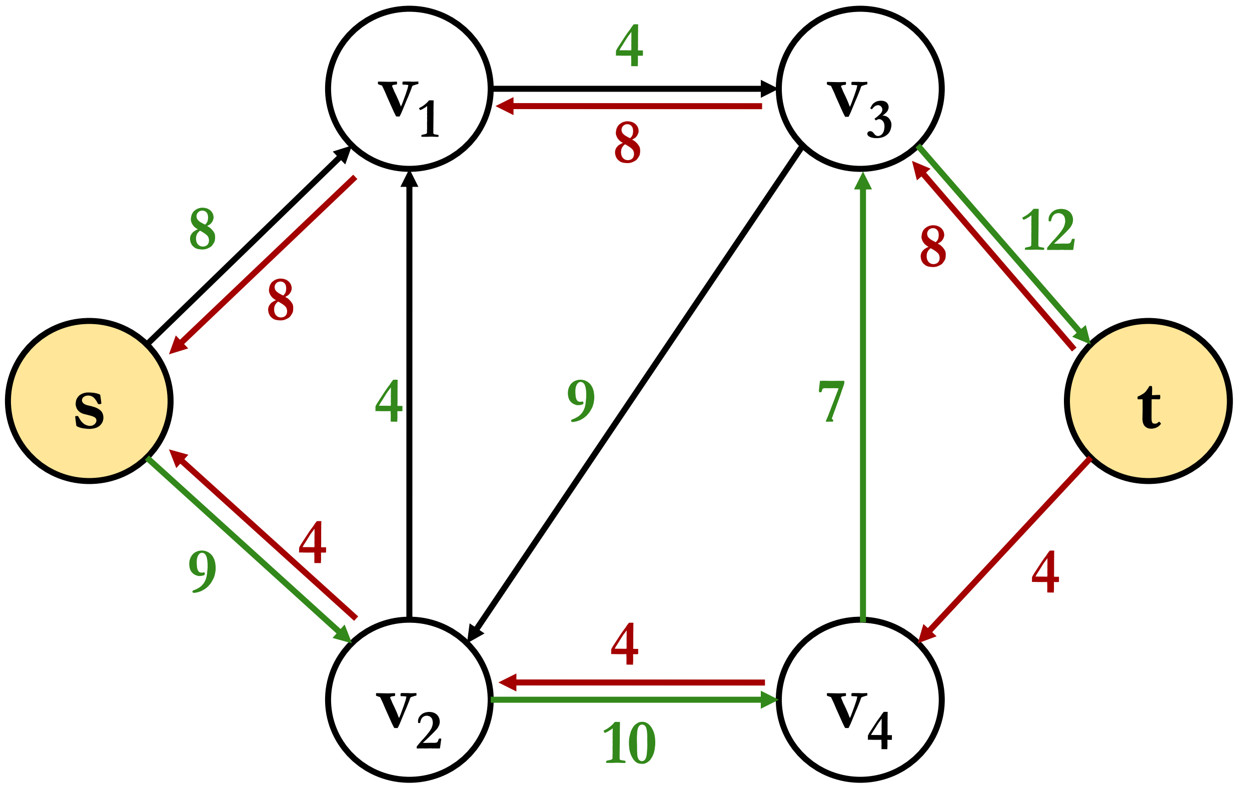
The Ford-Fulkerson method



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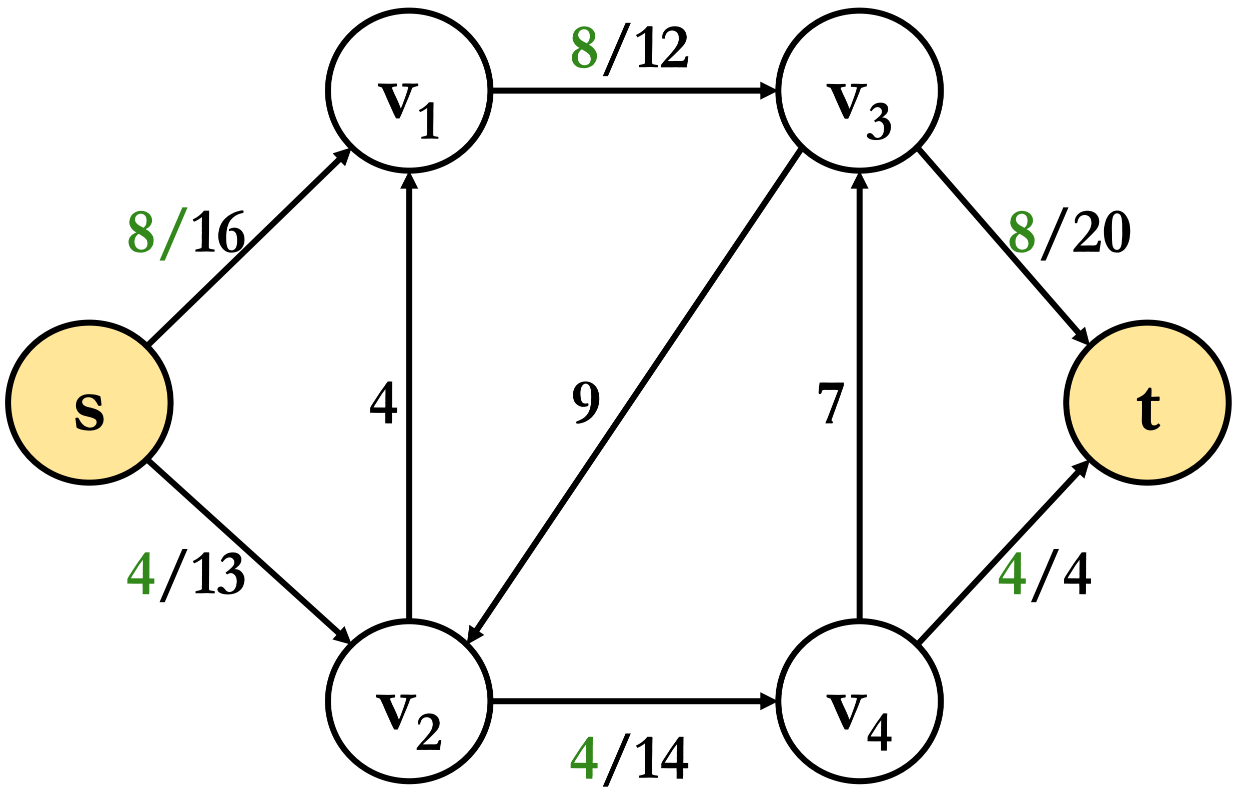

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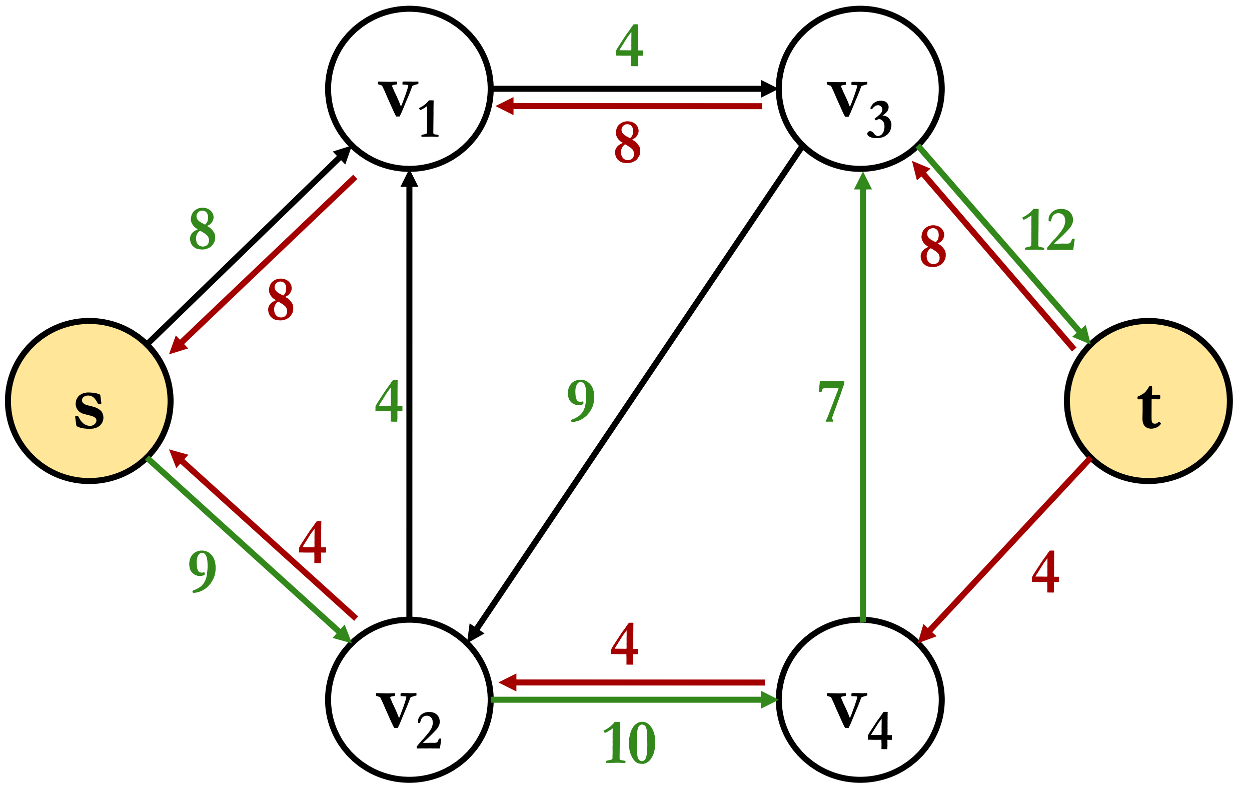

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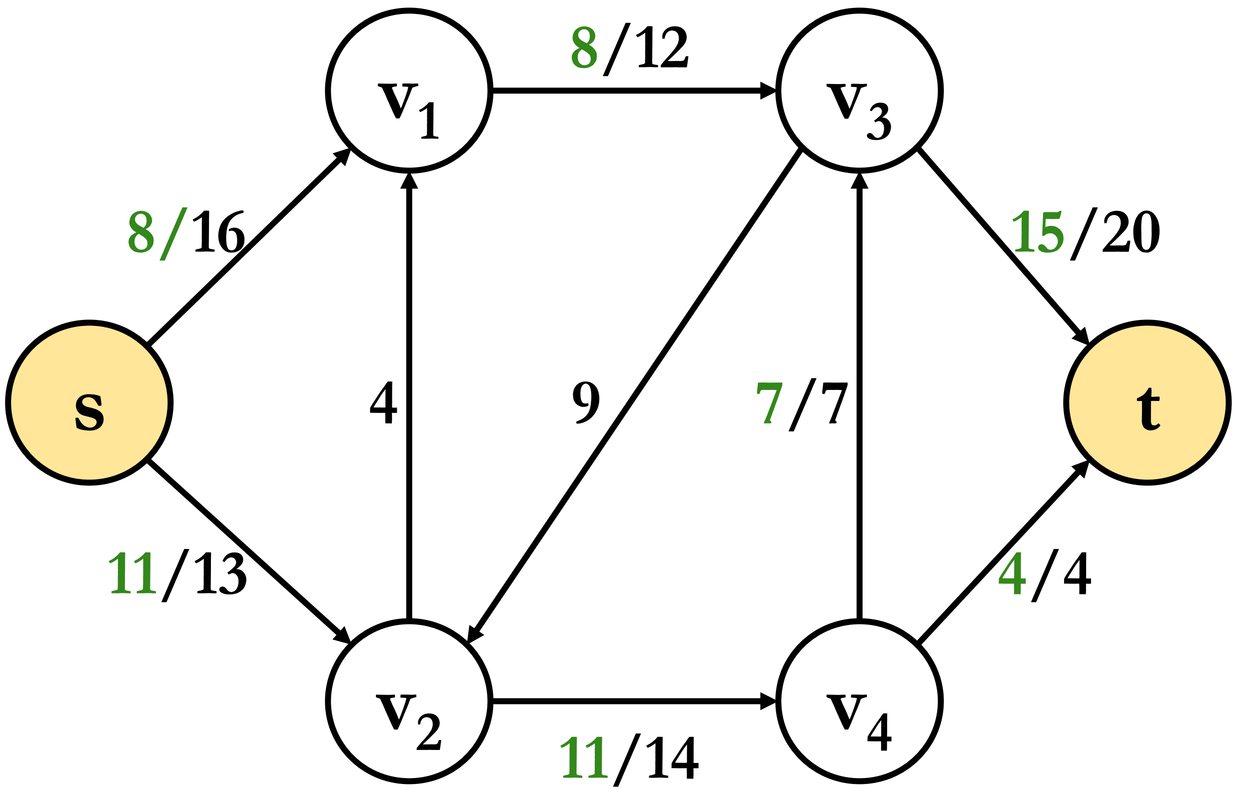

$G = (V, E)$

The Ford-Fulkerson method

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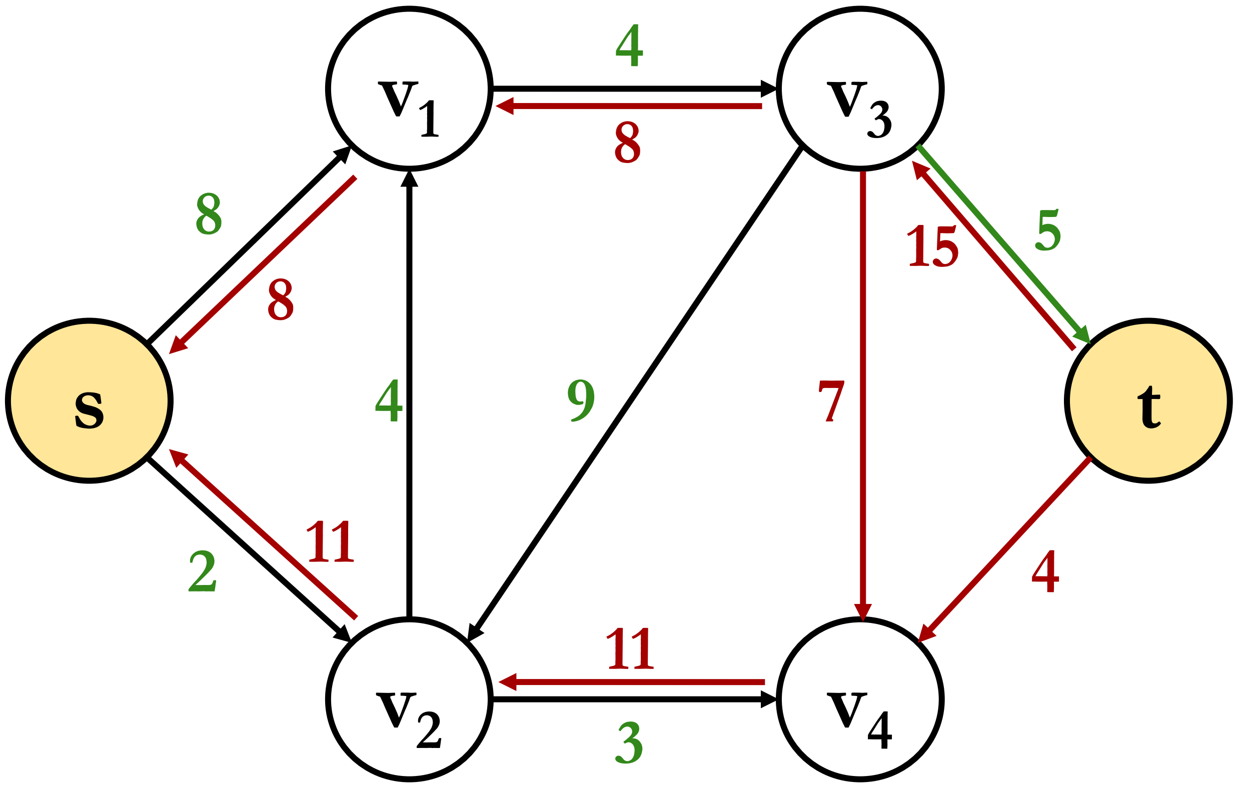

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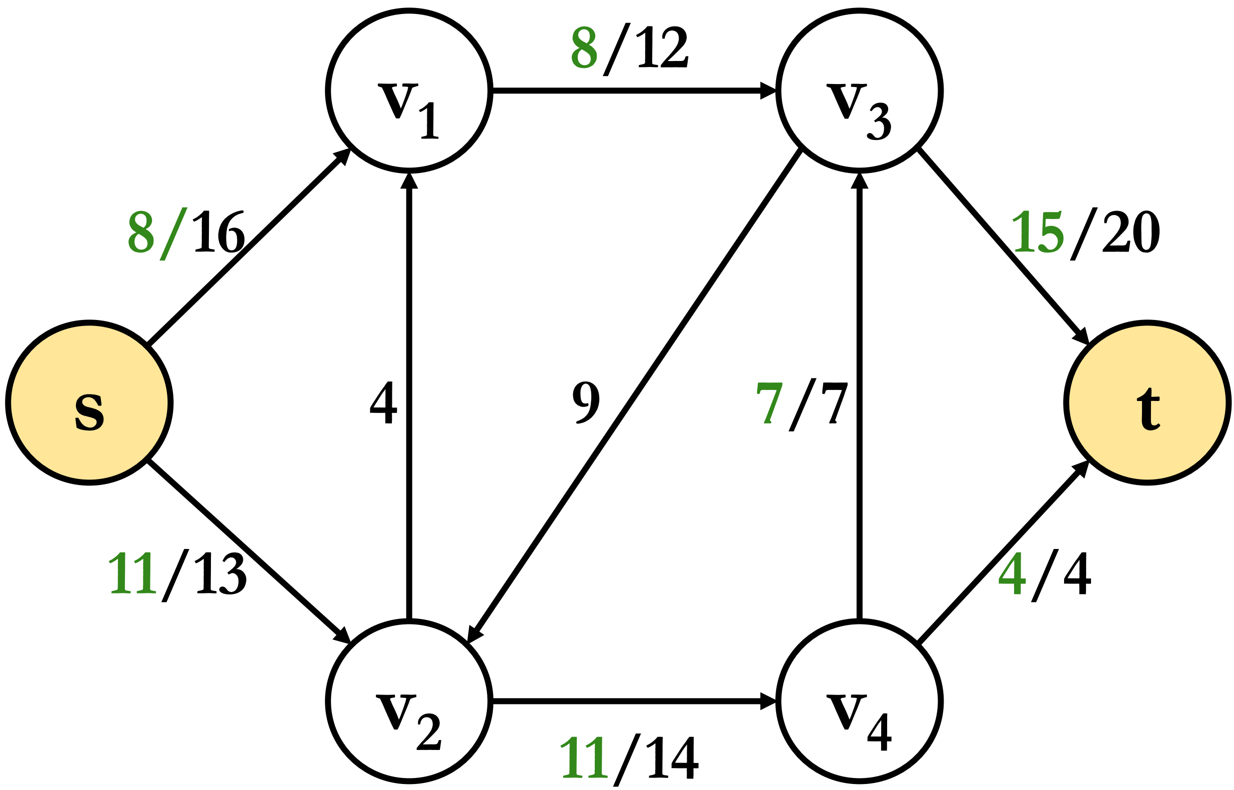

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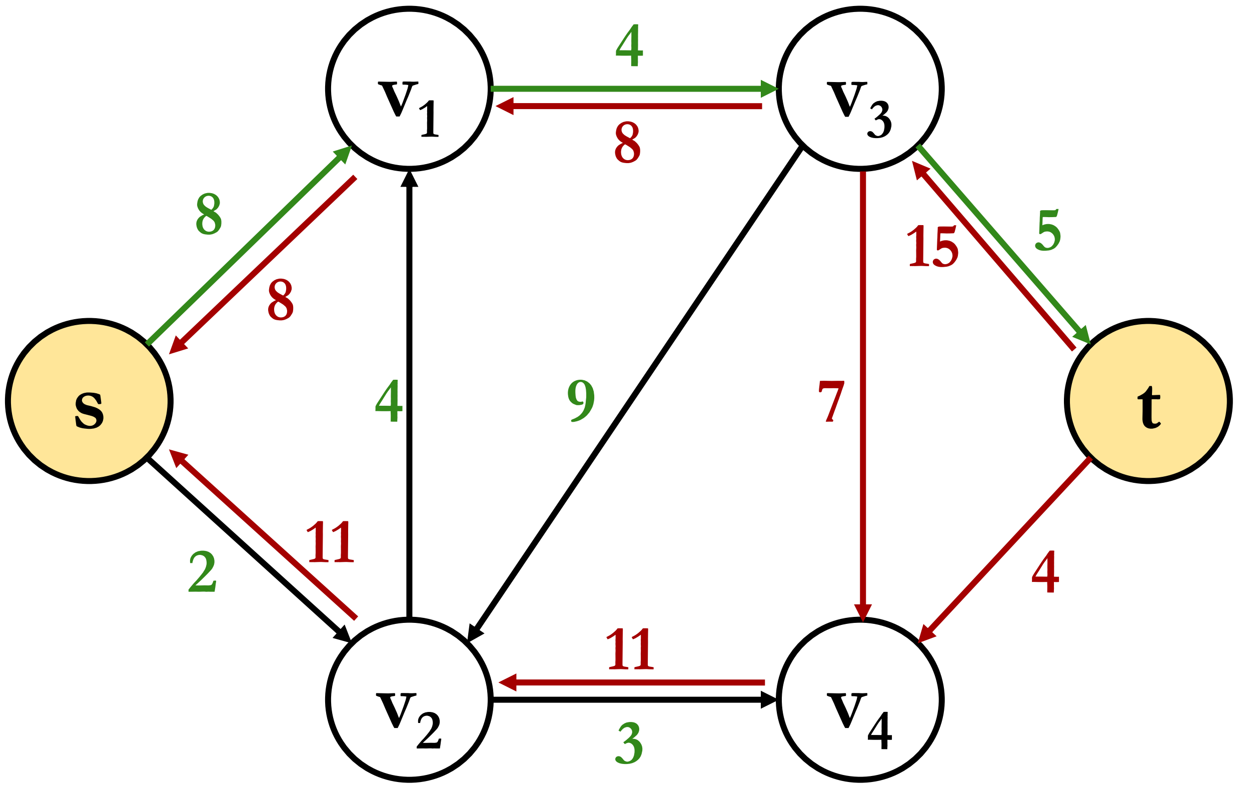

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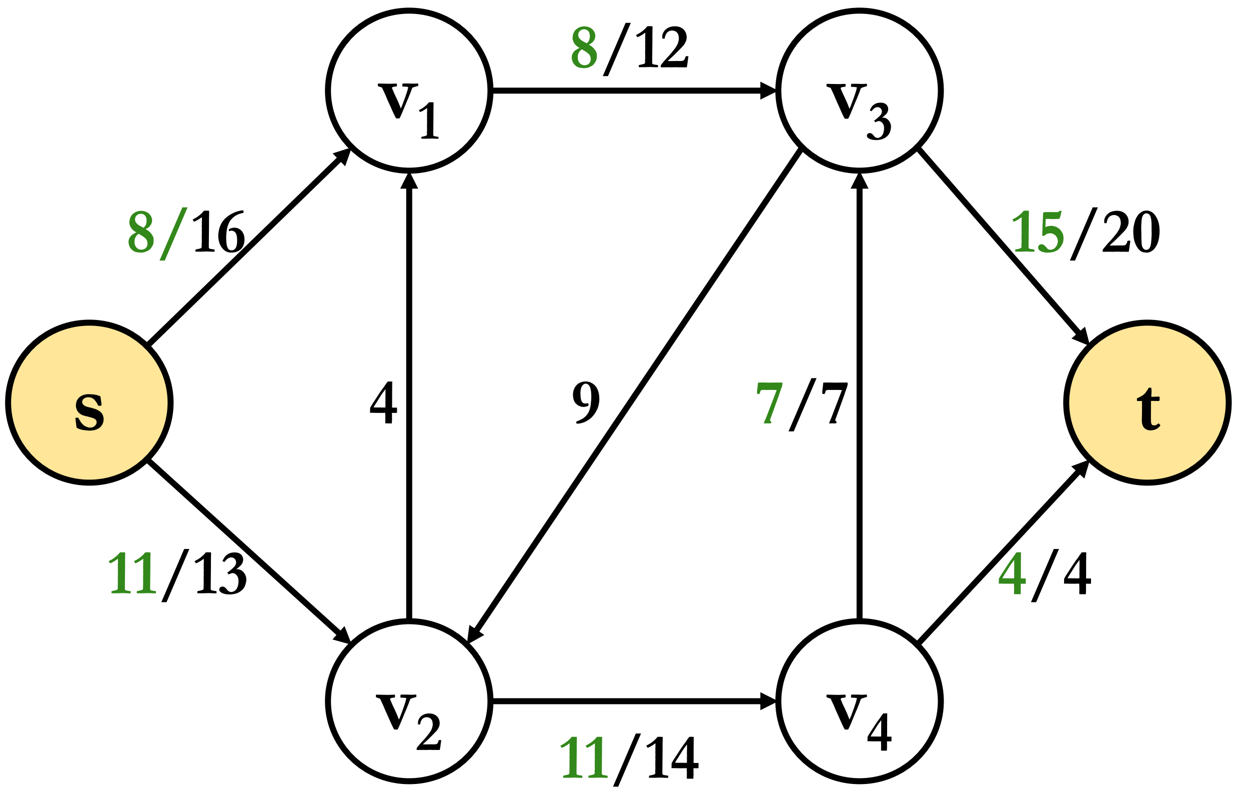

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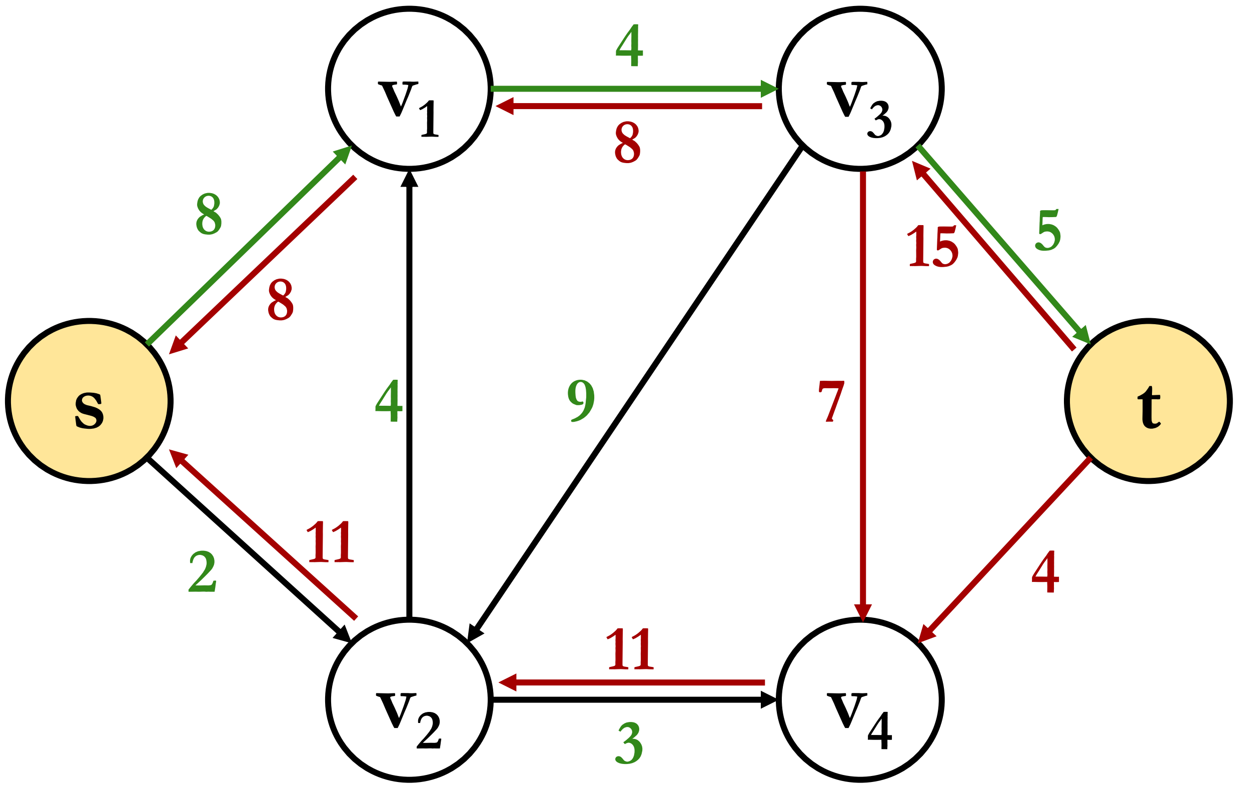
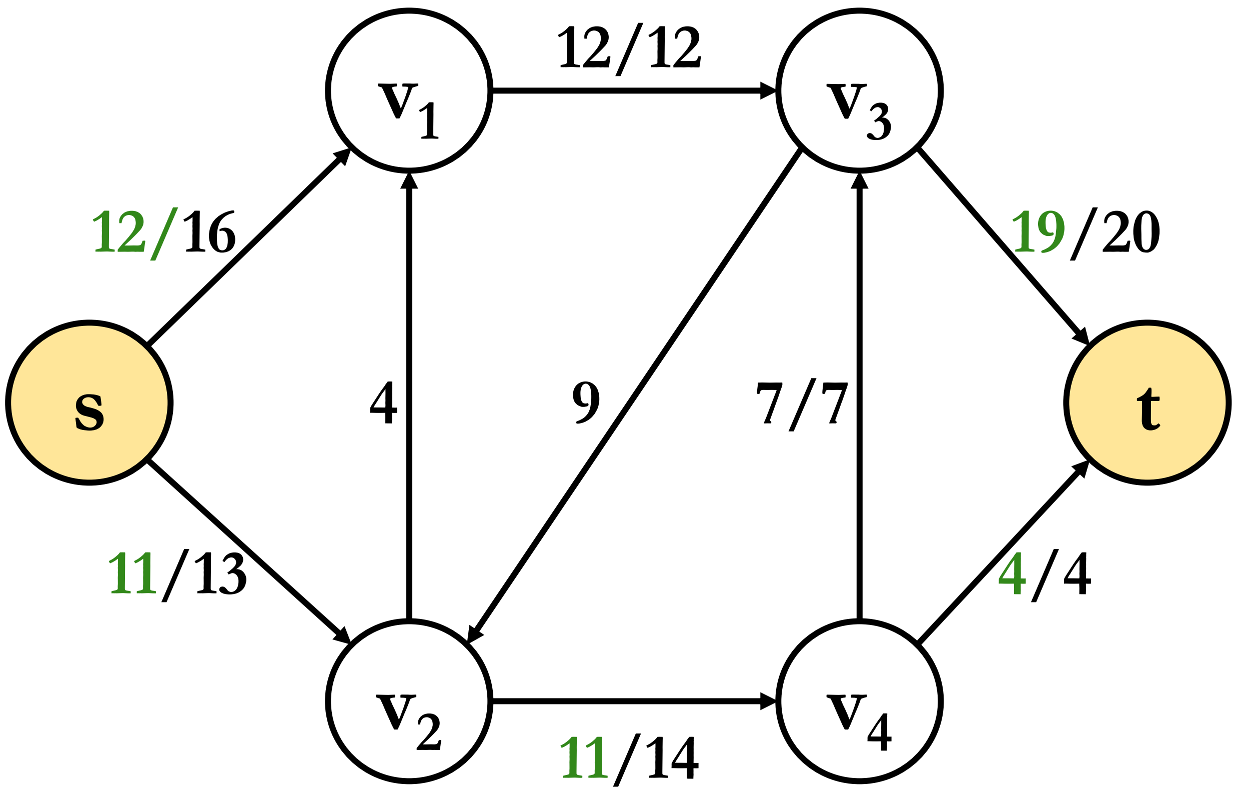
The Ford-Fulkerson method



$G_f = (V, E_f)$

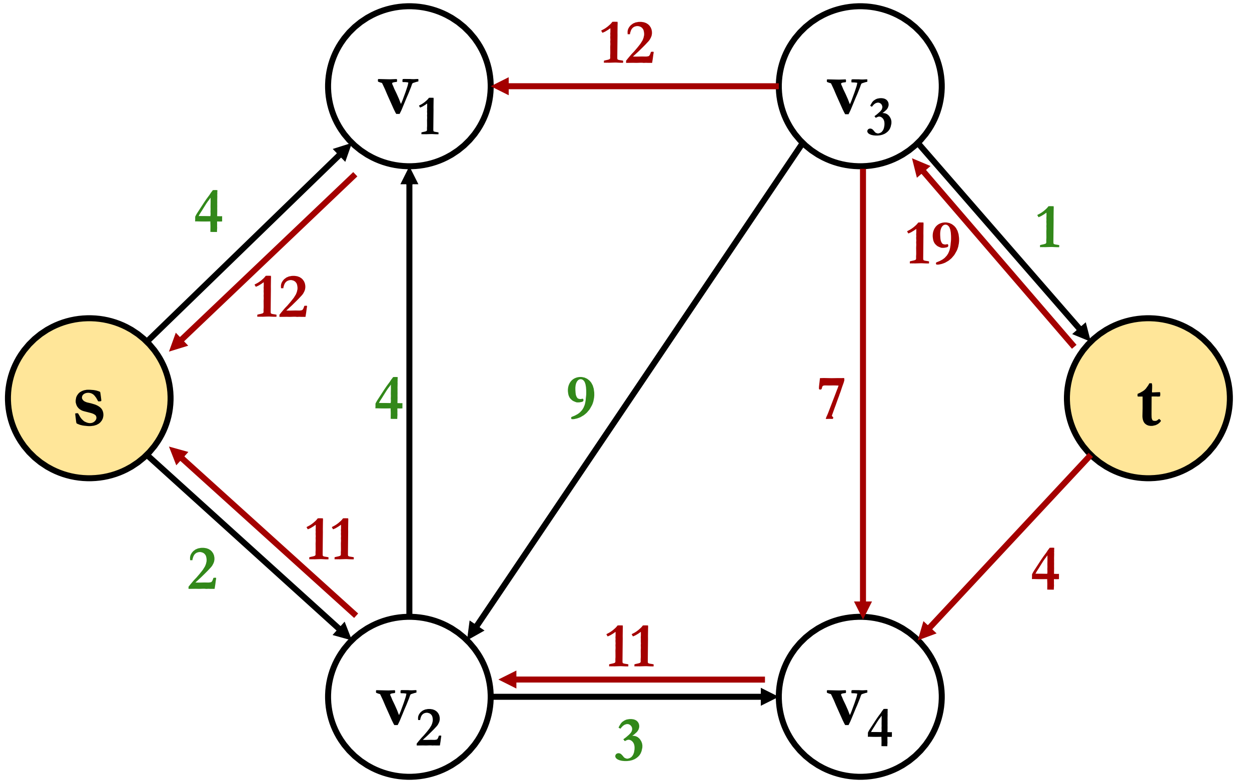
$|f| = 23$



$G = (V, E)$

The Ford-Fulkerson method

$\boxed{\textit{No further augmenting paths}} \Longrightarrow \boxed{\textit{maximum flow } |f| = 23}$



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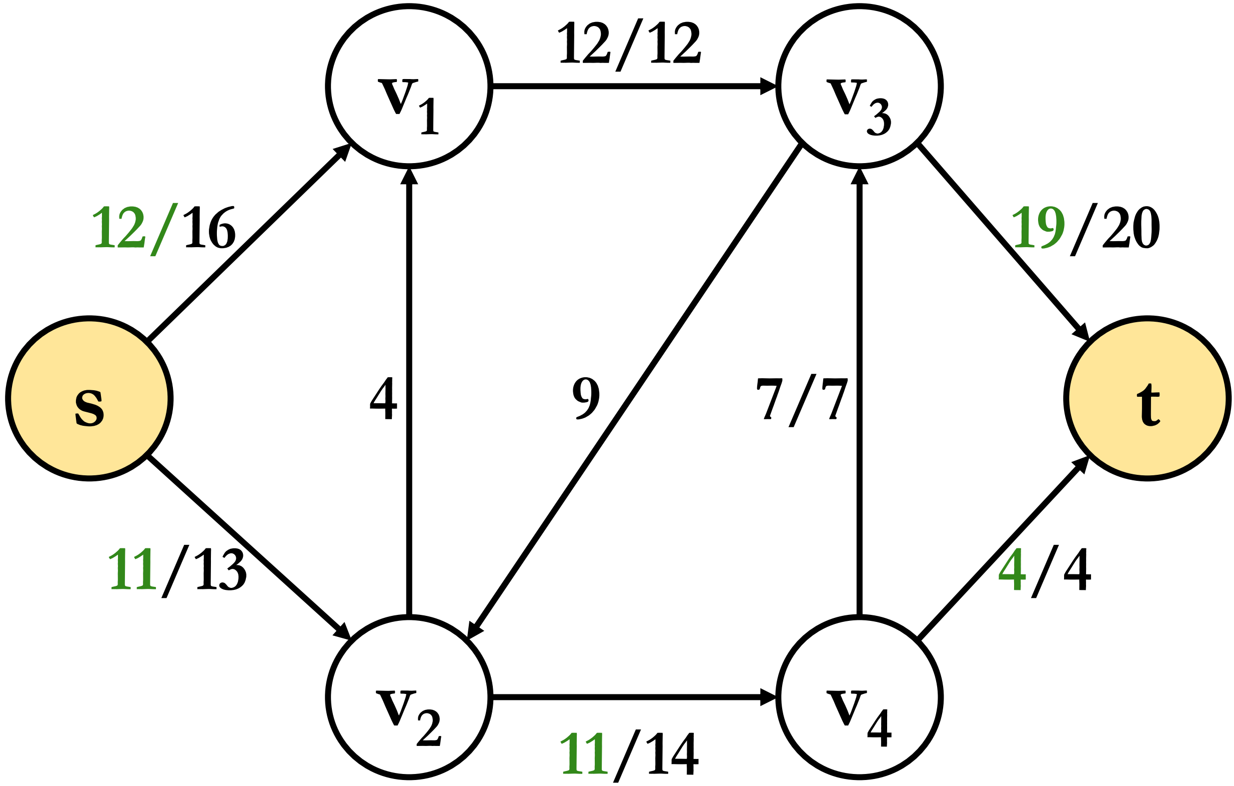


$G = (V, E)$

The Ford-Fulkerson method

```
FORD-FULKERSON(G, s, t)
for each edge (u, v) ∈ G.E
    (u, v).f = 0
while there exists a path p from s to t in the residual network G_f
    c_f(p) = min {c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) ∈ E
            (u, v).f = (u, v).f + c_f(p)
        else (v, u).f = (v, u).f − c_f(p)
```

The Edmonds-Karp algorithm uses BFS to find the shortest augmenting path… with running time $O(VE^2)$

How can we be sure that the Ford-Fulkerson method works?

Max-flow min-cut theorem:
The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

A 'cut' (S, T) of a flow graph $G = (V, E)$

Lemma 26.4

Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.

Corollary 26.5

The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

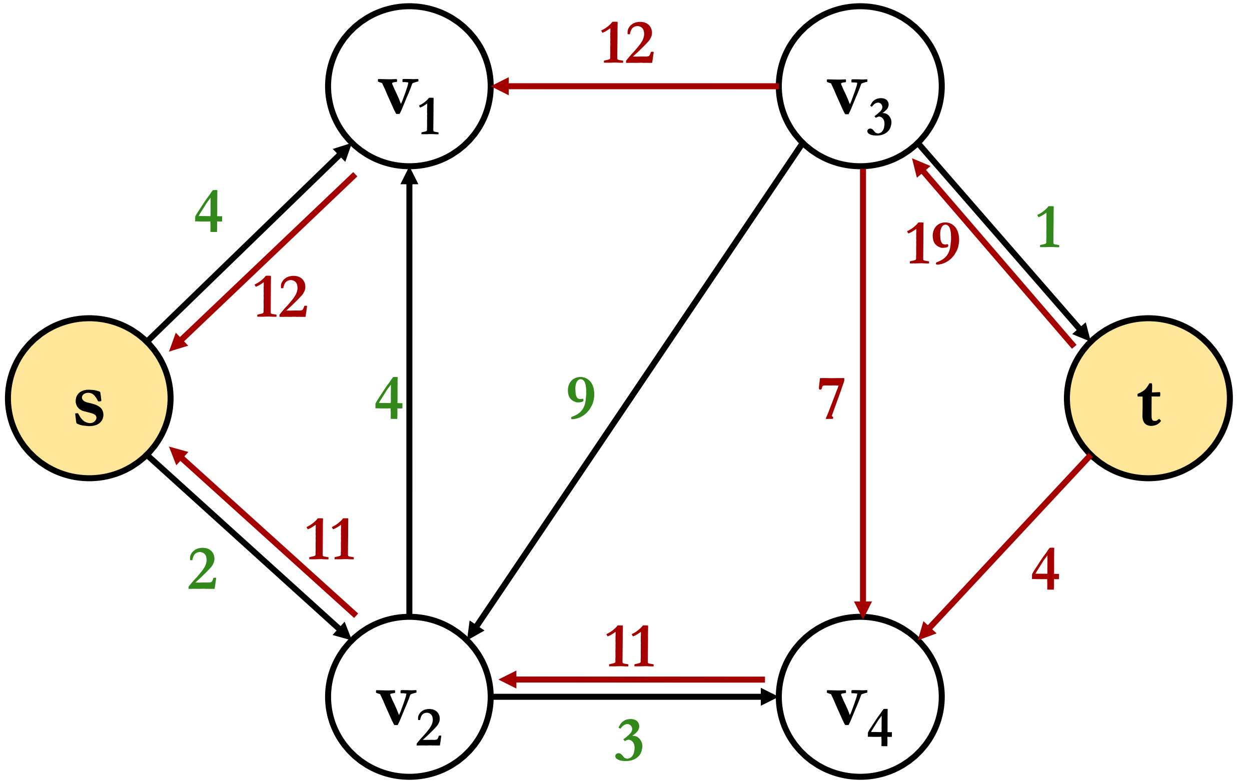
Theorem 26.6 (Max-flow min-cut theorem)

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

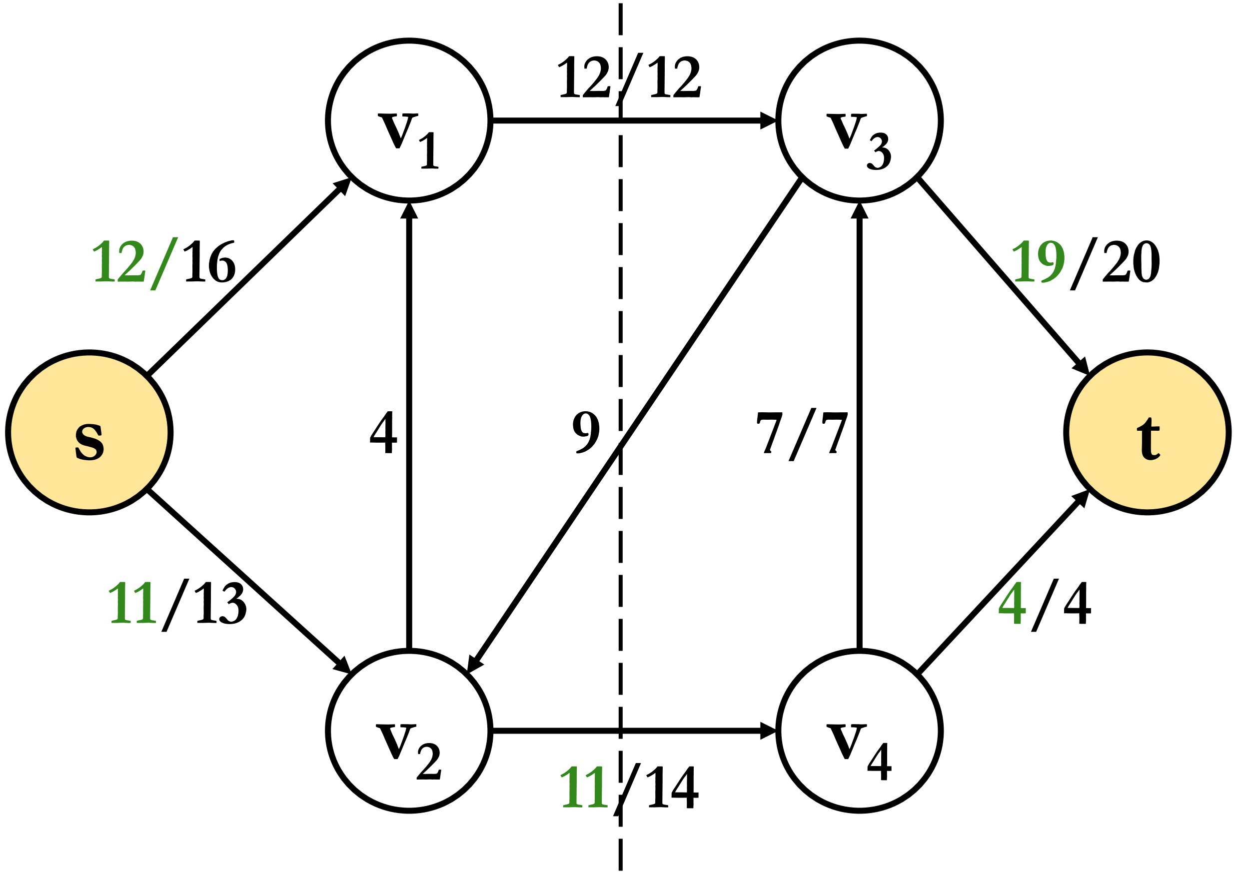
1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

The Ford-Fulkerson method

No further augmenting paths $\Longrightarrow$ $maximum\ flow\ |f| = 23$



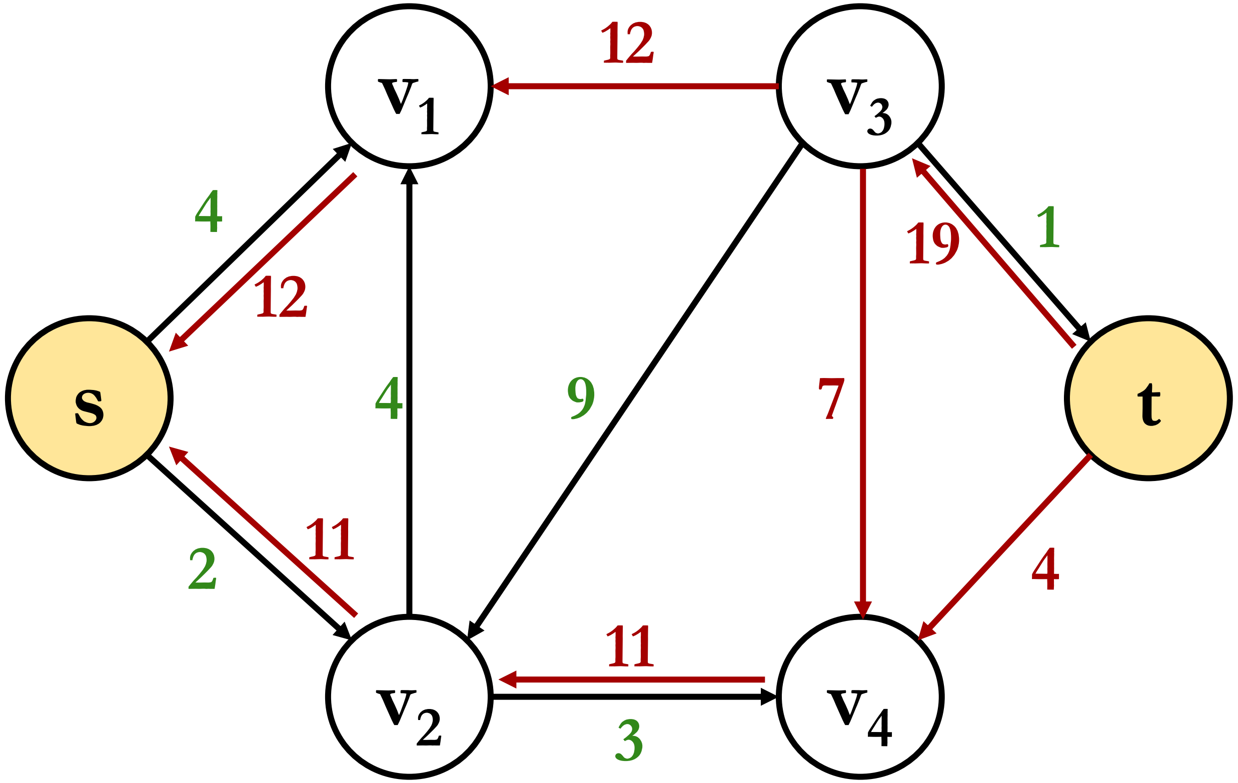
$G_f = (V, E_f)$



$G = (V, E)$

The min-cut max-flow Theorem

t is now unreachable from s in G_f $\Longrightarrow$ maximum flow = capacity of min cut



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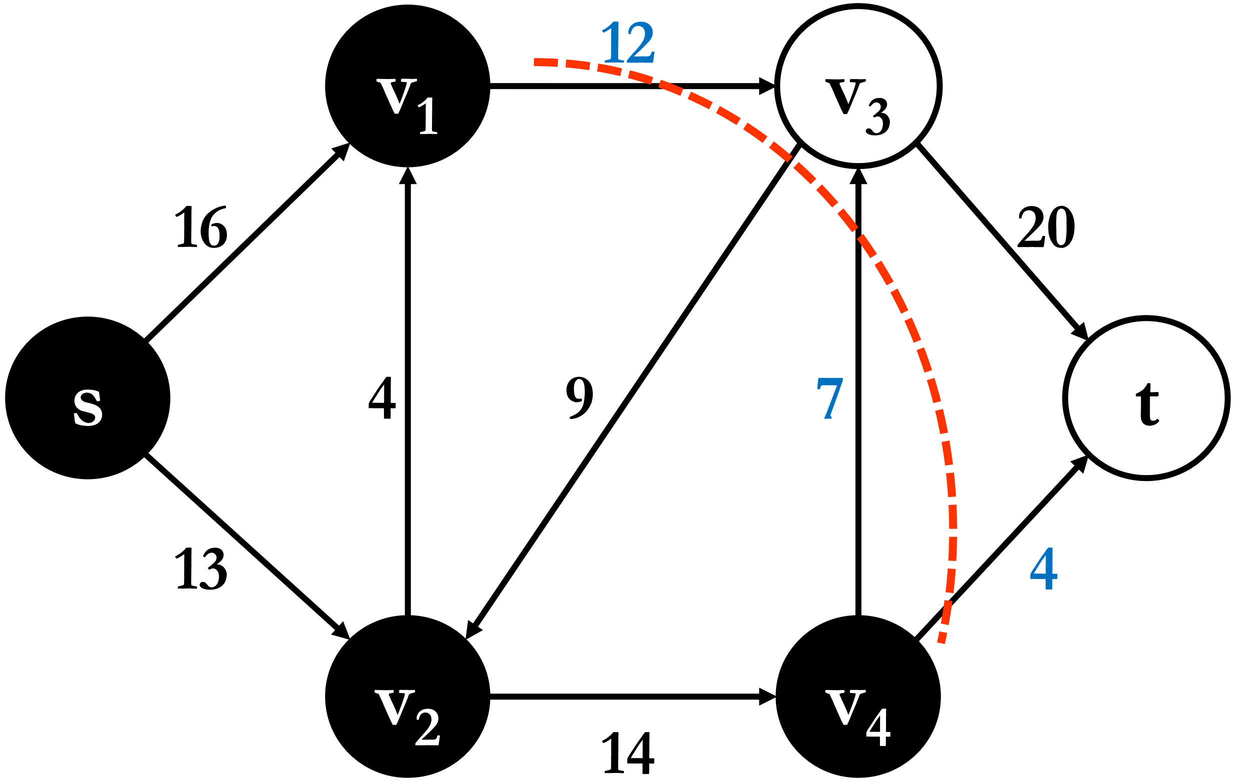
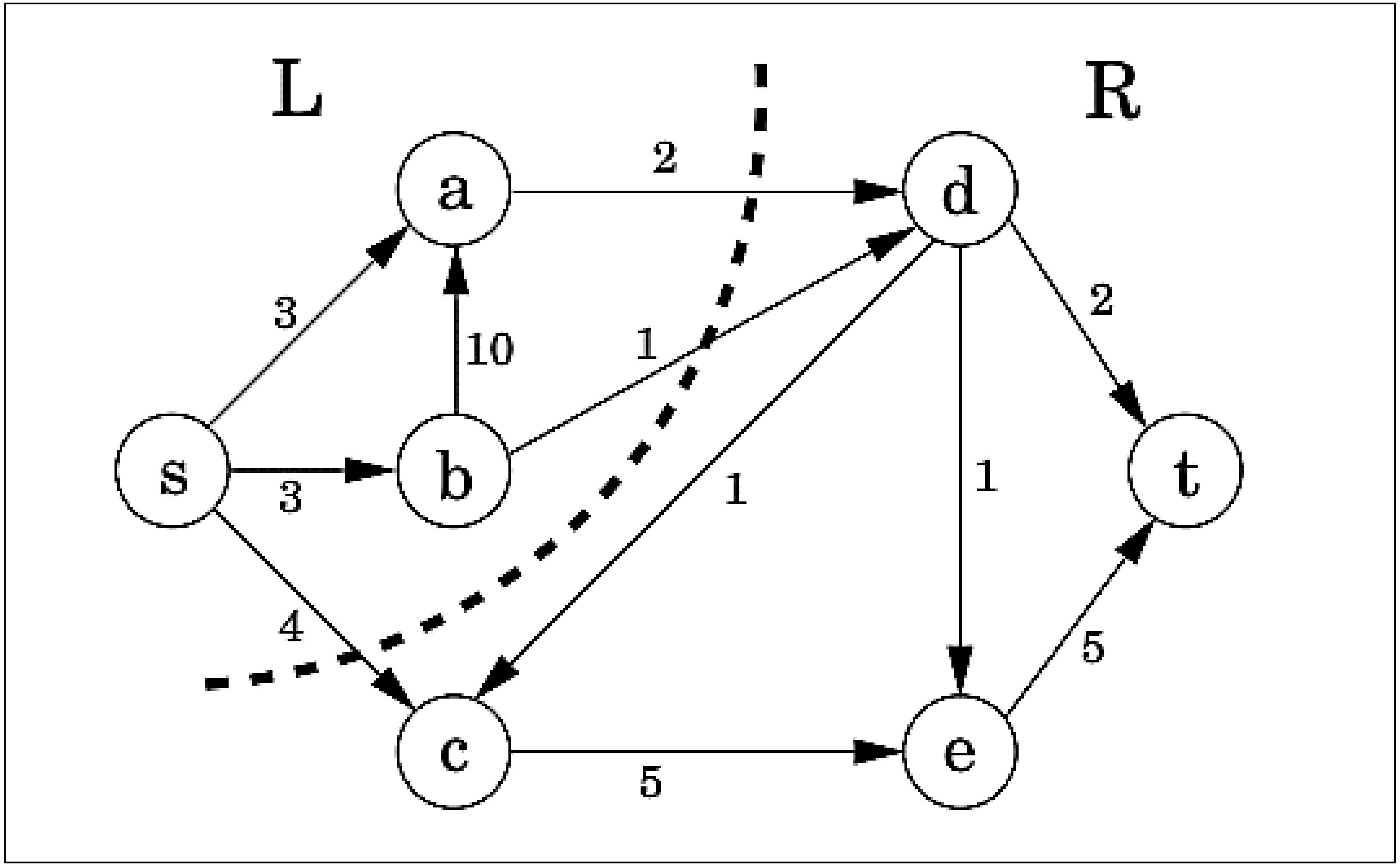


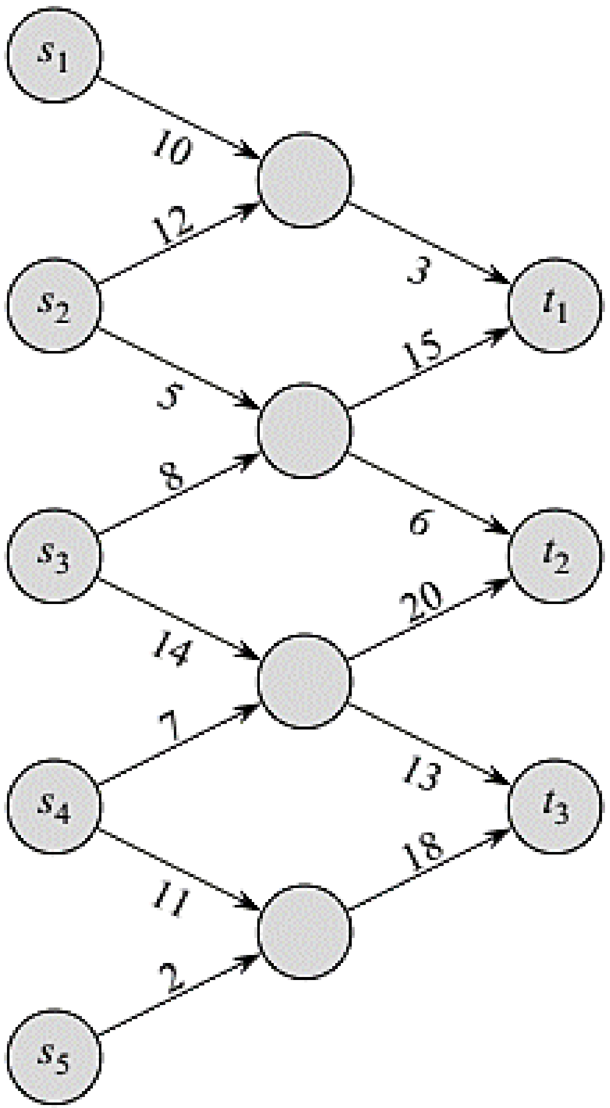
$G = (V, E)$



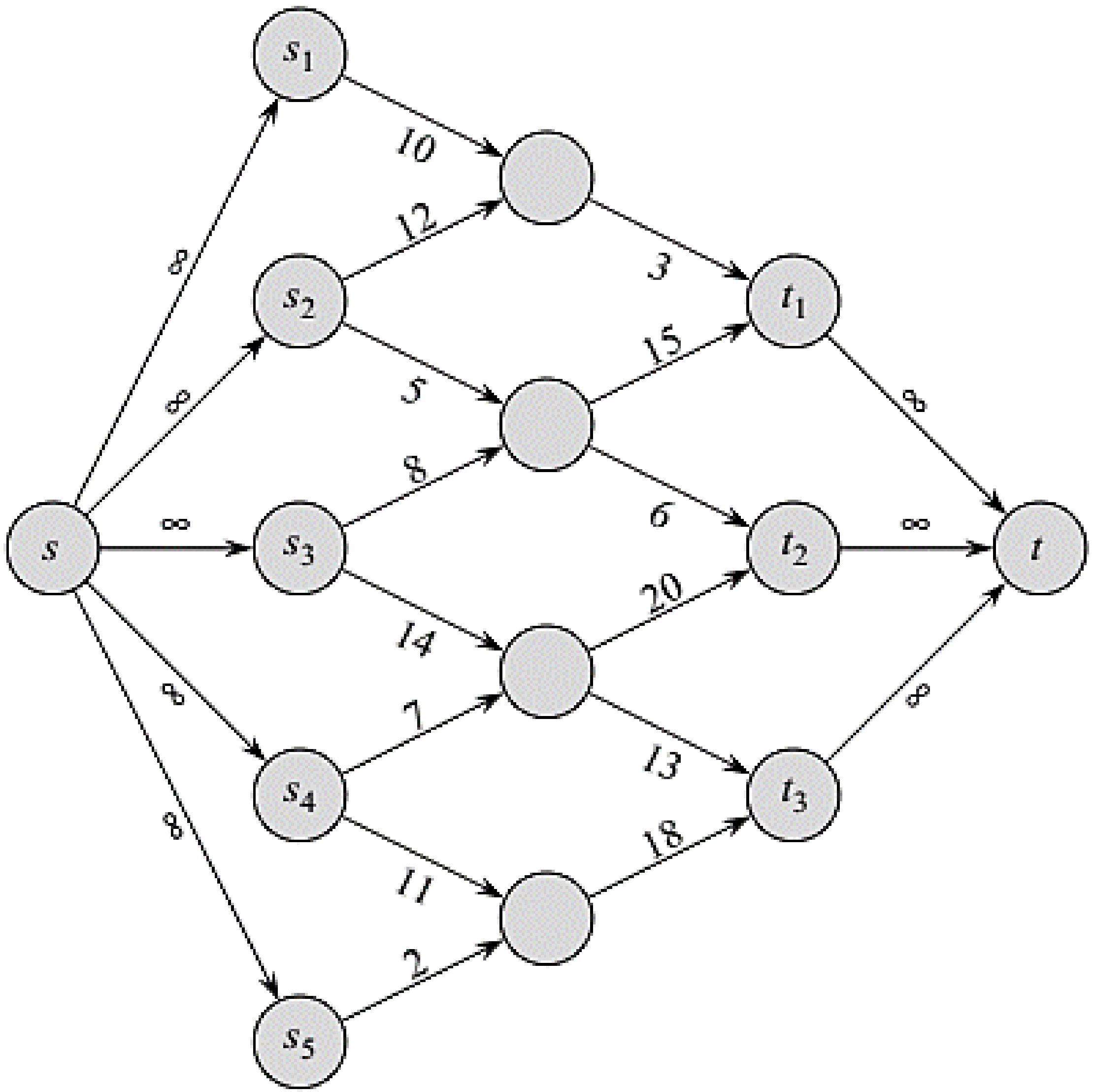
Pick any flow f and any (s,t)-cut (L,R). Then size$(f) \leq$ capacity(L,R).

Source: Dasgupta *et al.*

From *multiple-sources, multiple-sinks* to *single-source, single-sink*



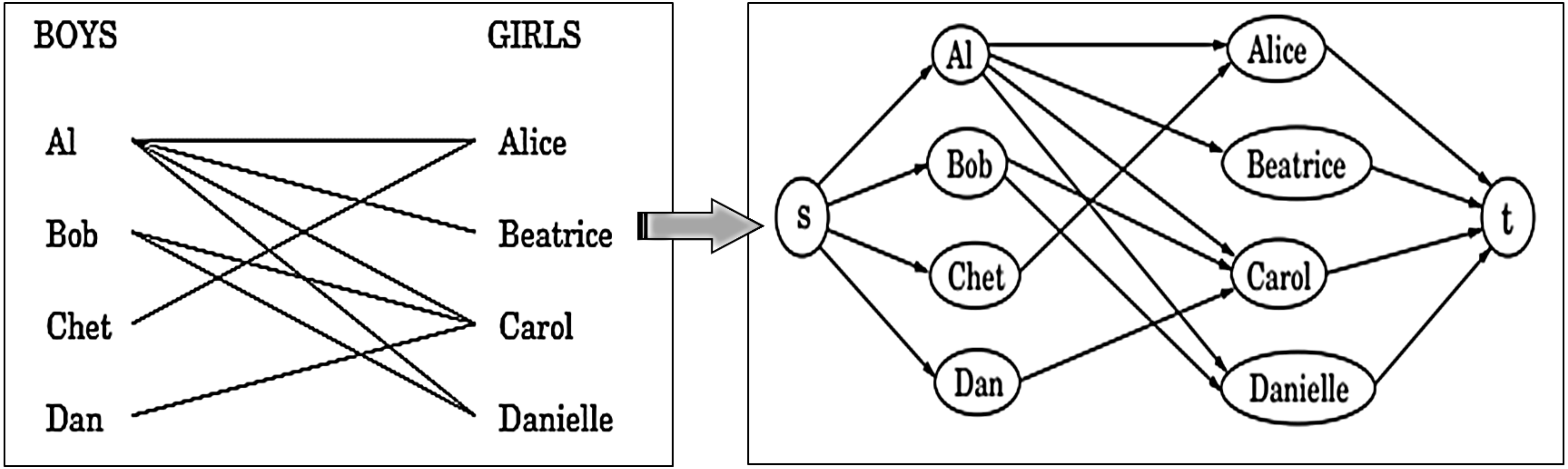
(a)



(b)

Source: Dasgupta *et al.*

Finding Perfect Bipartite Matching



Every edge in the flow network has unit (01) capacity.

A perfect matching exists *iff* the maximum flow $|f_{max}| = |V_1| = |V_2|$

https://www.cs.cmu.edu/~ckingsf/bioinfo-lectures/matching.pdf