Lecture 12

Thursday, February 17, 2022 7:06 AM

VECTOR SPACES:

WE SHALL STATE
SOME PROPERTIES,
IF SATISFIED BY A
SET OF OBJECTS, WILL ENTITLE
THOSE OBJECTS TO BE
CALLED $\boxed{\text{VECTORS}}$.

THESE NEW TYPE OF VECTORS WILL INCLUDE VARIOUS KINDS OF MATRICES AND FUNCTIONS ETC.

DEFINITION: LET $\boxed{V}$ BE A NON-EMPTY SET OF OBJECTS ON WHICH $\boxed{\text{TWO}}$ OPERATIONS ARE DEFINED, ADDITION AND MULTIPLICATION BY SCALARS (REAL NUMBERS).

(VECTOR ADDITION)

BY $\boxed{\text{ADDITION}}$ WE MEAN FOR $\underline{u}, \underline{v} \in V$ AN OBJECT $\underline{u} + \underline{v}$ (SOME AUTHORS WRITE $\underline{u} \oplus \underline{v}$ TO DISTINGUISH FROM ADDITION OF REAL NUMBERS)

BY SCALAR MULTIPLICATION WE MEAN FOR EACH SCALAR k AND EACH OBJECT $\underline{u} \in V$ AN OBJECT $k\underline{u}$, CALLED THE SCALAR MULTIPLE OF $\underline{u}$ BY k. (SOME AUTHORS WRITE $k \odot \underline{u}$ TO DISTINGUISH FROM MULTIPLICATION OF REAL NUMBERS). $\underline{u} + \underline{v} \longrightarrow$ VECTOR ADDITION

$k\underline{u} \longrightarrow$ SCALAR MULTIPLICATION

$\boxed{V}$ IS CALLED A VECTOR SPACE FOR $\underline{u}, \underline{v}, \underline{w} \in V$ AND FOR ALL SCALARS $\boxed{K \text{ AND } L}$ IF THE FOLLOWING $\boxed{10}$ PROPERTIES ARE SATISFIED:

(1) FOR ALL $\underline{u}, \underline{v} \in \boxed{V}$, $\underline{u} + \underline{v} \in \boxed{V}$ WHICH MEANS V IS CLOSED UNDER VECTOR ADDITION.

(2) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ {COMMUTATIVE PROPERTY OF VECTOR ADDITION}.

(3) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ {ASSOCIATIVE PROPERTY OF VECTOR ADDITION}

(4) THERE IS AN OBJECT $\underline{0}$ IN $\boxed{V}$, CALLED A ZERO VECTOR SUCH THAT $\underline{0} + \underline{u} = \underline{u} + \underline{0} = \underline{u}$ FOR ALL $\underline{u} \in \boxed{V}$

$\underline{0} \longrightarrow$ ADDITIVE IDENTITY OR IDENTITY ELEMENT WITH RESPECT TO VECTOR ADDITION.

(5) FOR EACH $\underline{u} \in \boxed{V}$ THERE IS AN OBJECT $-\underline{u} \in \boxed{V}$ CALLED A NEGATIVE OF $\underline{u}$ OR ADDITIVE INVERSE OF $\underline{u}$ SUCH THAT

$$\underline{u} + (-\underline{u}) = (-\underline{u}) + \underline{u} = \underline{0}$$

(6) IF K IS ANY SCALAR AND $\underline{u}$ IS ANY OBJECT IN $\boxed{V}$, THEN $K\underline{u}$ IS IN $\boxed{V}$. WHICH SHOWS THAT V IS CLOSED UNDER SCALAR MULTIPLICATION.

(7) $k(\underline{u}+\underline{v}) = k\underline{u} + k\underline{v}$

(8) $(k+l)\underline{u} = k\underline{u} + l\underline{u}$

(9) $k(l\underline{u}) = (kl)\underline{u}$

(10) $1\underline{u} = \underline{u}$ ONE

DUE TO PROPERTY NO. (4)

V i.e. VECTOR SPACE IS A

NONEMPTY SET.

DEFINITION

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars (numbers). By ***addition*** we mean a rule for associating with each pair of objects $\boldsymbol{u}$ and $\boldsymbol{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\boldsymbol{u}$ and $\boldsymbol{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\boldsymbol{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\boldsymbol{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V **vectors**.

1. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $0+\mathbf{u}=\mathbf{u}+0=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=0$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

EXAMPLE:

PROVE THAT THE SET $\boxed{V}$ OF ALL 2X2 MATRICES (VECTORS) WITH REAL ENTRIES IS A VECTOR SPACE UNDER $\boxed{\text{MATRIX ADDITION}}$ (AS VECTOR ADDITION) AND MATRIX SCALAR MULTIPLICATION (AS SCALAR MULTIPLICATION WITH VECTORS).

NOTE: IN ORDER TO AVOID THE CONFUSION WITH ORDINARY VECTORS USE α, β, γ AS ELEMENTS OF V.

SOLUTION:

LET $\alpha = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \beta = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

(1) $\alpha + \beta = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} \in V$

$\because$ $\alpha + \beta$ IS ALSO A 2X2 MATRIX

(2) $\boxed{\alpha+\beta} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$

$= \begin{bmatrix} b_1+a_1 & b_2+a_2 \\ b_3+a_3 & b_4+a_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} +$

$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \boxed{\beta + \alpha}$

(3) $\alpha + (\beta+\gamma) = (\alpha+\beta)+\gamma$ OBVIOUS

CHECK THIS BY TAKING $\gamma = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

(4) $\underline{0}+\alpha = \alpha + \underline{0} = \alpha$ OBVIOUS SINCE

$\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \underline{0}+\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha + \underline{0} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$= \alpha$

(5) $\alpha + (-\alpha) = (-\alpha)+\alpha = \underline{0} \rightarrow (*)$

FOR $-\alpha = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix}$ $(*)$ IS SATISFIED

$\hookrightarrow$ 2X2 MATRIX

(6) $k\alpha = k \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \in V$

(7) $k(\alpha+\beta) = k\alpha + k\beta$ OBVIOUS

CHECK

(8) $(k+l)\alpha = k\alpha + l\alpha$ OBVIOUS

CHECK

(9) $k(l\alpha) = k\begin{bmatrix} la_1 & la_2 \\ la_3 & la_4 \end{bmatrix}$

$$= (kl)\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (kl)\alpha$$

(10) $1\cdot\alpha = \alpha$ (OBVIOUS)

TRY THE FOLLOWING:

DETERMINE WHETHER THE SET OF ALL 2X2 MATRICES OF THE FORM $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ WITH MATRIX ADDITION AND SCALAR MULTIPLICATION IS A VECTOR SPACE.

ANSWER

NOT A VECTOR SPACE, AXIOM (1) FAILS, NO NEED TO CHECK THE REST.

EXAMPLE 4 A Vector Space of Real-Valued Functions

Let V be the set of real-valued functions defined on the entire real line $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two such functions and k is any real number, define the sum function $\mathbf{f} + \mathbf{g}$ and the scalar multiple $k\mathbf{f}$, respectively, by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \quad \text{and} \quad (k\mathbf{f})(x) = kf(x)$$

In other words, the value of the function $\mathbf{f} + \mathbf{g}$ at x is obtained by adding together the values of $\mathbf{f}$ and $\mathbf{g}$ at x (Figure 5.1.1*a*). Similarly, the value of $k\mathbf{f}$ at x is k times the value of $\mathbf{f}$ at x (Figure 5.1.1*b*). In the exercises we shall ask you to show that V is a vector space with respect to these operations. This vector space is denoted by $F(-\infty, \infty)$. If $\mathbf{f}$ and $\mathbf{g}$ are vectors in this space, then to say that $\mathbf{f} = \mathbf{g}$ is equivalent to saying that $f(x) = g(x)$ for all x in the interval $(-\infty, \infty)$.

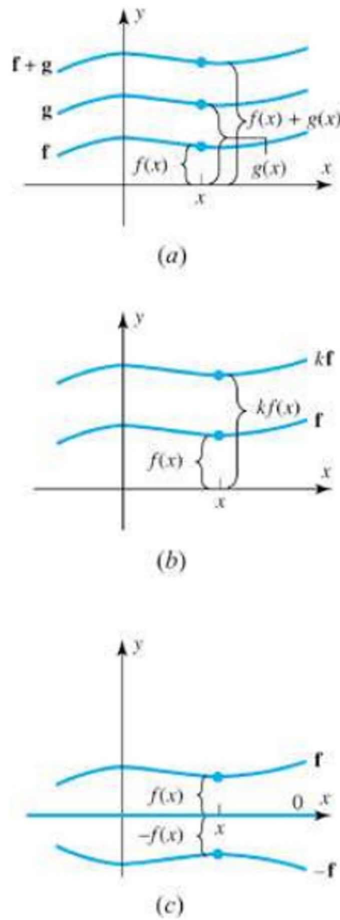

Figure 5.1.1

The vector $\mathbf{0}$ in $F(-\infty, \infty)$ is the constant function that is identically zero for all values of x. The graph of this function is the line that coincides with the x-axis. The negative of a vector $\mathbf{f}$ is the function $-\mathbf{f} = -f(x)$. Geometrically, the graph of $-\mathbf{f}$ is the reflection of the graph of $\mathbf{f}$ across the x-axis (Figure 5.1.1*c*).

Remark In the preceding example we focused on the interval $(-\infty, \infty)$. Had we restricted our attention to some closed interval $[a, b]$ or some open interval (a, b), the functions defined on those intervals with the operations stated in the example would also have produced vector spaces. Those vector spaces are denoted by $F[a, b]$ and $F(a, b)$, respectively.

EXAMPLE 6 Every Plane through the Origin Is a Vector Space

Let V be any plane through the origin in R^3. We shall show that the points in V form a vector space under the standard addition and scalar multiplication operations for vectors in R^3. From Example 1, we know that R^3 itself is a vector space under these operations. Thus Axioms 2, 3, 7, 8, 9, and 10 hold for all points in R^3 and consequently for all points in the plane V. We therefore need only show that Axioms 1, 4, 5, and 6 are satisfied.

Since the plane V passes through the origin, it has an equation of the form

$$ax + by + cz = 0 \tag{1}$$

(Theorem 3.5.1). Thus, if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are points in V, then $au_1 + bu_2 + cu_3 = 0$ and $av_1 + bv_2 + cv_3 = 0$. Additing these equations gives

$$a(u_1 + v_1) + b(u_2 + v_2) + c(u_3 + v_3) = 0$$

This equality tells us that the coordinates of the point

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

satisfy 1; thus $\mathbf{u} + \mathbf{v}$ lies in the plane V. This proves that Axiom 1 is satisfied. The verifications of Axioms 4 and 6 are left as exercises; however, we shall prove that Axiom 5 is satisfied. Multiplying $au_1 + bu_2 + cu_3 = 0$ through by -1 gives

$$a(-u_1) + b(-u_2) + c(-u_3) = 0$$

Thus $-\mathbf{u} = (-u_1, -u_2, -u_3)$ lies in V. This establishes Axiom 5.

Our final example will be an unusual vector space that we have included to illustrate how varied vector spaces can be. Since the objects in this space will be real numbers, it will be important for you to keep track of which operations are intended as vector operations and which ones as ordinary operations on real numbers.

EXAMPLE 8 An Unusual Vector Space

Let V be the set of positive real numbers, and define the operations on V to be

$$u + v = uv \quad [\text{Vector addition is numerical multiplication.}]$$

$$ku = u^k \quad [\text{Scalar multiplication is numerical exponentiation.}]$$

Thus, for example, $1 + 1 = 1$ and $(2)(1) = 1^2 = 1$—strange indeed, but nevertheless the set V with these operations satisfies the 10 vector space axioms and hence is a vector space. We will confirm Axioms 4, 5, and 7, and leave the others as exercises.

- Axiom 4—The zero vector in this space is the number 1 (i.e., $\mathbf{0} = 1$) since
$$u + 1 = u \cdot 1 = u$$
- Axiom 5—The negative of a vector u is its reciprocal (i.e., $-u = 1/u$) since
$$u + \frac{1}{u} = u\left(\frac{1}{u}\right) = 1 (= \mathbf{0})$$
- Axiom 7—$k(u + v) = (uv)^k = u^k v^k = (ku) + (kv)$

THEOREM 5.1.1

Let V be a vector space, $\mathbf{u}$ *a vector in V, and k a scalar; then:*

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) *If* $k\mathbf{u} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{u} = \mathbf{0}$.

We shall prove parts (a) and (c) and leave proofs of the remaining parts as exercises.

Proof (a) We can write

$$
\begin{aligned}
0\mathbf{u} + 0\mathbf{u} &= (0+0)\mathbf{u} && \text{[Axiom 8]} \\
&= 0\mathbf{u} && \text{[Property of the number 0]}
\end{aligned}
$$

By Axiom 5 the vector $0\mathbf{u}$ has a negative, $-0\mathbf{u}$. Adding this negative to both sides above yields

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = 0\mathbf{u} + (-0\mathbf{u})$$

or

$$
\begin{aligned}
0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] &= 0\mathbf{u} + (-0\mathbf{u}) && \text{[Axiom 3]} \\
0\mathbf{u} + \mathbf{0} &= \mathbf{0} && \text{[Axiom 5]} \\
0\mathbf{u} &= \mathbf{0} && \text{[Axiom 4]}
\end{aligned}
$$

■

Proof (c) To show that $(-1)\mathbf{u} = -\mathbf{u}$, we must demonstrate that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$. To see this, observe that

$$
\begin{aligned}
u + (-1)u &= 1u + (-1)u && \text{[Axiom10]} \\
&= (1 + (-1))u && \text{[Axiom 8]} \\
&= 0\mathbf{u} && \text{[Property of numbers]} \\
&= 0 && \text{[Part (a) above]}
\end{aligned}
$$

■

Exercise set 5.1
Q5, Q6, Q11, Q 17, Q 19, Q25, Q26, Q27, Q28, Q 29, Q31, Q32, Q33