Important Facts and Theorems 5

Tuesday, March 1, 2022 6:47 AM

DEFINITION

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a nonempty set of vectors, then the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

has at least one solution, namely

$$k_1 = 0, \qquad k_2 = 0, \dots, \qquad k_r = 0$$

If this is the only solution, then S is called a ***linearly independent*** set. If there are other solutions, then S is called a ***linearly dependent*** set.

EXAMPLE 1 A Linearly Dependent Set

If $\mathbf{v}_1 = (2, -1, 0, 3)$, $\mathbf{v}_2 = (1, 2, 5, -1)$, and $\mathbf{v}_3 = (7, -1, 5, 8)$, then the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, since $3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

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EXAMPLE 2 A Linearly Dependent Set

The polynomials

$$\mathbf{p}_1 = 1 - x, \qquad \mathbf{p}_2 = 5 + 3x - 2x^2, \qquad \text{and} \qquad \mathbf{p}_3 = 1 + 3x - x^2$$

form a linearly dependent set in P_2 since $3\mathbf{p}_1 - \mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{0}$.

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EXAMPLE 4 **Determining Linear Independence/Dependence**

Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \qquad \mathbf{v}_2 = (5, 6, -1), \qquad \mathbf{v}_3 = (3, 2, 1)$$

form a linearly dependent set or a linearly independent set.

Solution

In terms of components, the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

becomes

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

or, equivalently,

$$(k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) = (0, 0, 0)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

Thus $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly dependent set if this system has a nontrivial solution, or a linearly independent set if it has only the trivial solution. Solving this system using Gaussian elimination yields

$$k_1 = -\frac{1}{2}t, \qquad k_2 = -\frac{1}{2}t, \qquad k_3 = t$$

Thus the system has nontrivial solutions and $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2$ form a linearly dependent set. Alternatively, we could show the existence of nontrivial solutions without solving the system by showing that the coefficient matrix has determinant zero and consequently is not invertible (verify).

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THEOREM 5.3.1

A set S with two or more vectors is

(a) *Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

(b) *Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.*

We shall prove part (*a*) and leave the proof of part (*b*) as an exercise.

Proof (a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set with two or more vectors. If we assume that S is linearly dependent, then there are scalars $k_1, k_2, \ldots, k_r$, not all zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0} \tag{2}$$

To be specific, suppose that $k_1 \neq 0$. Then 2 can be rewritten as

$$\mathbf{v}_1 = \left(-\frac{k_2}{k_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\mathbf{v}_r$$

which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S. Similarly, if $k_j \neq 0$ in 2 for some $j = 2, 3, \ldots, r$, then $\mathbf{v}_j$ is expressible as a linear combination of the other vectors in S.
Conversely, let us assume that at least one of the vectors in S is expressible as a linear combination of the other vectors. To be specific, suppose that

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_r\mathbf{v}_r$$

so

$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \cdots - c_r\mathbf{v}_r = \mathbf{0}$$

It follows that S is linearly dependent since the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

is satisfied by

$$k_1 = 1, \qquad k_2 = -c_2, \ldots, \qquad k_r = -c_r$$

which are not all zero. The proof in the case where some vector other than $\mathbf{v}_1$ is expressible as a linear combination of the other vectors in S is similar.

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THEOREM 5.3.2

(a) *A finite set of vectors that contains the zero vector is linearly dependent.*

(b) *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

We shall prove part (*a*) and leave the proof of part (*b*) as an exercise.

Proof (a) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly dependent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero.

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EXAMPLE 8 **Using Theorem 5.3.2*b***

The functions $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ form a linearly independent set of vectors in $F(-\infty, \infty)$, since neither function is a constant multiple of the other.

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THEOREM 5.3.3

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

Proof Suppose that

$$\begin{aligned}
\mathbf{v}_1 &= (v_{11}, v_{12}, \ldots, v_{1n}) \\
\mathbf{v}_2 &= (v_{21}, v_{22}, \ldots, v_{2n}) \\
&\vdots \\
\mathbf{v}_r &= (v_{r1}, v_{r2}, \ldots, v_{rn})
\end{aligned}$$

Consider the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

If, as illustrated in Example 4, we express both sides of this equation in terms of components and then equate corresponding components, we obtain the system

$$\begin{aligned}
v_{11}k_1 + v_{21}k_2 + \cdots + v_{r1}k_r &= 0 \\
v_{12}k_1 + v_{22}k_2 + \cdots + v_{r2}k_r &= 0 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\ \ \vdots \\
v_{1n}k_1 + v_{2n}k_2 + \cdots + v_{rn}k_r &= 0
\end{aligned}$$

This is a homogeneous system of *n* equations in the *r* unknowns $k_1, \ldots, k_r$. Since $r > n$, it follows from Theorem 1.2.1 that the system has nontrivial solutions. Therefore, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set. ■

Exercise set 5.3
Q 10, Q 11, Q 12, Q 13, Q 14, Q 15, Q 16, Q 18, Q 24, Q 25