Linear Algebra – Math 205
Exercise Set of Lect 7 & 8
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Exercise Set 2.3 Solution

Question 03

By inspection, explain why $\det(A) = 0$.

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

Solution: We observe that $R_1 + R_2 = R_3$, hence two rows are identical then $\det(A) = 0$.

Question 04

Use Theorem 2.3.3 to determine which of the following matrices are invertible.

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$

Solution: (a) Determinant of a matrix is 54, Hence according to Theorem 2.3.3, *A square matrix A is invertible if and only if* $det(A) \neq 0$, it is invertible.

(b) Determinant of a matrix is 0, Hence it is not invertible. One can see by doing $C_3 := C_3 - C_1$; $\begin{bmatrix} 4 & 2 & 4 \\ -2 & 1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$. Hence two columns are same then determinant is zero.

(c) Determinant of a matrix is 0, Hence it is not invertible. One can see column(C_3) is entirely zero, Hence Determinant is zero.

Question 05

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Assuming that $\det(A) = -7$,
find
(a) $\det(3A)$
Solution: By $det(kA) = k^n det(A)$,

$$\det(3A) = 3^3 \det(A) = (27)(-7) = -189$$

(b) $\det\left(A^{-1}\right)$
Solution: By $det(A^{-1}) = \frac{1}{det(A)}$, So $\det(A^{-1}) = -1/7$

(c) $\det\left(2A^{-1}\right)$
Solution:

$$\det\left(2A^{-1}\right) = \frac{2^3}{\det(A)} = -\frac{8}{7}$$

(d) $\det\left((2A)^{-1}\right)$
Solution:

$$\det\left((2A)^{-1}\right) = \frac{1}{\det(2A)} = \frac{1}{2^3\det(A)} = -\frac{1}{8*7} = -\frac{1}{54}$$

(e) $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$
Solution:

$$\begin{aligned} \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} &= -\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} && \text{Interchange Columns 2 and 3.} \\ &= -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}. && \text{Take the transpose of the matrix.} \\ &= 7 \end{aligned}$$

Question 06

Without directly evaluating, show that $x = 0$ and $x = 2$ satisfy

$$\begin{vmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 0$$

Solution: First let $x = 0$, $\begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix}$. Hence Row 1 is a multiple of Row 3; $R_3 = \frac{-5}{2}R_1$.

Now let $x = 2$, $\begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix}$. Hence Row 1 is a multiple of Row 2; $R_1 = 2R_2$.

Question 07

Without directly evaluating, show that

$$\det\begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$$

Solution: If we replace Row 1 by Row 1 plus Row 2 , we obtain

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

because the first and third rows are proportional.

Question 08

In Exercises prove the identity without evaluating the determinants.
$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solution: Subtracting C_1 into C_3 and C_2 into C_3 , we will reach the right hand matrix.

Question 09

$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solution: Adding C_2 into C_1 and then taking 2 common in C_1 and then subtract C_1 into C_2 and then take -1 common in C_1, we will have right hand matrix.

Question 11

$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Solution: First $C_2 := C_2 - (t * C_1)$
then $C_3 := C_3 - (s * C_1) - (r * C_2)$

Question 13

Use Theorem 2.3.3 to show that

$$\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$$

is not invertible for any values of α, β, and γ.

Solution: By adding Row 1 to Row 2 and using the identity $\sin^2 x + \cos^2 x = 1$, we see that the determinant of the given matrix can be written as

$$\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

But this is zero because two of its rows are identical. Therefore the matrix is not invertible.

Question 16

Let A and B be $n \times n$ matrices. Show that if A is invertible, then $\det(B) = \det\left(A^{-1}BA\right)$.

Solution: Using *Theorem 2.3.4 and 2.3.5*
$det(A^{-1}BA) = det(A^{-1})det(B)det(A) = \frac{1}{det(A)}det(B)det(A) = det(B)$

Question 21

Let A and B be $n \times n$ matrices. You know from earlier work that AB is invertible if A and B are invertible. What can you say about the invertibility of AB if one or both of the factors are singular? Explain your reasoning.

Solution: If either A or B is singular, then either $\det(A)$ or $\det(B)$ is zero. Hence, $\det(AB) = \det(A)\,\det(B) = 0$. Thus AB is also singular.