Lecture 11

Tuesday, February 15, 2022 12:35 AM

4.1 EUCLIDEAN n-SPACE

Although our geometric visualization does not extend beyond 3-space, it is nevertheless possible to extend many familiar ideas beyond 3-space by working with analytic or numerical properties of points and vectors rather than the geometric properties. In this section we shall make these ideas more precise.

Vectors in n-Space

We begin with a definition.

DEFINITION

If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by R^n.

Two tuple (ordered pair)
e.g. $(2,3)$, $(5.1, 3.6)$

Three tuple $(1,1,5)$, $(3.1, 4, \sqrt{2})$

e.t.c

When $n = 2$ or 3, it is customary to use the terms ***ordered pair*** and ***ordered triple***, respectively, rather than *ordered 2-tuple* and *ordered 3-tuple*. When $n = 1$, each ordered n-tuple consists of one real number, so R^1 may be viewed as the set of real numbers. It is usual to write R rather than R^1 for this set.

It might have occurred to you in the study of 3-space that the symbol (a_1, a_2, a_3) has two different geometric interpretations (1): it can be interpreted as a point, in which case a_1, a_2, and a_3 are the coordinates (Figure 4.1.1a); (2) or it can be interpreted as a vector, in which case a_1, a_2, and a_3 are the components (Figure 4.1.1b). It follows, therefore, that an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ can be viewed either as a "generalized point" or as a "generalized vector"—the distinction is mathematically unimportant. Thus we can describe the 5-tuple $(-2, 4, 0, 1, 6)$ either as a point in R^5 or as a vector in R^5.

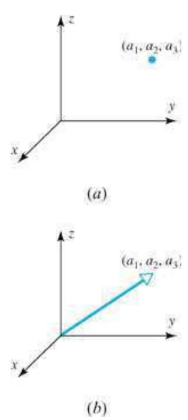

Figure 4.1.1 The ordered triple (a_1, a_2, a_3) can be interpreted geometrically as a point or as a vector.

DEFINITION

Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n are called ***equal*** if

$$u_1 = v_1, \quad u_2 = v_2, \ldots, \quad u_n = v_n$$

The ***sum*** $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

and if k is any scalar, the ***scalar multiple*** $k\mathbf{u}$ is defined by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

To be equal ALL corresponding coordinates must be equal!

The operations of addition and scalar multiplication in this definition are called the ***standard operations*** on R^n.

The ***zero vector*** in R^n is denoted by $\mathbf{0}$ and is defined to be the vector

$$\mathbf{0} = (0, 0, \ldots, 0)$$

We will use $\vec{0}$

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is any vector in R^n, then the ***negative*** (or ***additive inverse***) of $\mathbf{u}$ is denoted by $-\mathbf{u}$ and is defined by

$$-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$$

The ***difference*** of vectors in R^n is defined by

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u})$$

Definition!

or, in terms of components,

$$\mathbf{v} - \mathbf{u} = (v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n)$$

Read on your own

Some Examples of Vectors in Higher-Dimensional Spaces

- **Experimental Data** A scientist performs an experiment and makes n numerical measurements each time the experiment is performed. The result of each experiment can be regarded as a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in R^n in which $y_1, y_2, \ldots, y_n$ are the measured values.

- **Storage and Warehousing** A national trucking company has 15 depots for storing and servicing its trucks. At each point in time the distribution of trucks in the service depots can be described by a 15-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_{15})$ in which x_1 is the number of trucks in the first depot, x_2 is the number in the second depot, and so forth.

- **Electrical Circuits** A certain kind of processing chip is designed to receive four input voltages and produces three output voltages in response. The input voltages can be regarded as vectors in R^4 and the output voltages as vectors in R^3. Thus, the chip can be viewed as a device that transforms each input vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in R^4 into some output vector $\mathbf{w} = (w_1, w_2, w_3)$ in R^3.

- **Graphical Images** One way in which color images are created on computer screens is by assigning each pixel (an addressable point on the screen) three numbers that describe the ***hue, saturation***, and ***brightness*** of the pixel. Thus, a complete color image can be viewed as a set of 5-tuples of the form $\mathbf{v} = (x, y, h, s, b)$ in which x and y are the screen coordinates of a pixel and h, s, and b are its hue, saturation, and brightness.

- **Economics** Our approach to economic analysis is to divide an economy into sectors (manufacturing, services, utilities, and so forth) and to measure the output of each sector by a dollar value. Thus, in an economy with 10 sectors the economic output of the entire economy can be represented by a 10-tuple $\mathbf{s} = (s_1, s_2, \ldots, s_{10})$ in which the numbers $s_1, s_1, \ldots, s_{10}$ are the outputs of the individual sectors.

- **Mechanical Systems** Suppose that six particles move along the same coordinate line so that at time t their coordinates are $x_1, x_2, \ldots, x_6$ and their velocities are $v_1, v_2, \ldots, v_6$, respectively. This information can be represented by the vector

$$\mathbf{v} = (x_1, x_2, x_3, x_4, x_5, x_6, v_1, v_2, v_3, v_4, v_5, v_6, t)$$

in R^{13}. This vector is called the ***state*** of the particle system at time t.

- **Physics** In string theory the smallest, indivisible components of the Universe are not particles but loops that behave like vibrating strings. Whereas Einstein's space-time universe was four-dimensional, strings reside in an 11-dimensional world.

Properties of Vector Operations in n-Space

Properties of Vector Operations in n-Space

The most important arithmetic properties of addition and scalar multiplication of vectors in R^n are listed in the following theorem. The proofs are all easy and are left as exercises.

THEOREM 4.1.1

Properties of Vectors in R^n

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, *and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ *are vectors in* R^n *and k and m are scalars, then:*

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$; *that is,* $\mathbf{u} - \mathbf{u} = \mathbf{0}$

(e) $k(m\mathbf{u}) = (km)\mathbf{u}$

(f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(g) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

E.g. Prove (a) Let $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots, v_n)$

$\Rightarrow \vec{u} + \vec{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$

$= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ (by definition above)

$= (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$ (addition is commutative in $\mathbb{R}$)

$= (v_1, v_2, \ldots, v_n) + (u_1, u_2, \ldots, u_n)$ (definition again)

$= \boxed{\vec{v} + \vec{u}}$

$\therefore \vec{u} + \vec{v} = \vec{v} + \vec{u}$

(h) $1\mathbf{u} = \mathbf{u}$

This theorem enables us to manipulate vectors in R^n without expressing the vectors in terms of components. For example, to solve the vector equation $\mathbf{x} + \mathbf{u} = \mathbf{v}$ for $\mathbf{x}$, we can add $-\mathbf{u}$ to both sides and proceed as follows:

showing that this is true here

$$(\mathbf{x}+\mathbf{u}) + (-\mathbf{u}) = \mathbf{v} + (-\mathbf{u})$$
$$\mathbf{x} + (\mathbf{u}-\mathbf{u}) = \mathbf{v} - \mathbf{u}$$
$$\mathbf{x} + \mathbf{0} = \mathbf{v} - \mathbf{u}$$
$$\mathbf{x} = \mathbf{v} - \mathbf{u}$$

The reader will find it instructive to name the parts of Theorem 4.1.1 that justify the last three steps in this computation.

Euclidean n-Space

To extend the notions of distance, norm, and angle to R^n, we begin with the following generalization of the dot product on R^2 and R^3 [Formulas 3 and 4 of Section 3.3].

DEFINITION

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are any vectors in R^n, then the ***Euclidean inner product*** $\mathbf{u}\cdot\mathbf{v}$ is defined by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Multiply coordinate by coordinate and then add.
Like dot product.

Observe that when $n = 2$ or 3, the Euclidean inner product is the ordinary dot product.

EXAMPLE 1 Inner Product of Vectors in R^4

The Euclidean inner product of the vectors

$$\mathbf{u} = (-1, 3, 5, 7) \quad \text{and} \quad \mathbf{v} = (5, -4, 7, 0)$$

in R^4 is

$$\mathbf{u}\cdot\mathbf{v} = (-1)(5) + (3)(-4) + (5)(7) + (7)(0) = 18$$

◆

Since so many of the familiar ideas from 2-space and 3-space carry over to n-space, it is common to refer to R^n, with the operations of addition, scalar multiplication, and the Euclidean inner product, as ***Euclidean n-space***.

The four main arithmetic properties of the Euclidean inner product are listed in the next theorem.

THEOREM 4.1.2

Properties of Euclidean Inner Product

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n *and k is any scalar, then:*

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $(k\mathbf{u})\cdot\mathbf{v} = k(\mathbf{u}\cdot\mathbf{v})$

(d) $\mathbf{v}\cdot\mathbf{v} \geq 0$. *Further,* $\mathbf{v}\cdot\mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.

Note: With this and (a) we also know that $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{w}\cdot(\vec{u}+\vec{v}) = \vec{w}\cdot u + \vec{w}\cdot\vec{v}$

Impt. Because $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

We shall prove parts (*b*) and (*d*) and leave proofs of the rest as exercises.

i.e. DO THEM!

Read yourself

Application of Dot Products to ISBNs

Most books published in the last 25 years have been assigned a unique 10-digit number called an ***International Standard Book Number*** or ISBN. The first nine digits of this number are split into three groups—the first group representing the

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Most books published in the last 25 years have been assigned a unique 10-digit number called an ***International Standard Book Number*** or ISBN. The first nine digits of this number are split into three groups—the first group representing the country or group of countries in which the book originates, the second identifying the publisher, and the third assigned to the book title itself. The tenth and final digit, called a ***check digit***, is computed from the first nine digits and is used to ensure that an electronic transmission of the ISBN, say over the Internet, occurs without error.

To explain how this is done, regard the first nine digits of the ISBN as a vector ***b*** in R^9, and let ***a*** be the vector

$$\mathbf{a} = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

Then the check digit c is computed using the following procedure:

1. Form the dot product $\mathbf{a} \cdot \mathbf{b}$.

2. Divide $\mathbf{a} \cdot \mathbf{b}$ by 11, thereby producing a remainder c that is an integer between 0 and 10, inclusive. The check digit is taken to be c, with the proviso that $c = 10$ is written as X to avoid double digits.

For example, the ISBN of the brief edition of *Calculus*, sixth edition, by Howard Anton is

$$0 - 471 - 15307 - 9$$

which has a check digit of 9. This is consistent with the first nine digits of the ISBN, since

$$\mathbf{a} \cdot \mathbf{b} = (1, 2, 3, 4, 5, 6, 7, 8, 9) \cdot (0, 4, 7, 1, 1, 5, 3, 0, 7) = 152$$

Dividing 152 by 11 produces a quotient of 13 and a remainder of 9, so the check digit is $c = 9$. If an electronic order is placed for a book with a certain ISBN, then the warehouse can use the above procedure to verify that the check digit is consistent with the first nine digits, thereby reducing the possibility of a costly shipping error.

Proof (b) Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Then

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \cdot (w_1, w_2, \ldots, w_n) \\ &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \ldots + (u_n + v_n)w_n \\ &= (u_1w_1 + u_2w_2 + \ldots + u_nw_n) + (v_1w_1 + v_2w_2 + \ldots + v_nw_n) \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}\end{aligned}$$

■

Proof (d) We have $\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + \ldots + v_n^2 \geq 0$. Further, equality holds if and only if $v_1 = v_2 = \cdots = v_n = 0$—that is, if and only if $\mathbf{v} = \mathbf{0}$.

■

EXAMPLE 2 Length and Distance in R^4

Theorem 4.1.2 allows us to perform computations with Euclidean inner products in much the same way as we perform them with ordinary arithmetic products. For example,

$$\begin{aligned}(3\mathbf{u}+2\mathbf{v})\cdot(4\mathbf{u}+\mathbf{v}) &= (3\mathbf{u})\cdot(4\mathbf{u}+\mathbf{v}) + (2\mathbf{v})\cdot(4\mathbf{u}+\mathbf{v}) \\ &= (3\mathbf{u})\cdot(4\mathbf{u}) + (3\mathbf{u})\cdot\mathbf{v} + (2\mathbf{v})\cdot(4\mathbf{u}) + (2\mathbf{v})\cdot\mathbf{v} \\ &= 12(\mathbf{u}\cdot\mathbf{u}) + 3(\mathbf{u}\cdot\mathbf{v}) + 8(\mathbf{v}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) \\ &= 12(\mathbf{u}\cdot\mathbf{u}) + 11(\mathbf{u}\cdot\mathbf{v}) + 2(\mathbf{v}\cdot\mathbf{v})\end{aligned}$$

(property (h) i.e $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$)

(from what we derived from (a) and (b))

The reader should determine which parts of Theorem 4.1.2 were used in each step.

◆

Norm and Distance in Euclidean n-Space

By analogy with the familiar formulas in R^2 and R^3, we define the ***Euclidean norm*** (or ***Euclidean length***) of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n by

Need not have anything to do with visualizable length.

$$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \tag{1}$$

[Compare this formula to Formulas 1 and 2 in Section 3.2.]

Similarly, the ***Euclidean distance*** between the points $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{2}$$

[See Formulas 3 and 4 of Section 3.2.]

EXAMPLE 3 Finding Norm and Distance

If $\mathbf{u} = (1, 3, -2, 7)$ and $\mathbf{v} = (0, 7, 2, 2)$, then in the Euclidean space R^4,

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (7)^2} = \sqrt{63} = 3\sqrt{7}$$

and

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

◆

The following theorem provides one of the most important inequalities in linear algebra: the ***Cauchy–Schwarz*** inequality.

THEOREM 4.1.3

Very important inequality. This

PROOF LATER

BUT NOTE

Cauchy–Schwarz Inequality in R^n

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{3}$$

In terms of components, 3 is the same as

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{4}$$

We omit the proof at this time, since a more general version of this theorem will be proved later in the text. However, for vectors in R^2 and R^3, this result is a simple consequence of Formula 1 of Section 3.3: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors in R^2 or R^3, then

$$|\mathbf{u}\cdot\mathbf{v}| = |\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta| = \|\mathbf{u}\|\|\mathbf{v}\||\cos\theta| \leq \|\mathbf{u}\|\|\mathbf{v}\| \tag{5}$$

and if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then both sides of 3 are zero, so the inequality holds in this case as well.

The next two theorems list the basic properties of length and distance in Euclidean n-space.

Read if you care!

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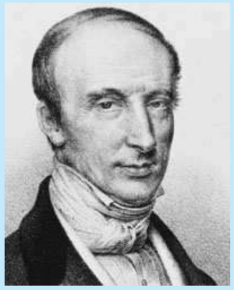
The next two theorems list the basic properties of length and distance in Euclidean *n*-space.

Read if you care!



Augustin Louis (Baron de) Cauchy

Augustin Louis (Baron de) Cauchy *(1789–1857),* French mathematician. Cauchy's early education was acquired from his father, a barrister and master of the classics. Cauchy entered L'Ecole Polytechnique in 1805 to study engineering, but because of poor health, he was advised to concentrate on mathematics. His major mathematical work began in 1811 with a series of brilliant solutions to some difficult outstanding problems.

Cauchy's mathematical contributions for the next 35 years were brilliant and staggering in quantity: over 700 papers filling 26 modern volumes. Cauchy's work initiated the era of modern analysis; he brought to mathematics standards of precision and rigor undreamed of by earlier mathematicians.

Cauchy's life was inextricably tied to the political upheavals of the time. A strong partisan of the Bourbons, he left his wife and children in 1830 to follow the Bourbon king Charles X into exile. For his loyalty he was made a baron by the ex-king. Cauchy eventually returned to France but refused to accept a university position until the government waived its requirement that he take a loyalty oath.

It is difficult to get a clear picture of the man. Devoutly Catholic, he sponsored charitable work for unwed mothers and criminals and relief for Ireland. Yet other aspects of his life cast him in an unfavorable light. The Norwegian mathematician Abel described him as “mad, infinitely Catholic, and bigoted.” Some writers praise his teaching, yet others say he rambled incoherently and, according to a report of the day, he once devoted an entire lecture to extracting the square root of seventeen to ten decimal places by a method well known to his students. In any event, Cauchy is undeniably one of the greatest minds in the history of science.

Herman Amandus Schwarz

Herman Amandus Schwarz *(1843–1921),* German mathematician. Schwarz was the leading mathematician in Berlin in the first part of the twentieth century. Because of a devotion to his teaching duties at the University of Berlin and a propensity for treating both important and trivial facts with equal thoroughness, he did not publish in great volume. He tended to focus on narrow concrete problems, but his techniques were often extremely clever and infiuenced the work of other mathematicians. A version of the inequality that bears his name appeared in a paper about surfaces of minimal area published in 1885.

THEOREM 4.1.4

Properties of Length in R^n

*If **u** and **v** are vectors in R^n and k is any scalar, then:*

(a) $\|\mathbf{u}\| \geq 0$

(b) $\|\mathbf{u}\| = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$

(c) $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$

(d) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ *(Triangle inequality)* → Another very impt. inequality

We shall prove (*c*) and (*d*) and leave (*a*) and (*b*) as exercises. → SO DO THEM!

Proof (c) If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, then $k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$, so

$$\begin{aligned}\|k\mathbf{u}\| &= \sqrt{(ku_1)^2 + (ku_2)^2 + \cdots + (ku_n)^2} \\ &= |k|\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \\ &= |k|\|\mathbf{u}\|\end{aligned}$$

■

Proof (d)

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 && \leftarrow \text{Property of absolute value} \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \leftarrow \text{Cauchy–Schwarz inequality} \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

The result now follows on taking square roots of both sides.

■

Part (*c*) of this theorem states that multiplying a vector by a scalar *k* multiplies the length of that vector by a factor of $|k|$ (Figure 4.1.2*a*). Part (*d*) of this theorem is known as the ***triangle inequality*** because it generalizes the familiar result from Euclidean geometry that states that the sum of the lengths of any two sides of a triangle is at least as large as the length of the third side (Figure 4.1.2*b*).

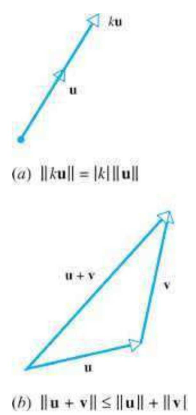


Figure 4.1.2

The results in the next theorem are immediate consequences of those in Theorem 4.1.4, as applied to the distance function $d(\mathbf{u}, \mathbf{v})$ on R^n. They generalize the familiar results for R^2 and R^3.

THEOREM 4.1.5

Properties of Distance in R^n

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n *and k is any scalar, then:*

(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(b) $d(\mathbf{u}, \mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(d) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{v}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ *(Triangle inequality)*

We shall prove part (*d*) and leave the remaining parts as exercises.

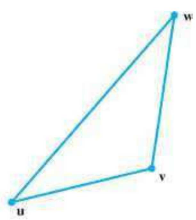
Clever trick. We use the triangle inequality for norms to prove it for distance.

Proof (d) From 2 and part (*d*) of Theorem 4.1.4, we have

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})\|$$
$$\leq \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| = d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$$

■

Part (*d*) of this theorem, which is also called the *triangle inequality*, generalizes the familiar result from Euclidean geometry that states that the shortest distance between two points is along a straight line (Figure 4.1.3).



$d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$

Figure 4.1.3

Formula 1 expresses the norm of a vector in terms of a dot product. The following useful theorem expresses the dot product in terms of norms.

THEOREM 4.1.6

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 \tag{6}$$

Proof

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$$
$$\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2$$

from which 6 follows by simple algebra.

■

Some problems that use this theorem are given in the exercises.

Orthogonality

Recall that in the Euclidean spaces R^2 and R^3, two vectors ***u*** and ***v*** are defined to be *orthogonal* (perpendicular) if $\mathbf{u}\cdot\mathbf{v}=0$ (Section 3.3). Motivated by this, we make the following definition.

DEFINITION

Two vectors ***u*** and ***v*** in R^n are called *orthogonal* if $\mathbf{u}\cdot\mathbf{v}=0$.

So here orthogonality is possible even when there is no geometric interpretation to it.

EXAMPLE 4 Orthogonal Vectors in R^4

In the Euclidean space R^4 the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v} = (1, 2, 0, -1)$$

are orthogonal, since

$$\mathbf{u}\cdot\mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

◆

Properties of orthogonal vectors will be discussed in more detail later in the text, but we note at this point that many of the familiar properties of orthogonal vectors in the Euclidean spaces R^2 and R^3 continue to hold in the Euclidean space R^n. For example, if ***u*** and ***v*** are orthogonal vectors in R^2 or R^3, then ***u***, ***v***, and $\mathbf{u}+\mathbf{v}$ form the sides of a right triangle (Figure 4.1.4); thus, by the Theorem of Pythagoras,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

The following theorem shows that this result extends to R^n.

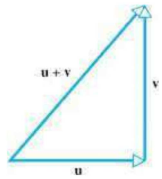


Figure 4.1.4

THEOREM 4.1.7

Pythagorean Theorem in R^n

If **u** *and* **v** *are orthogonal vectors in* R^n *with the Euclidean inner product, then*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 + \underbrace{2(\mathbf{u}\cdot\mathbf{v})}_{=0 \text{ as } \vec{u}, \vec{v} \text{ are orthogonal.}} + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Probably the shortest proof of the Pythagorean Theorem

Alternative Notations for Vectors in R^n

It is often useful to write a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n in matrix notation as a row matrix or a column matrix:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{or} \quad \mathbf{u} = [u_1 \quad u_2 \quad \ldots \quad u_n]$$

This is justified because the matrix operations

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \quad k\mathbf{u} = k\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$$

or

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= [u_1 \quad u_2 \quad \ldots \quad u_n] + [v_1 \quad v_2 \quad \ldots \quad v_n] \\ &= [u_1+v_1 \quad u_2+v_2 \quad \ldots \quad u_n+v_n] \\ k\mathbf{u} &= k[u_1 \quad u_2 \quad \ldots \quad u_n] = [ku_1 \quad ku_2 \quad \ldots \quad ku_n] \end{aligned}$$

produce the same results as the vector operations

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n) \\ k\mathbf{u} &= k(u_1, u_2, \ldots, u_n) = (ku_1, ku_2, \ldots, ku_n) \end{aligned}$$

The only difference is the form in which the vectors are written.

A Matrix Formula for the Dot Product

If we use column matrix notation for the vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

and omit the brackets on 1×1 matrices, then it follows that

$$\mathbf{v}^T\mathbf{u} = [v_1 \quad v_2 \quad \dots \quad v_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = [u_1v_1 + u_2v_2 + \cdots + u_nv_n] = [\mathbf{u} \cdot \mathbf{v}] = \mathbf{u} \cdot \mathbf{v}$$

Thus, for vectors in column matrix notation, we have the following formula for the Euclidean inner product:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{u} \tag{7}$$

For example, if

$$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 7 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 5 \\ -4 \\ 7 \\ 0 \end{bmatrix}$$

then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{u} = [5 \quad -4 \quad 7 \quad 0] \begin{bmatrix} -1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = [18] = 18$$

If A is an $n \times n$ matrix, then it follows from Formula 7 and properties of the transpose that

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^TA)\mathbf{u} = (A^T\mathbf{v})^T\mathbf{u} = \mathbf{u} \cdot A^T\mathbf{v}$$
$$\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T\mathbf{u} = (\mathbf{v}^TA^T)\mathbf{u} = \mathbf{v}^T(A^T\mathbf{u}) = A^T\mathbf{u} \cdot \mathbf{v}$$

The resulting formulas

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v} \tag{8}$$

$$\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \tag{9}$$

provide an important link between multiplication by an $n \times n$ matrix A and multiplication by A^T.

EXAMPLE 5 Verifying That $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

from which we obtain

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$

$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

Thus $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$ as guaranteed by Formula 8. We leave it for the reader to verify that 9 also holds.

A Dot Product View of Matrix Multiplication

Dot products provide another way of thinking about matrix multiplication. Recall that if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then the ijth entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

which is the dot product of the *i*th row vector of *A*

$$[a_{i1} \quad a_{i2} \quad \ldots \quad a_{ir}]$$

and the *j*th column vector of *B*

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

Thus, if the row vectors of *A* are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of *B* are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix} \tag{10}$$

In particular, a linear system $A\mathbf{x} = \mathbf{b}$ can be expressed in dot product form as

$$\begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{11}$$

where $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ are the row vectors of *A*, and $b_1, b_2, \ldots, b_m$ are the entries of ***b***.

EXAMPLE 6 A Linear System Written in Dot Product Form

The following is an example of a linear system expressed in dot product form 11.

Exercise Set 4.1

Q6, Q7, Q8, Q 10, Q 20, Q 21, Q 23, Q 24, Q 25, Q 26, Q 27, Q 29, Q 32, Q 34, Q 35, Q 36, Q 37