Homework Assignment 2

Ali Muhammad Asad
aa07190

Q1) 4 adults & 3 kids
(a) 3 member groups.
$4+3 = 7$ members, we need to chose 3

$$\text{Groups} = {}^7C_3 = \frac{7!}{3! \cdot 4!} = 35.$$

35 different 3-member groups can be made.

(b) Probability that a 3-member group has exaclthy one adult

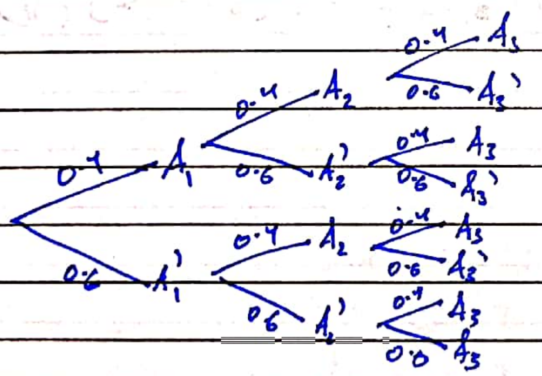
$$P(E) = \frac{\text{Group has one adult}}{\text{Total 3-member groups possible}}$$

$$= \frac{{}^4C_1 \times {}^3C_2}{35} = \frac{4 \times 3}{35} = \frac{12}{35}$$

$$\boxed{P(E) = 12/35}$$

(c) A = {child has to attend online class at 10 am}
$P(A) = 0.4$ for each kid of the three
Internet can only support one online class.
Probability internet overloaded at 10 am = ?

kid 1 kid 2 kid 3



$$P(E) = (A_1 \times A_2 \times A_3) + (A_1 \times A_2 \times A_3') + (A_1 \times A_2' \times A_3) + (A_1' \times A_2 \times A_3)$$

$$= (0.4 \times 0.4 \times 0.4) + (0.4 \times 0.4 \times 0.6) + (0.4 \times 0.6 \times 0.4) + (0.6 \times 0.4 \times 0.4)$$

$$= 0.4^3 + 3(0.4 \times 0.4 \times 0.6) = 0.352$$

$P(E) = 0.352$ Probability of overload is 0.352

Q2) Deck of cards

(a) 4 students, Probability each student gets an ace.

We can make partitions,

P(E) = Occurances of event [Number of elements required] / number of elements in Ω.

$$P(\mathcal{E}) = \frac{4!\left(\frac{48!}{12!\times 12!\times 12!\times 12!}\right)}{(52!)/(13!\times 13!\times 13!\times 13!)} \Rightarrow \underline{P(\mathcal{E}) = 0.105}$$

(b) We select 6, Probability that exactly 3 are aces.

$$P(\mathcal{E}) = \frac{{}^{4}C_3 \times {}^{48}C_3}{{}^{52}C_6} = \underline{0.0034}$$

[3 out of 4 Aces, & 3 out of remaining 48 cards]

Q3) (a) T = {Telemarketer has called} $P(T) = \frac{1}{8}$

X is number of telemarketers in next three calls.

PMF of X = ?

X can take values 0, 1, 2, 3 & is similar to tossing a coin, either telemarketer has called or it is not a telemarketer.

PMF: $P(X=0) = {}^{3}C_0\left(\frac{7}{8}\right)^3 = \underline{343/512}$

$P(X=1) = {}^{3}C_1\left(\frac{1}{8}\right)\left(\frac{7}{8}\right)^2 = \underline{147/512}$

$P(X=2) = {}^{3}C_2\left(\frac{1}{8}\right)^2\left(\frac{7}{8}\right) = \underline{21/512}$

$P(X=3) = {}^{3}C_3\left(\frac{1}{8}\right)^3 = \underline{1/512}$

(b) X denotes phone calls until telemarketer calls.

$P(X \geq 3) = 1 - P(X=1) - P(X=2)$

$P(X=1) = 1/8$ → first call $P(X=2) = \frac{7}{8}\times\frac{1}{8} = \frac{7}{64}$ → second call

$P(X \geq 3) = 1 - \frac{1}{8} - \frac{7}{64}$

$\underline{P(X \geq 3) = 49/64}$

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Q4) Random Variable X → outcome of roll of 6 sided die

(a) $E[X^2+1]$

Each outcome (1, 2, 3, 4, 5, 6) has equal probability of 1/6.

$E[X] = \sum_x x\, p_X(x)$

$\Rightarrow E[X^2+1] = \sum_x (x^2+1)\, p_X(x)$

$E[X^2+1] = (1+1)\frac{1}{6} + (2^2+1)\frac{1}{6} + (3^2+1)\frac{1}{6} + (4^2+1)\frac{1}{6} + (5^2+1)\frac{1}{6} + (6^2+1)\frac{1}{6}$

$= \frac{2}{6} + \frac{5}{6} + \frac{10}{6} + \frac{17}{6} + \frac{26}{6} + \frac{37}{6}$

$= 97/6.$

$\boxed{E[X^2+1] = 97/6}$

(b) $E[2X+1]$ Since we have linear function, we can use $E[X]$ directly.

$E[X] = 1(\frac{1}{6}) + 2(\frac{1}{6}) + 3(\frac{1}{6}) + 4(\frac{1}{6}) + 5(\frac{1}{6}) + 6(\frac{1}{6})$

$= 3.5$ [also the middle value]

$E[2X+1] = 2(3.5) + 1$

$= 7 + 1 = 8 \Rightarrow \boxed{E[2X+1] = 8}$

Q5) X = Magnitude of Difference between two rolls of fair 3-sided die

$Y = 2X$

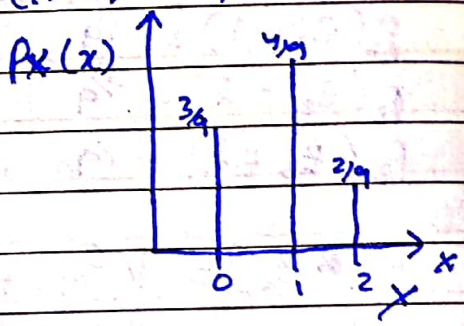
$R_2 \backslash R_1$	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

(a) PMF of X

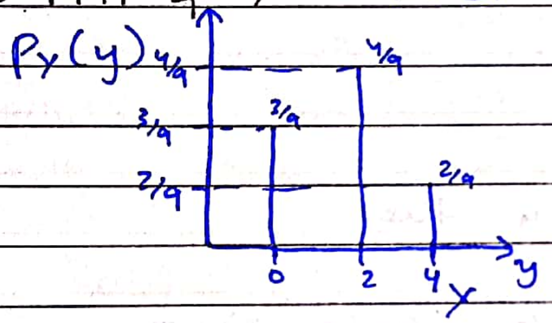
$X = 0, 1, 2.$

$P(X=0) = 3/9$ $\quad P(X=1) = 4/9$

$P(X=2) = 2/9$



(b) PMF of Y $\quad Y = 2X$, $Y = 0, 2, 4$



(c) Mean of Y $\quad E[Y] = 0\left(\frac{3}{9}\right) + 2\left(\frac{4}{9}\right) + 4\left(\frac{2}{9}\right)$

$= 8/9 + 8/9 = 16/9$

$\underline{E[Y] = 16/9}$

(d) Variance of Y $\quad Var(Y) = E[Y^2] - (E[Y])^2$

$E[Y^2] = 0^2(3/9) + 2^2(4/9) + 4^2(2/9)$

$= 16/9 + 32/9 = 48/9$

$Var(Y) = \frac{48}{9} - \left(\frac{16}{9}\right)^2 = \frac{176}{81} = 2.17$

$\underline{Var(Y) = 176/81}$

Q6) $P(X=x) = \begin{cases} kx & x = 2, 4, 6 \\ k(x-2) & x = 8, 9, 10 \\ 0 & \text{otherwise.} \end{cases}$

(a) $k = ?$ $\quad \sum_x P(X=x) = 1$

$\Rightarrow \quad 1 = 2k + 4k + 6k + k(8-2) + k(9-2) + k(10-2)$

$1 = 12k + 6k + 7k + 8k \Rightarrow \boxed{k = 1/33}$

(b) $\boxed{P(X=5) = 0}$ from PMF

(c) $E[X] = 2\left(\frac{2}{33}\right) + 4\left(\frac{4}{33}\right) + 6\left(\frac{6}{33}\right) + 8\left(\frac{8-2}{33}\right) + 9\left(\frac{9-2}{33}\right) + 10\left(\frac{10-2}{33}\right)$

$\Rightarrow \boxed{E[X] = \frac{247}{33} = 7.485}$

(d) $E[X^2] = 2^2\left(\frac{2}{33}\right) + 4^2\left(\frac{4}{33}\right) + 6^2\left(\frac{6}{33}\right) + 8^2\left(\frac{8-2}{33}\right) + 9^2\left(\frac{9-2}{33}\right) + 10^2\left(\frac{10-2}{33}\right)$

$\boxed{E[X^2] = \frac{2039}{33} = 61.788}$

(e) $Var[3-5X] = 5^2 Var(X)$

$Var[X] = E[X^2] - (E[X])^2 = \frac{2039}{33} - \left(\frac{247}{33}\right)^2 = 5.765$

$Var[3-5X] = 25 \times Var(X) = 144.123$

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$\boxed{Var[3-5X] = 144.123}$

Q7) Fair six sided Red die → X → number on red die.
Fair tetrahedral Green die

$$Y = \begin{cases} 0 & \text{two dice show same number} \\ 1 & \text{number on green die is greater} \\ 2 & \text{number on red die is greater} \end{cases}$$

(a) $P_{X,Y}(x,y)$

		X					
		1	2	3	4	5	6
Y	0	1/24	1/24	1/24	1/24	0	0
	1	3/24	2/24	1/24	0	0	0
	2	0	1/24	2/24	3/24	4/24	4/24

↑ $P_{X,Y}(x,y)$

	Red Die					
Green Die	1	2	3	4	5	6
1	1,1	2,1	3,1	4,1	5,1	6,1
2	1,2	2,2	3,2	4,2	5,2	6,2
3	1,3	2,3	3,3	4,3	5,3	6,3
4	1,4	2,4	3,4	4,4	5,4	6,4

↑ Rough work for idea

(b) $P_Y(y)$

$P_Y(y=0) = \frac{1}{24} + \frac{1}{24} + \frac{1}{24} + \frac{1}{24} = \underline{\frac{4}{24}}$

$P_Y(y=1) = \frac{3}{24} + \frac{2}{24} + \frac{1}{24} = \underline{6/24}$

$P_Y(y=2) = \frac{1}{24} + \frac{2}{24} + \frac{3}{24} + \frac{4}{24} + \frac{4}{24} = \underline{14/24}$

(c) $E[X] = (1+2+3+4+5+6)/6 = 21/6$

$\underline{E[X] = 3.5}$

(d) $P_{X|Y}(x|2)$ $\qquad P_{X|Y}(x|2) = \frac{P_{X,Y}(x,2)}{P_Y(2)}$

$P_{X|Y}(1|2) = 0$

$P_{X|Y}(2|2) = (1/24)/(14/24) = \underline{\frac{1}{14}}$

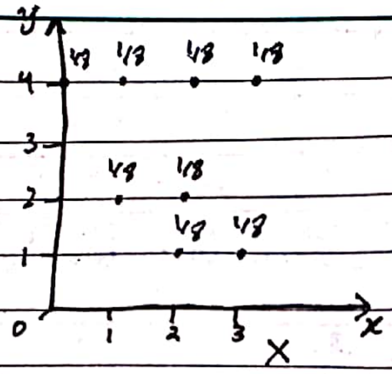
$P_{X|Y}(3|2) = \underline{2/14}$

$P_{X|Y}(4|2) = \underline{3/14}$

$P_{X|Y}(5|2) = \underline{4/14}$

$P_{X|Y}(6|2) = 4/14$

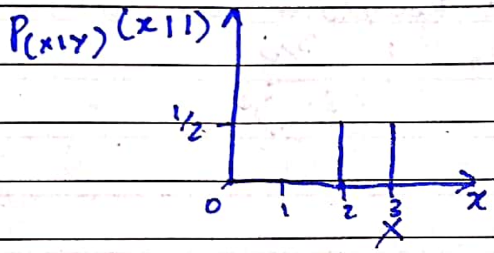
Q8)

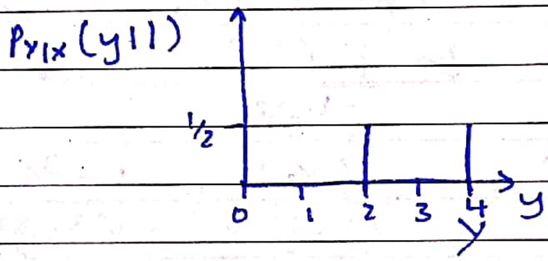


(a) Are X & Y independent?
It can be clearly observed from the PMF that they are not independent as X tends to take certain values on certain values of Y. A quick calculation verifies this.

$p_{X|Y}(x|y) = p_X(x)p_Y(y)$.
$p_{X|Y}(1|1) = 0$ $\quad p_X(1) = \frac{2}{8}$ $\quad p_Y(1) = \frac{2}{8}$
$p_{X|Y}(1|1) \neq p_X(1)p_Y(1)$. Hence not independent.

(b) $p_{X|Y}(x|1)$, $p_{Y|X}(y|1)$





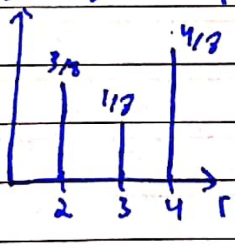
(c) $E[X|Y=4]$ $\quad p_{X|Y}(0|4) = \frac{1}{4}$ $\quad p_{X|Y}(1|4) = \frac{1}{4}$ $\quad p_{X|Y}(2|4) = \frac{1}{4}$
$p_{X|Y}(3|4) = \frac{1}{4}$

$E[X|Y=4] = 0(\frac{1}{4}) + 1(\frac{1}{4}) + 2(\frac{1}{4}) + 3(\frac{1}{4}) = 6/4$
$= 1.5$

(d) $R = \max(X, Y)$ $\qquad E[R]$

$P(R=0) = 0$ $\quad P(R=1) = 0$
$P(R=2) = 3/8$ $\quad P(R=3) = 1/8$ $\quad P(R=4) = 4/8$

$P_R(r)$



$E[R] = 2(\frac{3}{8}) + 3(\frac{1}{8}) + 4(\frac{4}{8})$
$= \frac{6}{8} + \frac{3}{8} + \frac{16}{8} = \frac{25}{8}$

$\underline{E[R] = 25/8}$

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(Q a) $P_X(x) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{x-1}$ $\quad x \geq 1$

$P_Y(y) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{y-1}$ $\quad y \geq 1$

(a) It is clear that X and Y are Geometric Random Variables.

X has 'p' = $1/3$ & Y has 'p' = $\frac{3}{4}$

where 'p' is the probability of success.

(b) $Var(X-Y)$ We know that $Var(X+Y) = Var(X) + Var(Y)$ for two independent random variables.

Since X & Y are independent, $Var(X-Y) = Var(X) + Var(Y)$

$Var(X) = 1-p/p^2$ [for Geometric Random Variables].

$Var(X) = \frac{1-(1/3)}{(1/3)^2}$ $\quad Var(Y) = \frac{1-3/4}{(3/4)^2}$

$Var(X-Y) = \frac{1-(1/3)}{(1/3)^2} + \frac{1-3/4}{(3/4)^2}$

$\underline{Var(X-Y) = 58/9}$

(c) Probability that X & Y are equal

$P[X=Y] = \sum_x (1-p)^{x-1} p (1-q)^{x-1} q$

$= \sum_x [(1-p)(1-q)]^{x-1} pq$

Then by the formula of sum of geometric series,

$P[X=Y] = \frac{pq}{p+q-pq}$

Then $P[X=Y] = \frac{(1/3)(3/4)}{1/3 + 3/4 - (1/3 \times 3/4)} = \frac{3}{10}$

$\underline{P[X=Y] = 0.3}$

Q10) $Z_n = \min(X_1, X_2, \ldots, X_n)$

X_i's are outcomes of independent rolls of fair 3-sided die.

$Z_2 = \min(X_1, X_2)$

$R_2 \backslash R_1$	1	2	3
1	1,1	2,1	3,1
2	1,2	2,2	3,2
3	1,3	2,3	3,3

(a) Mean of Z_2

$P_z(Z_2 = 1) = 5/9$

$P_z(Z_2 = 2) = 3/9$

$P_z(Z_3 = 3) = 1/9$

$E[Z_2] = 1(5/9) + 2(3/9) + 3(1/9)$

$= 5/9 + 6/9 + 3/9 = 14/9$

$\underline{E[Z_2] = 14/9}$

(b) $Y = 2Z_1 + Z_2$ $\quad E[Y] = ?$

Since we have a linear relation, then

$E[Y] = 2E[Z_1] + E[Z_2]$.

$Z_1 = \min(X_1)$ $\quad P_Z(Z_1 = 1) = 1/3$ $\quad P_Z(Z_1 = 2) = \frac{1}{3}$

$P_Z(Z_1 = 3) = 1/3$

$E[Z_1] = 1(1/3) + 2(1/3) + 3(1/3) = \frac{1}{3} + \frac{2}{3} + \frac{3}{3} = 2$

$E[Z_1] = 2$

$E[Y] = 2(2) + 14/9$

$E[Y] = 50/9$