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LINEAR ALGEBRA

FINAL EXAM [Total Marks: 100]

(SPRING 2022)

Question 1: [10 Marks]

Prove: if A is an n x n diagonalizable matrix, then A has n linearly independent eigenvectors.

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Question 2:

Part (a) If k is a positive integer, λ is an eigenvalue of matrix A, and X is a corresponding eigenvector, then prove that λ^k is an eigenvalue of A^k and X is a corresponding eigenvector.

[6 marks]

Part (b) Consider the matrix $A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$. Is A diagonalizable? Why or why not? **[4 marks]**

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Question 3: [10 Marks]

Let the vector space P_2 have the inner product $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$, apply the Gram-Schmidt process to transform the standard basis $S = \{\, 1\,,\ x\,,\ x^2 \,\}$ into an orthonormal basis.

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Question 4: [10 Marks]

Indicate whether each statement is always True, or sometimes False. Justify your answer by giving a logical argument or a counterexample:

(a) The intersection of two subspaces of a vector space *V* is also a subspace of *V*.

(b) A set that contains the zero vector is linearly dependent.

(c) Every nonzero finite dimensional inner product space has an orthonormal basis.

(d) If $Span(S_1) = Span(S_2)$, then $S_1 = S_2$.

(e) If *W* is a set of one or more vectors from a vector space *V*, and if $k\boldsymbol{u} + \boldsymbol{v}$ is a vector in *W* for all vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in *W* and for all scalars k, then *W* is a subspace of *V*.

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Question 5:

Part (a) Prove: If *V* is an *n*-dimensional vector space, and if *S* is a set in *V* with exactly *n* vectors, then *S* is a basis for *V* if either *S* spans *V* or *S* is linearly independent. **[5 marks]**

Part (b) Prove: If *W* is a subspace of a finite-dimensional vector space *V,* then $\dim(W) \leq \dim(V)$; moreover, if $\dim(W) = \dim(V)$, then *W* = *V*. **[5 marks]**

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Question 6

Given that A is diagonalizable and that $A^5 = 0$, show that A must also be a zero matrix.

[Warning: Do not assume that $(A^n = 0) \Longrightarrow (A = 0)$ holds true for matrices just because it holds true for real numbers. That is a major error.] **[10 marks]**

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Question 7

(a) Given a vector space V which has an inner product defined on it, prove that if $\|\boldsymbol{x}\| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x} \in V$, then $|\langle \boldsymbol{x_1}, \boldsymbol{x_2} \rangle| \leq \|\boldsymbol{x_1}\| \|\boldsymbol{x_2}\|$ for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in V$. [7 Marks]

(b) For the same vector space as in part (a) above, show that for any $\boldsymbol{a}, \boldsymbol{b} \in V$, we have $\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$. [3 Marks]

Question 8

(a) Prove that if $A\boldsymbol{x} = b$ is consistent for every $n \times 1$ matrix b, then the $n \times n$ matrix A is invertible. **[7 marks]**

(b) Prove that if an $n \times n$ matrix A is invertible, then $A\boldsymbol{x} = b$ is consistent for every $n \times 1$ matrix b. **[3 marks]**

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Question 9

(a) If $T\left(\underline{e_1}\right) = (1, 1)$, $T\left(\underline{e_2}\right) = (3, 0)$, and $T\left(\underline{e_3}\right) = (4, 7)$ then find $T(2, 4, 6)$, given that T is a linear transformation. **[3 marks]**

(b) For a general $(x, y, z) \in \mathbb{R}^3$, find $T(x, y, z)$. **[3 marks]**

(c) Find the matrix of the linear transformation in the parts above. **[4 marks]**

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Question 10

Let $A\underline{\boldsymbol{x}} = 0$ be a homogenous system where A is some $m \times n$ matrix. Prove that $\underline{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ is a solution of the system if and only if $\underline{\boldsymbol{x}}$ is orthogonal to every row vector of A (using the Euclidean inner product). **[10 marks]**

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