# Lecture 27

Thursday, April 28, 2022 10:48 AM

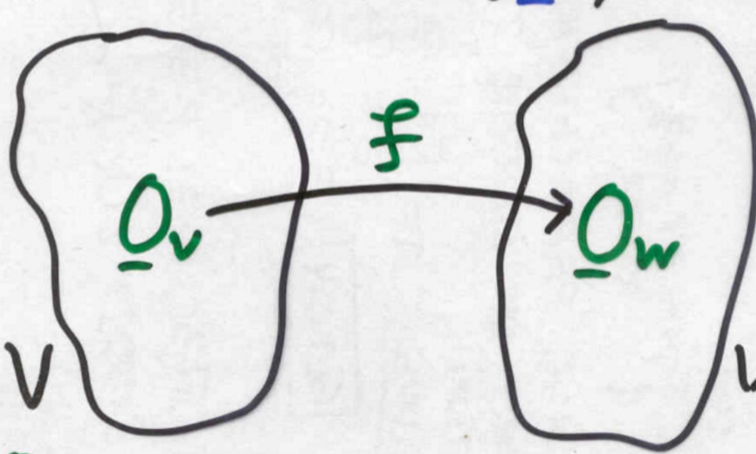
LINEAR ALGEBRA

# LECTURE 27



RESULT: IF $\boxed{f}$ IS A LINEAR TRANSFORMATION FROM $V \to W$, THEN THE $\boxed{\text{ZERO}}$ VECTOR OF THE VECTOR SPACE $\boxed{V}$ IS ALWAYS GOING TO $\boxed{\text{MAP}}$ ON THE $\boxed{\text{ZERO VECTOR}}$ OF THE VECTOR SPACE $\boxed{W}$ AS SHOWN BELOW:

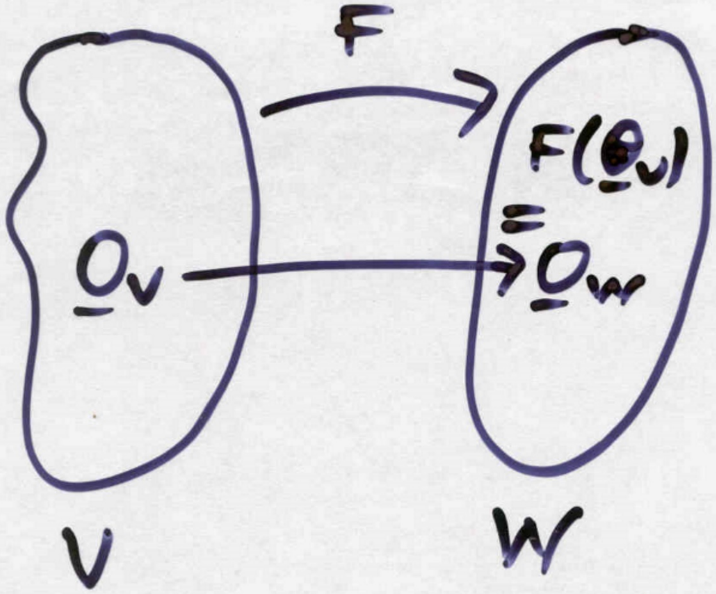
$$f(\underline{0}_v) = \underline{0}_w$$




$\underline{0}_v \to$ $\boxed{\text{ZERO}}$ VECTOR OF $\boxed{V}$

$\underline{0}_w \to$ $\boxed{\text{ZERO}}$ VECTOR OF $\boxed{W}$

 2




$F(\underline{0}_V) = \underline{0}_W$

<u>NOTE:</u> IN FUTURE WE SHALL USE $\underline{0}$ INSTEAD OF $\underline{0}_V$ OR $\underline{0}_W$.

## <u>EXAMPLES:</u>

① $T: R^n \longrightarrow R^m$

$T(\underline{x}) = A\underline{x}$ IS LINEAR.

$T(\underline{0}) = A\underline{0} = \underline{0}$

$\Rightarrow \boxed{T(\underline{0}) = \underline{0}}$

③ 13

13

WE ALREADY PROVED THE FOLLOWING TRANSFORMATIONS AS LINEAR.

② $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \\ 5x \end{bmatrix}$

IT IS EASILY SEEN THAT

$$\boxed{F\left(\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{ZERO OF } R^2}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \longrightarrow \text{ZERO OF } R^3$$

③ $J : V \longrightarrow R \qquad V = C[0,1]$

$$J(f) = \int_0^1 f(x)\,dx$$

$$\boxed{J(0) = \int_0^1 0\,dx = 0}$$

④ $D : W \longrightarrow V$

$$D(f) = f'(x) = \frac{d}{dx}(f(x))$$

$$\boxed{D(0) = \frac{d}{dx}(0) = 0}$$

BUT HOW TO PROVE IN

BUT HOW TO PROVE IN GENERAL?

$\boxed{\text{PROOF}}$ :- $\boxed{4}$

$\boxed{4}$ IF $T: V \longrightarrow W$ IS A LINEAR TRANSFORMATION, THEN

$$T(\underline{0}) = \underline{0}$$

VECTOR SCALAR

PROOF: $T(\underline{0}) = T(0\underline{v}) = 0\,T(\underline{v}) = \underline{0}$

$\underline{v} \in V$ $\quad \because\ T(k\underline{v}) = k\,T(\underline{v})$

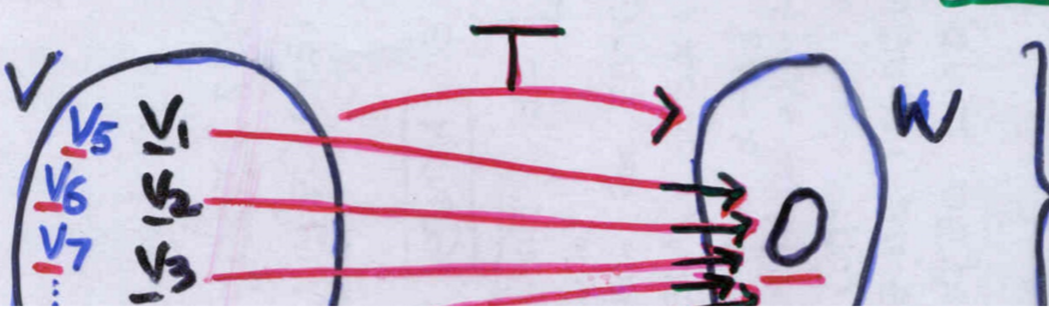
OR FOR ANY $\underline{v} \in V$

$$T(\underline{0}) = T(\underline{v} - \underline{v}) = T(\underline{v} + (-\underline{v}))$$

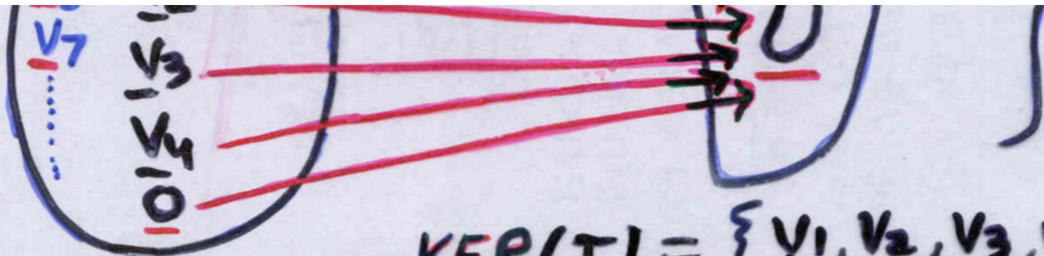
$$= T(\underline{v}) + T(-\underline{v}) = T(\underline{v}) - T(\underline{v}) = \underline{0}.$$

DEFINITION: P.316 (6th ED.) P.395 (7th ED.)

IF $T: V \rightarrow W$ IS A LINEAR TRANSFORMATION, THEN THE SET OF VECTORS IN $V$ THAT $\boxed{\text{MAPS}}$ INTO $\underline{0}$ IS CALLED THE KERNEL (OR NULLSPACE) OF $\boxed{T}$; IT IS DENOTED BY $\boxed{KER(T)}$.




EXAMPLE




$$\mathrm{KER}(T) = \{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{0}\}$$

$$\therefore\ T(\underline{v}_1) = T(\underline{v}_2) = T(\underline{v}_3) = T(\underline{v}_4) = T(\underline{0}) = \underline{0}$$

EXAMPLES: ① $D : V \to W$

$\Rightarrow \quad D(f) = f'(x)$

KER(D) = SET OF ALL FUNCTIONS S.t. $D(f) = 0$

$\Rightarrow \quad f'(x) = 0 \Rightarrow \boxed{f(x) = k}$

$\boxed{k \to \text{CONSTANT}}$

$\therefore$ KER(D) = SET OF ALL CONSTANT FUNCTIONS IN $\boxed{V}$.

② $J : P_1 \to R$ $\quad P_1(x)$ } POLYNOMIALS OF DEGREE $\leq 1$

FIND KER(J), WHERE

$$J(P) = \int_{-1}^{1} p(x)\,dx$$

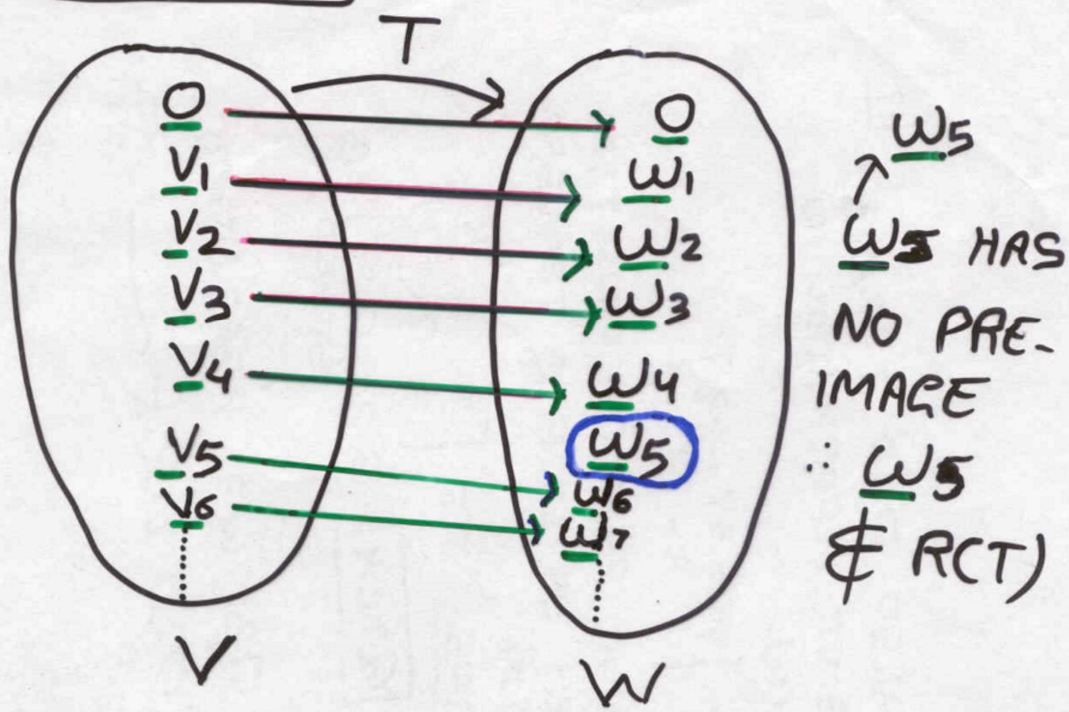
ANSWER: KER(J) CONSISTS OF ALL POLYNOMIALS OF THE FORM $\boxed{P(x) = kx}$

$k \to$ CONSTANT

# 6 RANGE OF LINEAR TRANSFORMATION:

IF $\boxed{T: V \to W}$ IS LINEAR, THEN THE SET OF ALL VECTORS IN $\boxed{W}$ THAT ARE IMAGES UNDER $\boxed{T}$ OF ATLEAST ONE VECTOR IN $\boxed{V}$ IS CALLED THE RANGE OF T; IT IS DENOTED BY $\boxed{R(T)}$.

EXAMPLE:

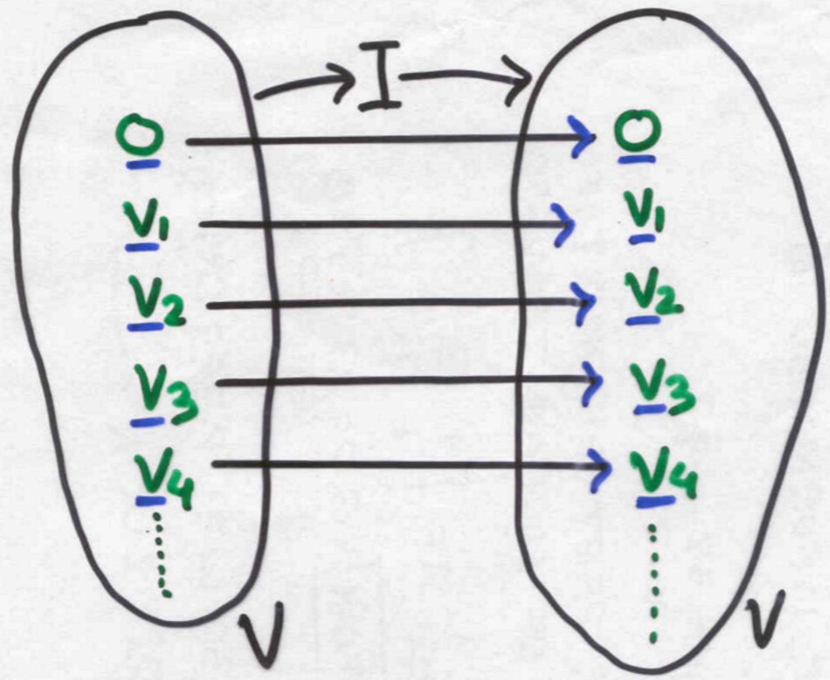



$$R(T) = \{\underline{0}, \underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4, \underline{w}_6, \underline{w}_7, \ldots\ldots\}$$

7

## IDENTITY TRANSFORMATION:

LET $\boxed{V}$ BE ANY VECTOR SPACE THE MAPPING $I : V \longrightarrow V$ DEFINED BY $\boxed{I(\underline{v}) = \underline{v}}$ IS CALLED THE IDENTITY OPERATOR AS SHOWN IN THE FIGURE:




$\Rightarrow I(\underline{v}_1) = \underline{v}_1, \quad I(\underline{v}_2) = \underline{v}_2,$

$I(\underline{v}_3) = \underline{v}_3, \quad I(\underline{v}_4) = \underline{v}_4, \ldots\ldots$

$I(\underline{0}) = \underline{0}$ ETC.

## TRY THE FOLLOWING:

IF $I : V \to V$ IS AN IDENTITY TRANSFORMATION i.e. $I(\underline{v}) = \underline{v} \ \forall \underline{v} \in V$ THEN

(1) $I$ IS LINEAR

(2) FIND $R(I)$, (3) $KER(I)$

## SOLUTION:

$k\underline{u} \in V,$

(1) LET $\underline{u}, \underline{v} \in \boxed{V} \Rightarrow \underline{u} + \underline{v} \in \boxed{V}$

$$\Rightarrow I(\underline{u}+\underline{v}) = \underline{u}+\underline{v} = I(\underline{u}) + I(\underline{v})$$

ALSO $I(k\underline{u}) = k\underline{u} = k\,I(\underline{u})$

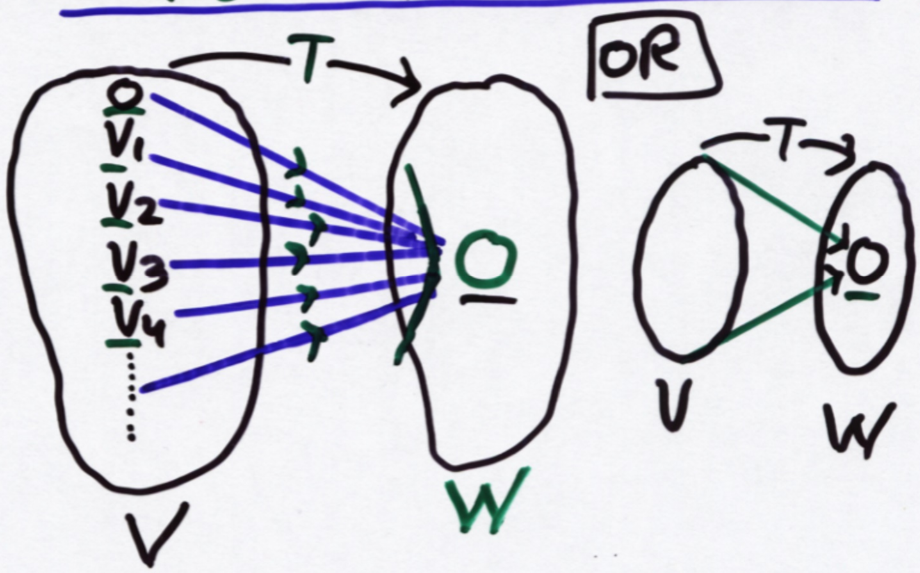
(2) $R(I) = V$ $\because$ EVERY VECTOR IN $\boxed{V}$ HAS A PREIMAGE

(3) $KER(I) = \underline{0}$ $\because$ $\underline{0}$ IS THE ONLY VECTOR WHICH MAPS INTO $\underline{0}$.

# ZERO TRANSFORMATION:



LET $\boxed{V}$ AND $\boxed{W}$ BE ANY **TWO** VECTOR SPACES. THE **MAPPING** $\boxed{T : V \rightarrow W}$ SUCH THAT $\boxed{T(\underline{v}) = \underline{0}}$ FOR **EVERY $\underline{v}$** IN $V$ IS CALLED THE ZERO TRANSFORMATION.




$$\Rightarrow T(\underline{v}_1) = T(\underline{v}_2) = \dots\dots = \underline{0}$$

## DEFINITION:

A LINEAR TRANSFORMATION $T: V \longrightarrow W$ IS SAID TO BE ONE-TO-ONE IF $\boxed{T}$ MAPS DISTINCT VECTORS IN $\boxed{V}$ INTO DISTINCT VECTORS IN $\boxed{W}$.

## EXAMPLE:

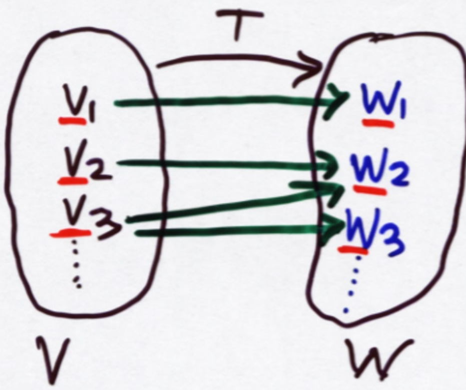
$\boxed{\text{IDENTITY}}$ TRANSFORMATION IS ONE-TO-ONE, $\forall\ \underline{v} \in V$

$$\boxed{I(\underline{v}) = \underline{v}},\quad I: V \longrightarrow V$$

NOTE: $\boxed{\text{ZERO}}$ TRANSFORMATION IS $\boxed{\text{NOT}}$ ONE-TO-ONE.

FOR DETAIL SEE SLIDES $\boxed{7}$ AND $\boxed{9}$.

NOTE: IN A MAPPING EVERY ELEMENT HAS ONLY ONE IMAGE, SEE BELOW



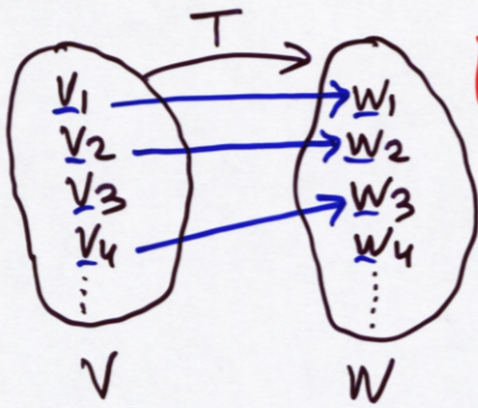

T IS NOT A MAPPING

$\because T(v_3) = w_2$

AND $T(v_3) = w_3$

$v_3$ HAS TWO IMAGES.

IN ADDITION EVERY ELEMENT IN DOMAIN MUST HAVE AN IMAGE.




T IS NOT A MAPPING $\because$ $v_3$ HAS NO IMAGE.

## INVERSE LINEAR TRANSFORMATION

8TH ED. P.385
P.402 7TH ED.

IF $T: V \rightarrow W$ IS LINEAR AND ONE-TO-ONE THEN THE INVERSE LINEAR TRANSFORMATION IS GIVEN BY

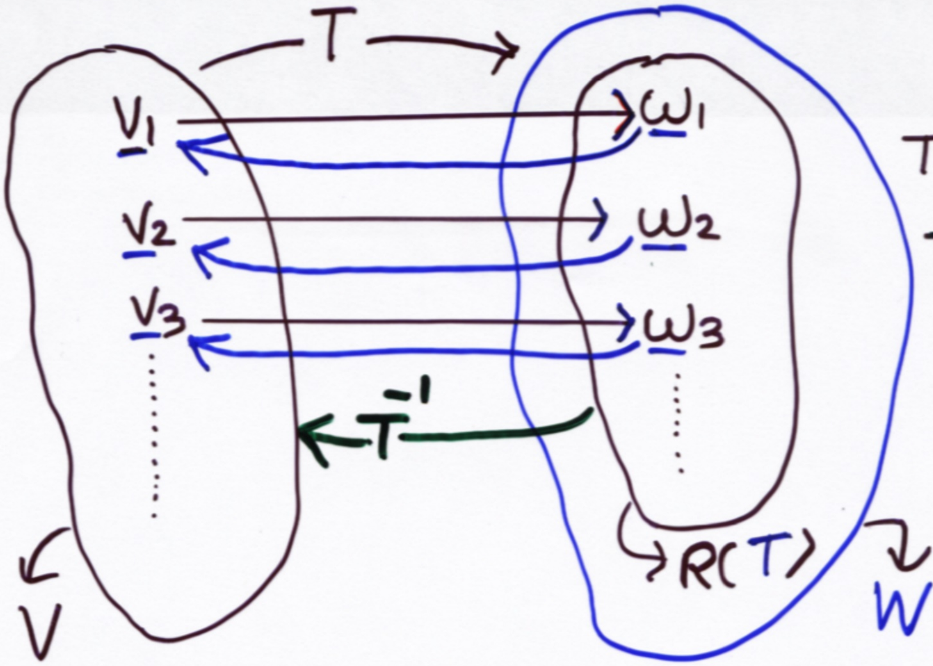
$$T^{-1}: R(T) \longrightarrow V$$

WHICH MAPS $w \in R(T)$ BACK INTO $v \in V$.

SEE THE FOLLOWING FIGURE

SEE THE FOLLOWING FIGURE

(12) (12)




$T(\underline{v}_1)=\underline{w}_1$

$T^{-1}(\underline{w}_1)=\underline{v}_1$

ETC.

$\because$ T IS ONE-TO-ONE $\therefore$ EACH VECTOR IN R(T) IS THE IMAGE OF A UNIQUE VECTOR IN V. R(T) MAY OR MAY NOT BE ALL OF W. FOR MORE DETAIL SEE Q.no.7 ASSIGNMENT 6(b).

RESULT: IF $T: R^n \to R^n$ IS MULTIPLICATION BY AN INVERTIBLE MATRIX A THEN THE INVERSE $T^{-1}: R^n \to R^n$ IS MULTIPLICATION BY $A^{-1}$. ( T IS LINEAR, $T^{-1}$ IS INVERSE LINEAR)

EXAMPLE: $T: R^2 \to R^2$ BE THE LINEAR OPERATOR THAT ROTATES EACH VECTOR IN $R^2$ THROUGH AN ANGLE $\theta$ GIVEN BY

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix},$$ ALSO

$T^{-1}: R^2 \to R^2$ IS $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

THAT ROTATES ...

$T : R^2 \rightarrow R^2$ IS $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

THAT ROTATES EACH VECTOR THROUGH AN ANGLE $-\theta$.