SI

Lecture 17
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Introduction

- One way to report the results of a hypothesis test is to state that the null hypothesis was or was not rejected at a specified -value or level of significance. This is called ***fixed significance level*** testing.
- The fixed significance level approach to hypothesis testing is useful because it leads directly to the concepts of type II error and power.
- But the fixed significance level approach does have a disadvantage. For example, in the rocket propellant problem earlier, we can say that the null hypothesis was rejected (for the given data) at the 0.05 level of significance. This statement of conclusions gives the decision maker no idea about whether the computed value of the test statistic was just barely in the rejection region or whether it was very far into this region. Furthermore, this approach may be unsatisfactory because some decision makers might be uncomfortable with the risks implied by .

P-Values in Hypothesis Tests

- To avoid these difficulties, the P-value approach has been adopted widely in practice.
- The *P-value* is the *probability that the test statistic will take on a value that is at least as extreme as the observed value of the statistic when the null hypothesis is true.*
- Thus, a P-value conveys much information about the weight of evidence against the null hypothesis, and so a decision maker can draw a conclusion at any specified level of significance.
- A formal definition of a *P-value.*

> ***P*-Value**
>
> The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

P-Values in Hypothesis Tests

- It is customary to consider the test statistic (and the data) significant when the null hypothesis is rejected; therefore, we may think of the P-value as the smallest level at which the data are significant.
- In other words, the P-value is the ***observed significance level***. Once the P-value is known, the decision maker can determine how significant the data are without the data analyst formally imposing a preselected level of significance.

Consider the two-sided hypothesis test for burning rate

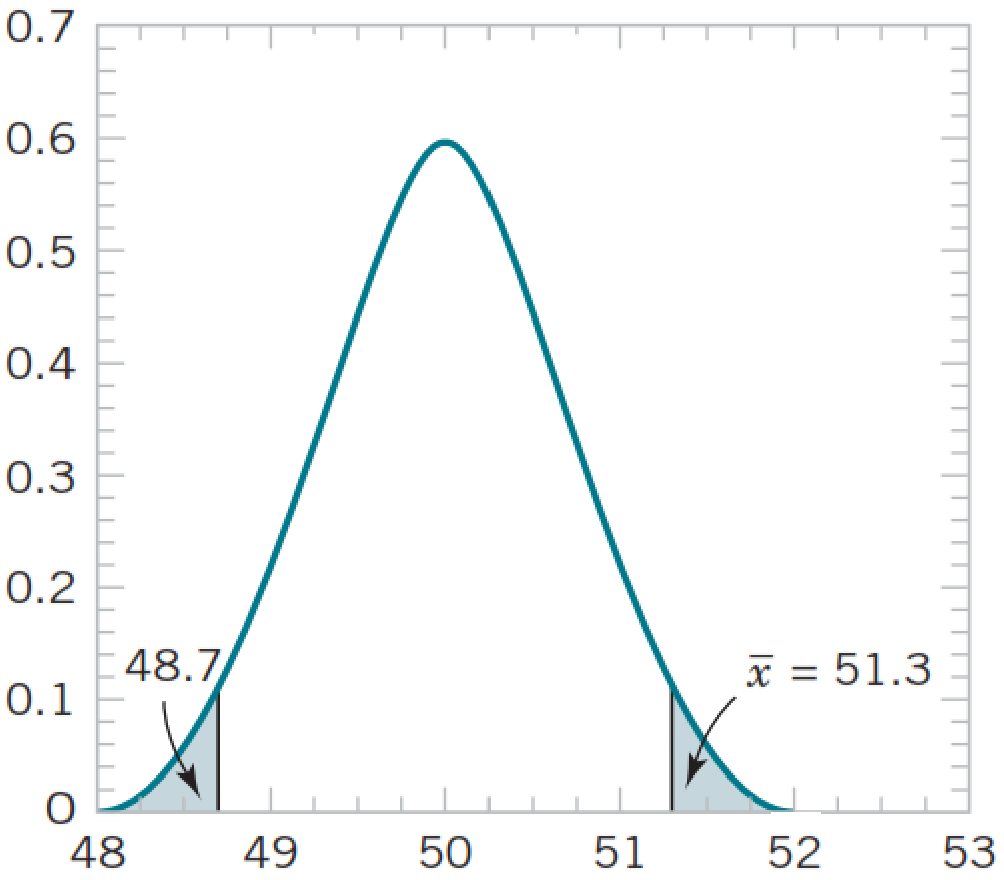
$$H_0\colon \mu = 50 \quad H_1\colon \mu \neq 50$$

with and . Suppose that the observed sample mean is cm/sec.

A critical region for this test with the value of and the symmetric value .

The P-value of the test is the probability above plus the probability below .

$$\begin{aligned} P\text{-value} &= P(\overline{X} > 51.3) + P(\overline{X} < 48.7) \\ &= 1 - P(48.7 < \overline{X} < 51.3) \\ &= 1 - P\left(\frac{48.7 - 50}{2.5/\sqrt{16}} < Z < \frac{51.3 - 50}{2.5/\sqrt{16}}\right) \\ &= 1 - P(-2.08 < Z < 2.08) \\ &= 1 - 0.962 = 0.038 \end{aligned}$$



The P-value tells us that if the null hypothesis is true, the probability of obtaining a random sample whose mean is at least as far from 50 as 51.3 (or 48.7) is .

Therefore, an observed sample mean of is a fairly rare event if the null hypothesis is really true.

Compared to the standard level of significance , our observed *P-value is smaller, so if we* were using a fixed significance level of , the null hypothesis would be rejected.

In fact, the null hypothesis would be rejected at *any level of significance greater than or equal to 0.038.*

This illustrates the previous boxed definition; the *P-value is the smallest level of significance that* would lead to rejection of .

Problem I

Suppose a null hypothesis is that the population mean is greater than or equal to 100. Suppose further that a random sample of 48 items is taken and the population standard deviation is 14. For each of the following α values, compute the probability of committing a Type II error if the population mean actually is 99.

a. $\alpha = .10$

b. $\alpha = .05$

c. $\alpha = .01$

d. Based on the answers to parts a, b, and c, what happens to the value of β as α gets smaller?

Problem II

For the last problem, use $\alpha = .05$ and solve for the probability of committing a Type II error for the following possible true alternative means.

a. $\mu_a = 98.5$

b. $\mu_a = 98$

c. $\mu_a = 97$

d. $\mu_a = 96$

e. What happens to the probability of committing a Type II error as the alternative value of the mean gets farther from the null hypothesized value of 100?

Finding the Probability of Type II Error β

Consider the two-sided hypotheses

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu \neq \mu_0$$

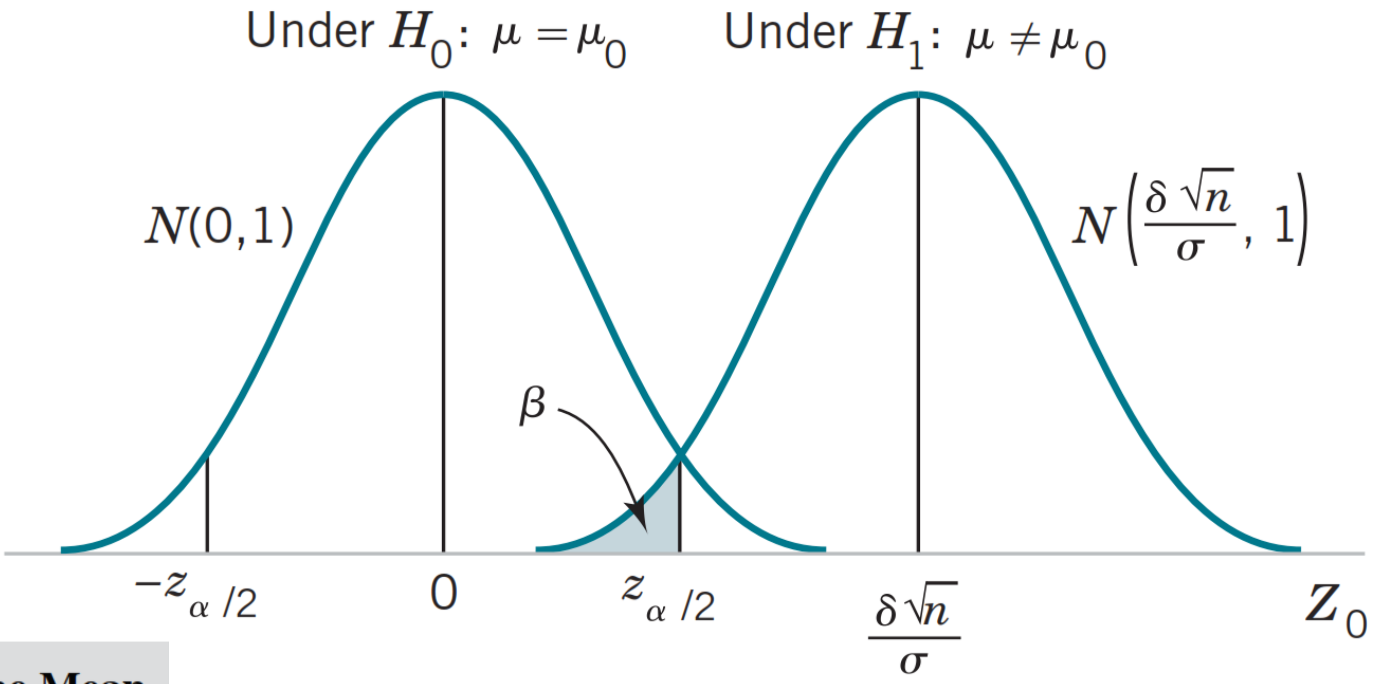
Suppose that the null hypothesis is false and that the true value of the mean is $\mu = \mu_0 + \delta$, say, where $\delta > 0$. The expected value of the test statistic Z_0 is

$$E(Z_0) = \frac{E(\overline{X}) - \mu_0}{\sigma/\sqrt{n}} = \frac{(\mu_0 + \delta) - \mu_0}{\sigma/\sqrt{n}} = \frac{\delta\sqrt{n}}{\sigma}$$

Therefore, the distribution of Z_0 when H_1 is true is

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right)$$

The distribution of the test statistic Z_0 under both the null hypothesis H_0 and the alternate hypothesis H_1 is shown in the Figure. From examining this figure, we note that if H_1 is true, a type II error will be made only if $-z_{\alpha/2} \leq Z_0 \leq z_{\alpha/2}$ where $Z_0 \sim N(\delta\sqrt{n}/\sigma, 1)$. That is, the probability of the type II error β is the probability that Z_0 falls between $-z_{\alpha/2}$ and $z_{\alpha/2}$ given that H_1 is true.



Probability of a Type II Error for a Two-Sided Test on the Mean, Variance Known

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$$