Weekly Challenge 14: Decidability

CS 212 Nature of Computation
Habib University
Ali Muhammad Asad - aa07190

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1. Deciding Primes

Show that a language, A, is decidable iff $A \leq_m PRIMES$, where $PRIMES$ is the set of all prime numbers.

Solution:

1. **If A is decidable, then A is mapping reducible to $PRIMES$**

   Assume A is decidable, then there exists a Turing Machine M that decides A. We can construct a computable function f such that for any string w, $w \in A$ iff $f(w) \in PRIMES$. We define f as follows: For each input w, compute $M(w)$. If $M(w)$ accepts, let $f(w)$ be a fixed prime number. If $M(w)$ rejects, let $f(w)$ be a fixed composite number. Then this function f is computable as M is a decider, and thus halts on all inputs.

   Thus, $w \in A \implies f(w)$ is prime, and $w \in A \implies f(w)$ is composite. Therefore, A is mapping reducible to $PRIMES$.

2. **If A is mapping reducible to $PRIMES$, then A is decidable**

   Assume A is mapping reducible to $PRIMES$. Then there exists a computable function f such that for any string w, $w \in A$ iff $f(w) \in PRIMES$.

   Since $PRIMES$ is decidable since there exists algorithms that can decide primes, we can construct a Turing Machine PM that decides A as follows:

   (i) On input w, compute $f(w)$.
   (ii) Run the algorithm that decides $PRIMES$ on $f(w)$.
   (iii) If the algorithm accepts, accept. If the algorithm rejects, reject

   So the computability of f ensures that PM can always compute $f(w)$ and the decidability of $PRIMES$ ensures that PM always halts. Thus, PM is a decider for A. Therefore, A is decidable.

Hence, we can conclude that A is decidable iff A is mapping reducible to $PRIMES$.

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