Homework 6b Updated

Question 01

Solution: $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

Question 06

Solution: Considering linearly dependent equation that holds for eigenvalues in R^3 is

$$k_1 v + k_2 Av + k_3 A^2 v + k^4 A^3 v = 0 \qquad \forall k_i \neq 0$$

Now let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $Av = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$, $A^2 v = \begin{bmatrix} 24 \\ 20 \\ 20 \end{bmatrix}$ and $A^3 v = \begin{bmatrix} 176 \\ 168 \\ 168 \end{bmatrix}$. So the matrix would be in the form of $k_1 v + k_2 Av + k_3 A^2 v + k^4 A^3 v = 0 \Rightarrow$

$$\begin{bmatrix} 1 & 4 & 24 & 176 \\ 0 & 2 & 20 & 168 \\ 0 & 2 & 24 & 176 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Echoleon form of above matrix leads to values for k_i.

$$\begin{aligned} k_1 &= 16k_3 + 160k_4 \\ k_2 &= -10k_3 - 84k_4 \\ k_3 &= t \\ k_4 &= s \end{aligned}$$

Lets take $s = 1$ and $t = -12$ for simplication of k_i. Hence $k_1 = 32, k_2 = 36, k_3 = -12$ and $k_4 = 1$.

Now substitute in above equation and solve for the roots of equation, we have 8,2,2.

Question 06

(a) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Solution: $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$

We have two eigenvalues $\lambda_1 = 4 \& \lambda_2 = 2$. First take λ_1

$$\begin{bmatrix} 3-\lambda_1 & 1 & 0 \\ 1 & 3-\lambda_1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3-4 & 1 & 0 \\ 1 & 3-4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence we have $-x_1 + x_2 = 0$ and $x_2 = x_2$, so first eigenvector would be $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly for λ_2, we have second eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For orthogonal, we need

to each vector by its norm, so it would be unit vector...
$v_1^* = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_2^* = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So, matrix P would be $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$
and $P^{-1} = P^t = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$P^{-1}AP = P^tAP = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution: Using similar procedure we have eigenvalues $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 0$
Eigenvalue: 2 , eigenvector: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so unit vector $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

Eigenvalue: 0 , eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so they are orthogonal hence we only need to make them unit vector. Unit eigenvectors are $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Form the matrix P, whose column i is eigenvector no. $i : P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Form the diagonal matrix D whose element at row, column is eigenvalue no. $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Question 07

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.
Solution: Proof. Let λ_1 and λ_2 be distinct eigenvalues with associated eigenvectors v_1 and v_2. Then, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Take the inner product of the first equation by ν_2 and the inner product of the second equation by ν_1 :

$$v_2^{\mathrm{T}} A v_1 = \lambda_1 v_2 v_1, \quad A v_2{}^{\mathrm{T}} v_1 = \lambda_2 v_2 v_1$$

In Equation, $(A\nu_2)^\top \nu_1 = v_2^\top A^\top v_1$, so becomes $v_2^{\mathrm{T}} A v_1 = \lambda_1 v_2 v_1$,$v_2^{\mathrm{T}} A^{\mathrm{T}} v_1 = \lambda_2 v_2 v_1$
Since $A^\top = A$, in Equation, we have $v_2^{\mathrm{T}} A v_1 = \lambda_1 v_2 v_1, \quad v_2^{\mathrm{T}} A v_1 = \lambda_2 v_2 v_1$ and

$$\lambda_1 v_2 v_1 = \lambda_2 v_2 v_1$$

Equation gives

$$(\lambda_1 - \lambda_2) v_2 v_1 = 0.$$

Since $\lambda_1 \neq \lambda_2, \langle v_2, v_1 \rangle = 0$, and v_1, v_2 are orthogonal.