Using Packages

SEL Activity 2

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We have two random variables defined as, $X \backsim \mathcal{U}[0,T]$ and $Y|X \backsim \mathcal{U}[X, X+\epsilon]$, where ϵ is defined as infinestimely small positive number. The pdf of X and $Y|X$ will be,

$$f_X(x) = \begin{cases} \frac{1}{T} & 0 \leq x \leq T \\ 0 & \text{else} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\epsilon} & x \leq y \leq x+\epsilon \\ 0 & \text{else} \end{cases}$$

The joint pdf of X and Y will be,

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\epsilon T} & 0 \leq y-\epsilon \leq x \leq y \leq T \\ 0 & \text{else} \end{cases}$$

The marginal pdf of Y will be,

$$f_Y(y) = \begin{cases} \int_0^y \frac{dx}{\epsilon T} & 0 \leq y \leq \epsilon \\ \int_{y-\epsilon}^y \frac{dx}{\epsilon T} & \epsilon \leq y \leq T \\ \int_{y-\epsilon}^T \frac{dx}{\epsilon T} & T \leq y \leq T+\epsilon \\ 0 & \text{else} \end{cases} \tag{1}$$

$$f_Y(y) = \begin{cases} \frac{y}{\epsilon T} & 0 \leq y \leq \epsilon \\ \frac{1}{T} & \epsilon \leq y \leq T \\ \frac{T-y+\epsilon}{\epsilon T} & T \leq y \leq T+\epsilon \\ 0 & \text{else} \end{cases}$$

The conditional pdf of X given Y will be,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{y} & 0 \leq x \leq y \\ \dfrac{1}{\epsilon} & y - \epsilon \leq x \leq y \\ \dfrac{1}{T - y + \epsilon} & y - \epsilon \leq x \leq T \\ 0 & \text{else} \end{cases}$$

The estimation of X given Y will be,

$$\begin{aligned}
\hat{X} = \mathrm{E}[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \\
\hat{X} &= \int_0^y \frac{x}{y} dx + \int_{y-\epsilon}^{y} \frac{x}{\epsilon} dx + \int_{y-\epsilon}^{T} \frac{x}{T + \epsilon - y} dx \\
\hat{X} &= \frac{1}{y} \int_0^y x dx + \frac{1}{\epsilon} \int_{y-\epsilon}^{y} x dx + \frac{1}{T + \epsilon - y} \int_{y-\epsilon}^{T} x dx \\
\hat{X} &= \left.\frac{x^2}{2y}\right|_0^y + \left.\frac{x^2}{2\epsilon}\right|_{y-\epsilon}^{y} + \left.\frac{x^2}{2(T + \epsilon - y)}\right|_{y-\epsilon}^{T} \\
\hat{X} &= \frac{y^2 - 0^2}{2y} + \frac{y^2 - (y-\epsilon)^2}{2\epsilon} + \frac{T^2 - (y-\epsilon)^2}{2(T + \epsilon - y)} \\
\hat{X} &= \frac{y}{2} + \frac{2y - \epsilon}{2} + \frac{(T - y + \epsilon)(T + y - \epsilon)}{2(T + \epsilon - y)} \\
\hat{X} &= 2y - \epsilon + \frac{T}{2}
\end{aligned} \tag{2}$$