Lecture 3

Monday, January 17, 2022 7:33 PM

1) LINEAR ALGEBRA

LECTURE 3

REVISION:

THEOREM

IF $\boxed{B}$ AND $\boxed{C}$ ARE BOTH INVERSES OF THE MATRIX $\boxed{A}$, THEN $\boxed{B=C}$

There is only one inverse

PROOF: $\because$ $\boxed{B}$ IS AN INVERSE OF $\boxed{A}$, WE HAVE $\boxed{BA=I} \rightarrow (1)$

MULTIPLYING BOTH SIDES (OF (1)) ON THE RIGHT BY $\boxed{C}$ GIVES

$$(BA)C = IC = C. \rightarrow (2)$$

BUT $(BA)C = B(AC)$, $\because \boxed{AC=I}$

$$= B(I) = B, \rightarrow (3)$$

SO THAT $C = B$ FROM (2) AND (3).

antonyms

invertible

singular

RESULT: INVERSE OF A NONSINGULAR MATRIX IS $\boxed{\text{UNIQUE}}$.

THEOREM

IF A AND B ARE INVERTIBLE MATRICES OF THE SAME SIZE, THEN $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

PROOF:

CONSIDER

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$$

$$= AA^{-1} = I. \text{ --- (1)}$$

ALSO

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB$$

$$= B^{-1}B = I \text{ --- (2)}$$

FROM (1) AND (2)

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

RESULT: IF $A_1, A_2, \ldots\ldots, A_n$ ARE INVERTIBLE MATRICES OF SAME SIZE THEN

$$\boxed{(A_1A_2 \cdots\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \cdots\cdots A_2^{-1}A_1^{-1}}$$

$A_1 A_2 \cdots A_n$ $(A_n^{-1} A_{n-1}^{-1} \cdots A_1^{-1})$

3) IN SHORT

(1) A PRODUCT OF ANY NUMBER OF INVERTIBLE MATRICES IS INVERTIBLE.

(2) THE INVERSE OF THE PRODUCT IS THE PRODUCT OF THE INVERSES IN THE $\boxed{\text{REVERSE ORDER.}}$

NOTATION: $A^n = \underbrace{A.A.A.\cdots\cdots A}_{n \text{ TIMES}}$ $(n>0)$

$A \rightarrow$ SQUARE MATRIX

TRY THE FOLLOWING:

(1) IF $\boxed{A}$ IS AN INVERTIBLE MATRIX, THEN:

$$\boxed{(A^{-1})^{-1} = A}$$

PROOF: $\because AA^{-1} = A^{-1}A = I$

$$\Rightarrow \boxed{(A^{-1})^{-1} = A}$$

(2) $(A^n)^{-1} = (A^{-1})^n$ FOR $n = 0, 1, 2, \ldots$

HINT: $A^n = \underbrace{AA\cdots\cdots A}$ $\rightarrow$ n TIMES

$(A^{-1})^{-1} = A$

$(A^n)^{-1}$

$(\underbrace{A A \cdots A}_{n\text{-times}})^{-1}$

A^{-1}

$= (\underbrace{A^{-1} A^{-1} \cdots A^{-1}}_{n \text{ times}})$

$= (A^{-1})^n$

4)

TRANSPOSE OF A MATRIX.

IF $\boxed{A}$ IS $\boxed{m \times n}$ MATRIX THEN TRANSPOSE OF $\boxed{A}$ (DENOTED BY A^T) IS OBTAINED BY INTERCHANGING ROWS AND COLUMNS OF $\boxed{A}$. ORDER (SIZE) OF $\boxed{A^T}$ $= \boxed{n \times m}$.

order of A
$m \times n$
order A^T $= n \times m$

EXAMPLE:

$$\underset{2 \times 3}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \rightarrow 3 \times 2$$

NOTICE THAT FIRST ROW OF $\boxed{A}$ BECOMES FIRST COLUMN OF $\boxed{A^T}$ ETC.

ALSO $(A^T)^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

illustrates, does not prove

$= A$.

RESULT: $\boxed{(A^T)^T = A}$ i.e. INTERCHANGING ROWS AND

5) COLUMNS TWICE LEAVES A MATRIX UNCHANGED.

RESULT: IF A AND B ARE MATRICES OF SAME SIZE THEN

$$\boxed{(A+B)^T = A^T + B^T}$$

EXAMPLE: LET $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

LHS

$$\boxed{(A+B)^T} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} \text{— (1)}$$

ALSO $\boxed{A^T + B^T}$ RHS

$$= \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix} \text{— (2)}$$

$\therefore (A+B)^T = A^T + B^T$ FROM (1) AND (2)

induction

check 1×1 $n \times m$ $k = n+m$

assume it is true for k

show using this that it is true for $k+1$

Why is a symmetric matrix a square matrix?

DEFINITION:

A $\boxed{\text{SQUARE}}$ MATRIX $\boxed{A}$ IS CALLED **SYMMETRIC** IF $\boxed{A^T = A.}$

EXAMPLES:

(1) IDENTITY MATRIX IS SYMMETRIC $\because I^T = I$ (ALWAYS)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) EVERY DIAGONAL MATRIX IS SYMMETRIC e.g.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}^T = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

(3) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix} = A$ IS SYMMETRIC SINCE

$A^T = \begin{bmatrix} 0 & 1 & \boxed{2} \\ 1 & 4 & 3 \\ \boxed{2} & 3 & 5 \end{bmatrix} = A$, HERE $\boxed{a_{12} = a_{21} = 1}$, $a_{13} = a_{31} = 2$

AND $a_{23} = a_{32} = 3$.

NOTE: WHILE TAKING THE TRANSPOSE, DIAGONAL ENTRIES OF A SQUARE MATRIX DON'T $\boxed{\text{CHANGE}}$ THEIR POSITIONS.

$$\text{IF } \boxed{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{A^t} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

DEFINITION:

(8)

A SQUARE MATRIX $\boxed{A}$ IS CALLED SKEW-SYMMETRIC IF $\boxed{A^T = -A}$

EXAMPLE:

$$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$ IS SKEW-SYMMETRIC

$$\because B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} = -B$$

HERE $a_{12} = +1 = -a_{21}$, $a_{21} = \overset{=-1}{1}$ ETC.

DIAGONAL ENTRIES IN A SKEW-SYMMETRIC MATRIX ARE ALWAYS $= \boxed{\text{ZERO}}$, $\because a_{11} = -a_{11} \Rightarrow a_{11} = 0$, ETC.

ASSIGNMENT NO. 1

Q.no.1

(a) LET $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$, FIND $\boxed{A^2}$ AND $\boxed{A^3}$

(i) Given $A^3 = A^2 + A - 5I$

Premultiply both sides by A

$A^4 = A^3 + A^2 - 5A$

$A^4 = (A^2 + A - 5I) + A^2 - 5A$

$= 2A^2 - 4A - 5I$

(b) SHOW THAT $A^3 = A^2 + A - 5I$ (9)

(c) USING (b) WITHOUT DOING ANY MORE MULTIPLICATION PROVE THAT

(i) $A^4 = 2A^2 - 4A - 5I$,

(ii) $A^{-1} = \frac{1}{5}(I + A - A^2)$

$\frac{1}{5}(I + A - A^2)A$

$\frac{1}{5}(A + A^2 - A^3)$

$\frac{1}{5}(\cancel{A} + \cancel{A^2} - A^2 - A + 5I)$

$= \frac{1}{5}(5I) = I$

Q.no.2

SHOW THAT THE MOST GENERAL MATRIX THAT $\boxed{\text{COMMUTES}}$ WITH

SHOW THAT THE MOST GENERAL MATRIX THAT COMMUTES WITH

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ IS OF THE FORM

$$\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

DON'T USE MATRIX ENTRIES.

Q.no.3

(a) IF A BE A SQUARE MATRIX THEN $A+A^T$ IS SYMMETRIC AND $A-A^T$ IS SKEW SYMMETRIC.

(b) IF A IS $m \times n$ MATRIX THEN PROVE THAT AA^T AND A^TA ARE BOTH SYMMETRIC. (SEE Q.no.7)

(c) IF $A^2 = A$, A^{-1} EXISTS THEN $A = I$.

(d) IF A IS INVERTIBLE THEN PROVE THAT $(A^{-1})^T = (A^T)^{-1}$

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Q. no. 4

Question 2 from Exercise 1.3

Q. no. 5

LET $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ AND $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$,

SHOW THAT IF THE MATRIX $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ SATISFIES THE EQUATION $\boxed{AX = XB}$ THEN $\boxed{X}$ IS A SCALAR MULTIPLE OF $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Q. no. 6

IF $\boxed{A}$ IS A SQUARE MATRIX OF ORDER 3 s.t. $\boxed{A^t = -A}$ THEN PROVE THAT THE DIAGONAL ENTRIES OF $\boxed{A = 0}$.

$\boxed{\text{Q.no.7}}$

IMPORTANT RESULT:

IF $\boxed{A}$, $\boxed{B}$ ARE MATRICES S.T. AB IS DEFINED THEN

$$\boxed{(AB)^T = B^T A^T}$$

CHECK THIS RESULT FOR $\boxed{2\times2}$ MATRICES BY TAKING GENERAL ENTRIES.

NOTE: THE TRANSPOSE OF A PRODUCT OF ANY NUMBER OF MATRICES IS EQUAL TO THE PRODUCT OF THEIR TRANSPOSES IN THE REVERSE ORDER.

i.e.

$$\boxed{(A_1 A_2 \ldots\ldots A_n)^T = A_n^T \ldots\ldots A_2^T A_1^T}$$

Exercise Set 1.4

Question 11 (Hint: remember cos^2 +sin^2 =1), Q 13, Q15, Q16 (It's not. Why?), Q 17, Q29