Weekly Challenge 06: Context-Free Languages

CS 212 Nature of Computation
Habib University
Ali Muhammad Asad (aa07190)

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1. Closure

Given the following theorem, prove or disprove the given claim.

Theorem 1. *The class of context-free languages is not closed under intersection.*

Claim 1. *The class of context-free languages is not closed under complementation.*

Solution: Let L_1 and L_2 be any arbitrary context-free languages (CFLs). By the theorem given above, we know that the intersection for the class of CFLs is not closed under intersection.

Then $L_1 \cap L_2$ is not closed. By De Morgan's Law, $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$. If CFLs were closed under both union and complement, then they would be closed under intersection.

Closure under Union

Let C_1 and C_2 be two CFLs recognized by $G_1 = (V_1, \sum, R_1, S_1)$ and $G_2 = (V_2, \sum, R_2, S_2)$ respectively. Assume that $V_1 \cap V_2 = \emptyset$; if this assumption is not true, rename the variables of one of the grammars to make this condition true.

We need to construct a grammar $G = (V, \sum, R, S)$ such that $L(G) = L(G_1) \cup L(G_2)$. We can construct G as follows:

- $V = V_1 \cup V_2 \cup \{S\}$, where $S \notin V_1 \cup V_2$ (and $V_1 \cap V_2 = \emptyset$)
- $R = R_1 \cup R_2 \cup \{S \to S_1 \mid S_2\}$

The above grammar combines the two grammars G_1 and G_2 into a single grammar G, by adding a new start variable S and a new production rule $S \to S_1 \mid S_2$, which allows us to derive strings in $L(G_1) \cup L(G_2)$.

Consider an arbitrary string $w \in L(G)$. Then either $w \in L(G_1)$, or $w \in L(G_2)$ which implies that either $S_1 \overset{*}{\Longrightarrow}_G w$ or $S_2 \overset{*}{\Longrightarrow}_G w$. Since G has the production $S \to S_1 \mid S_2$, we can derive w from S by using this rule. Hence $S \overset{*}{\Longrightarrow}_G w$, which implies that $w \in L(G)$. Hence, if $w \in L(G)$, then $w \in L(G_1) \cup L(G_2)$.

Conversely, we have $S \overset{*}{\Longrightarrow}_G w$. Then either $S_1 \overset{*}{\Longrightarrow}_G w$ or $S_2 \overset{*}{\Longrightarrow}_G w$. Since $V_1 \cap V_2 = \emptyset, w$ is either derived from S_1 using R_1, or from S_2 using R_2. Therefore, $w \in L_1 \cup L_2$. Hence if $w \in L(G_1) \cup L(G_2)$, then $w \in L(G)$.

Hence proved that CFLs are closed under Union.

Closure under Complementation

We already estbalished that $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$, and that the intersection is not closed.

Assume that CFLs were closed under complementation. Then $\overline{L_1}$ and $\overline{L_2}$ are closed. And since CFLs are closed under union, then $\overline{L_1} \cup \overline{L_2}$ is a CFL, which implies that $\overline{\overline{L_1} \cup \overline{L_2}}$ is a CFL.

However, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$, which means that CFLs are closed under intersection, hence we arrive at a contradiction, as by the theorem given above, CFLs are not closed under intersection.

Since CFLs are closed under union, then they must not be closed under complementation. Hence proved.

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