Statistics & Inferencing Homework #02

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Q3) (a) Estimate λ $\quad N = 400$

C_1 : $\bar{x}_1 = [(213)(0) + 128(1) + --- + 1(5)]/400 = 0.6825$

C_2 : $\bar{x}_2 = 1.3225$

C_3 : $\bar{x}_3 = 1.8$

C_4 : $\bar{x}_4 = 4.68$

$\hat{\lambda}_1 = 0.6825 \quad \hat{\lambda}_2 = 1.3225 \quad \hat{\lambda}_3 = 1.8 \quad \hat{\lambda}_4 = 4.68$

(b) 95% CI $\quad \hat{\lambda} \pm z_{\alpha/2} \cdot \sqrt{\hat{\lambda}/N}$

$\alpha = 0.05 \quad \alpha/2 = 0.025 \quad z_{\alpha/2} = 1.96$

C_1 : $0.6825 \pm 1.96\sqrt{0.6825/400} \Rightarrow 0.6825 \pm 0.08096$

C_2 : $\Rightarrow 1.3225 \pm 0.1127$

C_3 : $\Rightarrow 1.8 \pm 0.131$

C_4 : $\Rightarrow 4.68 \pm 0.212$

(c) $P(X=k) = \lambda^k/k! \cdot e^{-\lambda}$, $k \in \mathbb{N}_0$

$E_{k,i} = P(X=k)\cdot N = \lambda^k/k! \cdot e^{-\lambda_i} \cdot N \quad i \in \{1,2,3,4\}, N = 400$

$E_{0,1} = 0.6855^0/0! \cdot e^{-0.6825} \cdot 400 = 202.14$

The table attached below summarizes the expected & observed outcomes.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
Observed 1	213	128	37	18	3	1	0	0	0	0	0	0	0
Expected 1	202.14	137.96	47.08	10.71	1.83	0.25	0.03	0	0	0	0	0	0
Observed 2	103	143	98	42	8	4	2	0	0	0	0	0	0
Expected 2	106.59	140.96	93.21	41.09	1359	3.59	0.79	0.15	0.02	0	0	0	0
Observed 3	75	103	121	54	30	13	2	1	0	1	0	0	0
Expected 3	66.12	119.02	107.11	64.27	28.92	10.41	3.12	0.8	0.18	0.04	0.01	0	0
Observed 4	0	20	43	53	86	70	54	37	18	10	5	2	2
Expected 4	3.71	17.37	40.65	63.41	74.19	69.44	54.16	36.21	21.18	11.02	5.16	2.19	0.86

Q5) $X \to$ DRV $\quad P(X=1)=\theta \quad P(X=2)=1-\theta \quad x_1=1, x_2=2, x_3=2$

$E(X) = \sum_{k=1}^{2} k \cdot P(X=k) = 1\cdot\theta + 2(1-\theta) = 2-\theta$

$\theta = 2 - E[X] \qquad \theta = 2-\bar{x}$

$\bar{x} = (1+2+2)/3 = 5/3$

$\hat{\theta} = 2 - 5/3 = 1/3$

$lik(\theta) = f(x_1,\dots,x_n|\theta) = f(x_1|\theta)\dots f(x_n|\theta)$

$= \theta\cdot(1-\theta)^2$

$\log(lik(\theta)) = \ln(\theta) + 2\cdot\ln(1-\theta)$

$l'(\theta) = 1/\theta - 2/1-\theta = (1-3\theta)/(\theta(1-\theta))$

$l'(\theta) = 0 \Rightarrow \theta = 1/3$

$l''(\theta) = (-3\theta^2+2\theta-1)/[\theta\cdot(1-\theta)]^2$

$\hat{\theta} = 1/3$

$f_{\theta|X}(\theta|x) = [f_{X|\theta}(x|\theta)\cdot f_\theta(\theta)]/f_X(x)$

$f_{X|\theta}(x|\theta) = \theta(1-\theta)^2$

$f_X(x) = \int_0^1 f_{X|\theta}(x|\theta)$

$f_\theta(\theta) = 1, \ \theta \in [0,1]$.

Then $\int_0^1 x^{a-1}\cdot(1-x)^{b-1}dx = \Gamma(a)\cdot\Gamma(b)/\Gamma(a+b)$

$\Gamma(a) = (a-1)!$

$f_X(x) = \int_0^1 \theta^1\cdot(1-\theta)^2\cdot d\theta = 1/12$

$f_{\theta|X}(\theta|x) = 12\theta(1-\theta)^2$

$\hat{\theta} = 1/3 \qquad lik(\theta) = \theta(1-\theta)^2 \qquad \tilde{\theta} = 1/3$

$f_{\theta|X}(\theta|x) = 12\theta(1-\theta)^2$

Q7) $E(X) = 1/p \qquad p = 1/E(X) \qquad \hat{p} = 1/\bar{x}$

$lik(p) = p^n(1-p)^{\sum x_i - n}$

$l(p) = n\ln p + (\sum x_i - n)\ln(1-p)$

$l'(p) = \frac{n}{p} - (\sum x_i - n)\frac{1}{1-p}$

$l'(p) = 0 \qquad p = 1/\bar{x}$

$l''(p) = -\frac{n}{p^2} - (\sum x_i - n)\left(\frac{1}{(1-p)^2}\right) \qquad \tilde{p} = 1/\bar{x}$

$Var(\tilde{p}) \approx 1/n\cdot I(p) \qquad I(p) = E\left(\left[\frac{\partial}{\partial p}\ln f(X|p)\right]^2\right)$

$Var(\tilde{p}) = -1/E(l''(p))$

$E(l''(p)) = -n/p^2(1-p)$

$Var(\tilde{p}) \approx p^2(1-p)/n$

$f_{X|P}(x|p) = p^n(1-p)^{\sum x_i - n} = \int_0^1 f_{X|P}(x|p)$

$\int_0^1 x^{a-1}(1-x)^{b-1}\cdot dx \qquad E(Y) = a/a+b$

$E(P|X) = n/\sum x_i$

$\hat{p} = 1/\bar{x} \qquad \tilde{p} = 1/\bar{x} \qquad Var(\tilde{p}) = p^2(1-p)/n$

Q13) $\hat{\alpha} = 3\bar{X}$ $\quad E(X_i) = \alpha/3$

$E(\hat{\alpha}) = \frac{3}{n}\sum E(X_i) = \frac{3}{n}\sum\frac{\alpha}{3} = \frac{3}{n}\times n\frac{\alpha}{3} = \alpha$

$Var(X_1) = E(X_1^2) - [E(X_1)]^2$

$f(x) = 1+\alpha x/2 \quad , \quad x \in [-1,1]$

$E(X_1^2) = \int_{-1}^{1} x^2 \cdot f(x)\,dx = 1/3$

$Var(X_1) = 1/3 - (\alpha/3)^2 = 3-\alpha^2/9$

$Var(\hat{\alpha}) = 3-\alpha^2/n$

$\hat{\alpha} - E(\hat{\alpha})/\sqrt{Var(\hat{\alpha})} \approx N(0,1)$

$\hat{\alpha} \approx N(\alpha, \frac{3-\alpha^2}{n})$

$P(|\hat{\alpha}| > 0.5) = 2 - 2\Phi(0.5/\sqrt{\frac{3-\alpha^2}{n}}) = \underline{0.498}$

Q16) $E(X) = \int x \cdot f(x)\,dx = \int \frac{1}{2\sigma} x\, e^{-\frac{|x|}{\sigma}}\,dx = 0$

$E(X^2) = \int x^2 f(x)\,dx = \int \frac{1}{2\alpha} x^2 e^{-\frac{|x|}{\alpha}}\,dx = \frac{1}{\sigma}\int x^2 e^{-\frac{|x|}{\alpha}}\,dx$

$= 2\alpha^2$

$\sigma^2 = \frac{1}{2}E(X^2) \qquad \sigma = \sqrt{\frac{1}{2}E(X^2)}$

$\hat{\sigma} = \sqrt{\frac{1}{2n}\sum x_i^2}$

$lik(\sigma) = (\frac{1}{2\sigma})^n \cdot e^{-\frac{|x_1|+\dots+|x_n|}{\sigma}}$

$\ell(\sigma) = n\ln(2\sigma) - \frac{|x_1|+\dots+|x_n|}{\sigma}$

$\ell'(\sigma) = 0 \Rightarrow \sigma = \frac{1}{n}\sum|x_i|$

$\tilde{\sigma} = \frac{1}{2n}\sum|X_i|$

$Var(\tilde{\sigma}) \approx \frac{1}{nI(\sigma)} \qquad Var(\tilde{\sigma}) = -1/E(\ell''(\sigma))$

$\ell''(\sigma) = \frac{n}{\sigma^2} - 2\frac{|x_1|+\dots+|x_n|}{\sigma^3}$

$E(\ell''(\sigma)) = \frac{n}{\sigma^2} - 2\,E(|x_1|)+\dots-E(|x_n|)/\sigma^3$

$E(|X|) = \sigma\Gamma(2) = \sigma \qquad Var(\tilde{\sigma}) \approx \sigma^2/n$

$c(\sigma) = -1/\sigma \qquad T(x) = |x| \qquad d(\sigma) = \ln(\frac{1}{2\sigma}) \ , \ S(x) = 0$

$T = \sum T(X) = \sum|X_i|$

Q17) $E(X^2) = Var(X) + [E(X^2)]. \qquad E(X^2) = \alpha+1/2(2\alpha+1)$

$\alpha = 2E(X^2)-1/1-4E(X^2)$

$\hat{\alpha} = [\frac{2}{n}\sum X_i^2 - 1]/[1 - \frac{4}{n}\sum X_i^2]$

$\ell(\alpha) = n\ln(\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}) + (\alpha-1)\sum\ln(x_i(1-x_i))$

$\ell'(\alpha) = 0 \qquad = -\frac{1}{2n}\sum\ln(x_i(1-x_i))$

$$Var(\tilde{\alpha}) \approx \frac{1}{2n\frac{\Gamma''(\alpha)\Gamma(\alpha)-\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} - 4n\frac{\Gamma''(2\alpha)\Gamma(2\alpha)-\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2}}$$

$T = \sum T(X_i) = \sum\ln(X_i \cdot (1-X_i))$

Q19) $f(x) = \frac{1}{6\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/2\sigma^2}$

$lik(\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x_1-\mu)^2+\dots+(x_n-\mu)^2}{2\sigma^2}}$

$l(\sigma) = n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$

$l'(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2$

$l'(\sigma) = 0 \Rightarrow \hat{\sigma} = \sqrt{\frac{1}{n}\sum(x_i-\mu)^2}$

$l(\mu) = n\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$

$l'(\mu) = 0 \Rightarrow \mu = \frac{1}{n}\sum x_i$

$\hat{\mu} = \frac{1}{n}\sum X_i$

$I(\mu) = 1/\sigma^2 \Rightarrow \frac{1}{nI(\mu)} = \frac{\sigma^2}{n}$

Q21) $E(X) = \theta+1 \Rightarrow \theta = E(X)-1$ $\quad \hat{\theta} = \bar{X}-1$

$l(\theta) = n\theta - \sum x_i$ $\quad \tilde{\theta} = X_1$

$T = X_1 = \min X_i \Rightarrow T = X_1 \rightarrow$ sufficient.

Q32) $\mu = \bar{X} = \frac{1}{16}\sum X_i = 3.36109$

$\hat{\sigma}^2 = \frac{16-1}{16}S^2 = \frac{1}{16}\sum(X_i-\mu)^2 = 3.2045$

$N(3.36109, 3.2045)$

μ: CI: $\bar{X} \pm t_{n-1,\alpha/2} \cdot \frac{S}{\sqrt{n}}$ $\quad S = \sqrt{\frac{1}{16-1}\sum(X_i-\bar{X})^2} = 1.8488.$

90% CI: 3.6109 ± 0.8102

95% CI: 3.6109 ± 0.9849

99% CI: 3.6109 ± 1.3621

σ^2: 90% CI: $[2.0509, 7.0623]$

95% CI: $[1.8651, 8.1904]$

99% CI: $[1.05632, 11.1461]$

$\Rightarrow \sigma = \sqrt{\sigma^2} \Rightarrow$ 90% CI: $[1.4321, 2.6575]$

95% CI: $[1.3657, 2.8619]$

99% CI: $[1.2503, 3.3386]$

$m = 4n$

Q47) $E(X) = \int x f(x|x_0,\theta) = \int \theta \cdot x_0^{\theta} \cdot x^{-\theta} \cdot dx$

$E(X) = \theta x_0^{\theta} \cdot \frac{x_0^{-\theta+1}}{\theta-1} = \frac{\theta}{\theta-1} \cdot x_0$

$\Rightarrow \theta = \frac{E(X)}{E(X)-x_0}$ $\quad \hat{\theta} = \frac{\bar{X}}{\bar{X}-x_0}$

$\ell(\theta) = n\ln(\theta) + n\theta\ln(x_0) - (\theta+1)\sum\ln(x)$

$\ell'(\theta) = \frac{n}{\theta} + n\ln(x_0) - \sum\ln(x_i)$

$\ell'(\theta) = 0 \Rightarrow \theta = \frac{1}{\frac{1}{n}\sum\ln(x_i) - \ln(x_0)}$

$\ell''(\theta) = -\frac{n}{\theta^2} < 0$ $\quad \tilde{\theta} = \frac{1}{\frac{1}{n}\sum\ln(X_i) - \ln(x_0)}$

$Var(\tilde{\theta}) \approx \frac{1}{nI(\theta)}$ $\quad E(\ell''(\theta)) = -n/\theta^2$

$Var(\tilde{\theta}) \approx \theta^2/n$

$T = \sum\ln(X_i)$ is sufficient

Q50) $E(X) = \theta\sqrt{\pi/2} \Rightarrow \theta = E(X)\sqrt{2/\pi} \Rightarrow \hat{\theta} = \bar{X}\sqrt{2/\pi}$

$\ell(\theta) = \sum\ln(x_i) - 2n\ln(\theta) - \frac{1}{2\theta^2}\sum x_i^2$

$\ell'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^3}\sum x_i^2$ $\quad \ell'(\theta) = 0 \Rightarrow \theta = \sqrt{\frac{1}{2n}\sum X_i^2}$

$\tilde{\theta} = \sqrt{\frac{1}{2n}\sum X_i^2}$

$Var(\hat{\theta}) = -1/E(\ell''(\theta))$

$\ell''(\theta) = \frac{2n}{\theta^2} - 3/\theta^4 \sum X_i^2$

$E(X^2) = 2\theta^2 \Rightarrow Var(\tilde{\theta}) \doteq \theta^2/4n.$

Q52) $E(X) = (\theta+1)/(\theta+2)$ $\quad \theta = (2E(X)-1)/[1-E(X)]$

$\hat{\theta} = \frac{2\bar{x}-1}{1-\bar{x}}$ $\quad \ell(\theta) = n\ln(\theta+1) + \theta\sum\ln(x_i)$

$\ell'(\theta) = n/\theta+1 + \sum\ln(x_i) \Rightarrow \ell'(\theta) = 0$

$\Rightarrow \theta = -(1 + 1/\frac{1}{n}\sum\ln(x_i))$

$\ell''(\theta) = -n/(\theta+1)^2 < 0$

$\tilde{\theta} = -(1 + 1/\frac{1}{n}\sum\ln(X_i))$

$Var(\tilde{\theta}) = (\theta+1)^2/n$

$T = \sum\ln(X_i)$

Q53) $E(X) = \theta/2 \Rightarrow \theta = 2E(X)$

$\hat{\theta} = 2\bar{X} \qquad E(\hat{\theta}) = \theta$

$Var(\hat{\theta}) = 4/n \, Var(X_1)$

$Var(X_1) = E(X_1^2) - [E(X_1)]^2$

$= \theta^2/12 \Rightarrow Var(\hat{\theta}) = \theta^2/3n$

$lik(\theta) = 1/\theta^n \qquad \tilde{\theta} = \max X_i$

$F_{X_n}(x) = x/\theta \qquad f_x(x) = n \cdot x^{n-1}/\theta^n$

$E(X_n) = \frac{n}{n+1}\theta$

$E(X_n^2) = \frac{n}{n+2}\theta^2 \qquad Var(X_n) = n\theta^2/[(n+1)^2(n+2)]$

$Bias = E(\tilde{\theta}) - \theta = -\theta/n+1$

$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias^2 = \frac{\theta^2}{3n} + 0^2 = \theta^2/3n$

$MSE(\tilde{\theta}) = Var(\tilde{\theta}) + Bias^2 = \frac{n\theta^2}{(n+1)^2(n+2)} + \left(-\frac{\theta}{n+1}\right)^2$

$= \frac{2\theta^2}{(n+1)(n+2)}$

$E(\tilde{\theta}) = \frac{n}{n+1}\theta \Rightarrow \theta = \frac{n+1}{n}E(\tilde{\theta}) \Rightarrow \theta = E\left(\frac{n+1}{n}\tilde{\theta}\right)$

$\tilde{\theta}_2 = \frac{n+1}{n}\tilde{\theta}$

Q57) $S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2 = \frac{1}{n-1}\left(\sum X_i^2 - n\bar{X}^2\right)$

$Var(X_i) = E(X_i^2) - [E(X_i)]^2 \qquad E[X_i^2] = \sigma^2 + \mu^2$

$E(\bar{X}^2) = Var(\bar{X}) + [E(\bar{X})]^2$

$Var(\bar{X}) = \sigma^2/n \qquad E(\bar{X}) = \mu \qquad E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu$

$E(S^2) = \sigma^2 \qquad \hat{\sigma}^2 = \frac{n-1}{n}S^2$

$E(\hat{\sigma}^2) = E\left(\frac{n-1}{n}S^2\right) = \frac{n-1}{n}E(S^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$

$\hat{\sigma}^2$ ~~is~~ is biased for σ^2

$MSE(\hat{\theta}) = Var(\hat{\theta}) + Bias^2(\hat{\theta})$

$Y = \frac{(n-1)S^2}{\sigma^2} \qquad Var(Y) = 2(n-1)$

$Var(S^2) = 2\sigma^4/n-1$

$MSE(S^2) = \frac{2\sigma^4}{n-1} + 0^2 = 2\sigma^4/n-1$

$Var(\hat{\sigma}^2) = 2(n-1)\sigma^4/n^2$

$Bias(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = -\frac{1}{n}\sigma^2$

$MSE(\hat{\sigma}^2) = (2n-1)\sigma^4/n^2$

$MSE(\hat{\sigma}^2) < MSE(S^2)$

$P = \rho\sum(X_i - \bar{X})^2 \qquad P = \rho(n-1)S^2 \qquad Var(P) = 2\rho^2(n-1)\sigma^4$

$E[P] = \rho(n-1)\sigma^2 \qquad Bias(P) = \sigma^2[\rho(n-1) - 1]$

$MSE(P) = \cancel{2\rho^2(n-1)\sigma^4} \; \sigma^4(1 + 2\rho - 2n\rho - \rho^2 + n^2\rho^2)$

$f(\rho) = \sigma^4(1 + 2\rho - 2n\rho - \rho^2 + n^2\rho^2)$

$f'(\rho) = 0 \Rightarrow \rho = 1/n+1 \qquad f''(\rho) = 2\sigma^4(n^2-1) > 0$

$\rho = 1/n+1$

Q60) $lik(\tau) = 1/\tau^n \; e^{-\frac{x_1+\dots+x_n}{\tau}}$

$l(\tau) = n\ln(\tau) - \frac{1}{\tau}\sum x_i$

$l'(\tau) = -n/\tau + 1/\tau^2 \sum x_i$

$l'(\tau) = 0 \Rightarrow \tau = \frac{1}{n}\sum x_i$

$\hat{\tau} = \frac{1}{n}\sum X_i = \bar{X}$

$S = \sum X_i \sim \Gamma(n, \frac{1}{\tau})$ $\quad \bar{X} = S/n$

$F_{\bar{X}}(x) = F_S(n \cdot x)$ $\quad f_{\bar{X}}(x) = n \cdot \tau^n/\Gamma(n) \cdot (n \cdot x)^{n-1} e^{-\tau n x}$

$= \frac{(n\tau)^n}{\Gamma(n)} x^{n-1} e^{-(n\tau)x}$ $\quad \bar{X} \sim \Gamma(n, n/\tau)$

$E(X_1) = \tau$ $\quad Var(X_1) = \tau^2$

$((\bar{X}-\tau)/\tau)\sqrt{n} \approx N(0,1)$

$\bar{X} \approx N(\tau, \tau^2/n)$

$E(\bar{X}) = \tau$ $\quad Var(\bar{X}) = \frac{1}{n^2} n \, Var(X_1) = \frac{1}{n} Var(X_1) = \tau^2/n$

$I(\tau) = 1/\tau^2 \Rightarrow 1/nI(\tau) = \tau^2/n$

$\bar{X} \approx N(\tau, \tau^2/n)$

Then interval $\Rightarrow [\bar{X}/1+z_{\alpha/2}/\sqrt{n} \, , \, \bar{X}/1-z_{\alpha/2}/\sqrt{n}]$

$\bar{X} \sim \Gamma(n, \frac{n}{\tau})$ $\quad P(\gamma_{n,n/\tau}(1-\alpha/2) \le \bar{X} \le \gamma_{n,n/\tau}(\alpha/2)) = 1-\alpha,$

$[\gamma_{n,n/\tau}(1-\alpha/2), \gamma_{n,n/\tau}(\alpha/2)]$

Q68) $T = \sum X_i$ $\quad P(X=k) = \lambda^k/k! \cdot e^{-\lambda}$

$f_{X|T}(x|t) = \frac{t!}{x_1! \cdot \dots \cdot x_{n-1}! \cdot (t - x_1 - \dots - x_{n-1})! \cdot n!}$

$f_{X|X_1}(x|k) = \frac{\lambda^{x_2+\dots+x_n}}{x_2! \cdot \dots \cdot x_n!} e^{-\lambda(n-1)}$

$f(x_1,\dots,x_n|\lambda) = \lambda^{T(x_1,\dots,x_n)} \cdot e^{-n\lambda} \cdot \frac{1}{x_1! \cdot \dots \cdot x_n!}$

Q70) $f(x_1,\dots,x_n|\theta) = g(T(x_1,\dots,x_n),\theta) \cdot h(x_1,\dots,x_n)$

$f(x) = \lambda e^{-\lambda x}$

$= \lambda^n \cdot e^{-\lambda(x_1+\dots+x_n)}$

$T = \sum X_i \rightarrow$ is sufficient.

Q73) $f(x|\theta) = x/\theta^2 \cdot e^{-x^2/2\theta^2}$ $\quad$ ~~$f(x|\theta) = \frac{x}{\theta^2} e$~~

$f(x|\theta) = e^{-\frac{1}{2\theta^2} \cdot x^2 - 2\ln\theta + \ln x}$

$T = \sum X_i^2$ is sufficient.