Homework 1
Solution

Assigned on September 4, 2023
Due on September 18, 2023

Maximum Points: 100

Learning Outcomes

After this homework, you should be able to:

(a) assign frames and obtain expressions for physical quantities in different frames;

(b) obtain kinematic model of a wheeled robot.

Instructions

- The homework assignment can be attempted in groups of two.
- Each group will register themselves as a group on Canvas under People.
- The homework submission on Canvas will be set up for group submission, so each group needs to make only one submission.
- **If it appears that a group member has not contributed to a homework assignment, then each member will be graded individually.**

Tasks

Problem 1
CLO-1/C-2
15 points

In terms of the $\hat{x}_w$, $\hat{y}_w$, $\hat{z}_w$ coordinates of a fixed world frame $\{w\}$, the frame $\{a\}$ has its $\hat{x}_a$-axis pointing in the direction $(0, 0, 1)$ and its $\hat{y}_a$-axis pointing in the direction $(1, 0, 0)$, and the frame $\{b\}$ has its $\hat{x}_b$-axis pointing in the direction $(1, 0, 0)$ and its $\hat{y}_b$-axis pointing in the direction $(0, 0, 1)$. All frames are stationary.

(a) Draw by hand the three frames, at different locations so that they are easy to see.

(b) Write down the rotation matrices R_a^w and R_b^w.

(c) Given R_b^w, find $\left(R_b^w\right)^1$ without using a matrix inverse? Verify its correctness using your drawing.

(d) Given R_a^w and R_b^w, how do you calculate R_b^a?

(e) Use R_b^w to change the representation of the point ${}^b p = (1, 2, 3)$ (which is in $\{b\}$ coordinates) to $\{w\}$ coordinates.

(f) Choose a point p represented by ${}^w p = (1, 2, 3)$ in $\{w\}$ coordinates. Calculate $p = R_b^w\, {}^w p$ and $p'' = R_w^b\, {}^w p$. For each operation, should the result be interpreted as changing coordinates (from the $\{w\}$ frame to $\{b\}$) without moving the point p or as moving the location of the point without changing the reference frame of the representation?

(g) An angular velocity ω is represented in $\{w\}$ as ${}^w\omega = (3, 2, 1)$. What is its representation in $\{a\}$?

Solution 1

(a) Figure 1

(b)

$$R_a^w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad R_b^w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) $R_b^{w-1} = R_w^b = R_b^{wT}$

(d) $R_b^a = R_w^a\, R_b^w = R_a^{wT}\, R_b^w = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(e) ${}^w p = {}^w R_b\, {}^b p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

(f) $p'' = {}^b R_w\, {}^w p$. Here, we are changing coordinates to frame b.

For p', we can interpret it as rotating the point p

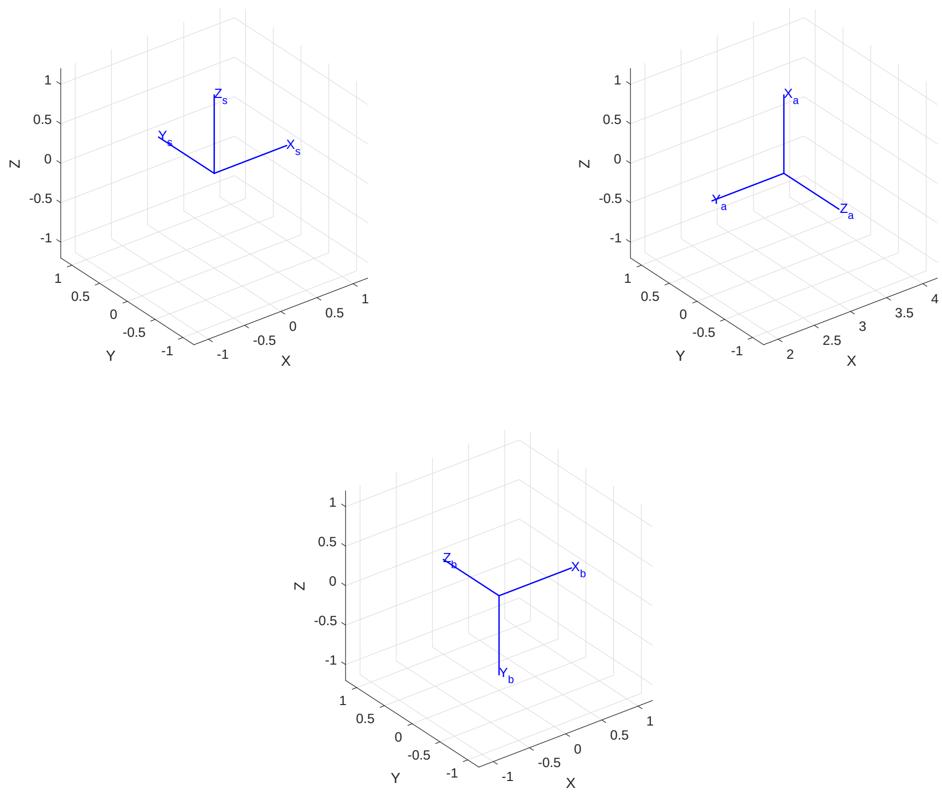

Figure 1: 3.1(a)

(g) Its representation in $\{a\}$ is:

$$^{a}\omega = {}^{a}R_{w}\ {}^{w}\omega = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Problem 2
CLO-1/C-3
20 points

An inertial measurement unit (IMU) is a popular sensor employed by a number of mobile robots for inertial navigation. It is comprised of at least two sensors - a 3-axis accelerometer and a 3-axis gyroscope [1]. Linear acceleration of an object can be determined from an accelerometer's measurements[1] and the gyroscope measures angular velocities. Historically, these sensors were mounted on mechanical structures, e.g. gimbals, isolating the sensors from the motion of the vehicle. In modern system, however, the sensors are *strapped down*

[1] I intentionally didn't say that an accelerometer measures acceleration, because it does not.

on the vehicle body. The laws of physics are such that the sensors can still only measure quantities with respect to an inertial reference frame, so the effects of the motion of the vehicle and motion of the earth have to be mathematically adjusted from the measurement data.

Highly sensitive IMUs can detect the rotation of the earth as well. Assume that you have access to such an IMU. A possible frame assignment for such an IMU is shown in Figure 2. Frame $\{i\}$ in Figure 2 is an inertial frame of reference at the center of earth. Frame $\{t\}$

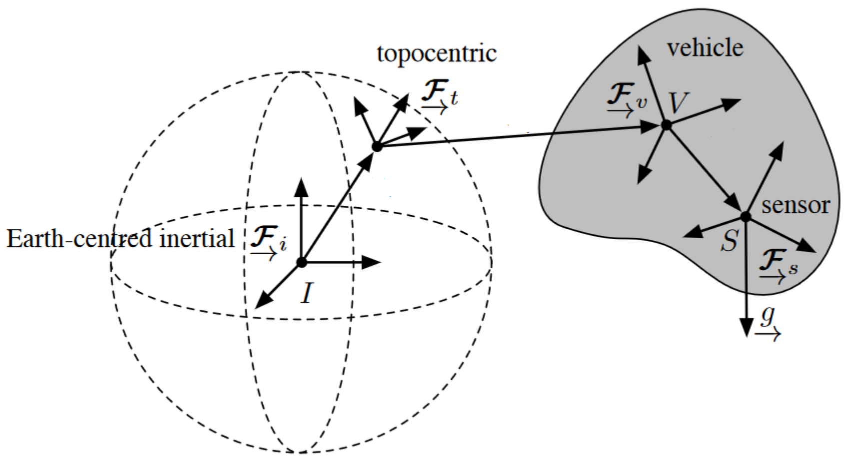

Figure 2: Frame assignment for an IMU [2]

is a frame located at the earth surface and incorporates the rotation of the earth. Frames $\{v\}$ and $\{s\}$ are the vehicle and sensor frames respectively. The IMU sensor located at the origin of $\{s\}$ is measuring ω_s^i and a_s^i, where ω and a are angular velocity and acceleration respectively. Find an expression for ${}^v a_v^t$, i.e. the acceleration of the robot with respect to earth surface written in the body coordinates.

Solution 2

$$r_s^i = r_t^i + r_s^t$$

Differentiating,

$$\begin{aligned} \left.\frac{d}{dt}\right|_i r_s^i &= \left.\frac{d}{dt}\right|_i r_t^i + \left.\frac{d}{dt}\right|_i r_s^t \\ v_s^i &= v_t^i + v_s^t + \omega_t^i \times r_s^t \end{aligned}$$

Let's differentiate it again,

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_i v_s^i &= \left.\frac{d}{dt}\right|_i v_t^i + \left.\frac{d}{dt}\right|_i v_s^t + \left.\frac{d}{dt}\right|_i \left(\omega_t^i \times r_s^t\right) \\
a_s^i &= a_t^i + a_s^t + \omega_t^i \times v_s^t + \left.\frac{d}{dt}\right|_i \omega_t^i \times r_s^t + \omega_t^i \times \left.\frac{d}{dt}\right|_i r_s^t \\
a_s^i &= a_t^i + a_s^t + \omega_t^i \times v_s^t + \alpha_t^i \times r_s^t + \omega_t^i \times \left(v_s^t + \omega_t^i \times r_s^t\right) \\
a_s^i &= a_t^i + a_s^t + 2\omega_t^i \times v_s^t + \alpha_t^i \times r_s^t + \omega_t^i \times \omega_t^i \times r_s^t
\end{aligned}
$$

We can simplify by noting that both the linear acceleration and angular acceleration of earth frame with respect to inertial frame is zero, i.e. $a_t^i = 0$ and $\alpha_t^i = 0$.

$$
\begin{aligned}
a_s^i &= a_s^t + 2\omega_t^i \times v_s^t + \omega_t^i \times \omega_t^i \times r_s^t \\
a_s^i &= a_s^t + 2\omega_t^i \times \left.\frac{d}{dt}\right|_t \left(r_v^t + r_s^v\right) + \omega_t^i \times \omega_t^i \times r_s^t \\
a_s^i &= a_s^t + 2\omega_t^i \times v_v^t + 2\omega_t^i \times \left(v_s^v + \omega_v^t \times r_s^v\right) + \omega_t^i \times \omega_t^i \times r_s^t
\end{aligned}
$$

Usually, the sensor is stationary with respect to the robot frame, i.e. $v_s^v = 0$

$$
a_s^i = a_s^t + 2\omega_t^i \times v_v^t + 2\omega_t^i \times \omega_v^t \times r_s^v + \omega_t^i \times \omega_t^i \times r_s^t
$$

The expression can be rearranged to find the quantity of interest:

$$
a_s^t = a_s^i - 2\omega_t^i \times v_v^t - 2\omega_t^i \times \omega_v^t \times r_s^v - \omega_t^i \times \omega_t^i \times r_s^t
$$

We're interested in a_v^t, so let's express a_s^t in terms of a_v^t:

$$
\begin{aligned}
r_s^t &= r_v^t + r_s^v \\
\left.\frac{d}{dt}\right|_t r_s^t &= \left.\frac{d}{dt}\right|_t r_v^t + \left.\frac{d}{dt}\right|_t r_s^v \\
v_s^t &= v_v^t + \underbrace{v_s^v}_{=0} + \omega_v^t \times r_s^v \\
\left.\frac{d}{dt}\right|_t v_s^t &= \left.\frac{d}{dt}\right|_t v_v^t + \left.\frac{d}{dt}\right|_t \omega_v^t \times r_s^v \\
a_s^t &= a_v^t + \alpha_v^t \times r_s^v + \omega_v^t \times \omega_v^t \times r_s^v
\end{aligned}
$$

Using this expression in the previous one:

$$
\begin{aligned}
a_v^t &= a_s^i - 2\omega_t^i \times v_v^t - 2\omega_t^i \times \omega_v^t \times r_s^v - \omega_t^i \times \omega_t^i \times r_s^t - \omega_v^t \times \omega_v^t \times r_s^v - \alpha_v^t \times r_s^v \\
a_v^t &= a_s^i - 2\omega_t^i \times v_v^t - 2\omega_t^i \times \omega_v^t \times r_s^v - \omega_t^i \times \omega_t^i \times r_v^t - 2\omega_v^t \times \omega_v^t \times r_s^v - \alpha_v^t \times r_s^v
\end{aligned}
$$

In this expression, ω_t^i is constant and r_s^v is a constant. As provided in the question, a_s^i is being measured by the accelerometer. Knowing the initial conditions, an inertial navigation loop can be established where a_v^t is integrated to obtain v_v^t and r_v^t, which are fed back in the a_v^t expression. As the question indicates, the installed gyroscope can be used to obtain ω_v^t and α_v^t. Since this is a coordinate-free expression, we can express all quantities in the coordinates of the robot frame.

Additionally, note that the sensor provides ω_s^i. We can obtain ω_v^t as follows:

$$^v\omega_s^i = {}^v\omega_t^i + {}^v\omega_v^t + {}^v\omega_s^v$$

Since the sensor frame is completely fixed in relation to the v frame, we have that $\omega_s^v = 0$.

$$\begin{aligned} ^v\omega_v^t &= {}^v\omega_s^i - {}^v\omega_t^i \\ &= {}^v\omega_s^i - R_t^v\, R_i^t\; {}^i\omega_t^i \end{aligned}$$

Further reading: [1, 6.3].

Problem 3
CLO-1/C-3

30 points

In this task, you're going to develop a kinematic model for CROMSCI, a climbing robot, illustrated in Figure 3, **using the equation for the standard steerable wheel**. This robot

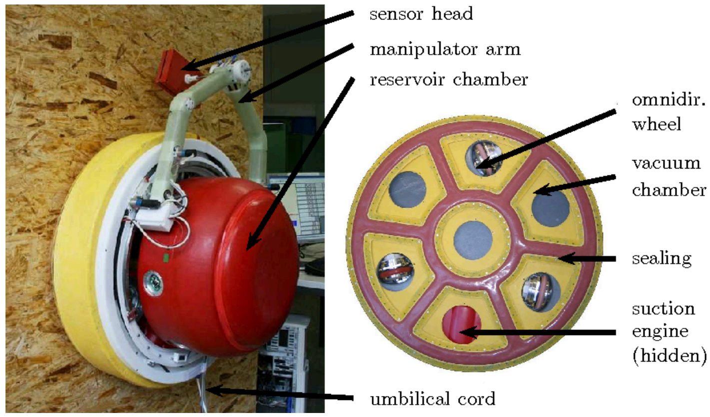

Figure 3: CROMSCI

has three wheels, each of which is both steerable and driven. The three wheels are arranged in a triangular configuration, as shown in Figure 4. This configuration allows the robot to be

Figure 4: Arrangement of three wheels on Cromsci

omnidirectional, i.e. it is able to move in any direction.

We'll set up our modeling in the following manner. The robot frame is to be placed at the center of the circular chassis. Let l_i be the line that runs through the center of a wheel and the kinematic center of the robot. Let α_i be the angle that the line l_i makes with x-axis of the robot frame, as shown in Figure 5. Then, the three wheels on this robot are located at $\alpha_1 = 0°$, $\alpha_2 = 120°$, and $\alpha_3 = -120°$, and at a distance d from the center of the robot. The x-axis of the wheel frame or contact frame is along the rolling direction of each wheel respectively. Let β_i be the steering angle of wheel i, defined as the angle formed by the current wheel axis (after steering) with l_i.
Obtain the differential kinematic model for this robot. You can view the equations obtained in [3] for reference.

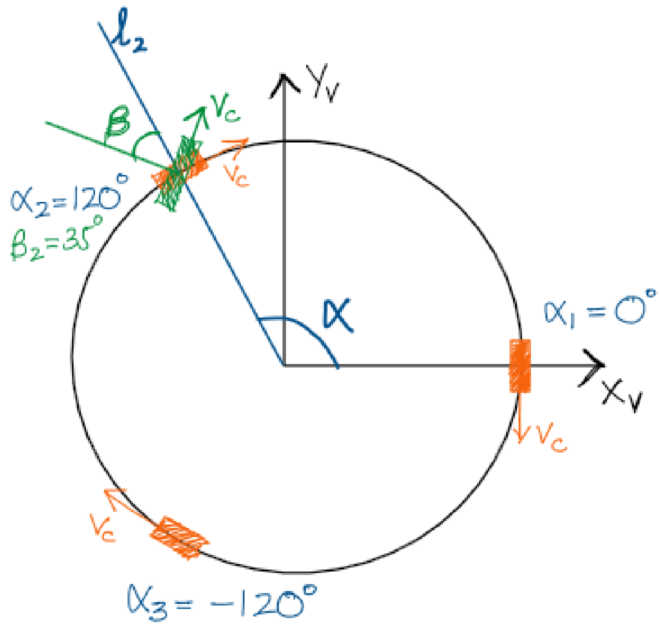

Figure 5: Modeling the Cromsci robot

Solution 3 We'll use the standard wheel equation on each wheel to obtain our kinematic model. Let v_1, v_2, v_3, and v be the velocities of wheel 1, wheel 2, wheel 3, and the robot respectively. Let ω be the angular velocity of the robot. We'll express all quantities in the robot frame, v.

Let's start by applying the wheel equation to wheel 2:

$$
\begin{aligned}
\begin{bmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{bmatrix} &= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} d\cos\alpha_2 \\ d\sin\alpha_2 \\ 0 \end{bmatrix} \\
\begin{bmatrix} v_2\cos(\beta_2+\alpha_2-90^\circ) \\ v_2\sin(\beta_2+\alpha_2-90^\circ \\ 0 \end{bmatrix} &= \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} -\omega d\sin\alpha_2 \\ \omega d\cos\alpha_2 \\ 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} v_2\sin(\alpha_2+\beta_2) \\ -v_2\cos(\alpha_2+\beta_2) \\ 0 \end{bmatrix} &= \begin{bmatrix} v_x - \omega d\sin\alpha_2 \\ v_y + \omega d\cos\alpha_2 \\ v_z \end{bmatrix}
\end{aligned}
$$

Similarly, the equations for other two wheels are:

$$\begin{bmatrix} v_1 \sin(\alpha_1 + \beta_1) \\ -v_1 \cos(\alpha_1 + \beta_1) \end{bmatrix} = \begin{bmatrix} v_x - \omega d \sin\alpha_1 \\ v_y + \omega d \cos\alpha_1 \end{bmatrix}$$
$$\begin{bmatrix} v_3 \sin(\alpha_3 + \beta_3) \\ -v_3 \cos(\alpha_3 + \beta_3) \end{bmatrix} = \begin{bmatrix} v_x - \omega d \sin\alpha_3 \\ v_y + \omega d \cos\alpha_3 \end{bmatrix}$$

Let's try to express these equations in terms of $v_i = r_i\omega_i$, the wheel velocities, from which the rotational speeds of each wheel can be obtained. Multiply the first equation in each set by $\sin(\alpha_i + \beta_i)$, the second equation by $-\cos(\alpha_i + \beta_i)$, and then add:

$$\begin{bmatrix} v_1 \sin^2(\alpha_1 + \beta_1) \\ v_1 \cos^2(\alpha_1 + \beta_1) \end{bmatrix} = \begin{bmatrix} v_x \sin(\alpha_1 + \beta_1) - \omega d \sin\alpha_1 \sin(\alpha_1 + \beta_1) \\ -v_y \cos(\alpha_1 + \beta_1) - \omega d \cos\alpha_1 \cos(\alpha_1 + \beta_1) \end{bmatrix}$$

Adding the two equations:

$$v_1 = v_x \sin(\alpha_1 + \beta_1) - v_y \cos(\alpha_1 + \beta_1) - \omega d \cos\beta_1$$

Similarly,

$$\begin{aligned} v_2 &= v_x \sin(\alpha_2 + \beta_2) - v_y \cos(\alpha_2 + \beta_2) - \omega d \cos\beta_2 \\ v_3 &= v_x \sin(\alpha_3 + \beta_3) - v_y \cos(\alpha_3 + \beta_3) - \omega d \cos\beta_3 \end{aligned}$$

Substituting the values of α_i:

$$\begin{aligned} v_1 &= v_x \sin\beta_1 - v_y \cos\beta_1 - \omega d \cos\beta_1 \\ v_2 &= \frac{v_x\sqrt{3}}{2}\cos\beta_2 - \frac{v_x}{2}\sin\beta_2 + \frac{v_y}{2}\cos\beta_2 + \frac{v_y\sqrt{3}}{2}\sin\beta_2 - \omega d\cos\beta_2 \\ v_3 &= \frac{v_x\sqrt{3}}{2}\cos\beta_3 - \frac{v_x}{2}\sin\beta_3 + \frac{v_y}{2}\cos\beta_3 + \frac{v_y\sqrt{3}}{2}\sin\beta_3 - \omega d\cos\beta_3 \end{aligned}$$

To obtain expressions for the angles β_i, we can consider our initial expressions:

$$\begin{bmatrix} v_i \sin(\alpha_i + \beta_i) \\ -v_i \cos(\alpha_i + \beta_i) \end{bmatrix} = \begin{bmatrix} v_x - \omega d \sin\alpha_i \\ v_y + \omega d \cos\alpha_i \end{bmatrix}$$

Thus,

$$\begin{aligned} \alpha_i + \beta_i &= \arctan 2\left(v_x - \omega d \sin\alpha_i,\ -v_y - \omega d \cos\alpha_i\right) \\ \Rightarrow \beta_i &= \arctan 2\left(v_x - \omega d \sin\alpha_i,\ -v_y - \omega d \cos\alpha_i\right) - \alpha_i \end{aligned}$$

Additional Content: We can also consider the constraint on the lateral velocity of the wheels, and use that to derive expressions for β_i. The components of the velocities v_{ix} and v_{iy} along the orthogonal direction to the wheel plane are:

$$\begin{aligned} v_{i\perp} &= -v_{ix}\sin(\beta_i + \alpha_i - 90^\circ) + v_{iy}\cos(\beta_i + \alpha_i - 90^\circ) \\ &= \sin\beta_i\left(v_{iy}\cos\alpha_i - v_{ix}\sin\alpha_i + \omega d\right) + \cos\beta_i\left(v_{iy}\sin\alpha_i + v_{ix}\cos\alpha_i\right) \end{aligned}$$

According to kinematic constraint, $v_{i\perp} = 0$. So,

$$\Rightarrow \frac{\sin\beta_i}{\cos\beta_i} = \frac{v_{ix}\cos\alpha_i + v_{iy}\sin\alpha_i}{v_{ix}\sin\alpha_i - v_{iy}\cos\alpha_i - \omega d}$$

Problem 4
CLO-1/C-3
15 points

Alphabot, our pet differential drive robot is currently located at coordinates $(1, 1)$ in the world frame with heading or orientation $\phi = 90°$. The robot width or distance between the wheels is 1. The commands that you can send to the vehicle are of the format (v_l, v_r, t), where v_l and v_r are the left and right wheel velocities, and t is the duration for which the robot has to follow this command.

(a) What command will you send to the robot to make it move in a circle of radius 1 in the clockwise direction from its present position and return to its starting position?

(b) Assume that the robot is to be moved to the location $(3, 1)$ and its final orientation is to be $\phi = 90°$. If you were constrained to use the least number of commands, what would be your strategy to execute the motion described above?

(c) For the same destination described in the previous part, what would be your strategy if your objective was to minimize the path length between the starting and ending positions.

Solution 4

(a) If the robot is currently located at $(1, 1)$ and we want it to move clockwise on a circle of radius 1, then the ICC is located at $(2, 1)$. Say, we want to move it at a constant angular speed of $\pi/10\, rad/s$, then the corresponding linear velocity is:

$$v(t) = \omega(t)R(t) = -\frac{\pi}{10}(-1m) = \frac{\pi}{10}\, m/s.$$

Subsequently,

$$\begin{aligned} v_r &= v + \frac{\omega L}{2} = \frac{\pi}{20}\, m/s \\ v_l &= v - \frac{\omega L}{2} = \frac{3\pi}{20}\, m/s \end{aligned}$$

So, the command is: $\left(\frac{3\pi}{20}, \frac{\pi}{20}, 20\right)$.

(b) Each command of a differential drive robot allows us to follow a portion of a circular trajectory, or to go straight, which is an infinite radius circular trajectory. It is possible to reach the desired location on a single circular trajectory [ref: (a)], but we won't be able to achieve the desired orientation. Thus, we require at least two commands. One possibility is to move on two semicircles - first clockwise to reach $(5, 1)$ on a circle of radius 2 and then again clockwise to reach $(3, 1)$ on a circle of radius 1:

1. $\left(\frac{5\pi}{20}, \frac{3\pi}{20}, 10\right)$.

2. $\left(\frac{3\pi}{20}, \frac{\pi}{20}, 10\right)$.

(c) The minimum length path between any two points is a straight line. So, we can use the strategy discussed in class. It can be achieved through the following three commands:

1. $\left(\frac{2\pi}{10}, \frac{\pi}{10}, 5\right)$.
2. $(1, 1, 3)$.
3. $\left(\frac{\pi}{10}, \frac{2\pi}{10}, 5\right)$.

Problem 5
CLO-1/C-2

20 points

Answer the following questions individually:

(a) How many hours did each of you spend on this homework? Answer as accurately as you can, as this will be used to structure next year's class.

(b) Each group member is to specifically state their contribution in this homework assignment.

(c) Each group member is to provide a note of at least one paragraph or a concept map, highlighting their understanding of the topics covered in this assignment, and a list of muddiest points/open questions. This should not be a chronological account of our classes, but a representation of the concepts as they exist in your mental model, i.e. How have you linked it to your prior knowledge? How have you linked the course concepts to each other?

Don't forget to indicate your name with your respective paragraph.

References

[1] A. Kelly, *Mobile robotics: mathematics, models, and methods*. Cambridge University Press, 2013.

[2] T. D. Barfoot, *State estimation for robotics*. Cambridge University Press, 2017.

[3] K. Berns and E. Von Puttkamer, "Autonomous land vehicles," *Vieweg+ Teubner GWV Fachverlage GmbH, Wiesbaden*, 2009.

Grading:

To obtain maximal score for each question, make sure to elaborate and include all the steps.