CS 314/PHYS 300: Quantum Computing: Homework #1

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Due on September 21, 2024, 11.59pm

Student 1 Name, ID **Student 2 Name, ID**

Problem 1

(10 points) Given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and an operator denoted by a matrix U, prove that

(a) (5 points) if U is unitary, then U preserves the norm of the qubit after application, i.e., $||\,|\psi\rangle\,|| = ||\,U|\psi\rangle\,||$.

(b) (5 points) alternately, if an operator U is applied on a qubit that preserves the norm of the qubit, then it must be unitary.

Solution Write solution here.

Problem 2

(10 points) Consider a Bloch Sphere representation of a qubit $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{(i\phi)}\sin(\theta/2)|1\rangle$ as shown here.

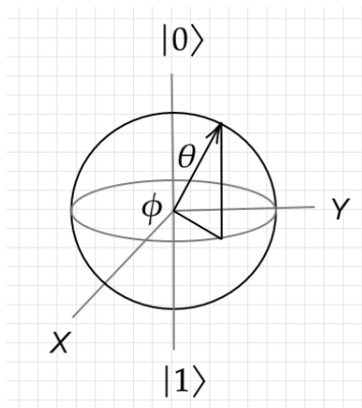

Figure 1: Bloch sphere representation of an arbitrary qubit state $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{(i\phi)}\sin(\theta/2)|1\rangle$

On the Bloch Sphere, plot the following qubits (show a plot similar to a diagram above):

(a) (2 points)

$$\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

(b) (3 points)

$$\frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle$$

Next, consider the following operations:

$$(i)\;\; R(\theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}, (ii)\;\; S(\phi_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi_2} \end{bmatrix}$$

Given an arbitrary qubit on the X-Z plane (i.e.) its amplitudes have no imaginary component,

(a) (2 points) what is the effect of applying $R(\theta_1)$ to it? In particular state the effect when $\theta_1 = \pi/2$?

(b) (3 points) what is the effect of applying $S(\phi_2)$ to it? In particular state the effect when $\phi_2 = \pi/2$?

Problem 3

(10 points) So far we have studied qubits defined using two basis states: $|0\rangle$ and $|1\rangle$. We have also seen two related states: the $|+\rangle$ state and the $|-\rangle$ state. Answer the following questions with reasons:

(a) (2 points) Do the $|+\rangle$ state and the $|-\rangle$ state form a pair of orthonormal states?

(b) (2 points) Can the $|+\rangle$ state and the $|-\rangle$ state be used as the basis states?

(c) If the answer to the both (a) and (b) is Yes, express the following qubits as a linear combination of the $|+\rangle$ state and the $|-\rangle$ state:

(i) (2 points) $|0\rangle$,

(ii) (2 points) $|1\rangle$,

(iii) (2 points) an arbitrary qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Problem 4

The Pauli Matrices are given by:

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that the following relations hold:
(a) (3 points) $HX = ZH$,
(b) (3 points) $HY = -YH$,
(c) (3 points) $HZ = XH$.
where H is the Hadamard operation.
(1 point) Using these properties, show that $HXHYHZ = ZYXH$

Problem 5

We have seen earlier that if we are given one of the two quantum states $|+\rangle$ and $|-\rangle$ such that it is not known whether it is a $|+\rangle$ and $|-\rangle$, we can distinguish between them *perfectly* by applying a Hadamard operation followed by a measurement.

Suppose we are randomly sent one of the following two states:

$$(a)\ \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, (b)\ \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle,$$

Devise a procedure to find out which one of the two states has been sent. This procedure should perform better than a random choice with a success probability as high as possible. More specifically, given one of the two qubits, by performing your operation(s) followed by a measurement, the probability of finding out which one of the states is should be as high as possible, with the lower limit being greater than a random guess, i.e. $> 1/2$.