Lecture 25

Tuesday, April 19, 2022 1:54 PM

1 MATH 221 $\boxed{\text{LECTURE 25}}$ LINEAR ALGEBRA

REVISION:

LAST TIME WE SAW THAT THE EIGENVALUES OF

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$ WERE $\boxed{\text{DISTINCT}}$ AND $= 1, 2, 3$

CORRESPONDING EIGENVECTORS ARE GIVEN BY

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \text{ AND } \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$ RESPECTIVELY

WHICH ARE LINEARLY INDEPENDENT $\therefore$ THEY FORM A $\boxed{\text{BASIS}}$ FOR $\boxed{R^3}$ BECAUSE $\boxed{A}$ IS DIAGONALIZABLE AS SHOWN BELOW:

$P^{-1}AP$

$$\overset{P^{-1}}{\begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \overset{P}{\begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{2}$ THEOREM 7.2.2: } P. 351 (8TH ED.) P. 369 (7TH ED.)

IF $\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_k$ ARE EIGENVECTORS OF $\boxed{A}$ CORRESPONDING TO DISTINCT EIGENVALUES $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$, THEN $\{\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_k\}$ IS A LINEARLY INDEPENDENT SET.

WHAT WILL HAPPEN IF EIGENVALUES ARE $\boxed{\text{NOT}}$ DISTINCT ?

CONSIDER THE FOLLOWING EXAMPLE :-

LET $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

EIGENVALUES OF $\boxed{A}$ ARE OBTAINED BY SOLVING $\boxed{\det(A-\lambda I)}$

$$= \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)[(4-\lambda)^2 - 4] - 2[2(4-\lambda)-4] + 2[4-2(4-\lambda)] = 0$$

SAME

$\boxed{3}$

$$\Rightarrow (4-\lambda)[\lambda^2 - 8\lambda + 12]$$
$$-4[2(4-\lambda)-4] = 0$$

$$\Rightarrow (4-\lambda)[\lambda-6)(\lambda-2)$$
$$- 8[4-\lambda-2] = 0$$

$$\Rightarrow (4-\lambda)(\lambda-6)\boxed{(\lambda-2)}$$
$$+ 8\boxed{(\lambda-2)} = 0$$

$$\Rightarrow \boxed{\lambda = 2} \text{ OR } (4-\lambda)(\lambda-6)+8 = 0$$

$$\Rightarrow -\lambda^2 + 10\lambda - 24 + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 16 = 0$$

$$\Rightarrow \boxed{(\lambda-8)(\lambda-2) = 0}$$

$$\Rightarrow \boxed{\lambda = 2, 2, 8 \rightarrow \text{REPEATED TWICE}}$$

EIGENVECTOR CORRESPONDING TO $\lambda = 8$, CONSIDER

$$\begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 4-8 \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x_1 + x_2 + x_3 = 0 \quad -(1)$$

$$x_1 - 2x_2 + x_3 = 0 \quad -(2)$$

$$x_1 + x_2 - 2x_3 = 0 \quad -(3)$$

$$(1) + 2(2) \Rightarrow x_2 - 4x_2 + x_3 + 2x_3 = 0$$

$$\Rightarrow -3x_2 = -3x_3 \Rightarrow \boxed{x_2 = x_3}$$

$$(2) + 2(3) \Rightarrow 3x_1 + x_3 - 4x_3 = 0$$

$$\Rightarrow \boxed{x_1 = x_3}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\text{CONSIDER FOR } \lambda = 2}$ REPEATED TWICE

$$\begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$

$$\Rightarrow x_1 = -t - s \text{ FOR } \boxed{\begin{array}{l} x_2 = t \\ x_3 = s \end{array}}$$

ONE EQUATION, THREE UNKNOWNS

$$\therefore \begin{bmatrix} x_1 \end{bmatrix} \quad \begin{bmatrix} -t-s \end{bmatrix} = t \begin{bmatrix} -1 \end{bmatrix} + s \begin{bmatrix} -1 \end{bmatrix}$$

UNKNOWNS

$$4 \leftarrow \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

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$\therefore$ EIGENVECTORS CORRESPONDING TO $\boxed{\lambda = 2}$ ARE OF THE FORM GIVEN BY 4 i.e. $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ FORM A BASIS FOR THE EIGENSPACE CORRESPONDING TO $\lambda = 2$, THEREFORE THEY ARE LINEARLY INDEPENDENT. THEREFORE EIGENVECTORS CORRESPONDING TO $\boxed{\lambda = 2}$ ARE

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (FOR $s = 0, t = 1$), $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (FOR $s = 1, t = 0$)

OR $\boxed{\text{LINEAR COMBINATION}}$ OF $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ AND $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. NOTICE THAT $\boxed{\lambda = 2}$ IS $\boxed{\text{REPEATED TWICE}}$ AND THE CORRESPONDING EIGENSPACE IS ALSO $\boxed{\text{TWO DIMENSIONAL}}$

$$\therefore \text{BASIS} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

IN THIS CASE $\boxed{A}$ IS DIAGONALIZABLE AND ONE COULD EASILY CHECK THAT $\boxed{P^{-1}AP = D}$

16 i.e,

$$\bar{P}^{1} A P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

NOTE: IF AN EIGENVALUE λ OF $\boxed{A}$ IS $\boxed{\text{REPEATED K TIMES}}$ AND THE $\boxed{\text{EIGENSPACE}}$ CORRESPONDING TO $\boxed{\lambda}$ IS K-DIMENSIONAL THEN THE SET CONSISTING OF THE $\boxed{\text{BASIS}}$ VECTORS $\boxed{\{\underline{v}_1, \ldots\ldots, \underline{v}_k\}}$ IS $\boxed{\text{LINEARLY INDEPENDENT}}$ AND THEY ARE ALSO $\boxed{\text{EIGENVECTORS}}$ CORRESPONDING TO $\boxed{\lambda}$ AS WE SAW IN THE LAST EXAMPLE.

7 7

EXAMPLE: EIGENVALUES OF

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = A$ ARE $\boxed{\lambda_1 = 3, \lambda_2 = \lambda_3 = 2}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = A \quad \text{ARE} \quad \boxed{\lambda_1 = 3, \lambda_2 = \lambda_3 = 2}$$

CHECK:

→ TRIANGULAR MATRIX. ITS EIGENVALUES ARE MAIN DIAGONAL ENTRIES.

FOR $\lambda = 2$: EIGENVECTORS

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 = 0$$

$x_3 = t$ (SAY)

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$ THEREFORE

THE EIGENSPACE CORRESPONDING TO $\lambda = 2$ IS ONE DIMENSIONAL, BUT $\lambda = 2$ IS REPEATED TWICE, SO A IS NOT DIAGONALIZABLE AS ONLY TWO OUT OF THREE EIGENVECTORS OF A ARE LINEARLY INDEPENDENT.

ASSIGNMENT 6(b) 8

1 FIND THE EIGENVALUES OF

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$ $\boxed{\text{WITHOUT}}$

FORMING THE CUBIC EQUATION

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

2 (SIMILAR MATRICES)

(i) IF $\boxed{A}$ IS SIMILAR TO MATRIX $\boxed{B}$ ($\boxed{A}$ & $\boxed{B}$ ARE SQUARE MATRICES) THEN $\boxed{B}$ IS ALSO SIMILAR TO $\boxed{A}$.

(ii) SIMILAR MATRICES HAVE THE SAME DETERMINANT (PROVE IT) i.e., IF $\boxed{A}$ IS SIMILAR TO $\boxed{B}$ THEN $\det(A) = \det(B)$

3 FIND A MATRIX $\boxed{P}$ THAT DIAGONALIZES $\boxed{A}$,

AND DETERMINE $P^{-1}AP$, WHERE

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$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

4 (Q. no. 11, 12) P. 345 8TH ED.
OR P. 363 (7TH ED.)

5 THEOREM 7.2.1 P. 347 8TH ED.
OR P. 365 7TH ED.

6 (a) FIND A MATRIX $\boxed{P}$ THAT ORTHOGONALLY DIAGONALIZES $\boxed{A}$, AND DETERMINE $P^{-1}AP$, WHERE $\boxed{A}$ IS

(i) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) WHAT IS THE SIGNIFICANCE OF THE COLUMN VECTORS OF $\boxed{P}$?

7 PROVE THAT IF $\boxed{A}$ IS A SYMMETRIC MATRIX THEN THE EIGENVECTORS FROM

DIFFERENT EIGENSPACES ARE ORTHOGONAL.

8 ARE THE FOLLOWING <u>TRUE</u> OR <u>FALSE</u>?

(i) IF $\boxed{A}$ IS DIAGONALIZABLE THEN $\boxed{A}$ HAS $\boxed{n}$ DISTINCT EIGENVALUES ($A \to n \times n$ MATRIX)

(ii) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ IS DIAGONALIZABLE.

(iii) IF $\boxed{A}$ IS A DIAGONALIZABLE MATRIX, THEN THE RANK OF $\boxed{A}$ IS THE NUMBER OF NONZERO EIGENVALUES OF $\boxed{A}$.

(iv) IF $\boxed{A}$ IS ANY $m \times n$ MATRIX, THEN $\boxed{A^tA}$ HAS AN ORTHONORMAL SET OF $\boxed{n}$ EIGENVECTORS.

(v) IF $\underline{v}$ IS ANY $n \times 1$ MATRIX AND $\boxed{I}$ IS THE $n \times n$ IDENTITY MATRIX, THEN $\boxed{I - \underline{v}\underline{v}^t}$ IS ORTHOGONALLY DIAGONALIZABLE.

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(9) Q. no. 1 (a, b) P. 360 8TH ED.
OR P. 378 (7TH ED.)

(LINEAR TRANSFORMATIONS)

① CONSIDER THE BASIS $S = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ FOR R^3, WHERE $\underline{v_1} = (1,1,1)$, $\underline{v_2} = (1,1,0)$, AND $\underline{v_3} = (1,0,0)$.

LET $T: R^3 \longrightarrow R^2$ BE THE LINEAR TRANSFORMATION SUCH THAT

$$T(\underline{v_1}) = (1,0),\ T(\underline{v_2}) = (2,-1),$$

$$T(\underline{v_3}) = (4,3)$$

FIND A FORMULA FOR $T(x_1, x_2, x_3)$; THEN USE THIS FORMULA TO COMPUTE $T(2,-3,5)$.

$\boxed{\text{Q.no.2}}$ 12

CHECK WHETHER THE FOLLOWING MAPPINGS ARE LINEAR?

(a) $F(x,y) = (2x, y)$

(b) $F(x,y) = (x^2, y)$

(c) $F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b + c$

(d) $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y+z \end{bmatrix}$

(e) $T(A) = AB$, $A, B \rightarrow$ MATRICES

$\boxed{\text{Q.no.3}}$

(a) IF $T(\underline{e}_1) = (1,1)$,
$T(\underline{e}_2) = (3,0)$ AND
$T(\underline{e}_3) = (4,-7)$ THEN
FIND $\boxed{T(1,3,8)}$ PROVIDED
T IS LINEAR. ($T: R^3 \rightarrow R^2$)

ANSWER: $T(1,3,8) = (42,-55)$

(b) ALSO FIND $T(x,y,z)$ by USING INFORMATION IN PART (a). 13

ANS. $\begin{bmatrix} x+3y+4z \\ x-7z \end{bmatrix}$

(c) USING PART (b) FIND THE MATRIX OF LINEAR TRANSFORMATION.

ANS. $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{bmatrix}$

$\boxed{\text{Q.no.4}}$ Q.29, Q.30 P.375 (8TH ED.)

OR Q.30, Q.31 P.395 (7th ED.)

$\boxed{\text{Q.no.5}}$ (PREVIOUS CONCEPT) REVISITED.

IF $\underline{u}_1 = \underline{v}_1$ AND $\underline{u}_2$ ARE EIGENVECTORS OF $\boxed{A}$ CORRESPONDING TO λ THEN PROVE THAT $\underline{v}_2 = \underline{u}_2 - \dfrac{(\underline{u}_2 . \underline{v}_1)\underline{v}_1}{\|\underline{v}_1\|^2}$ IS ALSO AN EIGENVECTOR OF $\boxed{A}$ CORRESPONDING TO λ.

$\boxed{Q. no. 6}$ 14

(a) IF $T: R^n \longrightarrow R^m$ BE THE MATRIX TRANSFORMATION $T(\underline{x}) = A\underline{x}$, FIND $\boxed{KER(T)}$ AND $\boxed{\text{RANGE OF } T}$ $(R(T))$.

(b) SEE THE DEFINITIONS OF NULLITY(T) AND RANK(T) (P. 378 8TH ED., P. 397 7TH ED.) WHERE T IS A LINEAR TRANSFORMATION.

(C) USING PARTS (a & b) TRY THE FOLLOWING

IF A IS AN $m \times n$ MATRIX AND $T: R^n \longrightarrow R^m$ IS MULTIPLICATION BY A, THEN

(i) NULLITY(A) = NULLITY(T)

(ii) RANK(A) = RANK(T)

(d) USING PART(a), TRY THE FOLLOWING:

LET $T: R^3 \longrightarrow R^3$ BE THE LINEAR OPERATOR DEFINED BY THE FORMULA

$T(x_1, x_2, x_3)$

$= (x_1 + x_2 + 2x_3,\ x_1 + x_3,\ 2x_1 + x_2 + 3x_3).$

FIND BASES FOR THE KERNEL AND RANGE OF T.

HINT: WRITE DOWN

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ x_1 + x_3 \\ 2x_1 + x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\underline{X}$$

AND PROCEED.

$\boxed{\text{Q.no.7}}$

Q.no.19 (P.389 8th ED.)

(P.410 7th ED.)

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$\boxed{\text{Q. no. 8}}$

SEE THE STATEMENT OF THE DIMENSION THEOREM FOR LINEAR TRANSFORMATIONS ON P. 379 (8th ED.) OR P. 398 (7TH ED.)

THEN PROVE THAT THIS THEOREM AGREES WITH DIMENSION THEOREM FOR MATRICES IF $\boxed{A}$ IS AN $m \times n$ MATRIX AND $T: R^n \to R^m$ IS MULTIPLICATION BY $\boxed{A}$.

~ GOOD LUCK ~

This is the part of this assignment that you must do now.

Assignment 6(b)
Q1, Q2, Q3.

Instead of Question 4, Do Exercise set 7.1 Q11, Q12

Instead of Question 5, do the Proof of Theorem 7.2.1 from our book

Q6, Q7, Q8

Instead of Q9, Do Exercise set 7.3, Q1

The rest of these questions are for Linear Transformations and will only be done after we have done the next two lectures.