$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X}) Y_i}{\sum_{i=1}^n (X_i - \bar{X})^2}$$

$$var(\hat{\beta}_1) = var\left(\frac{\sum_{i=1}^n (X_i - \bar{X}) Y_i}{\sum_{i=1}^n (X_i - \bar{X})^2} \right)$$

Recall the law of total variance.

$$var(Y) = E\left(var(Y|X)\right) + var\left(E(Y|X)\right)$$

$$Var(\hat{\beta}_1) = E\left(Var\left(\frac{\sum_{i=1}^n (X_i - \bar{X}) Y_i}{\sum_{i=1}^n (X_i - \bar{X})^2} \,\middle|\, X_i, \beta_i \right)\right)$$

$$+ var\left(E\left(\frac{\sum_{i=1}^n (X_i - \bar{X}) Y_i}{\sum_{i=1}^n (X_i - \bar{X})^2} \,\middle|\, X_i, \beta_i \right)\right)$$

$$= V_1 + Y_2$$

$$V_1 = E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2 var(Y_i | X_i, \beta_i)}{\left[\sum_{i=1}^n (X_i - \bar{X})^2\right]^2} \right)$$

$$= E\left(\frac{\sum_{i=1}^n (X_i - \bar{X})^2 \sigma^2}{S_X^4} \right)$$

$$= \sigma^2 E\left(\frac{S_X^2}{S_X^4} \right) = \frac{\sigma^2}{S_X^2}$$

$$V_2 = var\left(\frac{\sum_{i=1}^{n} (X_i - \bar{X})(\beta_0 + \beta_1 X_i)}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right)$$

$$= var\left(\frac{\beta_0 \sum_{i=1}^{n} (X_i - \bar{X}) + \beta_1 \sum_{i=1}^{n} (X_i^2 - \bar{X} X_i)}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right)$$

$$= var\left(\frac{\beta_1 \cdot \sum_{i=1}^{n} (X_i^2 - \bar{X} X_i)}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \right)$$

$$= var(\beta_1)$$

$$= 0$$

$$\Rightarrow \boxed{var(\hat{\beta}_1) = \frac{\sigma^2}{S_X^2}}$$