Lecture 7

Tuesday, February 1, 2022 12:59 PM

DETERMINANTS

$\det(A) =$ ______ (some real no.)

Determinant is a function from the set of square Matrices to the set of real nos. denoted by $\mathbb{R}$.

1) LINEAR ALGEBRA

LECTURE 7

REVISION:

IF $\boxed{A}$ IS AN SQUARE MATRIX (INVERTIBLE)

THEN $\boxed{A^{-1}}$ COULD BE EVALUATED BY THE FOLLOWING METHOD:

$$[A \mid I] \xrightarrow{\text{E.R.O.S.}} [I \mid A^{-1}]$$

$I \rightarrow$ IDENTITY MATRIX

$\boxed{\text{E.R.O.S}} \rightarrow$ ELEMENTARY ROW OPERATIONS

EXAMPLE:

LET $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ THEN USING THE TECHNIQUE

$$[A \mid I] \xrightarrow{\text{e.r.o.s}} [I \mid A^{-1}]$$

PROVE THAT

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \boxed{ad-bc \neq 0}$$

SOLUTION:

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$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$\boxed{R_1 \to \frac{R_1}{a}}$, $a \neq 0$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad \boxed{R_2 \to R_2 - cR_1}$$

$$d - \frac{cb}{a} \to \left(\frac{ad-bc}{a}\right)$$

$$= \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & b/a & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$\boxed{R_2 \to \frac{a}{(ad-bc)} R_2}$, $ad-bc \neq 0$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \leftarrow \text{inverse}$$

$\boxed{R_1 \to R_1 - \frac{b}{a} R_2}$

WHICH COMPLETES THE PROOF.

$\frac{a}{ad-bc}$

$\frac{\cancel{a}}{ad-bc} \times -\frac{c}{\cancel{a}}$

$\frac{a}{ad-bc} \times \frac{b}{\cancel{a}}$ $\quad \frac{b}{ad-bc}$

$-\frac{c}{ad-bc} \times \frac{b}{a}$

$\frac{1}{a} - \frac{-bc}{a(ad-bc)}$

$\frac{\cancel{(ad-bc) - (-bc)}}{\cancel{a}(ad-bc)}$

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

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RESULT: FOR $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

...cause $\frac{1}{...-bc}$ is undefined!

$$(ad-bc) \begin{bmatrix} \\ -c & a \end{bmatrix}$$

PROVIDED $ad-bc \neq 0$ → as otherwise $\frac{1}{ad-bc}$ is undefined!

THIS NUMBER $\boxed{ad-bc}$ IS CALLED THE DETERMINANT OF MATRIX $\boxed{A}$ AND IS WRITTEN AS

only for 2x2

$$ad-bc = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = |A|$$

HERE $\boxed{A}$ IS INVERTIBLE,

$\therefore$ $\boxed{A}$ IS A SQUARE MATRIX

SO WE HAVE THE FOLLOWING RESULT:

DETERMINANT OF A NONSQUARE MATRIX IS NOT DEFINED.

ALSO IF $\boxed{A^{-1}}$ EXISTS THEN $\det(A) \neq 0$

True for any order square matrix

but this is NO PROOF!

4) **RESULT:** IF $AX=B$ IS THE SYSTEM OF n LINEAR EQUATIONS IN n UNKNOWNS THEN:

(1) UNIQUE SOLUTION IF A IS INVERTIBLE $(\det(A) \neq 0)$ AND
$X = A^{-1}B$

(2) IF A^{-1} DOESN'T EXIST

- NO SOLUTION
- INFINITE SOLUTIONS

One sol.
No sol. (inconsistent)
∞ sol.

LET US CONSIDER EXAMPLES WHEN $AX=B$ IS THE SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

(1) FOR UNIQUE SOLUTION.

CONSIDER THE FOLLOWING SYSTEM: $(X = A^{-1}B)$

CONSIDER

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$$3x_1 + 4x_2 = 7 \quad \text{— ①}$$
$$x_1 - x_2 = 0 \quad \text{— ②}$$

$$\Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

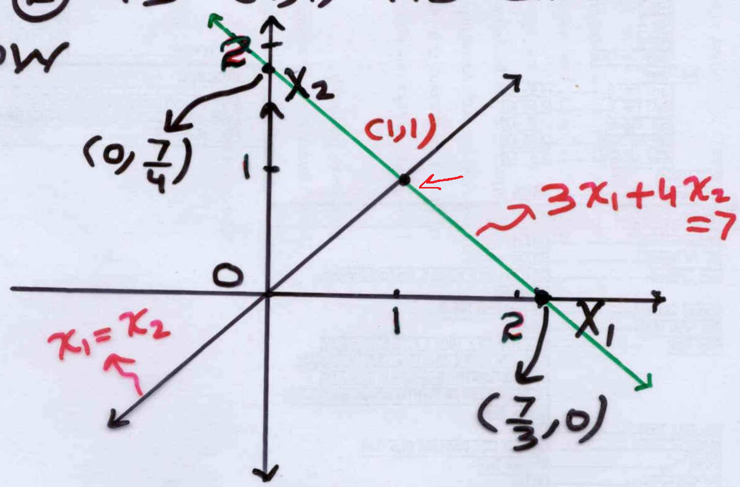
NOW $\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -3-4 = -7 \neq 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ 0 \end{bmatrix} \qquad (\bar{A}^1 \underline{B} = \underline{X})$$

$$= -\frac{1}{7}\begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \end{bmatrix} = -\begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ i.e. POINT

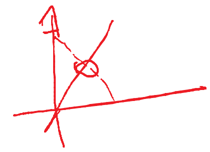
OF INTERSECTION OF LINES ① AND ② IS (1,1) AS SHOWN BELOW



$3x+4y=7$

$y = -\frac{3}{4}x + \frac{7}{4}$

$x-y=0 \quad y=x$



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(2) IF A^{-1} $\boxed{\text{DOESN'T}}$ EXIST

(a) $\boxed{\text{NO SOLUTION}}$ (b) $\boxed{\text{INFINITE SOLUTIONS}}$

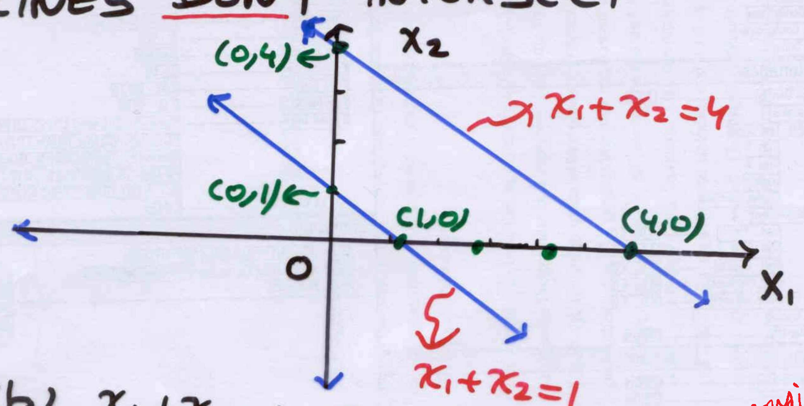
(a) CONSIDER $\boxed{\begin{matrix} x_1 + x_2 = 4 \\ x_1 + x_2 = 1 \end{matrix}} \rightarrow 4 \neq 1$ ∴ inconsistent

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ (NO SOLUTION)

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$ i.e. PARALLEL LINES DON'T INTERSECT

$x+y=4$

$y=-x+4$



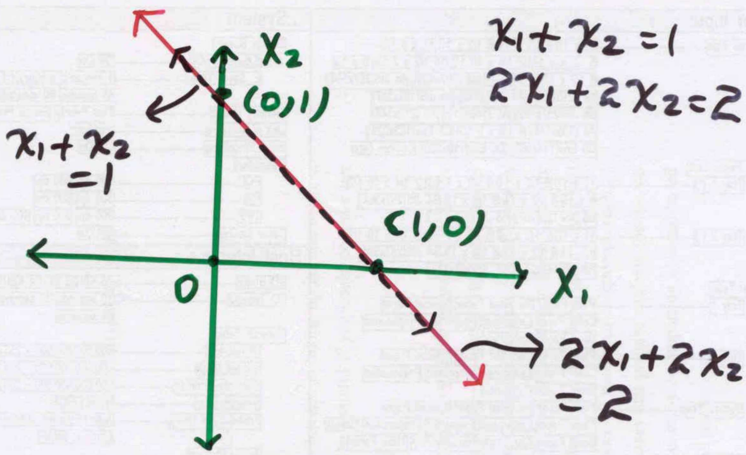
(b) $x_1 + x_2 = 1$, $2x_1 + 2x_2 = 2$ → consistent

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ (INFINITE SOLUTIONS)

$$\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$

7) BOTH LINES ARE $\boxed{\text{COINCIDENT.}}$

$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$



$\boxed{\text{ONE}}$ EQUATION AND $\boxed{\text{TWO}}$ UNKNOWNS.

$x_1 + x_2 = 1$, LET $\boxed{x_2 = t}$

$\Rightarrow \boxed{x_1 = 1 - t}$, INFINITE SOLUTIONS FOR DIFFERENT VALUES OF $\boxed{t}$.

NOTE: HERE $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] R_2 \to R_2 - 2R_1$

$\not\sim \left[\begin{array}{cc|c} 1 & 0 & A^{-1} \\ 0 & 1 & \end{array}\right]$, A^{-1} DOESN'T EXIST

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PROPERTIES OF DETERMINANTS (OF ORDER n) OR OF MATRICES OF ORDER n.

$\det(ABC) = \det(A)\det(B)\det(C)$

$\det(A_1 \dots A_n)$

$= \det(A_1) \dots \det(A_n)$

proof: induction

Note:
This is theorem 2.3.4. Proof is in The book. Will be Fully understood After we see how To compute Determinants.

① IF $\boxed{A}$ AND $\boxed{B}$ ARE SQUARE MATRICES OF THE SAME SIZE, THEN $\boxed{\det(AB) = \det(A)\det(B)}$

CHECK IT FOR $\boxed{2\times 2}$ MATRICES.

FOR $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, $\det(B) = b_{11}b_{22} - b_{12}b_{21}$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$|X|$

$[1]$

PROVE THAT

$\boxed{\det(AB)} = \boxed{\det(A)\det(B)}$

$= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$

② IF $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\boxed{\det(I) = 1}$

WHICH IS TRUE FOR $\boxed{\text{IDENTITY}}$ MATRIX OF ANY ORDER.

9)

PROVE THE FOLLOWING:

IF A SQUARE MATRIX $\boxed{A}$ IS INVERTIBLE THEN $\det(A) \neq 0$.

Note: This is Theorem 2.3.5

RESULT: IF A^{-1} EXISTS THEN

$$A^{-1}A = I \Rightarrow \det(A^{-1}A) = 1$$

$\det(A) \neq 0$

$$\Rightarrow \det(A^{-1})\det(A) = 1 \Rightarrow \det(A^{-1}) = 1/\det(A) \Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

This is theorem 2.2.2

3. $\boxed{\det(A) = \det(A^T)}$

FOR $\boxed{2 \times 2}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$\det(A) = \det(A^T) = a_{11}a_{22} - a_{21}a_{12}$$

4. IF TWO ROWS OR TWO COLUMNS OF A MATRIX ARE IDENTICAL THEN $\boxed{\det(A) = 0}$

CHECK FOR $\boxed{2 \times 2}$ MATRIX

10)

$$\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ba = 0$$

OR

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

⑤

ADDING $\boxed{\text{ROWS}}$ (OR $\boxed{\text{COLUMNS}}$) TOGETHER MAKES $\boxed{\text{NO}}$ DIFFERENCE TO THE DETERMINANT.

$$\begin{array}{cc} C_1 & C_2 \end{array}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

COLUMN 2

CONSIDER $(C_1 \to C_1 + C_2)$ $\begin{vmatrix} a_{11}+a_{12} & a_{12} \\ a_{21}+a_{22} & a_{22} \end{vmatrix}$

COLUMN 1

$$= (a_{11}+a_{12})a_{22} - a_{12}(a_{21}+a_{22})$$

$$= a_{11}a_{22} + \cancel{a_{12}a_{22}} - a_{12}a_{21} - \cancel{a_{12}a_{22}} = a_{11}a_{22} - a_{12}a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

11)

ALSO

$R_1 \to R_1 + R_2 \quad \begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} \\ a_{21} & a_{22} \end{vmatrix}$

$= (a_{11}+a_{21})a_{22} - (a_{12}+a_{22})a_{21}$

$= a_{11}a_{22} + \cancel{a_{21}a_{22}} - a_{12}a_{21} - \cancel{a_{22}a_{21}} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(6)

IF $\boxed{B}$ IS THE MATRIX THAT RESULTS WHEN A SINGLE ROW OR SINGLE COLUMN OF $\boxed{A}$ IS MULTIPLIED BY A SCALAR (K), THEN $\boxed{\det(B) = k\det(A)}$

$\det(kA) = k\det(A)$ X

$2\times 2 \quad \det(kA) = k^2 \det(A)$

$3\times 3 \quad \det(kA) = k^3 \det(A)$

LET $\det(B) = \begin{vmatrix} ka_{11} & a_{12} \\ ka_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & ka_{12} \\ a_{21} & ka_{22} \end{vmatrix}$

$\det(B)$

$= \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix}$

$= k(a_{11}a_{22} - a_{21}a_{12}) = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= k\,\det(A) = k\det(A).$

TRY THE FOLLOWING:

FIND $\det(A)$ WHERE

$$A = \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} =$$

$R_1 \to R_1 + R_2$

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= a+b+c \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

ANS: $\boxed{\det(A) = 0}$

(7) RESULT:

IF $\boxed{A}$ IS A SQUARE MATRIX WITH $\boxed{\text{TWO}}$ PROPORTIONAL ROWS OR $\boxed{\text{TWO}}$ PROPORTIONAL COLUMNS, THEN $\boxed{\det(A) = 0.}$

$\boxed{C_2 = 2C_3}$

EXAMPLE: $\det\left(\begin{bmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{bmatrix}\right) = 0$

$$\because \begin{vmatrix} -2 & 8 & 4 \\ 3 & 2 & 1 \\ 1 & 10 & 5 \end{vmatrix} = 2 \begin{vmatrix} -2 & 4 & 4 \\ 3 & 1 & 1 \\ 1 & 5 & 5 \end{vmatrix} = 0$$

$\because C_2$ IS IDENTICAL TO C_3.

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(8)

IF $\boxed{A}$ IS A SQUARE MATRIX SUCH THAT $\boxed{A}$ HAS A ROW OF ZEROS OR A COLUMN OF ZEROS THEN $\det(A) = 0$.

CONSIDER FOR 2X2 CASE

$$\begin{vmatrix} a & 0 \\ b & 0 \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}$$

$$= a(0) - b(0) = 0$$

NOTE: IF $\boxed{A}$ IS ANY SQUARE MATRIX THAT CONTAINS A ROW OF ZEROS OR A COLUMN OF ZEROS THEN $\boxed{A}$ IS SINGULAR.

Not invertible