Habib University
shaping futures

Longest Common Subsequence (LCS): an example

CS 412

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Source: *CLRS*

The LCS problem has an optimal-substructure property, however, as the following theorem shows. As we shall see, the natural classes of subproblems correspond to pairs of "prefixes" of the two input sequences. To be precise, given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, we define the ith ***prefix*** of X, for $i = 0, 1, \ldots, m$, as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$. For example, if $X = \langle A, B, C, B, D, A, B \rangle$, then $X_4 = \langle A, B, C, B \rangle$ and X_0 is the empty sequence.

Theorem 15.1 (Optimal substructure of an LCS)
Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

From: *CLRS.*

Let $c[i, j]$ be the **length** of the longest common subsequence (LCS) of two sequences X_i and Y_j. If $i = 0$ or $j = 0$, then $c[i, j] = 0$.

$$c[i,j] = \begin{cases} 0 & , if\ i = 0\ or\ j = 0 \\ c[i-1, j-1] + 1 & , if\ i, j > 0\ and\ x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & , if\ i, j > 0\ and\ x_i \neq y_j \end{cases}$$

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Example: X = ABCB and **Y = BDCAB**. We will fill-in the table of sub-problems, starting with the smallest optimal sub-problem.

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**B**
0	x_i						
1	A						
2	B						
3	C						
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Smallest sub-problems with an optimal solution.

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0	***x**$_i$*	0	0	0	0	0	0
1	**A**	0	0	0	0	1	1
2	**B**	0	1	1	1	1	2
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4	**B**	0	1	1	2	2	3

Let $c[i,j]$ be the **length** of the longest common sequence (LCS) of two sequences X_i and Y_j. If $i = 0$ or $j = 0$, then $c[i,j] = 0$.

$$c[i,j] = \begin{cases} 0 & , if\ i = 0\ or\ j = 0 \\ c[i-1, j-1] + 1 & , if\ i,j > 0\ and\ x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & , if\ i,j > 0\ and\ x_i \neq y_j \end{cases}$$

Example: X = ABCB and **Y = BDCAB**. We'll fill-in the table of sub-problems, starting with the smallest optimal sub-problem.

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**B**
0	x_i	0	0	0	0	0	0
1	**A**	0	0	0	0	1	1
2	**B**	0	1	1	1	1	2
3	**C**	0	1	1	2	2	2
4	**B**	0	1	1	2	2	3