Homework 12 Part A

Exercise 5.5

Question 02:

Part (a)

(a)

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 3\cdot 2 \\ (-1)\cdot 1 + 4\cdot 2 \end{bmatrix} = 1\cdot\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Part (b)

(b)

$$\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4\cdot(-2) + 0\cdot 3 + (-1)\cdot 5 \\ 3\cdot(-2) + 6\cdot 3 + 2\cdot 5 \\ 0\cdot(-2) + (-1)\cdot 3 + 4\cdot 5 \end{bmatrix} = (-2)\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

Question 06:

Part (a)

**Step 1**



$(a)$

Reducing matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \overset{(1)}{\rightarrow} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \overset{(2)}{\rightarrow} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \overset{(3)}{\rightarrow} \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Explanations:

$(1)$ Multiply row 1 by $-5$ and add to row 2. Multiply row 1 by $-7$ and add to row 3.

$(2)$ Subtract row 2 from row 3.

$(3)$ Add row 2 to row 1.

So if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is solution of the system $A\mathbf{x} = \mathbf{0}$ then

$$x_1 - 16x_3 = 0$$
$$x_2 - 19x_3 = 0$$

If we set $x_3 = t$ then $x_1 = 16t, x_2 = 19t$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}$$

is basis for the null space of $A$.

Part (c)

**Step 3**



$(c)$

Reducing matrix

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \overset{(1)}{\to} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \overset{(2)}{\to} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{(3)}{\to} \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Explanations:

$(1)$ Multiply row 1 by $-2$ and add to row 2.Add row 1 to row 3.

$(2)$ Subtract row 2 from row 3.

$(3)$ Add row 2 to doubled row 1.

So if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is solution of the system $A\mathbf{x} = \mathbf{0}$ then

is solution of the system $A\mathbf{x} = \mathbf{0}$ then

$$2x_1 + x_2 + 3x_3 = 0$$
$$-7x_2 - 7x_3 - 4x_4 = 0$$

If we set $x_2 = t, x_3 = s$ then $x_1 = -\frac{1}{2}t - \frac{3}{2}s, x_4 = -\frac{7}{4}t - \frac{7}{4}s$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t - \frac{3}{2}s \\ t \\ s \\ -\frac{7}{4}t - \frac{7}{4}s \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{7}{4} \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \\ -\frac{7}{4} \end{bmatrix}$$

Therefore

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ -\frac{7}{4} \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 1 \\ -\frac{7}{4} \end{bmatrix} \right\}$$

is basis for the null space of $A$.

Question 13:

**Solution** Verified

Let A be an $n \times n$ invertible matrix. Since $A^T$ is also invertible, it is row equivalent to $I_n$. It is clear that the column vectors of $I_n$ are linearly independent. Hence, by virtue of Theorem...(I), the column vectors of $A^T$, which are just the row vectors of A, are also linearly independent. Therefore the rows of A form a set of n linearly independent vectors in $R^n$, and consequently form a basis for $R^n$.

Theorem...(I) If A and B are row equivalent matrices, then (a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent. (b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

# Exercise 5.6

## Question 04:

**Step 1**



$(a)$

Dimensions of the row and column spaces of $A$ are equal to $\text{rank}(A) = 3$.

Dimension of the null space of $A$ is $\text{nullity}(A)$. We know that

$$\text{rank}(A) + \text{nullity}(A) = n$$

where $n$ is the number of columns of $A$. So

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 3 = 0$$

Also

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

where $m$ is the number of rows of $A$. Using $\text{rank}(A^T) = \text{rank}(A)$ we have

$$\text{rank}(A) + \text{nullity}(A^T) = m$$

Therefore dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 3 - 3 = 0$$

$(b)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 2$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 2 = 1$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 3 - 2 = 1$$

$(c)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 1$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 1 = 2$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 3 - 1 = 2$$

$(d)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 2$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 9 - 2 = 7$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 5 - 2 = 3$$

$(e)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 2$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 5 - 2 = 3$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 9 - 2 = 7$$

$(f)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 0$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 4 - 0 = 4$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 4 - 0 = 4$$

$(g)$

Dimensions of the row and column spaces of $A$ are $\text{rank}(A) = 2$.

Dimension of the null space of $A$ is

$$\text{nullity}(A) = n - \text{rank}(A) = 2 - 2 = 0$$

Dimension of the null space of $A^T$ is

$$\text{nullity}(A^T) = m - \text{rank}(A) = 6 - 2 = 4$$

**Result**

3 of 3

$(a)$ Dimensions of the row and column spaces of $A$ are 3. Dimension of the null space of $A$ is 0. Dimension of the null space of $A^T$ is 0.

$(b)$ Dimensions of the row and column spaces of $A$ are 2. Dimension of the null space of $A$ is 1. Dimension of the null space of $A^T$ is 1.

$(c)$ Dimensions of the row and column spaces of $A$ are 1. Dimension of the null space of $A$ is 2. Dimension of the null space of $A^T$ is 2.

$(d)$ Dimensions of the row and column spaces of $A$ are 2. Dimension of the null space of $A$ is 7. Dimension of the null space of $A^T$ is 3.

$(e)$ Dimensions of the row and column spaces of $A$ are 2. Dimension of the null space of $A$ is 3. Dimension of the null space of $A^T$ is 7.

$(f)$ Dimensions of the row and column spaces of $A$ are 0. Dimension of the null space of $A$ is 4. Dimension of the null space of $A^T$ is 4.

$(g)$ Dimensions of the row and column spaces of $A$ are 2. Dimension of the null space of $A$ is 0. Dimension of the null space of $A^T$ is 4.

## <u>Question 05:</u>

**Step 1**

1 of 2

* Largest possible value of the $Rank$ of any matrix of order m×n is min(m,n).

* Minimum possible value of the $Nullity$ of any matrix of order m×n is No of columns of Matrix − Rank of a Matrix.

(a) $A$ is $4 \times 4$

→Maximum possible $Rank$ of Matrix $A$ = $min(4, 4)$ = 4

→ Smallest possible $Nullity$ value of Matrix $A$=4 − 4 = 0

(b) $A$ is $3 \times 5$

→ Maximum possible $Rank$ of Matrix $A$ = $min(3, 5)$ = 3

→ Smallest possible $Nullity$ value of Matrix $A$=5 − 3 = 2

(c) $A$ is $5 \times 3$

→ Maximum possible $Rank$ of Matrix $A$ = $min(5, 3)$ = 3

→ Smallest possible $Nullity$ value of Matrix $A$=3 − 3 = 0

## Question 06:

### Step 1



The goal of the exercise is to find the largest possible value of the rank of the $m \times n$ matrix

$$[A]_{m\times n}$$

and to find the smallest possible value for its nullity.

### Step 2



Now since $A$ is a $m \times n$ matrix therefore the row vectors of the matrix lie in the space $\mathbb{R}^n$ and the column vectors of the matrix lie in space $\mathbb{R}^m$.

We know that the rank of a matrix is the common dimension of its column space and row space. This leads us to conclude that the rank of the matrix $A$ is less than or equal to the minimum of $m$ and $n$, that is

$$\operatorname{rank}(A) \leq \min\{m, n\}.$$

Thus we get the largest possible value of the rank of the matrix is

$$\min\{m, n\}.$$

### Step 3



Now let's recall that the **Rank-Nullity theorem** states that if $A$ is a $m \times n$ matrix then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Using this theorem we get the nullity of the matrix $A$ is equal to

$$\begin{aligned}\operatorname{nullity}(A) &= n - \operatorname{rank}(A) \\ &\geq n - \min\{m, n\}.\end{aligned}$$

Thus we get the smallest possible value of the matrix $A$ is

$$n - \min\{m, n\}.$$

# Question 07:

**Step 1**



$(a)$

Because $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 3$ the system is consistent. The number of parameters in general solution is equal to $\text{nullity}(A)$. We know that

$$\text{rank}(A) + \text{nullity}(A) = n$$

where $n$ is the number of columns of $A$. Therefore the number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 3 = 0$$

$(b)$

Because $\text{rank}(A) \neq \text{rank}[A|\mathbf{b}]$ the system is not consistent.

$(c)$

Because $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 1$ the system is consistent. The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 1 = 2$$

**Step 2**



$(d)$

Because $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 2$ the system is consistent. The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 9 - 2 = 7$$

$(e)$

Because $\text{rank}(A) \neq \text{rank}[A|\mathbf{b}]$ the system is not consistent.

$(f)$

Because $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 0$ the system is consistent. The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 4 - 0 = 4$$

$(g)$

Because $\text{rank}(A) = \text{rank}[A|\mathbf{b}] = 2$ the system is consistent. The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 2 - 2 = 0$$

**Result**



$(a)$ The system is consistent. The number of parameters in general solution is 0.

$(b)$ The system is not consistent.

$(c)$ The system is consistent. The number of parameters in general solution is 2.

$(d)$ The system is consistent. The number of parameters in general solution is 7.

$(e)$ The system is not consistent.

$(f)$ The system is consistent. The number of parameters in general solution is 4.

$(g)$ The system is consistent. The number of parameters in general solution is 0.

## Question 08:

### Step 1



$(a)$

The number of parameters in general solution is equal to $\text{nullity}(A)$. We know that

$$\text{rank}(A) + \text{nullity}(A) = n$$

where $n$ is the number of columns of $A$. Therefore the number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 3 = 0$$

$(b)$

The system is not consistent.

$(c)$

The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 3 - 1 = 2$$

### Step 2



$(d)$

The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 9 - 2 = 7$$

$(e)$

The system is not consistent.

$(f)$

The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 4 - 0 = 4$$

$(g)$

The number of parameters in general solution is

$$\text{nullity}(A) = n - \text{rank}(A) = 2 - 2 = 0$$

### Result



$(a)$ $0$

$(b)$ The system is not consistent.

$(c)$ $2$

$(d)$ $7$

$(e)$ The system is not consistent.

$(f)$ $4$

$(g)$ $0$

# Question 15:

## Step 1



Let

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

where $r_i$ for $i = \overline{1, n}$ is the row of the $A$ matrix and

$$kA = \begin{bmatrix} kr_1 \\ kr_2 \\ \vdots \\ kr_n \end{bmatrix}$$

where $kr_i$ for $i = \overline{1, n}$ is the row of the $kA$ matrix. Confirmation is proved by contradiction. Suppose it is rank $(A) = p$ and let $\{(r_1, r_2, \cdots r_p)\}$ are linearly independent. Since $k$ is a scalar then the set $\{(kr_1, kr_2, \cdots kr_p)\}$ are linearly independent. Let $kr_i$ be another row of the matrix $A$ so that it is the set $\{(kr_1, kr_2, \cdots kr_p, kr_i)\}$ are linearly independent for $i \neq \overline{1, p}$. Hence, the set $\{(r_1, r_2, \cdots r_p, r_i)\}$ is also linearly independent which is contrary to the assumption that it is rank$(A) = p$. Then, the maximum independent rows of the $kA$ matrix is $\{(kr_1, kr_2, \cdots kr_p)\}$, hence the matrices $A$ and $kA$ have the same rank.

## Result



$$\boxed{\text{The matrices } A \text{ and } kA \text{ have the same rank.}}$$