# **THEOREM 1.5.3**

is a square matrix

This is a square madrix! Short there are equivalent for rectangular modrices:

## **Equivalent Statements**

If A is an  $n \times n$  matrix, then the following statements are equivalent, that is, all true or all false.

(a) A is invertible.



- (b)  $A_{\mathbf{X}} = \mathbf{0}$  has only the trivial solution.
- (c) The reduced row-echelon form of A is  $I_n$ .
- (d) A is expressible as a product of elementary matrices.

**Proof** We shall prove the equivalence by establishing the chain of implications:  $(a) \rightleftharpoons (b) \rightleftharpoons (c) \rightleftharpoons (d) \rightleftharpoons (a)$ .

- (a)  $\Rightarrow$  (b) Assume A is invertible and let  $x_0$  be any solution of Ax = 0; thus  $Ax_0 = 0$ . Multiplying both sides of this equation by the matrix  $A^{-1}$  gives  $A^{-1}(Ax_0) = A^{-1}0$ , or  $(A^{-1}A)x_0 = 0$ , or  $Ix_0 = 0$ , or  $Ix_0 = 0$ . Thus, Ax = 0 has only the trivial solution.
- (b)  $\Rightarrow$  (c) Let  $A_{\mathbf{X}} = \mathbf{0}$  be the matrix form of the system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = 0$$
(1)

and assume that the system has only the trivial solution. If we solve by Gauss–Jordan elimination, then the system of equations corresponding to the reduced row-echelon form of the augmented matrix will be

$$\begin{array}{ccc}
x_1 & = 0 \\
x_2 & = 0 \\
& \ddots \\
& x_n = 0
\end{array}$$
(2)

Thus the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

for 1 can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

for 2 by a sequence of elementary row operations. If we disregard the last column (of zeros) in each of these matrices, we can conclude that the reduced row-echelon form of A is  $I_n$ .

(c)  $\Rightarrow$  (d) Assume that the reduced row-echelon form of A is  $I_n$ , so that A can be reduced to  $I_n$  by a finite sequence of elementary row operations. By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices  $E_1, E_2, ..., E_k$  such that

$$\underline{E_k \cdots E_2 E_1 A} = I_n \tag{3}$$

By Theorem 1.5.2,  $E_1$ ,  $E_2$ , ...,  $E_k$  are invertible. Multiplying both sides of Equation 3 on the left successively by  $E_k^{-1}$ , ...,  $E_2^{-1}$ ,  $E_1^{-1}$  we obtain

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$
(4)

By Theorem 1.5.2, this equation expresses A as a product of elementary matrices.

(d)  $\Rightarrow$  (a) If A is a product of elementary matrices, then from Theorems Theorem 1.4.6 and Theorem 1.5.2, the matrix A is a product of invertible matrices and hence is invertible.

A" = En ... Ei

#### **THEOREM 1.6.1**

Microsistent.

7/1 1/2 = 1 1/1 1/2 = 1

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

**Proof** If Ax = b is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that Ax = b has more than one solution, and let  $x_0 = x_1 - x_2$ , where  $x_1$  and  $x_2$  are any two distinct solutions. Because  $x_1$  and  $x_2$  are distinct, the matrix  $x_0$  is nonzero; moreover,

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$$Ax_0 = A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0$$

If we now let k be any scalar, then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + A(k\mathbf{x}_0) = \underline{A\mathbf{x}_1} + k(A\mathbf{x}_0)$$

$$= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

But this says that  $x_1 + kx_0$  is a solution of Ax = b. Since  $x_0$  is nonzero and there are infinitely many choices for k, the system  $A\mathbf{x} = \mathbf{b}$  has infinitely many solutions.

#### THEOREM 1.6.3

Let A be a square matrix.

- (a) If B is a square matrix satisfying BA = I, then  $B = A^{-1}$ .
- (b) If B is a square matrix satisfying AB = I, then  $B = A^{-1}$ .

We shall prove part (a) and leave part (b) as an exercise.

**Proof (a)** Assume that BA = I. If we can show that A is invertible, the proof can be completed by multiplying BA = I on both sides by  $A^{-1}$  to obtain

As in 1.9.3 above, part(b)  $BAA^{-1} = IA^{-1}$  or  $BI = IA^{-1}$  or  $B = A^{-1}$ 

To show that A is invertible, it suffices to show that the system Ax = 0 has only the trivial solution (see Theorem 3). Let  $x_0$  be any solution of this system. If we multiply both sides of  $Ax_0 = 0$  on the left by B, we obtain  $BAx_0 = B0$  or  $Ix_0 = 0$  or  $Ix_0 = 0$ . Thus, the system of equations Ax = 0 has only the trivial solution.

the LAS is not defined!

We cannot cose the same wick so part (A), because multiplying

And = 0 on the right by B gired

THAT THE PROOF FOR (a)

WORKS HERE THAT WAS

REACHED ASSUMING BA-I. HERE

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DK. Lets see if we can show B-1 exists. Let no be any solution to Bx=0. 50 Bx0 = 0 = 7 AB x0 = 0 => In0 = 0 => x0 = 0 (migne) 2. BT exist

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#### **THEOREM 1.6.4**

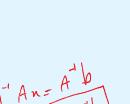
### **Equivalent Statements**

If A is an  $n \times n$  matrix, then the following are equivalent.

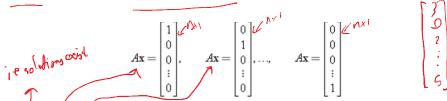
(a) A is invertible.

(a).

- (b) Ax = 0 has only the trivial solution.
- (c) The reduced row-echelon form of A is In.
- (d) A is expressible as a product of elementary matrices.
- (e) Ax = b is consistent for every  $n \times 1$  matrix b.
- (f) Ax = b has exactly one solution for every  $n \times 1$  matrix b.



- $(a) \Rightarrow (f)$  This was already proved in Theorem 1.6.2.
- This is self-evident: If  $A_{\mathbf{X}} = \mathbf{b}$  has exactly one solution for every  $n \times 1$  matrix  $\mathbf{b}$ , then  $A_{\mathbf{X}} = \mathbf{b}$  is consistent for every  $n \times 1$  matrix  $\mathbf{b}$ .
- (e)  $\Rightarrow$  (a) If the system Ax = b is consistent for every  $n \times 1$  matrix b, then in particular, the systems



are consistent. Let  $x_1, x_2, ..., x_n$  be solutions of the respective systems, and let us form an  $n \times n$  matrix C having these solutions as columns. Thus C has the form

$$C = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]$$

As discussed in Section 1.3, the successive columns of the product AC will be

$$Ax_1, Ax_2, ..., Ax_n$$

Thus

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are consistent. Let  $x_1$ ,  $x_2$ , ...,  $x_n$  be solutions of the respective systems, and let us form an  $n \times n$  matrix C having these solutions as columns. Thus C has the form

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As discussed in Section 1.3, the successive columns of the product AC will be

 $A\mathbf{x}_1, A\mathbf{x}_2, ..., A\mathbf{x}_n$ 

Thus

$$AC = [A\mathbf{x}_1 | A\mathbf{x}_2 | \cdots | A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

A  $[c_1|c_1|...|c_n]$   $= [Ac_1|Ac_1|...|f_n]$  A C= I

By part (b) of Theorem 1.6.3, it follows that  $C = A^{-1}$ . Thus, A is invertible