Statistics & Inferencing Homework#01

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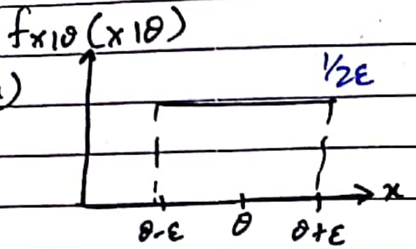
Q) Find the Bayes Estimator $\theta \rightarrow \boxed{\text{Sensor}} \rightarrow x \rightarrow \boxed{\text{Bayes}_{Est}} \rightarrow \hat{\theta}$

$\theta \sim U[0,T]$

(a)

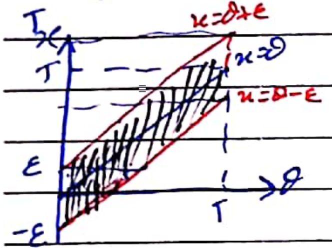


$X|\theta \sim U[\theta-\varepsilon, \theta+\varepsilon]$ $\varepsilon \in \mathbb{R}^+$

$\hat{\theta} = E(\theta|X) = \int \theta \cdot f_{\theta|x}(\theta|x)d\theta$

$\theta \sim U(0,T]$ $x|\theta \sim U[\theta-\varepsilon, \theta+\varepsilon]$

$$f_{x|\theta}(x|\theta) = \begin{cases} 1/2\varepsilon & \theta-\varepsilon \leq x \leq \theta+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



for θ, $\theta-\varepsilon \overset{-\varepsilon \leq x \leq \varepsilon}{\leq x \leq} \theta+\varepsilon$ for $T-\varepsilon \leq x \leq T+\varepsilon$, $x-\varepsilon \leq \theta \leq T$

for $-\varepsilon \leq x \leq \varepsilon$, $0 \leq \theta \leq x+\varepsilon$

for $\varepsilon \leq x \leq T-\varepsilon$, $x-\varepsilon \leq \theta \leq x+\varepsilon$

$f_{x,\theta}(x,\theta) = f_{x|\theta}(x|\theta)f_\theta(\theta) = \frac{1}{2\varepsilon} \times \frac{1}{T} = \frac{1}{2T\varepsilon}$ for $0 \leq \theta-\varepsilon \leq x < \theta+\varepsilon \leq T$

$$f_x(x) = \begin{cases} \int_0^{x+\varepsilon} \frac{1}{2T\varepsilon} \cdot d\theta = \frac{\theta}{2T\varepsilon}\Big|_0^{x+\varepsilon} = x+\varepsilon/2T\varepsilon & -\varepsilon \leq x \leq \varepsilon, 0 \leq \theta \leq x+\varepsilon, \\ \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{2T\varepsilon} \cdot d\theta = \frac{\theta}{2T\varepsilon}\Big|_{x-\varepsilon}^{x+\varepsilon} = \frac{1}{T} & \varepsilon \leq x \leq T-\varepsilon, x-\varepsilon \leq \theta \leq x+\varepsilon \\ \int_{x-\varepsilon}^{T} \frac{1}{2T\varepsilon} \cdot d\theta = \frac{\theta}{2T\varepsilon}\Big|_{x-\varepsilon}^{T} = \frac{T+\varepsilon-x}{2T\varepsilon} & T-\varepsilon \leq x \leq T+\varepsilon, x-\varepsilon \leq \theta \leq T \end{cases}$$

Then:

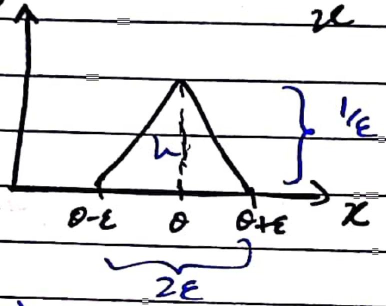
$$f_{\theta|x}(\theta|x) = \frac{f_{x,\theta}(x,\theta)}{f_x(x)} \begin{cases} \frac{1}{2T\varepsilon} \div \frac{x+\varepsilon}{2T\varepsilon} = \frac{1}{x+\varepsilon} & -\varepsilon \leq x \leq \varepsilon, 0 \leq \theta \leq x+\varepsilon \\ \frac{1}{2T\varepsilon} \div 1/T = \frac{1}{2\varepsilon} & \varepsilon \leq x \leq T-\varepsilon, x-\varepsilon \leq \theta < x+\varepsilon \\ \frac{1}{2T\varepsilon} \div \frac{T+\varepsilon-x}{2T\varepsilon} = 1/T+\varepsilon-x & T-\varepsilon \leq x \leq T+\varepsilon, x-\varepsilon \leq \theta \leq T \end{cases}$$

~~Then: $\hat{\theta} = E(\theta|x) = \int \theta \cdot f_{\theta|x}(\theta|x) d\theta = \int_0^{x+\varepsilon} \frac{\theta}{x+\varepsilon} = \frac{x+\varepsilon}{2}$~~

Then:

$$\hat{\theta} = E[\theta|x] = \int \theta \cdot f_{\theta|x}(\theta|x) \cdot d\theta = \begin{cases} \int_0^{x+\varepsilon} \frac{\theta}{x+\varepsilon} = \frac{x+\varepsilon}{2} \\ \int_{x-\varepsilon}^{x+\varepsilon} \frac{\theta}{2\varepsilon} = x \\ \int_{x-\varepsilon}^{T} \frac{\theta}{T+\varepsilon-x} = \frac{x-\varepsilon+T}{2} \end{cases}$$

(b) $f_{x|\theta}(x|\theta)$



x is given a triangular distribution

$h \Rightarrow \frac{1}{2} \times \cancel{2}\epsilon \times h = 1$

$\Rightarrow h = 1/\epsilon$

~~Cases~~ ① $\theta - \epsilon \leq x \leq \theta$: $(\theta - \epsilon, 0)$ $(\theta, 1/\epsilon)$

② $\theta \leq x \leq \theta + \epsilon$: $(\theta, 1/\epsilon)$ $(\theta + \epsilon, 0)$

① ~~$f_{x|\theta}(x|\theta) - 0 =$~~ $m = 1/\epsilon - 0 / \theta - (\theta - \epsilon) = \frac{1}{\epsilon} \div \epsilon = 1/\epsilon^2$

$f_{x|\theta}(x|\theta) - 0 = \frac{1}{\epsilon^2}(x - \theta + \epsilon)$

$f_{x|\theta} = x - \theta + \epsilon / \epsilon^2$

② $m = 1/\epsilon - 0 / \theta - \theta - \epsilon \Rightarrow m = -1/\epsilon^2$

$f_{x|\theta}(x|\theta) - 0 = -1/\epsilon^2 (x - \theta - \epsilon)$

$f_{x|\theta}(x|\theta) = -x + \theta + \epsilon / \epsilon^2$

$$f_{x|\theta}(x|\theta) = \begin{cases} \frac{x - \theta + \epsilon}{\epsilon^2} & \theta - \epsilon \leq x \leq \theta \\ \frac{-x + \theta + \epsilon}{\epsilon^2} & \theta \leq x \leq \theta + \epsilon \end{cases}$$

$f_\theta(\theta) = 1/T$

~~$f_{x,\theta}(x,\theta) = f_{x|\theta}(x|\theta) \cdot f_\theta(\theta) = \begin{cases} \theta(x - \theta + \epsilon) / \epsilon^2 \\ \theta(\theta + \epsilon - x) / \epsilon^2 \end{cases}$~~

$$f_{x,\theta}(x,\theta) = f_{x|\theta}(x|\theta)\ f_\theta(\theta) = \begin{cases} x - \theta + \epsilon / T\epsilon^2 & \theta - \epsilon \leq x \leq \theta \\ \theta + \epsilon - x / T\epsilon^2 & \theta \leq x \leq \theta + \epsilon \end{cases}$$