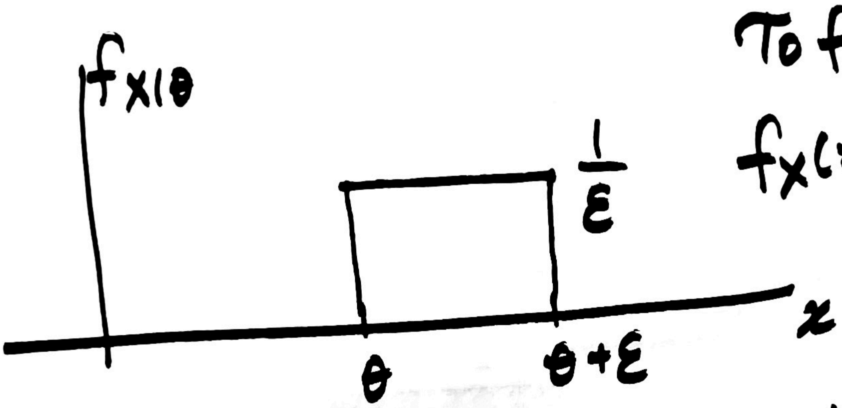
Problem 2:

Now consider another scenario.

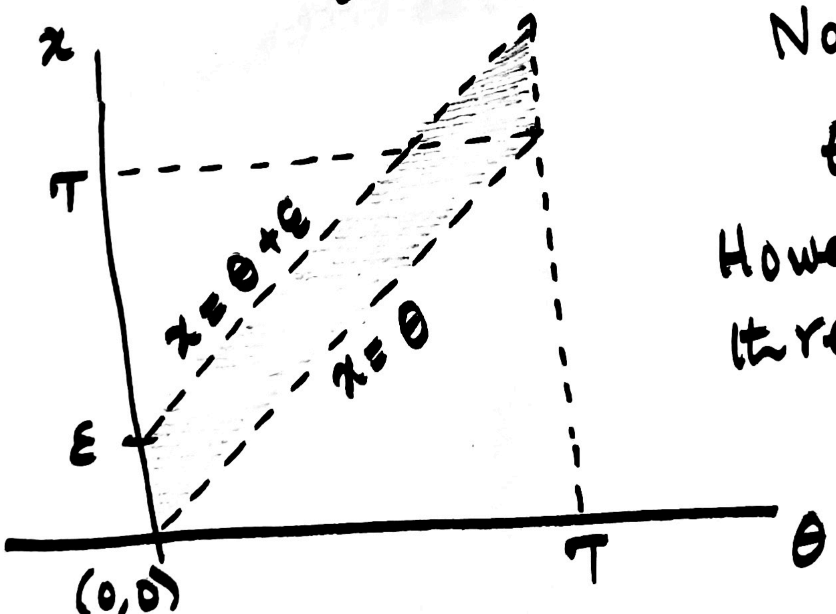
where $\theta \sim \mathcal{U}[0, T]$

and $X|\theta \sim \mathcal{U}[\theta, \theta+\varepsilon]$ where $\varepsilon \in \mathbb{R}^+$



To find $f_X(x)$, we solve

$$f_X(x) = \int f_{X|\theta} \cdot f_\theta \, d\theta$$



Note that

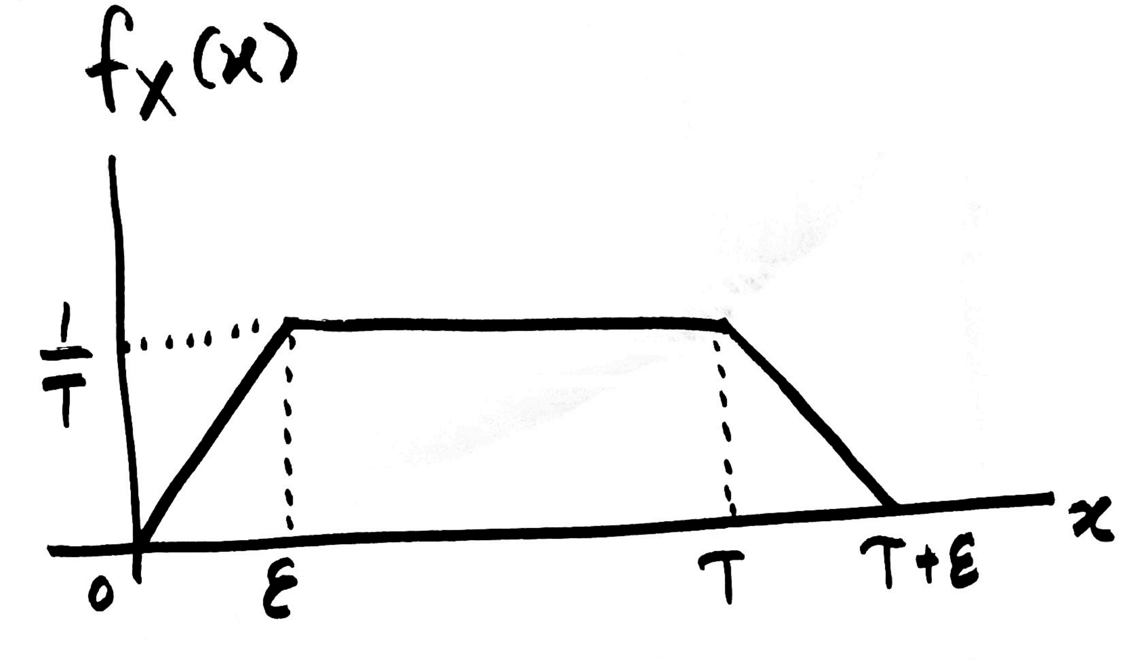
$\theta \leq x \leq \theta + \varepsilon$.

However, there are three regions.

1) When $0 \leq x \leq \varepsilon$, we have $0 \leq \theta \leq x$
2) When $\varepsilon \leq x \leq T$, we have $x - \varepsilon \leq \theta \leq x$
3) When $T \leq x \leq T + \varepsilon$, we have $x - \varepsilon \leq \theta \leq T$

$$
f_X(x) = \begin{cases}
\displaystyle\int_0^x \frac{1}{\varepsilon T}\, d\theta = \frac{x}{\varepsilon T}, & \text{for } 0 \le x \le \varepsilon, \\
\qquad (0 \le \theta \le x) & \\[2ex]
\displaystyle\int_{x-\varepsilon}^{x} \frac{1}{\varepsilon T}\, d\theta = \frac{1}{T}, & \text{for } \varepsilon \le x \le T, \\
\qquad (x-\varepsilon \le \theta \le x) & \\[2ex]
\displaystyle\int_{x-\varepsilon}^{T} \frac{1}{\varepsilon T}\, d\theta = \frac{T+\varepsilon-x}{\varepsilon T}, & \text{for } T \le x \le T+\varepsilon \\
\qquad (x-\varepsilon \le \theta \le T) &
\end{cases}
$$

Here, it was not necessary to mention ~~lim~~ ranges of θ because we are talking about marginal density of x.



$$f_{\theta|X} = \frac{f_{X,\theta}(x,\theta)}{f_X(x)}$$

$$= \begin{cases} \dfrac{1}{x}, & \text{for } \begin{array}{l} 0 \le x \le \varepsilon \\ 0 \le \theta \le x \end{array} \\[2em] \dfrac{1}{\varepsilon}, & \text{for } \begin{array}{l} \varepsilon \le x \le T \\ x-\varepsilon \le \theta \le x \end{array} \\[2em] \dfrac{1}{T+\varepsilon-x}, & \text{for } \begin{array}{l} T \le x \le T+\varepsilon \\ x-\varepsilon \le \theta \le T \end{array} \end{cases}$$

$$\hat{\theta} = E[\theta|X] = \begin{cases} \displaystyle\int_0^x \theta \frac{1}{x} d\theta = \frac{\theta^2}{2X}\Big|_0^x = \frac{x}{2} \\ \quad \text{for } 0 \le x \le \varepsilon \\[2em] \displaystyle\int_{x-\varepsilon}^x \theta \frac{1}{\varepsilon} d\theta = x - \frac{\varepsilon}{2}, \text{ for } \varepsilon \le x \le T \\[2em] \displaystyle\int_{x-\varepsilon}^T \theta \frac{1}{T+\varepsilon-x} d\theta = \frac{T+x-\varepsilon}{2}, \\ \quad \text{for } T \le x \le T+\varepsilon. \end{cases}$$