Deviation of $\hat{Y}_i$ from its mean value.
(variance)

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \qquad (i = 1, \cdots, n)$$

We want to see how much $\hat{Y}_i$ deviates from its mean value for a given $X_i = X_0$ value.

$$\operatorname{var}(\hat{Y}_i)\Big|_{X_i = X_0} = \operatorname{var}(\hat{\beta}_0 + \hat{\beta}_1 X_0)$$

value of $\hat{\beta}_0$ substituted.

$$= \operatorname{var}(\bar{Y} - \hat{\beta}_1 \bar{X} + \hat{\beta}_1 X_0)$$

$$= \operatorname{var}(\bar{Y} + \hat{\beta}_1 (X_0 - \bar{X}))$$

$$= \operatorname{var}(\bar{Y}) + \operatorname{var}(\hat{\beta}_1 (X_0 - \bar{X})) + \underbrace{2\operatorname{cov}(\bar{Y}, \hat{\beta}_1 (X_0 - \bar{X}))}$$

This is equal to zero because $\hat{\beta}_1$ and $\bar{Y}$ are independent.

$$= \operatorname{var}(\bar{Y}) + \operatorname{var}(\hat{\beta}_1 (X_0 - \bar{X}))$$

$$= \operatorname{var}(\bar{Y}) + (X_0 - \bar{X})^2 \operatorname{var}(\hat{\beta}_1)$$

$$= \frac{\sigma^2}{n} + (X_0 - \bar{X})^2 \frac{\sigma^2}{S_x^2}$$

$$\sigma_{\hat{Y}}^2 = \sigma^2 \left(\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{S_x^2} \right)$$

If we find a confidence interval on $E[\hat{Y}_i \mid X_i = X_0]$

then

$$\sigma_{\hat{Y}}^2 = \sigma^2 \left(\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{S_X^2} \right)$$

$$\sigma_{\hat{Y}} = \sigma \sqrt{\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{S_X^2}}$$

if σ^2 is given (which is the variance of noise in the data.) $\epsilon \sim N(0, \sigma^2)$

then $(1-\alpha)100\%$ of confidence interval on $E[\hat{Y}_i \mid X_i = X_0]$ is obtained as.

$$\hat{Y}_i(X_i = X_0) \pm Z_{\frac{\alpha}{2}} \, \sigma \sqrt{\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{S_X^2}}$$

If σ^2 is not given it will be estimated as follows:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$$

(biased estimate)

or

$$S^2 = \frac{1}{n-2} \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2$$

(unbiased estimate)

then $(1-\alpha)100\%$ of C.I. on $E[\hat{Y}_i \mid X_i = X_0]$ is obtained as:

$$\boxed{\hat{Y}_i(X_0) \pm t_{\frac{\alpha}{2}, n-2} \cdot S \cdot \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{X})^2}{S_X^2}}}$$