Homework 7 Solutions

Linear Algebra

Question 6

$$\mathbf{u} = (4,1,2,3), \mathbf{v} = (0,3,8,-2), \mathbf{w} = (3,1,2,2)$$

Part a

$\|\mathbf{u}+\mathbf{v}\|$

Solution:

$$\mathbf{u}+\mathbf{v} = (4,4,10,1)$$

$\therefore \|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}$

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{4^2+4^2+10^2+1^2} = \sqrt{16+16+100+1} = \sqrt{133}$$

Part b

$\|\mathbf{u}\| + \|\mathbf{v}\|$

Solution: $\therefore \|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}$

$$\|\mathbf{u}\| = \sqrt{4^2+1^2+2^2+3^2} = \sqrt{16+1+4+9} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{0^2+3^2+8^2+-2^2} = \sqrt{0+9+64+4} = \sqrt{77}$$

$$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{30} + \sqrt{77}$$

Part c

$\|-2\mathbf{u}\| + 2\|\mathbf{u}\|$

Soluiton: $\therefore \|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}$

For $\|-2\mathbf{u}\|$

$$\|-2\mathbf{u}\| = \|(-8,-2,-4,-6)\| = \sqrt{64+4+16+36} = \sqrt{150}$$

For $2\|\mathbf{u}\|$

$$2\|\mathbf{u}\| = 2(\|(4,1,2,3)\|) = 2(\sqrt{30})$$

$\|-2\mathbf{u}\| + 2\|\mathbf{u}\|$

$$\|-2\mathbf{u}\| + 2\|\mathbf{u}\| = \sqrt{150} + 2\sqrt{30}$$

Part d

$\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\|$

Soluiton: $\therefore \|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}$

$$3\mathbf{u} = (12,3,6,9), -5\mathbf{v} = (0,-15,-40,10), \mathbf{w} = (3,1,2,2)$$

$$3\mathbf{u} - 5\mathbf{v} + \mathbf{w} = (15,-11,-32,21)$$

$$\|3\mathbf{u} - 5\mathbf{v} + \mathbf{w}\| = \sqrt{225+122+1024+441} = \sqrt{1812}$$

Part e

$\frac{1}{\|\mathbf{w}\|}\mathbf{w}$

Soluiton: $\therefore \|\mathbf{u}\| = \sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2}$

$$\|\mathbf{w}\| = \sqrt{3^2 + 1^2 + 2^2 + 2^2} = \sqrt{9 + 1 + 4 + 4} = \sqrt{18}$$

$$\frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{(3, 1, 2, 2)}{\sqrt{18}}$$

Part f

$\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\|$

Soluiton :
We know that $\frac{1}{\|\mathbf{w}\|}$ is a constant and $\|k\mathbf{w}\| = |k|\|\mathbf{w}\|$ for some constant k.

$$\|\frac{1}{\|\mathbf{w}\|}\mathbf{w}\| = |\frac{1}{\|\mathbf{w}\|}|\|\mathbf{w}\| = 1$$

Question 7

$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ has euclidean norm 1

Solution: $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

$$\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\| = |\frac{1}{\|\mathbf{v}\|}|\|\mathbf{v}\| = 1$$

Question 8

$$\mathbf{v} = (-2, 3, 0, 6)$$

Find all scalars such that $\|k\mathbf{v} = 5\|$

Solution: $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

$$\|\mathbf{v}\| = \sqrt{-2^2 + 3^2 + 0^2 + 6^2} = \sqrt{4 + 9 + 0 + 36} = \sqrt{49} = 7$$

$$|k|\|\mathbf{v}\| = 5$$

$$|k| = \pm\frac{5}{7}$$

Question 10

Part a

Solution:

$$\mathbf{u} = (3, -1), \mathbf{v} = (x, y)$$

$$3u_1 - u_2 = 0 \implies 3u_1 = u_2$$

$$\mathbf{v} = (1, 3)$$

but,

$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

For norm = 1

$$\mathbf{v_1} = |\frac{1}{\|\mathbf{v}\|}|\|\mathbf{v}\| = (\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$$

The other vector

$$\mathbf{v_2} = (-\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}})$$

$$\mathbf{u} \cdot \mathbf{v_2} = 0, \|v_2\| = 1$$

part b

Show that there are infinitely many vectors in R^3 with Euclidean norm 1 whose Euclidean inner product with $(1, -3, 5)$ is zero.

Solution:

$$\mathbf{v} = (x, y, z)$$

$$\mathbf{u} \cdot \mathbf{v} = x - 3y + 5z = 0$$

v belongs to a plane in R3 with the equation above.
This plane contains the origin (0,0,0) and has infinitely many vectors that have norm 1 and initial point (0,0,0).
(Picture a circle of radius I in that plane. with center at the origin)
Thus, the statement is proved.

Question 20

Find $\mathbf{u} \cdot \mathbf{v}$ such that $\|\mathbf{u} + \mathbf{v}\| = 1$ and $\|\mathbf{u} - \mathbf{v}\| = 5$

Solution

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}(1^2) - \frac{1}{4}5^2 = -\frac{24}{4} = -6$$

Question 21

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\mathbf{u} \cdot \mathbf{v} = 0$

Solution: $\therefore \|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$$

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 = 0$$

Geometric Interpretition: Geometric interpretation:

If we draw $\mathbf{u}$ and $\mathbf{v}$ so that they have the same initial point, they correspond to two sides of a parallelogram

$(\mathbf{u} + \mathbf{v})$ and $(\mathbf{u} - \mathbf{v})$ correspond to diagonals of that parallelogram.

If the diagonals have same lengths, then the parallelogram is a RECTANGLE ($\mathbf{u}$ and $\mathbf{v}$ are orthogonal).

Question 23

Whether the two lines $r_1 = (3, 2, 3, -1) + t(4, 6, 4, -2)$ and $r_2 = (0, 3, 5, 4) + s(1, -3, -4, -2)$ intersect or not.

Solution: For it to have a solution (intersection point)

$$(3+4t, 2+6t, 3+4t, -1-2t) = (s, 3-3s, 5-4s, 4-2s) \tag{1}$$

From this we get:
$3+4t = s \implies 4t - s = -3$
$2+6t = 3-3s \implies 6t+3s = 1$
$3+4t = 5-4s \implies 4t+4s = 2$
$-1-2t = 4-2s \implies -2t+2s = 5$
The Augmented matrix is

$$\begin{bmatrix} 4 & -1 & -3 \\ 6 & 3 & 1 \\ 4 & 4 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

Which is reduced two

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third row indicates that the system is inconsistent since there is no value of s and t that satisfy equation (1). Hence, the lines don't intersect.

Question 24

$\|v_1 + v_2 + ... + v_r\|^2 = \|v_1\|^2 + \|v_2\|^2 + ... + \|v_r\|^2$

Solution:

$$\|v_1 + v_2 + ... + v_r\|^2 = (v_1 + v_2 + ...v_r) \cdot (v_1 + v_2 + ...v_r)$$

Which is equal to a sum of terms of teh form $v_i \cdot v_j$.
The terms for which $i \neq j$ are zero because v_i and v_j are orthogonal.
The non-zero terms are:

$$v_1 \cdot v_1 + v_2 \cdot v_2 + ... + v_r \cdot v_r = \|v_1\|^2 + \|v_2\|^2 + ... + \|v_r\|^2$$

Question 25

$(\mathbf{v}^T A^T A\mathbf{u})^2 \leqslant (\mathbf{u}^T A^T A\mathbf{u})(\mathbf{v}^T A^T A\mathbf{v})$

Solution: We have

$$v^T A^T Au = (Av)^T Au = (Av).(Au)$$
$$(v^T A^T Au)^2 = ((Av).(Au))^2 \tag{2}$$
$$u^T A^T Au = (Au)^T Au = \|Au\|^2 \tag{3}$$
$$v^T A^T Av = (Av)^T Av = \|Av\|^2 \tag{4}$$

By Cauchy-Schwartz inequality

$$((Av).(Au))^2 \leqslant \|Av\|^2 \cdot \|Au\|^2$$

From equations (2), (3) and (4), we have

$$(\mathbf{v}^T A^T A\mathbf{u})^2 \leqslant (\mathbf{u}^T A^T A\mathbf{u})(\mathbf{v}^T A^T A\mathbf{v})$$

Question 26

Use Cauchy-Schwartz inequality to prove

$$(acos\theta + bsin\theta)^2 \leqslant a^2 + b^2$$

For all real values of a and b

Solution: Cauchy-Schwartz inequality $|u.v| \leqslant \|u\| \cdot \|v\|$
Let

$$u = (a, b), v = (cos\theta, sin\theta)$$

Applying Cauchy-Schwartz inequality

$$acos\theta + bsin\theta \leqslant (a^2 + b^2)^{\frac{1}{2}}(cos^2\theta + sin^2\theta)^{\frac{1}{2}}$$

We know that $cos^2\theta + sin^2\theta = 1$

$$acos\theta + bsin\theta \leqslant (a^2 + b^2)^{\frac{1}{2}} \implies (acos\theta + bsin\theta)^2 \leqslant (a^2 + b^2)$$

Question 27

$\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are any vectors in R^n and k is any scalar

Part a

Solution: $\mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v})$

$$\mathbf{u} \cdot (k\mathbf{v}) = (u_1 + u_2 + ... + u_n).(kv_1 + kv_2 + ... + kv_n)$$

$$\mathbf{u} \cdot (k\mathbf{v}) = ku_1v_1 + ku_2v_2 + ... + ku_nv_n = k(u_1v_1 + u_2v_2 + ... + u_nv_n) = k(\mathbf{u} \cdot \mathbf{v})$$

Part b

Solution: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (u_1 + u_2 + ... + u_n).(v_1 + w_1 + v_2 + w_2 + ... + v_n + w_n)$$

$$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + ... + u_nv_n + u_nw_n = (u_1v_1 + u_2v_2 + ... + u_nv_n) + (u_1w_1 + u_2w_2 + ... + u_nw_n)$$
$$= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

Question 29

Part e

Solution: $k(m\mathbf{u}) = (km)\mathbf{u}$

$$k(m\mathbf{u}) = k(mu_1 + mu_2 + ... + mu_n)$$
$$= kmu_1 + kmu_2 + ... + kmu_n = (km)\mathbf{u}$$

Part f

Solution: $k(\mathbf{u} + \mathbf{v}) = k(\mathbf{u}) + k(\mathbf{v})$

$$k(\mathbf{u} + \mathbf{v}) = k(u_1 + v_1, u_2 + v_2, ..., u_n + v_n) = (ku_1 + kv_1, ku_2 + kv_2, ..., ku_n + kv_n)$$

$$= (ku_1, ku_2, ..., ku_n) + (kv_1, kv_2, ..., kv_n) = k(\mathbf{u}) + k(\mathbf{v})$$

Part g

Solution: $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

$$(k + m)\mathbf{u} = (k + m)(u_1 + u_2 + ... + u_n)$$

$$= (ku_1 + ku_2 + ... + ku_n) + (mu_1 + mu_2 + ... + mu_n) = k\mathbf{u} + m\mathbf{u}$$

Part h

Solution: $1\mathbf{u} = \mathbf{u}$

$$1\mathbf{u} = 1(u_1 + u_2 + ... + u_n) = (u_1 + u_2 + ... + u_n) = \mathbf{u}$$

Question 32

Part a

$d(u, v) \geqslant 0$

Solution: $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_n - v_n)^2}$
Since this is norm of $(u - v)$, by theorem 4.1.4 a this will be $\geqslant 0$

Part b

$d(u, v) = 0$ if and only if $u = v$

Solution: $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_n - v_n)^2}$
By theorem 4.1.4 the norm of $(u - v)$ will be zero if and only if $(u - v)$ is equal to zero vector i.e. $(u - v) = 0 \implies u = v$

Part c

$d(u, v) = d(v, u)$

Solution:

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + ... + (u_n - v_n)^2}$$
$$d(v, u) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + ... + (v_n - u_n)^2}$$

Since $(u_i - v_i)^2 = (v_i - u_i)^2$, $d(u, v) = d(v, u)$

Question 34

Part a

$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$

Solution:

$$\|u + v\|^2 = 2\|u\|^2 + 2u.v + \|v\|^2$$
$$\|u - v\|^2 = 2\|u\|^2 - 2u.v + \|v\|^2$$
$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2u.v + \|v\|^2 + 2\|u\|^2 - 2u.v + \|v\|^2$$
$$= \|u\|^2 + \|u\|^2 + \|v\|^2 + \|v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Part b

Solution: In a parallelogram with u and v as its sides, $(u + v)$ and $(u - v)$ are its diagonals.
The formula states that the sum of squares if lengths if the diagonals is equal to the sum of squares of lengths of all the sides.

Question 35

Part a

If u and v are orthogonal vectors and $\|u\| = 1$ and $\|v\| = 1$ then $d(u, v) =?$

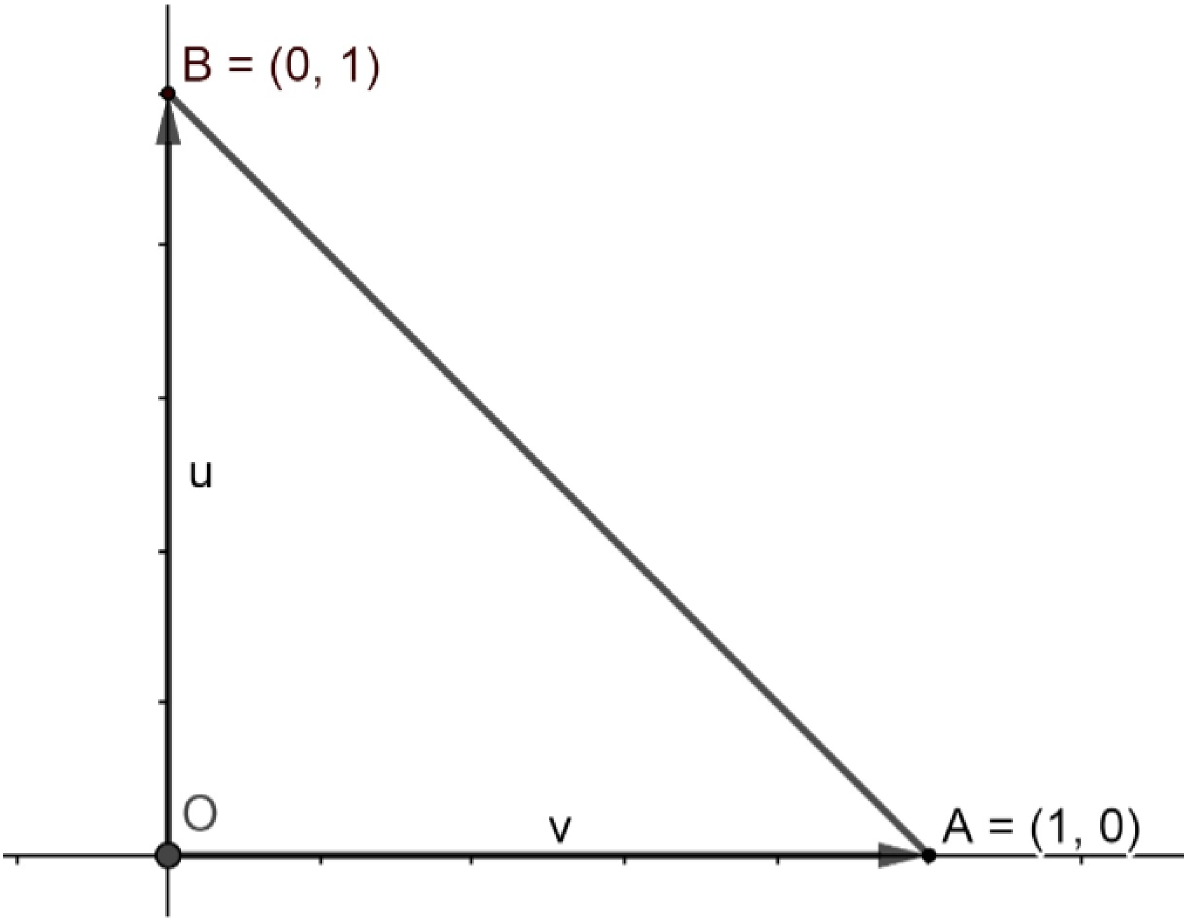
Solution: $d(u, v) = \|u - v\|$

$$\|u - v\| = \sqrt{\|u - v\|^2} = \sqrt{\|u\|^2 - 2u.v + \|v\|^2}$$

$\therefore u.v = 0$ and $\|u\| = \|v\| = 1$

$$= \sqrt{2} = d(u, v)$$

Part b



Result 3 of 3

Draw a diagram in XY plane with $\mathbf{u} = (0, 1)$ and $\mathbf{v} = (1, 0)$ and verify the result using Pythagoras' theorem.

Question 36

In the accompanying figure the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\mathbf{u}-\mathbf{v}$ form a triangle in R^2, and θ denotes the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. It follows from the law of cosines in trigonometry that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Do you think that this formula still holds if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in R^n? Justify your answer.

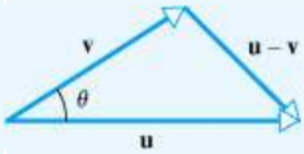


Figure Ex-36

Solution: If we consider the plane generated by the vectors u and v and define the angle between them in the same way as we define the angle between the vectors the vectors in the plane, then the formula will remain true.

Question 37

Part a

If $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ then u and v are orthogonal

Solution:

$$\|u+v\|^2 = 2\|u\|^2 + 2u.v + \|v\|^2$$

$u.v = 0$ if u and v are orthogonal. Hence the statement is always **true**

Part b

If u is orthogonal to v and w, then u is orthogonal to $v+w$

Solution: $u.(v+w) = u.v + u.w$ and $u.v = u.w = 0$, so $u.(v+w) = 0 + 0 = 0$. Hence, u is orthogonal to $v+w$.
Hence the statement is always **true**

Part c

If u is orthogonal to $v+w$, then u is orthogonal to v and w

Solution: We will disprove this using a counterexample

$$u = (1,0), v = (-1,1), w = (1,1), v+w = (0,2)$$

$u.(v+w) = 0$ but $u.v \neq 0$ and $u.w \neq 0$. Hence the statement is sometimes **false**

Part d

if $\|u-v\| = 0$, then $u = v$

Solution: According to theorem 4.1.4, $\|u-v\| = 0$ if and only if $u = v$. Hence the statement is always **true**

Part e

If $\|k\mathbf{u}\| = k\|\mathbf{u}\|$, then $k \geqslant 0$

Solution: $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$, then $|k| = k$ which means that k is greater than or equals to zero. Hence the statement is always **true**