Algorithms: Design and Analysis - CS 412

Problem Set 01: Asymptotic Analysis

1. Let

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

where $a_d > 0$, be a degree-d polynomial in n and let k be a constant. Use the definition of the asymptotic notations to prove the following properties:

(a) If $k \geq d$, then $p(n) = O(n^k)$.

(b) If $k \leq d$, then $p(n) = \Omega(n^k)$.

(c) If $k = d$, then $p(n) = \Theta(n^k)$.

(d) If $k > d$, then $p(n) = o(n^k)$.

(e) If $k < d$, then $p(n) = \omega(n^k)$.

Solution:

(a) If $k \geq d$, then $p(n) = O(n^k)$.

Definition of Big-Oh: $f(n) = O(g(n))$ if there exists positive constants c and n_0 such that $0 \leq f(n) \leq c.g(n) \ \ \forall n \geq n_0$

Proof. Choose $c = \sum_{i=0}^{d} |a_i|$ and $n_0 = 1$. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i \leq \sum_{i=0}^{d} |a_i| n^d \leq \left(\sum_{i=0}^{d} |a_i| \right) n^k = cn^k$$

Since $k \geq d, n^i \leq n^d \leq n^k \ \ \forall i \leq d$, thus $p(n) = O(n^k)$ □

(b) If $k \leq d$, then $p(n) = \Omega(n^k)$.

Definition of Big-Omega: $f(n) = \Omega(g(n))$ if there exists positive constants c and n_0 such that $0 \leq c.g(n) \leq f(n) \ \ \forall n \geq n_0$

Proof. Choose $c = a_d$ and $n_0 = 1$. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i \geq a^d n^d \geq a_d n^k = cn^k$$

Since $a_d > 0$ and $k \leq d, n^d \geq n^k \quad \forall n$, thus cn^k is a lower bound for $p(n)$, and $p(n) = \Omega(n^k)$. □

(c) If $k = d$, then $p(n) = \Theta(n^k)$.

Definition of Big-Theta: $f(n) = \Theta(g(n))$ if there exists positive constants c_1, c_2 and n_0 such that $0 \leq c_1.g(n) \leq f(n) \leq c_2.g(n) \ \ \forall n \geq n_0$. Or in other words, $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Proof. From parts (a) and (b), we have shown that if $k \geq d$, then $p(n) = O(n^k)$ and if $k \leq d$, then $p(n) = \Omega(n^k)$. When $k = d$, both conditions are satisfied, which means $p(n)$ is both upper and lower bounded by n^k, hence is both $O(n^k)$ and $\Omega(n^k)$, and therefore $p(n) = \Theta(n^k)$. □

(d) If $k > d$, then $p(n) = o(n^k)$.

Definition of Little-Oh: $f(n) = o(g(n))$ if for every positive constant c, there exists a constant n_0 such that $0 \leq f(n) < c.g(n) \ \ \forall n \geq n_0$

Proof. Given any $c > 0$, choose n_0 such that $n_0^k > \sum_{i=0}^{d} |a_i| n_0^i$. This is possible since $k > d$, and n^k grows faster than any n^i for $i < d$ as n approaches infinity. Then $\forall n \geq n_0$:

$$p(n) = \sum_{i=0}^{d} a_i n^i < \sum_{i=0}^{d} |a_i| n^i < \left(\sum_{i=0}^{d} |a_i| \right) n^k < cn^k$$

The above inequality holds because we can always find an n_0 such that the polynomial sum is less than cn^k for any c, thus $p(n) = o(n^k)$. □

(e) If $k < d$, then $p(n) = \omega(n^k)$.

Definition of Little-Omega: $f(n) = \omega(g(n))$ if for all constants $c > 0$, there exists some constant n_0 such that $0 \leq c.g(n) < f(n) \ \ \forall n \geq n_0$, or $p(n) > cn^k$.

Proof. Let $p(n) = a_d n^d + a_{d-1} n^{d-1} + ... + a_1 n + a_0$, with $a_d > 0$ and $k < d$. Consider the leading term $a_d n^d$, which dominates $p(n)$ as n grows large. For any $c > 0$, we can choose n_0 such that for all $n > n_0$, $a_d n^d > cn^k$. This is because the degree of n^d is higher than n^k, and $a_d > 0$.

Thus, as n approaches infinity, the ratio $p(n)/n^k$ approaches infinity which implies that $p(n)$ grows strictly faster than cn^k for any constant c, proving that $p(n) = \omega(n^k)$. □

2. Indicate for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Write your answer in the form of the table with "yes" or "no" written in each box.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

3. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

(a) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

(b) $f(n) + g(n) = \Theta(\min\{f(n), g(n)\})$.

(c) $f(n) = O(g(n))$ implies $\lg f(n) = O(\lg g(n))$, where $\lg g(n) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.

(d) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$

(e) $f(n) = O((f(n))^2)$.

(f) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.

(g) $f(n) = \Theta(f(\frac{n}{2}))$

(h) $f(n) + o(f(n))\Theta(f(n))$

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove the following identities.

(a) $\Theta(\Theta(f(n))) = \Theta(f(n))$

(b) $\Theta(f(n)) + O(f(n)) = \Theta(f(n))$

(c) $\Theta(f(n)) + \Theta(g(n)) = \Theta(f(n) + g(n))$

(d) $\Theta(f(n)).\Theta(g(n)) = \Theta(f(n).g(n))$