Exercise Set 1.2 Solution

Question 17

For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x+2y-\quad 3z &= 4 \\ 3x-y+\quad 5z &= 2 \\ 4x+\ y+\left(a^2-14\right)z &= a+2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & a+2 \end{array}\right]$$

$$R_2-3R_1, R_3-4R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-2) & a-14 \end{array}\right]$$

$$R_2-R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & (a^2-16) & a-4 \end{array}\right]$$

$$\frac{-1}{7}R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & (a^2-16) & a-4 \end{array}\right]$$

The Gauss-Jordan process will reduce this system to the equations

$$\begin{aligned} x+2y-3z &= 4 \\ y-2z &= 10/7 \\ \left(a^2-16\right)z &= a-4 \end{aligned}$$

If $a=4$, then the last equation becomes $0=0$, and hence there will be infinitely many solutions-for instance,

$$z=t, y=2t+\frac{10}{7}, x=-2\left(2t+\frac{10}{7}\right)+3t+4$$

. If $a=-4$, then the last equation becomes $0=-8$, and so the system will have no solutions.

Any other value of a will yield a unique solution for z and hence also for y and x.

Exercise Set 1.6 Solution

Question 10

Solution:

The coefficient matrix, augmented by the two **b** matrices, yields

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right]$$

Applying $R_2 + (-3)R_1$ This reduces to

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right]$$

and then applying $\frac{1}{17}R_2, R_1 + (-5)R_2$

$$\left[\begin{array}{cc|c|c} 1 & 0 & 22/17 & 21/17 \\ 0 & 1 & 1/17 & 11/17 \end{array}\right]$$

Thus the solution to Part (a) is $x_1 = 22/17, x_2 = 1/17$, and to Part (b) is $x_1 = 21/17$, $x_2 = 11/17$

Question 16

Find conditions that the b's must satisfy for the system to be consistent.

$$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

Solution:

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \xrightarrow{R_2 \leftrightarrows 2R_1} \left[\begin{array}{cc|c} 3 & -2 & b_2 \\ 6 & -4 & b_1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 3 & -2 & b_2 \\ 0 & 0 & b_1 - 2b_2 \end{array}\right]$$

One can see, for consistent second row has to get completely zeros, which deduce $b_1 = 2b_2$.

Question 23

Since $A\mathrm{x} =$ has only $\mathrm{x} = \mathbf{0}$ as a solution, Theorem 1.6.4 guarantees that A is invertible. By Theorem 1.4.8 (b), A^k is also invertible. In fact,

$$\left(A^k\right)^{-1} = \left(A^{-1}\right)^k$$

Since the proof of Theorem 1.4.8 (b) was omitted, we note that

$$\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{\substack{k \\ \text{factors}}} \underbrace{AA\cdots A}_{\substack{k \\ \text{factors}}} = I$$

Because A^k is invertible, Theorem 1.6.4 allows us to conclude that $A^k\mathrm{X} = \mathbf{0}$ has only the trivial solution.

Question 24

Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $A\mathbf{x} = \mathbf{0}$ has just the trivial solution if and only if $(QA)\mathbf{x} = 0$ has just the trivial solution.

Proof: First let $A\mathbf{x} = \mathbf{0}$ holds. Now we apply Q matrix from L.H.S we will have

$$\begin{aligned} Q(A\mathbf{x}) &= Q\mathbf{0} \\ (QA)\mathbf{x} &= \mathbf{0} \qquad \because \text{Assoociative property} \end{aligned}$$

Now we let $(QA)\mathbf{x} = \mathbf{0}$ and we apply Q^{-1} from L.H.S because Q is invert-able we will have

$$\begin{aligned} Q^{-1}(QA)\mathbf{x} &= Q^{-1}\mathbf{0} \\ (Q^{-1}Q)A\mathbf{x} &= Q^{-1}\mathbf{0} \qquad \because \text{Assoociative property} \\ IA\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} &= \mathbf{0} \end{aligned}$$

Question 25

Suppose that x_1 is a fixed matrix which satisfies the equation $A\mathrm{x}_1 = \mathbf{b}$. Further, let x be any matrix whatsoever which satisfies the equation $A\mathrm{x} = \mathbf{b}$. We must then show that there is a matrix x_0 which satisfies both of the equations $\mathrm{x} = \mathrm{x}_1 + \mathrm{x}_0$ and $A\mathrm{x}_0 = \mathbf{0}$. Clearly, the first equation implies that

$$\mathrm{x}_0 = \mathrm{X} - \mathrm{X}_1$$

This candidate for x_0 will satisfy the second equation because

$$A\mathrm{x}_0 = A\left(\mathrm{x} - \mathrm{x}_1\right) = A\mathrm{x} - A\mathrm{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

We must also show that if both $A\mathrm{x}_1 = \mathbf{b}$ and $A\mathrm{x}_0 = \mathbf{0}$, then $A\left(\mathrm{x}_1 + \mathrm{x}_0\right) = \mathbf{b}$. But

$$A\left(\mathrm{x}_1 + \mathrm{x}_0\right) = A\mathrm{x}_1 + A\mathrm{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Question 26

Solution:

For $B = A^{-1}$, we use part (a)

If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$

Given: $AB = I$ and $BA = I$.

Hence $AB = BA = I$.

Hence $B = A^{-1}$ for $AB = I$.

Another fact can be seen $A^{-1}A = AA^{-1} = I$.