Math Fundamentals

EE468/CE468: Mobile Robotics

Dr. Basit Memon

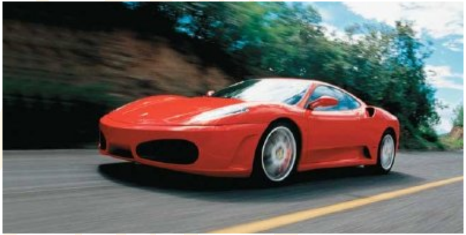
Electrical and Computer Engineering
Habib University

August 28, 2023

Table of Contents

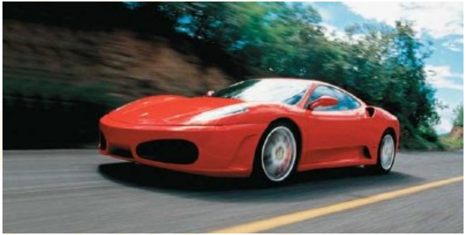
Table of Contents

Measurements require context.



- Speed of your car is measured as 45 km/h by the speed gun on Shuhada-e-ASF road.

Measurements require context.



- Speed of your car is measured as 45 km/h by the speed gun on Shuhada-e-ASF road.
- Every measurement requires context:
  - Unit system (e.g. meters, hour)
  - Number system (e.g. base 10)
  - Coordinate system (e.g. north, east)
  - Reference frame to which measurement is ascribed (e.g. car)
  - Reference system with respect to which measurement is made (e.g. speed gun)

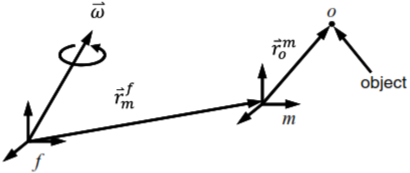


Figure: Transformation between frames

- **Coordinate systems** are conventions for representation.
- A **reference frame** is a state of motion, which is linked to a moving body for convenience.

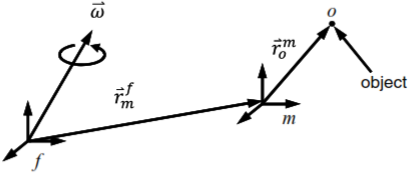


Figure: Transformation between frames

Is coordinate system same as frame of reference? [1, Section 4.1]

- **Coordinate systems** are conventions for representation.
- A **reference frame** is a state of motion, which is linked to a moving body for convenience.
- We use laws of physics to convert among frames, while laws of physics hold regardless of coordinate system.

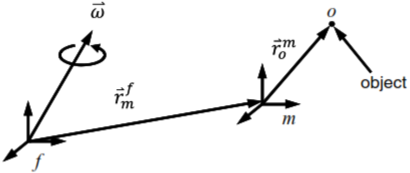
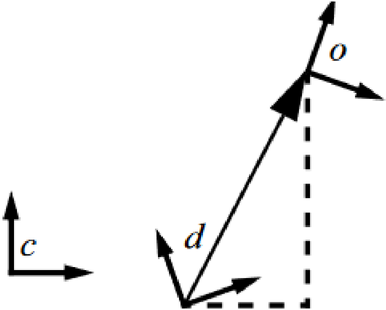
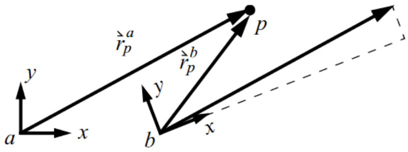


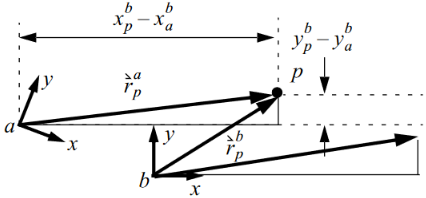
Figure: Transformation between frames

$${}^{c}r^{d}_{o}$$

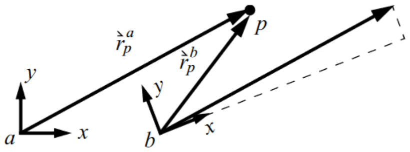


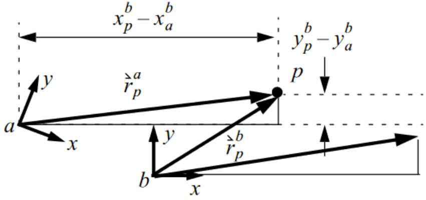
r: physical quantity / property
o: object possessing property
d: object whose state of motion serves as datum
c: object whose coordinate system is used to express result





- Datum with respect to which measurement is made has changed.
- This is change of reference frame.
- Requires laws of physics.





- Datum with respect to which measurement is made has changed.
- This is change of reference frame.
- Requires laws of physics.

- Change of coordinates. Quantity remains r_p^a.
- Quantity is free vector. Moved from origin of a to origin of b.
- Magnitude and direction remain the same.

Table of Contents

Robot is modeled as a rigid body on wheels.

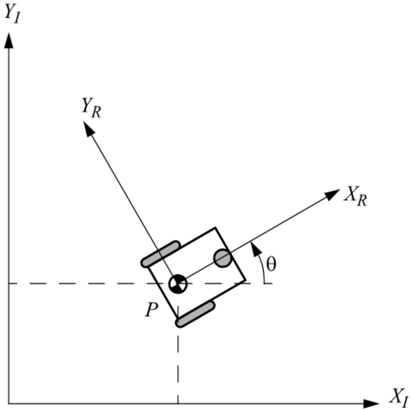


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

- Robot **chassis** is the rigid body minus joints and wheels with internal dof.

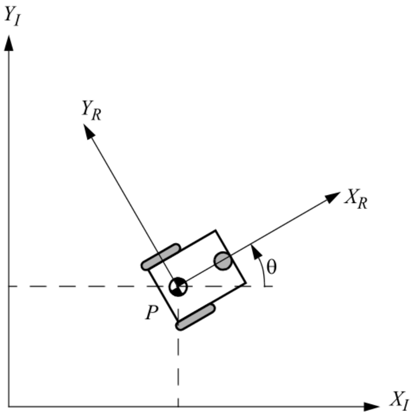


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

Robot is modeled as a rigid body on wheels.

- Robot **chassis** is the rigid body minus joints and wheels with internal dof.
- Assume robot moves on horizontal plane only.

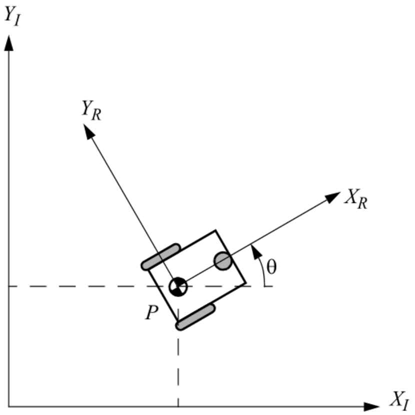


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

Robot is modeled as a rigid body on wheels.

- Robot **chassis** is the rigid body minus joints and wheels with internal dof.
- Assume robot moves on horizontal plane only.
- Describing motion of the frame $\{R\}$, rigidly attached to the chassis, with respect to a global inertial frame of reference $\{I\}$ completely captures the motion of chassis.

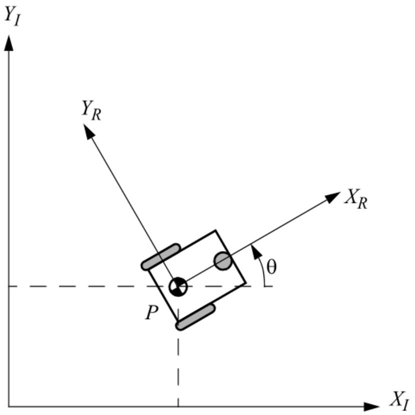


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

- Choose any point P on the robot chassis and attach a frame $\{R\}$ to it.

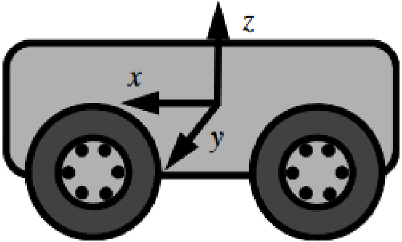


Figure: Embedded frame (Source: [1])

Embedded frames abstract the motion of a body.

- Choose any point P on the robot chassis and attach a frame $\{R\}$ to it.
- Frame moves with the robot.

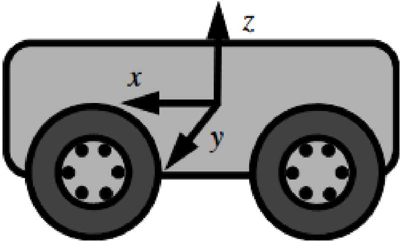


Figure: Embedded frame (Source: [1])

- Choose any point P on the robot chassis and attach a frame $\{R\}$ to it.
- Frame moves with the robot.
- It possesses properties of both reference frame and coordinate system.

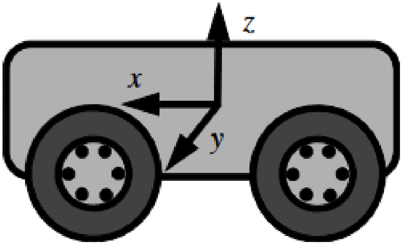


Figure: Embedded frame (Source: [1])

- w: world frame
- b: body frame
- c: wheel contact frame
- p: position estimator frame
- s: sensor frame
- h: sensor housing

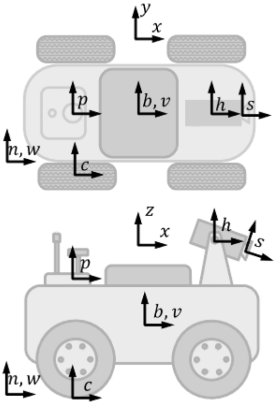


Figure 2.24 Standard Vehicle Frames. Many coordinate frames are commonly used when modelling vehicles.

Table of Contents

- 2×1 position vector

$$^{A}P = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

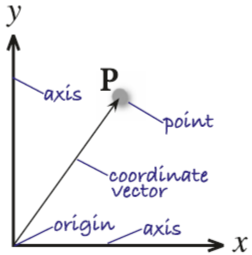


Figure: Source: Robotics, Vision, and Control

- 2×1 position vector

$$^{A}P = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

- Superscript indicates coordinate axes or frame information.

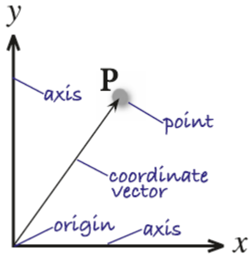


Figure: Source: Robotics, Vision, and Control

- $\hat{x}_b = \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s$

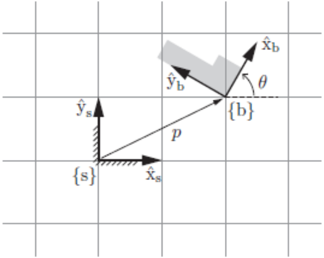


Figure: Source: Modern Robotics

We'll use rotation matrix to describe the orientation.

- $\hat{x}_b = \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s$
- $\hat{y}_b = -\sin\theta\, \hat{x}_s + \cos\theta\, \hat{y}_s$

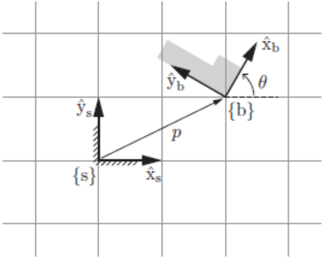


Figure: Source: Modern Robotics

We'll use rotation matrix to describe the orientation.

- $\hat{x}_b = \cos\theta\, \hat{x}_s + \sin\theta\, \hat{y}_s$
- $\hat{y}_b = -\sin\theta\, \hat{x}_s + \cos\theta\, \hat{y}_s$
- Orientation representation

$$
\begin{aligned}
{}^sR_b &= \begin{bmatrix} {}^s\hat{x}_b & {}^s\hat{y}_b \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_b \cdot \hat{x}_s & \hat{y}_b \cdot \hat{x}_s \\ \hat{x}_b \cdot \hat{y}_s & \hat{y}_b \cdot \hat{y}_s \end{bmatrix}
\end{aligned}
$$

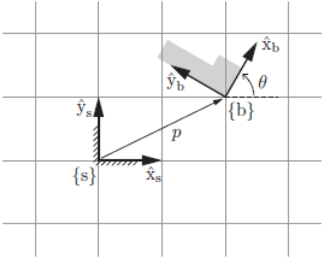


Figure: Source: Modern Robotics

We'll use rotation matrix to describe the orientation.

- $\hat{x}_b = \cos\theta\,\hat{x}_s + \sin\theta\,\hat{y}_s$
- $\hat{y}_b = -\sin\theta\,\hat{x}_s + \cos\theta\,\hat{y}_s$
- Orientation representation

$$
\begin{aligned}
{}^sR_b &= \begin{bmatrix} {}^s\hat{x}_b & {}^s\hat{y}_b \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \hat{x}_b \cdot \hat{x}_s & \hat{y}_b \cdot \hat{x}_s \\ \hat{x}_b \cdot \hat{y}_s & \hat{y}_b \cdot \hat{y}_s \end{bmatrix}
\end{aligned}
$$

- Called the **Rotation matrix**.

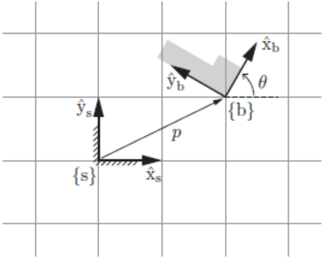


Figure: Source: Modern Robotics

Pair $({}^sP, {}^sR_b)$ can describe a reference frame relative to another.

Pair $({}^sP, {}^sR_b)$ can describe a reference frame relative to another.

- Dof of planar end-effector (rigid body) is 3, but we're using 6 numbers here!

Pair $({}^sP, {}^sR_b)$ can describe a reference frame relative to another.

- Dof of planar end-effector (rigid body) is 3, but we're using 6 numbers here!
- Any rotation matrix, $R \in \mathbb{R}^{2\times 2}$ with columns c_i, has 3 constraints.

Pair $({}^sP, {}^sR_b)$ can describe a reference frame relative to another.

- Dof of planar end-effector (rigid body) is 3, but we're using 6 numbers here!
- Any rotation matrix, $R \in \mathbb{R}^{2\times 2}$ with columns c_i, has 3 constraints.

Pair $({}^sP, {}^sR_b)$ can describe a reference frame relative to another.

- Dof of planar end-effector (rigid body) is 3, but we're using 6 numbers here!
- Any rotation matrix, $R \in \mathbb{R}^{2\times 2}$ with columns c_i, has 3 constraints.
  - Each column is a unit vector, i.e. $||c_i|| = 1, \text{ for } i \in \{1, 2\}$.
  - Two columns are orthogonal to each other, i.e. $c_1^T c_2 = 0$.

We want right-handed frames

- $\det R = +1$ corresponds to right-handed frame.
  - The $\hat{x}$, $\hat{y}$, and $\hat{z}$ of right-handed reference frame are aligned with index finger, middle finger, and thumb respectively.

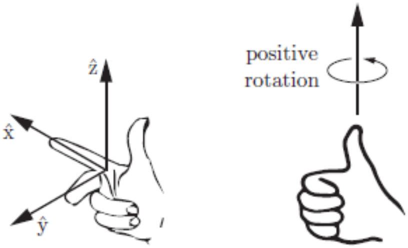


Figure: Source: Modern Robotics

- $\det R = +1$ corresponds to right-handed frame.
  - The $\hat{x}$, $\hat{y}$, and $\hat{z}$ of right-handed reference frame are aligned with index finger, middle finger, and thumb respectively.
- Positive rotation along an axis is in direction the fingers of right-hand curl when thumb is pointed along axis.

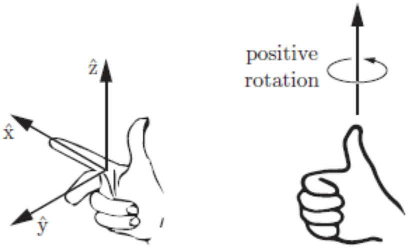


Figure: Source: Modern Robotics

Homogeneous Transformation

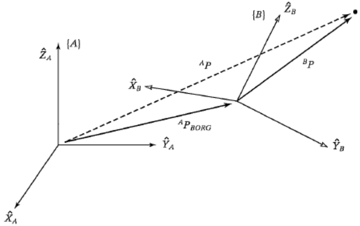


Figure: Source: Intro to Robotics, Mechanics and Control

- $$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_B & {}^{A}O_B \\ \mathbf{O} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

- 4×4 matrix is called **homogeneous transformation,** ${}^{A}T_B$.

- $\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$ is **homogeneous representation** of vector ${}^{B}P$.

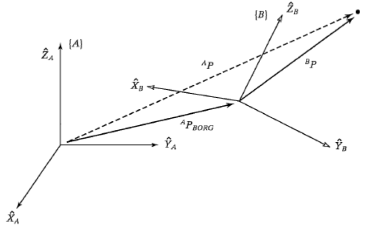


Figure: Source: Intro to Robotics, Mechanics and Control

Three interpretations of homogeneous transformation, T_B^A

- **Description of relationship between frames:** Moves frame A to be in coincidence with frame B.

Three interpretations of homogeneous transformation, T_B^A

- **Description of relationship between frames:** Moves frame A to be in coincidence with frame B.
- **Operator to move points or reorient directions:** Expression $T_B^A \, r_P^A$ moves the point P to the frame B, i.e. $r_{P_2}^A$.

Three interpretations of homogeneous transformation, T_B^A

- **Description of relationship between frames:** Moves frame A to be in coincidence with frame B.

- **Operator to move points or reorient directions:** Expression $T_B^A\, r_P^A$ moves the point P to the frame B, i.e. $r_{P_2}^A$.

- **Change of frame:** Same operator T_B^A acting on point r_P^B, vector relative to B and expressed in B, changes it to r_P^A, vector relative to A and expressed in A, i.e. $r_P^A = T_B^A\, r_P^B$

Rotation matrix can be used to transform only coordinates.

- Given some velocity ξ in the global frame, I, we can convert its coordinates to the local frame, R, as:

$$\dot{\xi}_R = {}^R R_I \, \dot{\xi}_I$$

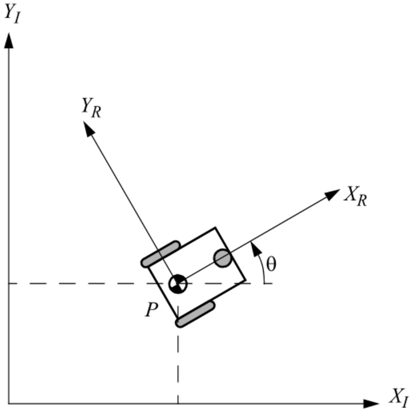


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

Inverse of a rotation matrix is its transpose.

- Interestingly,

$$^{R}R_{I}^{-1} = {}^{R}R_{I}^{T}$$

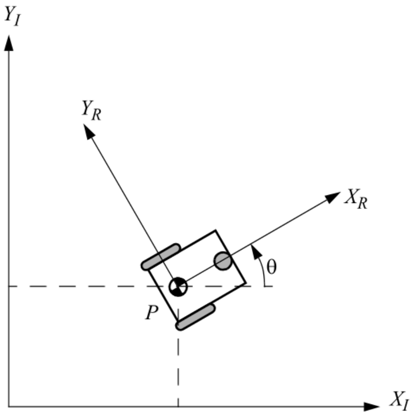


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

Inverse of a rotation matrix is its transpose.

- Interestingly,

$$^{R}R_{I}^{-1} = {}^{R}R_{I}^{T}$$

- Inverse of a homogeneous transformation is not its transpose. We have to find matrix inverse.

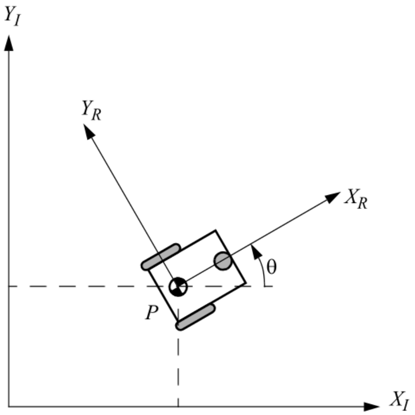


Figure: Pose of a mobile robot (Source: Autonomous Mobile Robots)

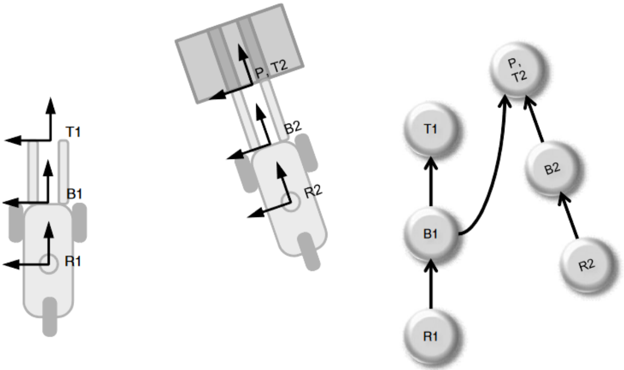


Figure 2.38 Kinematic Placement Problem. Where must *R2* be relative to *R1* if the tip frame (*T2*) is to be aligned with the pallet (*P*)?

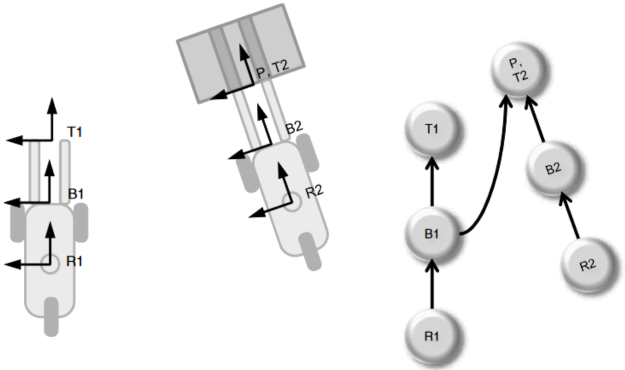


Figure 2.38 Kinematic Placement Problem. Where must $R2$ be relative to $R1$ if the tip frame ($T2$) is to be aligned with the pallet (P)?

$$T^{R1}_{R2} = T^{R1}_{B1}\, T^{B1}_{P}\, T^{P}_{T2}\, T^{T2}_{B2}\, T^{B2}_{R2} = T^{R}_{B}\, T^{B}_{P}\, (T^{B}_{T})^{-1}\, (T^{R}_{B})^{-1}$$

Table of Contents

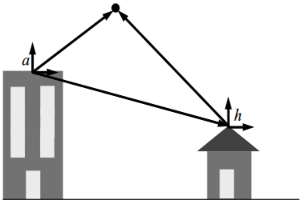


Figure 4.2 Mutually Stationary Frames. Observers in the two buildings will agree on the velocity of the particle but not on its position vector.

$$\vec{r}_p^{\,a} = \vec{r}_p^{\,h} + \vec{r}_h^{\,a}$$

Differentiating the expression,

$$\vec{v}_p^{\,a} = \vec{v}_p^{\,h}$$

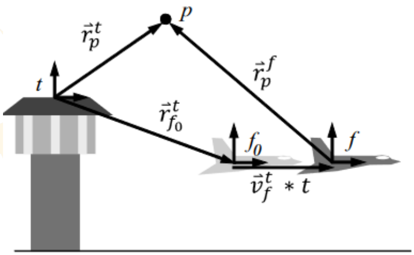


Figure 4.3 Frames Moving at Constant Velocity. Observers in the tower and airplane will agree on the acceleration of the particle but not on its velocity vector.

$$\vec{r}_p^{\,t} = \vec{r}_p^{\,f} + \vec{r}_{f0}^{\,t} + \vec{v}_f^{\,t} \cdot t$$

Differentiating the expression,

$$\vec{v}_p^{\,t} = \vec{v}_p^{\,f} + \vec{v}_f^{\,t}$$

Rotating Frames: Coriolis Equation or Transport Theorem

- Frame m is rotating with respect to frame f with instantaneous angular velocity $\vec{\omega}$.

- For any vector $\vec{u}$,

$$\left(\frac{d\vec{u}}{dt}\right)_f = \left(\frac{d\vec{u}}{dt}\right)_m + \vec{\omega} \times \vec{u}$$

General Relative Motion

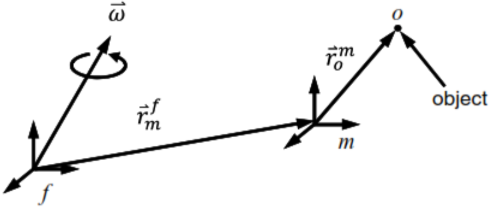


Figure: Frames in general motion

$$\vec{r}_o^f = \vec{r}_m^f + \vec{r}_o^m$$

Differentiating,

$$\begin{aligned}\vec{v}_o^f &= \left.\frac{d}{dt}\right|_f \vec{r}_m^f + \left.\frac{d}{dt}\right|_f \vec{r}_o^m \\ &= \vec{v}_m^f + \vec{v}_o^m + \vec{\omega}_m^f \times \vec{r}_o^m\end{aligned}$$

Table of Contents

[1] **Alonzo Kelly.**
Mobile robotics: mathematics, models, and methods.
Cambridge University Press, 2013.