$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\begin{aligned}
var(\hat{\beta}_0) &= var(\bar{Y} - \hat{\beta}_1 \bar{X}) \\
&= var(\bar{Y}) + var(\hat{\beta}_1 \bar{X}) \\
&= var(\bar{Y}) + \bar{X}^2 var(\hat{\beta}_1) \\
&= \frac{\sigma^2}{n} + \bar{X}^2 var(\hat{\beta}_1) \\
&= \frac{\sigma^2}{n} + \bar{X}^2 \left(\frac{\sigma^2}{S_X^2}\right) \\
&= \sigma^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{S_X^2}\right).
\end{aligned}$$

$$\bar{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}$$

$$var(\bar{Y}) = var(\hat{\beta}_0 + \hat{\beta}_1 \bar{X})$$

$$= var(\hat{\beta}_0) + var(\hat{\beta}_1 \bar{X}) + 2\,cov(\hat{\beta}_0, \hat{\beta}_1 \bar{X})$$

$$\cancel{\frac{\sigma^2}{n}} = \cancel{\frac{\sigma^2}{n}} + \frac{\sigma^2 \bar{X}^2}{S_X^2} + \frac{\bar{X}^2 \sigma^2}{S_X^2} + 2\,Cov(\hat{\beta}_0, \hat{\beta}_1 \bar{X})$$

$$2\bar{X}\,Cov(\hat{\beta}_0, \hat{\beta}_1) = -2\frac{\sigma^2 \bar{X}^2}{S_X^2}$$

$$\Rightarrow \boxed{Cov(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\sigma^2 \bar{X}}{S_X^2}}$$