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Theory of Influence Networks

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Abstract Influence networks are Bayesian networks whose probabilities are approximated via expert provided influence constants. They represent a modeling and analysis formalism for addressing complex decision problems. In this paper, we present a comprehensive theory of influence networks that incorporates design constraints for consistency, temporal issues and a dynamic programming evolution of the influence constants. We also include numerical evaluations for several example timed influence networks.

Keywords Bayesian networks · Influence networks · Timed influence networks

1 Introduction

The easy access to domain-specific information and cost-effective availability of high computational power have changed the way people think about complex decision problems in almost all areas of application, ranging from financial markets to regional and global politics. These decision problems often require modeling of informal, uncertain and unstructured domains, to allow the evaluation of alternatives and available courses of actions by a decision maker. The past decade has witnessed an emergence of several modeling and analysis formalisms that target this need, the

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most popular one being represented by probabilistic belief networks [19, 21], most commonly known as Bayesian networks (BNs).

BNs model uncertain domains probabilistically, by presenting the network nodes as random variables. The arcs (or directed edges) in the network represent the direct dependency relationships between the random variables. The arrows on the edges depict the *direction* of the dependencies. The strengths of these dependencies are captured as conditional probabilities associated with the connected nodes in a network. A complete BN model requires specification of all conditional probabilities prior to its use. The number of conditional probabilities on a node in a BN grows exponentially with the number of inputs to the node, which presents a computational challenge, at times. A major problem in BNs is the specification of the required conditional probabilities, especially when either objective values of these probabilities cannot be provided by experts or there exist insufficient empirical data to allow for their reliable estimation, or when newly obtain information may change the structural topology of the network. Although a pair-wise cause and effect relationship between two variables of a domain is easier to establish or extract from a domain expert, a BN of the domain requires prior knowledge of all the influencing causes to an effect as well as their aggregate influence on the effect variable, where the measures of influences are conditional probability values. To demonstrate cases where BN modeling may be problematic, we identify the following situations of practical significance: (1) When new, previously unknown, affecting variables to some effect event arise, there are no algorithms allowing easy pertinent adaptation of conditional probabilities. (2) When the need arises to develop a consolidated BN from partial fragments of separate BNs, there are no algorithms that utilize the parameters of the fragments to calculate the parameters of the consolidated structure.

Recognizing the problems in the construction of BNs, especially regarding the specification of the involved conditional probabilities, Chang et al. [2] developed a formalism, at George Mason University, named Causal Strength (CAST) logic, as an intuitive and approximate language. The logic utilizes a pair of parameter values to represent conditional dependency between a pair of random variables, where these parameter values model assessed (by experts) mutual influences between an affecting and an affected event. The CAST logic approximates conditional probabilities via influence relationships by employing an influence aggregation function. The approach provides the elicitation, update, reuse, and merge interface to an underlying BN, or multiple fragments of a BN, that only requires specification of individual influences between each pair of an affecting and an affected variables. The approach then combines these individual influences to calculate the aggregate effect of multiple affecting variables on an effect variable in terms of conditional probability values of a resulting BN. This pair-wise specification of influences provides us with the, albeit approximate, means to solve the three problems discussed earlier.

The CAST logic approach was later extended to represent relationships between events involved in network interconnections, as in BNs. The extension is basically a BN with conditional probabilities approximated via the use of influence parameters and was named influence nets (INs) [22–25]. INs require an expert who specifies the influence parameter values and their interrelationships, as well as some a priori probabilities, all needed for the approximation of the pertinent conditional probabilities. As basically modified BNs, the objective of INs is to compute the probabilities of occurrence of sequential dependent events, and do not provide recommendations for actions. However, the probabilities of occurrence computed by the INs may be utilized by activation networks towards the evaluation and recommendation of actions [20].

BNs and INs are designed to capture *static* interdependencies among variables in a system. A situation where the impact of a variable takes some *time* to reach the affected variable(s) cannot be modeled by either one. In the last several years, efforts have been made to integrate the notion of time and uncertainty. Wagenhals et al. [27, 28, 32] have added a special set of temporal constructs to the basic formalism of INs. The INs with these additional temporal constructs are called timed influence nets (TINs). TINs have been experimentally used in the area of Effects Based Operations (EBOs) for evaluating alternate courses of actions and their effectiveness to mission objectives in a variety of domains, e.g., war games [29–31, 34], and coalition peace operations [33], modeling adversarial behaviors [35], to name a few. The provision of time allows for the construction of alternate courses of action as timed sequences of actions or actionable events represented by nodes in a TIN [28, 29, 34]. A number of analysis tools have been developed over the years for TIN models, to help an analyst update beliefs [5, 6, 8–10] represented as nodes in a TIN, to map a TIN model to a Time Sliced Bayesian Network for incorporating feedback evidence, to determine best course of actions for both timed and un-timed versions of Influence Nets [7, 12] and to assess temporal aspects of the influences on objective nodes [11, 36].

The existing developments of INs and TINs suffer from a number of deficiencies: they do not represent scenarios encompassing dependent conditioning events and they utilize a priori probabilities inconsistently, in violation of the Bayes rule and the theory of total probability. The motivation behind the work presented in this paper is to address these shortcomings of INs and TINs by developing a correct analytical framework for the design and analysis of influences on some critical effects due to a set of external affecting events. We present a comprehensive theory of Influence Networks, which is free of restrictive independence assumptions, which is consistently observing the Bayes rule and the theorem of total probability. In this theory, we are concerned with the evaluation of cause–effect relationships between interconnected events. In particular, if the status of some event B is affected by the status of a set of events, A_1 to A_n, we are interested in a qualification and quantification of this effect. We first graph the relationships between events B and A_1 to A_n in a network format, as in Fig. 1 below, with each event being a node, with arcs indicating relationships and with arrows representing the cause–effect directions. This graphical representation is identical to that used in BNs.

Given the graph of Fig. 1, we next decide the metric to be used for the quantification of the effects of events A_1 to A_n on event B. As in BNs, modeling each of the involved events as binary random variables, we use conditional probabilities as effect metrics: in particular, we use the probabilities that event B occurs, given each of the 2^n scenarios regarding the occurrence or nonoccurrence of each one of the events A_1 to A_n.

Upon the decision to use conditional probabilities as the effect metrics, the issue of their computation arises. In most realistic scenarios, there exist insufficient amount of data for the reliable estimation of these probabilities. Instead, some influence indicators may be provided by experts. In the example of Fig. 1, for instance, for each one of the 2^n scenarios regarding the occurrence or nonoccurrence of each

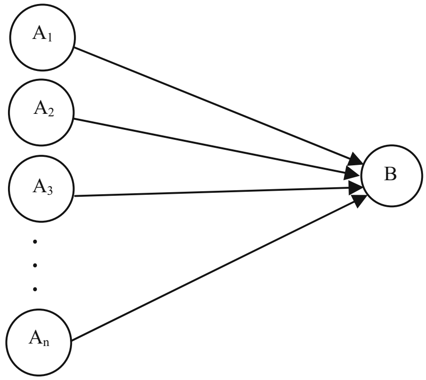


Fig. 1 Cause–effect relationships

on of the events A_1 to A_n, an expert may provide a number between -1 and 1, to reflect his assessment as to the effect of the above scenario on the occurrence of event B. The latter number is named *influence constant*. The objective at this point is to utilize the so provided influence constants for the approximate evaluation and computation of the required conditional probabilities, in a mathematically correct and consistent fashion. These conditional probabilities are subsequently utilized for the probabilistic evaluation of event occurrences in a network of events, giving rise to an influence network (IN). In different terms, a IN is a BN whose conditional probabilities are computed via the use of influence constants. The term IN should not be confused with a similarly named formalism called influence diagrams [13–15, 26]. Unlike INs, an influence diagram (ID) has different types of nodes (i.e., decision nodes, chance nodes, and utility nodes) and different types of influences (i.e., arcs between the nodes); and the decisions in an ID are assumed to have a certain precedence relationship among them. The IDs can be considered a BN extended with a utility function, while a IN, as noted above, is a special instance of a BN whose conditional probabilities are computed via the use of influence constants and which uses a set of special purpose algorithms for calculating the impact of a set of external affecting events on some desired effect/objective node.

Frequently, in several realistic scenarios, assessments of event occurrences may be needed at times when the status of all affecting events may not be known, while such assessments require sequential adaptation, as the status of more affecting events are revealed. For example, in Fig. 1, the evaluation of the probability of event B may be needed at times when the status of only some of the events A are known, while this probability need to be subsequently adapted when the status of the remaining A events become known. Such sequential adaptations require pertinent sequential computation methodologies for the approximation of conditional probabilities via influence constants and give rise to time influence networks (TINs). We present two different temporal models for the sequential computation of conditional probabilities in a timed influence nets. This enhances the capabilities of the timed influence nets in modeling domains of interest with different time characteristics.

The organization of the paper is as follows: In Section 2, we present the theoretical formalization and derive initial relationships. In Section 3, we derive the dynamic programming evolution of the influence constants. In Section 4, we examine the case where in the generic model, the affecting events are mutually independent, where in Section 5, the case where the latter events form a Markov chain is examined. In Section 6, temporal considerations are presented. In Section 7 we discuss decision model selection and testing. In Section 8, special forms of the influence constants are discussed. In Section 9, we discuss evaluation metrics. In Section 10, the experimental setup is laid out and some numerical results are presented. In Section 11, conclusions are drawn.

2 Initial Modeling and Relationships

In this section, we formalize our approach for the development of INs and TINs.

Let us consider an event B being potentially affected by events $\{A_i\}\ 1 \le i \le n$. In particular, we are interested in the effect the presence or absence of any of the events in the set $\{A_i\}\ 1 \le i \le n$ may have on the occurrence of event B.

Let us first define:

X_1^n An n-dimensional binary random vector whose j^{th} component is denoted X_j, where $X_j = 1$; if the event A_j is present, and $X_j = 0$; if the event A_j is absent. We will denote by x_1^n realizations or values of the random vector X_1^n.

A given realization x_1^n of the binary vector X_1^n describes precisely the status of the set $\{A_i\}\ 1 \le i \le n$ of events, regarding which events in the set are present. We name the vector X_1^n, the *status vector* of the affecting events. To quantify the effects of the status vector X_1^n on the event B, we define the *influence constant* $h_n\left(x_1^n\right)$ via the following quantitative properties

$$h_n\left(x_1^n\right) = \begin{cases} 1 \ ; \text{ if } & \begin{array}{l}\text{given n affecting events, given the status}\\ \text{vector } x_1^n \text{, event B occurs surely}\end{array} \\ -1 \ ; \text{ if } & \begin{array}{l}\text{given n affecting events, given the status}\\ \text{vector } x_1^n \text{, the nonoccurrence of event B is sure}\end{array} \\ 0 \ ; \text{ if } & \begin{array}{l}\text{given n affecting events, given the status}\\ \text{vector } x_1^n \text{, the occurrence of event B is unaffected}\end{array} \end{cases} \tag{1}$$

Let $P\left(B \middle| x_1^n\right)$ denote the probability of occurrence of event B, given the status vector x_1^n. Then, the quantitative definition of the influence constant $h_n\left(x_1^n\right)$ in (1) can be rewritten as follows, where $P(B)$ denotes the unconditional probability of occurrence of the event B.

$$P\left(B \middle| x_1^n\right) = \begin{cases} 1; \text{ if } & h_n\left(x_1^n\right) = 1 \\ P(B); \text{ if } & h_n\left(x_1^n\right) = 0 \\ 0; \text{ if } & h_n\left(x_1^n\right) = -1 \end{cases} \tag{2}$$

We now extend the definition of all values in [1,–1] of the influence constant, via linear interpolation from Eq. 2. In particular, we define the influence constant via its use to determine the derivation of the conditional probability $P\left(B|\, x_1^n\right)$ from the unconditional probabilities $P(B)$, where this derivation is derived via linear interpolation from Eq. 2. We thus obtain.

$$P\left(B|\, x_1^n\right) = \begin{cases} P(B) + h_n\left(x_1^n\right)[1 - P(B)]\,; & \text{if } h_n\left(x_1^n\right) \in [0, 1] \\ P(B) + h_n\left(x_1^n\right) P(B)\,; & \text{if } h_n\left(x_1^n\right) \in [-1, 0] \end{cases} \tag{3}$$

Defining $\operatorname{sgn}\gamma = \begin{cases} 1; \text{ if } \gamma \geq 0 \\ 0; \text{ if } \gamma < 0 \end{cases}$, we can finally write Eq. 3 as follows

$$P\left(B|\, x_1^n\right) = P(B)\left\{1 + h_n\left(x_1^n\right)[1 - P(B)]\, P^{-1}(B)\right\}^{\operatorname{sgn} h_n\left(x_1^n\right)} \times \left\{1 + h_n\left(x_1^n\right)\right\}^{1-\operatorname{sgn} h_n\left(x_1^n\right)} \tag{4}$$

At this point, we present a formal definition of INs and TINs.

Definition 1 An influence network (IN) is a Bayesian network mapping conditional probabilities $P\left(B|\, x_1^n\right)$ via the utilization of influence constants as in Eq. 4. Formally, an influence net is a tuple (**V**, **E**, **C**, **A**, **B**), with **G** = (**V**, **E**) representing a directed-acyclic graph satisfying the Markov condition (as in BN), where

- **V** set of nodes representing binary random variables,
- **E** set of edges representing causal influences between nodes,
- **C** set of causal strengths: $E \to \left\{\left[h_1^{(i)}(x_i = 1)\,, h_1^{(i)}(x_i = 0)\right] \text{such that } h_1\text{'s} \in [-1, 1]\right\}$,
- **A** a subset of **V** representing *external* affecting events$\{A_i\} 1 \leq i \leq n$ and a status of the corresponding vector X_1^n,
- **B** Probability distribution of the status vector X_1^n corresponding to the external affecting events $\{A_i\}\ 1 \leq i \leq n$.

A Timed Influence Network (TIN) adds two temporal parameters to the definition of a IN. Formally, a TIN is a tuple (**V**, **E**, **C**, **D**, $\mathbf{A_T}$, **B**), where **V**, **E**, **C**, and **B** are as defined for INs;

- **D** set of temporal delays on edges: **E** → N,
- $\mathbf{A_T}$ same as **A** with the addition that the status of each external affecting event is *time tagged* representing the time of realization of its status. In the IN/TIN literature [12, 27–30, 33, 34], A_T is also referred to as a course of action (COA). A COA is, therefore, a time-sequenced collection of external affecting events and their status.

Returning to the influence constant notion, we note that there exist 2^n distinct values of the status vector x_1^n; thus, there exist 2^n distinct values of the influence constant $h_n\left(x_1^n\right)$ as well as of the conditional probabilities in Eq. 4. In the case where the cardinality of the set $\{A_i\}\ 1 \leq i \leq n$ is one, the influence constant $h_1(x_1)$ equals the constant h in [22]; if $x_1 = 1$ and equals the constant g in [22]; if $x_1 = 0$.

We now proceed with a definition which will lead to a mathematically correct relationship between influence constants and unconditional probabilities.

Definition 2 A IN or TIN model is *consistent* if it observes the Bayes rule.

Let $P\left(x_1^n\right)$denote the probability of the status vector X_1^n at the value x_1^n. We can then express the following simple lemma.

Lemma 1 *Let the influence constant $h_n\left(x_1^n\right)$be accepted as reflecting accurately the relationship between the affecting events $\{A_i\}$ $1 \leq i \leq n$ and event B. Then the IN or TIN model is consistent iff:*

$$\sum_{x_1^n} P\left(x_1^n\right)\left\{1+h_n\left(x_1^n\right)[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_n\left(x_1^n\right)}\left\{1+h_n\left(x_1^n\right)\right\}^{1-\mathrm{sgn}h_n\left(x_1^n\right)}=1 \quad (5)$$

Or

$$\sum_{x_1^n} P\left(x_1^n\right) h_n\left(x_1^n\right)\left\{[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_n\left(x_1^n\right)}=0$$

Proof Substituting expression (4) in the Bayes' rule, $P(B)=\sum_{x_1^n} P\left(x_1^n\right) P\left(B|\,x_1^n\right)$, we obtain Eq. 5. □

Expression (5) relates the influence constant $h_n\left(x_1^n\right)$to the unconditional probabilities of event B and the status vector X_1^n. This relationship is necessary if the influence constant is accepted as accurately representing the conditional probability $P\left(B|\,x_1^n\right)$ in Eq. 3. Generally, the influence constant is selected based on a system design assessment provided by experts, while the a priori probabilities $P\left(x_1^n\right)$ are accepted to accurately represent the actual model.

Summary Given the events in Fig. 1, given well-established a priori probabilities of the cause events, given the influence constants, the cause–effect conditional probabilities are expressed as follows:

$$P\left(B|x_1^n\right)=\begin{cases} a+h_n\left(x_1^n\right)[1-a]\,; & \text{if } h_n\left(x_1^n\right)\in[0,1] \\ a+h_n\left(x_1^n\right)a; & \text{if } h_n\left(x_1^n\right)\in[-1,0] \end{cases}$$

where

$$a=P(B)=\left[\sum_{x_1^n:\mathrm{sgn}h_n\left(x_1^n\right)=1} P\left(x_1^n\right) h_n\left(x_1^n\right)\right]\left[\sum_{x_1^n} P\left(x_1^n\right)\left|h_n\left(x_1^n\right)\right|\right]^{-1}$$

Influence nets thus utilize expert-provided subjective influence constants, in conjunction with well-established objective a priori probabilities of cause events, to generate conditional probabilities of effect events.

3 Evolution of the Influence Constant

In Section 2, we derived the relationship between the conditional probability of event B, and the status x_1^n of its affecting events $\{A_i\}\, 1 \le i \le n$, via the influence constant $h_n\left(x_1^n\right)$. This relationship is based on the assumption that $\{A_i\}\, 1 \le i \le n$ is the maximum set of events affecting event B and that the value x_1^n of the status vector is given. In this section we investigate the case where the status of some of the affecting events may be unknown. Towards this direction, we derive a dynamic programming relationship between the influence constants $h_n\left(x_1^n\right)$ and $h_{n-1}\left(x_1^{n-1}\right)$, where $h_{n-1}\left(x_1^{n-1}\right)$ is the constant corresponding to the case where the status of the affecting event A_n is unknown. We express a lemma whose proof is in "Appendix." The proof is based on the observation of the Bayes' rule and the theorem of total probability.

Lemma 2 *Let the probability $P(B)$ be as in Section 2 and let $P\left(x_n|\, x_1^{n-1}\right)$ denote the probability of the value of the last bit in the status vector X_1^n being x_n, given that the reduced status vector value is x_1^{n-1}. Then, the influence constant $h_{n-1}\left(x_1^{n-1}\right)$ is given as a function of the influence constant $h_n(x_1^n)$, as shown below.*

$$h_{n-1}\left(x_1^{n-1}\right) = \begin{cases} Q_n & ; \quad Q_n \in [-1, 0] \\ P(B)\,[1 - P(B)]^{-1}\, Q_n & ; \quad Q_n \in \left[0, P^{-1}(B) - 1\right] \end{cases} \tag{6}$$

where

$$Q_n \triangleq \sum_{x_n=0,1} P\left(x_n|\, x_1^{n-1}\right) \left\{h_n\left(x_1^n\right)\right\} \left\{[1 - P(B)]\, P^{-1}(B)\right\}^{\mathrm{sgn} h_x\left(x_1^n\right)} \tag{7}$$

We note that the influence constants are deduced from the same constants of higher dimensionality, as shown in Lemma 2. In accordance, conditional probabilities of the event B are produced from the deduced influence constants, via expression (4), as:

$$\begin{aligned} P\left(B|\, x_1^{n-1}\right) = {} & P(B) \left\{1 + h_{n-1}\left(x_1^{n-1}\right)[1 - P(B)]\, P^{-1}(B)\right\}^{\mathrm{sgn} h_{n-1}\left(x_1^{n-1}\right)} \\ & \times \left\{1 + h_{n-1}\left(x_1^{n-1}\right)\right\}^{1 - \mathrm{sgn} h_{n-1}\left(x_1^{n-1}\right)} \end{aligned} \tag{8}$$

It is important to note that in the dynamic programming evolution of the influence constants $h_n\left(x_1^n\right)$, as well as in the evolution of the conditional probabilities in Eq. 7, knowledge of the joint probability $P\left(x_1^n\right)$ is assumed. This reflects a conjecture by the system designer, based on his /her previous experience regarding the a priori occurrence of the affecting events $\{A_i\}\, 1 \le i \le n$. Thus the probability $P\left(x_1^n\right)$ used for the construction exhibited by Lemma 2 is a design probability and it may not coincide with the actual probabilities of the status vector X_1^n. When full scale dependence of the components of the status vector X_1^n is incorporated within the design probability $P\left(x_1^n\right)$, then the relationship between the different dimensionality influence constants is that reflected by Lemma 2 and is of dynamic programming nature. In the case where the design probability $P\left(x_1^n\right)$ generically reflects either a Markov chain of events or mutually independent events, then the relationships between the different dimensionality influence constants may be also of recursive nature. The cases of Markovian or independent affecting events, as modeled by the system designer, are examined in Sections 4 and 5 below.

4 The Case of Independent Affecting Events

In this section, we consider the special case where the affecting events $\{A_i\}\ 1 \le i \le n$ are assumed to be generically mutually independent. Then, the components of the status vector X_1^n are mutually independent, and:

$$P\left(x_1^n\right) = \prod_{i=1}^{n} P\left(x_i\right)\ ;\ P\left(x_1^n \middle| B\right) = \prod_{i=1}^{n} P\left(x_i \middle| B\right) \tag{9}$$

Let us denote by $h_1^{(i)}\left(x_i\right)$the influence constant corresponding to the effect of the event A_i on the occurrence of the event B, when event A_i acts in isolation and when the status value of the event is x_i. Then, from expression (4) in Section 3, we have:

$$P\left(B \middle| x_i\right) = P(B)\left\{1 + h_1^{(i)}\left(x_i\right)\left[1 - P(B)\right] P^{-1}(B)\right\}^{\mathrm{sgn} h_1^{(i)}(x_i)} \bullet \left\{1 + h_1^{(i)}\left(x_i\right)\right\}^{1-\mathrm{sgn} h_1^{(i)}(x_i)} \tag{10}$$

We now express a lemma whose proof is in the Appendix.

Lemma 3 *Let the events $\{A_i\}\ 1 \le i \le n$ that affect event B be assumed to be generically mutually independent. Then*

$$P\left(B \middle| x_1^n\right) = P(B)\prod_{i=1}^{n}\left\{1 + h_1^{(i)}\left(x_i\right)\left[1 - P(B)\right] P^{-1}(B)\right\}^{\mathrm{sgn} h_1^{(i)}(x_i)} \bullet \left\{1 + h_1^{(i)}\left(x_i\right)\right\}^{1-\mathrm{sgn} h_1^{(i)}(x_i)} \tag{11}$$

Via the same logic as that in the last part in the proof of Lemma 2, we can show the result expressed in the corollary below.

Corollary 1 *When the affecting events are assumed to be generically mutually independent then, the influence constant $h_n\left(x_1^n\right)$ is given as a function of the single event influence constants $\left\{h_1^{(i)}\left(x_i\right)\right\}\ 1 \le i \le n$, as follows:*

$$h_n\left(x_1^n\right) = \begin{cases} R_n - 1 & ;\quad \text{if } R_n \in [0, 1] \\ P(B)\left[1 - P(B)\right]^{-1}\left[R_n - 1\right] & ;\ \text{if } R_n \in \left[1, P^{-1}(B)\right] \end{cases} \tag{12}$$

where

$$R_n \triangleq \prod_{i=1}^{n}\left\{1 + h_1^{(i)}\left(x_i\right)\left[1 - P(B)\right] P^{-1}(B)\right\}^{\mathrm{sgn} h_1^{(i)}(x_i)} \times \left\{1 + h_1^{(i)}\left(x_i\right)\right\}^{1-\mathrm{sgn} h_1^{(i)}(x_i)} \tag{13}$$

The sequence of expressions $\{R_i\}$ $1 \le i \le n$ in Eq. 13 is clearly recursively generated and the conditional probability $P\left(B|\, x_1^n\right)$ is given by $h_n\left(x_1^n\right)$ as in Eq. 4 in Section 2.

We note that the consistency condition in Lemma 1, Section 2 reduces in a straight forward fashion and by construction to the following condition here:

$$\sum_{x_i=0,1} P\left(x_i\right)\left\{1+h_{\cdot 1}\left(x_i\right)\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\operatorname{sgn}h_1\left(x_i\right)}\left[1+h_1\left(x_i\right)\right]^{1-\operatorname{sgn}h_1\left(x_i\right)}=1;\ \forall i$$

Or

$$\sum_{x_i=0,1} P\left(x_i\right)h_1\left(x_i\right)\left\{\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\operatorname{sgn}h_1\left(x_i\right)}=0;\ \forall i$$

5 The Case of A Markov Chain of Affecting Events

In this section, we consider the case where the affecting events $\{A_i\}$ $1 \le i \le n$ are assumed to form generically a Markov chain. In particular, we assume that the design probabilities $P\left(x_1^n\middle|\, B\right)$ and $P\left(x_1^n\right)$ are such that:

$$P\left(x_1^n\middle|\, B\right)=\prod_{i=1}^{n} P\left(x_i|\, x_{i-1}, B\right)$$
$$P\left(x_1^n\right)=\prod_{i=1}^{n} P\left(x_i|\, x_{i-1}\right) \tag{14}$$

where $P\left(x_1|\, x_0, B\right)\stackrel{\Delta}{=}P\left(x_1|\, B\right)$ and $P\left(x_1|\, x_0\right)\stackrel{\Delta}{=}P\left(x_1\right)$.

We denote by $h_1^{(1)}\left(x_1\right)$ the influence constant corresponding to the effect of the event A_1 on the occurrence of the event B, when the status value of A_1, is given by x_1. We denote by $h_2^{(i,i+1)}\left(x_i, x_{i+1}\right)$ the influence constant corresponding to the effect of the events A_i and A_{i+1} on the occurrence of the event B, when the status values of the (A_i, A_{i+1}) pair are given by (x_i, x_{i+1}). Then, via (4) in Section 2, we have

$$P\left(B|\, x_1\right)=P\left(B\right)\left\{1+h_1^{(1)}\left(x_1\right)\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\operatorname{sgn}h_1^{(1)}\left(x_1\right)}$$
$$\times\left\{1+h_1^{(1)}\left(x_1\right)\right\}^{1-\operatorname{sgn}h_1^{(1)}\left(x_1\right)} \tag{15}$$

$$P\left(B|\, x_i, x_{i+1}\right)=P\left(B\right)\left\{1+h_2^{(i,i+1)}\left(x_i, x_{i+1}\right)\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\operatorname{sgn}h_2^{(i,i+1)}\left(x_i,x_{i+1}\right)}$$
$$\times\left\{1+h_2^{(i,i+1)}\left(x_i, x_{i+1}\right)\right\}^{1-\operatorname{sgn}h_2^{(i,i+1)}\left(x_i,x_{i+1}\right)};\ i=1 \tag{16}$$

We now express a lemma whose proof is in the Appendix.

Lemma 4 *Let the affecting events $\{A_i\}\ 1 \le i \le n$ be assumed to generically form a Markov Chain; thus, $P\left(x_1^n\right)$is assumed to satisfy the equation in Eq. 14. Then,*

$$P\left(B|\,x_1^n\right) = P(B)\left\{1+h_1^{(1)}(x_1)\left[1-P(B)\right]P^{-1}(B)\right\}^{\mathrm{sgn}h_1^{(1)}(x_1)} \times \left\{1+h_1^{(1)}(x_1)\right\}^{1-\mathrm{sgn}h_1^{(1)}(x_1)}$$
$$\times \prod_{i=2}^{\mathrm{n}} \frac{\left\{1+h_2^{(i,i-1)}(x_i,x_{i-1})[1-P(B)]P^{-1}(B)\right\}^{\mathrm{sgn}h_2^{(i,i-1)}(x_i,x_{i-1})} \times \left\{1+h_2^{(i,i-1)}(x_i,x_{i-1})\right\}^{1-\mathrm{sgn}h_2^{(i,i-1)}(x_i,x_{i-1})}}{\left\{1+h_1^{(i-1)}(x_{i-1})[1-P(B)]P^{-1}(B)\right\}^{\mathrm{sgn}h_1^{(i-1)}(x_{i-1})} \times \left\{1+h_1^{(i-1)}(x_{i-1})\right\}^{1-\mathrm{sgn}h_1^{(i-1)}(x_{i-1})}}$$
$$\triangleq P(B)\,W_n \tag{17}$$

where,

$$h_1^{(i)}(x_i) = \begin{cases} Q_{i,i+1}-1 & ; \quad \text{if } Q_{i,i+1} \in [0,1] \\ P(B)\left[1-P(B)\right]^{-1}\left[Q_{i,i+1}-1\right]; & \text{if } Q_{i,i+1} \in \left[1, P^{-1}(B)\right] \end{cases} \tag{18}$$

$$Q_{i,i+1} \triangleq \sum_{x_{i+1}=0,1} P\left(x_{i+1}|\,x_i\right)\left\{1+h_2^{(i,i+1)}(x_i,x_{i+1})\left[1-P(B)\right]P^{-1}(B)\right\}^{\mathrm{sgn}h_2^{(i,i+1)}(x_i,x_{i+1})}$$
$$\times\left\{1+h_2^{(i,i+1)}(x_i,x_{i+1})\left[1+P(B)\right]P^{-1}(B)\right\}^{1-\mathrm{sgn}h_2^{(i,i+1)}(x_i,x_{i+1})} \tag{19}$$

As with Corollary 1 in Section 4, we can express the corollary below, in a direct fashion.

Corollary 2 *When the affecting events $\{A_i\}\ 1 \le i \le n$ are assumed to generically form a Markov chain, depicted by the expression in Eq. 14, then, the influence constant $h_n\left(x_1^n\right)$ is given as a function of the influence constants $\left\{h_1^{(i)}(x_i)\right\}$ and $\left\{h_2^{(i,i-1)}(x_i,x_{i-1})\right\}$, as below, where W_n is defined in Eq. 17.*

$$h_n\left(x_1^n\right) = \begin{cases} W_n-1 & ; \quad \text{if } W_n \in [0,1] \\ P(B)\left[1-P(B)\right]^{-1}\left[W_n-1\right]; & \text{if } W_n \in \left[1, P^{-1}(B)\right] \end{cases} \tag{20}$$

The sequence $\{W_i\}\ 1 \le i \le n$ in Eq. 17 is clearly recursively expressed; thus, $h_n\left(x_1^n\right)$ is recursively evolving. The consistency condition in Lemma 1, Section 2, takes here the following form, by construction.

$$\sum_{x_i=0,1}\sum_{x_{i-1}=0,1} P\left(x_i|\,x_{i-1}\right)\left\{1+h_2^{(i,i-1)}(x_ix_{i-1})\left[1-P(B)\right]P^{-1}(B)\right\}^{\mathrm{sgn}h_2^{(i,i-1)}(x_i,x_{i-1})}$$
$$\times\left\{1+h_2^{(i,i-1)}(x_i,x_{i-1})\right\}^{1-\mathrm{sgn}h_2^{(i,i-1)}(x_i,x_{i-1})} = 1;\ \forall i$$

6 Temporal Extension

In Sections 2 and 3, we presented our theoretical foundation for the development of INs and TINs, while in Sections 4 and 5, we focused on the special cases of independent and Markovian affecting events. In this section, we focus on the formalization of the temporal issues involved in the development of TINs. In particular, we are investigating the dynamics of the relationship of the affecting events $\{A_i\}$ $1 \leq i \leq n$to the affected event B, when the status of the former evens are learned asynchronously in time. Without lack in generality—to avoid cumbersome notation—let the affecting events $\{A_i\}$ $1 \leq i \leq n$ be ordered in the order representing the time when their status become known. That is, the status of events A_1 is first known, then that of event A_2, and so on. In general, the status of event A_k becomes known after the status of the events $A_1 \ldots, A_{k-1}$ are known, and this knowledge becomes available one event at the time.

Let us assume that the considered system model implies full dependence of the components of the status vector X_1^n. Then, the influence constants $\left\{h_i\left(x_1^n\right)\right\}$ $1 \leq i \leq n-1$ are first pre-computed via the dynamic programming expression in Lemma 2, Section 3, utilizing the pre-selected a priori probabilities $P\left(x_1^n\right)$ that are part of the given system parameters. The above influence constants can be recursively computed if the adopted system model implies either generically independent affecting events or affecting events that generically form a Markov chain, as shown in Sections 4 and 5.

Let T_0 denote the time when the computation of the system dynamics starts. Let T_1 denote the time when the status of event A_1, becomes known. Let T_k; $1 \leq k \leq n$denote the time when the status of event A_k becomes known. Then at time T_k, the conditional probabilities $P\left(B|\, x_1^k\right)$ are computed via expression (4), Section 2, as,

$$\underset{i \leq k \leq n}{P\left(B|\, x_1^k\right)} = P(B)\left\{1 + h_k\left(x_1^k\right)[1 - P(B)]\, P^{-1}(B)\right\}^{\operatorname{sgn} h_k\left(x_1^k\right)} \times \left\{1 + h_k\left(x_1^k\right)\right\}^{1-\operatorname{sgn} h_k\left(x_1^k\right)}; \tag{21}$$

where the probability $P(B)$ is computed via the consistency condition (5).

As the knowledge about the status of the affecting events unravels, the conditional probabilities of event B in Eq. 21 evolve dynamically in time and finally converge to the probability $P\left(B|\, x_1^n\right)$ at time T_n, when the status of all the affecting events become known.

It is important to point out that the conditional probability in Eq. 21 is sensitive to the time ordering of the affecting events. That is, for the same value x_1^k of a partial affecting vector, but different time ordering of events, different conditional probabilities values of the affected event B arise. Thus, the order by which the status of the affecting events become known is crucial in the evaluation of the conditional probabilities of event B.

7 Selection and Testing of the Decision Model

7.1 Model Selection

As we have discussed earlier, the unconditional probabilities $P\left(x_1^n\right)$ as well as the influence constant $h_n\left(x_1^n\right)$ are design parameters that may not represent the actual parameters correctly. Furthermore, as discussed in Section 2, the design parameters must be *consistent*, where consistency is represented by the satisfaction of condition

(5) in Lemma 1. Condition (5) can be rewritten as follows, in a straightforward fashion.

$$[1-P(B)]\sum_{x_1^n:\mathrm{sgn}h_n(x_1^n)=1} P\left(x_1^n\right)h_n\left(x_1^n\right) = P(B)\sum_{x_1^n:\mathrm{sgn}h_n(x_1^n)=0} P\left(x_1^n\right)\left|h_n\left(x_1^n\right)\right| \quad (22)$$

which gives:

$$P(B) = \left[\sum_{x_1^n:\mathrm{sgn}h_n(x_1^n)=1} P\left(x_1^n\right)h_n\left(x_1^n\right)\right]\left[\sum_{x_1^n} P\left(x_1^n\right)\left|h_n\left(x_1^n\right)\right|\right]^{-1};$$
$$\text{when } \sum_{x_1^n} P\left(x_1^n\right)\left|h_n\left(x_1^n\right)\right| \neq 0 \quad (23)$$

Example Let us consider the case where the only affecting event for B is A_1. Let $P(A_1) \triangleq P(X_1=1)=p$, where then, $P\left(A_1^C\right) \triangleq P(X_1=0)=1-p$. Define h and g as in [22] and let $P(B)$ be what has been called in [22] base probability for the event B. Then, due to Eq. 22 the above parameters must satisfy the following equation(s):

$$\begin{array}{ll} \textit{either} & [1-P(B)]\,ph = P(B)(1-p)\,|g|\ ;\ \text{if } h>0 \text{ and } g<0 \\ \textit{or} & [1-P(B)](1-p)\,g = P(B)\,p\,|h|\ ;\ \text{if } h<0 \text{ and } g>0 \end{array}$$

no other h and g combinations are acceptable.

Note that parameters h and g in [22] map to $h_1^{(i)}(x_i=1)$ and $h_1^{(i)}(x_i=0)$, respectively, in Definition 1, Section 2.

When new information about the a priori probability $P\left(x_1^n\right)$ is obtained, then, $P(B)$ and/or $h_n\left(x_1^n\right)$ need to be accordingly adjusted to satisfy the condition in Eq. 22. We note that the latter condition involves a number of free parameters; thus even specification of the probabilities $P(B)$ and $P\left(x_1^n\right)$ does not specify uniquely the values of the influence constant $h_n\left(x_1^n\right)$. Naturally, specification of $P\left(x_1^n\right)$and $h_n\left(x_1^n\right)$ uniquely determines the probability $P(B)$, however, as in Eq. 23.

In the case that the assumed system design model implies generically independent affecting events $\{A_i\}$ $1\le i\le n$, then, for consistency the probability $P(B)$, the probability $P\left(x_1^n\right)=\prod_{i=1}^{n}P(x_i)$ of the status vector and the influence constants $\left\{h_1^{(i)}(x_i)\right\}$ are constraint to satisfy the condition:

$$\sum_{x_i=0,1} P(x_i)\left\{1+h_1(x_i)[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_1(x_i)}\{1+h_1(x_i)\}^{1-\mathrm{sgn}h_1(x_i)}=1;\ \forall i \quad (24)$$

Or

$$\sum_{x_i=0,1} P(x_i)\,h_1(x_i)\left\{[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_1(x_i)}=0;\ \forall i$$

7.2 Model Testing

Since the "consistency" constraints allow for a number of free parameters, we will focus on the influence constant $h_n\left(x_1^n\right)$ as the constant to be tested, when information about the probabilities of the events $\{Ai\}\, 1 \leq i \leq n$ and B is obtained. Thus, model testing will involve comparison of the $P\left(x_1^n\right)$ and $P(B)$ probabilities assumed in the model with those computed, to test the validity of the assumed influence constant. When the computed $P\left(x_1^n\right)$ and $P(B)$ values do not satisfy Eq. 23 for the assumed $h_n\left(x_1^n\right)$, then a non valid model is declared and a new influence constant $h_n\left(x_1^n\right)$ is sought, in satisfaction of the consistency condition in Eq. 23.

8 Some Special Influence Constants

As noted at the end of Section 7, the influence constant is a important component of the system model: the appropriate choice of this constant needs to be carefully thought out, to accurately reflect the interleaving of partial influences. In this section, we study some specific influence constants, $h_n\left(x_1^n\right)$. In particular, we study such constants that are specific analytic functions of the one-dimensional components $h_i\left(x_i\right); 1 \leq i \leq n$. We note that we are not mapping the $\{h_i\left(x_i\right)\}_{1\leq i\leq n}$ constants onto conditional probabilities $\{P\left(B|\, x_i\right)\}_{1\leq i\leq n}$. Instead, we are using the constants $\{h_i\left(x_i\right)\}_{1\leq i\leq n}$ to construct a global $h_n\left(x_1^n\right)$ influence constant; it is the latter constant which is mapped onto the conditional probability $P\left(B|\, x_1^n\right)$, as in Section 2.

8.1 The $h_n\left(x_1^n\right)$ Corresponding to the CAST Logic

The influence constant presented below is that used by the CAST logic in [2, 22–25].

In the present case, given the constants $\left\{h_i^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$ the global influence constant, $h_n\left(x_1^n\right)$, is defined as follows

$$h_n\left(x_1^n\right) = \left[\prod_{i:h_1(x_i)<0}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right) - \prod_{i:h_1(x_i)>0}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right)\right] \times\left[\max\left(\prod_{i:h_1(x_i)<0}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right), \prod_{i:h_1(x_i)>0}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right)\right)\right]^{-1} \tag{25}$$

In agreement with the results in Section 2, and via Eq. 5 in Lemma 1, the global constants $h_n\left(x_1^n\right)$ and the probabilities $P\left(x_1^n\right)$ and $P(B)$ must satisfy the consistency condition

$$\sum_{x_1^n} P\left(x_1^n\right)\left\{1+h_n\left(x_1^n\right)\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\mathrm{sgn}h_n\left(x_1^n\right)}\left\{1+h_n\left(x_1^n\right)\right\}^{1-\mathrm{sgn}h_n\left(x_1^n\right)} = 1 \tag{26}$$

Via Eq. 4, the conditional probabilities $P\left(B|\, x_1^n\right)$ are then given, by the following expression:

$$P\left(B|\, x_1^n\right) = P\left(B\right)\left\{1+h_n\left(x_1^n\right)\left[1-P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\mathrm{sgn}h_n\left(x_1^n\right)} \times\left\{1+h_n\left(x_1^n\right)\right\}^{1-\mathrm{sgn}h_n\left(x_1^n\right)} \tag{27}$$

For maintaining the consistency condition in Eq. 26, the conditional probability $P\left(B|\,x_1^{n-1}\right)$ is defined via the influence constant $h_{n-1}\left(x_1^{n-1}\right)$ as in Lemma 2, Section 3, where,

$$P\left(B|\,x_1^{n-1}\right) = P\left(B\right)\left\{1 + h_{n-1}\left(x_1^{n-1}\right)\left[1 - P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\mathrm{sgn}h_{n-1}\left(x_1^{n-1}\right)} \times \left\{1 + h_{n-1}\left(x_1^{n-1}\right)\right\}^{1-\mathrm{sgn}h_{n-1}\left(x_1^{n-1}\right)}$$

and

$$h_{n-1}\left(x_1^{n-1}\right) = \begin{cases} Q_n - 1 & ; \quad Q_n \in [0,1] \\ P\left(B\right)\left[1 - P\left(B\right)\right]^{-1}\left[Q_n - 1\right] & ; \quad Q_n \in \left[1, P^{-1}\left(B\right)\right] \end{cases}$$

$$Q_n \triangleq \sum_{x_n=0,1} P\left(x_n|\,x_1^{n-1}\right)\left\{1 + h_n\left(x_1^n\right)\left[1 - P\left(B\right)\right]P^{-1}\left(B\right)\right\}^{\mathrm{sgn}h_n\left(x_1^n\right)} \times \left[1 + h_n\left(x_1^n\right)\right]^{1-\mathrm{sgn}h_n\left(x_1^n\right)}$$

8.2 A $h_n\left(x_1^n\right)$ Constant Representing Extreme Partial Values

In this part, we first define the effect of the constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\le i\le n}$ on the event B as follows:

a. If at least one of the constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\le i\le n}$ equals the value 1, then event B occurs surely, if in addition $\sum_{i=1}^{n} h_1^{(i)}\left(x_i\right) > 0$
b. If at least one of the constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\le i\le n}$ equals the value -1, then the nonoccurrence of event B is sure, if in addition $\sum_{i=1}^{n} h_1^{(i)}\left(x_i\right) < 0$
c. The events $\{A_i\}\ 1 \le i \le n$ do not affect the event B if $\sum_{i=1}^{n} h_1^{(i)}\left(x_i\right) = 0$

The above conditions translate to the following initial expressions for the conditional probability $P\left(B|\,x_1^n\right)$, where x_1^n is the value of the status vector of the affecting events $\{A_i\}\ 1 \le i \le n$:

$$P\left(B|\,x_1^n\right) = \begin{cases} 1 & ; \ if \quad \max\limits_{1\le i\le n} h_1^{(i)}\left(x_i\right) = 1 \text{ and } \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} h_1^{(i)}\left(x_i\right) > 0 \\ P\left(B\right) & ; \ if \quad \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} h_1^{(i)}\left(x_i\right) = 0 \\ 0 & ; \ if \quad \min\limits_{1\le i\le n} h_1^{(i)}\left(x_i\right) = -1 \text{ and } \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} h_1^{(i)}\left(x_i\right) < 0 \end{cases} \tag{28}$$

Via linear interpolation from the above expression we obtain the general expression of the conditional probability $P\left(B|\,x_1^n\right)$, as a function of the influence constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\le i\le n}$, as follows:

$$P\left(B|\,x_1^n\right)=\begin{cases}P\left(B\right)+\max\limits_{1\le i\le n}\left(h_1^{(i)}\left(x_i\right)\right)\left[1-P\left(B\right)\right]\,; & \sum\limits_{\mathrm{i=1}}^{\mathrm{n}}h_1^{(i)}\left(x_i\right))>0\\ P\left(B\right) & ;\ \sum\limits_{\mathrm{i=1}}^{\mathrm{n}}h_1^{(i)}\left(x_i\right)=0\\ P\left(B\right)+\min\limits_{1\le i\le n}\left(h_1^{(i)}\left(x_i\right)\right)P\left(B\right) & ;\ \sum\limits_{\mathrm{i=1}}^{\mathrm{n}}h_1^{(i)}\left(x_i\right)<0\end{cases}\tag{29}$$

Defining the operators $O\left(x\right)\triangleq\begin{cases}1\,;\ x>0\\0\,;\ x<0\end{cases}$ and $U(x)\triangleq\begin{cases}1\,;\ x\ge 0\\0\,;\ x<0\end{cases}$, we can rewrite Eq. 29 in a compressed form as follows.

$$P\left(B|\,x_1^n\right)=P\left(B\right)\left\{1+P^{-1}\left(B\right)\left[1-P\left(B\right)\right]\max_{1\le i\le n}h_1^{(i)}\left(x_i\right)\right\}^{O\left(\sum\limits_{\mathrm{i=1}}^{\mathrm{n}}h_1^{(i)}\left(x_i\right)\right)}$$
$$\times\left\{1+\min_{1\le i\le n}h_1^{(i)}\left(x_i\right)\right\}^{1-U\left(\sum\limits_{\mathrm{i=1}}^{\mathrm{n}}h_1^{(i)}\left(x_i\right)\right)}\tag{30}$$

Next, we express a lemma regarding the consistency condition for our present model, evolving from the application of the Bayes' rule and the theorem of total probability on Eq. 30. The lemma is the parallel to Lemma 1 in Section 2, for the model in the present case.

Lemma 5 *For the influence model expressed in Eq. 30, the probabilities $P(B)$, $P\left(x_1^n\right)$ and the influence constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\le i\le n}$ must satisfy the following condition:*

$$\left[1-P\left(B\right)\right]\sum_{x_1^n:\sum\limits_{i=1}^{n}h_1^{(i)}\left(x_i\right)>0}P\left(x_1^n\right)\max_{1\le i\le n}h_1^{(i)}\left(x_i\right)+P\left(B\right)\sum_{x_1^n:\sum\limits_{i=1}^{n}h_1^{(i)}\left(x_i\right)<0}P\left(x_1^n\right)\min_{1\le i\le n}h_1^{(i)}\left(x_i\right)=0\tag{31}$$

From the consistency condition in Eq. 31, we notice that when examining all the values of the status vector X_1^n, it is necessary that some x_1^nvector values exist such that $\max\limits_{1\le i\le n}h_1^{(i)}\left(x_i\right)$ is positive and that some x_1^n vector values exists such that $\min\limits_{1\le i\le n}h_1^{(i)}\left(x_i\right)$ is negative.

8.2.1 Temporal Issues

Here, we will assume that the very existence of the affecting events is revealed sequentially. Let then the existence and the status of the events $\{A_i\}\,1\le i\le n$ be revealed sequentially in time, from A_1 to A_n, where the status of events A_1 to A_k is known at time T_k. At time T_k, the partial status vector x_1^k is expressed and for each one of its values, the probability $P\left(x_1^k\right)$ and the quantities, $S_k\left(x_1^k\right)\triangleq\sum\limits_{i=1}^{k}h_1^{(i)}\left(x_i\right)$,

$F_k\left(x_1^k\right) \triangleq \max_{1\leq i\leq k} h_1^{(i)}\left(x_i\right)$ and $G_k\left(x_1^k\right) \triangleq \min_{1\leq i\leq k} h_1^{(i)}\left(x_i\right)$ are computed. Next, the probability $P(B)$ is computed from Eq. 31 as follows:

$$P\left(B\right) \triangleq P_k\left(B\right) = \left[\sum_{x_1^k : S_k\left(x_1^k\right)>0} P\left(x_1^k\right) F_k\left(x_1^k\right) - \sum_{x_1^k : S_k\left(x_1^k\right)<0} P\left(x_1^k\right) G_k\left(x_1^k\right)\right]^{-1} \times \sum_{x_1^k : S_k\left(x_1^k\right)>0} P\left(x_1^k\right) F_k\left(x_1^k\right) \tag{32}$$

Given each x_1^k value, the probability $P(B)$ in Eq. 32 is then used to compute the conditional probability $P\left(B|\, x_1^k\right)$, as,

$$P\left(B|\, x_1^k\right) = P_k\left(B\right)\left\{1 + P_k^{-1}\left(B\right)\left[1 - P_k\left(B\right)\right] F_k\left(x_1^k\right)\right\}^{O\left(S_k\left(x_1^k\right)\right)} \times \left\{1 + G_k\left(x_1^k\right)\right\}^{1-U\left(S_k\left(x_1^k\right)\right)} \tag{33}$$

At time T_{k+1}, upon the revelation of the existence and the status of the affecting event A_{k+1}, for each status vector x_1^{k+1}, the quantities, $S_{k+1}\left(X_1^{k+1}\right) = S_{k+1}\left(X_1^k\right) + X_{k+1}$, $F_{k+1}\left(X_1^{k+1}\right) = \max\left(F_k\left(X_1^k\right), h\left(X_{k+1}\right)\right)$, $G_{k+1}\left(x_1^{k+1}\right) = \min(G_k\left(x_1^k\right), h_1\left(x_{k+1}\right))$ are first recursively computed. Then, the probability $P(B)$ is recomputed as

$$P\left(B\right) \triangleq P_{k+1}\left(B\right) = \left[\sum_{x_1^{k+1} : S_{k+1}\left(x_1^{k+1}\right)>0} P\left(x_1^{k+1}\right) F_{k+1}\left(x_1^{k+1}\right) - \sum_{x_1^{k+1} : S_{k+1}\left(x_1^{k+1}\right)<0} P\left(x_1^{k+1}\right) G_{k+1}\left(x_1^{k+1}\right)\right]^{-1} \times \sum_{x_1^{k+1} : S_{k+1}\left(x_1^{k+1}\right)>0} P\left(x_1^{k+1}\right) F_{k+1}\left(x_1^{k+1}\right) \tag{34}$$

The probability in Eq. 31 is used to compute the conditional probability below.

$$P\left(B|\, x_1^{k+1}\right) = P_{k+1}\left(B\right)\left\{1 + P_{k+1}^{-1}\left(B\right)\left[1 - P_{k+1}\left(B\right)\right] F_{k+1}\left(x_1^{k+1}\right)\right\}^{O\left(S_{k+1}\left(x_1^{k+1}\right)\right)} \times \left\{1 + G_{k+1}\left(x_1^{k+1}\right)\right\}^{1-U\left(S_{k+1}\left(x_1^{k+1}\right)\right)} \tag{35}$$

We note that the time evolution of the conditional probabilities $P\left(B|\, x_1^k\right)$ is different for different time orderings of the affecting events $\{A_i\}\ 1 \leq i \leq n$.

8.3 A Linear $h_n\left(x_1^n\right)$ Constant

Here, we assume that the effects of events $\{A_i\}\ 1 \leq i \leq n$ on event B are weighted by a known set $\{w_i\}\ 1 \leq i \leq n$ of weights, such that $w_i \geq 0$; $\forall i$ and $\sum_{i=1}^{n} w_i = 1$. Then, given the constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$, we define $h_n\left(x_1^n\right)$ as follows, for some given value $\alpha : 0 \leq \alpha < 1$:

$$h_n\left(x_1^n\right) = \begin{cases} (1-\alpha)^{-1}\sum_{i=1}^{n} w_i h_1^{(i)}\left(x_i\right) ; & \left|\sum_{i=1}^{n} w_i h_1^{(i)}\left(x_i\right)\right| \leq 1-\alpha \\ 1 ; & \sum_{i=1}^{n} w_i h_1^{(i)}\left(x_i\right) \geq 1-\alpha \\ -1 ; & \sum_{i=1}^{n} w_i h_1^{(i)}\left(x_i\right) \leq -(1-\alpha) \end{cases}$$

A nonzero α value translates to the probability of event B being equal to one, not only when all the $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$ values equal one, but also when a predefined weighted majority exceeds a total weighted sum of $1-\alpha$. Similarly then, the event B occurs with zero probability when the weighted sum of the $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$ values is less than $-(1-\alpha)$, rather than only when it equals -1. The relationships between the $h_n\left(x_1^n\right)$ and $h_{n-1}\left(x_1^{n-1}\right)$ influence constants and the probabilities $P(B)$, $P\left(x_1^n\right)$ and $P\left(B|\, x_1^n\right)$ are as in Section 8.1.

8.4 A $h_n\left(x_1^n\right)$ Constant Representing Noisy OR Format

Given the constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$, we define here $h_n\left(x_1^n\right)$ as follows; where α is such that $0 \leq \alpha \leq 1$:

$$\begin{aligned} h_n\left(x_1^n\right) = &\left\{1-(1-\alpha)^{-1}\prod_{i=1}^{n}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right)\right\}^{\operatorname{sgn}\left(h_n\left(x_1^n\right)\right)} \\ &\times\left\{-1+\alpha^{-1}-\alpha^{-1}\prod_{i=1}^{n}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right)\right\}^{1-\operatorname{sgn}\left(h_n\left(x_1^n\right)\right)} \end{aligned} \tag{36}$$

Then, via Eqs. 3 and 5 in Section 2, we obtain:

$$P(B) = \alpha \tag{37}$$

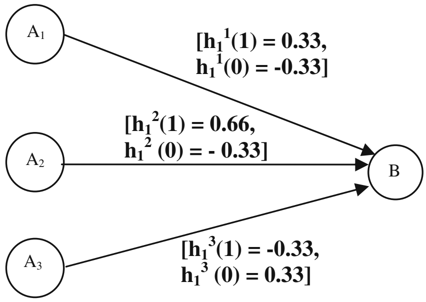
$$1-P\left(B|\, x_1^n\right) = \prod_{i=1}^{n}\left(1-\left|h_1^{(i)}\left(x_i\right)\right|\right) \tag{38}$$

The expression in Eq. 38 represents the noisy-OR format [1, 4], where the probabilities in the latter are here substituted by the absolute values of the one-dimensional influence constants $\left\{h_1^{(i)}\left(x_i\right)\right\}_{1\leq i\leq n}$.

8.5 Influence Constant Comparison

Figure 2 shows an example IN with a binary event B known to be affected by the events $\{A_i\}\ 1 \leq i \leq 3$. The edges connecting the external affecting events $\{A_i\}\ 1 \leq i \leq 3$

Fig. 2 Example TIN



to the event B are shown annotated with the constants $\left[h_1^{(i)}\left(x_i = 1\right), h_1^{(i)}\left(x_i = 0\right)\right]$ for each i, where $x_i = 0{,}1$ represents one of the two states of an affecting event A_i. A global influence constant $h_3\left(x_1^3\right)$ is then designed using all four (i.e., A-D) special influence functions presented in this section. Table 1 shows the computed values of $h_3\left(x_1^3\right)$ and corresponding $P\left(B|\, x_1^3\right); \forall x_1^3$ for each of the four cases. For illustration purposes, we also assume that the joint probability $P\left(x_1^3\right); \forall x_1^3$ values are computed by assigning $P\left(x_3 = 1\right) = 0.8$, $P\left(x_3 = 0\right) = 0.2$ and $P\left(x_i = 1\right) = P\left(x_i = 0\right) = 0.5;\ for\ i = 1, 2$ and by assuming $\{A_i\}\ 1 \le i \le 3$ to be mutually independent.

From the values included in Table 1, we notice the sensitivity of the computed probability of event B on the selected structure of the aggregate influence constant. Different such structures reflect different environments and their choice is at the discretion of an expert.

9 Evaluation Metrics

As already repeatedly stated, the INs and TINs studied in this paper are basically BNs whose conditional probabilities are approximated by expert provided influence constants. Thus, the architectural and computational complexities involved are similar to those in BNs [3, 16–18, 21], while the complexities involved in the computation of influence constants depend on the specific structure of the latter

Table 1

x_1	x_2	x_3	$h_3(x_1^3)$				$P(B\|x_1^3)$			
			A. CAST logic based	B. Extreme partials	C. Linear constant	D. Noisy-OR	A. CAST logic based	B. Extreme partials	C. Linear constant	D. Noisy-OR
0	0	0	−0.33	−0.33	−0.073	−0.095	0.335	0.335	0.452	0.463
0	0	1	−0.699	−0.33	−0.101	−0.095	0.150	0.335	0.452	0.449
0	1	0	0.66	0.66	0.498	0.326	0.83	0.83	0.663	0.749
0	1	1	0.242	0.0	0.0	0.326	0.621	0.5	0.663	0.5
1	0	0	0.33	0.33	0.176	−0.095	0.665	0.665	0.452	0.588
1	0	1	−0.33	−0.33	−0.044	−0.095	0.335	0.335	0.452	0.478
1	1	0	0.847	0.66	1.00	0.326	0.923	0.83	0.663	1.00
1	1	1	0.66	0.66	0.196	0.326	0.83	0.83	0.663	0.598

(see Section 8). The evolution of lower dimensionality conditional probabilities from high dimensionalities ones, as in Lemma 2, Section 3, is of dynamic programming nature inducing polynomial complexity. As stated in Section 7, the accuracy of a IN or TIN model is determined by the accuracy of the selected influence constants. The accuracy of the latter may be tested and they may be subsequently adjusted appropriately.

10 Experimental Setup and Numerical Evaluations

10.1 Experimental Setup

In this section, we lay out the steps involved in an experimental setup.

a. Given an event B, determine *all* the events $\{A_i\}\ 1 \le i \le n$ *known* to be affecting its occurrence.
b. Given B, all the known affecting events $\{A_i\}\ 1 \le i \le n$, and the causal strengths $\left[h_1^{(i)}(x_i = 1), h_1^{(i)}(x_i = 0)\right]$ between each A_i and B, design an influence constant $h_n\left(x_1^n\right)$, where x_1^n signifies the value of the status vector of the events $\{A_i\}\ 1 \le i \le n$, and where $-1 \le h_n\left(x_1^n\right) \le 1$; $\forall x_1^n$ values. The $h_n\left(x_1^n\right)$ constant may have one of the forms presented in Section 8.
c. If *all* in (b) is given, then upon a given probability of the status vector X_1^n, say $P\left(x_1^n\right)$;$\forall x_1^n$ values, the probability of event B is given by the following equation, named the consistency equation.

$$\sum_{x_1^n} P\left(x_1^n\right)\left\{1 + h_n\left(x_1^n\right)[1 - P(B)]\, P^{-1}(B)\right\}^{\mathrm{sgn} h_n\left(x_1^n\right)} \left\{1 + h_n\left(x_1^n\right)\right\}^{1-\mathrm{sgn} h_n\left(x_1^n\right)} = 1$$

whose equivalent form is:

$$P(B) = \left[\sum_{x_1^n : \mathrm{sgn} h_n\left(x_1^n\right)=1} P\left(x_1^n\right) h_n\left(x_1^n\right)\right]\left[\sum_{x_1^n} P\left(x_1^n\right)\left|h_n\left(x_1^n\right)\right|\right]^{-1},$$

if the denominator is non zero
d. When *all* the affecting events $\{A_i\}\ 1 \le i \le n$ are known, but the status of some of them are unknown, then, the probability $P(B)$, as computed in step (c) is used to compute the conditional probability $P\left(B \mid x_1^k\right)$, when the status vector of only k affecting events is known as:

$$P\left(B \mid x_1^k\right) = P(B)\left\{1 + h_k\left(x_1^k\right)[1 - P(B)]\, P^{-1}(B)\right\}^{\mathrm{sgn} h_k\left(x_1^k\right)} \bullet \left\{1 + h_k\left(x_1^k\right)\right\}^{1-\mathrm{sgn} h_k\left(x_1^k\right)}$$

where $h_k\left(x_1^k\right)$ is computed in a dynamic programming fashion from the influence constant $h_n\left(x_1^n\right)$in (b); as follows:

$$h_{n-1}\left(x_1^{n-1}\right) = \begin{cases} Q_n - 1 & ; \quad Q_n \in [0, 1] \\ P(B)\,[1 - P(B)]^{-1}\left[Q_n - 1\right] & ; \quad Q_n \in \left[1, P^{-1}(B)\right] \end{cases}$$

for

$$Q_n \triangleq \sum_{x_n=0,1} P\left(x_n|\, x_1^{n-1}\right)\left[1+h_n\left(x_1^n\right)[1-P(B)]\,P^{-1}(B)\right]^{\mathrm{sgn}h_n\left(x_1^n\right)}$$

$$\bullet\left[1+h_n\left(x_1^n\right)\right]^{1-\mathrm{sgn}h_n\left(x_1^n\right)}$$

We note that in the above expression, the affecting events $\{A_i\}\, 1 \le i \le n$ are assumed ordered as of the revealing of their status in time. Different such ordering results in different evolutions of the conditional probabilities $P\left(B|\, x_1^k\right)$.

e. When the existence as well as the status of the affecting events are sequentially revealed, then at time k, $P_k(B)$ and $P_k\left(B|\, x_1^k\right)$ are computed as in (c) and (d) where n is substituted by k in the latter.

Example 1 The following example illustrates the steps (a) to (e) with the help of an example TIN.

1. Figure 3 shows a IN with a binary event B known to be affected by the events $\{A_i\}\, 1 \le i \le 4$.
2. The edges connecting the external affecting events $\{A_i\}\, 1 \le i \le 4$ to the event B are shown in Fig. 3, annotated with the constants $\left[h_1^{(i)}(x_i=1),\, h_1^{(i)}(x_i=0)\right]$ for each i, where $x_i = 0, 1$ represents one of the two states of an affecting event A_i. A global influence constant $h_4\left(x_1^4\right)$ is then designed using the CAST logic expression (25) in Section 8. Table 2 shows the computed values for$h_4\left(x_1^4\right)$; $\forall x_1^n$.
3. The joint probability $P\left(x_1^4\right)$; $\forall\, x_1^4$ values are computed by assigning $P(x_i=1) = P(x_i=0) = 0.5$; $\forall\, i$ and by assuming $\{A_i\}\, 1 \le i \le 4$ to be mutually independent (Lemma 3). The probability of occurrence of event B, i.e., z = 1, is now calculated with the consistency equation, and is given as $P(z=1) = 0.5$. Assuming the status of all the affecting events to be known, the conditional probabilities $P\left(B|\, x_1^4\right)$; $\forall x_1^4$ are calculated via expression (26), and are shown in Table 2.

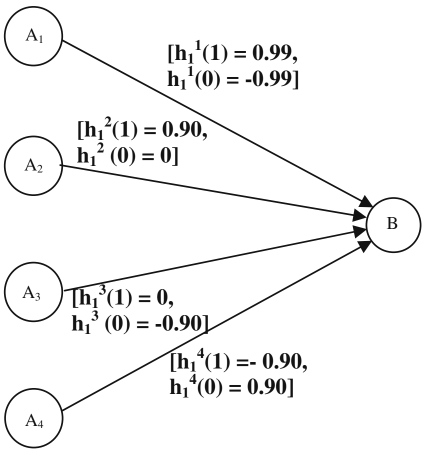


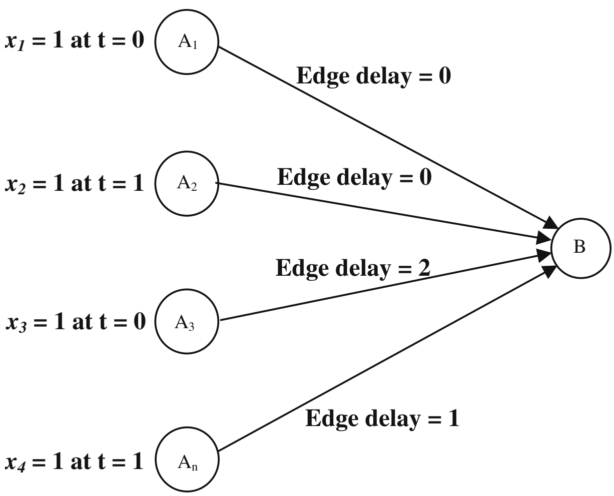
Fig. 3 Example TIN

Table 2

x_1	x_2	x_3	x_4	$h_4\left(x_1^4\right)$	$P\left(z=1\|x_1^4\right)$
0	0	0	0	−0.990000	0.005000
0	0	0	1	−0.999900	0.000050
0	0	1	0	−0.900000	0.050000
0	0	1	1	−0.999000	0.000500
0	1	0	0	−0.900000	0.050000
0	1	0	1	−0.999000	0.000500
0	1	1	0	−0.000001	0.499999
0	1	1	1	−0.990000	0.005000
1	0	0	0	0.990000	0.995000
1	0	0	1	0.000001	0.500001
1	0	1	0	0.999000	0.999500
1	0	1	1	0.900000	0.950000
1	1	0	0	0.999000	0.999500
1	1	0	1	0.900000	0.950000
1	1	1	0	0.999900	0.999950
1	1	1	1	0.990000	0.995000

4. The assumption in step (3), regarding the knowledge of the status of all the affecting events, may not be valid at times. Such is the case of a TIN with delays on edges (see Definition 1), reflecting variations in the times when the status of the affecting events become known. To illustrate this notion, we add temporal information to the IN in Fig. 3. The added temporal information together with the underlying graph is shown in Fig. 4. The time assigned to an affecting event A_i is the instance at when it assumes a state, i.e., $x_i = 0$ or 1. Prior to that time, the state of the event is assumed unknown. As stated in Definition 1, this combination of the external affecting events' status and their timing is also termed a course of action (COA), in the TIN literature.

Fig. 4 Example TIN with COA and edge delays



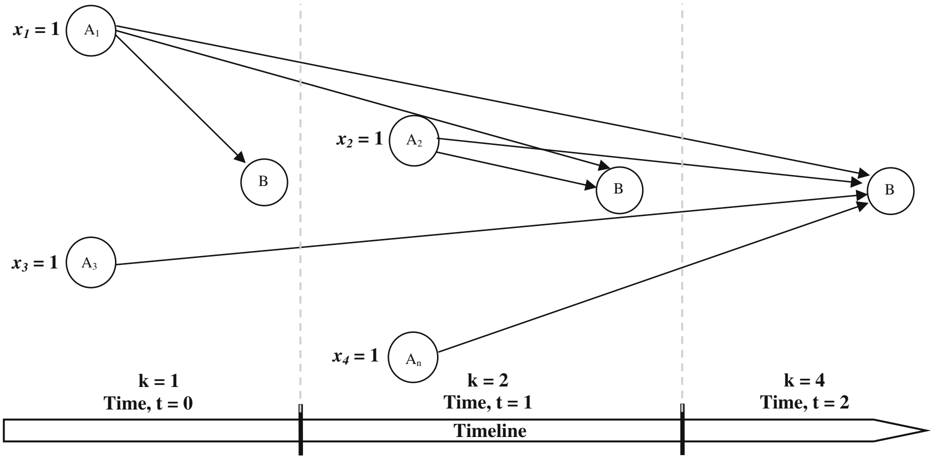
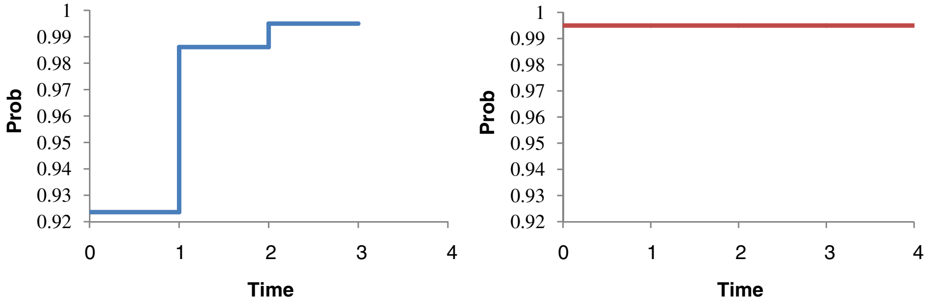


Fig. 5 Temporal model for the example TIN

5. The temporal information in the TIN, Fig. 4, determines the dynamics of the relationship between the affecting events and the affected event B; specifically, the times when the status of the affecting events are revealed to B. Figure 5 shows a IN equivalent, obtained by mapping the status of the affecting events and their effects on the event B, on a timeline. This mapping determines the number of affecting events 'k' at different time points (or time slices). For the temporal case presented in Section 6, the existence of all the affecting events is known to the event B a priori; their status, however, remain unknown until revealed, as determined by the COA and the delays on the edges. The probability $P(B)$, as calculated in step (c), is used to compute the conditional probabilities $P\left(B|\, x_1^k\right); k = 1, 2, 4$, i.e., $P\left(B|\, x_1^1\right)$, $P\left(B|\, x_1^2\right)$, and $P\left(B|\, x_1^4\right)$, as illustrated in the figure. Table 3 shows the values for $P\left(B|\, x_1^1\right)$ and $P\left(B|\, x_1^2\right)$, as computed by the corresponding $h_1\left(x_1^1\right)$ and $h_2\left(x_1^2\right)$. The posterior probability of B captures the impact of an affecting event on B and can be plotted as a function of time for a corresponding COA. This plot is called a probability profile [12, 27]. Figure 6 shows the resulting probability profile for the illustrative example. The plotted values in the profile are shown with bold letters in Tables 3 and 4. The overall complexity is polynomial.
6. For the temporal case presented in Section 9, the existence as well as the status of the affecting events are not known a priori but are determined by the given COA and the delays on the edges. At time k, $P_k(B)$ and $P_k\left(B|\, x_1^k\right)$ are computed

Table 3

x_1	$P\left(z = 1\|\, x_1^1\right)$	x_1	x_2	$P\left(z = 1\|\, x_1^2\right)$
0	0.076381	0	0	0.013887
1	0.923619	0	1	0.138875
		1	0	0.861125
		1	1	0.986113



(a) (b)

Fig. 6 Probability profile for the example COA, **a** For temporal case I. **b** For temporal case II

as in (c) and (d) where n is substituted by k in the latter. Table 4 shows the computed values of $P_k(B)$ and $P_k\left(B|\,x_1^k\right); k = 1, 2, 4$ and Fig. 6b shows the resulting probability profile.

Example 2 In multi-node connected network structures, given a set of external unaffected affecting events $\{A_i\}_i$, given influence constants $\left\{h_n\left(x_1^k\right)\right\}_k$, pertinent conditional probabilities are constructed hierarchically, as the structure of the network dictates. Consider, for example, the network in Fig. 7, below. In this network, the affecting events $A_i; i = 1, 2, 3, 4$ are external and unaffected by other events, while events B and C are affected, B being affecting as well. Let us denote the status of event $A_i; i = 1, 2, 3, 4$; by x_i, the status of event B by y and the status of event C by z, where y, z and $\{x_i\}_{1\le i\le 4}$ are 0–1 binary numbers. Let the influence constants $h(x_1, x_2)$, $h(x_3, x_4)$ and $h(y, x_3, x_4)$ be given. Let also the joint probability $P(x_1, x_2, x_3, x_4)$ be given.

Table 4

x_1	$P_1(B)$	$P\left(z=1\|\,x_1^1\right)$	x_1	x_2	$P_2(B)$	$P\left(z=1\|\,x_1^2\right)$	x_1	x_1	x_4	x_3	$P_4(B)$	$P\left(z=1\|\,x_1^4\right)$
0	0.5	0.005000	0	0	0.5	0.005000	0	0	0	0	0.5	0.005
1		0.995000	0	1		0.005000	0	0	0	1		0.005
			1	0		0.995000	0	0	1	0		0.005
			1	1		0.995000	0	0	1	1		0.005
							0	1	0	0		0.005
							0	1	0	1		0.005
							0	1	1	0		0.95
							0	1	1	1		0.005
							1	0	0	0		0.005
							1	0	0	1		0.995
							1	0	1	0		0.05
							1	0	1	1		0.995
							1	1	0	0		0.995
							1	1	0	1		0.995
							1	1	1	0		0.995
							1	1	1	1		0.995

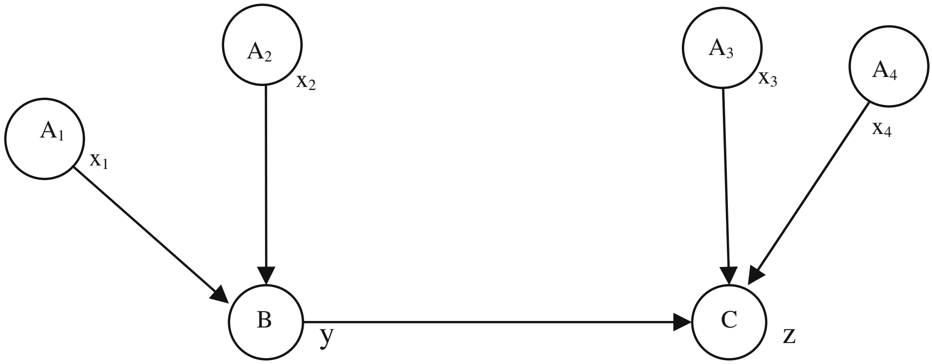


Fig. 7 A multi-node network

We then compute all the pertinent probabilities in the above network following the steps stated below:

1. Compute the probability $P(y)$ from the consistency condition:

$$\sum_{x_1,x_2} P(x_1,x_2)\left\{1+h(x_1,x_2)\left[1-P(y)\right]P^{-1}(y)\right\}^{\mathrm{sgn}h(x_1,x_2)} \times \{1+h(x_1,x_2)\}^{1-\mathrm{sgn}h(x_1,x_2)} = 1$$

where,

$$P(x_1,x_2) = \sum_{x_3,x_4} P(x_1,x_2,x_3,x_4)$$

2. Compute $P(y|x_1,x_2)$ from,

$$P(y|\,x_1,x_2) = P(y)\left\{1+h(x_1,x_2)\left[1-P(y)\right]P^{-1}(y)\right\}^{\mathrm{sgn}h(x_1,x_2)} \times \{1+h(x_1,x_2)\}^{1-\mathrm{sgn}h(x_1,x_2)}$$

3. Compute $P(y,x_3,x_4)$ as:

$$P(y,x_3,x_4) = \sum_{x_1,x_2} P(y|\,x_1,x_2)\,P(x_1,x_2,x_3,x_4)$$

4. Compute $P(z)$ from the consistency condition

$$\sum_{y,x_3,x_4} P(y,x_3,x_4)\left\{1+h(y,x_3,x_4)[1-P(z)]P^{-1}(z)\right\}^{\mathrm{sgn}h(y,x_3,x_4)} \times\{1+h(y,x_3,x_4)\}^{1-\mathrm{sgn}h(y,x_3,x_4)} = 1$$

5. Compute $P(z|y,x_3,x_4)$ from,

$$P(z|\,y,x_3,x_4) = P(z)\left\{1+h(y,x_3,x_4)[1-P(z)]P^{-1}(z)\right\}^{\mathrm{sgn}h(y,x_3,x_4)} \times\{1+h(y,x_3,x_4)\}^{1-\mathrm{sgn}h(y,x_3,x_4)}$$

6. Compute $P(z|x_1,x_2,x_3,x_4)$ from,

$$P(z|\, x_1, x_2, x_3, x_4) = \sum_y P(z, y|\, x_1, x_2, x_3, x_4) = \sum_y P(z|\, y, x_3, x_4)\, P(y,|\, x_1, x_2)$$

10.2 Numerical Evaluations

In this section, we apply the algorithms developed in this paper to an illustrative TIN. We also provide a comparison of the latter results with those previously obtained via the use of the CAST logic. The model used in this section was presented by Wagenhals et al. in 2001 [33] to address the following scenario: As described in [33], internal political instabilities in Indonesia have deteriorated and ethnic tensions between the multiple groups that comprise Indonesia have increased. Religion has been a major factor in these conflicts. Members of one of the minority (2%) religious groups have banded together to combat disenfranchisement. These members have formed a rebel militia group. Armed conflicts recently occurred between those rebels and the Indonesian military. The rebels fled to eastern Java where they have secured an enclave of land. This has resulted in a large number of Indonesian citizens being within the rebel-secured territory. Many of these people are unsympathetic to the rebels and are considered to be at risk. It is feared that they may be used as hostages if ongoing negotiations break down with the Indonesian government. The food and water supply and sanitation facilities are very limited within the rebel-secured territory.

Several humanitarian assistance (HA) organizations are on the island, having been involved with food distribution and the delivery of public health services to the urban poor for several years. So far, the rebels have not prevented HA personnel from entering the territory to take supplies to the citizens. The US and Australian embassies in Jakarta are closely monitoring the situation for any indications of increasing rebel activity. In addition, Thailand, which has sent several hundred citizens to staff numerous capital investment projects on Java, is known to be closely monitoring the situation. To reflect the situation stated above, a TIN was first created in [33] and is shown in Fig. 8.

The latter TIN models the causal and influencing relationships between (external) affecting events (on the left side and along the top of the model in Fig. 8) and the overall effect of concern which is the single node with no parents on the right-hand side of the model. In this case, the effect is "Rebels decide to avoid violence". The actionable (external) events in this model include a combination of potential coalition, UN, and rebel actions. The coalition actions include actions by the US government, its military instrument of national power, actions by the Government of Indonesia, and actions by Thailand.

For purposes of illustration and comparison of results, we have selected a part of this network, as shown in Fig. 9.

The (external) affecting events in the TIN of Fig. 9 are drawn as root nodes (nodes without incoming edges). The text in each node, e.g., "1—Coalition Deploys Forces to Indonesia," represents a node ID and a statement describing the binary proposition. In Fig. 9, $\{A_i\}\, 0 \leq i \leq 4$ represents the set of the external affecting events, where the index 'i' depicts the node ID. The marginal probabilities for the external affecting events are also shown inside each node. In this illustration, we assume all external affecting events to be mutually independent (Section 4). A

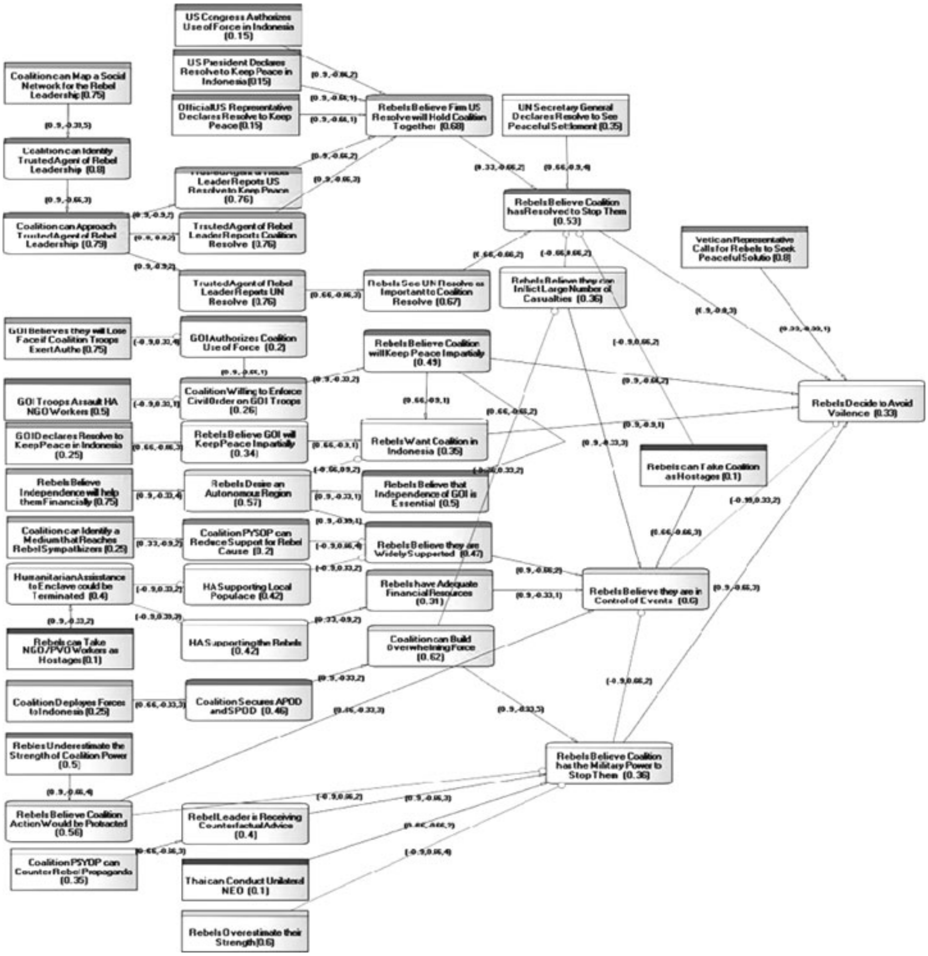


Fig. 8 Timed influence net of East Timor situation [33]

desired effect, or an objective which a decision maker is interested in, is modeled as a leaf node (node without outgoing edges). The node with ID ‘10’ in Fig. 9 represents the objective for the illustration. In both Figs. 8 and 9, the root nodes are drawn as rectangles while the non-root nodes are drawn as rounded rectangles. A directed edge with an arrowhead between two nodes shows the parent node promoting the chances of a child node being true, while the roundhead edge shows the parent node inhibiting the chances of a child node being true. The first two elements in the inscription associated with each arc quantify the corresponding strengths of the influence of a parent node’s state (as being either true or false) on its child node. The third element in the inscription depicts the time it takes for a parent node to influence a child node. For instance, in Fig. 9, event “1—Coalition Deploys Forces to Indonesia” influences the occurrence of event “7—Coalition Secures APOD and SPOD” after three time units.

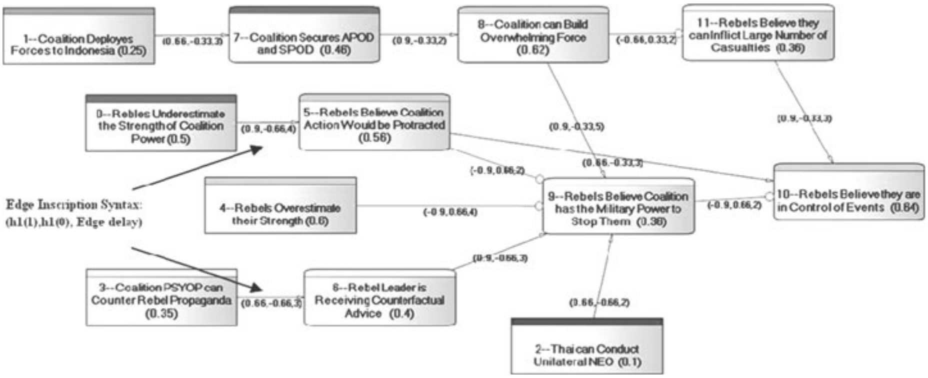


Fig. 9 Sample TIN for analysis

The purpose of building a TIN is to evaluate and compare the performances of alternative courses of actions described by the set $\mathbf{A_T}$ *in the definition of TINs* The impact of a selected course of action on the desired effect is analyzed with the help of a *probability profile*. The following is an illustration of such an analysis with the help of two COAs, given below:

COA1 All external affecting events are taken simultaneously at time 1 and are mutually independent.

COA2 Events {0, 2, 4} are taken at time 1, simultaneously, and events {1, 3} are taken at time 2, simultaneously.

The two COAs can also be described as in Table 5.

Note that the simultaneous occurrence of external affecting events does not necessarily imply simultaneous revealing of their status on an affected node; the time sequence of revealed affecting events is determined by both the time stamp on each affecting event and the delays on edges. Because of the propagation delay associated with each edge, influences of actions impact the affected event progressively in time. As a result, the probability of the affected event changes as time evolves. A probability profile draws these probabilities against the corresponding time line. In Fig. 10, probability profiles generated for nodes “9—Rebels Believe Coalition has the Military Power to Stop Them” and “10—Rebels Believe they are in Control of Events,” using the CAST logic based approach in [2, 22, 28, 29, 34] are shown.

Table 5

Event	COA1		COA2	
	Time	Status	Time	Status
0—Rebels underestimate the strength of coalition power	1	1 (=True)	1	1
1—Coalition deploys forces to Indonesia	1	1	2	1
2—Thai can conduct unilateral NEO	1	1	1	1
3—Coalition PSYOP can counter rebel propaganda	1	1	2	1
4—Rebels overestimate their strength	1	1	1	1

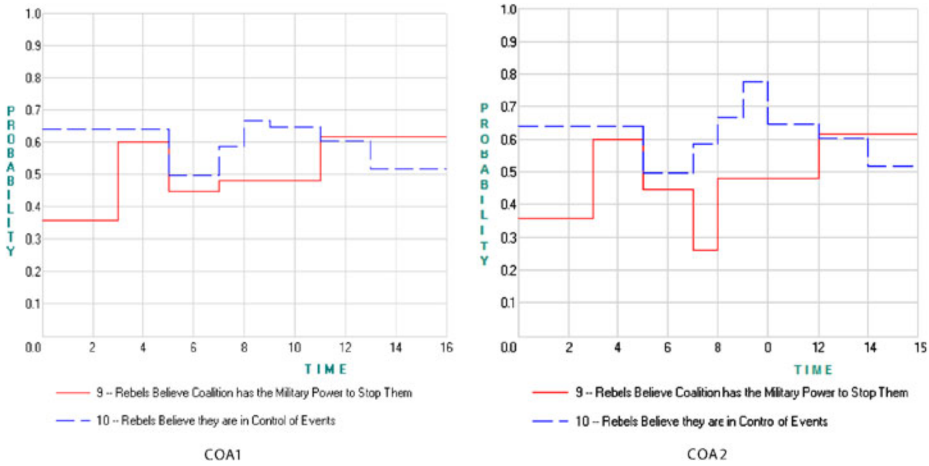


Fig. 10 Probability profiles generated by the CAST logic approach

For the same TIN model as in Fig. 9 and the corresponding course of actions, we used the approach presented in this paper and produced pertinent results for the following two cases:

Case 1 For this illustration, we utilize the influence constant model presented in Section 8.1 and the temporal case presented in Section 6. The influence constants $\{h_i(x_1^n)\}\ 1 \leq i \leq n-1$ are first pre-computed via the dynamic programming expression in Lemma 2, Section 3. The resulting probability profiles for the two affected events/propositions in the TIN are shown in Fig. 11.

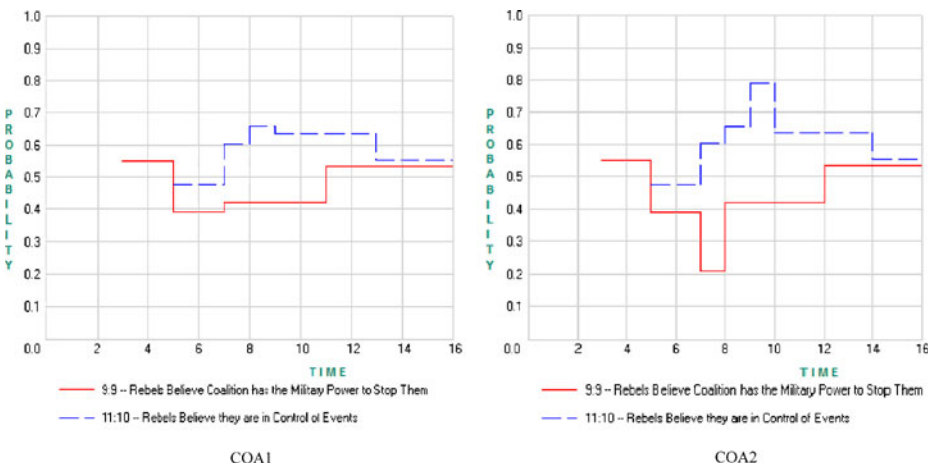


Fig. 11 Probability profiles for case I

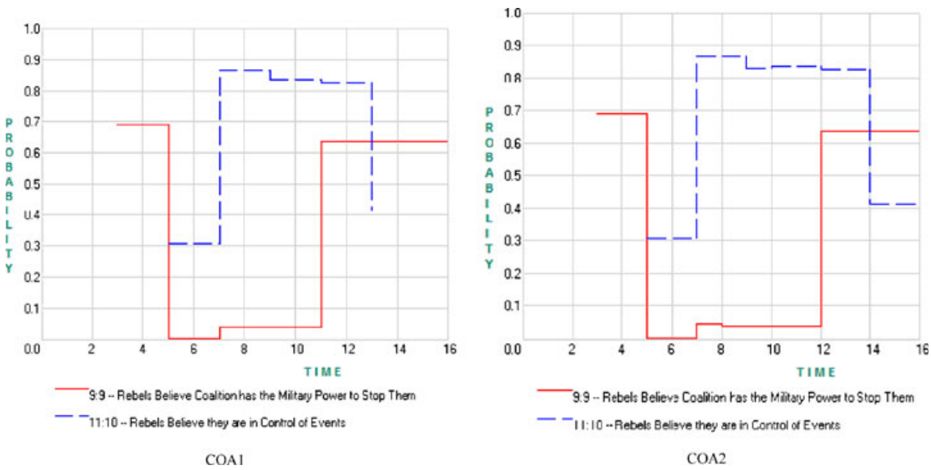


Fig. 12 Probability profiles for case II

Case 2 For this illustration, we utilize the influence constant model presented in Section 8.1 and the temporal case where the existence of an affecting event is assumed unknown to an affected event unless it reveals itself and makes its status known to the affected event. The conditional probabilities, in this case, are computed real-time by Eq. 33. The resulting probability profiles for the two affected events/propositions in the TIN are shown in Fig. 12.

Comparing Figs. 10 and 11, we note that when the existence of all the external affecting events are initially known, then the approach in this paper produces results that are more accurate and consistent than those produced by the CAST logic based approach. This was expected, since the present approach has eliminated the inconsistencies that the CAST logic based approach suffers from. Unlike the CAST logic based approach, the probability profiles generated by the new approach only record the posterior probabilities resulting from the impacts of the external affecting events and do not assume any default initial values; in profiles of Figs. 10 and 11 the first impact is recorded at time ‘3’. Comparing Figs. 11 and 12, we note that, as expected, when the existence of the external affecting events are revealed sequentially in time then, there is a relatively high level of instability in time evolution, as compared to the case where the existence of all the external affecting events is initially known. The selection of a influence constant and of temporal models for a TIN under construction/analysis is a design issue and is reflected by the differences in the resulting probability profiles.

11 Conclusion

In this paper, we presented a comprehensive approach to Influence Nets including conditions for model consistency and dynamic programming evolution of the influence constants, as well as temporal issues and model testing methodologies. We revisited the earlier CAST logic [2, 22] based approach to Timed Influence Network

(TIN) modeling [28, 29, 34], by redefining the design parameters for a TIN model, reevaluating the cases of independence and (partial) dependence among external affecting events, introducing new methods for aggregating joint influences from design parameters, and by offering new insights into the temporal aspects of causal influences modeled inside a TIN. The presented approach successfully overcomes the deficiencies in the CAST logic based TIN modeling and the inconsistencies therein. It also does not require any additional design information than that already available in a TIN constructed via CAST logic parameters; the entire repository of situational models developed earlier [29, 33, 34] may be simply reanalyzed (without any modifications) using the new set of computational tools introduced in this paper. We analyzed and evaluated our approach and tested it for a specific TIN. The approach produces consistent and stable in time results.

It is assumed the presented approach will benefit a growing community of Influence Net and Timed Influence Net users that include both government and private (e.g., Raytheon, ANSER, and other corporations supporting DOD and DHS) organizations for rapid construction and deployment of situational influence models for intelligence assessment, course of action planning and assessment in effects based operations, and adversarial modeling problems. The main reason for this growing acceptance is the fact that it can be effectively utilized to model, test and evaluate problem specific domains of interests where the estimation of influence metrics cannot be done due to lack of data, however subjective estimates, in the form of influence constants, can be obtained from subject matter experts.

Appendix

Proof of Lemma 2

Applying the Bayes rule, we write

$$P\left(B\middle|\, x_1^{n-1}\right) = \frac{P\left(x_1^{n-1}\middle|\, B\right) P\left(B\right)}{P\left(x_1^{n-1}\right)} \tag{39}$$

Where, due to the theorem of total probability, we have

$$P\left(x_1^{n-1}\middle|\, B\right) = \sum_{x_n=0,1} P\left(x_1^{n}\middle|\, B\right) \tag{40}$$

From the Bayes Rule we also have,

$$P\left(x_1^{n}\middle|\, B\right) = \frac{P\left(B\middle|\, x_1^{n}\right) P\left(x_1^{n}\right)}{P\left(B\right)} \tag{41}$$

Substituting Eqs. 40 and 41 in Eq. 39, we obtain:

$$P\left(B\middle|\, x_1^{n-1}\right) = \sum_{x_n=0,1} P\left(B\middle|\, x_1^{n}\right) P\left(x_1^{n}\right) P^{-1}\left(x_1^{n-1}\right) = \sum_{x_n=0,1} P\left(B\middle|\, x_1^{n}\right) P\left(x_n\middle|\, x_1^{n-1}\right) \tag{42}$$

Where in (42) we used the definition of conditional probability $P\left(x_n\middle|\, x_1^{n-1}\right) = P\left(x_1^{n}\right) P^{-1}\left(x_1^{n-1}\right)$.

Using expression Eq. 4 in Section 3 and substituting in Eq. 42, we obtain.

$$\begin{aligned}
&\left\{1+h_{n-1}\left(x_1^{n-1}\right)\left[1-P(B)\right]P^{-1}(B)\right\}^{\operatorname{sgn}h_{n-1}\left(x_1^{n-1}\right)}\bullet\left\{1+h_{n-1}\left(x_1^{n-1}\right)\right\}^{1-\operatorname{sgn}h_{n-1}\left(x_1^{n-1}\right)}\\
&\quad=\sum_{x_n=0,1}P\left(x_n|\,x_1^{n-1}\right)\left\{1+h_n\left(x_1^n\right)\left[1-P(B)\right]P^{-1}(B)\right\}^{\operatorname{sgn}h_n\left(x_1^n\right)}\bullet\left\{1+h_n\left(x_1^n\right)\right\}^{1-\operatorname{sgn}h_n\left(x_1^n\right)}\\
&\quad\triangleq Q_n+1
\end{aligned}\tag{43}$$

Observing Eq. 43, we notice that if $(Q_n+1)\in[0,1]$, then $h_{n-1}\left(x_1^{n-1}\right)$ must be necessarily negative, reducing the left part of the equality in Eq. 43 to $1+h_{n-1}\left(x_1^{n-1}\right)$. If, on the other hand, $(Q_n+1)\in\left[1,P^{-1}(B)\right]$ then the left part of Eq. 43 must necessarily reduce to $1+h_{n-1}\left(x_1^{n-1}\right)\left[1-P(B)\right]P^{-1}(B)$, with $h_{n-1}\left(x_1^{n-1}\right)$ positive. The above observations clearly lead to the result in the lemma. □

Proof of Lemma 3

Due to the Bayes rule, we have:

$$P\left(B|\,x_1^n\right)=\frac{P\left(x_1^n\middle|\,B\right)P(B)}{P\left(x_1^n\right)}\tag{44}$$

Due to the independence assumption, we have:

$$\frac{P\left(x_1^n\middle|\,B\right)}{P\left(x_1^n\right)}=\prod_{i=1}^{n}\frac{P\left(x_i|\,B\right)}{P\left(x_i\right)}\tag{45}$$

where

$$\frac{P\left(x_i|\,B\right)}{P\left(x_i\right)}=\frac{P\left(B|x_i\right)}{P(B)}\tag{46}$$

Substituting Eq. 46 in Eq. 45 and then Eq. 44, we obtain:

$$P\left(B|\,x_1^n\right)=P^{-(n-1)}(B)\prod_{i=1}^{n}P\left(B|\,x_i\right)\tag{47}$$

Substituting expression (10) in Eq. 47 we obtain the expression in the lemma. □

Proof of Lemma 4

Due to the Bayes rule, we have:

$$\frac{P\left(B|\,x_1^n\right)}{P(B)}=\frac{P\left(x_1^n\middle|\,B\right)}{P\left(x_1^n\right)}\tag{48}$$

Then, due to Eq. 13, we obtain from Eq. 48,

$$\frac{P\left(B|\,x_1^n\right)}{P(B)}=\frac{P\left(x_1|\,B\right)}{P\left(x_1\right)}\prod_{i=2}^{n}\frac{P\left(x_i|\,x_{i-1},B\right)}{P\left(x_i|\,x_{i-1}\right)}\tag{49}$$

Applying the Bayes chain rule, we have:

$$P(x_i|\,x_{i-1},B)\,P(x_{i-1}|\,B)\,P(B)=P(B|\,x_i,x_{i-1})\,P(x_i|x_{i-1})\,P(x_{i-1}) \tag{50}$$

where,

$$P(x_{i-1}|\,B)\,P(B)=P(B|\,x_{i-1})\,P(x_{i-1}) \tag{51}$$

Substituting Eq. 51 in Eq. 50 we then obtain:

$$\begin{aligned}\frac{P(x_i|\,x_{i-1},B)}{P(x_i|\,x_{i-1})}&=\frac{P(B|\,x_i,x_{i-1})}{P(B|\,x_{i-1})};i\geq 2\\ \frac{P(x_1|\,B)}{P(x_1)}&=\frac{P(B|\,x_1)}{P(B)}\end{aligned} \tag{52}$$

Where, directly from the results in Lemma 1, we have:

$$P(B|\,x_i)=\left\{1+h_1^{(i)}(x_i)\,[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_1^{(i)}(x_i)}\left\{1+h_1^{(i)}(x_i)\right\}^{1-\mathrm{sgn}h_1^{(i)}(x_i)} \tag{53}$$

where

$$\underset{i\geq 2}{h_1^{(i)}}(x_i)=\begin{cases}Q_{i,i+1}-1 & ;\quad \mathrm{if}\,Q_{i,i+1}\in[0,1]\\ P(B)\,[1-P(B)]^{-1}\left[Q_{i,i+1}-1\right]; & \mathrm{if}\,Q_{i,i+1}\in\left[1,P^{-1}(B)\right]\end{cases} \tag{54}$$

$$\begin{aligned}Q_{i,i+1}\triangleq\sum_{x_{i+1}=0,1}P(x_{i+1}|\,x_i)&\left\{1+h_2^{(i,i+1)}(x_i,x_{i+1})\,[1-P(B)]\,P^{-1}(B)\right\}^{\mathrm{sgn}h_2^{(i,i+1)}(x_i,x_{i+1})}\\ &\bullet\left\{1+h_2^{(i,i+1)}(x_i,x_{i+1})\right\}^{1-\mathrm{sgn}h_2^{(i,i+1)}(x_i,x_{i+1})}\end{aligned} \tag{55}$$

Substitution of expression (52) and (55) in Eq. 49, in conjunction with expression (14), give the result in the lemma. □

Proof of Lemma 5

Due to the Bayes rule and the theorem of total probability, we have:

$$P(B)=\sum_{x_1^n}P\left(B|\,x_1^n\right)P\left(x_1^n\right) \tag{56}$$

Substituting in Eq. 56 the expression (27) for the conditional probability $P\left(B|\,x_1^n\right)$, we obtain:

$$\begin{aligned}&\sum_{x_1^n:\sum_{i=1}^n h_1(x_i)>0}P\left(x_1^n\right)\left\{1+P^{-1}(B)\,[1-P(B)]\max_{1\leq i\leq n}h_1(x_i)\right\}\\ &+\sum_{x_1^n:\sum_{i=1}^n h_1(\backslash x_i)=0}P\left(x_1^n\right)+\sum_{x_1^n:\sum_{i=1}^n h_1(x_i)<0}P\left(x_1^n\right)\left\{1+\min_{1\leq i\leq n}h_1(x_i)\right\}=1\end{aligned} \tag{57}$$

which gives after simplification:

$$[1 - P(B)] \sum_{x_1^n : \sum_{i=1}^{n} h_1(x_i) > 0} P\left(x_1^n\right) \max_{1 \le i \le n} h_1(x_i) + P(B) \sum_{x_1^n : \sum_{i=1}^{n} h_1(x_i) < 0} P\left(x_1^n\right) \min_{1 \le i \le n} h_1(x_i) = 0 \tag{58}$$

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