Habib University - City Campus
Instructors: Aeyaz Jamil Keyani
Course: MATH 307 Mathematical Foundations and Reasoning
Examination: Test 1 – Spring 2025
Exam Date: Tuesday, February 18, 2025
Exam Time: 10:00 – 11:15

Total Marks: 50 Marks Duration: 75 Minutes

Name: ____________ Student ID: ____________ Section: ____________

1. $\boxed{\text{3 points}}$ Consider the following proof that shows that all horses are of same color:

Theorem 1. *In any finite set of horses all horses are always of the same color.*

Proof. Let H be a finite set of horses, we will prove by induction (induct on $\#(H)$) that all horses in H are of same color.

Base case: $\#(H) = 1$, if $\#(H) = 1$ then there is one horse in the set and so trivially all the horses in the set have same color.

Induction hypothesis: Suppose for $\#(H) = k$ all horses in H are of same color.

Induction step: Suppose $\#(H) = k$, lets take out a horse x from H the set $\#(H \setminus \{x\}) = k$, by induction hypothesis we know that all horses in $H \setminus \{x\}$ are of same color and as $\{x\}$ is a set of one horse all horses in $\{x\}$ are of same color. Now we consider the set $H \setminus \{y\}$ for $y \neq x$ then by the induction hypothesis again all horses in $H \setminus \{y\}$ are of the same color and now as $x \in H \setminus \{y\}$, x has the same color as all the horses in $H \setminus \{y\}$, and therefore x has the same color as all the horses in $H \setminus \{x\}$. And so all the horses in H have the same color.

So by the principles of Mathematical induction we have that in any finite set of horses, all horses have the same color. □

Is th above proof correct? If not explain what is the error in the above reasoning.

Solution: Induction fails at $k = 2$, that is $p(1)$ does not imply $p(2)$. The inductive step assumes $k \geq 3$ to create the equivalence between $\{x\}$ and $\{y\}$. When we remove x from H and put it back ans remove y to claim that color of x and y are the same there should have been a horse z left in H such that $z \neq x$ and $y \neq$. z creates the equivalence of color between x and y. But this means that there should be at least three horses in H. And therefore we skipped $p(2)$ so $p(1)$ doesn't imply $p(2)$, and so the induction fails.

2. $\boxed{\text{7 points}}$ Show that addition in natural numbers is commutative.

Solution: We shall use induction on n (keeping m fixed). First we do the base case $n = 0$, i.e., we show $0 + m = m + 0$. By the definition of addition, $0 + m = m$, while by Lemma 2.2.2, $m + 0 = m$. Thus the base case is done. Now suppose inductively that $n + m = m + n$, now we have to prove that $(++) + m = m + (n\,++)$ to close the induction. By the definition of addition, $(n\,++) + m = (n + m)\,++$. By Lemma 2.2.3, $m + (n\,++) = (m + n)\,++$, but this is equal to $(n + m)\,++$ by the inductive hypothesis $n + m = m + n$. Thus $(n\,++) + m = m + (n\,++)$ and we have closed the induction.

□

3. $\boxed{\text{10 points}}$ Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Solution: By definition of exponentiation,

$$(a+b)^2 = (a+b)^1(a+b) = (a+b)(a+b)$$

By distributivity

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = (aa+ab) + (ba+bb)$$

By associativity

$$(a+b)^2 = (aa+ab) + (ba+bb) = aa + (ab+ba) + bb$$

By commutativity

$$(a+b)^2 = aa + (ab+ba) + bb = aa + (ab+ab) + bb$$

By distributivity, definition of multiplication and commutativity

$$(a+b)^2 = aa + (ab+ab) + bb = aa + ab(1+1) + bb = aa + 2ab + bb = aa + 2ab + bb$$

By definition of exponentiation

$$(a+b)^2 = aa + 2ab + bb = a^2 + 2ab + b^2$$

□

4. $\boxed{\text{10 points}}$ Let A, B, C and D be some sets. Show that $A \times B \subseteq C \times D \iff A \subseteq C$ and $B \subseteq D$, and furthermore $A \times B = C \times D \iff A = C$ and $B = D$.

Solution: Let $a \in A$ and $b \in B$, then by definition of cartesian product we have that $a \in A \wedge b \in B \iff (a, b) \in A \times B$. As $A \times B \subseteq C \times D$, we have that $(a, b) \in C \times D$ and so by the definition of cartesian product $a \in C$ and $b \in D$. Therefore $A \subseteq C$ and $B \subseteq D$. Conversely let $(a, b) \in A \times B$ then by definition of cartesian product, $a \in A$ and $b \in B$, as $A \subseteq C$ and $B \subseteq D$, we have that $a \in C$ and $b \in D$. And by the definition of cartesian product $(a, b) \in C \times D$. Therefore $A \times B \subseteq C \times D$. So we have that $A \times B \subseteq C \times D \iff A \subseteq C$ and $B \subseteq D$

Now we show that $A \times B = C \times D \iff A = C$ and $B = D$. Consider $A \times B = C \times D$ so by the definition of set equality and subset relation we have that: $A \times B = C \times D \iff (A \times B \subseteq C \times D) \wedge (C \times D \subseteq A \times B) \iff (A \subseteq C \wedge B \subseteq D) \wedge (C \subseteq A \wedge D \subseteq B) \iff (A \subseteq C \wedge C \subseteq A) \wedge (B \subseteq D \wedge D \subseteq B) \iff A = C \wedge B = D$. Therefore $A \times B = C \times D \iff A = C$ and $B = D$.

□

5. $\boxed{\text{20 points}}$ Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijections then so is $g \circ f$, and we have that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution: Let X, Y and Z be some sets. And let $f : X \to Y$ and $g : Y \to Z$ be bijections.

We'll show that the function $g \circ f$ is also a bijection. First we show that $g \circ f$ is an injection. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$ then as f is an injection, $f(x_1) \neq f(x_2)$, and as $f(x_1), f(x_2) \in Y$ and g is injective we have that $g(f(x_1)) \neq g(f(x_2))$. And therefore $g \circ f$ is injective. Now we show that $g \circ f$ is surjective. Let $z \in Z$ as g is surjective, we have that there exists $y \in Y$ such that $g(y) = z$. And as f is surjective, we have that there exists $x \in X$ such that $f(x) = y$, and so $g(f(x)) = z$. So $g \circ f$ is surjective. Therefore $g \circ f$ is a bijection.

Next we'll show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Let $z \in Z$ as $g \circ f$ is a bijection from X to Y we have that there exists $x \in X$ such that $g \circ f(x) = z$. By the definition of inverse we have that, $g \circ f(x) = z \iff (g \circ f)^{-1}(z) = x \iff g(f(x)) = z \iff g^{-1}(g(f(x))) = g^{-1}(z) \iff f(x) = g^{-1}(z) \iff f^{-1}(f(x)) = f^{-1}(g^{-1}(z)) \iff x = f^{-1}(g^{-1}(z))$. So we have that $(g \circ f)^{-1}(z) = x \iff f^{-1} \circ g^{-1}(z) = x$. Therefore $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

$\square$

6. $\boxed{\text{7 points}}$ Show that cardinality is an equivalence relation, that is to say show that:

If X, Y, Z are sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Solution: Let X, Y and Z be some sets.

Reflexivity: The identity map $f(x) = x$ is a bijection from X to itself. As for each $x \in X$, x is the preimage of x so f is surjective. And for $x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, as $f(x_1) = x_1$ and $f(x_2) = x_2$. And so X has the same cardinality as X.

Symmetry: If cardinality of X is equal to the cardinality of Y then there is a bijection $f : X \to Y$. Now the the inverse map, f^{-1}, such that $f^{-1}(y) = x$ such that $f(x) = y$ is a bijection from Y to X. As f is a bijection from X to Y then for each $y \in Y$, there is a preimage of y (due to surjectivity of f). Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ then $f^{-1}(y_1) \neq f^{-1}(y_2)$, as if $f^{-1}(y_1) = f^{-1}(y_2) \in X$, then $f(f^{-1}(y_1)) = f(f^{-1}(y_2))$ and so $y_1 = y_2$ which is a contradiction. And so f^{-1} is also injective. Therefore f^{-1} is a bijection from Y to X. And so Y has the same cardinality as X.

Transitivity: If X has the same cardinality as Y and Y has the same cardinality as Z then we have a bijections $f : X \to Y$, $g : Y \to Z$. And from problem 5 we have that $g \circ f$ is a bijection from X to Z. And therefore X has the same cardinality as Z.

$\square$

7. $\boxed{\text{3 points}}$ For any natural number n, let S_n be the set of all bijections $\phi : \{i \in \mathbb{N} | 1 \leq i \leq n\} \to \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$. Show that for any natural number n, S_n is finite and $\#(S_{n++}) = n ++ \times \#(S_n)$

Solution: For any $n \in \mathbb{N}$, let $[n]$ denote the set $[n] = \{i \in \mathbb{N} | 1 \leq i \leq n\}$.

As $[n]$ is a finite set by Proposition 3.6.14 we know that $[n]^{[n]}$ (set containing all functions from $[n]$ to $[n]$) is finite and $\#([n]^{[n]}) = \#([n])^{\#([n])}$. For any $n \in \mathbb{N}$, $S_n \subseteq [n]^{[n]}$, and so the cardinality of S_n is less than equal to the cardinality of $[n]^{[n]}$ (The identity map from S_n to $[n]^{[n]}$ is an injection from S_n to $[n]^{[n]}$). And as $[n]^{[n]}$ is finite, then so is S_n.

First we partition S_{n++} into $n ++$ subsets such that: $A_i = \{\phi \in S_{n++} \mid \phi(n ++) = i\}$, for each $i \in [n]$. As each $\phi \in S_{n++}$ is a bijection we have that all A_i are disjoint and $\bigcup_{i \in [n++]} A_i = S_{n++}$. So we have that $\#(S_{n++}) = \sum\limits_{i \in [n]} \#(A_i)$. We do so by constructing a bijection from A_i to S_n.

Let $i \in [n ++]$ we have that f_i is a bijection from A_i to S_n such that for any $\phi \in A_i$, $f(\phi) = \psi$ such that

$$\psi(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \neq n ++ \\ i & \text{if } \phi(x) = n ++ \end{cases}$$

f is a surjection, as for any $\psi \in S_n$ we have the function $\phi \in A_i$ such that

$$\phi(x) = \begin{cases} \psi(x) & \text{if } x \neq n\,{++} \\ n\,{++} & \text{if } x = n\,{++} \end{cases}$$

f is an injection as if for $\phi, \phi' \in A_i$, $\phi \neq \phi'$ then there is $x \in [n\,{++}] \setminus \{i\}$ such that $\phi(x) \neq \phi'(x)$ and then $f(\phi)(x) \neq f(\phi')(x)$ therefore $f(\phi) \neq f(\phi')$.

So for each A_i, $\#(A_i) = \#(S_n)$. So we have that $\#(S_{n++}) = \sum_{i \in [n]} \#(A_i) = \sum_{i \in [n]} \#(S_n)$. By induction on n it is easy to see that $\#(S_{n++}) = \sum_{i \in [n++]} \#(S_n) = (n\,{++}) \times \#(S_n)$.

$\square$