# Lecture 8

Tuesday, February 1, 2022 1:07 PM

1)

LECTURE 8 LINEAR ALGEBRA

PROPERTIES OF DETERMINANTS (OF MATRICES OF ORDER n). REVISITED

① $\det(AB) = \det(A)\det(B)$

A, B ARE SQUARE MATRICES OF THE SAME SIZE.

② $\det(I) = 1$, $I \to$ IDENTITY MATRIX

③ IF $A^{-1}$ EXISTS THEN $\det(A) \neq 0$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

④ IF TWO ROWS/COLUMNS OF A MATRIX A ARE IDENTICAL THEN $\det(A) = 0$.

⑤ ADDING ROWS (OR COLUMNS) TOGETHER MAKES NO DIFFERENCE TO THE DETERMINANT.

⑥
$$\begin{vmatrix} ka_{11} & a_{12} \cdots\cdots & a_{1n} \\ ka_{21} & a_{22} \cdots\cdots & a_{2n} \\ \vdots & & \\ ka_{n1} & a_{n2} \cdots\cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{vmatrix}$$

2)

ETC. WHERE $\boxed{k}$ IS ANY SCALAR.

(7) IF $\boxed{A}$ IS A SQUARE MATRIX WITH TWO PROPORTIONAL ROWS OR TWO PROPORTIONAL COLUMNS, THEN $\boxed{\det(A)=0}$.

(8) IF $\boxed{A}$ IS A SQUARE MATRIX SUCH THAT $\boxed{A}$ HAS A ROW OF $\boxed{\text{ZEROS}}$ OR A COLUMN OF ZEROS THEN $\boxed{\det(A)=0}$.

NOTATION: FOR $\boxed{2\times 2}$ CASE

IF $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, THEN

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = |A|$$

MORE PROPERTIES

(1) IF $\boxed{A}$ AND $\boxed{B}$ ARE SQUARE MATRICES OF SAME SIZE THEN $\boxed{\det(A)+\det(B)\neq\det(A+B)}$ (IN GENERAL).

Does not mean this never happens, it means it will not always happen.

3)

EXAMPLE: FOR $A=\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A+B=\begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$

VERIFY $\boxed{\det(A)+\det(B)\neq\det(A+B)}$

(P.93 7th ED.) P.96 8th ED.

always happen.

3) **EXAMPLE:** FOR $A=\begin{bmatrix}1&2\\2&5\end{bmatrix}$, $B=\begin{bmatrix}3&1\\1&3\end{bmatrix}$, $A+B=\begin{bmatrix}4&3\\3&8\end{bmatrix}$

VERIFY $\boxed{\det(A)+\det(B) \neq \det(A+B)}$ (P.93 7th ED) (P.96 8th ED)

**HINT:** $\det(A)=1$, $\det(B)=8$, $\det(A+B)=23$, $9\neq 23$

TRY THE FOLLOWING:

LET $A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$, $B=\begin{bmatrix}a_{11}&a_{12}\\b_{21}&b_{22}\end{bmatrix}$

FIND $\det(A)+\det(B)=?$

**HINT:** $\det(A)=\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

$\det(B)=\det\begin{bmatrix}a_{11}&a_{12}\\b_{21}&b_{22}\end{bmatrix}$

$=\begin{vmatrix}a_{11}&a_{12}\\b_{21}&b_{22}\end{vmatrix}=a_{11}b_{22}-a_{12}b_{21}$

**ANS:** $\boxed{a_{11}(a_{22}+b_{22})-a_{12}(a_{21}+b_{21})}$

WHICH CAN BE WRITTEN AS

$\det(A)+\det(B) =$

4) $\begin{vmatrix}a_{11}&a_{12}\\a_{21}+b_{21}&a_{22}+b_{22}\end{vmatrix}$, $\therefore$ WE HAVE

$\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}+\begin{vmatrix}a_{11}&a_{12}\\b_{21}&b_{22}\end{vmatrix}$

det (A) + de...

4) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}$, ∴ WE HAVE

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{vmatrix}$$

(2) RESULT:
(ADDITION RULE)

LET $\boxed{A}$, $\boxed{B}$, AND $\boxed{C}$ BE $\boxed{n \times n}$ MATRICES THAT DIFFER ONLY IN A SINGLE ROW (SAY $r$th), AND ASSUME THAT THE $\boxed{r\text{th}}$ ROW OF $\boxed{C}$ CAN BE OBTAINED BY ADDING CORRESPONDING ENTRIES IN THE $\boxed{r\text{th}}$ ROWS OF $\boxed{A}$ AND $\boxed{B}$. THEN

$$\boxed{\det(C) = \det(A) + \det(B)}$$

IN THE LAST EXAMPLE

$$C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 4 \\ 5 & 3 & 6 & 2 \\ 3 & 5 & 7 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 4 \\ 3 & 8 & 4 & 1 \\ 3 & 5 & 7 & 1 \end{pmatrix}$$

$$\det(A) + \det(B) = \begin{vmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 4 \\ 8 & 11 & 10 & 3 \\ 3 & 5 & 7 & 1 \end{vmatrix}$$

$\boxed{5}$

THE SAME $\boxed{\text{RESULT}}$ HOLDS FOR $\boxed{\text{COLUMNS}}$.

(Follows from the fact that $\det(A) = \det(A^T)$)

③

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

IF $\boxed{B}$ IS THE MATRIX THAT

(3)

IF $\boxed{B}$ IS THE MATRIX THAT RESULTS WHEN $\boxed{\text{TWO ROWS}}$ OR $\boxed{\text{TWO COLUMNS}}$ OF $\boxed{A}$ ARE INTERCHANGED, THEN

$$\boxed{\det(B) = -\det(A)}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

CHECK: FOR 2X2 CASE

IF $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$a_{21}a_{12} - a_{22}a_{11}$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$= -(\underbrace{a_{12}a_{21} - a_{11}a_{22}})$$

$$= -\boxed{\det(B)}$$

$$\Rightarrow \boxed{\det(A) = -\det(B)}$$

$\det(B) = -\det(A)$

TRY THE FOLLOWING:

IF $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, LET $\det(A) = -7$

FIND $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} = \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = ?$

SOLUTION:

$$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} \quad \text{TAKING TRANSPOSE}$$

$$\underset{R_2 \leftrightarrow R_3}{=} - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\det(A) = -(-7) = 7$$

(4)

IF $\boxed{B}$ IS THE MATRIX THAT RESULTS WHEN $\boxed{A}$ $\boxed{\text{MULTIPLE}}$ OF $\boxed{\text{ONE}}$ ROW OF $\boxed{A}$ IS ADDED TO ANOTHER ROW OR WHEN A MULTIPLE OF ONE COLUMN IS ADDED TO ANOTHER COL-

7) UMN, THEN $\boxed{\det(A) = \det(B)} \to (*)$

CHECK: FOR 2X2 CASE

TAKE $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{bmatrix}$

$= \begin{bmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{bmatrix}$, $k$ IS ANY SCALAR, $k \neq 0$

$C_1 \to C_1 + kC_2$

DETAIL:

$$\det(B) = \begin{vmatrix} a_{11} + k a_{12} & a_{12} \\ a_{21} + k a_{22} & a_{22} \end{vmatrix}$$

$$= (a_{11} + k \cancel{a_{12}}) a_{22} - a_{12}(a_{21} + k \cancel{a_{22}})$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(A)$$

8

# ASSIGNMENT 3(a)

Exercise Set 2.3
Questions 3,4,5,6,7,8,9,11,13,16,21

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

row 1 expansion

column 2 expansion

$(-1)^{i+j}$

$$\begin{vmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} + (-1)h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}$$