Important facts and Theorems 2

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7.1.5

This is a square matrix!

Don't make the mistake of thinking that these are equivalent for rectangular matrices!

THEOREM 1.5.3

Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(a) *A is invertible.*

(b) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(c) *The reduced row-echelon form of A is I_n.*

(d) *A is expressible as a product of elementary matrices.*

$a \to b \to c \to d \to a$

Proof We shall prove the equivalence by establishing the chain of implications: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

$(a) \Rightarrow (b)$ Assume A is invertible and let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = \mathbf{0}$; thus $A\mathbf{x}_0 = \mathbf{0}$. Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{0}$, or $(A^{-1}A)\mathbf{x}_0 = \mathbf{0}$, or $I\mathbf{x}_0 = \mathbf{0}$, or $\mathbf{x}_0 = \mathbf{0}$. Thus, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$(b) \Rightarrow (c)$ Let $A\mathbf{x} = \mathbf{0}$ be the matrix form of the system

$$\begin{array}{cccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & a_{nn}x_n & = & 0 \end{array} \tag{1}$$

and assume that the system has only the trivial solution. If we solve by Gauss–Jordan elimination, then the system of equations corresponding to the reduced row-echelon form of the augmented matrix will be

$$
\begin{aligned}
x_1 \quad\quad\quad\quad &= 0 \\
x_2 \quad\quad &= 0 \\
\ddots \quad\quad & \\
x_n &= 0
\end{aligned}
\tag{2}
$$

Thus the augmented matrix

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0
\end{bmatrix}
$$

for 1 can be reduced to the augmented matrix

$$
\left[\begin{array}{ccccc|c}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{array}\right]
$$

for 2 by a sequence of elementary row operations. If we disregard the last column (of zeros) in each of these matrices, we can conclude that the reduced row-echelon form of A is I_n.

(c)* ⇒ *(d) Assume that the reduced row-echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$
E_k \cdots E_2 E_1 A = I_n \tag{3}
$$

By Theorem 1.5.2, $E_1, E_2, \ldots, E_k$ are invertible. Multiplying both sides of Equation 3 on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ we obtain

$$
A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \tag{4}
$$

By Theorem 1.5.2, this equation expresses A as a product of elementary matrices.

(d)* ⇒ *(a) If A is a product of elementary matrices, then from Theorems Theorem 1.4.6 and Theorem 1.5.2, the matrix A is a product of invertible matrices and hence is invertible.

$A = E_1 \cdots E_n$

$A^{-1} = E_n^{-1} \cdots E_1^{-1}$

■

$x_1 + x_2 = 1$

$x_1 + x_2 = 2$

THEOREM 1.6.1

inconsistent

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

Proof If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover, Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$ $\rightarrow A x_0 = \mathbf{0}$

If we now let k be any scalar, then

$$A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0)$$
$$= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

$A(x_1 + kx_0) = b$

$x_1 + kx_0$ is a sol. for all $k \in \mathbb{R}$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. ■

THEOREM 1.6.3

Let A be a square matrix.

(a) *If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.*

(b) *If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.*

We shall prove part (*a*) and leave part (*b*) as an exercise.

Proof (a) Assume that $BA = I$. If we can show that A is invertible, the proof can be completed by multiplying $BA = I$ on both sides by A^{-1} to obtain

As in 1.5.3 above, part (b)

$$BAA^{-1} = IA^{-1} \quad \text{or} \quad BI = IA^{-1} \quad \text{or} \quad B = A^{-1}$$

To show that A is invertible, it suffices to show that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (see Theorem 3). Let $\mathbf{x}_0$ be any solution of this system. If we multiply both sides of $A\mathbf{x}_0 = \mathbf{0}$ on the left by B, we obtain $BA\mathbf{x}_0 = B\mathbf{0}$ or $I\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_0 = \mathbf{0}$. Thus, the system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. ■

Proof (b)

We cannot use the same trick as part (a), because multiplying $Ax_0 = 0$ on the right by B gives

$(Ax_0)B = 0B$

Now, if Ax_0 is a column, then unless B is a row, the product on the LHS is not defined!

[Note: DO NOT MAKE THE ERROR OF JUST THINKING THAT THE PROOF FOR (a) WORKS HERE. THAT WAS REACHED ASSUMING $BA=I$. HERE WE HAVE $AB=I$]

OK. *Lets see if we can show B^{-1} exists. Let x_0 be any solution to $Bx=0$.

So $Bx_0 = 0 \Rightarrow ABx_0 = 0 \Rightarrow Ix_0 = 0 \Rightarrow x_0 = 0$ (unique)

$\therefore B^{-1}$ exists

Now take $AB = I$ (given)

$AB B^{-1} = I B^{-1} \Rightarrow AI = B^{-1} \Rightarrow A = B^{-1} \Rightarrow (A)^{-1} = (B^{-1})^{-1}$

$= \boxed{B = A^{-1}}$

THEOREM 1.6.4

Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent.

(a) *A is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row-echelon form of A is* I_n.

(d) *A is expressible as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* ***b***.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* ***b***.

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[r] & c \arrow[d] \\ f \arrow[r] & e \arrow[ul] & d \arrow[ull] \end{tikzcd}$$

$Ax = b$

$A^{-1}Ax = A^{-1}b$

$\boxed{x = A^{-1}b}$

Proof Since we proved in Theorem 3 that (*a*), (*b*), (*c*), and (*d*) are equivalent, it will be sufficient to prove that (*a*) $\rightleftharpoons$ (*f*) $\rightleftharpoons$ (*e*) $\rightleftharpoons$ (*a*).

(*a*) ⇒ (*f*) This was already proved in Theorem 1.6.2.

(*f*) ⇒ (*e*) This is self-evident: If $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix ***b***, then $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix ***b***.

(*e*) ⇒ (*a*) If the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix ***b***, then in particular, the systems

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow n\times 1, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow n\times 1, \ldots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \leftarrow n\times 1 \qquad \begin{bmatrix} 3 \\ 0 \\ 2 \\ \vdots \\ 5 \end{bmatrix}$$

i.e solutions exist

are consistent. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]$$

$A[c_1|c_2|\cdots|c_n]$

As discussed in Section 1.3, the successive columns of the product AC will be

$$A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n$$

$= [Ac_1 | Ac_2 \cdots | Ac_n]$

Thus

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

i.e solutions exist

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$n \times 1$ $n \times 1$

$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ 5 \end{bmatrix}$

are consistent. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]$$

$A[C_1|C_2|\cdots|C_n]$

As discussed in Section 1.3, the successive columns of the product AC will be

$$\boxed{A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n}$$

$=[AC_1|AC_2|\cdots|AC_n]$

Thus

$$AC = [A\mathbf{x}_1 | A\mathbf{x}_2 | \cdots | A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

$AC = I$

By part (*b*) of Theorem 1.6.3, it follows that $\boxed{C = A^{-1}}$. Thus, A is invertible.