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Linear Algebra – Math 205
Exercise Set of Lect 13
(SPRING 2023)
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Homework 8: Exercise Set 5.2 Solution

Question 01

Use Theorem 5.2.1 to determine which of the following are subspaces of R^3.
(a) All vectors of the form $(a, 0, 0)$.
Solution: Using Theorem 5.2.1
Let $u = (a, 0, 0)$ and $v = (b, 0, 0)$

$$u + v = (a, 0, 0) + (b, 0, 0) = (a + b, 0, 0) = (c, 0, 0)$$

$$ku = k(a, 0, 0) = (ka, 0, 0) = (d, 0, 0)$$

Hence, it is a vector space.
(b) All vectors of the form $(a, 1, 1)$.
Solution: Using Theorem 5.2.1
$u = (a, 1, 1)$ and $v = (b, 1, 1)$

$$u + v = (a, 1, 1) + (b, 1, 1) = (a + b, 2, 2) = (c, 2, 2)$$

Hence, it is not a vector space.
(c) All vectors of the form (a, b, c) , where $b = a + c$.
Solution: Using Theorem 5.2.1
Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$

$$u + v = (a_1, b_1, c_1) + (a_2, b_2, c_2) \Rightarrow (b_1 + b_2) = (a_1 + a_2) + (c_1 + c_2) \Rightarrow b_3 = a_3 + c_3$$

$$ku = k(a, b, c) \Rightarrow k(b = a + c) \Rightarrow kb = ka + kc$$

Hence, it is a vector space.
(d) All vectors of the form (a, b, c) , where $b = a + c + 1$.
Solution: Using Theorem 5.2.1
Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$

$$u+v = (a_1, b_1, c_1)+(a_2, b_2, c_2) \Rightarrow (b_1+b_2) = (a_1+a_2)+(c_1+c_2)+(1+1) \Rightarrow b_3 = a_3+c_3+2$$

Hence, it is not a vector space.
(e) All vectors of the form $(a, b, 0)$.
Solution: Using Theorem 5.2.1
Let $u = (a_1, b_1, 0)$ and $v = (a_2, b_2, 0)$

$$u + v = (a_1, b_1, 0) + (a_2, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0) = (a_3, b_3, 0)$$

$$ku = k(a_1, b_1, 0) = (ka_1, kb_1, 0) = (a_4, b_4, 0)$$

Hence, it is a vector space.

Question 02

2. Use Theorem 5.2.1 to determine which of the following are subspaces of $M_{2\times 2}$
(a) All 2×2 matrices with integer entries
Solution: To be a subspace it needs to be closed under scalar multiplication (and addition); and the set of 2×2 matrices with integer entries is not. For example, take any 2×2 matrix with non-zero integer entries and multiply it by 1/2; the resulting matrix will not be a matrix with integer entries and so it will not be in our set. Mathematically,

$$\frac{1}{2}u = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

(b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

Solution: Let $u = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $v = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ where $a+b+c+d=0$ and $a_1+b_1+c_1+d_1=0$, so by using Theorem 5.2.1

$$u+v = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} a+a_1 & b+b_1 \\ c+c_1 & d+d_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

where $(a+a_1)+(b+b_1)+(c+c_1)+(d+d_1) = (a+b+c+d)+(a_1+b_1+c_1+d_1) = 0+0 = 0$

$$ku = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

where $(ka)+(kb)+(kc)+(kd) = k(a+b+c+d) = k(0) = 0$. Hence it is a subspace.
(c) All 2×2 matrices such that $det(A) = 0$. It is a subspace.
Solution: Let $u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, hence $det(u) = det(v) = 0$ but $det(u+v) = 1$. Hence it is not a subspace.
(d) all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

Solution: Let $u = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $v = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$. It is a subspace, it is closed under addition and scalar multiplication , $u+v$ and ku.

$$u+v = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} a+a_1 & b+b_1 \\ 0 & d+d_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$$

$$ku = k\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kd \end{bmatrix}$$

(e) all matrices of the form $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$
Solution: It is not a subspace, it is not closed under scalar multiplication when k is negative.

Question 05

Use Theorem 5.2.1 to determine which of the following are subspaces of M_{nn}.
(a) all $n \times n$ matrices A such that $tr(A) = 0$
Solution: It is a subspace, one can show the following conditions.
Trace of a zero matrix is zero, so zero matrix must belongs here.

Any two matrices A_1, A_2, it is closed under addition. $A_1 + A_2 \Rightarrow tr(A_1 + A_2) = tr(A_1) + tr(A_2) = 0 + 0 = 0$. Addition is matrices is component-wise.
Scalar multiplication is also closed since $kA_1 \Rightarrow tr(kA_1) = 0$, $k(tr(A_1)) = k0 = 0$
(b) all $n \times n$ matrices A such that $A^T = -A$
Solution:

We will prove that it is a subspace. The zero vector 0 is the space, and it is skew-symmetric because $\mathbf{0}^T = \mathbf{0} = -\mathbf{0}$ Thus it is not empty set.

For condition, take arbitrary elements A, B. The matrices A, B are skew-symmetric, namely, we have $A^T = -A$ and $B^T = -B$. We show that $A + B$ belongs here. , or equivalently we show that the matrix A+B is skew-symmetric.

We have

$$(A + B)^T = A^T + B^T = -A + (-B) = -(A + B)$$

. Therefore the matrix A+B is skew-symmetric and condition 2 is met.

To prove the last condition, We show that kA is skew-symmetric.

$$(kA)^T = kA^T = k(-A) = -kA$$

. Hence kA is skew-symmetric.
(c) All $n \times n$ matrices A such that the linear system $Ax = 0$ has only the trivial solution.
Solution: It is not a subspace.
Since $Ax = 0$ has only trivial solution implies A^{-1} exists.
Let $u = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}$ are both invert-able but $u + v$ is not invert-able.
(d) All $n \times n$ matrices A such that $BA = AB$ for a fixed $n \times n$ matrix B
Solution: It is a subspace.
Zero matrix belongs to the subspace.

$$B\mathbf{0} = \mathbf{0}B = \mathbf{0}$$

Closed under addition

$$BA_1 + BA_2 = A_1B + A_2B \Rightarrow B(A_1 + A_2) = (A_1 + A_2)B \Rightarrow BA_3$$

Closed under scalar multiplication.

$$B(kA_1) = (kA_1)B = k(AB)$$