

Lecture 22

Sunday, April 10, 2022 3:04 PM

1)

LINEAR
ALGEBRA

LECTURE 22

RESULT:

IF A IS AN ORTHOGONAL MATRIX THEN
 $\det(A) = \pm 1$

PROOF:

$$\therefore \bar{A}^T = A^T$$

$$\Rightarrow \det(\bar{A}^T) = \det(A^T)$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A^T) = \det(A)$$

$$\Rightarrow \frac{1}{\det(A)} = \det(A)$$

$$\Rightarrow 1 = [\det(A)]^2$$

$$\Rightarrow \boxed{\det(A) = \pm 1}$$

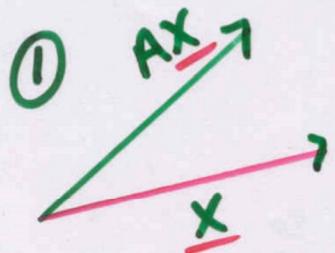
NOTE: IF $\bar{A}^T = A^T$ AND
 $\det(A) = 1$ THEN A
IS CALLED PROPER ORTHOGONAL MATRIX.

2

EIGENVALUES / EIGENVECTORS:

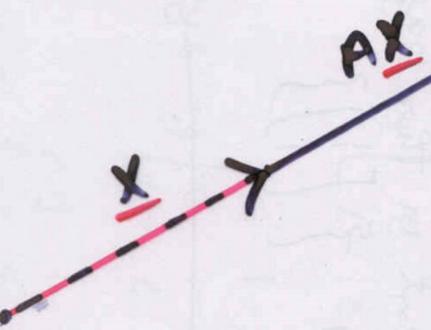
NOTE: IF A IS AN $n \times n$ MATRIX AND $X \neq 0$ IS A COLUMN VECTOR OF ORDER $n \times 1$, THEN $AX = b$ IS ALSO A COLUMN VECTOR OF ORDER $n \times 1$. BUT IS THERE ANY GEOMETRIC RELATIONSHIP BETWEEN X AND AX ?

SEE THE FOLLOWING FIGURES:



} **NO GENERAL GEOMETRIC RELATIONSHIP BETWEEN X AND AX .**

②



} **X AND AX ARE SCALAR MULTIPLES OF EACH OTHER.**

 **DEFINITION:** IF A IS AN $n \times n$

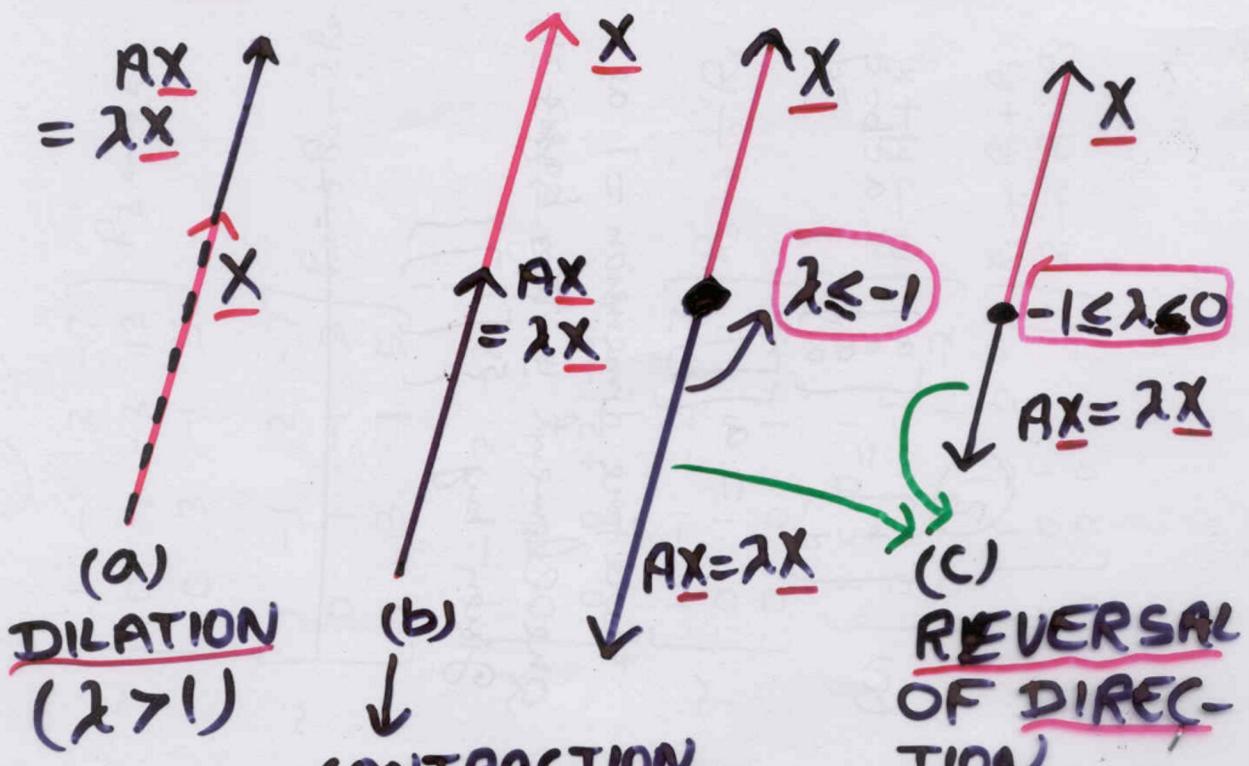
MATRIX, THEN A **NONZERO** VECTOR X IS CALLED AN EIGENVECTOR OF A IF AX IS A SCALAR MULTIPLE OF X .

OF X .

3) THAT IS, $\underline{AX} = \lambda \underline{X}$ FOR SOME SCALAR λ . THE SCALAR λ IS CALLED AN EIGENVALUE OF A , AND \underline{X} IS SAID TO BE AN EIGENVECTOR OF A CORRESPONDING TO λ .

GEOMETRIC INTERPRETATION

IF $\underline{AX} = \lambda \underline{X}$, MULTIPLICATION BY A DILATES \underline{X} , CONTRACTS \underline{X} , OR REVERSES THE DIRECTION OF \underline{X} , DEPENDING ON THE VALUE OF λ (AS SHOWN BELOW)



(2 > 1)

↓
CONTRACTION
(0 < λ < 1)

OF DIRECTION
(λ < 0)

4

TWO PROBLEMS

HOW TO FIND
EIGENVALUES ?

HOW TO
FIND
EIGENVECTORS ?

METHOD TO FIND EIGENVALUES

$\therefore A\underline{x} = \lambda \underline{x} \Rightarrow A\underline{x} = \lambda I \underline{x} \quad (1)$,
WHERE I IS THE IDENTITY
MATRIX

$$(1) \Rightarrow (A - \lambda I)\underline{x} = \underline{0} \quad (2), \underline{x} \neq \underline{0}$$

SINCE EIGENVECTOR IS A NONZERO VECTOR THEREFORE
FOR λ TO BE AN EIGENVALUE
OF A , NONZERO (NONTRIVIAL)
SOLUTIONS OF (2) EXIST IF

$$\det(A - \lambda I) = 0 \rightarrow (*)$$

(*) IS CALLED THE CHARACTERISTIC EQUATION OF A .

(*) IS CALLED THE CHARACTERISTIC EQUATION OF A .
AFTER SOLVING (*), WE GET
THE EIGENVALUES (λ 's) OF A .

(5)

EXAMPLE:FIND THE EIGENVALUES

(5)

OF

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

SOLUTION:

CONSIDER

$$A - \lambda I$$

$$= \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \quad \text{NOTE THIS STEP}$$

$A - \lambda I$ IS OBTAINED BY SUBTRACTING λ FROM THE DIAGONAL ENTRIES OF A .

THE CHARACTERISTIC EQUATION OF A IS

$$\det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

6

6

$$\Rightarrow (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0 \rightarrow ①$$

NOTE: THIS IS AN EQUATION OF

DEGREE 3 = ORDER OF A.

THEREFORE NO. OF EIGENVALUES OF A ARE = ORDER OF A.

$$① \Rightarrow (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[(4-\lambda)(1-\lambda) + 2] = 0$$

↳ NOTE THIS STEP

$$\Rightarrow 1-\lambda = 0 \Rightarrow \lambda = 1$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 2, \lambda = 3$$

OR $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

ARE THE EIGENVALUES OF A.

NOTE: FOR AN $n \times n$ MATRIX A, THE CHARACTERISTIC EQUATION $\det(A - \lambda I) = 0$ IS OF DEGREE n AND NUMBER OF

EIGENVALUES OF $A = n$ (A FEW MAY BE REPEATED). (7)

HOW TO FIND EIGENVECTORS OF A ?

EXAMPLE: FIND THE EIGENVECTORS OF $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

SOLUTION: IF $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ IS AN EIGENVECTOR OF A CORRESPONDING TO λ THEN WE HAVE

$$\underline{AX} = \lambda \underline{x} = \lambda \underline{IX}$$
$$\Rightarrow (A - \lambda I) \underline{x} = \underline{0} \quad \rightarrow (*)$$

THEREFORE IN ORDER TO OBTAIN THE EIGENVECTORS WE HAVE TO SOLVE (*)

$$\therefore (*) \Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

NOW WE HAVE TO OBTAIN THE EIGENVECTORS CORRESPONDING TO $\lambda = 1, 2, \text{ AND } 3$. (ALREADY OBTAINED)

$\lambda = 1, 2, \text{ AND } 3$. (ALREADY OBTAINED)

[8]

$$\Rightarrow \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow ①$$

STEP (1) :- EIGENVECTOR CORRESPONDING TO $\lambda=1$

PUTTING $\lambda=1$ IN ①

$$\Rightarrow \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 + x_3 = 0, -2x_1 = 0$$

$$\Rightarrow x_1 = x_3 = 0, x_2 = t, \therefore X \neq 0$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, t \neq 0$$

\therefore FOR $\lambda=1$, EIGENVECTOR IS $\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$

OR ANY OF ITS NONZERO
MULTIPLE.

CHECK:

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\overset{\textcolor{green}{\uparrow}}{A} \underset{\textcolor{red}{\underline{x}}}{X} = \lambda \underset{\textcolor{red}{\underline{x}}}{X}$$

STEP (2)

FOR $\lambda = 2$

⑨

$$\begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_3 = 0 \Rightarrow x_3 = -2x_1$$

$$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1$$

$$\downarrow -2x_1 - x_3 = 0$$

SAME

FOR $x_1 = t, x_2 = -2t, x_3 = -2t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ -2t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, t \neq 0$$

(10)

$\therefore \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ OR ANY OF ITS \downarrow MULTIPLES
 \downarrow NONZERO
 ARE EIGENVECTORS OF
A CORRESPONDING TO $\lambda = 2$

STEP (3) FINALLY FOR
 $\lambda = 3$, EIGENVECTOR

$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ OR ANY OF ITS
 \downarrow MULTIPLES (CHECK)
 \downarrow NONZERO

NOTE:

$$(1) \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$(2) \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

\uparrow \uparrow
 $\text{AX} = \lambda X$