Exercise Set 1.1 Solution

Question 1

Which of the following are linear equations in x_1, x_2, and x_3?

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$.
Solution: Each variable x_1, x_2 and x_3 has power one. Hence it is linear.

(b) $x_1 + 3x_2 + x_1x_3 = 2$.
Solution: First two variable x_1 and x_2 have power one but third variable is a product of two variables x_1x_3 or one can say its collective power is 2. Hence it is not linear.

(c) $x_1 = -7x_2 + 3x_3$.
Solution: Each variable x_1, x_2 and x_3 has power one. Hence it is linear.

(d) $x_1^{-2} + x_2 + 8x_3 = 5$.
Solution: Two variables x_2 and x_2 have power one but first variable has power -2. Hence it is not linear.

(e) $x_1^{\frac{3}{5}} - 2x_2 + x_3 = 4$.
Solution: Two variables x_2 and x_2 have power one but first variable has power $\frac{3}{5}$. Hence it is not linear.

(f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{\frac{1}{3}}$.
Solution: Each variable x_1, x_2 and x_3 has power one. Hence it is linear.

Question 2

Given that k is a constant, which of the following are linear equations?

(a) $x_1 - x_2 - x_3 = k$.
Solution: Each variable x_1, x_2 and x_3 has power one. Hence it is linear.

(b) $kx_1 + -\frac{1}{2}kx_2 = 9$.
Solution: Each variable x_1 and x_2 has power one. Hence it is linear.

(c) $2^k x_1 + 7x_2 - x_3 = 0$.
Solution: Each variable x_1, x_2 and x_3 has power one. Hence it is linear.

Question 3

Find the solution set of each of the following linear equations.

(a) $7x - 5y = 3$.
Solution: We can assign an arbitrary value to x and solve for y, or choose an arbitrary value for y and solve for x.

We solve for x

$7x - 5y = 3 \implies 7x = 3 + 5y$.

Let $y = t$, where t is an arbitrary number or parameter. So the solution is $7x = 3 + 5t, y = t$.

(b) $3x_1 - 5x_2 + 4x_3 = 7$.

Solution: We solve for x_1

$3x_1 - 5x_2 + 4x_3 = 7 \implies 3x_1 = 7 + 5x_2 - 4x_3 \implies x_1 = \frac{7}{3} + \frac{5}{3}x_2 - \frac{4}{3}x_3$.

Let $x_2 = t$ and $x_3 = s$, where t and s are an arbitrary numbers or parameters. So the solution is $x_1 = \frac{7}{3} + \frac{5}{3}t - \frac{4}{3}s, x_2 = t, x_3 = s$.

(c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$.

Solution: We solve for x_1

$-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1 \implies -8x_1 = 1 - 2x_2 + 5x_3 - 6x_4 \implies x_1 = -\frac{1}{8} + \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4$.

Let $x_2 = t, x_3 = s$ and $x_4 = u$, where t, s and u are an arbitrary numbers or parameters. So the solution is $x_1 = -\frac{1}{8} - \frac{1}{4}x_2 - \frac{5}{8}x_3 + \frac{3}{4}x_4, x_2 = t, x_3 = s, x_4 = u$.

(d) $3v - 8w + 2x - y + 4z = 0$.

Solution: We solve for x

$3v - 8w + 2x - y + 4z = 0 \implies 2x = y - 4z - 3v + 8w \implies x = \frac{1}{2}y - 2z - \frac{3}{2}v + 4w$.

Let $y = a, z = b, v = c$ and $w = d$, where a, b, c and d are an arbitrary numbers or parameters. So the solution is $x = \frac{1}{2}a - 2b - \frac{3}{2}c + 4d, y = a, z = b, v = c, w = d$.

Question 4

Find the augmented matrix for each of the following systems of linear equations.

(c)
$$\begin{aligned} x_1 + 2x_2 \qquad\quad - x_4 + x_5 &= 1 \\ 3x_2 + x_3 \qquad - x_5 &= 2 \\ x_3 + 7x_4 \qquad &= 1 \end{aligned}$$

Solution: We can write the above system as follow

$$\begin{aligned} x_1 + 2x_2 + 0x_3 - x_4 + x_5 &= 1 \\ 0x_1 + 3x_2 + x_3 + 0x_4 - x_5 &= 2 \\ 0x_1 + 0x_2 + x_3 + 7x_4 + 0x_5 &= 1 \end{aligned}$$

Here its augmented matrix can be written as follows

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 3 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 7 & 0 & 1 \end{bmatrix}$$

(d) $x_1 = 1$
$x_2 = 2$
$x_3 = 3$

Solution: We can write the above system as follow

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + x_2 + 0x_3 &= 2 \\ 0x_1 + 0x_2 + x_3 &= 3 \end{aligned}$$

Here its augmented matrix can be written as follows

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Question 5

Find a system of linear equations corresponding to the augmented matrix.

(c) $\begin{bmatrix} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$

Solution: We can write the above system as follow

$$\begin{aligned} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ 1x_1 + 2x_2 + 4x_3 + 0x_4 &= 1 \end{aligned}$$

Here its system of linear equations can be written as follows

$$\begin{aligned} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ x_1 + 2x_2 + 4x_3 \qquad &= 1 \end{aligned}$$

Question 7

Solution: Since each of the three given points($(x_1, y_1), (x_2, y_2)$, and (x_3, y_3)) passes through the curve ($y = ax^2 + bx + c$), so it must satisfy the equation of the curve. We have $l_1 = ax_1^2 + bx_1 + c = y_1, l_2 = ax_2^2 + bx_2 + c = y_2$ and $l_3 = ax_3^2 + bx_3 + c = y_3$. Now we take these equations as a system of linear in terms of a, b, c.

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3 \end{aligned}$$

One can represent above system of equations into matrix form

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Here all x_1, x_2, x_3 are the component of coefficient matrix and a, b, c are the unknowns component of variable vector and y_1, y_2, y_3 are the components of resultant vector.

If we consider this to be a system of equations, the augmented matrix is clearly the one(three equations with three unknowns).

Here is the augmented matrix

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

Note: Keep in mind $x_1, y_1, x_2, y_2, x_3, y_3$ are any fixed real numbers and a, b, c are the unknown and the solution of system of linear equations.

Question 8

Solution: (**if there is at least one solution of the system, it is called consistent**)

$$\begin{aligned} x + y + 2z &= a \\ x + 0y + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix}$$

Apply $R3 - R1$ (Row 3 minus Row 1)

$$\begin{aligned} x + y + 2z &= a \\ 0x + -y - z &= b \\ x + 0y + z &= c - a \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b \\ 1 & 0 & 1 & c - a \end{bmatrix}$$

Apply $R3 - R2$ (Row 3 minus Row 2)

$$\begin{aligned} x + y + 2z &= a \\ 0x + -y - z &= b \\ 0x + 0y + 0z &= c - a - b \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b \\ 0 & 0 & 0 & c - a - b \end{bmatrix}$$

Hence this system is consistent if $c = a + b$, which implies infinitely many solution.

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 9

Rewriting the equations as:

$$x_1 = -kx_2 + c$$
$$x_1 = -lx_2 + d$$

For $x_2 = 0, (1)$ gives $x_1 = c$. The pair $(0, c)$ also satisfies (2):

$$c = 0 + d$$

$$\text{or } c = d$$

Therefore

$$x_1 + kx_2 = x_1 + lx_2$$
$$\text{or } k = l$$

Since $k = l$ and $c = d$, the equations are identical.