# Probabilistic Graph Models CS-452

## Assignment 02

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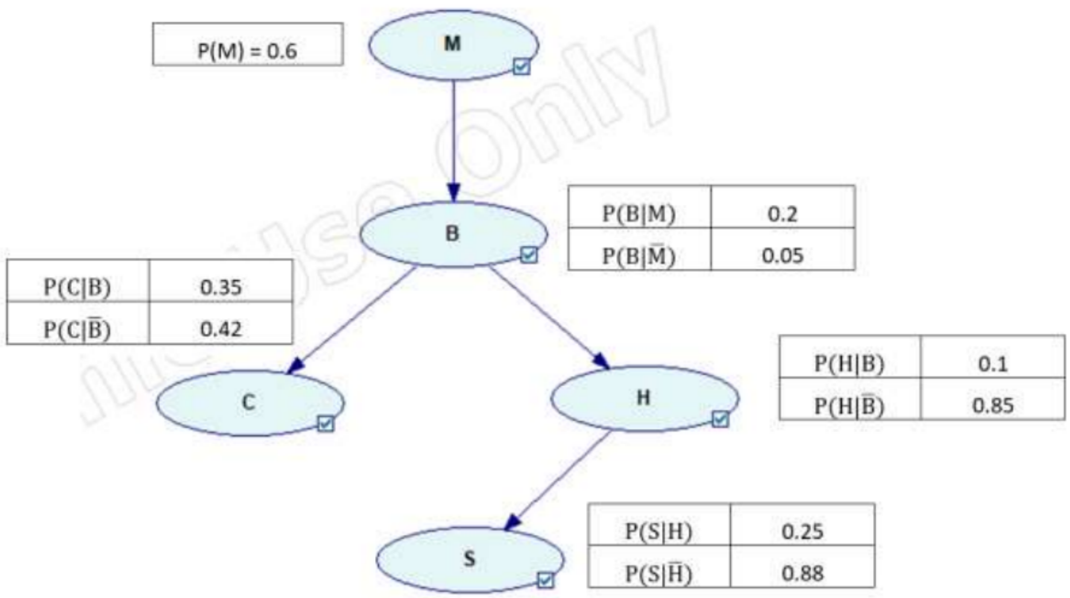
| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 5 | 15 | 15 | 15 | 50 |
| Score: | | | | | |

You will attempt Q1 - Q3 using the Bayesian Network given below:




## Q1 - [05 points] Apply Bayes theorem and the law of total probability to:

a) compute the marginal probabilities of all variables

b) given that there is evidence on ‘S’, compute posterior probabilities of all variables

*You have to show all intermediate steps of calculating probabilities.*

**Solution:** a) To compute the marginal probabilities, we will use the law of total probability.

P(M) = 0.6
P(B | M) = 0.2 P(B | M’) = 0.05
P(C | B) = 0.35 P(C | B’) = 0.42
P(H | B) = 0.1 P(H | B’) = 0.85
P(S | H) = 0.25 P(S | H’) = 0.88

P(M’) = 1 - 0.6 = 0.4
P(B’ | M) = 0.8 P(B’ | M’) = 0.95
P(C’ | B) = 0.65 P(C’ | B’) = 0.58
P(H’ | B) = 0.9 P(H’ | B’) = 0.15
P(S’ | H) = 0.75 P(S’ | H’) = 0.12

P(M) = 0.6

P(B) = P(B | M)P(M) + P(B | M')P(M') = 0.2*0.6 + 0.05*0.4 = 0.14
P(B') = 1 - P(B) = 1 - 0.14 = 0.86

P(C) = P(C | B)P(B) + P(C | B')P(B') = 0.35*0.14 + 0.42*0.86 = 0.4102
P(C') = 1 - P(C) = 1 - 0.4102 = 0.5898

P(H) = P(H | B)P(B) + P(H | B')P(B') = 0.1*0.14 + 0.85*0.86 = 0.745
P(H') = 1 - P(H) = 1 - 0.745 = 0.255

P(S) = P(S | H)P(H) + P(S | H')P(H') = 0.25*0.745 + 0.88*0.255 = 0.41065
P(S') = 1 - P(S) = 1 - 0.41065 = 0.58935

Then the marginal probabilities are as so:

| **Variable** | **Probability** | | **Variable** | **Probability** |
|---|---|---|---|---|
| P(M) | 0.6 | | P(M') | 0.4 |
| P(B) | 0.14 | | P(B') | 0.86 |
| P(C) | 0.4102 | | P(C') | 0.5898 |
| P(H) | 0.745 | | P(H') | 0.255 |
| P(S) | 0.41065 | | P(S') | 0.58935 |

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b) To compute the posterior probabilities, we will use Bayes theorem.

P*(H) = P(H | S)P(S) + P(H | S')P(S')
P(S') = 0, hence the other term becomes 0
P(H | S) = P(S | H)P(H) / P(S) = 0.25*0.745 / 0.41065 = 0.45356
P*(H) = 0.45356(1) + 0 = 0.45356
P*(H) = 0.45356    P*(H') = 1 - 0.45356 = 0.54644

P*(B) = P(B | H)P*(H) + P(B | H')P*(H')
P(B | H) = P(H | B)P(B) / P(H) = 0.1*0.14 / 0.745 = 0.01879
P(B | H') = P(H' | B)P(B) / P(H') = 0.9*0.14 / 0.255 = 0.49412
P*(B) = 0.01879(0.45356) + 0.49412(0.54644) = 0.27853
P*(B) = 0.27853    P*(B') = 1 - 0.27853 = 0.72147

P*(C) = P(C | B)P*(B) + P(C | B')P*(B')
P(C | B) = 0.35    P(C | B') = 0.42
P*(C) = 0.35(0.27853) + 0.42(0.72147) = 0.40050
P*(C) = 0.40050    P*(C') = 1 - 0.40050 = 0.59950

P*(M) = P(M | B)P*(B) + P(M | B')P*(B')
P(M | B) = P(B | M)P(M) / P(B) = 0.2*0.6 / 0.14 = 0.85714
P(M | B') = P(B' | M)P(M) / P(B') = 0.8*0.6 / 0.86 = 0.55814
P*(M) = 0.85714(0.27853) + 0.55814(0.72147) = 0.64142
P*(M) = 0.64142    P*(M') = 1 - 0.64142 = 0.35858

Then the posterior probabilities, given evidence on 'S' are as so:

| Variable | Probability | | Variable | Probability |
|---|---|---|---|---|
| P*(M) | 0.64142 | | P*(M') | 0.35858 |
| P*(B) | 0.27853 | | P*(B') | 0.72147 |
| P*(C) | 0.40050 | | P*(C') | 0.59950 |
| P*(H) | 0.45356 | | P*(H') | 0.54644 |

## Q2 - [15 points] Apply Kim and Pearl's Message passing algorithm to:

a) compute the marginal probabilities of all variables

b) given that there is evidence on 'S', compute posterior probabilities of all variables

*You have to show all intermediate steps of passing $\pi$ and $\lambda$ messages and computing their values at each node.*

**Solution:** To apply Kim and Pearl's message passing algorithm, we first set all $\lambda$ values, $\lambda$ messages, and $\pi$ messages to 1.

Then, $\lambda(M) = \lambda(M') = \lambda(B) = \lambda(B') = \lambda(C) = \lambda(C') = \lambda(H) = \lambda(H') = \lambda(S) = \lambda(S') = 1$

The root node is initialized with $\pi(M)$ = P(M) = 0.6

a) We find the marginal probabilities of all variables as so:

$\pi(M)$ = 0.6 $\quad$ $\pi(M')$ = 1 - 0.6 = 0.4

We then send a $\pi$ message from M to B.

$\pi_B(M) = \pi(M) = 0.6 \qquad \pi_B(M') = \pi(M') = 0.4$
$\pi(B) = P(B \mid M)\pi_B(M) + P(B \mid M')\pi_B(M') = 0.2 * 0.6 + 0.05 * 0.4 = 0.14$
$\pi(B') = P(B' \mid M)\pi_B(M) + P(B' \mid M')\pi_B(M') = 0.8 * 0.6 + 0.95 * 0.4 = 0.86$
$P(B \mid \emptyset) = \dfrac{0.14\alpha}{0.14\alpha + 0.86\alpha} = 0.14 \qquad P(B' \mid \emptyset) = \dfrac{0.86\alpha}{0.14\alpha + 0.86\alpha} = 0.86$

We then send a $\pi$ message from B to C.

$\pi_C(B) = \pi(B)\lambda_C(B) = 0.14(1) = 0.14 \qquad \pi_C(B') = \pi(B')\lambda_C(B) = 0.86(1) = 0.86$
$\pi(C) = P(C \mid B)\pi_C(B) + P(C \mid B')\pi_C(B') = 0.35 * 0.14 + 0.42 * 0.86 = 0.4102$
$\pi(C') = P(C' \mid B)\pi_C(B) + P(C' \mid B')\pi_C(B') = 0.65 * 0.14 + 0.58 * 0.86 = 0.5898$
$P(C \mid \emptyset) = \dfrac{0.4102\alpha}{0.4102\alpha + 0.5898\alpha} = 0.4102 \qquad P(C' \mid \emptyset) = \dfrac{0.5898\alpha}{0.4102\alpha + 0.5898\alpha} = 0.5898$

We then send a $\pi$ message from B to H

$\pi_H(B) = \pi(B)\lambda_H(B) = 0.14(1) = 0.14 \qquad \pi_H(B') = \pi(B')\lambda_H(B) = 0.86(1) = 0.86$
$\pi(H) = P(H \mid B)\pi_H(B) + P(H \mid B')\pi_H(B') = 0.1 * 0.14 + 0.85 * 0.86 = 0.745$

$\pi(H') = P(H' \mid B)\pi_H(B) + P(H' \mid B')\pi_H(B') = 0.9 * 0.14 + 0.15 * 0.86 = 0.255$

$P(H \mid \emptyset) = \dfrac{0.745\alpha}{0.745\alpha + 0.255\alpha} = 0.745 \qquad P(H' \mid \emptyset) = \dfrac{0.255\alpha}{0.745\alpha + 0.255\alpha} = 0.255$

We then send a $\pi$ message from H to S

$\pi_S(H) = \pi(H)\lambda_S(H) = 0.745(1) = 0.745 \qquad \pi_S(H') = \pi(H')\lambda_S(H) = 0.255(1) = 0.255$

$\pi(S) = P(S \mid H)\pi_S(H) + P(S \mid H')\pi_S(H') = 0.25 * 0.745 + 0.85 * 0.255 = 0.41065$

$\pi(S') = P(S' \mid H)\pi_S(H) + P(S' \mid H')\pi_S(H') = 0.75 * 0.745 + 0.15 * 0.255 = 0.58935$

$P(S \mid \emptyset) = \dfrac{0.41065\alpha}{0.41065\alpha + 0.58935\alpha} = 0.41065 \qquad P(S' \mid \emptyset) = \dfrac{0.58935\alpha}{0.41065\alpha + 0.58935\alpha} = 0.58935$

Then the marginal probabilities are as follows:

| Variable | Marginal Probability |
|---|---|
| P(M) | 0.6 |
| P(B) | 0.14 |
| P(C) | 0.4102 |
| P(H) | 0.745 |
| P(S) | 0.41065 |

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b) Given that we have evidence on 'S', we can find the posterior probabilities as so:

$\lambda(S) = 1; \pi(S) = 1; P(S \mid \{s\}) = 1$

$\lambda(S') = 0; \pi(S') = 0; P(S \mid \{s\}) = 0$

We then send a $\lambda$ message from S to H

$\lambda_S(H) = P(S \mid H)\lambda(S) + P(S' \mid H)\lambda(S') = 0.25(1) + 0.75(0) = 0.25$

$\lambda_S(H') = P(S \mid H')\lambda(S) + P(S' \mid H')\lambda(S') = 0.88(1) + 0.12(0) = 0.88$

$\lambda(H) = \lambda_S(H) = 0.25 \qquad \lambda(H') = \lambda_S(H') = 0.88$

$P(H \mid \{s\}) = \alpha\lambda(H)\pi(H) = \alpha(0.25)(0.745) = 0.18625\alpha$

$P(H' \mid \{s\}) = \alpha\lambda(H')\pi(H') = \alpha(0.88)(0.255) = 0.2244\alpha$

$P(H \mid \{s\}) = \dfrac{0.18625\alpha}{0.18625\alpha + 0.2244\alpha} = 0.4535$

$P(H' \mid \{s\}) = \dfrac{0.2244\alpha}{0.18625\alpha + 0.2244\alpha} = 0.5464$

We send a $\lambda$ message from H to B

$\lambda_H(B) = P(H \mid B)\lambda(H) + P(H' \mid B)\lambda(H') = 0.1(0.25) + 0.9(0.88) = 0.817$

$\lambda_H(B') = P(H \mid B')\lambda(H) + P(H' \mid B')\lambda(H') = 0.85(0.25) + 0.15(0.88) = 0.3445$

$\lambda(B) = \lambda_H(B) = 0.817 \qquad \lambda(B') = \lambda_H(B') = 0.3445$

$P(B \mid \{H\}) = \alpha\lambda(B)\pi(B) = \alpha(0.817)(0.14) = 0.11438\alpha$

$P(B' \mid \{H\}) = \alpha\lambda(B')\pi(B') = \alpha(0.3445)(0.86) = 0.29657\alpha$

$P(B \mid \{H\}) = \dfrac{0.11438\alpha}{0.11438\alpha + 0.29657\alpha} = 0.27853$

$P(B' \mid \{H\}) = \dfrac{0.29657\alpha}{0.11438\alpha + 0.29657\alpha} = 0.72147$

We send a $\lambda$ message from B to M

$\lambda_B(M) = P(B \mid M)\lambda(B) + P(B' \mid M)\lambda(B') = 0.2(0.817) + 0.8(0.3445) = 0.439$
$\lambda_B(M') = P(B \mid M')\lambda(B) + P(B' \mid M')\lambda(B') = 0.05(0.817) + 0.95(0.3445) = 0.368125$
$\lambda(M) = \lambda_B(M) = 0.439 \quad \lambda(M') = \lambda_B(M') = 0.368125$
$P(M \mid \{B\}) = \alpha\lambda(M)\pi(M) = \alpha(0.439)(0.6) = 0.2634\alpha$
$P(M' \mid \{B\}) = \alpha\lambda(M')\pi(M') = \alpha(0.368125)(0.4) = 0.14725\alpha$
$P(M \mid \{B\}) = \dfrac{0.2634\alpha}{0.2634\alpha + 0.14725\alpha} = 0.6414$
$P(M' \mid \{B\}) = \dfrac{0.14725\alpha}{0.2634\alpha + 0.14725\alpha} = 0.3586$

We send a $\pi$ message from B to C

$\pi_C(B) = \pi(B)\lambda_H(B) = 0.14(0.817) = 0.11438$
$\pi_C(B') = \pi(B')\lambda_H(B') = 0.86(0.3445) = 0.29627$
$\pi(C) = P(C \mid B)\pi_C(B) + P(C \mid B')\pi_C(B') = 0.35(0.11438) + 0.42(0.29627) = 0.1644664$
$\pi(C') = P(C' \mid B)\pi_C(B) + P(C' \mid B')\pi_C(B') = 0.65(0.11438) + 0.58(0.29627) = 0.2461836$
$P(C \mid \{B\}) = \dfrac{0.1644664\alpha}{0.1644664\alpha + 0.2461836\alpha} = 0.40050$
$P(C' \mid \{B\}) = \dfrac{0.2461836\alpha}{0.1644664\alpha + 0.2461836\alpha} = 0.59950$

Then the posterior probabilities are as follows:

| Variable | Posterior Probability |
|---|---|
| P(M) | 0.6414 |
| P(B) | 0.2785 |
| P(C) | 0.4005 |
| P(H) | 0.4535 |

## Q3 - [15 points] Apply i) Logic sampling and ii) Likelihood weighting to:

a) compute the marginal probabilities of all variables

b) compute posterior probabilities of all variables given there is evidence on 'S'.

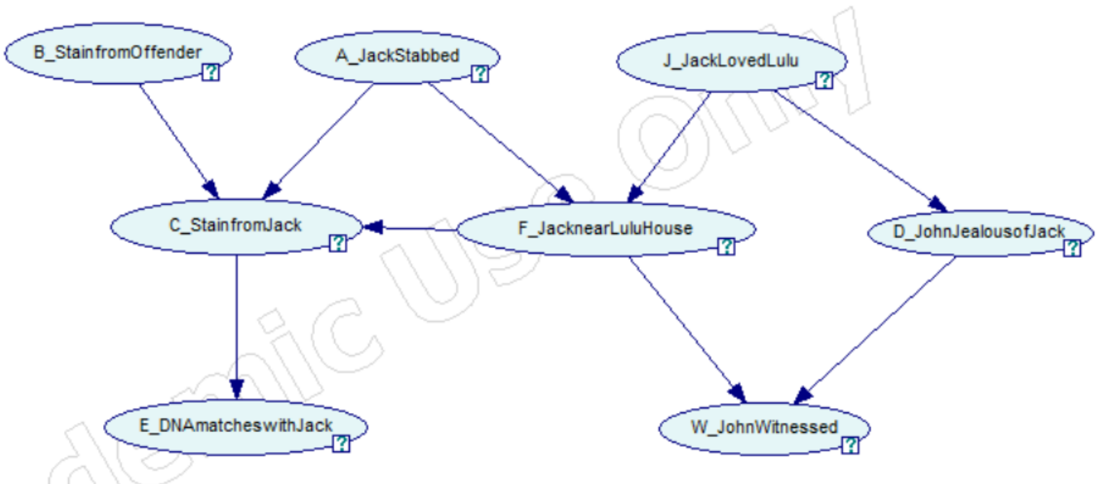
You should do this question in excel showing all calculations. Start with a sample size of 100 and increase it to 1000+ to see how probabilities based on sampling get closer to the actul probability as sample size increases.

**Solution:** The link to the excel sheet can be found `https://habibuniversity-my.sharepoint.com/:x:/g/personal/aa07190_st_habib_edu_pk/EdH8O2CrV3hGoMDSJdE_8mUB6S44d6kZ0PgMKTuYTlsp5A?e=lvM7dk`

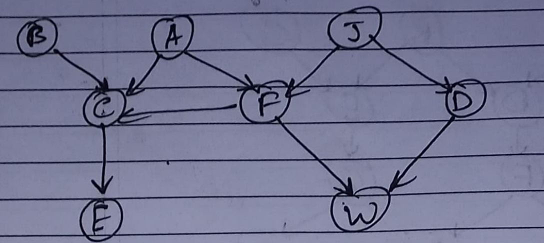
**Note: In the link, both parts (a) and (b) have been done on the same sheet for its respective method (Logic Sampling and Likelihood Weighting), with the marginal probabilities referring to part (a) and the posterior probability referring to part (b).*

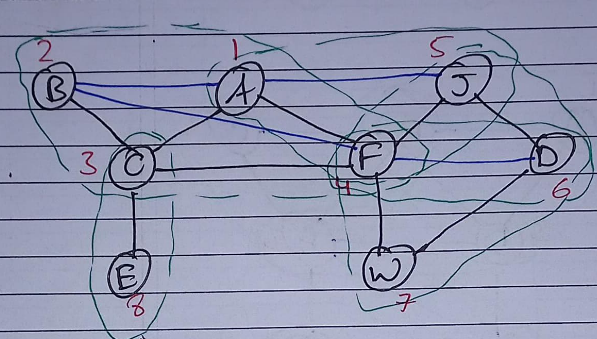
**Q4 - [15 points] Transform the Bayesian networks given in parts (a) to (c) into Junction Trees. Show intermediate steps of Junction tree algorithm and clearly show the final Junction tree.**

**a)**

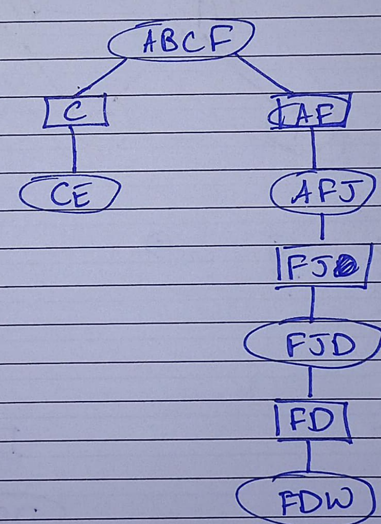


Q4) (a)

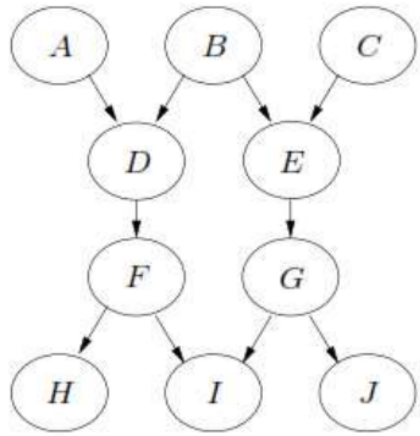


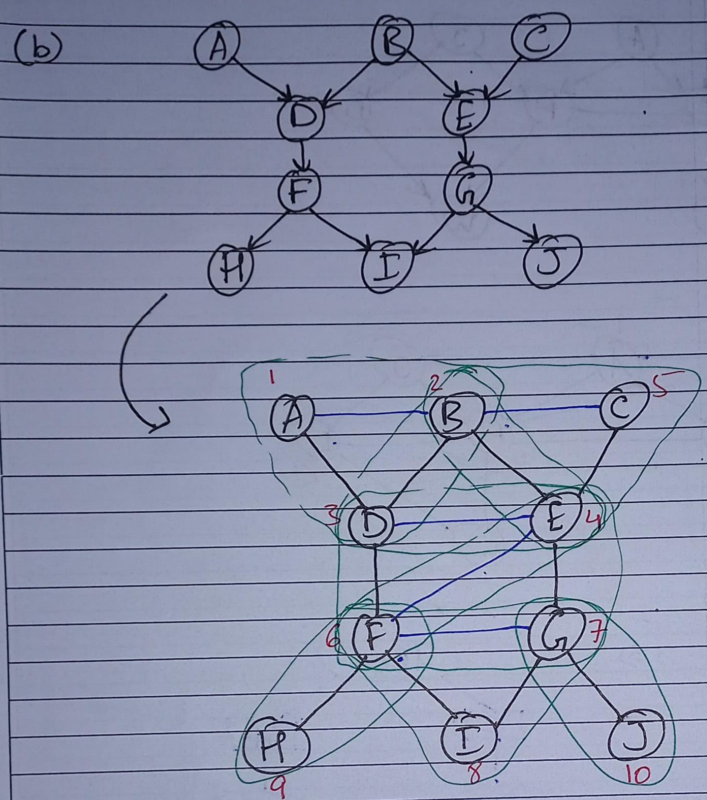

| | | |
|---|---|---|
| CE ABCF | 4 | ⇒ |
| AFJ | 3 | |
| FJD | 6 | |
| FD W | 7 | |



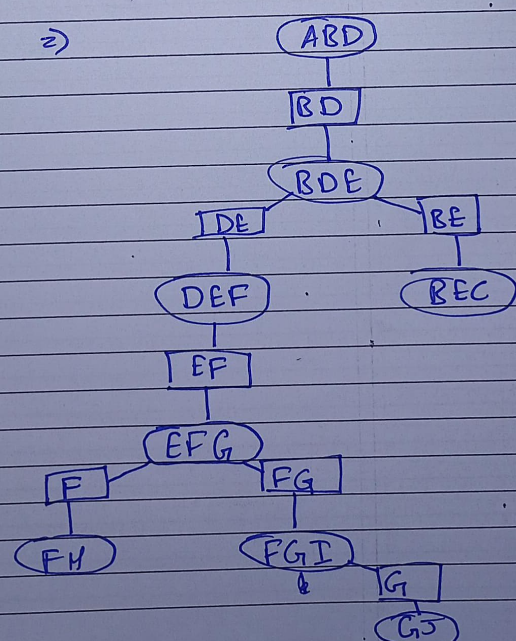
**b)**



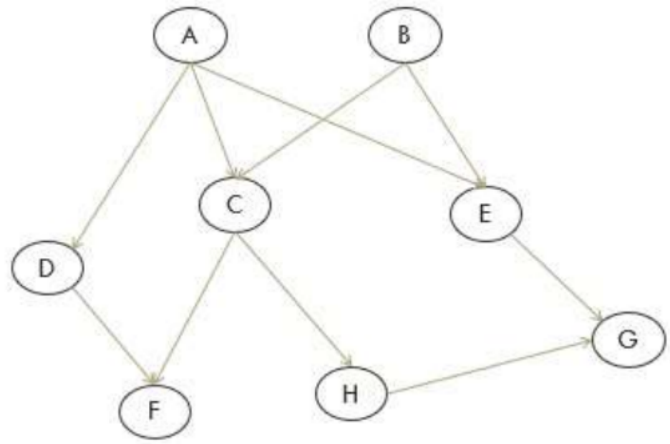
(b)



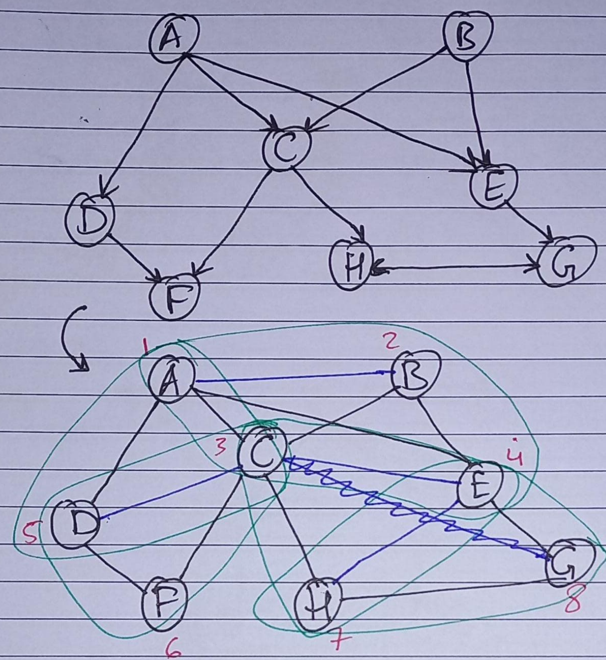

ABD 3 =>
BDE 4
BCE
DEF
EFG
FGI
FH
GJ



**c)**



(c)




| | |
|---|---|
| ABCE | 4 |
| ACD | 5 |
| DCF | 6 |
| CEH | 7 |
| EGH | 8 |

⇒