# Solution: MidTerm-2 (Spring 2023)

**Intro to Probability and Statistics - EE 354 / CE 361 / MATH 310**

**Allowed Time: 50 Minutes**

**Notes:**

1. To ensure partial credit, all answers must be supported by proper justification.
2. This is an open-book, open-notes exam. However, you are not allowed to consult internet-based resources, your peers, and classmates.

**Question 1 -** (9 points)

Rahim Bhai has the following flavors available for fries: Ketchup, Chilli Sauce, BBQ Sauce, Poodina Chatni, Raita, and Chaat Masala.

a) How many 3-flavored fries are possible?
b) If you randomly order 3-flavored fries, what is the probability of getting Raita in it?
c) If one of your friends randomly orders 4-flavored fries, what is the probability of getting Raita and BBQ Sauce in it?

**Solution:**

(a)

$$\binom{6}{3} = \frac{6!}{3!\,2!} = 20$$

(b)

Total number of 3-flavored fries $= \binom{6}{3} = 20$

Number of 3-flavored fries with Raita $= 1.\binom{5}{2} = 10$

Req. Probability $= \frac{10}{20} = 1/2$

(c)

$$\text{Req. Probability} = \frac{1.1.\binom{4}{2}}{\binom{6}{4}}$$

$$= \frac{6}{15} = 2/5$$

**Question 2 -** (5 points)
A project group consists of 4 students. What is the probability that no two students share a birthday? Assume that no student was born in a leap year. Also, assume that every student has an equal probability of being born on any day during the year.

**Solution:**

$$\text{Req Probability} = \frac{(365)(365-1)(365-2)(365-3)}{(365)^4}$$

**Question 3 -** (6 points)
From the Bernoulli, Binomial, Geometric, Uniform, and Poisson random variables, which one will be the most appropriate choice to model the following:

a) Number of games Karachi Kings will win in the 2024 edition of Pakistan Super League
b) Outcome of rolling a fair 6-sided die
c) Number of balls that will be bowled by a new player in international cricket before getting his/her first wicket.

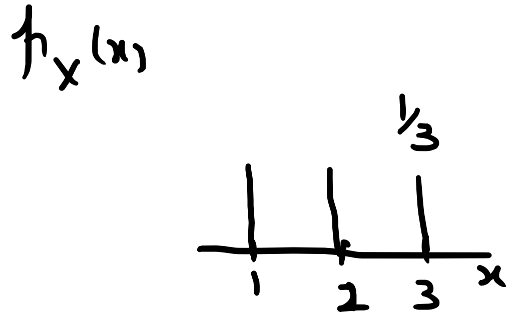
**Solution:**

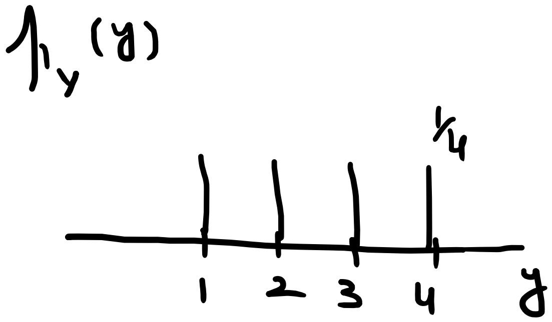
a) Binomial
b) Uniform
c) Geometric

**Question 4 -** (21 points)
Suppose you roll a fair 3-sided die and a fair 4-sided die simultaneously. Assume that the outcome of each die is independent of the other. Consider the following three random variables:
X = outcome of the fair 3-sided die
Y = outcome of the fair 4-sided die
Z = Y - X

a) (3 points) What is $E[X^3]$?
b) (3 points) What is $E[3Z]$?
c) (3 points) What is $E[3Z + 2]$?
d) (3 points) What is $Var(3Z)$?
e) (3 points) What is $Var(3Z + 2)$?
f) (3 points) What is $p_{X,Y}(x, y)$?
g) (3 points) What is $p_{X|Y}(x|2)$?

Solution:




$\mu_X = E[X] = 2$

$\mu_Y = E[Y] = 2.5$

$$E[X^2] = \sum_x x^2 p_X(x)$$

$$= 1^2\left(\frac{1}{3}\right) + 2^2\left(\frac{1}{3}\right) + 3^2\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{4}{3} + \frac{9}{3}$$

$$E[X^2] = \frac{14}{3} = 4.666$$

$$Var(X) = E[X^2] - \mu_X^2$$

$$= 4.666 - 4$$

$$Var(X) = 0.666$$

$$E[Y^2] = \sum_y y^2 p_Y(y)$$

$$= 1^2\left(\frac{1}{4}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{4}\right) + 4^2\left(\frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4}$$

$$= \frac{30}{4} = 7.5$$

$$Var(Y) = E[Y^2] - \mu_Y^2$$

$$= 7.5 - (2.5)^2$$

$$Var(Y) = 1.25$$

(a)

Using Expected Value Rule:

$$E[X^3] = \sum_x x^3 p_X(x)$$

$$= 1^3\left(\frac{1}{3}\right) + 2^3\left(\frac{1}{3}\right) + 3^3\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{8}{3} + \frac{27}{3}$$

$$= \frac{36}{3}$$

$$E[X^3] = 12$$ — Ans

(b)

$$E[3Z] = E[3Y - 3X]$$

$= 3E[Y] - 3E[X]$

$= 3(2.5) - 3(2)$

$= 1.5$ — Ans

(c)

$$E[3Z+2] = E[3Y - 3X + 2]$$

$$= 3E[Y] - 3E[X] + 2$$

$$= 3.5$$ — Ans

(d)

$$Var(3Z) = Var(3Y - 3X)$$

$$Var(2Z) = Var\{3Y + (-3)X\} \quad \text{— (1)}$$

X and Y are indefendent

$\Rightarrow -2X$ and $2Y$ are indefendent

$$\Rightarrow Var\{3Y + (-3)X\} = Var\{3Y\} + Var\{(-3)X\} \quad \text{— (2)}$$

Now

$$Var(3Y) = 3^2 \, Var\, Y$$

$$= 9(1.25) = 11.25$$

$$Var\{(-3)X\} = (-3)^2 \, Var\, X$$

$$= 9 \operatorname{Var} X$$

$$= 6$$

From (1) and (2):

$$\operatorname{Var}(3Z) = 11.25 + 6$$

$$= 17.25 \quad \text{— Ans}$$

(e)

$$\operatorname{Var}(3Z+2) = \operatorname{Var}(3Z)$$

$$= 17.25 \quad \text{— Ans}$$

(f)

Since X and Y are independent

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

| Y \ X | 1 | 2 | 3 |
|---|---|---|---|
| 4 | 1/12 | 1/12 | 1/12 |
| 3 | 1/12 | 1/12 | 1/12 |
| 2 | 1/12 | 1/12 | 1/12 |
| 1 | 1/12 | 1/12 | 1/12 |

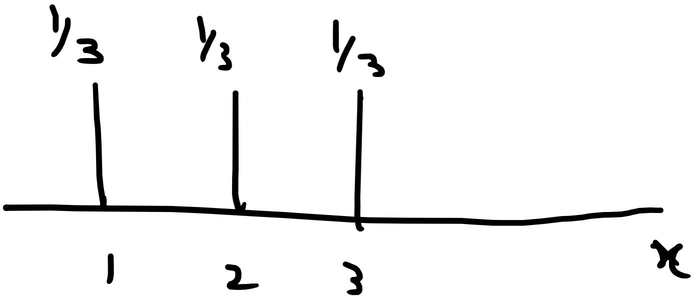
(g)

Since X and Y are independent

$$p_{X|Y}(x|y) = p_X(x)$$

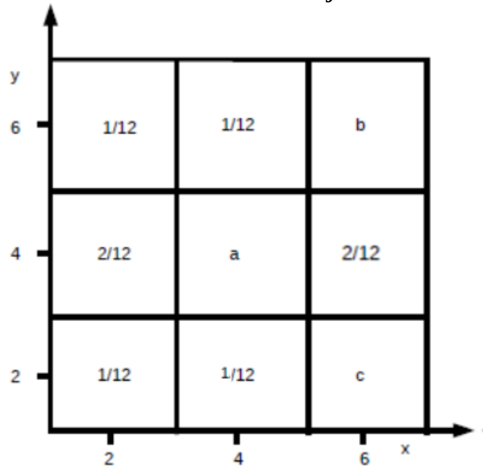
$$\Rightarrow \quad p_{X|Y}(x|2) = p_X(x)$$

$p_{X|Y}(x|2)$



**Question 5 -** (9 points)

Let X and Y be two discrete random variables with their joint PMF shown below:




a) Calculate $p_X(2)$.
b) Calculate $E[Y|X=2]$
c) Determine the values of a, b, and c, under which X and Y are independent random variables. Justify your answer.

(a)

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$\Rightarrow p_X(2) = p_{X,Y}(2,2) + p_{X,Y}(2,4) + p_{X,Y}(2,6)$$

$$= \frac{1}{12} + \frac{2}{12} + \frac{1}{12}$$

$$p_X(2) = \frac{1}{3} \quad \text{— Ans}$$

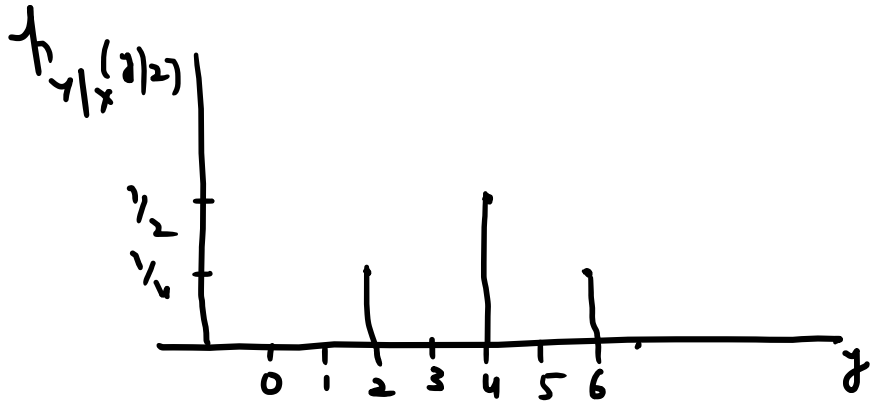
(b)

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

$$p_{Y|X}(2|2) = \frac{p_{X,Y}(2,2)}{p_X(2)} = \frac{1/12}{1/3} = \frac{1}{4}$$

$$p_{Y|X}(4/2) = \frac{p_{X,Y}(2,4)}{p_X(2)} = \frac{2/12}{1/3} = \frac{1}{2}$$

$$p_{Y|X}(6|2) = \frac{p_{X,Y}(2,6)}{p_X(2)} = \frac{1/12}{1/3} = \frac{1}{4}$$




(c)

$$E[Y|X=2] = 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{2}\right) + 6\left(\frac{1}{4}\right)$$

$$= \frac{1}{2} + 2 + \frac{3}{2}$$

$$E[Y|X=2] = 4 \quad \text{— Ans}$$

(d)

$a = \frac{2}{12}$ $\quad b = \frac{1}{12}$ $\quad c = \frac{1}{12}$

With these choices, all the rows have the same relative likelihood among the members of the row. This implies that having information about y does not change the relative likelihoods of x's. This implies independence. Similar argument can be applied about columns as well.