Assignment 6a

Question 01:

$\mathbf{1.}$ The terminal point $\mathbf{e_1} \in \mathbf{R}^2$ is $\mathbf{e_1} = (1,0)$.

$$\therefore \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Question 02:

$\mathbf{2.}$ The angle rotation took place is $\theta = \frac{\pi}{4}$.

$$\because \begin{bmatrix} \frac{\cos\pi}{4} & -\frac{\sin\pi}{4} \\ \frac{\sin\pi}{4} & \frac{\cos\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Question 03:

$\mathbf{3.}$ The rotation matrix when rotation is about $\mathbf{z}-$axis in three dimension is,

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 04:

Yes, since
a Transition Matrix from one orthonormal base to another is a an Orthogonal Matrix
And
Rotation Matrices are Orthogonal Matrices

Question 05:

Matrix R_1 gives rotation through θ(counter clockwise), R_2 gives rotation through ϕ.

Then, the geometrical signinificance of R_1R_2 is that it gives rotation through $(\theta + \phi)$ and that too counter clockwise.

Now taking an example to understand it.

Let, $R_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $R_2 = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

Now, $R_1R_2 = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$

Thus, we can see that this R_1R_2 is giving us rotation through $(\theta + \phi)$.

Hence, we can say that the geometrical significance of R_1R_2 is that, it's giving us rotation through $(\theta + \phi)$.

Question 06:

Proving that AB is orthogonal:

$(AB)^T$

(B^TA^T)

$(B^{-1}A^{-1})$

$(AB)^{-1}$

Hence, now by definition of orthogonality of square matrices, AB is an orthogonal matrix if A, B are orthogonal.

Proving that A^T is orthogonal:

from Theorem 1.6.3 that a square matrix A is orthogonal if *either* $\overline{AA^T = I}$ or $\overline{A^TA = I}$.

Replacing A in AA^T with A^T,

$A^T(A^T)^T = A^TA$

Proving that A^{-1} is orthogonal:

A square matrix A with the property

$$A^{-1} = A^T$$

is said to be an ***orthogonal matrix***.

Since A^T is orthogonal,

And $A^{-1} = A^T$

A^{-1} is orthogonal

Question 07:

$< \mathbf{Av}, \mathbf{Au} >= (\mathbf{Av})^\mathbf{T}\mathbf{Au} = \mathbf{v}^\mathbf{T}\mathbf{A}^\mathbf{T}\mathbf{Au}$ and $< \mathbf{u}, \mathbf{v} >= \mathbf{v}^\mathbf{T}\mathbf{u}$

$< \mathbf{Av}, \mathbf{Au} >=< \mathbf{v}, \mathbf{u} > \mapsto \mathbf{v}^\mathbf{T}\mathbf{A}^\mathbf{T}\mathbf{Au} = \mathbf{v}^\mathbf{T}\mathbf{u}$

for all $\mathbf{u}, \mathbf{v} \epsilon \mathbf{R}^\mathbf{n}$,

$\mathbf{v}^\mathbf{T}\mathbf{A}^\mathbf{T}\mathbf{Au} - \mathbf{v}^\mathbf{T}\mathbf{u} = 0$

$\mathbf{v}^\mathbf{T}(\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u} = 0$

If you put $\mathbf{v} = (\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u}$
in this identity, then we got

$((\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u})^\mathbf{T}(\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u} = 0$

$(\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u} \quad ^2 = 0$

which implies that $(\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I})\mathbf{u} = 0.$

This must hold for all $\mathbf{u} \epsilon \mathbf{R}^\mathbf{n}$,

which implies that

$\mathbf{A}^\mathbf{T}\mathbf{A} - \mathbf{I} = 0$, or equivalently $\mathbf{A}^\mathbf{T}\mathbf{A} = \mathbf{I}$,

making A orthogonal.

If $< \mathbf{Au}, \mathbf{Av} >=< \mathbf{u}, \mathbf{v} >$ it is true for the orthogonal matrix.

That means $\mathbf{A}^\mathbf{T}\mathbf{A} = \mathbf{I}$.

Question 08:

We have given $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

The charecteristic equation of the matrix is $\det(A-\lambda)I = 0$

$$\Rightarrow \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow -\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda) = 0$$

$$\Rightarrow (\lambda-2)^2(\lambda-1) = 0$$

$$\Rightarrow \lambda = 2, 2 \quad \text{and} \quad 1$$

Calculating eigenspaces:

the eigenspace corresponding to the eigenvalue 1:

$$E_1 = \{x \in \mathbb{R}^3 | (A - I)x = 0\}$$

$$\Rightarrow E_1 = \left\{ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$

$$\Rightarrow E_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x = -2z \quad \text{and} \quad x + y + z = 0 \right\}$$

$$\Rightarrow E_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x = -2z \quad \text{and} \quad y = z \right\}$$

$$\Rightarrow E_1 = \left\{ \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} \middle| z \in \mathbb{R} \right\}$$

$$\Rightarrow E_1 = \left\{ z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \middle| z \in \mathbb{R} \right\}$$

$$\Rightarrow E_1 = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Bases for the eigenspace E_1 Is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

the eigenspace corresponding to the eigenvalue 2:

$$\mathrm{E_1} = \{\mathrm{x} \in \mathbb{R}^3 | (\mathrm{A} - 2\mathrm{I})\mathrm{x} = 0\}$$

$$\Rightarrow \mathrm{E_2} = \left\{ \mathrm{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} | \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \right\}$$

$$\Rightarrow \mathrm{E_2} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} | \mathrm{x} + \mathrm{z} = 0 \right\}$$

$$\Rightarrow \mathrm{E_1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} | \mathrm{x} = -\mathrm{z} \right\}$$

$$\Rightarrow \mathrm{E_1} = \left\{ \begin{bmatrix} -z \\ y \\ z \end{bmatrix} | \mathrm{y}, \mathrm{z} \in \mathbb{R} \right\}$$

$$\Rightarrow \mathrm{E_2} = \left\{ \mathrm{z} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \mathrm{y} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} | \mathrm{y}, \mathrm{z} \in \mathbb{R} \right\}$$

$$\Rightarrow \mathrm{E_2} = \mathrm{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Bases for the eigenspace $\mathrm{E_1}$ Is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

We have the required answer:

Bases for the eigenspace $\mathrm{E_1}$ Is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ And Bases for the eigenspace $\mathrm{E_1}$ Is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

<u>Question 09:</u>

1.) $Ax = \lambda x$, given.

2.) Inductive hypothesis: $A^k x = \lambda^k x$ for some positive integer k.

3.) Operate on the equation of (2) with A: $A^{k+1}x = A(A^k x) = A(\lambda^k)x = \lambda^k(Ax) = \lambda^{k+1}x$.

4.) Conclude that $A^k x = \lambda^k x$ for all positive integers k.

Question 10 :

Assume eigen vectors of A corresponding to distinct eigen values are equal.

$Ax = \lambda_1 x$; $Ax = \lambda_2 x$

then

$$Ax - Ax = \lambda_1 x - \lambda_2 x$$

$$0 = \lambda_1 x - \lambda_2 x$$

$$(\lambda_1 - \lambda_2) x = 0$$

but since λ_1 & λ_2 are distinct, $\lambda_1 - \lambda_2 \neq 0$ that implies x is 0, which can't be true for eigen vectors. Hence the eigen vectors of A corresponding to distinct eigen values will also be distinct.

Question 11:

Q11) The entries in the leading diagonals are the eigenvalues for an upper triangular, a lower triangular and a diagonal matrix.

eg $\begin{pmatrix} x & \cdot & \cdot \\ 0 & y & \cdot \\ 0 & 0 & z \end{pmatrix} \rightarrow \begin{pmatrix} x-\lambda & \cdot & \cdot \\ 0 & y-\lambda & \cdot \\ 0 & 0 & z-\lambda \end{pmatrix} \rightarrow$ determinant $= 0$

$\hookrightarrow (x-\lambda)\big((y-\lambda)(z-\lambda) - 0\big) - 0 + 0 = 0$

same is the case for the other two kinds.

Question 12:

Q12) $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \Rightarrow A - \lambda I \Rightarrow \begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix}$

$|A - \lambda I| = 0 \rightarrow (4-\lambda)\big((1-\lambda)(1-\lambda)\big) - 0 + 1\big(+(1-\lambda)(2)\big) = 0$

$\Rightarrow (4-\lambda)(1 - 2\lambda + \lambda^2) + 2 - 2\lambda = 0$

$\Rightarrow 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$

$\therefore \lambda = 1, 2, 3.$

eigenvalues for A are 1, 2, 3. $\therefore$ eigenvalues for A^{100} are $1, 2^{100}, 3^{100}$.

Ans

Question 13:

Q.13) $\rightarrow$ $AX = \lambda X$.

multiplying by A twice on both sides.

$\Rightarrow A^3 X = A^2 \lambda X \rightarrow A^3 X = A\lambda(AX) \rightarrow A^3 X = \lambda^2 (AX)$.

$\Rightarrow A^3 X = \lambda^3 X$

$\therefore$ the eigenvalue corresponding to X for A^3 is λ^3.