Lecture 4

Monday, January 17, 2022 8:09 PM

LINEAR ALGEBRA

$\boxed{\text{LECTURE 4}}$

TRY THE FOLLOWING:

IF $\underline{A}\underline{X}=\underline{B}$ REPRESENTS A SYSTEM OF $\boxed{n}$ EQUATIONS IN n VARIABLES THEN PROVE THAT SOLUTION IS UNIQUE IF $\boxed{A}$ IS INVERTIBLE.

SOLUTION:

LET $\underline{X}_1$, $\underline{X}_2$ BE $\boxed{\text{TWO}}$ SOLUTIONS s.t. $\boxed{A\underline{X}_1=\underline{B},}$ $A\underline{X}_2=\underline{B} \Rightarrow A\underline{X}_1=A\underline{X}_2$, $\because$ $\boxed{A}$ IS INVERTIBLE.

$$\Rightarrow A^{-1}(A\underline{X}_1)=A^{-1}(A\underline{X}_2)$$

$$\Rightarrow I\underline{X}_1=I\underline{X}_2 \Rightarrow \underline{X}_1=\underline{X}_2$$

$\boxed{\text{OR}}$

$\because$ A^{-1} IS $\boxed{\text{UNIQUE}}$

$$\therefore A\underline{X}=\underline{B} \Rightarrow A^{-1}(A\underline{X})=A^{-1}\underline{B}$$

$$\Rightarrow \boxed{\underline{X}=A^{-1}\underline{B}} \text{ IS UNIQUE.}$$

$\therefore$ $X = A^{-1}B$ IS A SOLUTION OF $AX = B$ (PROVIDED $\boxed{A}$ IS INVERTIBLE).

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$\boxed{\text{HOW TO FIND } A^{-1}}$?

FOR THIS WE START ELEMENTARY ROW OPERATIONS.

ELEMENTARY ROW OPERATIONS ARE THE FOLLOWING:

(1) MULTIPLY A $\boxed{\text{ROW}}$ BY A NONZERO CONSTANT

$R_1 \to 3R_1$

E.G. IF $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 & 6 & 9 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

You can see an ERO as a function θ on matrices. $\theta(A_1) = A_2$

CONSIDER $\boxed{R_2 \longrightarrow 2R_2}$ GIVES

$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 6 & 12 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

(2) INTERCHANGE TWO ROWS:

E.G. CONSIDER $(R_1 \longleftrightarrow R_2)$ GIVES

$C = \begin{bmatrix} 2 & -1 & 3 & 6 \\ 1 & 0 & 2 & 3 \\ 1 & 4 & 4 & 0 \end{bmatrix}$ FROM $\boxed{B}$

(3) ADD A MULTIPLE OF ONE ROW TO ANOTHER ROW.

E.G. FOR $B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

CONSIDER $\boxed{R_1 \longrightarrow R_1 + 2R_2}$ GIVES

$[4 \;\; -2 \;\; 6 \;\; 12]$

$$D = \begin{bmatrix} 5 & -2 & 8 & 15 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

EQUIVALENT MATRICES:

IF $\boxed{B}$ IS A MATRIX AND $\boxed{A}$ IS A MATRIX OBTAINED FROM $\boxed{B}$ BY ONE OR MORE ELEMENTARY ROW OPERATIONS THEN $\boxed{A}$ IS CALLED ROW EQUIVALENT (OR JUST EQUIVALENT) TO $\boxed{B}$ AND VICE VERSA AND IS DENOTED BY $\boxed{A \sim B}$ OR $\boxed{B \sim A}$.

IN LAST THREE EXAMPLES WE HAVE

(1) $B \sim A$ (SLIDE 2)

(2) $B \sim C$ (SLIDE 2)

(3) $B \sim D$ (SLIDE 3)

Don't confuse with EROs!

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow C \begin{bmatrix} 0 & 1 \\ 1024 & 0 \end{bmatrix}$

ELEMENTARY MATRICES

AN $\boxed{n \times n}$ MATRIX IS CALLED AN ELEMENTRY MATRIX IF IT CAN BE OBTAINED FROM THE

$R_2 \rightarrow 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \rightarrow R_2 + 2R_1$ $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ ← Not elementary!

CAN BE OBTAINED FROM THE $\boxed{n \times n}$ IDENTITY MATRIX I_n BY PERFORMING A $\boxed{\text{SINGLE}}$ ELEMENTARY ROW OPERATION.

EXAMPLE: $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ IS AN ELEMENTARY MATRIX

$$\because \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \underset{R_3 + 3R_1}{R_3 \to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

i.e. $\boxed{E}$ IS OBTAINED FROM $\boxed{I_3}$ BY PERFORMING THE E.R.O.

$$\boxed{R_3 \longrightarrow R_3 + 3R_1}$$

$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ ← Not ele

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix}$

$R_1 \to 1024R_1$

5) LET $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 1 & 4 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 4 & 10 \end{bmatrix} = B \quad R_3 \to R_3 + 3R_1 \quad \text{— (1)}$

3 0 6

CONSIDER EA → ELEMENTARY MATRIX

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 1 & 4 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 4 & 10 \end{bmatrix} = B \quad \text{— (2)}$

FROM (1) AND (2), WE GET THE FOLLOWING RESULT:

THE ELEMENTARY ROW OPERATION HAS THE SAME EFFECT ON A MATRIX AS PREMULTIPLICATION OF AN ELEMENTARY MATRIX (CORRESPONDING TO SAME E.R.O.)

OR IN MATHEMATICAL LANGUAGE IF θ BE ANY E.R.O. AND E BE THE ELE_

MENTARY MATRIX CORRESPONDING TO θ THEN

6) $$\theta(A) = EA = B \text{ (SAY)}, \quad \boxed{A \sim B}$$

LET $\boxed{\theta_1, \theta_2, \ldots, \theta_n}$ BE $\boxed{n}$ NUMBER OF E.R.O.S. AND $\boxed{E_1, E_2, \ldots, E_n}$ ARE THE CORRESPONDING ELEMENTARY MATRICES SUCH THAT WHEN APPLIED ON AN INVERTIBLE MATRIX $\boxed{A}$ GIVE $\boxed{I}$ (IDENTITY MATRIX)

i.e. $\theta_n \ldots \theta_2 \theta_1 (A) = E_n \ldots E_2 E_1 A = I$

OR $\boxed{PA = I}$, $P = E_n \ldots E_2 E_1$

$\because \boxed{A}$ IS INVERTIBLE — i.e. A^{-1} exists

$$\Rightarrow PAA^{-1} = IA^{-1} \Rightarrow \boxed{PI = A^{-1}}$$

NOW $PA = I \Rightarrow \boxed{A \sim I}$

AND $PI = A^{-1} \Rightarrow \boxed{I \sim A^{-1}}$

$\therefore$ E.R.O.S. WHICH TRANSFORMED $\boxed{A}$ INTO $\boxed{I}$ ALSO TRANSFORMED $\boxed{I}$ INTO $\boxed{A^{-1}}$ $\therefore$ A^{-1} CAN BE FOUND BY

$$[A \mid I] \xrightarrow{\text{E.R.O.S.}} [I \mid A^{-1}]$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{\theta_3 \theta_2 \theta_1 (A)}$
$\downarrow$
$E_3 E_2 E_1$

$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

θ_1 $\underline{R_2 \to R_2 - 3R_1}$ $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

θ_2 $\underline{R_2 \to -\frac{1}{2} R_2}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

θ_3 $\underline{R_1 \to R_1 - 2R_2}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\theta_3 \theta_2 \theta_1 (A)$
$= \underline{E_3 E_2 E_1 A}$

METHOD TO FIND A^{-1}

$$\boxed{[A \,\vdots\, I] \xrightarrow{E.R.O.S} [I \,\vdots\, A^{-1}]}$$

DEFINITIONS (1) $A\underline{X} = \underline{B}$ IS CALLED A **CONSISTENT** SYSTEM OF LINEAR EQUATIONS IF THERE IS ATLEAST ONE SOLUTION, OTHERWISE ITS CALLED INCONSISTENT.

IF $\underline{B} \neq \underline{0}$ IN $A\underline{X} = \underline{B}$ THEN $A\underline{X} = \underline{B}$ IS CALLED A NON-HOMOGENEOUS SYSTEM,

IF $\underline{B} = \underline{0}$ THEN $\boxed{A\underline{X} = \underline{B} = \underline{0}}$ IS CALLED A HOMOGENEOUS SYSTEM.

NOTE: HOMOGENEOUS SYSTEM IS ALWAYS CONSISTENT $\because \underline{X} = \underline{0}$ IS ALWAYS A SOLUTION OF $A\underline{X} = \underline{0}$ CALLED A **ZERO** SOLUTION OR TRIVIAL SOLUTION.

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$x_1 + x_2 = 1$

$2x_1 + 2x_2 = 3$

2

$x_1 + x_2 = 1$

$x_1 + 2x_2 = 3$

$x_1 + x_2 = 0$

$x_1 + 2x_2 = 0$

x_1

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Q: SOLVE THE FOLLOWING SYSTEM (NON-HOMOGENEOUS) OF LINEAR EQUATIONS BY FINDING THE INVERSE OF THE COEFFICIENT MATRIX

$$3x_1 + 4x_2 + 5x_3 = 12$$
$$x_1 - x_2 + 2x_3 = 2$$
$$2x_1 + x_2 + 3x_3 = 6$$

$$\Rightarrow \begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix}$$

CONSIDER

$$[A / I] = \left[\begin{array}{ccc|ccc} 3 & 4 & 5 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

STEP (1) MAKE THE ENTRY (FIRST ROW) (1ST COLUMN)

$(1,1) = 1$ BY ANY E.R.O.

FOR THIS PERFORM $\boxed{R_1 \longleftrightarrow R_2}$

TO GET

$$[A/I] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 5 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

STEP (2) MAKE ENTRIES (2,1) AND (3,1) = 0, FOR THIS PERFORM THE FOLLOWING $\boxed{\text{TWO}}$ ELEMENTARY ROW OPERATIONS

$$\boxed{R_2 \longrightarrow R_2 - 3R_1}, \quad \boxed{R_3 \rightarrow R_3 - 2R_1}$$

TO GET

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 7 & -1 & 1 & -3 & 0 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right]$$

STEP (3) NEXT TO MAKE THE ENTRY (2,2) = 1, FOR THIS PERFORM $\boxed{R_2 \longrightarrow R_2 - 2R_3}$ TO GET

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 3 & -1 & 0 & -2 & 1 \end{array}\right]$$

STEP (4) MAKE ENTRIES (1,2) AND (3,2) = 0, FOR THIS PERFORM

$$\boxed{R_1 \longrightarrow R_1 + R_2}, \quad \boxed{R_3 \rightarrow R_3 - 3R_2}$$

TO GET

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & -4 & -3 & -5 & 7 \end{array}\right]$$

STEP (5) MAKE ENTRY $(3,3)=1$, FOR THIS PERFORM $\boxed{R_3 \to -\frac{1}{4} R_3}$ TO GET

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 1 & \frac{3}{4} & \frac{5}{4} & -7/4 \end{array}\right]$$

STEP (6) IN THE FINAL STEP PERFORM $\boxed{R_1 \to R_1 - 3R_3}$ AND $\boxed{R_2 \to R_2 - R_3}$ TO GET

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/4 & -7/4 & 13/4 \\ 0 & 1 & 0 & \frac{1}{4} & -1/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & 5/4 & -7/4 \end{array}\right]$$

$= [I \,|\, A^{-1}]$, THEREFORE

$[A \,|\, I] \sim [I \,|\, A^{-1}]$

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$$A^{-1} = \frac{1}{4}\begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}$$

$$\therefore \boxed{\underline{X} = A^{-1}\underline{B}}$$

$$= \frac{1}{4}\begin{bmatrix} -5 & -7 & 13 \\ 1 & -1 & -1 \\ 3 & 5 & -7 \end{bmatrix}\begin{bmatrix} 12 \\ 2 \\ 6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -60 - 14 + 78 \\ 12 - 2 - 6 \\ 36 + 10 - 42 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow \boxed{x_1 = 1,\ x_2 = 1,\ x_3 = 1}$ IS THE REQUIRED SOLUTION OF THE GIVEN LINEAR SYSTEM.

Exercise Set 1.5

Q2, Q3, Q9 (a) (b),Q 10, Q 16 (Hint: The row containing b must become the first row after ERO have been carried out and the one with g must be the last row--Why?-- this caused problems when trying to reduce the middle column to only one non-zero entry)