Exercise 6.1

Question 03:

Step 1

According to example 7, the inner product of $\mathbf{u}, \mathbf{v}$ is given by $Tr(\mathbf{u}^t\mathbf{v})$. Hence

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= Tr\left(\begin{pmatrix}3 & -2\\ 4 & 8\end{pmatrix}^t \begin{pmatrix}-1 & 3\\ 1 & 1\end{pmatrix}\right)\\ &= Tr\left(\begin{pmatrix}3 & 4\\ -2 & 8\end{pmatrix} \begin{pmatrix}-1 & 3\\ 1 & 1\end{pmatrix}\right)\\ &= Tr\left(\begin{pmatrix}1 & 13\\ 10 & 2\end{pmatrix}\right)\\ &= 3.\end{aligned}$$

Result

We have $\langle \mathbf{u}, \mathbf{v}\rangle = 3$.

Result

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We have $\langle \mathbf{u}, \mathbf{v}\rangle = 3$.

Step 1

According to example 7, the inner product of $\mathbf{u}, \mathbf{v}$ is given by $Tr(\mathbf{u}^t\mathbf{v})$. Hence

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= Tr\left(\begin{pmatrix}1 & 2\\ -3 & 5\end{pmatrix}^t \begin{pmatrix}4 & 6\\ 0 & 8\end{pmatrix}\right)\\ &= Tr\left(\begin{pmatrix}1 & -3\\ 2 & 5\end{pmatrix} \begin{pmatrix}4 & 6\\ 0 & 8\end{pmatrix}\right)\\ &= Tr\left(\begin{pmatrix}4 & -18\\ 8 & 52\end{pmatrix}\right)\\ &= 56.\end{aligned}$$

Result

We have $\langle \mathbf{u}, \mathbf{v}\rangle = 56$.

Question 04:

Step 1

According to example 8, the inner product of $a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2$ is given by $a_0b_0 + a_1b_1 + a_2b_2$. Hence

$$\begin{aligned}\langle -2 + x + 3x^2, 4 - 7x^2\rangle &= (-2)\cdot(4) + (1)\cdot(0) + (3)\cdot(-7)\\ &= -8 + 0 - 21\\ &= -29.\end{aligned}$$

Result

We have $\langle -2 + x + 3x^2, 4 - 7x^2\rangle = -29$.

Step 1

According to example 8, the inner product of $a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2$ is given by $a_0b_0 + a_1b_1 + a_2b_2$. Hence

$$\begin{aligned}\langle -5 + 2x + x^2, 3 + 2x - 4x^2\rangle &= (-5)\cdot(3) + (2)\cdot(2) + (1)\cdot(-4)\\ &= -15 + 4 - 4\\ &= -15.\end{aligned}$$

Result

We have $\langle -5 + 2x + x^2, 3 + 2x - 4x^2\rangle = -15$.

Question 10:

Step 1

(a)
Recall that the definition of Euclidean inner product of vectors $\mathbf{u} = (u_1, \dots u_n)$ and $\mathbf{v} = (v_1, \dots v_n)$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$

and the norm generated by the product is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

Therefore

$$\|\mathbf{w}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

(b)
The norm generated by this product is

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{3u_1^2 + 2u_2^2}$$

Therefore

$$\|\mathbf{w}\| = \sqrt{3(-1)^2 + 2 \cdot 3^2} = \sqrt{3 + 18} = \sqrt{21}$$

(c)
Recall that the inner product generated by matrix A can be computed as follows

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^T A^T A \mathbf{u}$$

The norm is given by

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{\mathbf{u}^T A^T A \mathbf{u}} = \sqrt{(A\mathbf{u})^T (A\mathbf{u})}$$

We have

$$A\mathbf{w} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Therefore

$$\|\mathbf{w}\| = \sqrt{\begin{bmatrix} 5 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix}} = \sqrt{25 + 100} = \sqrt{125} = 5\sqrt{5}$$

Result

(a) $\sqrt{10}$

(b) $\sqrt{21}$

(c) $5\sqrt{5}$

Question 11:

Step 1

We know that, for two vectors $\mathbf{u}, \mathbf{v}$, the distance is givne by

$$\begin{aligned} d(\mathbf{u}, \mathbf{v}) &= ||\mathbf{u} - \mathbf{v}|| \\ &= ||(-1, 2) - (2, 5)|| \\ &= ||(-3, -3)||. \end{aligned}$$

(a) The Euclidean norm on $\mathbb{R}^2$ is given by

$$||(u_1, u_2)|| = \sqrt{u_1^2 + u_2^2}.$$

Hence in this case $||(-3, -3)|| = \sqrt{9 + 9} = 3\sqrt{2}$.

(b) For the weighted inner product $\langle (u_1, u_2), (v_1, v_2) \rangle = 3u_1v_1 + 2u_2v_2$, we have

$$||(u_1, u_2)|| = \sqrt{3u_1^2 + 2u_2^2}.$$

Hence in this case $||(-3, -3)|| = \sqrt{3 \cdot 9 + 2 \cdot 9} = 3\sqrt{5}$.

(c) For the inner product generated by

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

, we have

$$\begin{aligned} ||(u_1, u_2)|| &= \sqrt{\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}} \\ &= \sqrt{(u_1 + 2u_2, -u_1 + 3u_2) \cdot (u_1 + 2u_2, -u_1 + 3u_2)} \\ &= \sqrt{2u_1^2 - 2u_1u_2 + 13u_2^2}. \end{aligned}$$

Hence in this case $||(-3, -3)|| = \sqrt{2(9) - 2(9) + 13(9)} = 3\sqrt{13}$.

Result

We have

$$\begin{aligned} &(a) d(\mathbf{u}, \mathbf{v}) = 3\sqrt{2} \\ &(b) d(\mathbf{u}, \mathbf{v}) = 3\sqrt{5} \\ &(c) d(\mathbf{u}, \mathbf{v}) = 3\sqrt{13}. \end{aligned}$$

EXERCISE 6.1

Question 17:

Step 1

(a)

Let $\mathbf{p} = 1$.

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} \mathrm{d}x} = \sqrt{x\Big|_{-1}^{1}} = \sqrt{1-(-1)} = \sqrt{2}$$

Let $\mathbf{p} = x$.

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} x^2 \mathrm{d}x} = \sqrt{\frac{x^3}{3}\Big|_{-1}^{1}} = \sqrt{1/3-(-1/3)} = \sqrt{2/3}$$

Let $\mathbf{p} = x^2$.

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} x^4 \mathrm{d}x} = \sqrt{\frac{x^5}{5}\Big|_{-1}^{1}} = \sqrt{1/5-(-1/5)} = \sqrt{2/5}$$

(b)

$$\begin{aligned} d(\mathbf{p}, \mathbf{q}) &= \|\mathbf{p} - \mathbf{q}\| \\ &= \|1 - x\| \\ &= \sqrt{\int_{-1}^{1} (1-x)^2 \mathrm{d}x} \\ &= \sqrt{\int_{-1}^{1} (1-2x+x^2) \mathrm{d}x} \\ &= \sqrt{x\Big|_{-1}^{1} + (-x^2)\Big|_{-1}^{1} + \frac{x^3}{3}\Big|_{-1}^{1}} \\ &= \sqrt{(1-(-1)) + (-1+1) + (1/3-(-1/3))} \\ &= \sqrt{8/3} \end{aligned}$$

Result

(a) For $\mathbf{p} = 1$, the norm is $\sqrt{2}$.

For $\mathbf{p} = x$, the norm is $\sqrt{2/3}$.

For $\mathbf{p} = x^2$, the norm is $\sqrt{2/5}$.

(b) $\sqrt{8/3}$

Question 20:

Step 1

Let consider the left side of the equation:

$$\| u+v \|^2 + \| u-v \|^2 = (u+v, u+v) + (u-v, u-v)$$

So we have

$$(u+v, u+v) + (u-v, u-v) = (u, u+v) + (v, u+v) + (u, u-v) - (v, u-v)$$
$$= (u,u) + (u,v) + (v,u) + (v,v) + (u,u) - (u,v) - (v,u) + (v,v)$$

By using $(u,u) = \| u \|^2$ and $(u,v) = (v,u)$ we get

$$(u+v, u+v) + (u-v, u-v) = 2(u,u) + 2(u,v) - 2(u,v) + 2(v,v)$$

$$\boxed{\| u+v \|^2 + \| u-v \|^2 = 2 \| u \|^2 + 2 \| v \|^2}$$

Result

Hint:Use $(u,u) = \| u \|^2$ and $(u,v) = (v,u)$.

Question 27:

Step 1

The goal is to compute $\langle \mathbf{p} \cdot \mathbf{q} \rangle$ using the given inner product for parts **(A)** and **(B)**.

Step 2

Part (A): In our case we have that $\mathbf{p} = 1 - x + x^2 + 5x^3$ and $\mathbf{q} = x - 3x^2$. Using the definition of the inner product we obtain the following:

$$\begin{aligned}
\langle \mathbf{p} \cdot \mathbf{q} \rangle &= \int_{-1}^{1} p(x)q(x)dx \\
&= \int_{-1}^{1} (1 - x + x^2 + 5x^3)(x - 3x^2)dx \\
&= \int_{-1}^{1} \left[(x - x^2 + x^3 + 5x^4) - (3x^2 - 3x^3 + 3x^4 + 15x^5) \right] dx \\
&= \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)dx
\end{aligned}$$

Using the power rule for integration, we'll have:

$$\begin{aligned}
\langle \mathbf{p} \cdot \mathbf{q} \rangle &= \int_{-1}^{1} (x - 4x^2 + 4x^3 + 2x^4 - 15x^5)dx \\
&= \left(\frac{x^2}{2} - \frac{4}{3}x^3 + \frac{4}{4}x^4 + \frac{2}{5}x^5 - \frac{15}{6}x^6 \right) \Big|_{-1}^{1} \\
&= \left(\frac{x^2}{2} - \frac{4}{3}x^3 + x^4 + \frac{2}{5}x^5 - \frac{5}{2}x^6 \right) \Big|_{-1}^{1} \\
&= \left[\frac{(1)^2}{2} - \frac{4}{3}(1)^3 + (1)^4 + \frac{2}{5}(1)^5 - \frac{5}{2}(1)^6 \right] \\
&\quad - \left[\frac{(-1)^2}{2} - \frac{4}{3}(-1)^3 + (-1)^4 + \frac{2}{5}(-1)^5 - \frac{5}{2}(-1)^6 \right] \\
&= -\frac{29}{15} - \left(-\frac{1}{15} \right) \\
&= \boxed{-\frac{28}{15}}
\end{aligned}$$

Step 3

Part (B): In this case we are given that $\mathbf{p} = x - 5x^3$ and $\mathbf{q} = 2 + 8x^2$. Let us again find the required inner product:

$$\begin{aligned} \langle \mathbf{p} \cdot \mathbf{q} \rangle &= \int_{-1}^{1} p(x)q(x)dx \\ &= \int_{-1}^{1} (x - 5x^3)(2 + 8x^2)dx \\ &= \int_{-1}^{1} \left[(2x - 10x^3) + (8x^3 - 40x^5)\right] dx \\ &= \int_{-1}^{1} (2x - 2x^3 - 40x^5)dx \end{aligned}$$

Using the power rule for integration, we'll have:

$$\begin{aligned} \langle \mathbf{p} \cdot \mathbf{q} \rangle &= \int_{-1}^{1} (2x - 2x^3 - 40x^5)dx \\ &= \left(\frac{2}{2}x^2 - \frac{2}{4}x^4 - \frac{40}{6}x^6\right)\Big|_{-1}^{1} \\ &= \left(x^2 - \frac{1}{2}x^4 - \frac{20}{3}x^6\right)\Big|_{-1}^{1} \\ &= \left[(1)^2 - \frac{1}{2}(1)^4 - \frac{20}{3}(1)^6\right] - \left[(-1)^2 - \frac{1}{2}(-1)^4 - \frac{20}{3}(-1)^6\right] \\ &= -\frac{37}{6} - \left(-\frac{37}{6}\right) \\ &= \boxed{0} \end{aligned}$$

Question 28:

Step 1

(a)

Using trigonometric formula $\sin(2x) = 2\sin(x)\cos(x)$, we get

$$\cos(2\pi x)\sin(2\pi x) = \frac{1}{2}\sin(4\pi x)$$

We have

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle &= \int_{0}^{1} \cos(2\pi x)\sin(2\pi x)\mathrm{d}x \\ &= \int_{0}^{1} \frac{1}{2}\sin(4\pi x)\mathrm{d}x \\ &= \frac{-\cos(4\pi x)}{8\pi}\Big|_{0}^{1} \\ &= \frac{-1}{8\pi} + \frac{-1}{8\pi} \\ &= 0 \end{aligned}$$

(b)

$$\begin{aligned} \langle \mathbf{f}, \mathbf{g} \rangle &= \int_{0}^{1} xe^x \mathrm{d}x \\ &= \left[u = x, \mathrm{d}u = \mathrm{d}x \;\middle|\; \mathrm{d}v = e^x\mathrm{d}x, v = e^x\right] \\ &= xe^x\Big|_{0}^{1} - \int_{0}^{1} e^x \mathrm{d}x \\ &= e - e^x\Big|_{0}^{1} \\ &= e - (e - 1) \\ &= 1 \end{aligned}$$

Step 2

(c)

$$
\begin{aligned}
\langle \mathbf{f}, \mathbf{g} \rangle &= \int_0^1 \tan\left(\frac{\pi}{4}x\right) \mathrm{d}x \\
&= \int_0^1 \frac{\sin \frac{\pi}{4}x}{\cos \frac{\pi}{4}x} \mathrm{d}x \\
&= \left[u = \cos \frac{\pi}{4}x, \mathrm{d}u = -\frac{\pi}{4} \sin \frac{\pi}{4}x \mathrm{d}x \;\middle|\; 0 \to 1,\ 1 \to \frac{\sqrt{2}}{2} \right] \\
&= \frac{4}{\pi} \int_{\sqrt{2}/2}^1 \frac{\mathrm{d}u}{u} \\
&= \frac{4}{\pi} \ln u \Big|_{\sqrt{2}/2}^1 \\
&= \frac{-4}{\pi} \ln\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{-4}{\pi} \ln\left(2^{-1/2}\right) \\
&= \frac{-4}{\pi} \frac{-1}{2} \ln(2) \\
&= \frac{2 \ln 2}{\pi}
\end{aligned}
$$

Result

(a) 0

(b) 1

(c) $\frac{2 \ln 2}{\pi}$