Let $$\boxed{\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + \epsilon_i \\ \hat{Y}_i &= \hat{\beta}_0 + \hat{\beta}_1 X_i \end{aligned}}$$ — A

(1)

$\epsilon_i \sim N(0, \sigma^2)$

$Y_i \mid X_i, \beta_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$

$$\boxed{E(Y_i \mid X_i, \beta_i) = \beta_0 + \beta_1 X_i}$$ — B

$$E(\bar{Y}) = E_X\left(E_{Y|X}(\bar{Y} \mid X_i, \beta_i)\right) = E_X\left(E_{Y|X}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i \,\Big|\, X_i, \beta_i\right)\right)$$

$$= E_X\left(\frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 X_i)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 E(X_i))$$

$$= \frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 \bar{X}) = \beta_0 + \beta_1 \bar{X}$$

Since $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

$$E(\hat{\beta}_0) = E(\bar{Y} - \hat{\beta}_1 \bar{X})$$

$$= E(\bar{Y}) - E(\hat{\beta}_1 \bar{X})$$

$$= \beta_0 + \beta_1 \bar{X} - E(\hat{\beta}_1)\bar{X}$$

$$= \beta_0 + \bar{X}(\beta_1 - E(\hat{\beta}_1))$$

If $E(\hat{\beta}_1) = \beta_1$,

Then $E(\hat{\beta}_0) = \beta_0$.

Now, we have to prove that $E(\hat{\beta}_1) = \beta_1$.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$E(\hat{\beta}_1) = E_{X,Y}\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]$$

$$= E_X\left[E_{Y|X}\left(\frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\,\middle|\, X_i\right)\right]$$

$$= E_X\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})\,E(Y_i|X_i)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]$$

From eq (B)

$$= E_X\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})(\beta_0 + \beta_1 X_i)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]$$

$$= E_X\left[\frac{\sum_{i=1}^{n}(\beta_1 X_i^2 - \beta_1 X_i \bar{X} - \bar{X}\beta_0 + X_i\beta_0)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]$$

$$= \beta_1 \underbrace{E_X\left[\frac{\sum_{i=1}^{n}(X_i^2 - \bar{X}X_i)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]}_{\text{(numerator is equal to the denominator)}} + \beta_0 \underbrace{E_X\left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right]}_{\text{(numerator becomes zero)}}$$

$$\Rightarrow E(\hat{\beta}_1) = \beta_1$$