Q2) $\mu_0 = 50$

$H_0: \mu = \mu_0$

$H_a: \mu \neq \mu_0$

$n = 40$

$\beta = ?$

$\sigma = 8$

$\mu_a = 54$

$\alpha = 0.01 \Rightarrow \frac{\alpha}{2} = 0.005$

$\therefore$ ~~$z = \bar{x}$~~

From the z-table we have

$\pm z_c^0 = \pm 2.5758$

Now, we find critical mean under H_0

$$\because \pm z_c^0 = \frac{\bar{x}_c^0 - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

$$\bar{x}_c^0 = \cancel{\pm z_c^0} \times$$

$$= \pm 2.5758 \times \frac{8}{\sqrt{40}} + 50$$

$\frac{30}{30}$

$\bar{x}_c^0 = 46.742, 53.258$

Now we find z critical under H_1 using:

$$\pm z_c^1 = \frac{\bar{x}_c^0 - \mu_a}{\frac{\sigma}{\sqrt{n}}}$$

$\cancel{\pm z_c^+}$

$$z_{c,u}^1 = \frac{53.258 - 54}{\frac{8}{\sqrt{40}}}, \quad z_{c,l}^1 = \frac{46.742 - 54}{\frac{8}{\sqrt{40}}}$$

$$z_{c,u}^1 = -0.5866, \quad z_{c,l}^1 = -5.74$$

$$\therefore \beta = \Phi(z_{c,u}^1) - \Phi(z_{c,l}^1)$$

$$= 0.2810 - 0$$

$$\boxed{\beta = 28.10\%}$$

7) By applying anova on the given data, we get

$H_0 : \mu_1 = \mu_2 = \mu_3 \ldots\ldots = \mu_K$
H_a : at least one mean is different.

$SSC = 100.783$
$SSE = 31.324$
$SST = 132.107$
$MSC = 25.1958$
$MSE = 1.3619$
$df_c = 4$
$df_E = 23$
$df_T = 27$

$\boxed{F = 18.5}$

$p = 6.33788 \times 10^{-07}$

$\alpha = 0.01$

50/50.

Critical values of F ~~are~~ is,

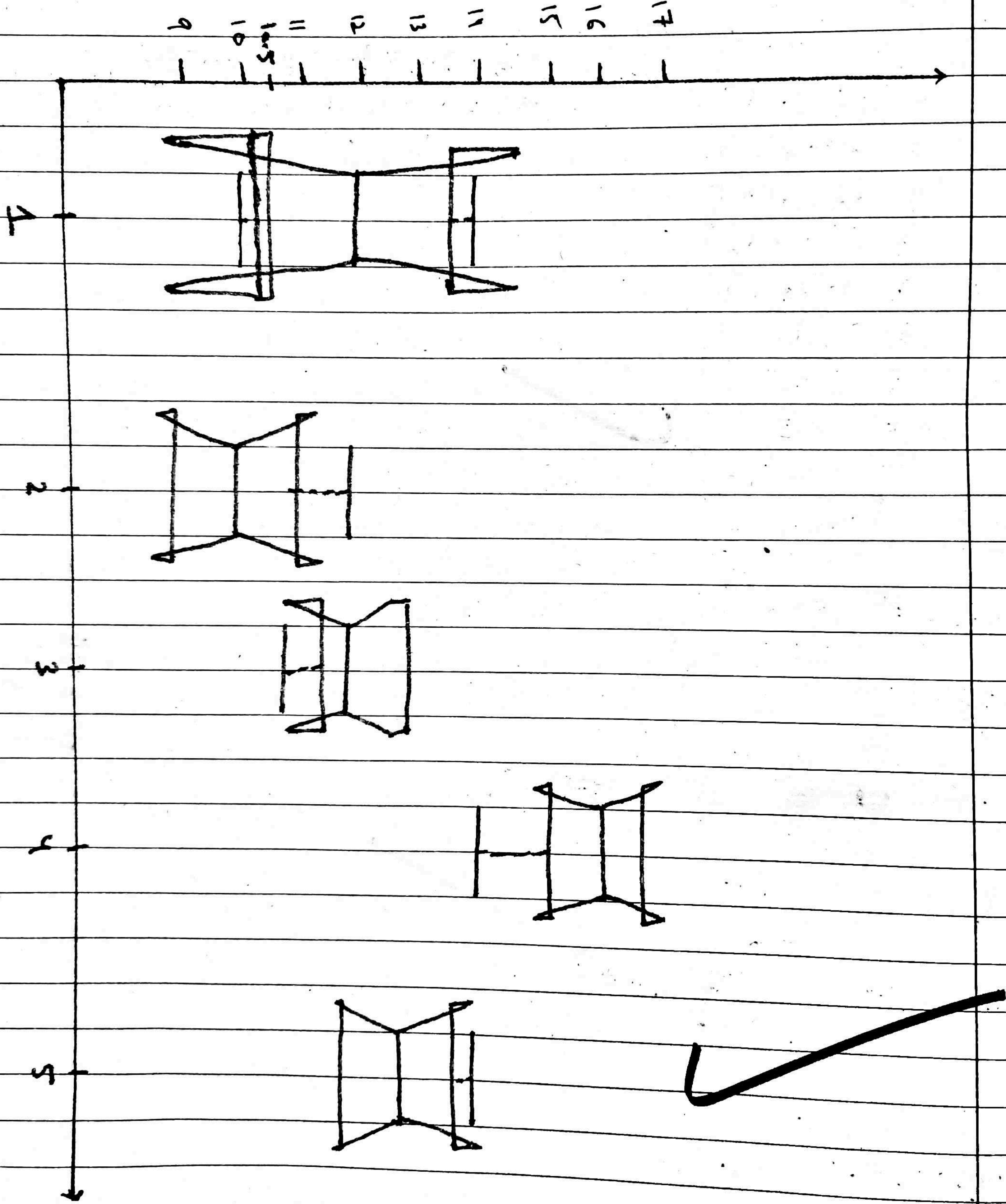
$F_{0.01, 4, 23} = 4.26$

~~$F_{0.99, 23, 4} = \frac{1}{4.26}$~~

Since $F > F_c$
i.e. $18.5 > 4.26$

$\Rightarrow$ The decision is to reject the null hypothesis

Instructor
comments-
do not write
in this sectio



Instructor comments do not write in this section

Q1)

$f(x|a) = ax^{a-1}$, for $0<x<1$ and $a>0$

$\Rightarrow f(X_i|\theta) = \cancel{\ } \; a x_i^{a-1}$

a)

$$f(X_1, X_2, \ldots, X_n | a) = \prod_{i=1}^{n} a X_i^{a-1}$$

$$lik(a) = \underbrace{a^n \cdot \prod_{i=1}^{n} X_i^{a-1}}_{\cancel{g(X,\theta)}\; g(X,a)} \times \underbrace{1}_{h(X)}$$

Here $g(X,a)$ provides the sufficient statistics.

To extract T from $g(X,a)$ we take log.

$$\Rightarrow \cancel{\ln\left(\prod_{i=1}^{n} X_i^{a-1}\right)} =$$

$$\ln(g(X,a)) = n\ln(a) + \sum_{i=1}^{n} \ln(X_i^{a-1})$$

$$= n\ln(a) + (a-1)\underbrace{\sum_{i=1}^{n} \ln(X_i)}$$

$\Rightarrow$ $\boxed{T(X) = \sum_{i=1}^{n} \ln(X_i)}$ is sufficient statistic for a.

b)

$$\ell(a) = n\ln(a) + (a-1)\sum_{i=1}^{n} \ln(X_i)$$

Take 1^{st} derivative and equate to 0

$$\ell'(a) = 0$$

$$\Rightarrow \frac{n}{a} + \sum_{i=1}^{n} \ln(X_i) = 0$$

$$\Rightarrow a = \cancel{\sum_{i=1}^{n}} \frac{-n}{\sum_{i=1}^{n} \ln(X_i)}$$

Therefore, the MLE estimate is,

$$\boxed{\hat{a} = -\frac{n}{\sum_{i=1}^{n} \ln(X_i)}}$$

OR

$$\boxed{\hat{a} = \frac{-1}{\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)}}$$

Instructor comments do not write in this section

d)

$$\because \quad Var(\hat{a}) \approx \frac{1}{E(l''(a))}$$

$$l'(a) = \frac{n}{a} + \sum_{i=1}^{n} \ln(X_i)$$

$$l''(a) = -\frac{n}{a^2}$$

$$\Rightarrow Var(\hat{a}) = -\frac{1}{-n/a^2}$$

$$\Rightarrow \boxed{Var(\hat{a}) = \frac{a^2}{n}}$$

e): $E[X] = \int x f(x)\, dx$

$$\Rightarrow E[X] = \int_0^1 x \cdot a x^{a-1}\, dx$$

$$\Rightarrow E[X] = \int_0^1 a x^a\, dx = a\left[\frac{x^{a+1}}{a+1}\right]_0^1$$

Instructor comments do not write in this section

$$E[X] = \frac{a}{a+1}\left(1^{a+1} - 0^{a+1}\right)$$

$$E[X] = \frac{a}{a+1} \quad (\text{assuming } a \neq -1)$$

$$\Rightarrow aE[X] + E[X] - a = 0$$

$$a(E[X] - 1) + E[X] = 0$$

$$a = \frac{-E[X]}{E[X] - 1}$$

$$\Rightarrow \boxed{\hat{a} = \frac{-\frac{1}{n}\sum_{i=1}^{n} X_i}{\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] - 1}}$$

OR

$$\boxed{\hat{a} = \frac{\sum_{i=1}^{n} X_i}{n - \sum_{i=1}^{n} X_i}} \quad — \; 1$$

Instructor comments do not write in this section

f) Approximate method states that,

For $Y = g(Z)$

$$E[Y] = g(\mu_Z) + \frac{1}{2}\sigma_Z^2 g''(\mu_Z)$$

and $$var(Y) = \sigma_Z^2 [g'(\mu_Z)]^2$$

So we have,

$$\hat{a} = g(T) = \frac{T}{1 - T}$$

Where, $T = \frac{1}{n}\sum_{i=1}^{n} X_i$

and,

$$E[\hat{a}] = g(\mu_T) + \frac{1}{2}\sigma_T^2 g''(\mu_T) \quad — ②$$

$$var(\hat{a}) = \sigma_T^2 [g'(\mu_T)]^2 \quad — ③$$

$$\mu_T = E[T] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

$$\mu_T = \frac{1}{n} \times \frac{(na)}{(a+1)}$$

$$\Rightarrow \mu_T = \frac{a}{a+1}$$

Similarly,

$$\sigma_T^2 = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\left(n\sigma_x^2\right)$$

$$\sigma_T^2 = \frac{\sigma_x^2}{n} \quad — \ (4)$$

Finding σ_x^2,

$$\sigma_x^2 = E[X^2] - \left(E[X]\right)^2$$

$$= \int_0^1 x^2 . a x^{a-1}\, dx - \left(\frac{a}{a+1}\right)^2$$

$$= a\left[\frac{x^{a+2}}{a+2}\right]_0^1 - \left(\frac{a}{a+1}\right)^2$$

$$= \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$$

$$= \frac{a(a+1)^2 - a^2(a+2)}{(a+2)(a+1)^2}$$

$$= \frac{a(a^2+2a+1) - a^3 + 2a^2}{(a+2)(a+1)^2} = \frac{\cancel{a^3} + \cancel{2a^2} + a - \cancel{a^3} - \cancel{2a^2}}{(a+2)(a+1)^2}$$

$$\cancel{\sigma_x^2 = 4a^2 + a}$$

$$\sigma_x^2 = \frac{a}{(a+2)(a+1)^2}$$

(4)

$$\Rightarrow \sigma_T^2 = \frac{a}{n(a+2)(a+1)^2}$$

since, $g(T) = \dfrac{T}{1-T}$

$$\Rightarrow g'(T) = \frac{1}{(1-T)^2}$$

and $g''(T) = \cancel{\dfrac{-2}{(1-T)^3}} \quad \dfrac{-2}{(T-1)^3}$

(2) $\Rightarrow E[\hat{a}] = g\left(\dfrac{a}{a+1}\right) + \dfrac{1}{2}\dfrac{a}{n(a+2)(a+1)^2}$ $\cancel{\dfrac{-2}{(T-1)^3}}$

$$\times\, g''\left(\frac{a}{a+1}\right)$$

$$[\hat{a}] = \frac{\frac{a}{a+1}}{1-\frac{a}{a+1}} + \frac{1}{\cancel{2}}\frac{a}{n(a+2)(a+1)^2} \times \frac{-\cancel{2}}{\left(\frac{a}{a+1}-1\right)^3}$$

$=$

$$E[\hat{a}] = \frac{\frac{a}{\cancel{a+1}}}{\frac{1}{\cancel{a+1}}} - \frac{a}{n(a+2)(a+1)^2} \times \left(\frac{-1}{a+1}\right)^3$$

$$= a + \frac{a(a+1)^{\cancel{3}}}{n(a+2)\cancel{(a+1)^2}}$$

$$E[\hat{a}] = a + \frac{a(a+1)}{n(a+2)}$$

$$\cancel{E[\hat{a}] = \frac{na(a+2) + a(a+1)}{n(a+2)}}$$

$$\boxed{E[\hat{a}] = a\left(1 + \frac{(a+1)}{n(a+2)}\right)}$$

For $n \to \infty$,

$$\boxed{E[\hat{a}] \to a}$$

$\Rightarrow$ estimator is asymptotically unbiased

$$Var(\hat{a}) = \sigma_T^2 \left[g'(\mu_T)\right]^2$$

$$= \frac{a}{n(a+2)(a+1)^2} \times \left[g'\left(\frac{a}{a+1}\right)\right]^2$$

Instructor comments do not write in this section

$$var(\hat{a}) = \frac{a}{n(a+2)(a+1)^2} \times \left[\frac{1}{\left(1-\frac{a}{a+1}\right)^2}\right]^2$$

$$var(\hat{a}) = \frac{a}{n(a+2)(a+1)^2} \times \left[\frac{1}{\left(\frac{1}{a+1}\right)^2}\right]^2$$

$$var(\hat{a}) = \frac{a(a+1)^{\cancel{4}\,2}}{n(a+2)\cancel{(a+1)^2}}$$

$$\Rightarrow \boxed{var(\hat{a}) = \frac{a(a+1)^2}{n(a+2)}}$$

For large a, i.e., $a \to \infty$

we have,

$$\boxed{var(\hat{a}) \to \frac{a^2}{n}}$$

Which shows the same behaviorr as shown by MLE

Instructor comments: do not write in this section

d) we have $\hat{a} = g(T) = \dfrac{-1}{\underbrace{\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)}_{\to T}}$

$$E[\hat{a}] = g(\mu_T) + \frac{1}{2}\sigma_T^2 g''(\mu_T) \quad \text{— } 5$$

$$var(\hat{a}) = \sigma_T^2 [g'(\mu_T)]^2 \quad \text{— } 6$$

$$\mu_T = E[T] = E\left[\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E[\ln(X_i)]$$

$\cancel{E[\ln(X)] =}$

$\cancel{\ln(X) =}$

$$E[\ln(X)] = \int_0^1 \ln(x) \cdot a x^{a-1} \, dx$$

Using Integral calculator, we have

$$E[\ln(X)] = \cancel{\ln(a)} \; \frac{-1}{a}$$

$$\Rightarrow E[T] = \mu_T = \frac{1}{n}\sum_{i=1}^{n} \frac{-1}{a}$$

Instructor comments do not write in this section

$$\Rightarrow \quad \mu_T = \frac{-1}{a}$$

$$\sigma_T^2 = var\left(\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} var(\ln(X_i))$$

$$var(\ln(X)) = E\left[[\ln(X)]^2\right] - \left(E[\ln(X)]\right)^2$$

$$= \int_0^1 (\ln x)^2 a x^{a-1}\, dx - \left(\frac{-1}{a}\right)^2$$

$\downarrow$ From online tool

$$= \frac{2}{a^2} - \left(\frac{1}{a^2}\right)$$

$$\Rightarrow var(\ln(X)) = \frac{1}{a^2}$$

$$\therefore \sigma_T^2 = \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{a^2} = \frac{1}{na^2}$$

$$g'(T) = \frac{1}{T^2}$$

$$g''(T) = \frac{-2}{T^3}$$

eq 5 =>

$$E[\hat{a}] = \frac{-1}{-1/a} + \frac{1}{\cancel{2}} \frac{1}{na^2} \frac{-\cancel{2}}{(-1/a)^3}$$

$$= a \cancel{5} + \frac{a^3}{na^2}$$

$$E[\hat{a}] = a + \frac{a}{n}$$

$$\boxed{E[\hat{a}] = a\left(1 + \frac{1}{n}\right)}$$

For larger, n
i.e $n \to \infty$

$$\Rightarrow \boxed{E[\hat{a}] \to a}$$

estimator is asymptotically unbiased.

g) Taking $a = 2.18$, The table is filled:

For $n = 500$, i.e relatively small n, we see that the obtained values from both the estimators deviate slightly from the mean valve.

~~ore~~ Moreover, the true mean of the estimator also deviates slightly from the true value of a. Hence, in both the estimators, we can see a slight bias. However, the variance in the case of MLE is less than that of MoM, indicating that MLE is a better estimator than MoM.

For n = 500,000, We see that the values of a obtained from both the estimators is very much close to the true value of mean.

We also see that the estimated value of mean ~~also~~ ~~is also equal to~~ and the theoritically obtained mean of estimator both are very close to ~~a~~ true value of a for large n, i.e. n = 500,000. This proves our fact that the estimators are asymptotically unbiased. This is also seen by observing that both the variance of estimator and the estimated variance for both the estimator is (almost) 0.

The MLE estimator still stands better than MoM since MLE's obtained variance is in power of -6 and MoM's obtained variance is in power of -5.

120/120

Name: Hamad Abdul Razzaq		
ID No.: hr06899		
	True value of a = 2.18	
Experiment number Assume $n = 500$	$\hat{a}$ MoM estimate of a	$\hat{a}$ ML estimate of a
1	2.1275	2.1665
2	2.1269	2.1142
3	2.3208	2.3381
4	2.0352	2.0179
5	2.0822	2.1108
6	2.0937	2.0813
7	2.1595	2.2083
8	2.3043	2.2747
9	2.3966	2.3471
10	2.2891	2.2852
E[$\hat{a}$] = Estimated Mean value of $\hat{a}$ (the average of 10 experiment values)	2.1936	2.1944
E[$\hat{a}$] = Mean value of Estimator (the derived theoretical value)	2.1933	2.18436
Var($\hat{a}$) = Estimated Variance of $\hat{a}$ (the variance of 10 experiment values)	0.0151	0.013
Var($\hat{a}$) = True Variance of Estimator (the derived theoretical value)	0.0105	0.0095

Figure 1: **Table 1**

Name: Hamad Abdul Razzaq

ID No.: hr06899

	True value of a = 2.18	
Experiment number Assume $n = 500{,}000$	$\hat{a}$ MoM estimate of a	$\hat{a}$ ML estimate of a
1	2.1817	2.1829
2	2.1842	2.1831
3	2.1798	2.1792
4	2.1807	2.1809
5	2.1772	2.1779
6	2.1792	2.1783
7	2.1778	2.1765
8	2.1837	2.1845
9	2.1811	2.1815
10	2.1810	2.1821
E[$\hat{a}$] = Estimated Mean value of $\hat{a}$ (the average of 10 experiment values)	2.1805	2.1807
E[$\hat{a}$] = Mean value of Estimator (the derived theoretical value)	2.180003317	2.18000436
Var($\hat{a}$) = Estimated Variance of $\hat{a}$ (the variance of 10 experiment values)	1.87×10^{-5} ~~0.000~~	1.07×10^{-5} ~~0.000~~
Var($\hat{a}$) = True Variance of Estimator (the derived theoretical value)	$1.055 \times 10^{-5} \approx 0$	$9.5 \times 10^{-6} \approx 0$

Figure 2: Table 2

Q2 **[30 points]**: Suppose a hypothesis states that the mean is exactly 50. If a random sample of 40 items is taken to test this hypothesis, what is the value of Type II error if the population standard deviation is 8 and the alternative mean is 54? Use Type-I error equal to 0.01.

Q3 [50 points]: Compute a one-way ANOVA on the following data.

1	2	3	4	5
14	10	11	16	14
13	9	12	17	12
10	12	13	14	13
11	9	12	16	13
	10	13	17	12
	11		15	14
				12

Determine the observed F value. Compare the observed F value with the critical table F value and decide whether to reject the null hypothesis. Use $\alpha = 0.01$. Provide a neat, full-page size labeled box-plot diagram in your answer copy.

Name: ~~hr06899~~ Hamad Abdul Razzaq

ID No.: hr06899

Q3	1	2	3	4	5
Any upper outlier? Mention values	N/A	N/A	N/A	N/A	N/A
Upper adjustment	14	12	13	17	14
75th percentile	13.5	11	13	17	13.75
Median	12	10	12	16	13
25th percentile	10.5	9	11.75	15	12
Lower adjustment	10	9	11	14	12
Any lower outlier? Mention values	N/A	N/A	N/A	N/A	N/A

Figure 3: **Table 3**