Habib University
shaping futures

Linear Algebra – Math 205
Exercise Set of Lect 22 & 23
(SPRING 2023)
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Homework 6a

Question 04

Solution: To make notation simpler, let $\mathcal{S}$ and $\mathcal{T}$ be the bases. Form matrices S and T from the respective basis vectors. Since P is an orthogonal matrix, we have

$$PP^T = P^T P = I$$

Since $\mathcal{S}$ is an orthonormal basis, S is an orthogonal matrix and

$$SS^T = S^T S = I$$

Using that P is the transition matrix from S to T,

$$\begin{aligned} T &= PS \\ T^T T &= T^T PS = (PS)^T PS \\ T^T T &= S^T P^T PS = S^T S = I \end{aligned}$$

Similarly,

$$\begin{aligned} T &= PS \\ TT^T &= PST^T = PS(PS)^T \\ TT^T &= PSS^T P^T = P^T P = I. \end{aligned}$$

Since $T^T T = TT^T = I$, we have that T is an orthogonal matrix. Since T was formed from the basis vectors of $\mathcal{T}$, we have that $\mathcal{T}$ is an orthonormal basis.

Question 06

Solution:

$$\begin{aligned} AB(AB)^T = ABB^T A^T = AIA^T = AA^T = I \\ A^T(A^T)^T = A^T A = I \\ A^{-1}(A^{-1})^T = A^T(A^T)^T = A^T A = I \end{aligned}$$

Question 07

Solution:

$$< Au, Av >= (Av)^T(Au) = v^T A^T Au = v^T Iu = v^T u =< u, v >$$

This conditions holds when A is an orthogonal matrix, i.e $A^T A = I$.
One can find this condition by equating $< Au, Av >$ and $< u, v >$, so
$v^T A^T Au = v^T u \Rightarrow= v^T A^T Au = v^T Iu - v^T u = 0 \Rightarrow v^T(A^T A - I)u = 0$.

Question 10

Solution: Assume matrix A has two different eigenvalues corresponding to some eigenvector, such that

$$Ax = \lambda_1 x \qquad \& \qquad Ax = \lambda_2 x$$

so, $Ax = \lambda_1 x = \lambda_2 x \longrightarrow \lambda_1 x = \lambda_2 x \longrightarrow \lambda_1 x - \lambda_2 x = 0 \longrightarrow (\lambda_1 - \lambda_2)x = 0$
Hence, this implies $(\lambda_1 - \lambda_2) = 0$ so $\lambda_1 = \lambda_2$.

Question 11

Solution: Eigenvalues of Diagonal & Triangular matrices are the diagonal entries of a matrix.

Question 11

Solution: Since $Ax = \lambda x$ so, $A^k x = \lambda^k x$. By computing eigenvalues of a matrix A has 1,2,3 so the eigenvalues of a matrix A^{100} is $1^{100}, 2^{100}, 3^{100}$.

Question 13

Solution: $Ax = \lambda x \longrightarrow A^2 x = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$, similarly for $A^3 x = \lambda^3 x$.