$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq NPSPACE \subseteq EXP$

Class NL : $PATH = \{(G,s,t) \mid$ Graph G has a path from s to $t\}$

$PATH \in NL$

logspace on any of its branches

→ Nondeterministically guess a path starting from s, that is at most n vertices long.

→ If the path reaches t, accept

$PATH \in L$?

↳ nondeterminism helps guess all the paths.

↳ deterministically, need to make sure all paths are explored

↳ there can be an exponential number of paths

$PATH \in NL$

$PATH \in NL$-Hard : $A \leq_p B$

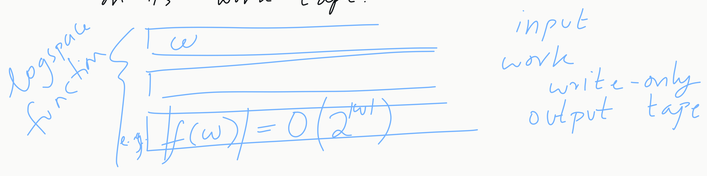
logspace reduction.

① $A \in NL$

② $A_i \in NL \leq_L A$ ($NL \stackrel{?}{=} L$)

logspace reduction → to be used to show NL-Hardness

we say f is logspace computable iff there is a TM with ① read-only input tape ② write-only output tape ③ read/write work tape that computes $f: \Sigma^* \to \Sigma^*$ using $O(\log n)$ cells on its work tape.



If $A \leq_L B$ and $B \in L$, then $A \in L$

$f: \Sigma^* \to \Sigma^*$

$x \in A \iff f(x) \in B$

$PATH \in NL$-Hard : ..

Proof Idea: Suppose $A \in NL$.

$A \leq_f PATH$, f is logspace computable

There is a configuration graph of A!

each node is a snapshot of a TM at some step of computation

each node has: ① all tape contents ② all pointer positions. ③ state

space required for a node is $O(s(n))$ where the TM is in $SPACE(s(n))$ i.e. each node has $O(\log n)$ length

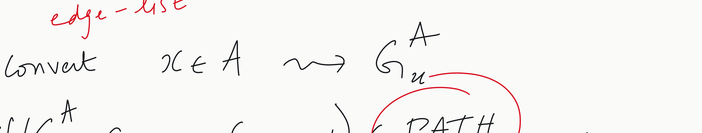
node: $x_1 x_2 \ldots\ldots x_{\log n}$ — log n length bitstring

$2^{\log n}$ strings

the graph has $O(n)$ nodes.

convert/reduce $x \stackrel{?}{\in} A \rightsquigarrow G_x^A$

$C_i \rightsquigarrow C_j$



☆ convert $x \in A \rightsquigarrow G_x^A$

☆ $if(G_x^A, C_{x,start}, C_{x,accept}) \in PATH$

NL Complete

2 Important problems:

① Is $NSPACE = SPACE$? ($NP = P$?)

② Is $NSPACE = coNSPACE$? ($NP = coNP$?)

if the answer 'no' $NSPACE \neq coNSPACE \Rightarrow NSPACE \neq SPACE$?

i.e. NSPACE is not closed under complementation

PSPACE = NPSPACE ✗

YES ✓

All deterministic complexity classes (decision classes) are closed under complementation!

Immerman – Szelepcsényi

$\overline{PATH} \in NL$

$\therefore NL = coNL$

$NSPACE = coNSPACE$

Savitch's Theorem

$NSPACE(f) \subseteq SPACE(f^2)$

i.e. $PSPACE = NPSPACE$

$P \stackrel{?}{=} NP$

$L \subseteq NL = coNL \subseteq P \subseteq NP \cap coNP \subseteq NP \cup coNP \subseteq PSPACE = NPSPACE \subseteq EXP$

$NL \subseteq P$?

$P \subset EXP$? ✓

$L \subsetneq PSPACE$

$NSPACE(f) \subseteq DTIME(2^f)$

$P \subsetneq EXP$? ✓

$NL \subseteq SPACE(\log^2 n)$

$EXPTIME \subseteq EXPSPACE$?

$L \subseteq P$