SOLUTION

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INSTRUCTOR:

MATH 205 LINEAR ALGEBRA

MIDTERM PART A

[Total Marks: 50]

FALL 2022

Question 1: [10 Marks]

(a) Suppose $AX = B$ is the matrix equation for a system of equations in m variables with A being an $m \times m$ coefficient matrix. Prove that if A is not singular, then this system has only one solution.

SOLUTION 1 Let $\underline{x}_1, \underline{x}_2$ be two solutions s.t.

$A\underline{x}_1 = \underline{B}$ and $A\underline{x}_2 = \underline{B}$

Since A is not singular, A^{-1} exists

$\Rightarrow A^{-1}(A\underline{x}_1) = A^{-1}(A\underline{x}_2)$

$I\underline{x}_1 = I\underline{x}_2 \Rightarrow \underline{x}_1 = x_2 \therefore$ unique solution

OR

SOLUTION 2 Since A^{-1} is unique:

$A\underline{x} = \underline{B} \Rightarrow A^{-1}(A\underline{x}) = A^{-1}\underline{B}$

$\underline{x} = A^{-1}\underline{B}$ is also unique

(b) List the three possible elementary row operations and define an elementary matrix.

① Swap 2 rows

② Multiplying a row by a non-zero constant

③ Adding non-zero multiple of 1 row to another

An elementary matrix is obtained by performing a single row operation on an identity matrix.

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Question 2: [10 Marks]

Let Y and Z be $n \times n$ matrices.

(a) Prove that if $I_n = YZ$ then $Z = Y^{-1}$.

$I_n = YZ \Rightarrow \det(I_n) = \det(Y)\det(Z) = 1$

$\therefore \det(Y) \neq 0, \det(Z) \neq 0 \therefore Y^{-1} \& Z^{-1}$ exist

$(Y^{-1}YZ = Y^{-1}I$

$IZ = Y^{-1}$

$Z = Y^{-1}$ (Proved)

(b) Prove that if $I_n = ZY$ then $Z = Y^{-1}$.

Since Y^{-1} exists (proved in part a):

$I = ZY$

$IY^{-1} = ZYY^{-1}$

$Y^{-1} = ZI$

$Z = Y^{-1}$ (Proved)

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Question 3: [10 Marks]

Consider a homogenous system in n variables for which the coefficient matrix is B. Given a non-singular $n \times n$ matrix M, show that the $BX = 0$ has only the trivial solution if and only if $(MB)X = 0$ only has a trivial solution.

$A \Leftrightarrow B$

B ($\Rightarrow$) MB is invertible $\Rightarrow \det(MB) \neq 0$
$\Rightarrow \det(B) \neq 0$ $\therefore$ B^{-1} exists and so, $BX = 0$
has only the trivial solution A

A ($\Rightarrow$) Suppose $(MB)X = 0$ for non-zero vector X
$\because M^{-1}$ exists, $(M^{-1}M)BX = M^{-1}0$
$BX = 0 \Rightarrow$ this only
has a trivial solution, which contradicts
what we supposed. So, we can
conclude $MBX = 0$ has only the
trivial solution.

Question 4: [10 Marks]

Show that the diagonal entries of a 3×3 skew-symmetric matrix are all zero.

Skew-Symmetric means $A^T = -A$

$$A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$$

$\Rightarrow a_{11} = -a_{11}, \; a_{22} = -a_{22}, \; a_{33} = -a_{33}$

This is only possible if $a_{11} = a_{22} = a_{33} = 0$

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Question 5: [10 Marks]

Prove the following identity WITHOUT evaluating determinants.

$$\begin{vmatrix} a_1 & b_1 + ta_1 & c_1 + rb_1 + sa_1 \\ a_2 & b_2 + ta_2 & c_2 + rb_2 + sa_2 \\ a_3 & b_3 + ta_3 & c_3 + rb_3 + sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

LHS:

$$= \begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix} + \underbrace{\begin{vmatrix} a_1 & ta_1 & c_1+rb_1+sa_1 \\ a_2 & ta_2 & c_2+rb_2+sa_2 \\ a_3 & ta_3 & c_3+rb_3+sa_3 \end{vmatrix}}_{=0 \text{ (since } C_2 = tC_1)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \underbrace{\begin{vmatrix} a_1 & b_1 & rb_1 \\ a_2 & b_2 & rb_2 \\ a_3 & b_3 & rb_3 \end{vmatrix}}_{=0 \ (\because C_3 = rC_2)} + \underbrace{\begin{vmatrix} a_1 & b_1 & sa_1 \\ a_2 & b_2 & sa_2 \\ a_3 & b_3 & sa_3 \end{vmatrix}}_{=0 \ (\because C_3 = sC_2)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \Rightarrow \text{RHS}$$

Since $\det(A) = \det(A^T)$

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MIDTERM PART B

[Total Marks: ~~40~~] 50

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Question 6: [10 Marks]

Check whether the set of all triples of real numbers with the standard vector addition and with scalar multiplication defined by $k(x, y, z) = (k^2x, k^2y, k^2z)$ is a vector space or not. If not, identify all axioms that fail to hold.

Axioms 1-5 all work because they are simply the same as addition axioms on the Euclidean space $\mathbb{R}^3$.

Axioms 6 and 10 also hold (by inspection)

Axiom 8 Let $\underline{u} = (x, y, z)$, $\underline{v} = (a, b, c)$

$(k+m)\underline{u} = k\underline{u} + m\underline{u}$

LHS: $(k+m)(x, y, z) = ((k^2+2km+m^2)x, (k^2+2km+m^2)y, (k^2+2km+m^2)z)$

RHS: $k\underline{u} + m\underline{u} = ((k^2+m^2)x, (k^2+m^2)y, (k^2+m^2)z)$

$\therefore$ Axiom 8 fails. Hence set not a vector space.

Axiom 7 holds $\because$ $k(\underline{u}+\underline{v}) = k\underline{u} + k\underline{v}$

$= (k^2(x+a), k^2(y+b), k^2(z+c))$

Axiom 9 holds: $k(l\underline{u}) = k(l^2x, l^2y, l^2z)$

$= (k^2l^2x, k^2l^2y, k^2l^2z) = (kl)(\underline{u})$

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Question 7: [10 Marks]

A

Prove: If W is a set of one or more vectors from a vector space V, then W is a subspace of V **if and only if** the following conditions hold:

(a) If $\underline{u}$ and $\underline{v}$ are vectors in W, then $\underline{u} + \underline{v} \in W$.

(b) If k is any scalar and $\underline{u}$ is any vector in W, then $k\underline{u} \in W$.

$\Big\}$ B

[3] A ⇒ B If W is a subspace of V, then all vector space axioms are satisfied, including (1) and (6), which are same as (a) & (b)

B ⇒ A Assume (a) and (b) hold. We now show the other 8 axioms are satisfied.

[1] Axioms 2, 3, 7, 8, 9 and 10 automatically satisfied by vectors in W, as they're satisfied by all vectors in V.

[3] Axiom 4: Let $\underline{u} \in W$. By (b), $k\underline{u} \in W$ for every scalar k. So, for $k = 0$,

$k\underline{u} = 0\underline{u} = \underline{0}$

But $k\underline{u} \in W$. Therefore, $\underline{0} \in W$

[3] Axiom 5: Similarly, for $k = -1$,

$k\underline{u} = (-1)\underline{u} = -\underline{u} \in W$

$\therefore$ W is a subspace of V.

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Question 8: [10 Marks]

(a) Indicate whether each of the following statements is always true or sometimes false. Justify your answers:

(i) If W is a set of one or more vectors from a vector space V, and if $k\underline{u} + \underline{v}$ is a vector in W for all vectors $\underline{u}$ and $\underline{v}$ in W and all $k \in \mathbb{R}$, then W is a subspace of V. [3]

TRUE $\underline{0} \in W$, since if $\underline{u} = \underline{v}$, and $k = -1$, then $-\underline{u} + \underline{u} = \underline{0}$

Axiom 1 Let $k = 1 \Rightarrow \underline{u} + \underline{v} \in W$

Axiom 6 Let $\underline{v} = 0 \Rightarrow k\underline{u} \in W$

(ii) If $span\ (S_1) = span\ (S_2)$ then $S_1 = S_2$. [3]

FALSE Give example of any 2 unequal sets that have same span. e.g $S_1 = \{(1,0), (0,1)\}$ and $S_2 = \{(2,0), (0,2)\}$ under standard addition & multiplication in $\mathbb{R}^2$.

(b) In words, describe a set of matrices that spans M_{nn}. [4]

This can be any set of matrices containing n^2 elements, each matrix having all entries zero except one, which is non-zero (such that they will be bases for M_{nn})

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Question 9: [10 Marks]

Let $\underline{u}$ and $\underline{v}$ be vectors in the Euclidean space $\mathbb{R}^n$. Show the following:

(a) $A\underline{u} \cdot \underline{v} = \underline{u} \cdot A^T \underline{v}$.

$$A\underline{u} \cdot \underline{v} = \underline{v}^T (A\underline{u}) \quad (\text{using eq. } \underline{u} \cdot \underline{v} = \underline{v}^T \underline{u})$$
$$= (\underline{v}^T A)\underline{u}$$
$$= (A^T \underline{v})^T \underline{u} = \underline{u} \cdot A^T \underline{v}$$

(b) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.

$$\|\underline{u} + \underline{v}\|^2 = (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v})$$
$$= (\underline{u} \cdot \underline{u}) + 2(\underline{u} \cdot \underline{v}) + (\underline{v} \cdot \underline{v})$$
$$= \|\underline{u}\|^2 + 2(\underline{u} \cdot \underline{v}) + \|\underline{v}\|^2$$
$$\leq \|\underline{u}\|^2 + 2|\underline{u} \cdot \underline{v}| + \|\underline{v}\|^2 \quad (\text{Absolute value property})$$
$$\leq \|\underline{u}\|^2 + 2\|\underline{u}\|\|\underline{v}\| + \|\underline{v}\|^2 \quad (\text{CS Inequality})$$
$$\|\underline{u} + \underline{v}\|^2 = (\|\underline{u}\| + \|\underline{v}\|)^2$$

Taking square root on both sides:

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\| \quad (\text{Proved})$$

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Question 10: [10 Marks]

Let V be the subspace of $F(-\infty, \infty)$ given by $V = span\{\sin x, \cos x\}$. Show that for any given value of θ, $\boldsymbol{f_1} = \cos(x + \theta)$ and $\boldsymbol{g_1} = \sin(x + \theta)$, we have that $\boldsymbol{f_1} \in V$ and $\boldsymbol{g_1} \in V$.

$f_1 = \cos(x+\theta) = \cos\theta \cos x - \sin\theta \sin x$

$f_2 = \sin(x+\theta) = \sin x \cos\theta + \cos x \sin\theta$

For any given value of θ, both $\sin\theta$ and $\cos\theta$ are also simply constant real numbers (and hence scalars).

$\therefore$ f_1 and f_2 are both simply linear combinations of $\sin x$ and $\cos x$, and therefore in their span i.e. both $\in V$.