Lecture 1

Tuesday, January 11, 2022 8:19 AM

1) LINEAR ALGEBRA

LECTURE NO. 1

MATH 205

LINEAR ALGEBRA:

THE BRANCH OF MATHEMATICS CONCERNED WITH LINEAR EQUATIONS, MATRICES, DETERMINANTS, VECTOR SPACES, ETC.

LINEAR $\longrightarrow$ RELATING TO THE FIRST DEGREE; HAVING NO VARIABLE RAISED TO ANY POWER.

(P.2)/8th ED.

INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS: (P.1)/7th ED.

A LINE IN THE XY-PLANE IS AN EQUATION OF THE FORM

$$\boxed{a_1x + a_2y = b \quad (1)}$$

EQUATION (1) IS CALLED A LINEAR EQUATION IN THE VARIABLES $\boxed{x}$ AND $\boxed{y}$.

WE DEFINE A LINEAR EQUATION IN THE $\boxed{n}$ VARIABLES $x_1, x_2, \ldots, x_n$ TO BE ONE THAT CAN

BE EXPRESSED IN THE FORM

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad -(2)$$

WHERE $a_1, a_2, \ldots, a_n$, AND $\boxed{b}$ ARE REAL CONSTANTS. THE VARIABLES IN A LINEAR EQUATION ARE CALLED THE $\boxed{\text{UNKNOWNS}}$ AND IN (2) UNKNOWNS ARE $x_1, x_2, \ldots, x_n$.

THE REAL CONSTANTS $a_1, a_2, \ldots, a_n$ ARE ALSO CALLED COEFFICIENTS.

QUESTION:

ARE THE FOLLOWING LINEAR EQUATIONS ?

(1) $x + 3y^2 = 7$

(2) $y - \sin x = 0$

(3) $3x + 2y - z + xz = 4$

(4) $\sqrt{x_1} + 2x_2 + x_3 = 1$

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SOLUTION: (NONE OF THEM)

(1) NO. $\because$ IN $x+3y^2=7$, $\boxed{y^2}$ IS PRESENT BUT IN A LINEAR EQUATION ALL VARIABLES OCCUR ONLY TO THE FIRST POWER.

(2) $y - \sin x = 0$, NO, $\because$ IN A LINEAR EQUATION VARIABLES DO NOT APPEAR AS ARGUMENTS FOR TRIGNOMETRIC, LOGARITHMIC, OR EXPONENTIAL FUNCTIONS.

(3) AND (4) NO, $\because$ IN $3x+2y-z+xz=4$ & $\sqrt{x_1}+2x_2+x_3=1$, $\boxed{xz}$ AND $\boxed{\sqrt{x_1}}$ ARE PRESENT BUT A LINEAR EQUATION DOES NOT INVOLVE ANY PRODUCTS OR ROOTS OF VARIABLES.

Note: A solution is a set.

NOTE: A SOLUTION OF A LINEAR EQUATION $\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$ IS A SET OF NUMBERS $\boxed{\{s_1, s_2, \ldots, s_n\}}$ SUCH THAT THE EQUATION IS SATISFIED WHEN WE SUBSTITUTE

$$\boxed{x_1 = s_1,\ x_2 = s_2,\ \ldots\ldots,\ x_n = s_n.}$$

DEFINITION

A SYSTEM OF LINEAR EQUATIONS IT IS A $\boxed{\text{FINITE SET}}$ OF LINEAR EQUATIONS IN THE VARIABLES $\boxed{x_1, x_2, \ldots\ldots, x_n.}$ (or other variables)

A GENERAL $\boxed{\text{SYSTEM}}$ OF $\boxed{m}$ LINEAR EQUATIONS IN $\boxed{n}$ UNKNOWNS WILL BE WRITTEN

$$\begin{array}{ll} & \quad\ \text{Col 1} \qquad\quad \text{Col-2} \qquad\qquad\qquad\quad \text{Col-n} \\ \text{Row 1} & [\, a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ \text{Row 2} & [\, a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ & \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots \qquad\quad \vdots \\ \text{Row m} & [\, a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = b_m \end{array}$$

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THE FIRST SUBSCRIPT $\boxed{i}$ $(1 \le i \le m)$ ON THE COEFFICIENT $\boxed{a_{ij}}$ INDICATES THE EQUATION IN WHICH THE

COEFFICIENT OCCURS, AND THE SECOND SUBSCRIPT $j\ (1 \le j \le n)$ INDICATES WHICH UNKNOWN IT MULTIPLIES.

LINEAR SYSTEM (3) CAN BE ABBREVIATED BY WRITING ONLY THE RECTANGULAR ARRAY OF NUMBERS:

→ ARRANGEMENT

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

THIS IS CALLED THE AUGMENTED MATRIX DUE TO $\boxed{b_i\text{'s}}$ $(1 \le i \le m)$.

AUGMENT ⟶ INCREASE/ENLARGE

DUE TO THE PRESENCE OF THE ENTRIES ON R.H.S. OF THE LINEAR SYSTEM 3.

SO IT'S IMPORTANT TO STUDY MATRICES IN DETAIL.

LATER ON WE SHALL SEE THAT THEY HELP US IN SOLVING THE SYSTEMS OF LINEAR EQUATIONS

MATRIX: (P.24)/8th ED. (P.25)/7th ED.

① A MATRIX IS A RECTANGULAR ARRAY OF NUMBERS ENCLOSED IN BRACKETS. THE NUMBERS IN THE ARRAY ARE CALLED THE ENTRIES IN THE MATRIX.

② THE SIZE OF A MATRIX IS DESCRIBED BY SPECIFYING THE NUMBER OF ROWS (HORIZONTAL LINES) AND COLUMNS (VERTICAL LINES).

③ PLURAL OF A MATRIX IS MATRICES.

EXAMPLES: (OF MATRICES)

(1) $A = \begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix} \rightarrow$ 2 ROWS

AND $\begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix} \rightarrow$ 3 COLUMNS

SIZE: 2×3

2 → ROWS

3 → COLUMNS

NOTE: $2 \times 3 = 6$ GIVES TOTAL NUMBER OF ENTRIES.

NOTE: A MATRIX IN WHICH NO. OF ROWS $\neq$ NO. OF COLUMNS IS CALLED A RECTANGULAR MATRIX.

$\therefore A = \begin{bmatrix} 2 & 4 & 8 \\ 5 & -3 & 0 \end{bmatrix}$ IS A RECTANGULAR MATRIX $\because$ SIZE: 2×3

$\boxed{2 \neq 3}$.

(2) LET $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left.\right\}$ 2 ROWS, 2 COLUMNS

SIZE: 2×2

HERE $\boxed{B}$ IS A SQUARE MATRIX OF ORDER $\boxed{2}$ OR 2×2 $\because$ NUMBER OF ROWS ARE = NUMBER OF COLUMNS AND ARE $= \boxed{2}$.

(3) $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ } SIZE: 2×1 : ORDER

THIS TYPE OF MATRIX WHICH HAS ONLY ONE COLUMN IS ALSO CALLED A COLUMN VECTOR AND SIMILARLY A MATRIX WHICH HAS ONLY ONE ROW IS CALLED A ROW VECTOR E.G. $[a_1 \; a_2 \; a_3]$ } 1×3 SIZE OR ORDER 1×3.

General for Col. vector: Matrix of size $n \times 1$, where $n \in \mathbb{N}$

General for Row vector: matrix of size $1 \times m$, where $m \in \mathbb{N}$

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IN GENERAL $m \times n$ MATRIX OR A MATRIX OF ORDER $m \times n$ MIGHT BE WRITTEN

$$\begin{array}{cc} & \text{COLUMN 1} \\ \text{ROW 2} & \\ B = & \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \end{array}$$

WITH m ROWS AND n COLUMNS

OR $[b_{ij}]_{m \times n}$

Where i goes from 1 to m, i.e. $1 \leq i \leq m$

and j goes from 1 to n i.e. $1 \leq j \leq n$

$m \in \mathbb{N}$ $n \in \mathbb{N}$

WHERE b_{ij} IS USED TO DENOTE THE ENTRY THAT OCCURS IN ROW i AND COLUMN j OF B.

DEFINITIONS:

① THE ENTRIES $a_{11}, a_{22}, \ldots, a_{nn}$ IN A SQUARE MATRIX A OF ORDER n, ARE SAID TO BE ON THE MAIN DIAGONAL OF A AS SHOWN HERE IN THE FOLLOWING MATRIX

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

DIAGONAL OF A.

(2) A SQUARE MATRIX IN WHICH ALL ENTRIES OFF THE MAIN DIAGONAL ARE ZERO IS CALLED A DIAGONAL MATRIX.

QUESTION:

ARE THE FOLLOWING MATRICES DIAGONAL?

(1) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $\boxed{\text{YES}}$

(2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\boxed{\text{YES}}$

3 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ → NOT A DIAGONAL

NO ∵ THIS IS NOT A SQUARE MATRIX.

4 $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

YES THIS IS A DIAGONAL MATRIX. ITS ALL ENTRIES ARE ZERO AND IS ALSO CALLED A ZERO OR NULL MATRIX, DENOTED BY 0

DEFINITION:

IF A AND B ARE ANY TWO MATRICES OF THE SAME SIZE, THEN THE SUM A+B IS THE MATRIX OBTAINED BY ADDING TOGETHER THE CORRESPONDING ENTRIES IN THE TWO MATRICES.

CONSIDER THE FOLLOWING EXAMPLE:

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$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & -2 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

FIND $\boxed{A+B}$ AND $\boxed{B+C}$

SOLUTION: $\boxed{A+B}$ IS NOT DEFINED $\because$ A & B ARE OF DIFFERENT SIZE.

$$B+C = \begin{bmatrix} 3+1 & 2+0 \\ 5+0 & 6+1 \end{bmatrix}$$

$$\Rightarrow \boxed{B+C = \begin{bmatrix} 4 & 2 \\ 5 & 7 \end{bmatrix}}$$

TRY THE FOLLOWING:

SALES FIGURES FOR THREE PRODUCTS I, II, III IN STORE A ON MONDAY (M), TUESDAY (T), ... GIVEN BY

12)

M	T	W	Th	F	S	
40	30	81	0	21	47	I
0	12	78	50	50	96	II
10	0	0	27	43	78	III

SIMILARLY THE DATA FOR STORE B IS

M	T	W	Th	F	S	
20	19	18	1	12	74	I
2	21	87	50	49	96	II
10	1	2	72	34	87	III

AND FOR STORE C IS

M	T	W	Th	F	S	
20	3	18	1	12	45	I
1	12	78	5	0	6	II
1	2	3	6	8	99	III

BY USING MATRICES FIND THE TOTAL SALES OF EACH PRODUCT EACH DAY.

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SOLUTION:

$$\begin{bmatrix} 40 & 30 & 81 & 0 & 21 & 47 \\ 0 & 12 & 78 & 50 & 50 & 96 \\ 10 & 0 & 0 & 27 & 43 & 78 \end{bmatrix} + \begin{bmatrix} 20 & 19 & 18 & 1 & 12 & 74 \\ 2 & 21 & 87 & 50 & 49 & 96 \\ 10 & 1 & 2 & 72 & 34 & 87 \end{bmatrix} + \begin{bmatrix} 20 & 3 & 18 & 1 & 12 & 45 \\ 1 & 12 & 78 & 5 & 0 & 6 \\ 1 & 2 & 3 & 6 & 8 & 99 \end{bmatrix}$$

$$= \begin{array}{c c} \begin{array}{cccccc} \boxed{M} & \boxed{T} & \boxed{W} & \boxed{Th} & \boxed{F} & \boxed{S} \end{array} & \\ \begin{bmatrix} 80 & 52 & 117 & 2 & 45 & 166 \\ 3 & 45 & 243 & 105 & 99 & 198 \\ 21 & 3 & 5 & 105 & 85 & 264 \end{bmatrix} & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{array} \end{array}$$

THIS MATRIX GIVES THE TOTAL SALES OF EACH PRODUCT EACH DAY.

14) DEFINITION

TWO MATRICES ARE DEFINED TO BE EQUAL (1) IF THEY HAVE THE SAME SIZE (ORDER) AND (2) THE CORRESPONDING ENTRIES IN THE TWO MATRICES ARE EQUAL.

Two conditions

E.g. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

HERE $A \neq B$, BUT $\boxed{A}$ AND $\boxed{C}$ ARE EQUAL OR $\boxed{A = C.}$

Exercise Set 1.1

Questions 1,2,3,4(c),4(d), 5(c), 7,8,9