NP is asymmetric. “Yes” instances of a language can be easily verified. “No” instances not so easily.

What about the class where “No” answers can be easily verified?

TAUT = { φ | φ is a tautology } vs TAUT’ = { φ | φ is not a tautology}.

HAMCYCLE = { G | G contains a hamiltonian cycle }, vs HAMCYCLE’.

2.6.1 coNP

If $L \subseteq \{0, 1\}^*$ is a language, then we denote by $\overline{L}$ the *complement* of L. That is, $\overline{L} = \{0, 1\}^* \setminus L$. We make the following definition:

Definition 2.19 $\mathbf{coNP} = \left\{L : \overline{L} \in \mathbf{NP}\right\}$. ◇

Theorem: Each co-NP-complete problem is the complement of an NP-complete problem.

Prove that TAUT is coNP-Complete.
SAT’ $\leq_p$ TAUT

Definition 2.20 (**coNP**, *alternative definition*) For every $L \subseteq \{0, 1\}^*$, we say that $L \in$ **coNP** if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M such that for every $x \in \{0, 1\}^*$,

$$x \in L \Leftrightarrow \forall u \in \{0, 1\}^{p(|x|)}, \quad M(x, u) = 1$$ ◇

Note the use of the “∀” quantifier in this definition where Definition 2.1 used ∃.

We can define **coNP**-completeness in analogy to **NP**-completeness: A language is **coNP**-complete if it is in **coNP** and every **coNP** language is polynomial-time Karp reducible to it.

Theorem. P ⊆ NP ∩ coNP

Theorem. If P = NP, then NP=coNP.

FACTOR = { (m, r) | r is prime, ∃ s < r, s is prime, s divides m}

Integer Factorization is both in NP and co-NP but not known to be in P.

Proof that *Each co-NP-complete problem is the complement of an NP-complete problem:*

Consider L $\in$ coNP-complete, i.e. L $\in$ coNP and all problems in coNP reduce to L

L' $\in$ NP by definition. The definition of karp-reduction ensures that a valid function f is one such that

$\forall x \quad x \in A$ iff $f(x) \in L$

which is logically equivalent to

$x \notin A$ iff $f(x) \notin L$, i.e. $x \in A'$ iff $f(x) \in L'$

Therefore, the same reduction function can be used to reduce A' to L', and L' is NP-Hard as well.