Friday, Aug 23, 2024

Lecture 1 + 2A

Logistics:

1. Attendance policy
2. The course will go online in end of september, all exams will happen in person
3. NOC is a strict prerequisite, tell students who may still be considering
4. What skills do you learn :
   a. classify any given problem into a number of classes based on complexity,
   b. write and understand complicated proofs (become great at language manipulation),
   c. understand some fundamental concepts in mathematics such as P vs NP, and the relation between time and space complexity,
   d. become better at algorithmic thinking because it is intricately linked with complexity.
5. Type of course : theoretical, proofs, abstract
6. Type of attention : ready to play with language, leave world behind
7. Type of teacher : not a standup comic, just straight up teaching, contrary to what may appear I do put in effort in preparing lectures 😛

Bar Fight Prevention Problem

Imagine that you are an exceptionally tech-savvy security guard of a bar in a small town. Every Friday, half of the inhabitants of the town go out, and the bar you work at is well known for its nightly brawls. This of course results in an excessive amount of work for you; having to throw out intoxicated guests is tedious and rather unpleasant labor. Thus you decide to take preemptive measures. As the town is small, you know everyone in it, and you also know who will be likely to fight with whom if they are admitted to the bar. So you wish to plan ahead, and only admit people if they will not be fighting with anyone else at the bar. At the same time, the management wants to maximize profit and is not too happy if you on any given night reject more than *k* people at the door. Thus, you are left with the following optimization problem. You have a list of all of the n people who will come to the bar, and for each pair of people a prediction of whether or not they will fight if they both are admitted. You need to figure out whether it is possible to admit everyone except for at most k troublemakers, such that no fight breaks out among the admitted guests. Let us call this problem the **Bar Fight Prevention** problem.

What we know:

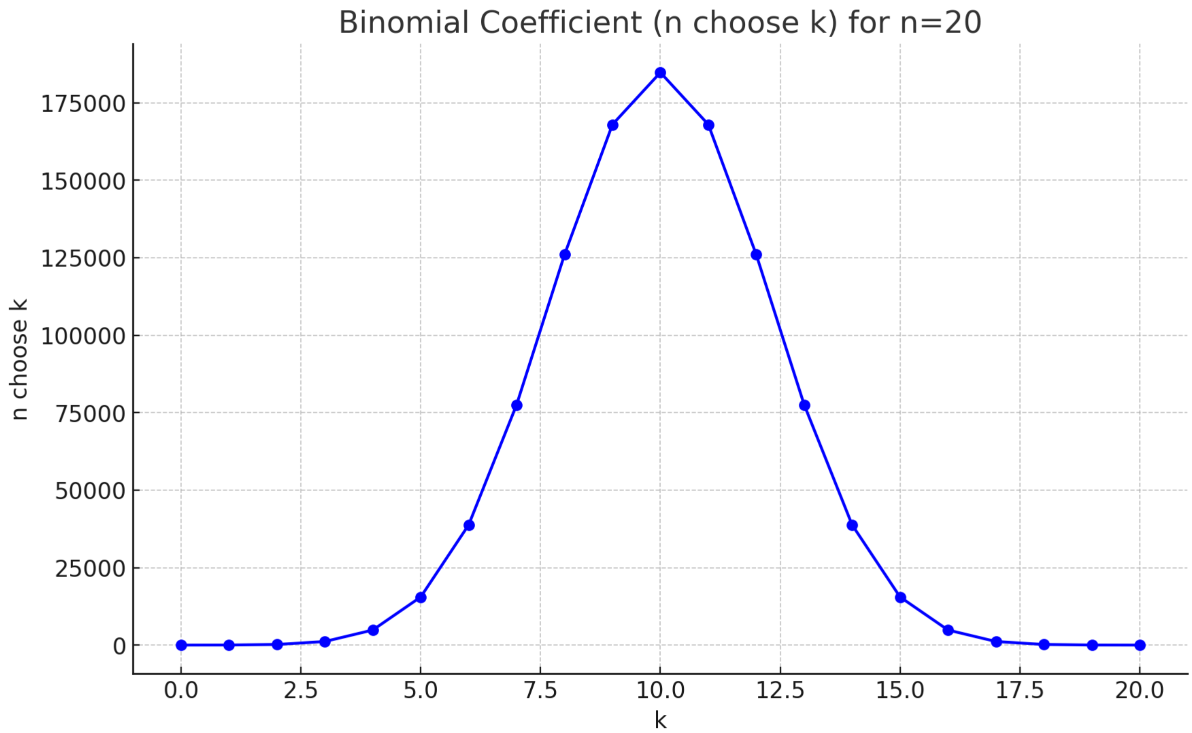
1. n guests
2. each pair of guests, whether there is a conflict

Our goal:

3. Accept guests such that there are no conflicts by refusing at most k guests.

Strategies:

1. Check all the 2^n combinations, to see if any of them meet the criterion. $\Omega(2^n p)$ where p is the minimum time taken per combination.
2. Choose the k people to reject. Ω(n choose k) let's verify that this number is exponential in n using sterling's approximation: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$



(Stirling's approximation given at the end)

3. Accept all people who have 0 conflicts (suppose x people), reject all people who have more than k conflicts (suppose y people). Ω((n-x-y) choose (k-y)).
4. Since all people have between 1 and k conflicts. At most k^2 conflicts can be removed. $\Omega(2k^2$ choose k). e.g. Ω (200 choose 10).

Vertex Cover Problem :
a set of vertices (of size k) in a graph where each edge of the graph is incident to at least one vertex from the set.

Q: How to map this problem to the vertex cover problem?

Independent Set Problem :
a set of vertices (of size k) in a graph, no two of which are adjacent.

This problem hsa $\Omega(2^n)$
NP-Complete (in the general case, we do not have an efficient solution).

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&\approx \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi k}(\frac{k}{e})^k\sqrt{2\pi(n-k)}(\frac{(n-k)}{e})^(n-k)} && \text{(Stirling's approximation)} \\
&\approx \frac{\sqrt{2\pi n}(n^n)}{\sqrt{2\pi k}(k^k)\sqrt{2\pi(n-k)}(n-k)^{n-k}} && \text{(e's cancel out)} \\
&\approx \frac{(n^n)}{(k^k)(n-k)^{n-k}}\frac{\sqrt{2\pi n}}{\sqrt{2\pi k}\sqrt{2\pi(n-k)}} && \text{(re-arranged terms)} \\
&\approx \frac{(n^n)}{(k^k)(n-k)^{n-k}}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{n}}{\sqrt{k(n-k)}} && \text{(re-arranged terms)} \\
&\approx \frac{(n^n)}{(k^k)(n-k)^{n-k}}\frac{1}{\sqrt{2\pi}}\frac{\sqrt{k+(n-k)}}{\sqrt{k(n-k)}} && (\sqrt{n} = \sqrt{k+(n-k)}) \\
&\approx \frac{(n^n)}{(k^k)(n-k)^{n-k}}\frac{1}{\sqrt{2\pi}} && (\frac{\sqrt{k+(n-k)}}{\sqrt{k(n-k)}} \leq 1^*) \\
&\approx \frac{(n^n)}{(k^k)(n-k)^{n-k}} && \text{(ignoring constant term)}
\end{aligned}
$$

*The sum of two positive integers greater than 1 is less then or equal to their product.

Taking logarithm on both sides:

$$
\begin{aligned}
\log\binom{n}{k} &\approx \frac{\log n^n}{\log k^k \log(n-k)^{n-k}} \\
&\approx \frac{n\log n}{k\log k + (n-k)\log(n-k)} \\
&\approx n\log n - k\log k - (n-k)\log(n-k)
\end{aligned}
$$

Taking the value of $k = \frac{n}{2}$:

$$
\begin{aligned}
\log\binom{n}{k} &\approx n\log n - \frac{n}{2}\log\frac{n}{2} - \frac{n}{2}\log\frac{n}{2} \\
&\approx n\log n - n\log\frac{n}{2} \\
&\approx \log 2^n
\end{aligned}
$$

Therefore, $\binom{n}{k} \approx \log 2^n$.