Homework 11

Exercise 5.4

Question 03:

Solution Verified

Step 1

(a)

We need to check linear independence and spanning of the given vectors i.e. we need to solve two systems

$$c_1(1,0,0)+c_2(2,2,0)+c_3(3,3,3)=(0,0,0) \quad c_1(1,0,0)+c_2(2,2,0)+c_3(3,3,3)=(b_1,b_2,b_3)$$

$$\begin{aligned} c_1+2c_2+3c_3&=0\\ 2c_2+3c_3&=0\\ 3c_3&=0 \end{aligned}$$

$$\begin{aligned} c_1+2c_2+3c_3&=b_1\\ 2c_2+3c_3&=b_2\\ 3c_3&=b_3 \end{aligned}$$

If we prove that $\det A \neq 0$, where

$$A=\begin{bmatrix} 1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 3 \end{bmatrix}$$

(note that columns of matrix A are coordinates of given vectors), we will prove that homogeneous system has trivial solution and nonhomogeneous system has solution i.e. given vectors will form basis for $\mathbb{R}^3$. Let's compute determinant

$$\begin{vmatrix} 1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 3 \end{vmatrix}=1\cdot 2\cdot 3=6\neq 0$$

we used property of upper triangular matrices whose determinant is equal to product of diagonal elements. We proved that $\det A \neq 0$ so given vectors form basis for $\mathbb{R}^3$.

(b)

We will use procedure from part (a)

$$\begin{aligned} \det A &= \begin{vmatrix} 3 & 2 & 1\\ 1 & 5 & 4\\ -4 & 6 & 8 \end{vmatrix}\\ &=3\begin{vmatrix} 5 & 4\\ 6 & 8 \end{vmatrix}-\begin{vmatrix} 2 & 1\\ 6 & 8 \end{vmatrix}-4\begin{vmatrix} 2 & 1\\ 5 & 4 \end{vmatrix}\\ &=3(40-24)-(16-6)-4(8-5)\\ &=26 \end{aligned}$$

Since $\det A \neq 0$, given vectors form basis for $\mathbb{R}^3$.

(c)

We will use procedure from part (a)

$$\begin{aligned}\det A &= \begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & -7 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & -7 \\ 1 & 1 \end{vmatrix} \\ &= 2(1+7) - 4(-3+7) \\ &= 0 \end{aligned}$$

. Since $\det A = 0$, given vectors don't form basis for $\mathbb{R}^3$.

(d)

We will use procedure from part (a)

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 2 \\ -1 & 5 \end{vmatrix} - 2\begin{vmatrix} 6 & 2 \\ 4 & 5 \end{vmatrix} - \begin{vmatrix} 6 & 4 \\ 4 & -1 \end{vmatrix} \\ &= 20 + 2 - 2(30-8) - (-6-16) \\ &= 0 \end{aligned}$$

. Since $\det A = 0$, given vectors don't form basis for $\mathbb{R}^3$.

Result

(a) Yes.

(b) Yes.

(c) No.

(d) No.

Question 04:

Solution Verified

Step 1

(a)

We need to check linear independence and spanning of the given vectors i.e. we need to solve two systems

$$c_1(1-3x+2x^2) + c_2(1+x+4x^2) + c_3(1-7x) = 0$$

$$c_1(1-3x+2x^2) + c_2(1+x+4x^2) + c_3(1-7x) = b_1 + b_2x + b_3x^2$$

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -3c_1 + c_2 - 7c_3 &= 0 \\ 2c_1 + 4c_2 &= 0 \end{aligned}$$

$$\begin{aligned} c_1 + c_2 + c_3 &= b_1 \\ -3c_1 + c_2 - 7c_3 &= b_2 \\ 2c_1 + 4c_2 &= b_3 \end{aligned}$$

If we prove that $\det A \neq 0$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{bmatrix}$$

(note that columns of matrix A are coefficients of given vectors), we will prove that homogeneous system has trivial solution and nonhomogeneous system has solution i.e. given vectors will form basis for P_2. Let's compute determinant

$$\begin{aligned}\det A &= \begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 1 \\ 1 & -7 \end{vmatrix} - 4\begin{vmatrix} 1 & 1 \\ -3 & -7 \end{vmatrix} \\ &= 2(-7-1) - 4(-7+3) \\ &= 0\end{aligned}$$

Since $\det A = 0$, given vectors don't form basis for P_2.

Step 2

(b)

We will use procedure from part (a)

$$\begin{aligned}\det A &= \begin{vmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 5 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 6 & 2 \end{vmatrix} - \begin{vmatrix} 4 & -1 \\ 6 & 4 \end{vmatrix} \\ &= -2 - 20 - 2(8-30) - (16+6) \\ &= 0\end{aligned}$$

. Since $\det A = 0$, given vectors don't form basis for P_2.

(c)

We will use procedure from part (a)

$$\det A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

we used property of down triangular matrices whose determinant is equal to product of diagonal elements.

Since $\det A \neq 0$, given vectors form basis for P_2.

(d)

We will use procedure from part (a)

$$\begin{aligned}\det A &= \begin{vmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} \\ &= -4\begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 6 & 8 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & 8 \\ 5 & 4 \end{vmatrix} \\ &= -4(5-8) - (6-16) + 3(24-40) \\ &= -26\end{aligned}$$

we used property of down triangular matrices whose determinant is equal to product of diagonal elements.

Since $\det A \neq 0$, given vectors form basis for P_2.

Result

(a) No.

(b) No.

(c) Yes.

(d) Yes.

Question 06:

Step 1 1 of 2

Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$ and $\mathbf{v}_3 = \cos 2x$.

(a) Observe that $v_1 - v_2 - v_3 = \cos^2 x - \sin^2 x - \cos 2x = \cos^2 x - \sin^2 x - (\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x - \cos^2 x + \sin^2 x = 0$for all $x \in (-\infty, \infty)$ which gives us that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set on V and therefore S is not a basis for V.

Step 2 2 of 2

(b) Consider the set $S' = \{v_1, v_2\}$.

Take scalars a and b such that $a\cos^2 x + b\sin^2 x = 0$ for all $x \in (-\infty, \infty)$.

Taking $x = 0$ we have that $a = 0$ and taking $x = \frac{\pi}{2}$ we have that $b = 0$. Hence the set S' is a linearly independent set in V.

Consider a vector $f(x) = a\cos^2 x + b\sin^2 x + c\cos 2x \in V$.

Since $\cos 2x = \cos^2 x - \sin^2 x$ for all $x \in (-\infty, \infty)$ we have that $f(x) = a\cos^2 x + b\sin^2 x + c(\cos^2 x - \sin^2 x) = a\cos^2 x + b\sin^2 x + c\cos^2 x - c\sin^2 x = (a + c)\cos^2 x + (b - c)\sin^2 x$ which gives us that $f(x) \in \text{span}(S')$ and therefore S' spans V.

Hence S' is a linearly independent set of V which spans V and therefore it is a basis for V.

Question 13:

Solution Verified

Step 1 1 of 3

The given system can be expressed in the form $Ax = 0$ where,

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The matrix A can be reduced as

$$\begin{aligned} A &= \begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \\ C_1 \leftrightarrow C_2 &: \begin{bmatrix} 1 & 3 & 1 & 1 \\ -1 & 5 & 1 & -1 \end{bmatrix} \\ R_2 \to R_2 + R_1 &: \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 8 & 2 & 0 \end{bmatrix} \end{aligned}$$

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Thus, the given system becomes

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 8 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies

$$\begin{aligned}x_1 + 3x_2 + x_3 + x_4 &= 0\\ 8x_2 + 2x_3 &= 0\\ \implies x_3 &= -4x_2\\ x_1 + 3x_2 - 4x_2 + x_4 &= 0\\ x_1 - x_2 + x_4 = 0 \implies x_1 &= x_2 - x_4\end{aligned}$$

Thus,

$$\begin{aligned}x &= \begin{bmatrix} x_2 - x_4 \\ x_2 \\ -4x_2 \\ x_4 \end{bmatrix}\\ &= x_2 \begin{bmatrix} 1 \\ 1 \\ -4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\end{aligned}$$

Step 2

2 of 3

Since the vectors $(1, 1, -4, 0), (-1, 0, 0, 1)$ are linearly independent in $\mathbb{R}^4$ as they are not multiple of each other. So dimension of solution space is 2 and its basis is given by $\{(1, 1, -4, 0), (-1, 0, 0, 1)\}$.

Result

3 of 3

The basis is given by $\{(1, 1, -4, 0), (-1, 0, 0, 1)\}$.

Question 15:

Step 1

1 of 2

Note that the given system of homogeneous equation can be written as $Ax = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let us row reduce the matrix A, before proceeding further. We have

$$\begin{aligned}A &= \begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix}\\ &\sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(applying the operations } R_2 - 2R_1, R_3 - 3R_1\text{).}\end{aligned}$$

Hence we have that the system of equation reduces to

$$\begin{aligned}&\begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\\ &\implies x_1 - 3x_2 + x_3 = 0\\ &\implies x_3 = -x_1 + 3x_2\end{aligned}$$

Hence the solution set is

$$\begin{aligned}&\{(x_1, x_2, x_3) : x_3 = -x_1 + 3x_2, x_i \in \mathbb{R}\}\\ &= \{(x_1, x_2, -x_1 + 3x_2) : x_1, x_2 \in \mathbb{R}\}\\ &= \{x_1(1, 0, -1) + x_2(0, 1, 3) : x_1, x_2 \in \mathbb{R}\},\end{aligned}$$

which is two dimensional and a basis for the same is given by $\{(1, 0, -1), (0, 1, 3)\}$, as they are linearly independent.

Question 18:

Part (a)

Step 1 1 of 2

Let us denote the given subspace by W. Then we have

$$\begin{aligned} W &= \{(x,y,z) \in \mathbb{R}^3 : 3x - 2y + 5z = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : 2y = 3x + 5z\} \\ &= \left\{(x,y,z) \in \mathbb{R}^3 : y = \frac{3}{2}x + \frac{5}{2}z\right\} \\ &= \left\{\left(x, \frac{3}{2}x + \frac{5}{2}z, z\right) : x, z \in \mathbb{R}\right\} \\ &= \left\{x\left(1, \frac{3}{2}, 0\right) + z\left(0, \frac{5}{2}, 1\right) : x, z \in \mathbb{R}\right\}. \end{aligned}$$

$$\begin{aligned} & k_1\left(1, \frac{3}{2}, 0\right) + k_2\left(0, \frac{5}{2}, 1\right) = (0,0,0) \\ \implies & \left(k_1, \frac{3}{2}k_1 + \frac{5}{2}k_2, k_2\right) = (0,0,0) \\ \implies & k_1 = k_2 = 0. \end{aligned}$$

Result 2 of 2

The set $\left\{\left(1, \frac{3}{2}, 0\right), \left(0, \frac{5}{2}, 1\right)\right\}$ is a basis for the given subspace.

Part (b)

Step 1 1 of 2

Let us denote the given subspace by W. Then we have

$$\begin{aligned} W &= \{(x,y,z) \in \mathbb{R}^3 : x - y = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : x = y\} \\ &= \{(x,x,z) : x, z \in \mathbb{R}\} \\ &= \{x(1,1,0) + z(0,0,1) : x, z \in \mathbb{R}\}. \end{aligned}$$

$$\begin{aligned} & k_1(1,1,0) + k_2(0,0,1) = (0,0,0) \\ \implies & (k_1, k_1, k_2) = (0,0,0) \\ \implies & k_1 = k_2 = 0. \end{aligned}$$

Result 2 of 2

The set $\{(1,1,0), (0,0,1)\}$ is a basis for the given subspace.

Part (c)

Step 1

1 of 2

Let us denote the given subspace by W. Then we have

$$\begin{aligned} W &= \{(x,y,z) \in \mathbb{R}^3 : x = 2t, y = -t, z = 4t\} \\ &= \{(2t, -t, 4t) : t \in \mathbb{R}\} \\ &= \{t(2, -1, 4) : t \in \mathbb{R}\} \end{aligned}$$

Result

2 of 2

The set $\{(2, 1, -4)\}$ is a basis for the given subspace.

Part (d)

Step 1

1 of 2

Let us denote the given subspace by W. Then we have

$$\begin{aligned} W &= \{(a,b,c) \in \mathbb{R}^3 : b = a + c\} \\ &= \{(a, a+c, c) : a, c \in \mathbb{R}\} \\ &= \{a(1,1,0) + c(0,1,1) : a, c \in \mathbb{R}\} \end{aligned}$$

$$\begin{aligned} & k_1(1,1,0) + k_2(0,1,1) = (0,0,0) \\ \implies & (k_1, k_1 + k_2, k_2) = (0,0,0) \\ \implies & k_1 = k_2 = 0. \end{aligned}$$

. Hence $\{(1,1,0), (0,1,1)\}$ is a basis for W.

Result

2 of 2

The set $\{(1,1,0), (0,1,1)\}$ is a basis for the given space.

Question 25:

Step 1

1 of 2

Let $\{v_1, v_2, \cdots, v_r\}$ be a linearly independent set of vectors of V. If possible assume $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ to be linearly dependent. Then there exists $k_1, k_2, \cdots, k_r \in \mathbb{R}$, not all zero such that

$$\begin{aligned} & k_1(v_1)_S + k_2(v_2)_S + \cdots + k_r(v_r)_S = 0 \\ \implies & (k_1v_1 + k_2v_2 + \cdots + k_rv_r)_S = 0_S \qquad (\text{as } k(v)_S = (kv)_S, \\ & \qquad k(v)_S + l(w)_S = (kv + lw)_S, k, l \in \mathbb{R}) \\ \implies & k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0 \\ \implies & k_1 = k_2 = \cdots = k_r = 0 \qquad (\text{as } \{v_1, v_2, \cdots, v_r\} \text{ is linearly independent}). \end{aligned}$$

Hence $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ is linearly independent.

Conversely suppose $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ is linearly independent and $\{v_1, v_2, \cdots, v_r\}$ be linearly dependent. Then there exist $k_1, k_2, \cdots, k_r \in \mathbb{R}$, not all zero such that

$$\begin{aligned} & k_1v_1 + k_2v_2 + \cdots + k_rv_r = 0 \\ \implies & (k_1v_1 + k_2v_2 + \cdots + k_rv_r)_S = 0_S \\ \implies & (k_1v_1)_S + (k_2v_2)_S + \cdots + (k_rv_r)_S = 0_S \\ \implies & k_1(v_1)_S + k_2(v_2)_S + \cdots + k_r(v_r)_S = 0_S \\ \implies & k_1 = k_2 = \cdots = k_r = 0 \qquad (\text{as } \{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\} \text{ is linearly independent}). \end{aligned}$$

Hence $\{v_1, v_2, \cdots, v_r\}$ is linearly independent.

Result

We prove that $\{v_1, v_2, \cdots, v_r\}$ is linearly independent if and only if $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ is linearly independent.

Question 26:

Step 1

Let $v_1, v_2, \cdots, v_r$ span V and $w \in \mathbb{R}^n$. Since S is a basis of V, we have that there exists $v \in V$, such that $(v)_S = w$. Since $v_1, v_2, \cdots, v_r$ span V, we have $k_1, k_2, \cdots, k_r \in \mathbb{R}$, such that $k_1v_1 + k_2v_2 + \cdots + k_rv_r = v$. Hence we have

$$\begin{aligned} & k_1v_1 + k_2v_2 + \cdots + k_rv_r = v \\ \Longrightarrow\ & (k_1v_1 + k_2v_2 + \cdots + k_rv_r)_S = (v)_S \\ \Longrightarrow\ & k_1(v_1)_S + k_2(v_2)_S + \cdots + k_r(v_r)_S = w. \end{aligned}$$

Hence $\{(v_1)_S, (v_2)_S, \cdots, (v_r)_S\}$ spans $\mathbb{R}^n$.

Conversely assume $(v_1)_S, (v_2)_S, \cdots, (v_r)_S$ spans $\mathbb{R}^n$ and $v \in V$. Then $(v)_S = w \in \mathbb{R}^n$. Hence there exists $k_1, k_2, \cdots, k_r \in \mathbb{R}$, such that $k_1(v_1)_S + k_2(v_2)_S + \cdots + k_r(v_r)_S = w$. Hence we have

$$\begin{aligned} & k_1(v_1)_S + k_2(v_2)_S + \cdots + k_r(v_r)_S = (v)_S \\ \Longrightarrow\ & (k_1v_1 + k_2v_2 + \cdots + k_rv_r)_S = (v)_S \\ \Longrightarrow\ & k_1v_1 + k_2v_2 + \cdots + k_rv_r = v. \end{aligned}$$

Hence $v_1, v_2, \cdots, v_r$ span V.

Result

We proev that $v_1, v_2, \cdots, v_r$ span V if and only if $(v_1)_S, (v_2)_S, \cdots, (v_r)_S$ spans $\mathbb{R}^n$.

Question 33:

Step 1

1 of 2

Let us first prove that M_{33} has dimension 9. Note that for any

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in M_{33}$$

, we have

$$\begin{aligned}
&\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\
=&a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ d\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ g\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Hence the set consisting of the elements

Hence the set consisting of the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

, spans $M_{3,3}$. Also the elements are linearly independent. Thus M_{33} has dimension 9. Hence if the set $\{I_3, A, A^2, \cdots, A^9\}$ has cardinality 10, it must be linearly dependent by theorem $5.4.2(a)$.

Result

2 of 2

We prove that M_{33} has dimension 9 and conclude the result.

Step 1

Just as done in $33(a)$, if we take for $1 \leq i, j \leq n$, the element E_{ij}, which has 1 at (i, j)-th position and 0 elsewhere, then the set $\mathcal{B} = \{E_{ij} : 1 \leq i, j \leq n\}$ spans M_{nn} and is linearly independent. Hence if the collection $\{I_n, A, A^2, \cdots, A^{n^2}\}$ has $n^2 + 1$ different elements, then it will be linearly dependent.

Result

We prove that if the collection $\{I_n, A, A^2, \cdots, A^{n^2}\}$ has $n^2 + 1$ different elements, then it will be linearly dependent