Important facts and Theorems 3

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LEMMA 2.3.2

> *If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*
>
> $$\det(EB) = \det(E)\det(B)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I_n$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} = k \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = k \times 1 = k$$

Proof We shall consider three cases, each depending on the row operation that produces matrix E.

Case 1. If E results from multiplying a row of I_n by k, then by Theorem 1.5.1, EB results from B by multiplying a row by k; so from Theorem 2.2.3*a* we have

$$\det(EB) = \overset{\det(E)}{k}\det(B)$$

But from Theorem 2.2.4*a* we have $\det(E) = k$, so

$$\det(EB) = \det(E)\det(B)$$

Cases 2 and 3. The proofs of the cases where E results from interchanging two rows of I_n or from adding a multiple of one row to another follow the same pattern as Case 1 and are left as exercises.

Remark It follows by repeated applications of Lemma 2.3.2 that if B is an $n \times n$ matrix and $E_1, E_2, \cdots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1E_2 \cdots E_rB) = \det(E_1)\det(E_2) \cdots \det(E_r)\det(B) \tag{3}$$

$\det(E_rB)$

$\det(E_{r-1}E_rB)$

For example,

$$\det(E_1E_2B) = \det(E_1)\det(E_2B) = \det(E_1)\det(E_2)\det(B)$$

THEOREM 2.2.4

> *Let E be an $n \times n$ elementary matrix.*
>
> (a) *If E results from multiplying a row of I_n by k, then* $\det(E) = k$.
>
> (b) *If E results from interchanging two rows of I_n, then* $\det(E) = -1$.
>
> (c) *If E results from adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Determinant Test for Invertibility

The next theorem provides an important criterion for invertibility in terms of determinants, and it will be used in proving 2.

THEOREM 2.3.3

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof Let R be the reduced row-echelon form of A. As a preliminary step, we will show that $\det(A)$ and $\det(R)$ are both zero or both nonzero: Let $E_1, E_2, \cdots, E_r$ be the elementary matrices that correspond to the elementary row operations that produce R from A. Thus

$$R = E_r \cdots E_2 E_1 A$$

and from 3,

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A) \tag{4}$$

But from Theorem 2.2.4 the determinants of the elementary matrices are all nonzero. (Keep in mind that multiplying a row by zero is *not* an allowable elementary row operation, so $k \neq 0$ in this application of Theorem 2.2.4.) Thus, it follows from 4 that $\det(A)$ and $\det(R)$ are both zero or both nonzero. Now to the main body of the proof.

If A is invertible, then by Theorem 1.6.4 we have $R = I$, so $\det(R) = 1 \neq 0$ and consequently $\det(A) \neq 0$. Conversely, if $\det(A) \neq 0$, then $\det(R) \neq 0$, so R cannot have a row of zeros. It follows from Theorem 1.4.3 that $R = I$, so A is invertible by Theorem 1.6.4.

Note that this uses part k) of theorem 1.6.4 (or of 2.3.6 below!)

■

THEOREM 2.3.4

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

Proof We divide the proof into two cases that depend on whether or not A is invertible. If the matrix A is not invertible, then by Theorem 1.6.5 neither is the product AB. Thus, from Theorem 2.3.3, we have $\det(AB) = 0$ and $\det(A) = 0$, so it follows that $\det(AB) = \det(A)\det(B)$.

Now assume that A is invertible. By Theorem 1.6.4, the matrix A is expressible as a product of elementary matrices, say

$$A = E_1 E_2 \cdots E_r \tag{5}$$

so

$$AB = E_1 E_2 \cdots E_r B$$

Applying 3 to this equation yields

$$\det(AB) = \det(E_1)\det(E_2) \cdots \det(E_r)\det(B)$$

and applying 3 again yields

$$\det(AB) = \det(E_1 E_2 \cdots E_r)\det(B)$$

which, from 5, can be written as $\det(AB) = \det(A)\det(B)$.

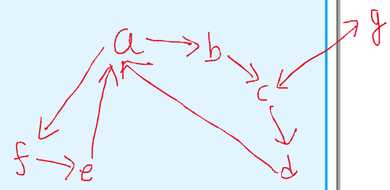
■

THEOREM 2.3.6

Equivalent Statements

Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.



(a) *A is invertible.*

(b) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(c) *The reduced row-echelon form of A is* I_n.

(d) *A can be expressed as a product of elementary matrices.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* b.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* $\mathbf{b}$.

(g) $\det(A) \neq 0$.