Lecture 10

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DEFINITION:

IF A IS A SQUARE MATRIX THEN THE MINOR OF ENTRY a_{ij} IS DENOTED BY M_{ij} AND IS DEFINED TO BE THE DETERMINANT THAT REMAINS AFTER THE iTH ROW AND jTH COLUMN ARE DELETED FROM A. THE NUMBER $(-1)^{i+j} M_{ij}$ IS DENOTED BY C_{ij} AND IS CALLED THE COFACTOR OF ENTRY a_{ij}.

EXAMPLES:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{11} = MINOR OF a_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

C_{11} = COFACTOR OF a_{11}

$$= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$(-1)^{i+j} M_{ij} = C_{ij}$$

cofactor of a_{ij}

$$C_{32} = (-1)^5 M_{32} = -M_{32}$$

M_{12} = MINOR OF a_{12}

$$= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$ WHICH IS THE DETERMINANT OBTAINED AFTER IGNORING ROW 1 AND COLUMN 2.

$$\Rightarrow \boxed{C_{12}} = (-1)^{1+2} M_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

a_{33}

SIMILARLY

$$\boxed{C_{13}} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

ETC.

NOTE: DET(A) = DET(A)

$$= a_{11}\left\{(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right\} + a_{12}\left((-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right)$$

$$+ a_{13}\left\{(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right\}$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

THIS IS EXPANSION BY FIRST ROW.

$[d]$

$d[1]$

TRY THE FOLLOWING:

FIND $C_{11}, C_{12}, C_{21}, C_{22}$ FOR

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

SOLUTION: $\boxed{C_{11}} = (-1)^{1+1} d = \boxed{d}$

$\boxed{C_{12}} = (-1)^{1+2} c = \boxed{-c}$,

$\boxed{C_{21}} = (-1)^{1+2} b = \boxed{-b}$, $\boxed{C_{22}} = (-1)^{2+2} a$

$\Rightarrow \boxed{C_{22}} = \boxed{a}$.

CONSIDER $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, ① RECALL THAT ②

$$A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$ WHERE $ad - bc = \det(A) \neq 0$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \boxed{\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T}$$ FROM ① AND ②.

THIS MATRIX IS CALLED $\boxed{\mathrm{Adj}(A)}$ OR $\boxed{\text{Adjoint}}$ of $\boxed{A}$.

NOTATION

INVERSE OF A MATRIX USING ITS ADJOINT:

IF A IS AN INVERTIBLE MATRIX, THEN

$$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

EXAMPLE: FIND THE INVERSE OF $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$ BY ADJOINT METHOD

$-1[(-1)^{1+3}(-4-12)]$

$-1(-4-12)$

SOLUTION: STEP (1): FIND $\det(A)$ WHICH IS GIVEN BY

$$3(0+12)-2(-6)-1(-4-12) = 36+12+16$$
$$= 64 = \det(A) \neq 0.$$

$C_{ij} = (-1)^{i+j} M_{ij}$

STEP (2): FIND $\boxed{\operatorname{Adj}(A)}$ WHICH IS

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \operatorname{adj}(A)$$

C_{11} = COFACTOR OF $(a_{11} = 3)$
IS GIVEN BY

$$\boxed{C_{11}} = (-1)^{1+1} M_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = \boxed{12}$$

SIMILARLY

$$\boxed{C_{12}} = (-1)^{1+2} M_{12} = - \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = \boxed{6},$$

CHECK THE FOLLOWING:

$$\boxed{C_{13}} = \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -4 - 12 = \boxed{-16},$$

$$\boxed{C_{21}} = - \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = -(-4) = \boxed{4},$$

$$\boxed{C_{22}} = \boxed{2}, \quad \boxed{C_{23}} = -(-16) = \boxed{16}, \quad \boxed{C_{31} = 12},$$

$$\boxed{C_{32}} = -(10) = \boxed{-10}, \quad \boxed{C_{33} = 16},$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A) = \frac{1}{64} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$\Rightarrow A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

TRIANGULAR MATRICES:

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & 0 \\ a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} 1 & 0 \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33}$$

(1) A SQUARE MATRIX IN WHICH ALL THE ENTRIES ABOVE THE MAIN DIAGONAL ARE ZERO IS CALLED LOWER TRIANGULAR.

EXAMPLE:

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

IS LOWER TRIANGULAR.

HERE $\det(A) = a_{11} a_{22} a_{33}$

i.e. PRODUCT OF DIAGONAL ENTRIES. WHICH IS TRUE FOR ANY LOWER TRIANGULAR MATRIX.

(2) SIMILARLY A SQUARE MATRIX IN WHICH ALL THE ENTRIES BELOW THE MAIN DIAGONAL ARE ZERO IS CALLED UPPER TRIANGULAR MATRIX.

EXAMPLE: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

IS UPPER TRIANGULAR MATRIX.

HERE $\boxed{\det(A) = a_{11} a_{22} a_{33}}$

i.e. PRODUCT OF DIAGONAL ENTRIES.

RESULT: A MATRIX THAT IS EITHER UPPER TRIANGULAR OR LOWER TRIANGULAR IS CALLED TRIANGULAR AND DETERMINANT OF ANY TRIANGULAR MATRIX IS EQUAL TO THE PRODUCT OF ITS DIAGONAL ENTRIES.

NOTE: A SQUARE MATRIX IN ROW-ECHELON FORM IS UPPER TRIANGULAR SINCE IT HAS ZEROS BELOW THE MAIN DIAGONAL. SEE THE FOLLOWING EXAMPLES:

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&4&-3\\0&1&6\\0&0&1\end{bmatrix},$$

AND

$$\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}$$

ARE IN ROW-ECHELON FORM

AND ALSO UPPER TRIANGULAR MATRICES.

NOTE: DIAGONAL MATRICES ARE BOTH UPPER TRIANGULAR AND LOWER TRIANGULAR, E.G. $\boxed{I} \rightarrow$ IDENTITY MATRIX ETC

LU Decomposition Shortcut

When given a square matrix A we want to find L (a lower triangular matrix) and U (an upper triangular matrix such that

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & * & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

$\det(A) = \det(LU)$

$= \cancel{\det(L)}^{1} \det(U)$

$\det(A) = \det(U)$

- *A* must be able to be reduced to row-echelon form, *U*, without interchanging any rows.
- L and U are not unique.
- **Using the opposites of the multiplies used in the row operations to obtain U, we can build L.**

Determine the LU Decomposition of

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} \quad \underbrace{2R_1 + R_2}_{R_2 \to}$$

We need to keep track of the elementary row operations to write A as a upper triangular matrix.

opposite

Obtain U here

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 3 & 4 & 7 \end{bmatrix} \quad \underbrace{-3R_1 + R_3}_{R_3 \to}$$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & -8 & 16 \end{bmatrix} \quad \tfrac{1}{2}R_2 + R_3$$

$$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 15.5 \end{bmatrix} = U$$

Build L here

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -\tfrac{1}{2} & 1 \end{bmatrix}$$

L

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 15.5 \end{bmatrix}$$

Determine the LU Decomposition of

$$A = \begin{bmatrix} 2 & 4 & -4 \\ 1 & -4 & 3 \\ -6 & -9 & 5 \end{bmatrix} \quad R_2 \to -\tfrac{1}{2}R_1 + R_2$$

We need to keep track of the elementary row operations to write A as a upper triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -1/2 & 1 \end{bmatrix}$$

Obtain U here

$$\begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ -6 & -9 & 5 \end{bmatrix} \quad R_3 \to 3R_1 + R_3$$

$$\begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 3 & -7 \end{bmatrix} \quad \tfrac{1}{2}R_2 + R_3$$

$$\begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} = U$$

Build L here

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 2 & 4 & -4 \\ 1 & -4 & 3 \\ -6 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 0 & -9/2 \end{bmatrix}$$

$$A \quad = \quad L \quad \cdot \quad U$$