Lecture 13

Tuesday, February 22, 2022 8:27 AM

1 LINEAR ALGEBRA $\boxed{\text{LECTURE 13}}$

$\boxed{\text{Q.no.3}}$

DETERMINE WHETHER THE GIVEN (V) SET IS A VECTOR SPACE UNDER THE GIVEN OPERATIONS.
THE SET OF ALL PAIRS OF REAL NUMBERS (x,y) WITH THE OPERATIONS

$(x,y)+(x',y')=(x+x',y+y')$ AND

$\boxed{\text{VECTOR ADDITION}}$

$k(x,y)=(2kx,2ky)$

$\boxed{\text{SCALAR MULTIPLICATION}}$

SOLUTION:

(1) $(x,y)+(x',y')=(x+x',y+y')\in V$
SINCE THIS IS ALSO AN ORDER PAIR OF REAL NUMBERS

(2) $(x,y)+(x',y')=(x+x',y+y')$
$=(x'+x,y'+y)=(x',y')+(x,y)$

$\Rightarrow \boxed{\underline{u}+\underline{v}=\underline{v}+\underline{u}}$ FOR $\boxed{\underline{u}=(x,y)}$ AND $\boxed{\underline{v}=(x',y')}$.

(3) $(x,y)+[(x',y')+(x'',y'')]$
$=(x,y)+(x'+x'',y'+y'')$
$=\{x+(x'+x''),y+(y'+y'')\}$

$$= \{x + (x' + x''),\ y + (y' + y'')\}$$

$$= \{(x + x') + x'',\ (y + y') + y''\}$$

$$= (x + x',\ y + y') + (x'',\ y'')$$

$$= [(x, y) + (x', y')] + (x'', y'')$$

(4) $\Rightarrow \boxed{\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}}$

FOR $\underline{w} = (x'', y'')$

(4) $(x, y) + (0, 0) = (0, 0) + (x, y)$

$$= (x + 0,\ y + 0) = (x, y)$$

$\Rightarrow \boxed{\underline{0} = (0, 0)}$

(5) IF $\boxed{\underline{u} = (x, y)}$, $\boxed{-\underline{u} = (x', y')}$

THEN

$(x, y) + (x', y') = (0, 0)$ — ①

BUT $(x, y) + (x', y') = (x + x', y + y')$ ②

FROM ① AND ②

$x + x' = 0 \Rightarrow \boxed{x' = -x}$

$y + y' = 0 \Rightarrow \boxed{y' = -y}$

$\therefore -\underline{u} = (-x, -y)$

$\because (x, y) + (-x, -y) = (-x, -y) + (x, y)$

$= (0, 0)$

(6) $k(x, y) = (2kx, 2ky) \in V$

(OBVIOUS) $\Rightarrow k\underline{u} \in V$

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(7) $k(\underline{u}+\underline{v}) = k[(x,y)+(x',y')]$

$= k[(x+x', y+y')]$

$= \{2k(x+x'), 2k(y+y')\}$

$= (2kx + 2kx', 2ky + 2ky')$

$= (2kx, 2ky) + (2kx', 2ky')$

$= k(x,y) + k(x',y')$

$= k\underline{u} + k\underline{v}$

(8) $(k+l)\underline{u} = (k+l)(x,y)$

$= [2(k+l)x, 2(k+l)y]$ — ①

ALSO

$k\underline{u} + l\underline{u} = k(x,y) + l(x,y)$

$= (2kx, 2ky) + (2lx, 2ly)$

$= [2(k+l)x, 2(k+l)y]$ — ②

FROM ① AND ②

$\boxed{(k+l)\underline{u} = k\underline{u} + l\underline{u}}$

(9) $k(l\underline{u}) = k[l(x,y)]$

$= k(2lx, 2ly) = (4lkx, 4kly)$ — ①

$(kl)\underline{u} = (kl)(x,y) = (2klx, 2kly)$ — ②

FROM ① AND ② $k(l\underline{u}) \neq (kl)\underline{u}$

SO $\boxed{\text{NOT}}$ A VECTOR SPACE.

(X) $1\underline{u} = 1(x,y) = (2x,2y) \neq (x,y)$

$\Rightarrow 1\underline{u} \neq \underline{u}$

$\because k(x,y) = (2kx, 2ky)$

SO GIVEN SET IS NOT A VECTOR SPACE $\because$ AXIOMS (9) AND (10) FAIL.

SUBSPACES

DEFINITION: A SUBSET $\boxed{W}$ OF A VECTOR SPACE $\boxed{V}$ IS CALLED A SUBSPACE OF $\boxed{V}$ IF $\boxed{W}$ IS ITSELF A VECTOR SPACE UNDER THE ADDITION AND SCALAR MULTIPLICATION DEFINED ON $\boxed{V}$.

THEOREM 5.2.1.

IF $\boxed{W}$ IS A SET OF ONE OR MORE VECTORS FROM A VECTOR SPACE V, THEN $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$ IF AND ONLY IF THE FOLLOWING CONDITIONS HOLD.

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(a) IF $\underline{u}$ AND $\underline{v}$ ARE VECTORS IN $\boxed{W}$ THEN $\underline{u}+\underline{v}$ IS IN $\boxed{W}$.

(b) IF $\boxed{k}$ IS ANY SCALAR AND $\underline{u}$ IS ANY VECTOR IN $\boxed{W}$, THEN $k\underline{u} \in \boxed{W}$.

PROOF. IF $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$, THEN ALL THE VECTOR SPACE AXIOMS OR PROPERTIES ARE SATISFIED INCLUDING (1) AND (6) WHICH ARE SAME AS (a) AND (b) ABOVE.

CONVERSELY, ASSUME CONDITIONS $\boxed{(a)}$ AND $\boxed{(b)}$ HOLD. SINCE THEY ARE VECTOR SPACE AXIOMS 1 AND 6, WE NEED ONLY SHOW THAT OTHER 8 AXIOMS ARE SATISFIED.

AXIOMS $\boxed{2, 3, 7, 8, 9}$ AND $\boxed{10}$ ARE AUTOMATICALLY SATISFIED BY THE VECTORS IN $\boxed{W}$ SINCE THEY ARE SATISFIED BY ALL VECTORS IN $\boxed{V}$.

THEREFORE TO COMPLETE THE PROOF, WE NEED ONLY VERIFY

THAT AXIOMS $\boxed{4 \text{ AND } 5}$ ARE SATISFIED BY VECTORS IN $\boxed{W}$.

LET $\underline{u}$ BE ANY VECTOR IN $\boxed{W}$. BY CONDITION (b), $k\underline{u} \in \boxed{W}$ FOR EVERY SCALAR $\boxed{k}$.

SETTING $k=0$, $k\underline{u} = 0\underline{u} = \underline{0}$

BUT $k\underline{u} \in W \Rightarrow \underline{0} \in W$,

AND SETTING $k=-1$, IT FOLLOWS THAT

$$(-1)\underline{u} = -\underline{u} \in W.$$

<u>RESULT</u>. $\boxed{W}$ IS A SUBSPACE OF $\boxed{V}$ <u>IF AND ONLY IF</u> $\boxed{W}$ IS <u>CLOSED UNDER ADDITION</u> AND <u>CLOSED</u> UNDER <u>SCALAR MULTIPLICATION</u>.

<u>EXAMPLE</u>:

SHOW THAT THE SET $\boxed{W}$ OF ALL 2X2 MATRICES HAVING <u>ZEROS</u> ON THE <u>MAIN DIAGONAL</u> IS A SUBSPACE OF THE VECTOR SPACE M_{22} (OF ALL 2X2 MATRICES).

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SOLUTION:

LET $\underline{u} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$

$\underline{u} + \underline{v} = \begin{bmatrix} 0 & a+c \\ b+d & 0 \end{bmatrix} \in \boxed{W}$ AND

$k\underline{u} = \begin{bmatrix} 0 & ka \\ kb & 0 \end{bmatrix} \in \boxed{W}$, SINCE BOTH $u+v$ AND ku CONTAIN $\boxed{\text{ZEROS}}$ ON THE MAIN DIAGONAL, $\therefore W$ IS A SUBSPACE OF M_{22}.

TRY THE FOLLOWING:

SHOW THAT THE SET $\boxed{W}$ OF ALL THE POLYNOMIALS OF DEGREE $\leq n$ (INCLUDING THE ZERO POLYNOMIAL) IS A SUBSPACE OF REAL-VALUED FUNCTIONS UNDER ADDITION AND SCALAR MULTIPLICATION.

$\boxed{\text{HINT:}}$ CHECK $\underline{u} + \underline{v} \in W$, $k\underline{u} \in W$

TAKE

$\underline{u} = P(x) = a_0 + a_1 x + \cdots + a_n x^n$

AND $\underline{v} = q(x) = b_0 + b_1 x + \cdots + b_n x^n$

TRY THE FOLLOWING:

CHECK WHETHER THE FOLLOWING SET OF VECTORS GIVEN BY

$$\boxed{W = \{ (a,b,c) \mid b = a+c \}}$$

IS A SUBSPACE OF R^3 ?

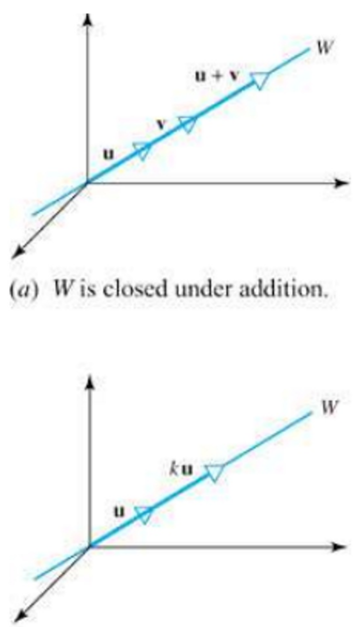
ANSWER: $\boxed{\text{YES}}$

EXAMPLE 2 Lines through the Origin Are Subspaces

Show that a line through the origin of R^3 is a subspace of R^3.

Solution

Let W be a line through the origin of R^3. It is evident geometrically that the sum of two vectors on this line also lies on the line and that a scalar multiple of a vector on the line is on the line as well (Figure 5.2.2). Thus W is closed under addition and scalar multiplication, so it is a subspace of R^3. In the exercises we will ask you to prove this result algebraically using parametric equations for the line.



(a) W is closed under addition.

(b) W is closed under scalar multiplication.

Figure 5.2.2

EXAMPLE 3 Subset of R^2 That Is Not a Subspace

Let W be the set of all points (x, y) in R^2 such that $x \geq 0$ and $y \geq 0$. These are the points in the first quadrant. The set W is *not* a subspace of R^2 since it is not closed under scalar multiplication. For example, $\mathbf{v} = (1, 1)$ lies in W, but its negative $(-1)\mathbf{v} = -\mathbf{v} = (-1, -1)$ does not (Figure 5.2.3).

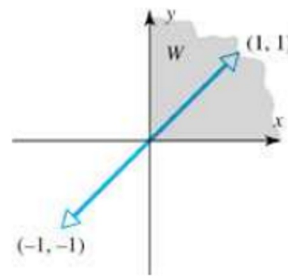


Figure 5.2.3 W is not closed under scalar multiplication.

Every nonzero vector space V has at least two subspaces: V itself is a subspace, and the set $\{\mathbf{0}\}$ consisting of just the zero vector in V is a subspace called the ***zero subspace***. Combining this with Examples Example 1 and Example 2, we obtain the following list of subspaces of R^2 and R^3:

Solution Spaces of Homogeneous Systems

If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, then each vector $\mathbf{x}$ that satisfies this equation is called a ***solution vector*** of the system. The following theorem shows that the solution vectors of a *homogeneous* linear system form a vector space, which we shall call the ***solution space*** of the system.

THEOREM 5.2.2

If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of R^n.

Proof Let W be the set of solution vectors. There is at least one vector in W, namely $\mathbf{0}$. To show that W is closed under addition and scalar multiplication, we must show that if $\mathbf{x}$ and $\mathbf{x}'$ are any solution vectors and k is any scalar, then $\mathbf{x} + \mathbf{x}'$ and $k\mathbf{x}$ are also solution vectors. But if $\mathbf{x}$ and $\mathbf{x}'$ are solution vectors, then

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad A\mathbf{x}' = \mathbf{0}$$

from which it follows that

$$A(\mathbf{x} + \mathbf{x}') = A\mathbf{x} + A\mathbf{x}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and

$$A(k\mathbf{x}) = kA\mathbf{x} = k\mathbf{0} = \mathbf{0}$$

which proves that $\mathbf{x} + \mathbf{x}'$ and $k\mathbf{x}$ are solution vectors. ■