Assignment # 6(b)

Thursday, 5 May 2022 8:23 pm

Q1) $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \to A$ $\to$ post multiplied with $v \leftarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\hookrightarrow AV \leftarrow \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \to$ premultiplied with $A \Rightarrow A^2v \leftarrow \begin{bmatrix} 24 \\ 20 \\ 20 \end{bmatrix}$

A^2v premultiplied with $A \to \begin{bmatrix} 176 \\ 168 \\ 168 \end{bmatrix} \to A^3v$

as these vectors are from $\mathbb{R}^3$, they must be linearly dependent.

$$\therefore \; k_1 V + k_2 A v + k_3 A^2 v + k_4 A^3 v = 0$$

where not all k_1, k_2, k_3 and k_4 are 0.

$\begin{bmatrix} 1 & 4 & 24 & 176 \\ 0 & 2 & 20 & 168 \\ 0 & 2 & 20 & 168 \end{bmatrix} \to$ row echelon

$\downarrow$

$$\left.\begin{aligned} k_1 &= 16k_3 + 160k_4 \\ k_2 &= -10k_3 - 84k_4. \\ k_3 &= k_3 \\ k_4 &= k_4 \end{aligned}\right\}$$ for $k_3, k_4 = 1$

$k_1 = 176,\ k_2 = -94,\ k_3 = 1$
$k_4 = 1$

$$\rightarrow \left(176 - 94A + A^2 + A^3\right)\vec{v} = 0$$

$$(A+11)(A-8)(A-2)\vec{v} = 0$$

$\therefore$ eigenvalues :- $8, 2 \rightarrow -11$ rejected.

$\downarrow$
repeated

Q2) i) $\rightarrow B = P^{-1}AP$ where P nonsingular.

$\rightarrow Q^{-1}BQ = (P^{-1})^{-1} B P^{-1}$ $\quad Q = P^{-1}$

$$= PBP^{-1}$$

$$= P(P^{-1}AP)P^{-1} \longrightarrow \boxed{A.}$$ $\therefore$ B is similar to A.

ii) $B = P^{-1}AP$

$$\det(B) = \det(P^{-1}) \cdot \det(A) \cdot \det(P).$$

$$= \frac{1}{k} \cdot \det(A) \cdot k \cdot \rightarrow \det(A)$$

$$= \frac{1}{K} \cdot \det(A) \cdot K \cdot \rightarrow \det(A) \quad \blacksquare$$

Q6) (a) (i) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow (3-\lambda)^2 - 1 = 0$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\boxed{\lambda = 4, 2}$$

$V_1 = \begin{vmatrix} x & y \\ -1 & 1 \end{vmatrix} \rightarrow \langle x, y \rangle$

$= (1, 1).$

$V_2 = \begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix} = \langle x, -y \rangle \rightarrow (1, -1)$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} A P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ Ans

Q6) (b) They are the eigenvectors associated with A.

Q7) A → symmetric

eigenvalue λ_1 , λ_2
$\downarrow$ $\vec{x}$ $\downarrow$ $\vec{y}$

$\Rightarrow$ $Ax = \lambda_1 x$.

multiply with y^T on both sides.

$y^T A x = \lambda_1 y^T x$ → (i)

similarly..

$x^T A y = \lambda_2 x^T y$ → (ii)

→ subtracting transpose of (i) from (ii)

$(i)^T \rightarrow x^T A y = \lambda_1 x^T y$

$(ii) - (i)^T \Rightarrow 0 = (\lambda_2 - \lambda_1) x^T y$.

$\lambda_2 - \lambda_1 \neq 0$ hence $x^T \cdot y$ are orthogonal.

as it was assumed that λ_2 and λ_1 are distinct.

x, y are from different eigenspaces.

Q8) (i) False → Identity Matrix.

(ii) False.

(iii) True.

(iv) True.

Question 3

Q3) Find a matrix P that diagonalizes A and determine $P^{-1}AP$

where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\det(\lambda I - A) = \left| \begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{bmatrix} \right| = \lambda-1 \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)\left[(\lambda-1)^2-1\right]$

$= (\lambda-1)\left[\lambda^2 - 2\lambda\right]$

$= (\lambda-1)(\lambda-2)\lambda$

$\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2$

if $\lambda_1 = 0$ $P_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ | if $\lambda_2 = 1$ $P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ | if $\lambda_3 = 2$ $P_3 = \begin{bmatrix} 0 \\ +1 \\ +1 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ i.e. $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

11. By Theorem 7.1.1, the eigenvalues of A are 1, 1/2, 0, and 2. Thus by Theorem 7.1.3, the eigenvalues of A^9 are $1^9 = 1$, $(1/2)^9 = 1/512$, $0^9 = 0$, and $2^9 = 512$.

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Q12) $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ Eigen value and Bases of eigenspace

using theorem 7.1.2d $\Rightarrow \det(\lambda I - A) = 0$

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & +2 & +2 \\ -1 & \lambda-2 & -1 \\ +1 & +1 & \lambda \end{vmatrix} \Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$\lambda = -1, \lambda = 1$

if $\lambda = 1$

$$(\lambda I - A) = \left[\begin{array}{ccc|c} \lambda & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \xrightarrow[\text{elimination}]{\text{Guassian}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

let $x_2 = t$ $x_3 = u$

Parametric equations

$x_1 = -t + -u$

$x_2 = t +$

$x_3 = u$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

if $\lambda = 1$ bases $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

if $\lambda = -1$

$$= \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{elemination}]{\text{Guassian}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2x_3$

$x_2 = -x_3$

$x_3 = x_3$

$$\Rightarrow \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Question 5

Proof *(a)* ⇒ *(b)* Since A is assumed diagonalizable, there is an invertible matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

such that $P^{-1}AP$ is diagonal, say $P^{-1}AP = D$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

It follows from the formula $P^{-1}AP = D$ that $AP = AD$; that is,

$$AP = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} \tag{1}$$

If we now let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ denote the column vectors of P, then from 1, the successive columns of AP are $\lambda_1\mathbf{p}_1, \lambda_2\mathbf{p}_2, \ldots, \lambda_n\mathbf{p}_n$. However, from Formula 6 of Section 1.3, the successive columns of AP are $A\mathbf{p}_1, A\mathbf{p}_2, \ldots, A\mathbf{p}_n$. Thus we must have

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, \qquad A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, \qquad A\mathbf{p}_n = \lambda_n\mathbf{p}_n \tag{2}$$

Since P is invertible, its column vectors are all nonzero; thus, it follows from 2 that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, and $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are corresponding eigenvectors. Since P is invertible, it follows from Theorem 7.1.5 that $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are linearly independent. Thus A has n linearly independent eigenvectors.

(b) ⇒ *(a)* Assume that A has n linearly independent eigenvectors, $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

be the matrix whose column vectors are $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. By Formula 6 of Section 1.3, the column vectors of the product AP are

$$A\mathbf{p}_1, A\mathbf{p}_2, \ldots, A\mathbf{p}_n$$

But

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, \qquad A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, \qquad A\mathbf{p}_n = \lambda_n\mathbf{p}_n$$

so

$$\begin{aligned} AP &= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} \\ &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD \end{aligned} \tag{3}$$

where D is the diagonal matrix having the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the main diagonal. Since the column vectors of P are linearly independent, P is invertible. Thus 3 can be rewritten as $P^{-1}AP = D$; that is, A is diagonalizable.

■

Procedure for Diagonalizing a Matrix

Question 9

1. **(a)** The characteristic equation is $\lambda(\lambda - 5) = 0$. Thus each eigenvalue is repeated once and hence each eigenspace is 1-dimensional.

(c) The characteristic equation is $\lambda^2(\lambda - 3) = 0$. Thus the eigenspace corresponding to $\lambda = 0$ is 2-dimensional and that corresponding to $\lambda = 3$ is 1-dimensional.

(e) The characteristic equation is $\lambda^3(\lambda - 8) = 0$. Thus the eigenspace corresponding to $\lambda = 0$ is 3-dimensional and that corresponding to $\lambda = 8$ is 1-dimensional.

exercise 7.3

characteristic polynomial : $\det(\lambda I - A) = 0$

1 b) $A = \begin{bmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{bmatrix} \Rightarrow \left| \begin{bmatrix} \lambda-1 & 4 & -2 \\ 4 & \lambda-1 & 2 \\ -2 & 2 & \lambda-2 \end{bmatrix} \right| = \cancel{(\lambda-1)^2(\lambda-2) + (-16) + (-16)} - \cancel{[4(\lambda-1) + (-4)(\lambda-1) + 16(\lambda-2)]}$

$\det(\lambda I - A) = \lambda^3 - 27\lambda - 54$

$\Rightarrow (\lambda-6)(\lambda+3)^2 = 0$

$\lambda_1 = -3,\ \lambda_2 = 6$

eigenspace for $\lambda_1 = -3$ has dimension $= 2$

" " for $\lambda_2 = 6$ has dimension $= 1$

d) Characteristic polynomial :

$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ $\quad \Rightarrow (\lambda-8)(\lambda-2)^2 = 0$

$\lambda = 2 \Rightarrow \dim(\text{eigenspace}) = 2$

$\lambda = 8 \Rightarrow \dim(\text{eigenspace}) = 1$

f) $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ characteristic polynomial

$\quad (\lambda-1)^2(\lambda-3)^2 = 0$

$\lambda = 1 \Rightarrow \dim(\text{eigenspace}) = 2$

$\lambda = 3 \Rightarrow \dim(\text{eigenspace}) = 2$