Kinematics of Wheeled Mobile Robots

EE468/CE468: Mobile Robotics

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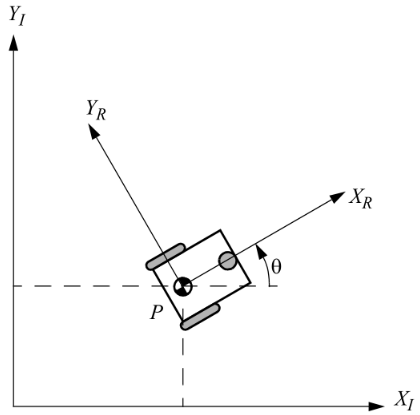
Revisiting reference frames

- Quiz 'Frames of reference' on Canvas

Pose of the robot

- Choose any point P on the robot chassis and attach a frame $\{R\}$ to it. X_R is conventionally chosen along forward direction of robot.



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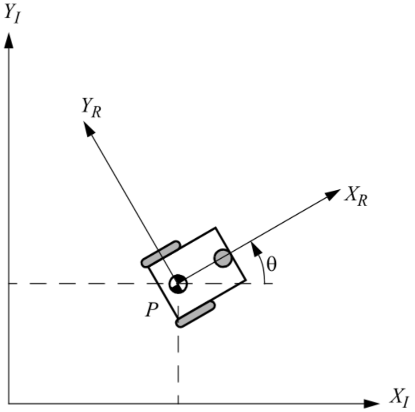

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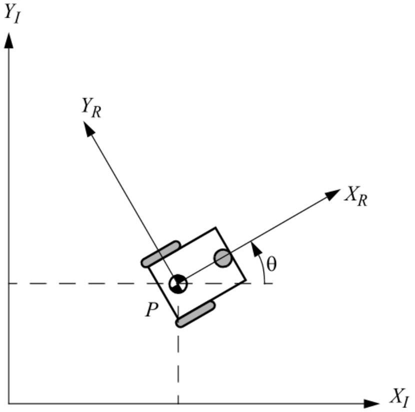

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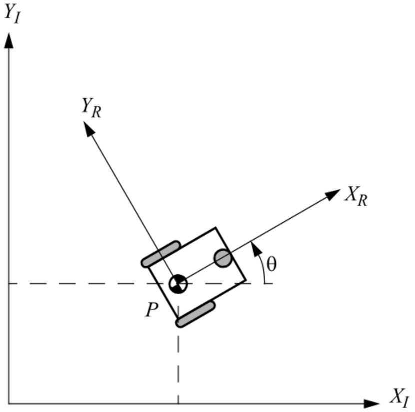

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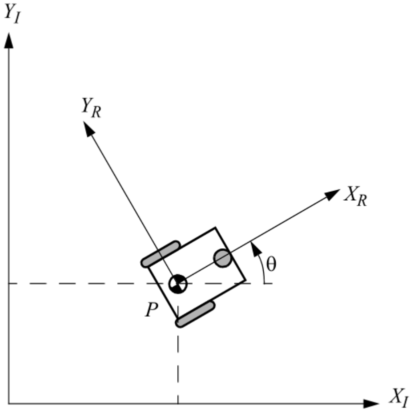
- $\xi = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$ is called **pose** of the robot.



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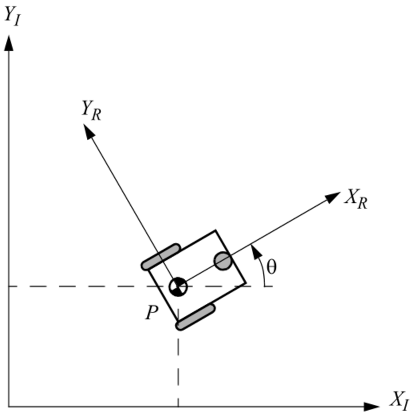
Configuration is description of all points of a rigid body.



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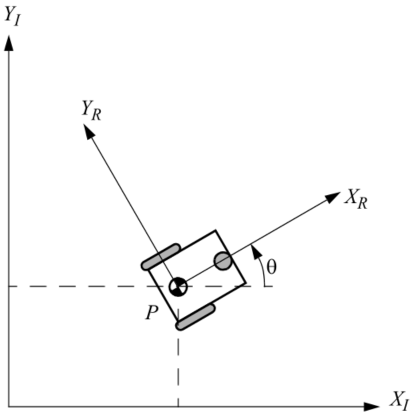
- We're using configuration of chassis as pose of the wheeled robot.



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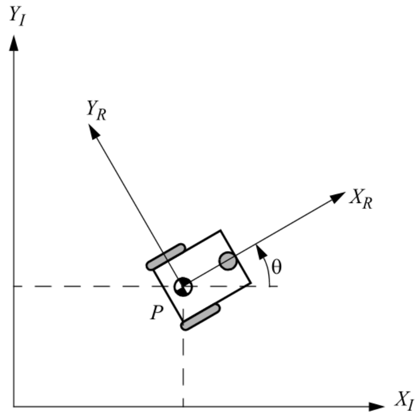
- We're using configuration of chassis as pose of the wheeled robot.
- Configuration of the robot will include positions of the wheels as well as other internal degrees of freedom.



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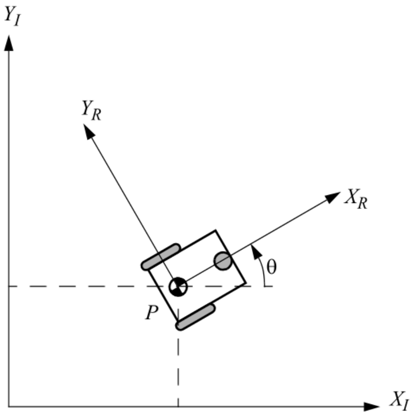
Configuration is description of all points of a rigid body.

- We're using configuration of chassis as pose of the wheeled robot.
- Configuration of the robot will include positions of the wheels as well as other internal degrees of freedom.
- But we're only interested in the chassis position and orientation.



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- **Find a model that relates robot's internal variables to external variables of interest, such as its position in the world.**
- **Kinematics** is the study of motion without considering the forces causing the motion.
- **Dynamics** includes the forces in the model as well.
- We'll build kinematic model of our robots.

Form of the **Differential Kinematic** model

- Kinematic equations for WMRs are of the form

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\begin{aligned}
\dot{x} &= f(x, u) \\
w(x)\dot{x} &= 0.
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- Wheel variables that can be controlled are usually the wheel speeds and steering angles.

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Plan for building kinematic model

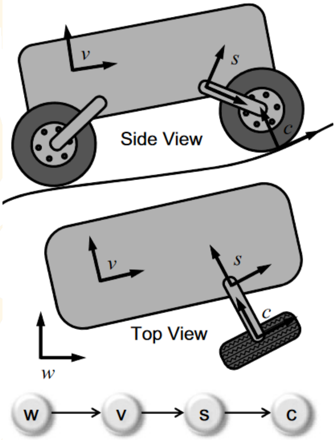

Figure: Frames for modeling of a wheel [2]

- Build model step by step.
- Relate single wheel's velocity to robot velocity.
- Study wheel constraints.
- Relate multiple wheels to robot velocity.

Mutually Stationary Frames are equivalent wrt velocity.

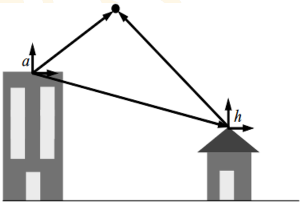

Figure 4.2 Mutually Stationary Frames. Observers in the two buildings will agree on the velocity of the particle but not on its position vector.

$$\vec{r}_p^{\,a} = \vec{r}_p^{\,h} + \vec{r}_h^{\,a}$$

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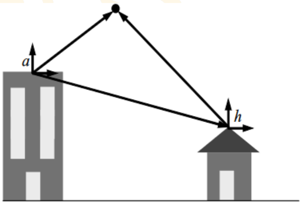

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$$\vec{r}_p^a = \vec{r}_p^h + \vec{r}_h^a$$

Differentiating the expression with respect to time,

$$\vec{v}_p^a = \vec{v}_p^h$$

Uniform Velocity Frames are equivalent wrt acceleration.

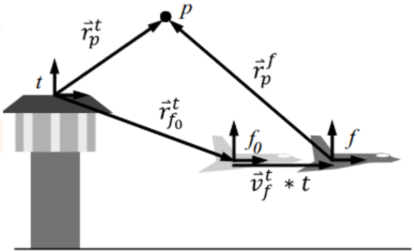
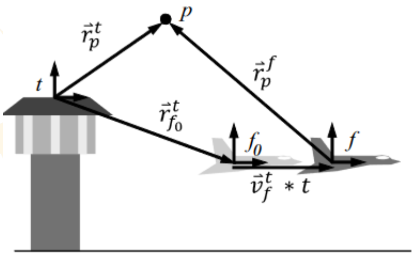


Figure 4.3 Frames Moving at Constant Velocity. Observers in the tower and airplane will agree on the acceleration of the particle but not on its velocity vector.

$$\vec{r}_p^{\,t} = \vec{r}_p^{\,f} + \vec{r}_{f0}^{\,t} + \vec{v}_f^{\,t} \cdot t$$



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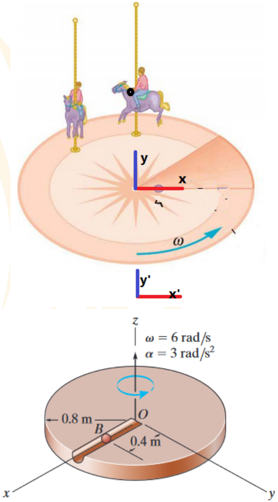
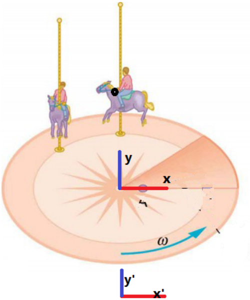
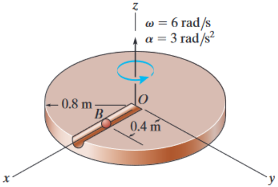


If a frame m is rotating with respect to a frame f, with instantaneous angular velocity, $\vec{\omega}$, then the rate of change of any vector $\vec{u}$ can be expressed as:

$$\left(\frac{d\vec{u}}{dt}\right)_f = \left(\frac{d\vec{u}}{dt}\right)_m + \vec{\omega} \times \vec{u}$$





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- Note that this is a coordinate-free expression.

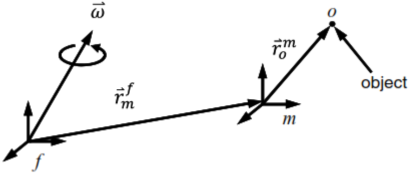
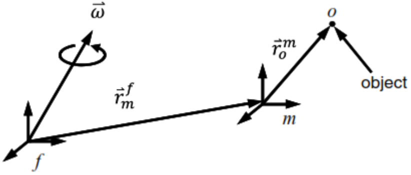


Figure: Frames in general motion

$$\vec{r}_o^f = \vec{r}_m^f + \vec{r}_o^m$$

Differentiating,

$$\vec{v}_o^f = \left.\frac{d}{dt}\right|_f \vec{r}_m^f + \left.\frac{d}{dt}\right|_f \vec{r}_o^m$$



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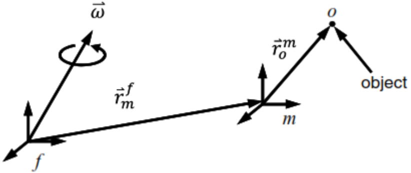

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$\vec{v}_o^m$: Velocity of o measured by moving observer

$\vec{v}_m^f$: Linear velocity of moving frame

$\vec{\omega}_m^f$: Angular velocity of moving frame

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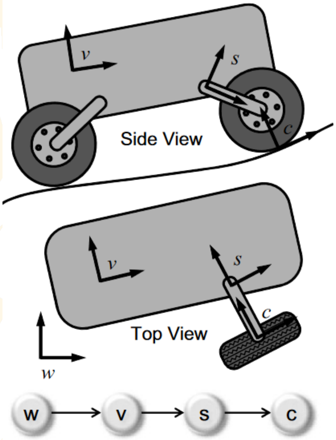

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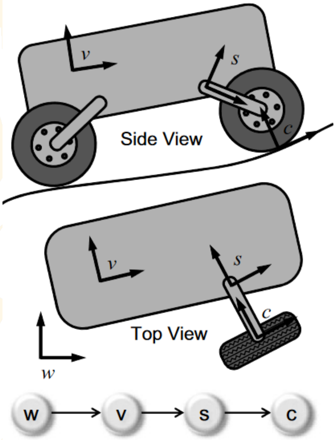

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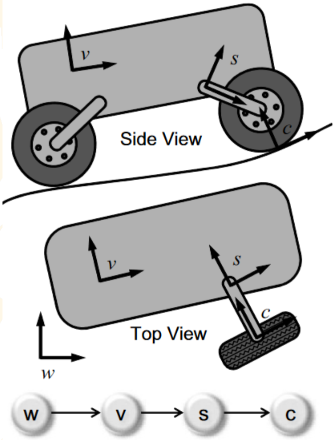
- **Frames:**

w	:	World frame
v	:	Robot frame
s	:	Steering frame
c	:	Wheel or Contact Point frame

Attach frames at points of interest, isolating degrees of freedom:



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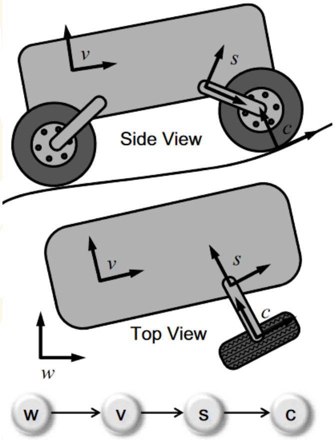
- Wheel is steered around an axis offset from contact point.



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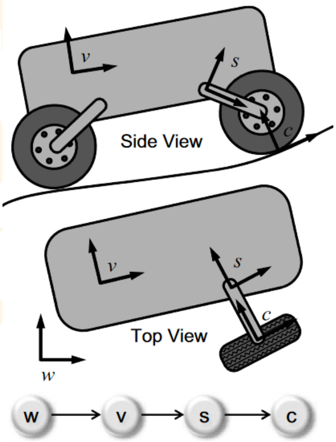
- s can move arbitrarily with respect to v. Imagine suspension.

What is the velocity of wheel contact point wrt global frame, $\mathbf{v}_c^w$?



- Note that the position vector,

$$\mathbf{r}_c^w = \mathbf{r}_v^w + \mathbf{r}_s^v + \mathbf{r}_c^s$$

- Differentiate the expression for position,

$$\left.\frac{d}{dt}\right|_w \mathbf{r}_c^w = \left.\frac{d}{dt}\right|_w = \mathbf{r}_v^w + \left.\frac{d}{dt}\right|_w \mathbf{r}_c^v$$

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Coriolis Equation:

$$\left.\frac{d}{dt}\right|_f \vec{u} = \left.\frac{d}{dt}\right|_m \vec{u} + \vec{\omega}_m^f \times \vec{u}$$

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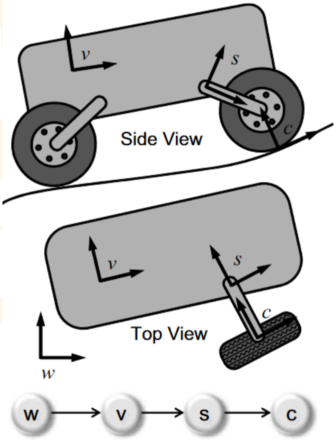

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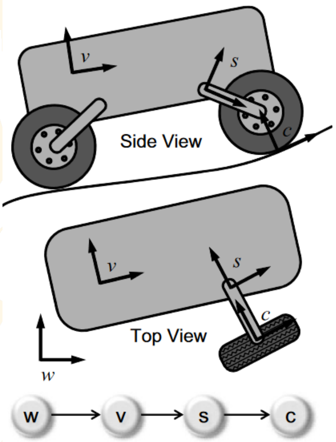

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- c is rigidly attached to s, and has same orientation as s at all times. So, $\mathbf{v}_c^s = 0$.
- If there is no suspension, $\mathbf{v}_s^v = 0$.

$$\mathbf{v}_c^w = \mathbf{v}_v^w + (\omega_s^v \times \mathbf{r}_c^s) + (\omega_v^w \times \mathbf{r}_c^v)$$

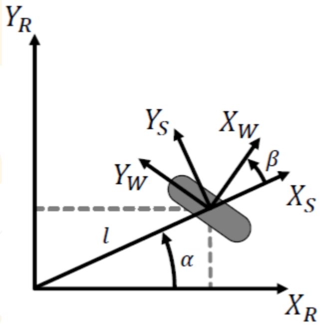
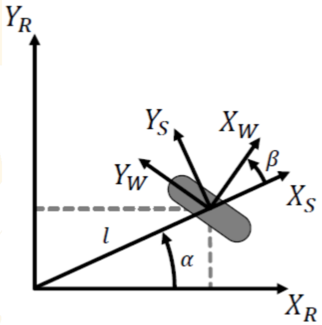


Figure: Fixed Standard Wheel and its parameters

- Start with general equation of wheel:

$$\mathbf{v}_c^w = \mathbf{v}_v^w + (\omega_s^v \times \mathbf{r}_c^s) + (\omega_v^w \times \mathbf{r}_c^v)$$

What will be the equation for a standard wheel, with no steering?



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- No wheel offset, i.e. $\mathbf{r}_c^s = 0$ and

$$\mathbf{v}_c^w = \mathbf{v}_v^w + (\omega_v^w \times \mathbf{r}_c^v)$$

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- **Assumption:** Wheel motions are already consistent with rigid body motion.
- **Assumption:** Idealized rolling wheel model.
  - Motion by rolling only.
  - No slip in the driving or lateral direction.

Rolling Contact

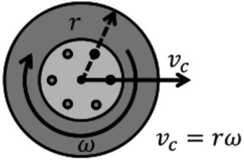

Figure: Contact point idealization

- Wheel has single point of contact with flat terrain, which means $\mathbf{v}_c^s = 0$.

Rolling Contact

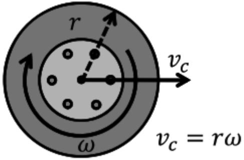

Figure: Contact point idealization

- Wheel has single point of contact with flat terrain, which means $\mathbf{v}_c^s = 0$.
- Under these nominal conditions, $v_c = r\omega$.

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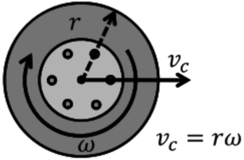

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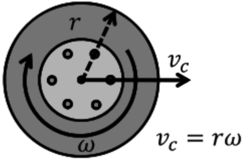

Figure: Contact point idealization

- Wheel has single point of contact with flat terrain, which means $\mathbf{v}_c^s = 0$.
- Under these nominal conditions, $v_c = r\omega$.
- Wheel velocity normal to the terrain is zero.
- On uneven terrain, the contact point moves wrt wheel center. Because wheels and terrain are not perfectly rigid, we have contact patch instead of point.

Rolling without slipping [2, Section 4.2.1]

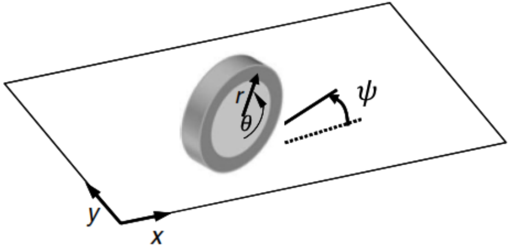

Figure: No slipping

- Direction of wheel velocity is consistent with direction of rolling.

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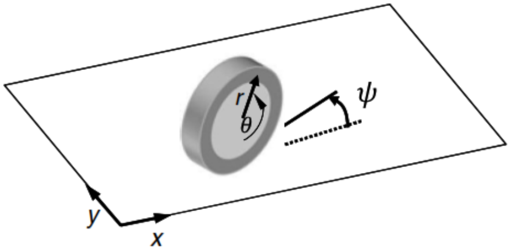


Figure: No slipping

- Direction of wheel velocity is consistent with direction of rolling.
- Velocity in the direction normal to wheel plane, on terrain, is zero.

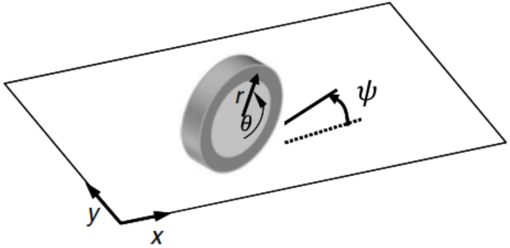


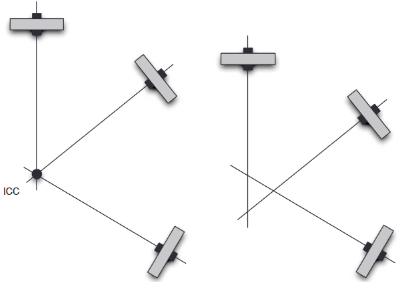
Figure: No slipping

- Direction of wheel velocity is consistent with direction of rolling.
- Velocity in the direction normal to wheel plane, on terrain, is zero.
- If wheel velocity vector is $(\dot{x}, \dot{y})$, this constraint can be expressed as the dot product between the velocity and unallowed direction to be zero, i.e.

$$\dot{x}\sin\psi - \dot{y}\cos\psi = 0.$$

How to make wheel motions consistent with robot motion?

All wheels in contact with ground should exhibit rolling motion about the same point in the environment at any instant, and the entire robot is also consistently rotating about this same point at that instant. This point is called **Instantaneous center of rotation (ICR)** or **Instantaneous center of curvature (ICC)**.



All rigid body motions in the plane can be considered to be rotation about some point. This point is ICR.

- A particle p is rotating in the plane.
- $\vec{v}_p = \vec{\omega} \times \vec{r}_p$

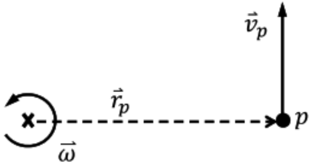


Figure: Rotation of point in plane

- A rigid body is executing a general motion.

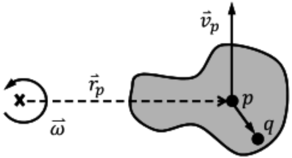
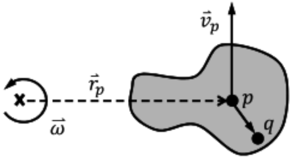


Figure: Rotation of point in plane

Proof of ICR theorem

- A rigid body is executing a general motion.
- Let v_p^w and ω be velocities of a point p with respect to some world frame.



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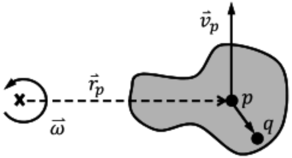
- A rigid body is executing a general motion.
- Let v_p^w and ω be velocities of a point p with respect to some world frame.
- Let $r = \frac{v_p^w}{\omega}$.



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Proof of ICR theorem

- A rigid body is executing a general motion.
- Let v_p^w and ω be velocities of a point p with respect to some world frame.
- Let $r = \frac{v_p^w}{\omega}$.
- Equation $\vec{\mathbf{v}}_p^{\text{icr}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_p^{\text{icr}}$ describes motion of point p.

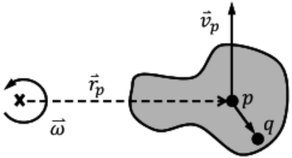


Figure: Rotation of point in plane

Proof of ICR theorem

- For another point q,

$$\vec{\mathbf{r}}_q^{\text{icr}} = \vec{\mathbf{r}}_p^{\text{icr}} + \vec{\mathbf{r}}_q^{p}.$$

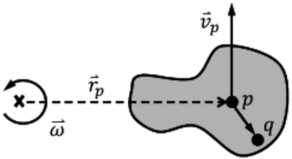


Figure: Rotation of point in plane

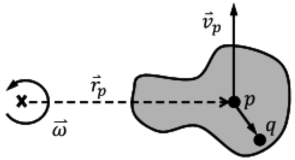
- For another point q,

$$\vec{\mathbf{r}}_q^{icr} = \vec{\mathbf{r}}_p^{icr} + \vec{\mathbf{r}}_q^{p}.$$

- Take time derivatives in a frame fixed to the world:

$$\left.\frac{d}{dt}\right|_w \vec{\mathbf{r}}_q^{icr} = \left.\frac{d}{dt}\right|_w \vec{\mathbf{r}}_p^{icr} + \underbrace{\left.\frac{d}{dt}\right|_w \vec{\mathbf{r}}_q^{p}}_{\left.\frac{d}{dt}\right|_b \vec{\mathbf{r}}_q^{p} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_q^{p}}$$

$$\begin{aligned}\vec{\mathbf{v}}_q^{icr} &= \vec{\mathbf{v}}_p^{icr} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_q^{p} \\ &= \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_p^{icr} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_q^{p} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_q^{icr}\end{aligned}$$



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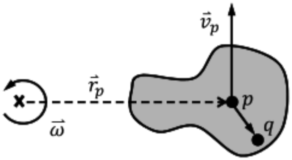


Figure: Rotation of point in plane

Velocity of any point q on rigid body is consistent with rotation with the same speed about the same point as p.

- When wheels are in steering configuration consistent with rigid body motion, the wheels don't slip and we have same ICR.

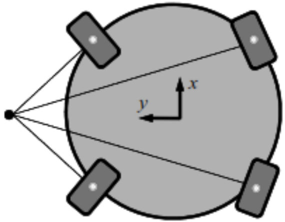
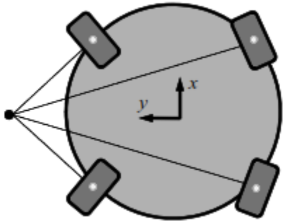


Figure: Jeantaud Diagram

How to find the ICR of a wheeled robot?

- When wheels are in steering configuration consistent with rigid body motion, the wheels don't slip and we have same ICR.
- ICR is intersection point of normals of linear velocities of all wheels.



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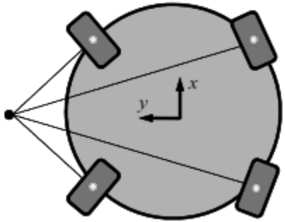

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How to find the ICR of a wheeled robot?

- When wheels are in steering configuration consistent with rigid body motion, the wheels don't slip and we have same ICR.
- ICR is intersection point of normals of linear velocities of all wheels.
- If directions are same, magnitudes are required.
- If magnitudes are also same, ICR is at infinity.

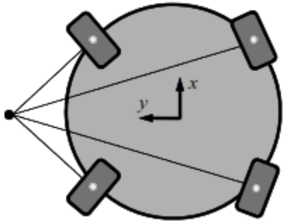
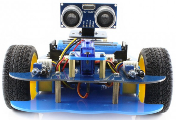
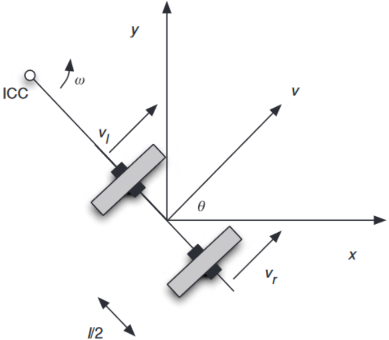


Figure: Jeantaud Diagram

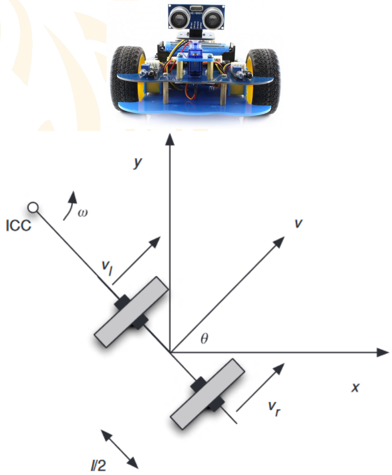
Table of Contents

Construction of a differential drive robot





- Two main wheels (shown in figure) are on same axis and actuated.
- There is one castor wheel or free wheel.
- L is distance between wheels.



- Two main wheels (shown in figure) are on same axis and actuated.
- There is one castor wheel or free wheel.
- L is distance between wheels.
- Velocity of each actuated wheel is controlled by separate motor.
- v_l and v_r are the velocities of left and right wheel contact points.

We want to relate wheel velocities to robot velocities.

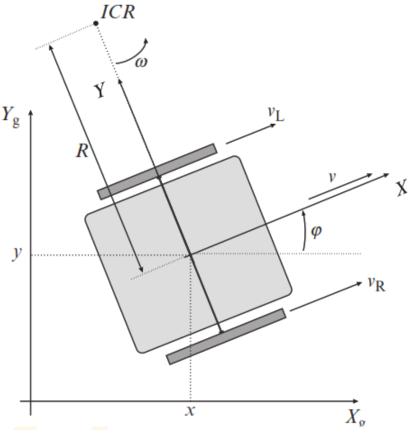
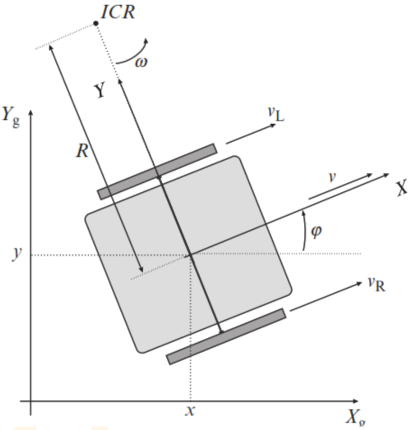


Figure: Differential Drive

- Attach the robot frame $\{v\}$ to the point in the middle of driven wheels axle. $\hat{x}_v$ is in the forward direction of motion.



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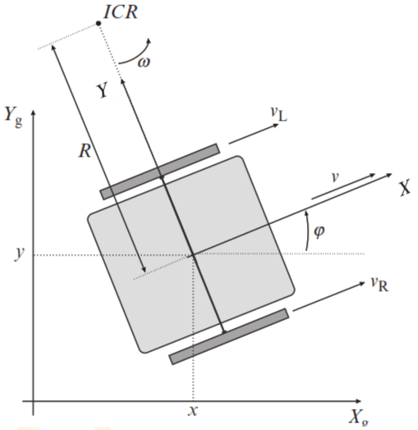

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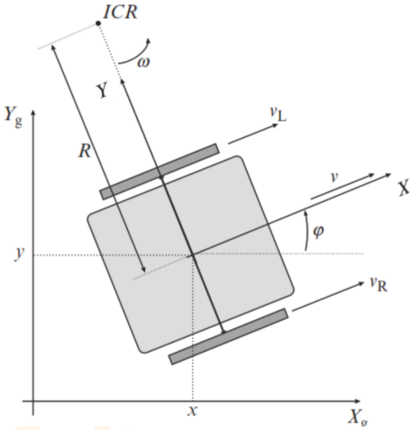
- At any instant, motion of robot can be considered as rotation about ICR with velocities $\vec{\boldsymbol{\omega}}(t)$ and $\vec{\mathbf{v}}(t)$.
- These velocities are with respect to the world frame, i.e. $\vec{\mathbf{v}}_v^w$ and $\vec{\boldsymbol{\omega}}_v^w$.



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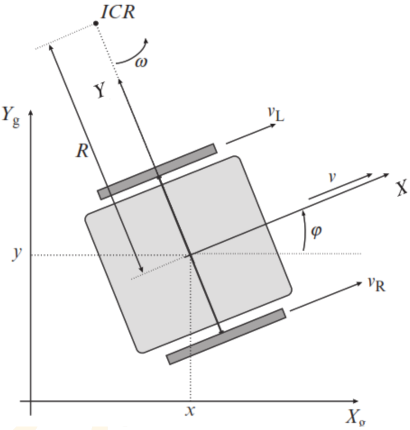
- Recall the standard wheel equation:

$$\mathbf{v}_c^w = \mathbf{v}_v^w + (\omega_v^w \times \mathbf{r}_c^v)$$



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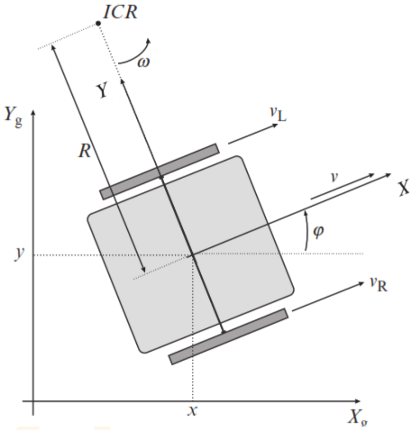
- We'll write this equation for differential drive in the coordinates of body frame, $\{v\}$.



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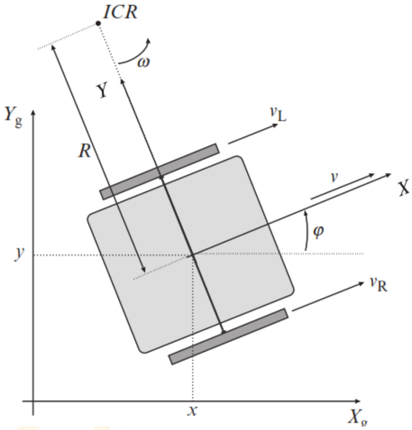
- So,

$$^v\mathbf{r}_l^v = \begin{bmatrix} 0 \\ L/2 \end{bmatrix} \quad \text{and} \quad ^v\mathbf{r}_r^v = \begin{bmatrix} 0 \\ -L/2 \end{bmatrix}$$

Using wheel equation to obtain differential drive model (2)



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- $\boldsymbol{\omega}_v^w$ is in the $\hat{z}_v$ direction.

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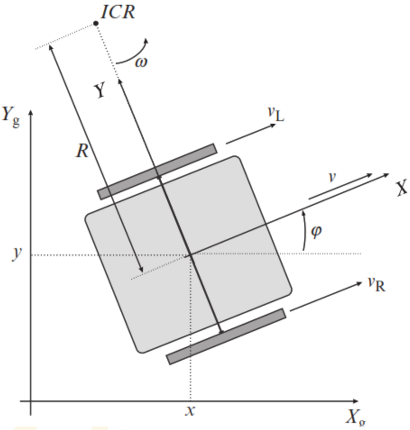

Figure: Differential Drive

- $\boldsymbol{\omega}_v^w$ is in the $\hat{z}_v$ direction.
- The model for velocities in the $\hat{x}_v$ direction is:

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & L/2 \\ 1 & -L/2 \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

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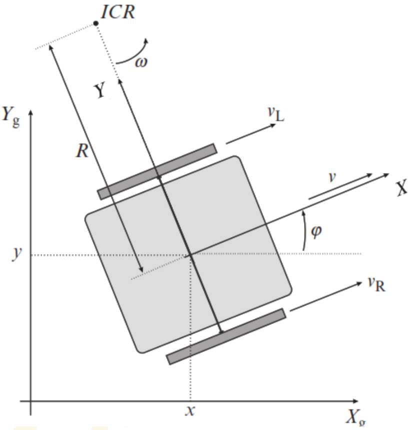

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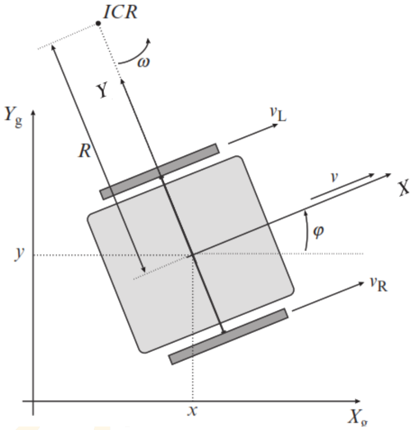
- The velocity in the lateral direction or y is assumed zero because of the no lateral slip assumption.



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- The velocity in the lateral direction or y is assumed zero because of the no lateral slip assumption.
- This is **Inverse Kinematics** model.

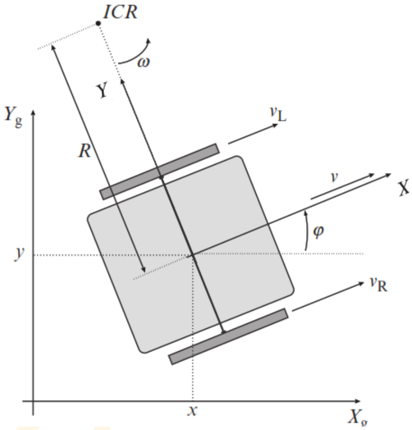

Figure: Differential Drive

- Invert the previous equation:

$$\begin{bmatrix} v_x \\ \omega \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \frac{2}{L} & -\frac{2}{L} \end{bmatrix} \begin{bmatrix} v_r \\ v_l \end{bmatrix}$$

Forward differential kinematic model of differential drive

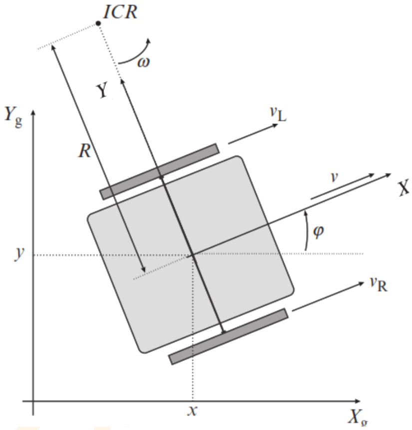

Figure: Differential Drive

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If both wheels have radius r, then

$$\begin{bmatrix} v_x \\ \omega \end{bmatrix} = \frac{r}{2} \begin{bmatrix} 1 & 1 \\ \frac{2}{L} & -\frac{2}{L} \end{bmatrix} \begin{bmatrix} \omega_r \\ \omega_l \end{bmatrix}$$

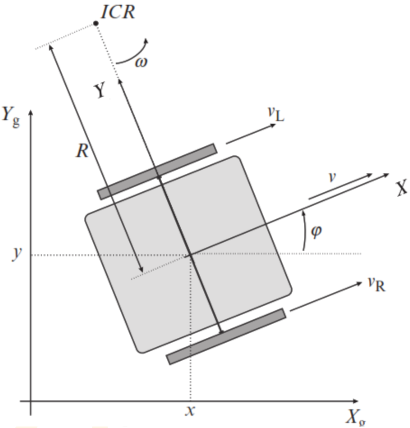


Figure: Differential Drive

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & L/2 \\ 1 & -L/2 \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

- How to move it straight in forward direction?

Driving strategies for a differential drive

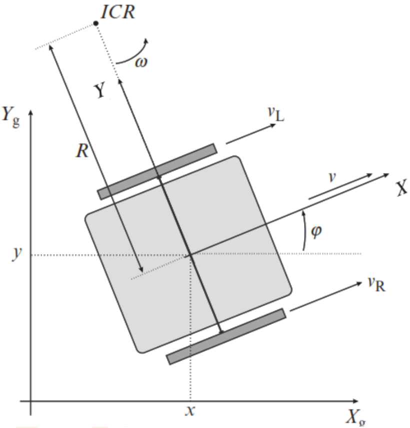
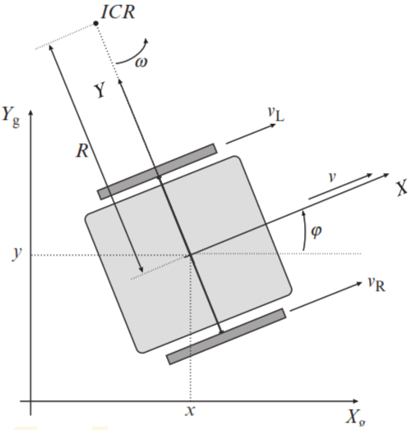


Figure: Differential Drive

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & L/2 \\ 1 & -L/2 \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

- How to move it straight in forward direction?
- How to make it turn left?



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- How to move it straight in forward direction?
- How to make it turn left?
- Can it rotate without changing its position, i.e. rotation at a point?

Driving strategies for a differential drive

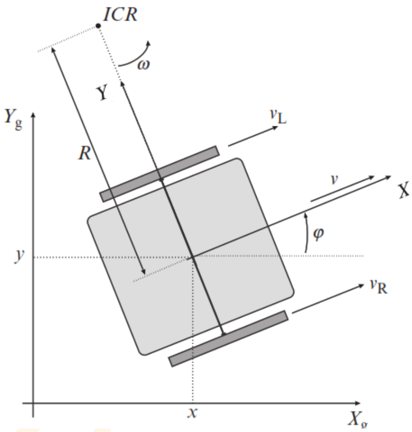

Figure: Differential Drive

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & L/2 \\ 1 & -L/2 \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

- How to move it straight in forward direction?
- How to make it turn left?
- Can it rotate without changing its position, i.e. rotation at a point?
- Can you make it move in any direction?

Assumptions

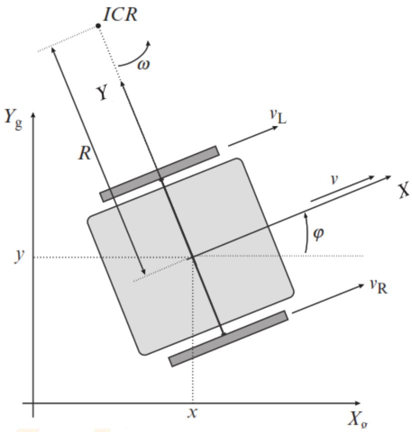

Figure: Differential Drive

- Robot is a rigid body
- Symmetry about longitudinal axis
  - Identical wheels ($r_l = r_r = r$)
  - Equidistant wheels (Axle length = L)
- Pure rolling
  - No lateral motion
  - No wheel slip ($v = r\omega$)
- Unmodeled
  - Castor wheel

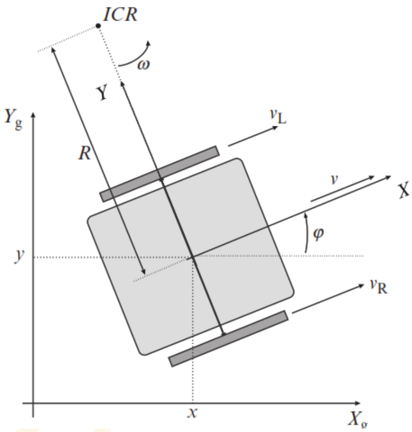

Figure: Differential Drive

- Let $R(t)$ be normal distance from ICR at time instant t, then tangential velocity of vehicle is:

$$v(t) = \omega(t)\, R(t).$$

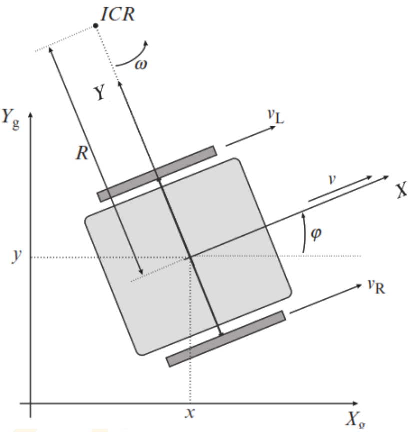

Figure: Differential Drive

- Let $R(t)$ be normal distance from ICR at time instant t, then tangential velocity of vehicle is:

$$v(t) = \omega(t)\, R(t).$$

- All points on the robot, including wheels, are rotating about ICR with $\omega(t)$ at time t. Then,

$$\omega(t) = \frac{v_L(t)}{R(t) - \frac{L}{2}}$$

$$\omega(t) = \frac{v_R(t)}{R(t) + \frac{L}{2}}$$

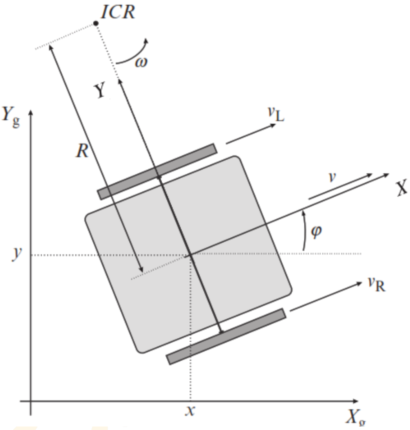

Figure: Differential Drive

$$\omega(t) = \frac{v_L(t)}{R(t) - \frac{L}{2}}$$

$$\omega(t) = \frac{v_R(t)}{R(t) + \frac{L}{2}}$$

$$\Rightarrow R(t) = \frac{L}{2}\,\frac{v_R(t) + v_L(t)}{v_R(t) - v_L(t)}$$

$$\omega(t) = \frac{v_R(t) - v_L(t)}{L}$$

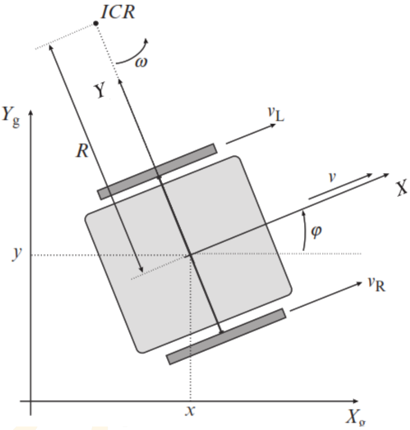

Figure: Differential Drive

-

$$v(t) = \omega(t)R(t)$$

$$v(t) = \frac{v_R(t) + v_L(t)}{2}$$

- Given, r is the wheel radius.

$$\begin{bmatrix} \dot{x}_m(t) \\ \dot{y}_m(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{bmatrix} v_L(t) \\ v_R(t) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ 0 & 0 \\ -\frac{r}{L} & \frac{r}{L} \end{bmatrix} \begin{bmatrix} \omega_L(t) \\ \omega_R(t) \end{bmatrix}$$

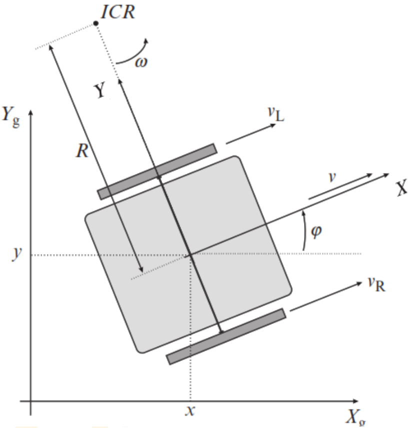

Figure: Differential Drive

- Note that v is the velocity of mobile robot with respect to the global frame, expressed in coordinates of the local mobile robot frame. Written precisely,

$$ {}^{V}\vec{\mathbf{v}}_{V}^{W} $$

What is kinematic model in global frame?

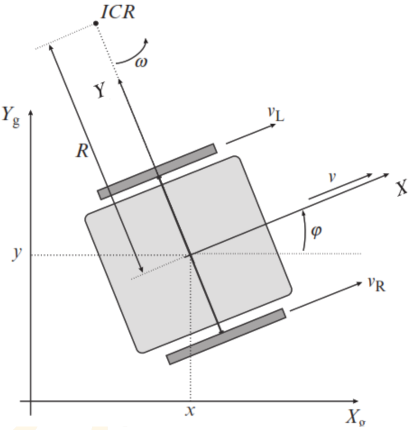

Figure: Differential Drive

- Note that v is the velocity of mobile robot with respect to the global frame, expressed in coordinates of the local mobile robot frame. Written precisely,

$$^{v}\vec{\mathbf{v}}_{v}^{w}$$

- We can convert it to coordinates in the global frame by

$$^{w}\mathbf{v}_{v}^{w} = {}^{w}R_{v}\ {}^{v}\mathbf{v}_{v}^{w}$$

Kinematic model in global frame:

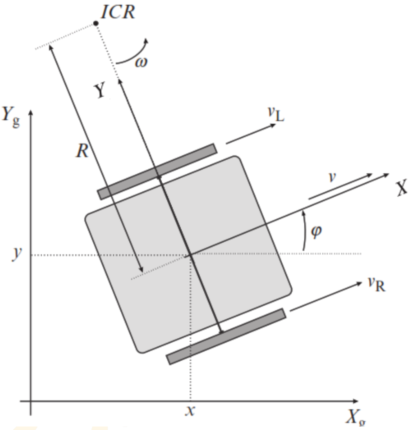

Figure: Differential Drive

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}\cos\phi(t) & -\sin\phi(t)\\ \sin\phi(t) & \cos\phi(t)\end{bmatrix}\begin{bmatrix}\dot{x}_v(t)\\ \dot{y}_v(t)\end{bmatrix}$$

$$= \begin{bmatrix}\cos\phi(t) & -\sin\phi(t)\\ \sin\phi(t) & \cos\phi(t)\end{bmatrix}\begin{bmatrix}\frac{r}{2} & \frac{r}{2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\omega_l(t)\\ \omega_r(t)\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}\frac{r}{2}\cos\phi(t) & \frac{r}{2}\cos\phi(t)\\ 0 & 0\end{bmatrix}}_{J}\begin{bmatrix}\omega_l(t)\\ \omega_r(t)\end{bmatrix}$$

Find differential kinematic model for Ackerman Drive

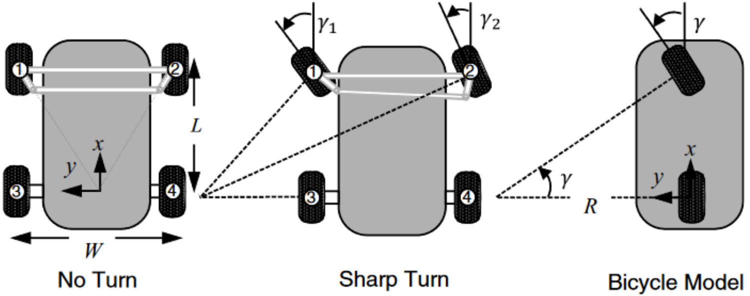

Figure 4.13 Ackerman Steer. The two front wheels are connected by a mechanism that maintains consistent steering angles. In this way, wheel slip is minimized. The difference between left and right steering angles increases with path curvature.

The bicycle model is a simplified model of the Ackerman steering drive. Find kinematic model for it.

Find differential kinematic model for Ackerman Drive

- Center of the $\{v\}$ frame is chosen at the center of the rear wheel. It can be placed at another location of our choice.
- The $\hat{x}$ direction of each wheel frame is chosen along the rolling direction of that wheel.
- Write a wheel equation for each wheel in the bicycle model.
- The wheel equations are being written in coordinates of the v frame, but we can choose to write them in any frame.

Equation for the rear wheel,

$$
{}^{v}v_{r}^{w} = {}^{v}v_{v}^{w} + {}^{v}\omega_{v}^{w} \times {}^{v}r_{r}^{v}
$$

$$
\begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}
$$

Note that subscript x in v_{rx} or v_x indicates x-direction of the v frame. The rotation of the v frame is about the z-axis only, so the angular velocity vector is only pointed in the $\hat{z}_v$-direction. a is the radius of the wheel.

$$
\begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
$$

Equation for the rear wheel,

$$\begin{bmatrix} v_{rx} \\ v_{ry} \\ v_{rz} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

From wheel constraints, $v_{ry} = 0$ and $v_{rz} = 0$. So,

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_{rx} \\ 0 \\ 0 \end{bmatrix}$$

Equation for the front wheel,

$$
{}^{v}v_r^{w} = {}^{v}v_v^{w} + {}^{v}\omega_v^{w} \times {}^{v}r_r^{v}
$$

$$
\begin{bmatrix} v_{fx} \\ v_{fy} \\ v_{fz} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} L \\ 0 \\ a \end{bmatrix}
$$

$$
\begin{bmatrix} v_{fx} \\ v_{fy} \\ v_{fz} \end{bmatrix} = \begin{bmatrix} v_x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega L \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ \omega L \\ 0 \end{bmatrix}
$$

Why do we have the component v_{fy}? Wasn't lateral motion to be equal to zero?

Note that v_{fy} is along the y-direction of v-frame and not the component orthogonal to the plane of the wheel.

How do you find the steering angle?

- Steering angle is formed by the vector along rolling direction of front wheel with v_x.
- We know that front wheel can also not move in the direction orthogonal to its plane.
- So, (v_{fx}, v_{fy}) velocity of front wheel is all along the rolling direction. Consequently, the angle that this velocity vector makes with v_x is:

$$\tan(\gamma) = \frac{v_{fy}}{v_{fx}} = \frac{\omega L}{v_x}.$$

- We'll also see the same if we try to find velocity in direction orthogonal to plane of front wheel. This will turn out to be:

$$\omega L \cos\gamma - v_x \sin\gamma.$$

- Since this component (lateral velocity) has to be zero, we get

$$\omega L \cos\gamma - v_x \sin\gamma = 0$$
$$\frac{\omega L}{v_x} = \frac{\sin\gamma}{\cos\gamma} = \tan\gamma$$

Acerman Model

$$\begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

$$\tan(\gamma) = \frac{\omega L}{v_x}$$

Final Comments

- While we call v_v^w the velocity of the robot, it is the velocity of the origin (a point) of the $\{v\}$ frame to be precise.
- Why do we label it as velocity of the robot?
- Because our robot is rigid and if we know this velocity and angular velocity we can work out the velocity of every point on the robot.
- In the case of this bicycle, if we find out the velocity of other points we'll see that the robot velocity vectors have components in the y direction as well, otherwise the bicycle will never turn.

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How do you obtain position of robot using FK differential model?

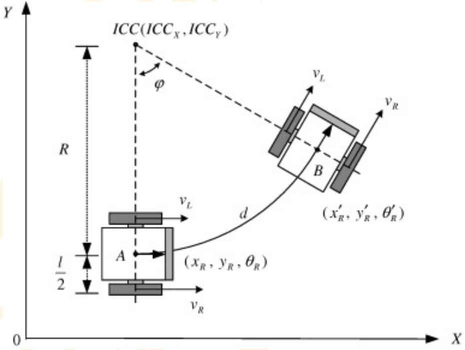


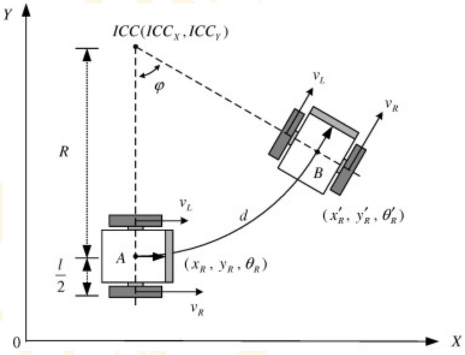
Figure: Differential Drive

- We have,

$$\dot{x}(t) = v(t)\cos(\phi(t))$$
$$\dot{y}(t) = v(t)\sin(\phi(t))$$
$$\dot{\phi}(t) = \omega(t)$$

(x, y, ϕ) is pose of robot in global coordinates, while (v, ω) are the velocities of the robot in robot coordinate system.

How do you obtain position of robot using FK differential model?



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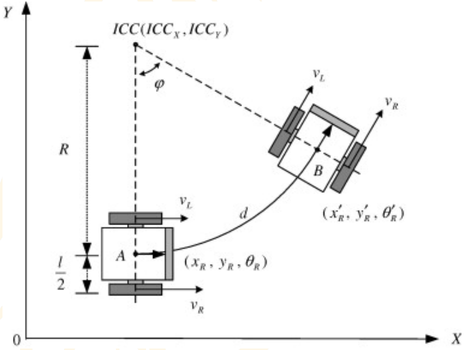
- **Forward Kinematics:** What is the robot pose, given controls u?

How do you obtain position of robot using FK differential model?



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  (x, y, ϕ) is pose of robot in global coordinates, while (v, ω) are the velocities of the robot in robot coordinate system.

- **Forward Kinematics:** What is the robot pose, given controls u?

- Is this enough information?

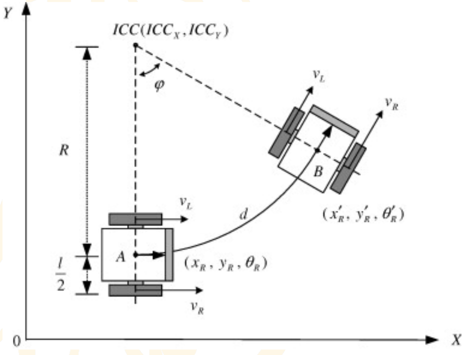


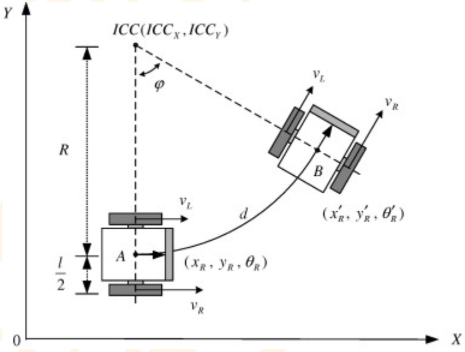
Figure: Differential Drive

- That is,

$$x(t) = \int_0^t v(t)\cos(\phi(t))dt$$

$$y(t) = \int_0^t v(t)\sin(\phi(t))dt$$

$$\phi(t) = \int_0^t \omega(t)dt$$



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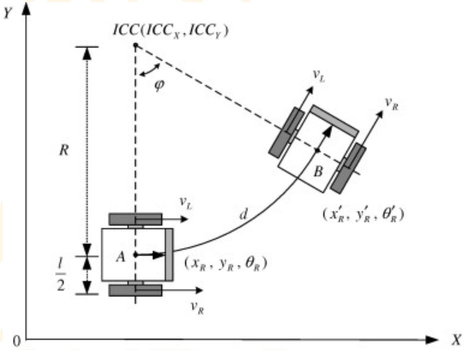
$$\phi(t) = \int_0^t \omega(t)dt$$

- We also need the initial pose of robot.

FK: Robot pose is obtained by integration.



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- We also need the initial pose of robot.
- This is called **Dead Reckoning**. The velocities of robot are measured using sensors.

FK: Euler or Rectangular Integration

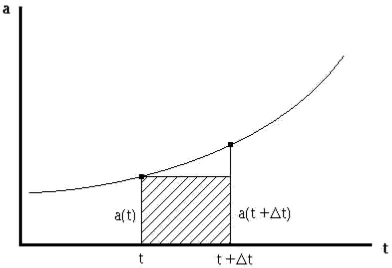

Figure: Euler Integration

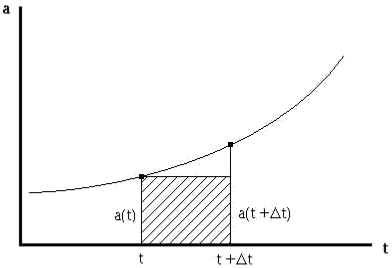

Figure: Euler Integration

- Sample the integrand at regular intervals, say of width $\Delta t = T_s$.

FK: Euler or Rectangular Integration

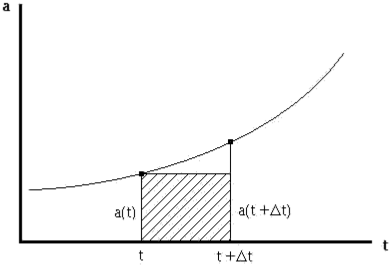

Figure: Euler Integration

- Sample the integrand at regular intervals, say of width $\Delta t = T_s$.
- The value of integrand is assumed to be constant in each sampling interval.

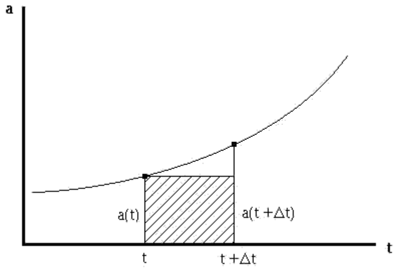

Figure: Euler Integration

- Sample the integrand at regular intervals, say of width $\Delta t = T_s$.
- The value of integrand is assumed to be constant in each sampling interval.
- Pose can be found iteratively as:

$$x(k+1) = x(k) + T_s\, v(k) \cos(\phi(k))$$
$$y(k+1) = y(k) + T_s\, v(k) \sin(\phi(k))$$
$$\phi(k+1) = \phi(k) + T_s\, \omega(k)$$

FK: Trapezoidal Integration

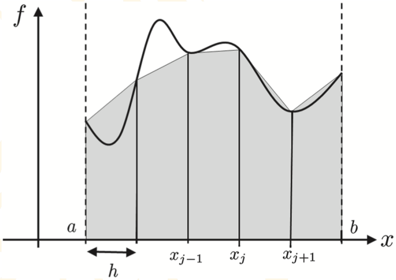

Figure: Trapezoidal Integration

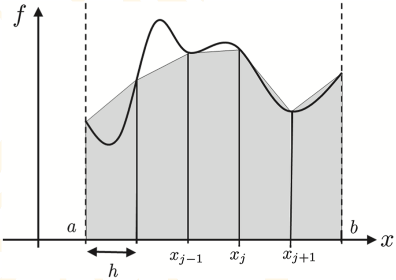

Figure: Trapezoidal Integration

- Integrand is approximated by a line instead of constant in each sampling period.

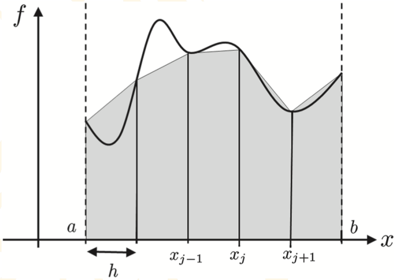

Figure: Trapezoidal Integration

- Integrand is approximated by a line instead of constant in each sampling period.
- Pose can be found iteratively as:

$$x_{k+1} = x_k + \frac{v_k T_s}{2}\left[\cos(\phi_k) + \cos(\phi_k + T_s\omega_k)\right]$$
$$y_{k+1} = y_k + \frac{v_k T_s}{2}\left[\sin(\phi_k) + \sin(\phi_k + T_s\omega_k)\right]$$
$$\phi_{k+1} = \phi_k + T_s\,\omega_k$$

- If $f(k+1)$ appears in intermediate expressions, it is assumed to be constant, i.e. $f(k+1) = f(k)$ in sampling period.

FK: Exact Integration

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- v and ω are assumed constant in the sampling period, but

$$\phi(t) = \phi_k + \int_{kT_s}^{(k+1)T_s} \omega_k dt, \quad \forall t \in [kT_s, (k+1)T_s]$$

FK: Exact Integration

- v and ω are assumed constant in the sampling period, but

$$\phi(t) = \phi_k + \int_{kT_s}^{(k+1)T_s} \omega_k dt, \quad \forall t \in [kT_s, (k+1)T_s]$$

- So,

$$x_{k+1} = x_k + v_k \int_{kT_s}^{(k+1)T_s} \cos\left(\phi_k + \omega_k(t - kT_s)\right) dt$$

- Pose can be found iteratively as:

$$\begin{aligned} x_{k+1} &= x_k + \frac{v_k}{\omega_k}\left[\sin(\phi_k + \omega_k T_s) - \sin(\phi_k)\right] \\ y_{k+1} &= y_k + \frac{v_k}{\omega_k}\left[\cos(\phi_k + \omega_k T_s) - \cos(\phi_k)\right] \\ \phi_{k+1} &= \phi_k + T_s \omega_k \end{aligned}$$

FK: Geometrical solution [1, Section 2.3]

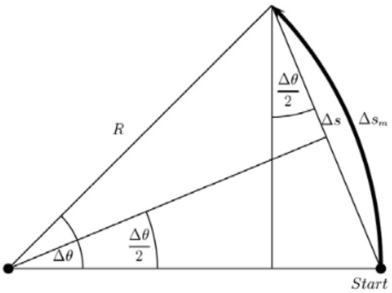

Figure: Geometrical interpretation

FK: Geometrical solution [1, Section 2.3]

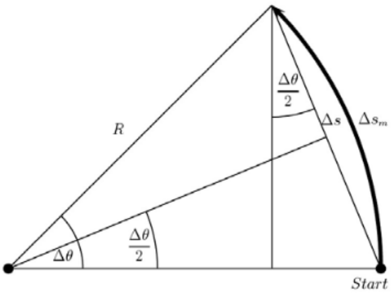

Figure: Geometrical interpretation

- If v_l and v_r are wheel velocities, then incremental distances are:

$$\Delta s_l = v_l \, \Delta t$$
$$\Delta s_r = v_r \, \Delta t$$

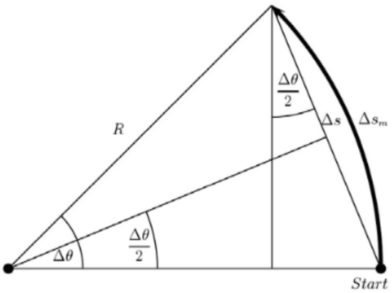

Figure: Geometrical interpretation

- If v_l and v_r are wheel velocities, then incremental distances are:

$$\Delta s_l = v_l \, \Delta t$$
$$\Delta s_r = v_r \, \Delta t$$

- Instantaneous motion of any point of robot is about ICR. So incremental change in angle about ICR for any point on the robot is the same, i.e.

$$\Delta\phi = \frac{\Delta s_l}{R - L/2} = \frac{\Delta s_r}{R + L/2} = \frac{\Delta s_m}{R},$$

where Δs_m corresponds to point between wheels.

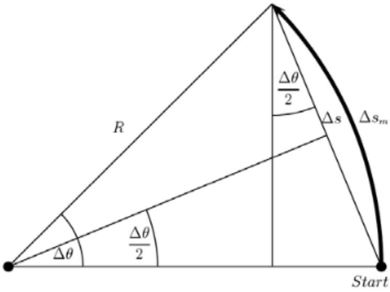

Figure: Geometrical interpretation

- Using the previous expressions,

$$R = \frac{L}{2}\frac{\Delta s_r + \Delta s_l}{\Delta s_r - \Delta s_l}$$

$$\Delta\phi = \frac{\Delta s_r - \Delta s_l}{L}$$

- Finally,

$$\Delta s = 2R\sin\left(\frac{\Delta\phi}{2}\right)$$

$$\Delta x = \Delta s \cdot \cos\left(\frac{\Delta\theta}{2} + \phi\right)$$

$$\Delta y = \Delta s \cdot \sin\left(\frac{\Delta\theta}{2} + \phi\right)$$

Inverse Kinematics Problem

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- It is not easy. No closed-form solution. Not unique.

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- **Special Cases:**

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- **Special Cases:**

- **Straight-line motion**: $\omega(t) = 0$ and $v(t)$ is constant.

$$x(t) = x(0) + vt\cos(\phi(0))$$
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- **Rotation**: ω is constant and $v = 0$.

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- **Straight-line motion**: $\omega(t) = 0$ and $v(t)$ is constant.

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- **Rotation**: ω is constant and $v = 0$.

$$\begin{aligned} x(t) &= x(0) \\ y(t) &= y(0) \\ \phi(t) &= \phi(0) + \frac{2v_r t}{L} \end{aligned}$$

- Possible strategy is to orient first, move in a straight line to desired position, and then reorient to desired pose.

IK for desired smooth trajectory

- Given desired smooth trajectory $(x(t), y(t))$, make the robot follow this trajectory such that its orientation is always tangent to it.

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IK for desired smooth trajectory

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- Assume robot's initial pose is on trajectory.

- Then,

$$
\begin{aligned}
v(t) &= \pm\sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \\
\phi(t) &= \operatorname{arctan2}\left(\dot{y}(t), \dot{x}(t)\right) + l\pi,
\end{aligned}
$$

where the $+$ sign of velocity and $l = 0$ correspond to forward driving direction and $l = 1$ to reverse.

IK for desired smooth trajectory

- Differentiating the expression for ϕ,

$$\omega(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}^2(t) + \dot{y}^2(t)}$$

IK for desired smooth trajectory

- Differentiating the expression for ϕ,

$$\omega(t) = \frac{\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)}{\dot{x}^2(t) + \dot{y}^2(t)}$$

- Notice that this requires a twice differentiable path and non-zero tangential velocity.

Table of Contents

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