Lecture 21

Sunday, April 10, 2022 3:00 PM

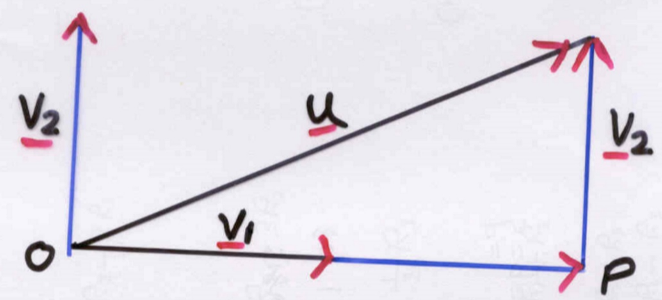
1 LINEAR ALGEBRA | LECTURE 21 | MATH 221

PREVIOUS RESULTS:

(1) IF $\boxed{v_2}$ IS THE VECTOR PROJECTION OF $\boxed{u}$ ORTHOGONAL TO $\boxed{v_1}$ THEN $\boxed{v_2}$ IS GIVEN BY

$$v_2 = u - \underbrace{\frac{\langle u, v_1 \rangle v_1}{\|v_1\|^2}}_{\to\ \overrightarrow{OP}}$$

, AS SHOWN BELOW:



(2) SIMILARLY THE VECTOR PROJECTION $\boxed{v_3}$ OF $\boxed{v}$, WHICH IS ORTHOGONAL TO BOTH $\boxed{v_1}$ AND $\boxed{v_2}$ IS GIVEN BY

$$\boxed{v_3 = v - \frac{\langle v, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle v, v_2 \rangle v_2}{\|v_2\|^2}}$$

$\hookrightarrow$ (*)

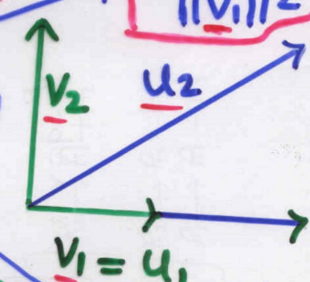
SUMMARY:

2 IF $\boxed{V}$ IS AN INNER PRODUCT SPACE AND $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n\}$ IS A $\boxed{\text{BASIS}}$ FOR $\boxed{V}$ THEN WE CAN FIND THE ORTHOGONAL BASIS BY FOLLOWING THESE STEPS:

(1) LET $\boxed{\underline{v}_1 = \underline{u}_1} \rightarrow$ ①

(2) TO FIND $\underline{v}_2$ ORTHOGONAL TO $\underline{v}_1$ BY COMPUTING THE COMPONENT OF $\underline{u}_2$ THAT IS ORTHOGONAL TO $\underline{v}_1$.

$$\boxed{\underline{v}_2 = \underline{u}_2 - \underbrace{\text{Proj}_{\underline{v}_1} \underline{u}_2}_{\frac{\langle \underline{u}_2, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}}} \rightarrow ②$$



(3) TO FIND $\underline{v}_3$ ORTHOGONAL TO BOTH $\underline{v}_1$ AND $\underline{v}_2$ BY COMPUTING THE COMPONENT OF $\underline{u}_3$ ORTHOGONAL TO THE $\boxed{\text{PLANE}}$ SPANNED BY $\underline{v}_1$ AND $\underline{v}_2$ AND IS GIVEN BY

$$\boxed{\underline{v}_3 = \underline{u}_3 - \frac{\langle \underline{u}_3, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2} - \frac{\langle \underline{u}_3, \underline{v}_2 \rangle \underline{v}_2}{\|\underline{v}_2\|^2}} \rightarrow ③$$

3. REPLACE $\underline{v}$ BY $\underline{u}_3$ IN (*) ON SLIDE ONE IN ORDER TO GET (3). SO WE OBTAINED $\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots$ SO ON UNTIL WE GET $\underline{v}_n$.

THE PRECEDING STEP-BY-STEP CONSTRUCTION FOR CONVERTING AN ARBITRARY BASIS INTO AN ORTHOGONAL BASIS IS CALLED THE GRAM-SCHMIDT PROCESS.

→ P. 304 (8TH ED.) OR P. 318 (7TH ED.)

EXAMPLE:

LET THE VECTOR SPACE $\boxed{P_2}$ HAVE THE INNER PRODUCT

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

APPLY THE GRAM-SCHMIDT PROCESS TO TRANSFORM THE STANDARD BASIS $S = \{1, x, x^2\}$ INTO AN ORTHONORMAL BASIS.

4)

SOLUTION: (HINTS)

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

$S = \{1, x, x^2\}$, HERE

$\underline{u}_1 = 1$, $\underline{u}_2 = x$, $\underline{u}_3 = x^2$

① $\underline{u}_1 = \underline{v}_1 = 1$

② $\|\underline{v}_1\| = \|1\| = \langle \underline{v}_1, \underline{v}_1\rangle^{\frac{1}{2}}$

$= \langle 1, 1\rangle^{1/2} = \left(\int_{-1}^{1} 1\,dx\right)^{\frac{1}{2}} = \sqrt{2}$

③ $\underline{v}_2 = \underline{u}_2 - \dfrac{\langle \underline{u}_2, \underline{v}_1\rangle \underline{v}_1}{\|\underline{v}_1\|^2}$

$= x - \dfrac{\overbrace{\langle x, 1\rangle}^{0}}{\|1\|^2} = x$

④ $\|\underline{v}_2\| = \|x\| = \left(\int_{-1}^{1} x^2\,dx\right)^{\frac{1}{2}}$

$= \sqrt{\frac{2}{3}}$

⑤ $\underline{v}_3 = \underline{u}_3 - \dfrac{\langle \underline{u}_3, \underline{v}_1\rangle \underline{v}_1}{\|\underline{v}_1\|^2} - \dfrac{\langle \underline{u}_3, \underline{v}_2\rangle \underline{v}_2}{\|\underline{v}_2\|^2}$

5)

BUT $\langle \underline{u}_3, \underline{v}_1 \rangle = \int_{-1}^{1} x^2 dx$

$$= \left. \frac{x^3}{3} \right|_{-1}^{1} = \frac{1}{3}[1-(-1)] = \frac{2}{3}$$

AND $\langle \underline{u}_3, \underline{v}_2 \rangle = \langle x^2, x \rangle$

$$= \int_{-1}^{1} x^3 dx = \left. \frac{x^4}{4} \right|_{-1}^{1} = 0$$

$$\therefore \underline{v}_3 = \underline{u}_3 - \frac{\langle \underline{u}_3, \underline{v}_1 \rangle}{\|\underline{v}_1\|^2} \underline{v}_1 - \frac{\langle \underline{u}_3, \underline{v}_2 \rangle}{\|\underline{v}_2\|^2} \underline{v}_2$$

0

$$= x^2 - \frac{2}{3} \cdot \frac{1}{2} = x^2 - \frac{1}{3} = \underline{v}_3$$

(6) $\|\underline{v}_3\| = \langle \underline{v}_3, \underline{v}_3 \rangle^{\frac{1}{2}}$

$$= \left[\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx \right]^{1/2} = \sqrt{\frac{8}{45}} \text{ (CHECK)}$$

(7) REQUIRED <u>ORTHONORMAL</u> BASIS

IS $= \left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \frac{\underline{v}_3}{\|\underline{v}_3\|} \right\}$

$$= \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\, x, \left(x^2 - \frac{1}{3}\right)\sqrt{\frac{45}{8}} \right\}$$

6)

$$= \left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ \frac{(3x^2-1)}{\cancel{3}} \sqrt{\frac{\cancel{9}\times 5}{8}} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ (3x^2-1)\frac{\sqrt{5}}{2\sqrt{2}} \right\}$$

TRY THE FOLLOWING:

IF $S = \{ \underline{V_1}, \underline{V_2}, \ldots, \underline{V_n} \}$ IS AN ORTHOGONAL SET OF $\boxed{\text{NONZERO}}$ VECTORS IN AN INNER PRODUCT SPACE, THEN $\boxed{S}$ IS LINEARLY INDEPENDENT.

HINT: ASSUME

$$k_1\underline{V_1} + k_2\underline{V_2} + \cdots + k_n\underline{V_n} = \underline{0}$$

AND PROVE

$$k_1 = k_2 = \cdots = k_n = 0$$

ALSO THIS IS TH. 6.3.3. (P.301 8TH ED.) OR (P. 315 7TH ED.)

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PROOF: LET $k_1\underline{v_1} + k_2\underline{v_2} + \cdots + k_n\underline{v_n} = \underline{0}$

TAKING THE INNER PRODUCT WITH $\underline{v_i}$ ON BOTH SIDES, $(1 \le i \le n)$

$$\langle k_1\underline{v_1} + k_2\underline{v_2} + \cdots + k_n\underline{v_n}, \underline{v_i} \rangle = 0$$

$$\because \langle \underline{0}, \underline{v_i} \rangle = \langle \underline{0}+\underline{0}, \underline{v_i} \rangle = \langle \underline{0}, \underline{v_i} \rangle + \langle \underline{0}, \underline{v_i} \rangle \Rightarrow \langle \underline{0}, \underline{v_i} \rangle = 0$$

$$k_1 \langle \underline{v_1}, \underline{v_i} \rangle + k_2 \langle \underline{v_2}, \underline{v_i} \rangle + \cdots + k_i \langle \underline{v_i}, \underline{v_i} \rangle + \cdots + k_n \langle \underline{v_n}, \underline{v_i} \rangle = 0 \quad (1)$$

BUT $S = \{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$ IS AN ORTHOGONAL SET THEREFORE

$$\langle \underline{v_i}, \underline{v_j} \rangle = 0 \text{ WHEN } i \neq j$$

SO THAT THE LAST EQUATION (1) REDUCES TO $k_i \langle \underline{v_i}, \underline{v_i} \rangle = 0$

BUT $\underline{v_i} \neq \underline{0}$ (GIVEN)

THEREFORE $\langle \underline{v_i}, \underline{v_i} \rangle = \|\underline{v_i}\|^2 > 0$

SO THAT $\boxed{k_i = 0}$. SINCE THE SUBSCRIPT $\boxed{i}$ IS ARBITRARY, WE HAVE $\boxed{k_1 = k_2 = \cdots = k_n = 0}$;

THUS, $\boxed{S}$ IS LINEARLY INDEPENDENT.

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ASSIGNMENT 5(b)

Not Include

(1) Q 17 (P.[illegible] 8TH ED) / P.256 7TH ED.

Already done earlier.

(2) PROVE CAUCHY-SCHWARZ INEQUALITY IN CASE OF EUCLIDEAN INNER PRODUCT.

HINT: SEE P.30[illegible] 7TH ED. OR P.287 8TH ED.

See note at the end of this lecture for this.

(3) ALSO PROVE (2) IN GENERAL.

See the following Link for the General Cauchy Schwarz Inequality:

6_7.pdf (berkeley.edu)

(4) IF $\underline{u}$ AND $\underline{v}$ ARE TWO VECTORS IN AN INNER PRODUCT SPACE THEN

(i) $\|\underline{u}+\underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$ } TRIANGLE INEQUALITY

(ii) USING THE FACT THAT $\underline{u}$ AND $\underline{v}$ ARE ORTHOGONAL PROVE THAT $\|\underline{u}+\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2$ WHICH IS ALSO CALLED A GENERALIZED THEOREM OF PYTHAGORAS.

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(5) ~~Q [illegible] P.295 (8TH ED)~~
OR P.310 (7TH ED)

Do Exercise set 6.2 Q6

(6) LET $f(x)$ AND $g(x)$ BE CONTINUOUS FUNCTIONS ON $[0,1]$ PROVE

$$\left[\int_0^1 f(x)g(x)\,dx\right]^2 \le \left[\int_0^1 f^2(x)\,dx\right]\left[\int_0^1 g^2(x)\,dx\right]$$

$$= \left[\int_0 f(x)\,dx\right]\left[\int_0 g^2(x)\,dx\right]$$

(7) GRAM-SCHMIDT PROCESS AND ORTHONORMALITY

17(a), 18, 25, 26, 27, 29 From Exercise set 6.3

(8) Q. no. 32, P. 297 (8th ED.) OR
Q. no. 30, P. 311 (7th ED.)

HINT: $\cos(\alpha+B) = \cos\alpha\cos B - \sin\alpha\sin B$
$\cos(\alpha-B) = \cos\alpha\cos B + \sin\alpha\sin B$

ADDING $\cos(\alpha+B) + \cos(\alpha-B)$
$= 2\cos\alpha\cos B$

ROTATION OF AXES:

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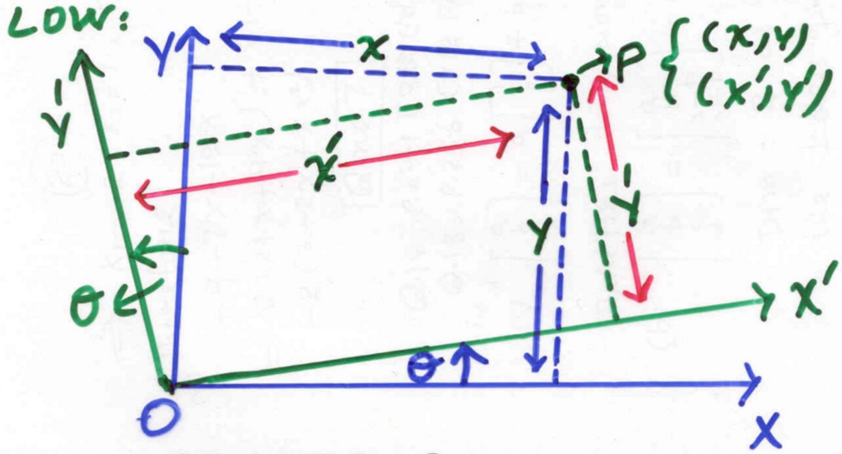
CONSIDER A ROTATION OF THE AXES ABOUT THE ORIGIN AS SHOWN BELOW:



X COORDINATE GIVES DISTANCE FROM Y AXIS, Y COORDINATE GIVES DISTANCE FROM X AXIS. IF THE AXES ARE ROTATED THROUGH AN ANGLE θ, THEN EVERY POINT OF THE PLANE HAS $\boxed{\text{TWO}}$ REPRESENTATIONS: (x, y) IN THE ORIGINAL COORDINATE SYSTEM AND (x', y') IN THE NEW COORDINATE SYSTEM.

PROBLEM: WHAT IS THE RELATIONSHIP BETWEEN THE $\boxed{x}$ AND $\boxed{y}$ OF ONE COORDINATE SYSTEM

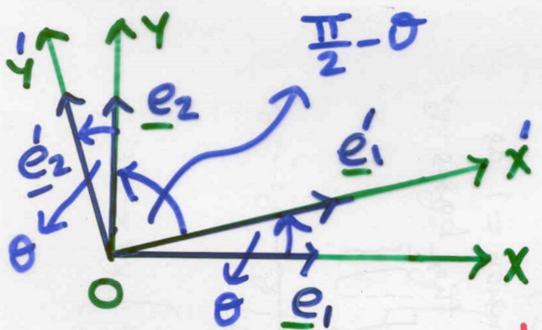
AND THE $\boxed{x'}$ AND $\boxed{y'}$ OF THE OTHER?

(11) CONSIDER THE VECTOR $\vec{OP}$ WHICH IS GIVEN BY $\boxed{\vec{OP} = (x,y) = x\underline{e}_1 + y\underline{e}_2}$ IN THE ORIGINAL COORDINATE SYSTEM AND ALSO

$\boxed{\vec{OP} = (x',y') = x'\underline{e}_1' + y'\underline{e}_2'}$ IN

THE NEW COORDINATE SYSTEM ($\underline{e}_1'$ AND $\underline{e}_2'$ ARE ALSO UNIT VECTORS).

CONSIDER THE FOLLOWING FIGURE:



$\because \{\underline{e}_1, \underline{e}_2\}$ IS A BASIS

$\therefore$

$$\boxed{\underline{e}_1' = k_1\underline{e}_1 + k_2\underline{e}_2}$$

$$\Rightarrow \underline{e}_1' \cdot \underline{e}_1 = k_1 \overbrace{\cancel{\underline{e}_1 \cdot \underline{e}_1}}^{1} + k_2 \overbrace{\cancel{\underline{e}_2 \cdot \underline{e}_1}}^{0}$$

$$\Rightarrow \boxed{k_1 = \underline{e}_1' \cdot \underline{e}_1 = \cos\theta}$$

SIMILARLY $\boxed{k_2 = \sin\theta}$ (CHECK).

$$\therefore \boxed{\underline{e}_1' = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2}$$

$$= (\cos\theta, \sin\theta)$$

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SIMILARLY

$\boxed{\underline{e}_2' = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2}$ (CHECK)

$\to = (-\sin\theta, \cos\theta)$

THEREFORE

$$\overrightarrow{OP} = x'\underline{e}_1' + y'\underline{e}_2'$$
$$= x'(\cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2)$$
$$+ y'(-\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2)$$
$$= (x'\cos\theta - y'\sin\theta)\underline{e}_1 + (x'\sin\theta + y'\cos\theta)\underline{e}_2$$
$$= x\underline{e}_1 + y\underline{e}_2 = \overrightarrow{OP} \quad \text{—①}$$

① SHOWS THAT

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$$\Rightarrow \boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}$$

THE MATRIX $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R$ (SAY)

WHICH GIVES ROTATION THROUGH AN ANGLE $\boxed{\theta}$ (COUNTER-CLOCKWISE) IS CALLED A ROTATION MATRIX.

ITS COLUMN VECTORS ARE NEW BASIS VECTORS i.e. $\left[[\underline{e}'_1] \; [\underline{e}'_2] \right]$ AND ARE ORTHONORMAL WITH THE EUCLIDEAN INNER PRODUCT. ALSO ITS $\boxed{\text{ROW}}$ VECTORS ARE ORTHONORMAL WITH THE EUCLIDEAN INNER PRODUCT.

CHECK: $(\cos\theta, -\sin\theta) \cdot (\sin\theta, \cos\theta) = 0$

NOTES: ① $RR^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ INTO

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= R^T R \implies \boxed{R^T = R^{-1}}$$

② $\det(R) = \cos^2\theta + \sin^2\theta = 1$

③ WHEN THERE IS NO ROTATION THEN $R = I = \left[[\underline{e}_1][\underline{e}_2] \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ FOR $\theta = 0$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

DEFINITION:

A SQUARE MATRIX $\boxed{A}$ WITH THE PROPERTY $\boxed{A^{-1} = A^T}$ IS SAID TO BE AN ORTHOGONAL MATRIX.

TRY THE FOLLOWING:

IF $A^T = A^{-1}$ THEN WHAT ARE THE POSSIBLE VALUES OF $\det(A)$?

ANSWER: ± 1

6.7 Cauchy-Schwarz Inequality

Recall that we may write a vector $\mathbf{u}$ as a scalar multiple of a nonzero vector $\mathbf{v}$, plus a vector orthogonal to $\mathbf{v}$:

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} + \left(\mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v}\right). \tag{1}$$

The equation (1) will be used in the proof of the next theorem, which gives one of the most important inequalities in mathematics.

Theorem 16 (Cauchy-Schwarz Inequality)**.** *If $\mathbf{u}, \mathbf{v} \in V$, then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{2}$$

This inequality is an equality if and only if one of $\mathbf{u}, \mathbf{v}$ is a scalar multiple of the other.

Proof. Let $\mathbf{u}, \mathbf{v} \in V$. If $\mathbf{v} = \mathbf{0}$, then both sides of (2) equal 0 and the desired inequality holds. Thus we can assume that $\mathbf{v} \neq \mathbf{0}$. Consider the orthogonal decomposition

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} + \mathbf{w},$$

where $\mathbf{w}$ is orthogonal to $\mathbf{v}$ (here $\mathbf{w}$ equals the second term on the right side of (1)). By the Pythagorean theorem,

$$\begin{aligned}
\|\mathbf{u}\|^2 &= \left\| \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{v}\|^2}\mathbf{v} \right\|^2 + \|\mathbf{w}\|^2 \\
&= \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2} + \|\mathbf{w}\|^2 \\
&\geq \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\|\mathbf{v}\|^2}.
\end{aligned}$$

Multiplying both sides of this inequality by $\|\mathbf{v}\|^2$ and then taking square roots gives the Cauchy-Schwarz inequality (2).

Looking at the proof of the Cauchy-Schwarz inequality, note that (2) is an equality if and only if the last inequality above is an equality. Obviously this happens if and only if $\mathbf{w} = \mathbf{0}$. But $\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{u}$ is a multiple of $\mathbf{v}$. Thus the Cauchy-Schwarz inequality is an equality if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ or $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$ (or both; the phrasing has been chosen to cover cases in which either $\mathbf{u}$ or $\mathbf{v}$ equals $\mathbf{0}$). □

The next result is called the triangle inequality because of its geometric interpretation that the length of any side of a triangle is less than the sum of the lengths of the other two sides. Consider a triangle with sides consisting of vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u} + \mathbf{v}$.

Theorem 17 (Triangle Inequality)**.** *If $\mathbf{u}, \mathbf{v} \in V$, then*

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{3}$$

This inequality is an equality if and only if one of $\mathbf{u}, \mathbf{v}$ is a nonnegative multiple of the other.

Proof. Let $\mathbf{u}, \mathbf{v} \in V$. Then

$$\begin{aligned}
\|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
&= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle \\
&\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| \\
&\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| \\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Taking square roots of both sides of the inequality above gives the triangle inequality (3).

We have equality in the triangle inequality if and only if

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\|. \tag{4}$$

If one of $\mathbf{u}, \mathbf{v}$ is a nonnegative multiple of the other, then (4) holds. Conversely, suppose (4) holds.Then the condition for equality in the Cauchy-Schwarz inequality implies that one of $\mathbf{u}, \mathbf{v}$ must be a scalar multiple of the other. Clearly (4) forces the scalar in question to be nonnegative, as desired. $\square$

Example 1. *Suppose $p(t) = 3t - t^2$ and $q(t) = 3 + 2t^2$. For p and q in $\mathbb{P}_2$, define*

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

a) Compute $\langle p, q \rangle$.

b) Compute the orthogonal projection of q onto the subspace spanned by p.

Example 2. *Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use the Cauchy-Schwarz inequality to show that*

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}.$$