CS/Math 113 - Problem Set 8

Dead TAs Society
Habib University - Spring 2023

Week 12

Problems

Problem 1. **[Chapter 2.5, Question 1]** Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

(a) the negative integers
(b) the even integers
(c) the integers less than 100
(d) the real numbers between 0 and $\frac{1}{2}$
(e) the positive integers less than 1,000,000,000
(f) the integers that are multiples of 7

Solution:

(a) Countably infinite, as a one-to-one correspondence can be made by mapping each negative integer to its positive or absolute value: $1 \leftrightarrow -1, 2 \leftrightarrow -2, 3 \leftrightarrow -3, \ldots$

(b) Countably infinite, as a one-to-one correspondence can be made by first reordering the even integers like so; $0, -2, 2, -4, 4, -6, 6, ...$ and then mapping each even integer with the set of positive integers.

(c) Countably infinite, as a one-to-one correspondence can be made by ordering the elements in descending order; $99, 98, 97, 96, \ldots$ and then mapping each integer with the set of positive integers.

(d) Uncountable. A subset of an interval of real numbers is uncountable because there is always a real number in between any two real numbers.

(e) It is a finite set ranging from 1 to 999,999,999.

(f) Countably infinite by the same logic as in part (b). We can make a sequence: $0, -7, 7, -14, 14, \ldots$ and make a mapping to the positive integers in the natural order.

Problem 2. [Chapter 2.5, Questions 10] Given an example of two uncountable sets A and B such that $A - B$ is

(a) finite

(b) countably infinite

(c) uncountable

Solution:

(a) Let A and B both be the set of real numbers. Then $A - B = \emptyset$which is finite

(b) Let A be the set of real numbers and B be the set of real numbers that are not integers. Then $B = A - \mathbb{Z}$. Then $A - B = \mathbb{Z}$ which is countably infinite.

(c) Let A be the set of real numbers and B be the set of real numbers except numbers between 0 and 1. Then $A - B$ gives us the set of real numbers between 0 and 1 which is uncountable.

Problem 3. [Chapter 2.5, Questions 11] Given an example of two uncountable sets A and B such that $A \cap B$ is

(a) finite

(b) countably infinite

(c) uncountable

Solution:

(a) Let A be the interval $(0, 1)$ and B be the interval $(2, 3)$. They are both uncountable, however, $A \cap B = \emptyset$ which is finite.

(b) We need to find some common elements in between the two uncountable sets such that their intersection remains countable. Then let A be set in part (a) adjoined with the set of integers; $A = (0, 1) \cup \mathbb{Z}$ and B be the set in part (a) adjoined with the set of integers; $B = (2, 3) \cup \mathbb{Z}$. Then $A \cap B = \mathbb{Z}$ which is countably infinite.

(c) Just find any interval that overlaps, then their intersection will also be uncountabe. Let A be the interval $(0, 1)$ and B be the interval $(0, 2)$. Then $A \cap B = (0, 1)$ which is uncountable.

Problem 4. **[Chapter 2.5, Questions 12]** Show that if A and B are sets and $A \subset B$ then $\mid A \mid \leq \mid B \mid$

Solution: If $A \subset B$, then we can establish a one-to-one function from A to B such that each element in A is mapped onto an element in B; $f : A \to B$. Since a one-to-one function can be made from A to B, this fits the definiton of $|A| \leq |B|$. Hence shown.

Problem 5. **[Chapter 2.5, Questions 14]** Show that if A and B are sets with the same cardinality, then $\mid A \mid \leq \mid B \mid$ and $\mid B \mid \leq \mid A \mid$

Solution: If A and B have the same cardinality, then there exists a one-to-one function from A to B such that $f : A \to B$. Then f meets the definiton of $|A| \leq |B|$.

Now consider f^{-1} such that $f^{-1} : B \to A$. Since the cardinalities of A and B are the same, then there also exists a one-to-one function from B to A defined by f^{-1}. Then f^{-1} meets the definition of $|B| \leq |A|$.

Hence proved.