Midterm Exam B
CS/MATH 113 Discrete Mathematics

Habib University — Spring 2023

11 March, 2023. 1300-1415h.

This exam consists of 4 questions for a total of 40 points on 3 pages. Attempt all problems and submit this sheet with your answer sheet by the end of the exam.

Student ID: ______________________________

Student Name: ______________________________

1. Tautologies (10 points)

Prove or disprove that the following are tautologies.

(a) $(a \implies b) \iff (\neg b \implies \neg a)$

Solution: This can be proved easily using a truth table. We provide a proof using logical equivalences.

Proof.

$$\begin{aligned} \text{LHS} &\equiv \neg a \vee b && \text{implication} \\ &\equiv b \vee \neg a && \text{commutativity} \\ &\equiv \neg b \implies \neg a && \text{implication} \end{aligned}$$

□

(b) $(a \wedge b \wedge c \wedge d) \implies (c \vee b)$

Solution: We use a sequence of inferences to prove the claim.

Proof.

c	simplification on LHS	(1)
$c \vee b$	addition on (1)	

□

2. Logical Equivalence (10 points)

Show that $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$.

Solution: We prove that LHS $\implies$ RHS, and vice versa, through a sequence of inferences.

Proof. Case: LHS $\implies$ RHS

$P(c) \vee Q(c)$	EI on LHS, c is an specific element	(1)

For (1) to be True, $P(c)$ is True or $Q(c)$ is True. We look at each case.
Sub-case: $P(c)$ is True

$P(c)$	premise	(2)
$\exists x P(x)$	EG on (2)	(3)
$\exists x P(x) \vee \exists x Q(x)$	addition on (3)	(4)

Sub-case: $Q(c)$ is True

$Q(c)$	premise	(5)
$\exists x Q(x)$	EG on (5)	(6)
$\exists x Q(x) \vee \exists x P(x)$	addition on (6)	(7)
$\exists x P(x) \vee \exists x Q(x)$	commutativity on (7)	(8)

Case: RHS $\implies$ LHS

$P(c_1) \vee Q(c_2)$	EI on RHS, c_1, c_2 are specific elements	(9)

For (9) to be True, $P(c_1)$ is True or $Q(c_2)$ is True. We look at each case.
Sub-case: $P(c_1)$ is True

$P(c_1)$	premise	(10)
$P(c_1) \vee Q(c_1)$	addition on (10)	(11)
$\exists x(P(x) \vee Q(x))$	EG on (11)	(12)

Sub-case: $Q(c_2)$ is True

$Q(c_2)$	premise	(13)
$Q(c_2) \vee P(c_2)$	addition on (13)	(14)
$P(c_2) \vee Q(c_2)$	commutativity on (14)	(15)
$\exists x(P(x) \vee Q(x))$	EG on (15)	

□

3. Subsets (10 points)

Prove that $\{30n \mid n \in \mathbb{Z}\}$ is a subset of $\{3n \mid n \in \mathbb{Z}\} \cap \{5n \mid n \in \mathbb{Z}\}$.

Solution: Let $A = \{30n \mid n \in \mathbb{Z}\}, B = \{3n \mid n \in \mathbb{Z}\}, C = \{5n \mid n \in \mathbb{Z}\}$.
Then we have to show that $A \subseteq (B \cap C)$.
We do so by showing that $A \subseteq B$ and $A \subseteq C$.

Proof. Case: $A \subseteq B$.
Consider $a \in A$.
Then $a = 30k$ for some $k \in \mathbb{Z}$.
Then $a = 3 \cdot 10k$.
$\therefore a \in B$.

Case: $A \subseteq C$.
Consider $a \in A$.
Then $a = 30k$ for some $k \in \mathbb{Z}$.
Then $a = 5 \cdot 6k$.
$\therefore a \in C$. □

4. Function Properties (10 points)

Given $g : A \to B$ and $f : B \to C$, prove that: if $(f \circ g)$ is one-to-one, then so is g.

Solution: We prove the above by contradiction.

Proof. Assume that $(f \circ g)$ is one-to-one and g is not.
Then, $\exists a_1, a_2 \in A \ni (a_1 \neq a_2 \wedge g(a_1) = g(a_2))$.
Now, $f \circ g : A \to C$.
$(f \circ g)(a_1) = f(g(a_1)) = f(g(a_2)) = (f \circ g)(a_2)$.
$\therefore f \circ g$ is not one-to-one. $\perp$ $\square$

Good luck!