State Estimation

EE468/CE468: Mobile Robotics

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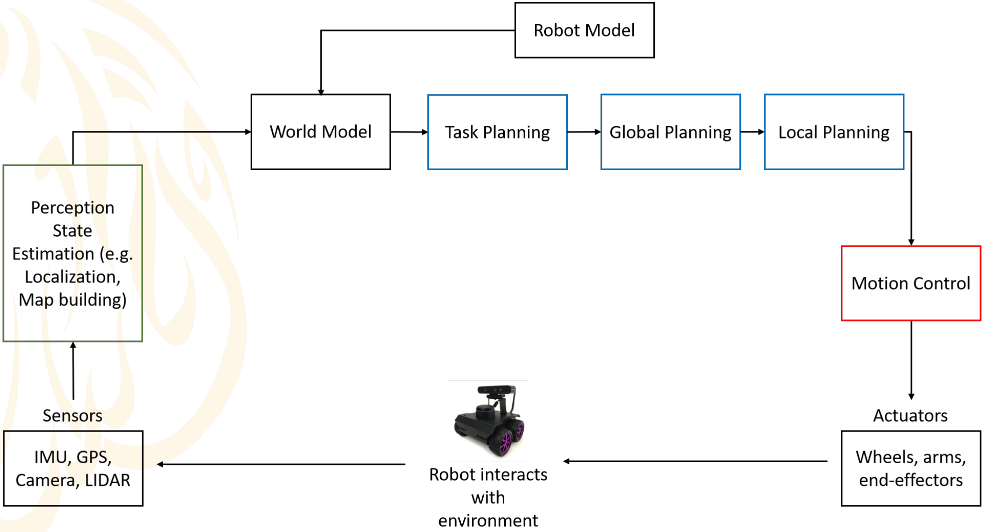
Navigation problem for mobile robots



Four fundamental blocks in navigation are:

Remember the pose for a planar robot is $\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$ with respect to a global or inertial frame.

- **Perception:** Extract meaningful information from sensors
- **Localization:** Where am I? A robot must determine its pose in its environment.
- **Cognition:** Decide a plan for achieving its goals.
- **Motion Control:** Modulate motor outputs to achieve desired trajectory.

Example 1: Self-driving car

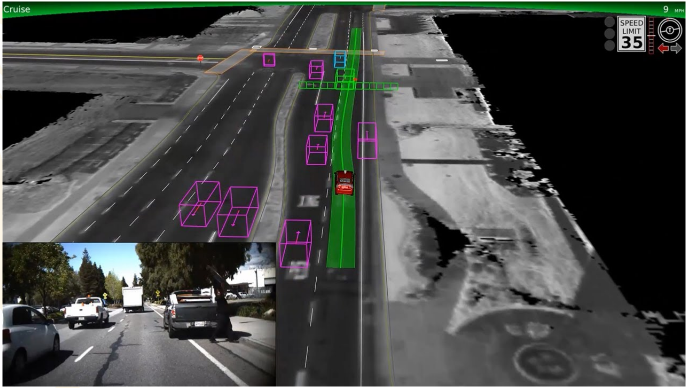


Figure: Source: https://blog.google/alphabet/the-latest-chapter-for-self-driving-car/

Sensors: GPS, LiDAR, Camera, IMU

World Model: Location, People, Cars, Signs

Example 2: GPS-denied forest harvesting

Figure: Source: https://youtu.be/1FLD0djPFgU

Sensors: LiDAR, IMU

World Model: Location, Trees

State Estimation is generalization of Localization.

- We may want to estimate more to add to the world model, e.g. velocities, location of other objects, etc.

Given data of measurements and actions $\longrightarrow$ Estimate $\longrightarrow$ State of the system

What is state of system?

- State of system, $(x(t))$, is set of variables sufficient to predict the future of the system (at least predict what we care about).

- Examples:
  - Pose of robot;
  - Configuration of manipulator;
  - Location of cars, people, position of car on the road;

What is measurement?

- Measurements of system, $(z(t))$, are sensor readings that provide information about the state at current time.
- It is seldom possible for state to be directly measurable.
- Examples:
  - Wheel encoder data;
  - Laser scan - relative positions of objects in environment;
  - Camera image - information about color, texture, relative positions.

What are actions?

- Actions on system, $(u(t))$, are the influence under which the state of the system evolves.
- Examples:
  - Actions taken by the robot - change velocity of wheels;
  - Disturbances - effects of weather and other agents;
  - Doing nothing

State estimation solves ...

- Fundamental problem: State is unknown, but all decision making depends on it.
- Robot can only see measurements and actions!
- Can you always estimate state?
- No! Observability of a system tells us whether the state can be determined or not.

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Our robot has wheel encoders. [2, 5.2.4]

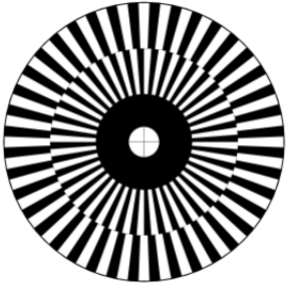

Figure: Quadrature Encoder Disk

- Assume that initial pose of the robot is known.
- Wheel encoders provide information about incremental wheel motion, i.e. Δs_r and Δs_l.
- If r is radius of wheel, n_0 is number of ticks of encoder per revolution, then

$$\Delta s_l = n_l \frac{2\pi r}{n_0}.$$

Use kinematic model to find change in pose. [2, 5.2.4]

- $$\Delta x = \Delta s \cos\left(\phi + \frac{\Delta\phi}{2}\right)$$
  $$\Delta y = \Delta s \sin\left(\phi + \frac{\Delta\phi}{2}\right),$$

  where

  $$\Delta\phi = \frac{\Delta s_r - \Delta s_l}{L}$$
  $$\Delta s = \frac{\Delta s_r + \Delta s_l}{2}$$

Let $\mathbf{x} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$. Then,

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{x}_k + \begin{bmatrix} \Delta s \cos\left(\phi + \frac{\Delta\phi}{2}\right) \\ \Delta s \sin\left(\phi + \frac{\Delta\phi}{2}\right) \\ \Delta\phi \end{bmatrix} \\
&= \mathbf{x}_k + \begin{bmatrix} \frac{\Delta s_r + \Delta s_l}{2} \cos\left(\phi + \frac{\Delta s_r - \Delta s_l}{2L}\right) \\ \frac{\Delta s_r + \Delta s_l}{2} \sin\left(\phi + \frac{\Delta s_r - \Delta s_l}{2L}\right) \\ \frac{\Delta s_r - \Delta s_l}{L} \end{bmatrix} \\
&= f(\mathbf{x}_k, \Delta s_l, \Delta s_r).
\end{aligned}$$

Is our pose estimate accurate? [2, 5.2]

- No! There are many sources of error.
- Error in our knowledge of initial state
- Sensor Errors (Resolution, Noise)
- Limited resolution during integration (time increments)
- Control signal errors (voltage discretization, communication lag)
- Unknown parameters (Inaccurate wheel radii, friction of carpet,)
- Incorrect physics (Misalignment of wheels, ignoring tire deformation, ignoring slippage, nonplanar surface)

What to do?

- True value of any error is unknown.
- By understanding different types of error, we can be better equipped to tolerate them.

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Uncertainty is modeled by adding an unknown ϵ.

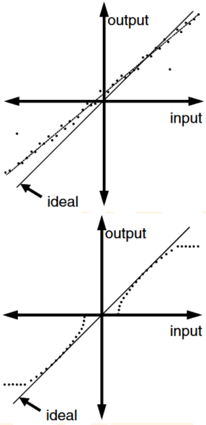


Figure: Types of error

- Measurements are generally modeled with an additive error model:

$$x_{meas} = x_{true} + \epsilon.$$

- ϵ could be deterministic or stochastic or combination of both.

- Types of error:
  - **Outliers:** Error is very far off from true value occasionally.
  - **Systematic Error:** Error follows a deterministic relationship.
  - **Random:** Error possesses a random distribution.

Removing Uncertainty

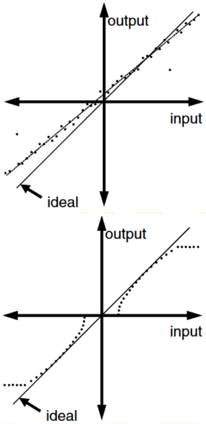


Figure: Types of error

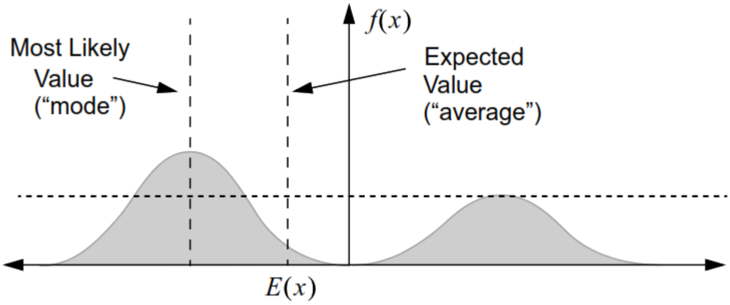
- Error in figure can be modeled as:

$$\epsilon = a + b\theta + N(\mu, \sigma).$$

  (a, b) are parameters of systematic error and $N(\mu, \sigma)$ represents a random Gaussian variable.

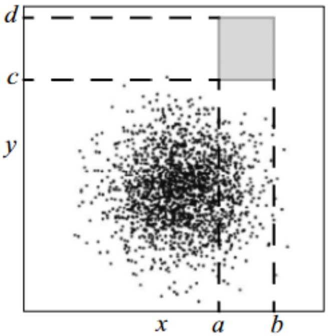
- Random error is unbiased (mean is zero), if systematic error is removed.
- Systematic error can be removed through calibration.
- Some random errors and outliers are removed by filtering. Otherwise, they are tolerated.

Probability Digression



Samples Representation of a 2D PDF

Probability Digression



Covariance [1, 5.1]

Probability Digression

- Let x be a vector of random variables and $\mu = E[x]$. Then, covariance is a matrix defined as:

$$\Sigma_{xx} = E\left[(x-\mu)(x-\mu)^T\right].$$

- Note that diagonal entries are the variances of different components of vector x.

- Covariance matrix provides information about spread of data or how uncertain are we about each component of the vector x.

Why use Gaussian to model random errors?

Probability Digression

- *The sum of a number of independent variables has a Gaussian distribution, regardless of their individual distributions, by Central Limit Theorem.*
- A Gaussian pdf is completely characterized by its expectation and variance. Higher moments are all zero.

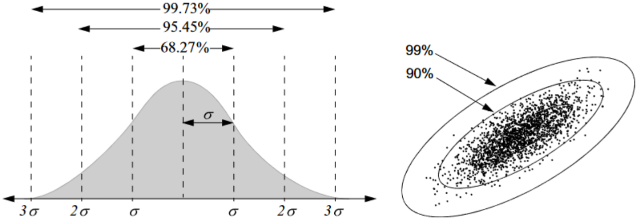


Figure: Contours of Constant Probability

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Model the error in encoder measurements as random.

- Let's model Δs_l and Δs_r, as random variables whose mean is the true value and they have finite variance.

$$\Delta s_l = \Delta s_l^{true} + \epsilon_l,$$

where $\epsilon_l \sim \mathcal{D}(0, \sigma^2)$.

- $E\left[\Delta s_l\right] =?$ and $Var\left[\Delta s_l\right] =?$

$$\begin{aligned} E\left[\Delta s_l\right] &= \Delta s_l^{true} \\ Var\left[\Delta s_l\right] &= Var[\epsilon_l] \end{aligned}$$

We usually know statistics about the errors.

- Assumed that two errors are independent of each other.
- Variance of error is proportional to absolute traveled distance. Let $u = (\Delta s_l, \Delta s_r)$. Then,

$$\Sigma_{uu} = \begin{bmatrix} k_l|\Delta s_l| & 0 \\ 0 & k_r|\Delta s_r| \end{bmatrix}.$$

- How does the uncertainty in Δs_l and Δs_r affect the pose x_k?

Underlying PDF for pose estimation is: $p(x_k | x_{k-1}, u_k)$

If $\mathbf{x} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$. Then,

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \begin{bmatrix} \frac{\Delta s_r + \Delta s_l}{2} \cos\left(\phi + \frac{\Delta s_r - \Delta s_l}{2L}\right) \\ \frac{\Delta s_r + \Delta s_l}{2} \sin\left(\phi + \frac{\Delta s_r - \Delta s_l}{2L}\right) \\ \frac{\Delta s_r - \Delta s_l}{L} \end{bmatrix}$$

$$= f(\mathbf{x}_k, \Delta s_l, \Delta s_r).$$

- We know uncertainty in $(\Delta s_l, \Delta s_r)$, can we find uncertainty in x, $\Sigma_{\mathbf{xx}} = ?$

Probability Digression

- Let x and y be vectors of random variables, such that $y = Fx$ and F is independent of x.

- The mean of y is:

$$E[y] = \mu_y = E[Fx] = F\,E[x].$$

- Covariance of y is:

$$\begin{aligned}\Sigma_{yy} &= E\left[(y-\mu_y)(y-\mu_y)^T\right] \\ &= E[yy^T] - E[y]\mu_y^T - \mu_y E[y^T] + \mu_y\mu_y^T \\ &= E[Fxx^TF^T] - F\mu_x\mu_x^TF^T \\ &= F\left(E[xx^T] - E[x]E[x^T]\right)F^T = F\Sigma_{xx}F^T\end{aligned}$$

Probability Digression

- What if $y = f(x)$?
- Linearly approximate y with first term of Taylor series about some reference value x', i.e.

$$y = f(x' + \epsilon) \approx f(x') + J\epsilon,$$

where $J = \frac{\partial f}{\partial x}\Big|_{x'}$ is Jacobian.

- Mean of y is:

$$\mu_y = E[f(x')] + J\,E[\epsilon]$$

Transformation of uncertainty [1, 5.1.3]

Probability Digression

- If $x' = \mu_x$ and ϵ is unbiased, i.e. $E[\epsilon] = 0$ then $\mu_y = f(\mu_x)$.
- You can show that

$$\Sigma_{yy} \approx J\,\Sigma_{xx}\,J^T.$$

- If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a partition of x, and x_1 and x_2 are uncorrelated, then

$$\Sigma_{yy} = J_1\,\Sigma_{x_1x_1}\,J_1^T + J_2\,\Sigma_{x_2x_2}\,J_2^T,$$

where $J_1 = \nabla_{x_1} f$ and $J_2 = \nabla_{x_2} f$.

How does uncertainty in $(\Delta s_l, \Delta s_r)$ impacts (x, y, ϕ)?

- Recall $\mathbf{x}_{k+1} = f(\mathbf{x}_k, u)$.
- Assume that covariance of $\mathbf{x}_k$ is known, say P_k.
- Reasonable to assume that $\mathbf{x}_k$ and measurement errors in $\Delta s_l, \Delta s_r$ are uncorrelated.
- Then,

$$P_{k+1} = J_1\, P_k\, J_1^T + J_2\, \Sigma_{uu}\, J_2^T$$

What are the Jacobians?

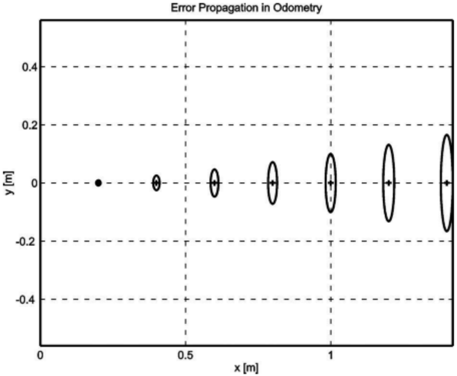
$$J_1 = \nabla_{\mathbf{x}_k} f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \phi} \end{bmatrix}$$

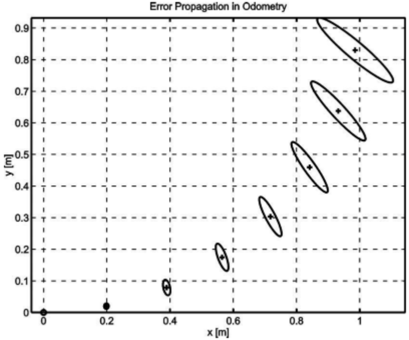
$$= \begin{bmatrix} 1 & 0 & -\Delta s \sin\left(\phi + \frac{\Delta\phi}{2}\right) \\ 0 & 1 & \Delta s \cos\left(\phi + \frac{\Delta\phi}{2}\right) \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_2 = \nabla_u f$$

$$= \begin{bmatrix} \frac{\Delta s}{2L} \sin\left(\phi + \frac{\Delta\phi}{2}\right) + \frac{1}{2} \cos\left(\phi + \frac{\Delta\phi}{2}\right) & -\frac{\Delta s}{2L} \sin\left(\phi + \frac{\Delta\phi}{2}\right) + \frac{1}{2} \cos\left(\phi + \frac{\Delta\phi}{2}\right) \\ -\frac{\Delta s}{2L} \cos\left(\phi + \frac{\Delta\phi}{2}\right) + \frac{1}{2} \sin\left(\phi + \frac{\Delta\phi}{2}\right) & \frac{\Delta s}{2L} \cos\left(\phi + \frac{\Delta\phi}{2}\right) + \frac{1}{2} \sin\left(\phi + \frac{\Delta\phi}{2}\right) \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

Uncertainty in our pose estimate grows unbounded.





Center indicates the ideal path and ellipses contain samples from distribution.

Small errors in orientation have a larger cumulative effect on position than longitudinal errors.

Evolution of state of robot system

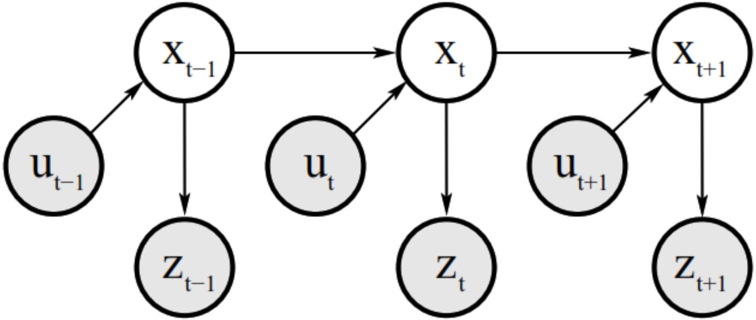


Figure: Relationship between state, action, measurements [3]

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Obtain the best estimate of a variable.

- A robot is measuring temperature of the ocean with two sensors.
- Both sensors take a reading at the same time.
- Two readings, x_1 and x_2, are noisy and $x_1 \neq x_2$.
- What is the best estimate of the true value of temperature, which is not changing?

What do we know?

- Suppose noise model for both sensors is known:

$$x_1 = x + \epsilon_1$$
$$x_2 = x + \epsilon_2$$

  where $\epsilon_i \sim \mathcal{D}(0, \sigma_i^2)$.

- Reasonable to assume that two errors are independent.

Idea: Take average of the two measurements.

- See MATLAB live script *averaging*.
- Is average the optimal estimate? What do we mean by optimal?
  - Estimate can also be viewed as a random variable
  - We want the mean of estimate distribution to be equal to true value of unknown. Unbiased estimator.
  - We want the variance to be small or the expected value of square of the error.

Idea: Take weighted average of the two measurements.

- Take a weighted average. Linear Estimator

$$\hat{x} = a_1 x_1 + a_2 x_2$$

- **Idea:** Choose weights that reduces the variance. Best Linear Unbiased Estimator

- Unbiased, i.e. $E[x - \hat{x}] = 0$

$$\begin{aligned} \Rightarrow x - a_1 x - a_2 x &= 0 \\ \Rightarrow a_1 + a_2 &= 1 \end{aligned}$$

$$
\begin{aligned}
E\left[(x-\hat{x})^2\right] &=? \\
&= E[x^2 - 2x\hat{x} + \hat{x}^2] \\
&= x^2 - 2(a_1 + a_2)x^2 + a_1^2 E[x_1^2] + a_2^2 E[x_2^2] + 2a_1 a_2 E[x_1 x_2] \\
&= -x^2 + a_1^2(Var[x_1] + x^2) + a_2^2(Var[x_2] + x^2) \\
&\quad + 2a_1 a_2 E[x_1]E[x_2] \\
&= a_1^2\, Var[x_1] + a_2^2\, Var[x_2] - x^2 + (a_1 + a_2)^2 x^2 \\
&= a_1^2\, Var[x_1] + a_2^2\, Var[x_2]
\end{aligned}
$$

as x_1 and x_2 are independent.

$$= (1-a)^2\, \sigma_1^2 + a^2\, \sigma_2^2$$

Choose a that minimizes the mean square error.

$$C = E\left[(x - \hat{x})^2\right] = (1-a)^2\,\sigma_1^2 + a^2\,\sigma_2^2$$

$$\frac{\partial C}{\partial a} = -2(1-a)\sigma_1^2 + 2a\sigma_2^2 = 0$$

$$\Rightarrow a = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

Consequently,

$$\hat{x} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\,x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\,x_2$$

Error variance can be computed as:

$$
\begin{aligned}
E\left[(x-\hat{x})^2\right] &= (1-a)^2\,\sigma_1^2 + a^2\,\sigma_2^2 \\
&= \frac{\sigma_2^4}{(\sigma_1^2+\sigma_2^2)^2}\sigma_1^2 + \frac{\sigma_1^4}{(\sigma_1^2+\sigma_2^2)^2}\sigma_2^2 \\
&= \frac{\sigma_1^2\,\sigma_2^2}{\sigma_1^2+\sigma_2^2}
\end{aligned}
$$

This can also be written as:

$$
E\left[(x-\hat{x})^2\right] = \sigma_1^2 - \frac{\sigma_1^4}{\sigma_1^2+\sigma_2^2}
$$

Recap

- x_1 and x_2 are noisy versions of x with variances σ_1^2 and σ_2^2 respectively.

- The best linear estimate ($\hat{x} = a_1 x_1 + a_2 x_2$) is:

$$\hat{x} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2$$

- Best is defined as $\hat{x}$ such that $E[\hat{x}] = E[x]$ and $E[(x - \hat{x})^2]$ is minimized.

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