Deviation of $\hat{Y}_i$ from the true value Y_i

That is the mean-square error value of prediction (orestimation) $\tilde{Y} = Y - \hat{Y}$.

We want to evaluate $\mathbf{E}\left((Y-\hat{Y})^2\right) = E(\tilde{Y}^2)$

$$E\left((Y-\hat{Y})^2\right) = E\left(Y - \bar{Y} + \bar{Y} - \hat{Y}\right)^2$$

$$= E\left(Y-\bar{Y}\right)^2 + E\left(\hat{Y}-\bar{Y}\right)^2$$

$$-2\underbrace{E\left(Y-\bar{Y}\right)\left(\hat{Y}-\bar{Y}\right)}_{\text{equal to zero b/c the two factors are independent of each other.}}$$

$$= E\left(Y-\bar{Y}\right)^2 + E\left(\hat{Y}-\bar{Y}\right)^2$$

$$= var(Y) + var(\hat{Y})$$

$$= \sigma^2 + var(\hat{\beta}_0 + \hat{\beta}_1 x)$$

$$= \sigma^2 + var(\hat{\beta}_0) + var(\hat{\beta}_1 x) + 2\,Cov(\hat{\beta}_0, x\hat{\beta}_1)$$

$$= \sigma^2 + \frac{\sigma^2}{n} + \frac{\sigma^2\bar{x}^2}{S_x^2} + \frac{x^2\sigma^2}{S_x^2} + 2x\left(\frac{-\sigma^2\bar{x}}{S_x^2}\right)$$

$$\boxed{\sigma_{\tilde{Y}}^2 = \sigma^2 + \frac{\sigma^2}{n} + \frac{\sigma^2}{S_x^2}\left(x-\bar{x}\right)^2}$$

$$MSE = \sigma^2\left(1+\frac{1}{n}+\frac{(x-\bar{x})^2}{S_x^2}\right) = \sigma_{\tilde{Y}}^2$$

for the given value of $X = X_0$.

$$\sigma_{\tilde{Y}} = \sigma\sqrt{1+\frac{1}{n}+\frac{(x-\bar{x})^2}{S_x^2}}$$

Under the assumption of normality of error term

$$Z = \frac{Y(X=X_0) - \hat{Y}(X=X_0)}{\sigma\sqrt{1+\frac{1}{n}+\frac{(X_0-\bar{X})^2}{S_x^2}}} \sim N(0,1)$$

If σ^2 is known (which is not possible in real life Scenarios)

then C.I. is obtained by Z-statistic.

If σ^2 is to be estimated, then we use the following unbiased formula:

$$S^2 = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$$

Then $(1-\alpha)100\%$ of C.I. on $E(Y-\hat{Y})^2$ is obtained as (for the given value of $X_i = X_0$)

$$\hat{Y}(X_0) \pm t_{\frac{\alpha}{2}, n-2} \cdot S \cdot \sqrt{1+\frac{1}{n}+\frac{(X_0-\bar{X})^2}{S_x^2}}$$

where $S_x^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.