Statistics & Inferencing Activity #03

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Q) $f(x|a) = a x^{a-1}$ for $0 < x < 1$, $a > 0$

(i) $E[x] = \int_0^1 x(a x^{a-1}) \cdot dx \Rightarrow E[x] = \int_0^1 a x^a \cdot dx$

$= a \int x^a \cdot dx = a \left| \frac{x^{a+1}}{a+1} \right|_0^1 = a\left[\frac{1}{a+1} - 0\right]$

$\Rightarrow E[x] = a/_{a+1}$

$\Rightarrow E[x] + aE[x] = a$

$\Rightarrow a(E[x] - 1) = -E[x]$

$\Rightarrow a = \frac{-E[x]}{E[x]-1} \Rightarrow a = \frac{E[x]}{1-E[x]}$

Then $E[x] = T(x) = \frac{1}{n}\sum_{i=1} x$

$\hat{a}_{mom} = \frac{T(x)}{1-T(x)}$ $\quad [T(x) = \hat{\mu}]$

Then $\boxed{\hat{a}_{mom} = \frac{\hat{\mu}}{1-\hat{\mu}}}$

(ii) $E[T(x)] = E[x] = a/_{a+1}$

$Var(T(x)) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} x_i\right) = \frac{Var(x)}{n}$

$Var(x) = E[x^2] - (E[x])^2$

$E[x^2] = \int_0^1 x^2(a x^{a-1}) \cdot dx = \int_0^1 a x^{a+1} \cdot dx$

$E[x^2] = a\int x^{a+1} \cdot dx = a \left| \frac{x^{a+2}}{a+2} \right|_0^1 = a\left[\frac{1}{a+2} - 0\right]$

$\Rightarrow E[x^2] = a/_{a+2}$

$Var(x) = \frac{a}{a+2} - a^2/_{(a+1)^2} = \frac{a(a+1)^2 - a^2(a+2)}{(a+2)(a+1)^2} = \frac{a}{(a+1)^2(a+2)}$

$Var(T(x)) = \frac{a}{n(a+1)^2(a+2)}$ $\quad \hat{a}_{mom} = \frac{\hat{\mu}}{1-\hat{\mu}}$

$Var(\hat{a}_{mom}) = Var\left(\frac{T(x)}{1-T(x)}\right)$

Let $f(T) = T(x)/_{1-T(x)}$ $\quad$ Then $Var(\hat{a}_{mom}) = Var(f(T))$

$f'(T) = \frac{T}{(T-1)^2} - \frac{1}{T-1}$

Then $f'(\mu_T) = \frac{\mu_T}{(\mu_T - 1)^2} - \frac{1}{\mu_T - 1}$ $\quad [\mu_T = a/_{(a-1)}]$

Then $f'(\mu_T) = (a+1)^2 \quad a$

$\therefore Var(\hat{a}_{mom}) = \frac{a}{n(a+1)^2(a+2)} \times \left[(a+1)^2\right]^2 \Rightarrow \boxed{Var(\hat{a}_{mom}) = \frac{a(a+1)^2}{n(a+2)}}$

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$$f'(T) = \frac{T}{(T-1)^2} - \frac{1}{T-1} \Rightarrow f''(T) = \frac{-2}{(T-1)^3}$$

Then $$f''(\mu_T) = \frac{-2}{(\mu_T - 1)^3} \Rightarrow f''(\mu_T) = 2(a+1)^3$$

$$E[\hat{a}_{mom}] \approx \frac{\hat{\mu}}{1-\hat{\mu}} + \frac{1}{\cancel{2}} \frac{a}{n\cancel{(a+1)^2}(a+2)} \times \cancel{2}(a+1)^{\cancel{3}\,1}$$

$$\approx \frac{(a/a+1)}{1-(a/(a+1))} + \frac{a(a+1)}{n(a+2)}$$

$$\boxed{E[\hat{a}_{mom}] \approx a + \frac{a(a+1)}{n(a+2)}}$$

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