Linear Transformation

Question 01:

Let us consider the basis $\mathbf{S} = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ for $\mathbb{R}^3$, where
$\mathbf{v_1} = (1,1,1), \mathbf{v_2} = (1,1,0) \quad \text{and} \quad \mathbf{v_3} = (1,0,0)$.
Let $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation such that
$\mathbf{T}(\mathbf{v_1}) = (1,0), \mathbf{T}(\mathbf{v_2}) = (2,-1), \mathbf{T}(\mathbf{v_3}) = (4,3)$.
Let $(\mathbf{x},\mathbf{y},\mathbf{z})$ be a vector in $\mathbb{R}^3$, Since the set $\mathbf{S}$ is basis for $\mathbb{R}^3$, then there exists scalars $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3} \in \mathbb{R}$ such that

$$
\begin{aligned}
& c_1v_1 + c_2v_2 + c_3v_3 = (x,y,z) \\
& \Rightarrow c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0) = (x,y,z) \\
& \Rightarrow (c_1,c_1,c_1) + (c_2,c_2,0) + (c_3,0,0) = (x,y,z) \\
& \Rightarrow c_1 + c_2 + c_3 = x \\
& c_1 + c_2 = y \\
& c_1 = z \Rightarrow c_2 = y - c_1 = y - z \quad \text{and} \quad c_3 = x - (c_1 + c_2) = x - y.
\end{aligned}
$$

Therefore $(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{z}(1,1,1) + (\mathbf{y}-\mathbf{z})(1,1,0) + (\mathbf{x}-\mathbf{y})(1,0,0)$.
Therefore the linear transformation is given by

$$
\begin{aligned}
T(x,y,z) &= T\{z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)\} \\
&= T\{z(1,1,1)\} + T\{(y-z)(1,1,0)\} + T\{(x-y)(1,0,0)\}\{\because T \text{ is linear mapping}\} \\
&= zT(1,1,1) + (y-z)T(1,1,0) + (x-y)T(1,0,0)\{\because T \text{ is linear mapping}\} \\
&= zT(v_1) + (y-z)T(v_2) + (x-y)T(v_3) \\
&= z(1,0) + (y-z)(2,-1) + (x-y)(4,3) \\
&= (z,0) + (2(y-z), -(y-z)) + (4(x-y), 3(x-y)) \\
&= (z,0) + (2y-2z, z-y) + (4x-4y, 3x-3y) \\
&= (z+2y-2z+4x-4y, 0+z-y+3x-3y) \\
&= (4x-2y-z, 3x-4y+z)
\end{aligned}
$$

The linear transformation is given by $\mathbf{T}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = (4\mathbf{x_1} - 2\mathbf{x_2} - \mathbf{x_3}, 3\mathbf{x_1} - 4\mathbf{x_2} + \mathbf{x_3})$.

We know that when $\mathbf{T}$ is a linear transformation then $\mathbf{T}(\mathbf{x}\mathbf{v_1} + \mathbf{y}\mathbf{v_2}) = \mathbf{x}\mathbf{T}(\mathbf{v_1}) + \mathbf{y}\mathbf{T}(\mathbf{v_2})$, where $\mathbf{x},\mathbf{y} \in \mathbb{R} \quad \text{and} \quad \mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^3$.

From the above problem we get $\mathbf{T}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = (4\mathbf{x_1} - 2\mathbf{x_2} - \mathbf{x_3}, 3\mathbf{x_1} - 4\mathbf{x_2} + \mathbf{x_3}), (\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) \in \mathbb{R}^3$.
Then from the above formula we get

$$
\begin{aligned}
T(2,-3,5) &= (\{4 \times 2 - 2 \times (-3) - 5\}, \{3 \times 2 - 4 \times (-3) + 5\}) \\
&= (\{8 + 6 - 5\}, \{6 + 12 + 5\}) \\
&= (9, 23).
\end{aligned}
$$

The final answers are given by
* The linear transformation is given by $\mathbf{T}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}) = (4\mathbf{x_1} - 2\mathbf{x_2} - \mathbf{x_3}, 3\mathbf{x_1} - 4\mathbf{x_2} + \mathbf{x_3})$.
* $\mathbf{T}(2,-3,5) = (9,23)$.

Question ~~02:~~

Note that a map $\mathbf{f}:\mathbf{A}(\neq\phi)\rightarrow\mathbf{B}(\neq\phi)$ is linear $\Leftrightarrow$
$\mathbf{f}(\alpha\mathbf{a}+\mathbf{b})=\alpha\mathbf{f}(\mathbf{a})+\mathbf{f}(\mathbf{b})\ \forall\ \mathbf{a},\mathbf{b}\in\mathbf{A}$ and for any scalar
$\alpha\in\mathbf{F}$ where $\mathbf{F}$ is a field.

Here I suppose the field $\mathbb{R}$.

a) This is linear. Becuse for any $\mathbf{X}=(\mathbf{x},\mathbf{y}),\mathbf{Y}=(\mathbf{r},\mathbf{z})\quad\mathbf{and}\quad\mathbf{any\ a}\in\mathbb{R}$,we have
$\mathbf{F}(\mathbf{aX}+\mathbf{Y})=\mathbf{F}(\mathbf{ax}+\mathbf{r},\mathbf{y}+\mathbf{z})=(2(\mathbf{ax}+\mathbf{r}),\mathbf{y}+\mathbf{z})$
$\Rightarrow\mathbf{F}(\mathbf{aX}+\mathbf{Y})=(2(\mathbf{ax}),\mathbf{y})+(2\mathbf{r},\mathbf{z})=\mathbf{a}(2\mathbf{x},\mathbf{y})+(2\mathbf{r},\mathbf{z})$
$\Rightarrow\mathbf{F}(\mathbf{aX}+\mathbf{Y})=\mathbf{aF}(\mathbf{X})+\mathbf{F}(\mathbf{Y})$
$\Rightarrow\mathbf{F}$ is linear.

b) This is not linear. For example,
$\mathbf{F}((1,1)+(1,0))=\mathbf{F}(2,1)=(4,1)$ but
$\mathbf{F}(1,1)+\mathbf{F}(1,0)=(1,1)+(1,0)=(2,1)\neq(4,1)$.

c) This is linear. Because for any $\alpha\in\mathbb{R}$,
and for any $\mathbf{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\quad\mathbf{and}\quad\mathbf{B}=\begin{bmatrix}e & f\\ g & h\end{bmatrix}$, we have

$$\mathbf{F}(\alpha\mathbf{A}+\mathbf{B})=\mathbf{F}\left(\begin{bmatrix}\alpha a+e & \alpha b+f\\ \alpha c+g & \alpha d+h\end{bmatrix}\right)$$
$$=(\alpha\mathbf{b}+\mathbf{f})+(\alpha\mathbf{c}+\mathbf{g})$$
$$=\alpha(\mathbf{b}+\mathbf{c})+(\mathbf{f}+\mathbf{g})$$
$$=\alpha\mathbf{F}\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)+\mathbf{F}\left(\begin{bmatrix}e & f\\ g & h\end{bmatrix}\right)=\alpha\mathbf{F}(\mathbf{A})+\mathbf{F}(\mathbf{B})$$

$\Rightarrow\mathbf{F}$ is linear.

d) This is not linear becuause under linear map, image of zero element is zero element but here
$\cancel{\mathbf{F}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\3\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\mathbf{F}\text{ is not linear.}}$ It is linear, zero maps to zero.

e) This is linear. Because for any $\mathbf{a}\in\mathbb{R}$,and any matrices $\mathbf{C},\mathbf{D}$, we have
$\mathbf{T}(\mathbf{aC}+\mathbf{D})=(\mathbf{aC}+\mathbf{D})\mathbf{B}=(\mathbf{aC})\mathbf{B}+\mathbf{DB}$
$\Rightarrow\mathbf{T}(\mathbf{aC}+\mathbf{D})=\mathbf{a}(\mathbf{CB})+\mathbf{DB}=\mathbf{aT}(\mathbf{C})+\mathbf{T}(\mathbf{D})$
$\Rightarrow\mathbf{T}$ is linear.

I proved that the maps given in a) , c) and e) are LINEAR and the maps given in b) and d) are NOT LINEAR.

Question 03:

Q.3.(a) Let $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that
$\mathrm{T}(e_1) = (1,1), \mathrm{T}(e_2) = (3,0) \quad \text{and} \quad \mathrm{T}(e_3) = (4,-7)$, where
$\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the standard basis for $\mathbb{R}^3$.
Let $(1,3,8) \in \mathbb{R}^3$, then
$(1,3,8) = (1,0,0) + (0,3,0) + (0,0,8) = (1,0,0) + 3(0,1,0) + 8(0,0,1) = e_1 + 3e_2 + 8e_3.$

$$\begin{aligned}
\therefore \mathrm{T}(1,3,8) &= \mathrm{T}(e_1 + 3e_2 + 8e_3) \\
&= \mathrm{T}(e_1) + \mathrm{T}(3e_2) + \mathrm{T}(8e_3)\{\because \mathrm{T} \text{ is a linear transformation}\} \\
&= \mathrm{T}(e_1) + 3\mathrm{T}(e_2) + 8\mathrm{T}(e_3)\{\because \mathrm{T} \text{ is a linear transformation}\} \\
&= (1,1) + 3(3,0) + 8(4,-7) \\
&= (1,1) + (9,0) + (32,-56) \\
&= (1+9+32, 1+0-56) \\
&= (42,-55)
\end{aligned}$$

(b). Let (x,y,z) be a vector in $\mathbb{R}^3$, Since $\{e_1,e_2,e_3\}$ be a standard basis for $\mathbb{R}^3$, then there exists scalars c_1, c_2, c_3 such that

$$\begin{aligned}
(x,y,z) &= c_1e_1 + c_2e_2 + c_3e_3 \\
&= (c_1,0,0) + (0,c_2,0) + (0,0,c_3) \\
&= (c_1, c_2. c_3) \\
\Rightarrow c_1 &= x \\
c_2 &= y \\
c_3 &= z
\end{aligned}$$

Therefore the linear transformation is given by

$$\begin{aligned}
\mathrm{T}(x,y,z) &= \mathrm{T}(xe_1 + ye_2 + ze_3) \\
&= \mathrm{T}(xe_1) + \mathrm{T}(ye_2) + \mathrm{T}(ze_3)\{\because \mathrm{T} \text{ is a linear mapping}\} \\
&= x\mathrm{T}(e_1) + y\mathrm{T}(e_2) + z\mathrm{T}(e_3)\{\because \mathrm{T} \text{ is a linear mapping}\} \\
&= x(1,1) + y(3,0) + z(4,-7) \\
&= (x,x) + (3y,0) + (4z,-7z) \\
&= (x+3y+4z, x-7z)
\end{aligned}$$

Since $\mathbb{R}^2$ be a vector space , then
$c(x,y) = (cx,cy) \quad \text{and} \quad (x,y) + (z,w) = (x+z, y+w) \forall (x,y),(z,w) \in \mathbb{R}^2, c \in \mathbb{R}.$

(c). The linear transformation is given by $\mathrm{T}(x,y,z) = (x+3y+4z, x-7z)$.
Let $\{e_1,e_2,e_3\}$ be the standard basis for $\mathbb{R}^3$,and $\{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$.
Now

$$
\begin{aligned}
\mathbf{T}(\mathbf{e}_1) &= \mathbf{T}(1,0,0) \\
&= (1\times 1 + 3\times 0 + 4\times 0, 1 - 7\times 0) \\
&= (1,1) = 1(1,0) + 1(0,1) \\
\mathbf{T}(\mathbf{e}_2) &= \mathbf{T}(0,1,0) \\
&= (1\times 0 + 3\times 1 + 4\times 0, 0 - 7\times 0) \\
&= (0+3+0, 0-0) \\
&= (3,0) = 3(1,0) + 0(0,1) \\
\text{and}\quad \mathbf{T}(\mathbf{e}_3) &= \mathbf{T}(0,0,1) \\
&= (1\times 0 + 3\times 0 + 4\times 1, 0 - 7\times 1) \\
&= (0+0+4, 0-7) \\
&= (4,-7) = 4(1,0) - 7(0,1)
\end{aligned}
$$

The matrix of the linear transformation is $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{bmatrix}$.

The final answers are given by

(a). $\mathbf{T}(1,3,8) = (42,-55).$

(b). $\mathbf{T}(x,y,z) = (x + 3y + 4z, x - 7z).$

(c). The matrix of the linear transformation is $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 0 & -7 \end{bmatrix}$.

Question 06:

Part(a)

DEFINITION 11.7. *The* ***kernel*** *of a linear transformation* $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ *is the set of all* $\mathrm{x} \in \mathbb{R}^n$ *such that* $\mathbf{Tx} = \mathbf{0} \in \mathbb{R}^m$.

$$\ker(T) = \{\mathrm{x} \in \mathbb{R}^m \mid T(\mathrm{x}) = \mathbf{0}\}$$

LEMMA 11.8. *The kernel of a linear transformation* $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ *is a subspace of* $\mathbb{R}^m$.

Proof. $\ker(T)$ is obviously a subset of $\mathbb{R}^m$. We need to show that it's closed under scalar multiplication and vector addition. Let $\lambda \in \mathbb{R}$ and $\mathrm{x} \in \ker(T)$ be arbitary elements of their respective sets. Then $T(\lambda\mathrm{x}) = \lambda T(\mathrm{x})$, since T is a linear transformation. But $T(\mathrm{x}) = 0$ since $\mathrm{x} \in \ker(T)$. So $T(\lambda\mathrm{x}) = \mathbf{0}$. We conclude that if $\lambda \in \mathbb{R}$, and $\mathrm{x} \in \ker(T)$, then $\lambda\mathrm{x} \in \ker(T)$, and so $\ker(T)$ is closed under scalar multiplication.

Now let $\mathrm{x}_1, \mathrm{x}_2$ be arbitrary vectors in $\ker(T)$. Then since T is a linear transformation, $T(\mathrm{x}_1 + \mathrm{x}_2) = T(\mathrm{x}_1) + T(\mathrm{x}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and so $\mathrm{x}_1 + \mathrm{x}_2 \in \ker(T)$. Thus, $\ker(T)$ is closed under vector addition.

Since $\ker(T)$ is a subset of $\mathbb{R}^m$ that is closed under both scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^m$. □

DEFINITION 11.9. *The* ***image*** *or* ***range*** *of* $\mathbf{T}$ *is the set of all* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y} = \mathbf{T}(\mathrm{x})$ *for some* $\mathrm{x} \in \mathbb{R}^n$.

$$range(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathrm{x}) \quad \textit{for some } \mathrm{x} \in \mathbb{R}^m\}$$

LEMMA 11.10. *The range of a linear transformation* $T : \mathbb{R}^m \to \mathbb{R}^n$ *is a subspace of* $\mathbb{R}^n$.

Proof. We need to show that $range(T)$ is closed under both scalar multiplication and vector addition.

Suppose $\mathbf{y} \in range(T)$. Then there must be an $\mathrm{x} \in \mathbb{R}^m$ such that $\mathbf{y} = T(\mathrm{x})$. But then $\lambda\mathrm{x} \in \mathbb{R}^m$ and

$$T(\lambda\mathrm{x}) = \lambda T(\mathrm{x}) = \lambda\mathbf{y}$$

and so $\lambda\mathbf{y}$ is in $range(T)$. Hence, $range(T)$ is closed under scalar multiplication.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in range(T)$. Then there must be vectors $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{R}^m$ such that $\mathbf{y}_1 = T(\mathrm{x}_1)$ and $\mathbf{y}_2 = T(\mathrm{x}_2)$. Now apply T to the vector sum $\mathrm{x}_1 + \mathrm{x}_2$:

$$T(\mathrm{x}_1 + \mathrm{x}_2) = T(\mathrm{x}_1) + T(\mathrm{x}_2) = \mathbf{y}_1 + \mathbf{y}_2$$

This displays $\mathbf{y}_1 + \mathbf{y}_2$ as an element of $range(T)$.

Since $range(T) \subset \mathbb{R}^n$ is closed under both scalar multiplication and vector addition, it is a subspace of $\mathbb{R}^n$. □

Now let $\mathbf{A}$ be the $m \times n$ matrix corresponding to a linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$. Then

$$\begin{aligned} \ker(\mathbf{T}) &= \{\mathrm{x} \in \mathbb{R}^n \mid \mathbf{T}(\mathrm{x}) = \mathbf{0}\} \\ &= \{\mathrm{x} \in \mathbb{R}^n \mid \mathbf{A}_T\mathrm{x} = \mathbf{0}\} = \text{ Null space of } \mathbf{A}_T \end{aligned}$$

$$\begin{aligned} range(\mathbf{T}) &= \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{T}(\mathrm{x}) \quad , \quad \text{for some } \mathrm{x} \in \mathbb{R}^n\} \\ &= \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = \mathbf{A}_T\mathrm{x} \quad , \quad \text{for some } \mathrm{x} \in \mathbb{R}^n\} = \text{ column space of } \mathbf{A}_T \end{aligned}$$

Part(b)

DEFINITION

If $T: V \longrightarrow W$ is a linear transformation, then the dimension of the range of T is called the ***rank of T*** and is denoted by $\text{rank}(T)$; the dimension of the kernel is called the ***nullity of T*** and is denoted by $\text{nullity}(T)$.

If A is an $m \times n$ matrix and $T_A: R^n \longrightarrow R^m$ is multiplication by A, then we know from Example 1 that the kernel of T_A is the nullspace of A and the range of T_A is the column space of A. Thus we have the following relationship between the rank and nullity of a matrix and the rank and nullity of the corresponding matrix transformation.

Part(c)

If A is the matrix representaiton of a linear transformation T, then

1. $\mathcal{N}(T) = \mathcal{N}(A)$ and $\mathcal{R}(T) = \mathcal{R}(A)$.
2. The nullity of T is the same as the nullity of A.
3. The rank of T is the same as the rank of A.

Part(d)

for finding out the kernel we have to do
$\mathrm{T}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = (0,0,0)$
so $\mathrm{x}_1 + 2\mathrm{x}_3 + \mathrm{x}_2 = 0, \mathrm{x}_1 + \mathrm{x}_3 = 0, 2\mathrm{x}_1 + \mathrm{x}_2 + 3\mathrm{x}_3 = 0$
so $\mathrm{x}_1 = -\mathrm{x}_3$
implies also we get , $\mathrm{x}_2 + \mathrm{x}_3 = 0$
so $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = (-\mathrm{x}_3, -\mathrm{x}_3, \mathrm{x}_3)$
so dimension of kernel is 1 and bases is {(-1,-1,1)}
and for range we know we have to collect linearly independent coulmns

here columns are $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

so linearly independent columns is first and secound column

so basis for range is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right.$

dimesion of range space is 2

Question 08:

Let A denote an $m \times n$ matrix of rank r and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ denote the corresponding matrix transformation given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. It follows from Example 7.2.1 and Example 7.2.2 that $\text{im } T_A = \text{col } A$, so $\dim(\text{im } T_A) = \dim(\text{col } A) = r$. On the other hand Theorem 5.4.2 shows that $\dim(\ker T_A) = \dim(\text{null } A) = n - r$. Combining these we see that

$$\dim(\text{im } T_A) + \dim(\ker T_A) = n \quad \text{for every } m \times n \text{ matrix } A$$

The main result of this section is a deep generalization of this observation.

Theorem 7.2.4: Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that $\ker T$ and $\text{im } T$ are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\text{im } T)$$

In other words, $\dim V = \text{nullity}(T) + \text{rank}(T)$.

Proof. Every vector in $\text{im } T = T(V)$ has the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V. Hence let $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ be a basis of $\text{im } T$, where the $\mathbf{e}_i$ lie in V. Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ be any basis of $\ker T$. Then $\dim(\text{im } T) = r$ and $\dim(\ker T) = k$, so it suffices to show that $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{f}_1, \dots, \mathbf{f}_k\}$ is a basis of V.

1. *B spans V.* If $\mathbf{v}$ lies in V, then $T(\mathbf{v})$ lies in $\text{im } V$, so

$$T(\mathbf{v}) = t_1T(\mathbf{e}_1) + t_2T(\mathbf{e}_2) + \cdots + t_rT(\mathbf{e}_r) \quad t_i \text{ in } \mathbb{R}$$

This implies that $\mathbf{v} - t_1\mathbf{e}_1 - t_2\mathbf{e}_2 - \cdots - t_r\mathbf{e}_r$ lies in $\ker T$ and so is a linear combination of $\mathbf{f}_1, \dots, \mathbf{f}_k$. Hence $\mathbf{v}$ is a linear combination of the vectors in B.

2. B is *linearly independent*. Suppose that t_i and s_j in $\mathbb{R}$ satisfy

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r + s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0} \tag{7.1}$$

Applying T gives $t_1T(\mathbf{e}_1) + \cdots + t_rT(\mathbf{e}_r) = \mathbf{0}$ (because $T(\mathbf{f}_i) = \mathbf{0}$ for each i). Hence the independence of $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ yields $t_1 = \cdots = t_r = 0$. But then (7.1) becomes

$$s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0}$$

so $s_1 = \cdots = s_k = 0$ by the independence of $\{\mathbf{f}_1, \dots, \mathbf{f}_k\}$. This proves that B is linearly independent.

□

Note that the vector space V is not assumed to be finite dimensional in Theorem 7.2.4. In fact, verifying that $\ker T$ and $\text{im } T$ are both finite dimensional is often an important way to *prove* that V is finite dimensional.

Note further that $r + k = n$ in the proof so, after relabelling, we end up with a basis

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$$

of V with the property that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$ and $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\text{im } T$. In fact, if V is known in advance to be finite dimensional, then *any* basis $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of $\ker T$ can be extended to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of V by Theorem 6.4.1. Moreover, it turns out that, no matter how this is done, the vectors $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ will be a basis of $\text{im } T$. This result is useful, and we record it for reference. The proof is much like that of Theorem 7.2.4 and is left as Exercise 7.2.26.