Deep Neural Networks

Abdul Samad

Adopted from Prof. Simon Prince

Deep neural networks

- Networks with more than one hidden layer
- Intuition becomes more difficult

Deep neural networks

- Composing two networks
- Combining the two networks into one
- Hyperparameters
- Notation change and general case
- Shallow vs. deep networks

Composing two networks.

Network 1:

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

Network 2:

$$h'_1 = \mathrm{a}[\theta'_{10} + \theta'_{11}y]$$
$$h'_2 = \mathrm{a}[\theta'_{20} + \theta'_{21}y]$$
$$h'_3 = \mathrm{a}[\theta'_{30} + \theta'_{31}y]$$

$$y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$$

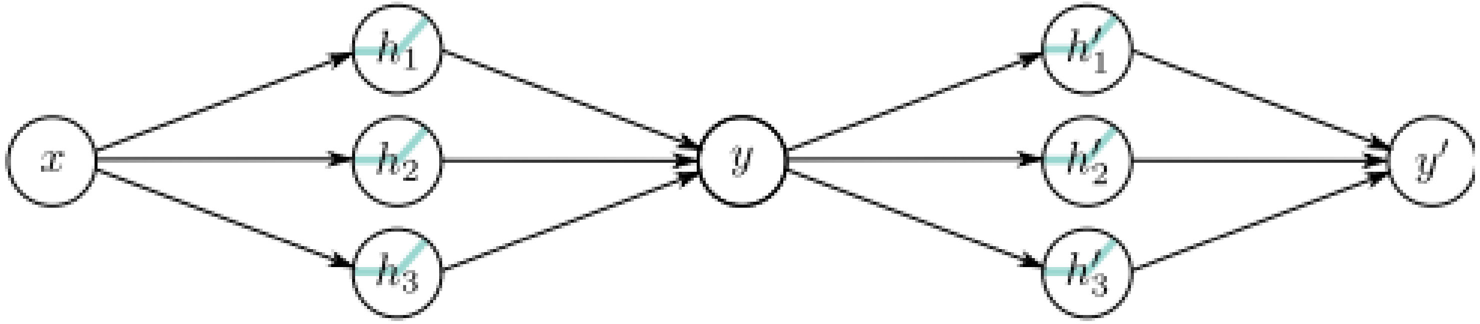
Composing two networks.

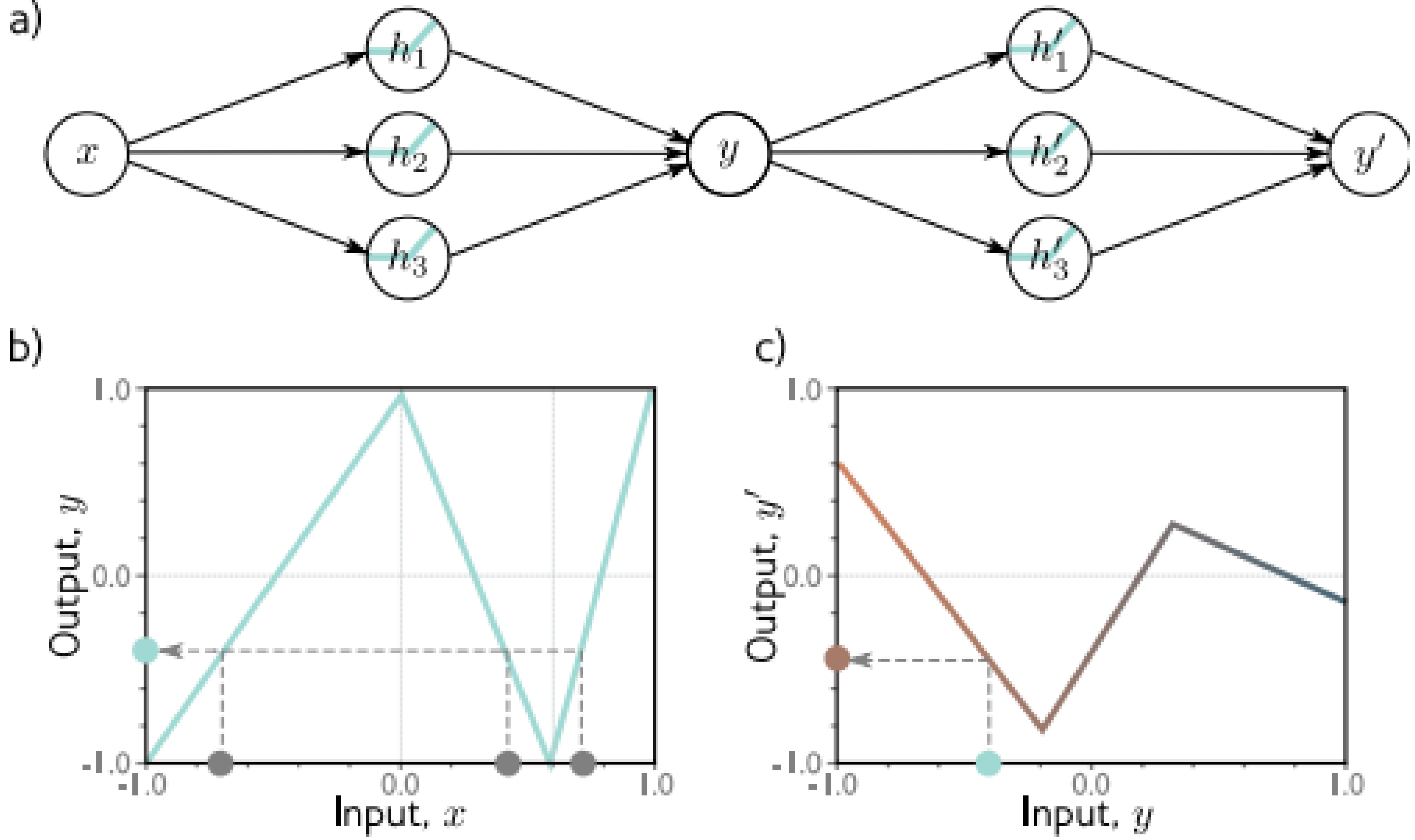
Network 1:

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x] \qquad y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

Network 2:

$$h'_1 = \mathrm{a}[\theta'_{10} + \theta'_{11}y]$$
$$h'_2 = \mathrm{a}[\theta'_{20} + \theta'_{21}y] \qquad y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$$
$$h'_3 = \mathrm{a}[\theta'_{30} + \theta'_{31}y]$$





a)
x
h_1
h_2
h_3
y
h'_1
h'_2
h'_3
y'
b)
1.0
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1.0
Output, y
Input, x
c)
1.0
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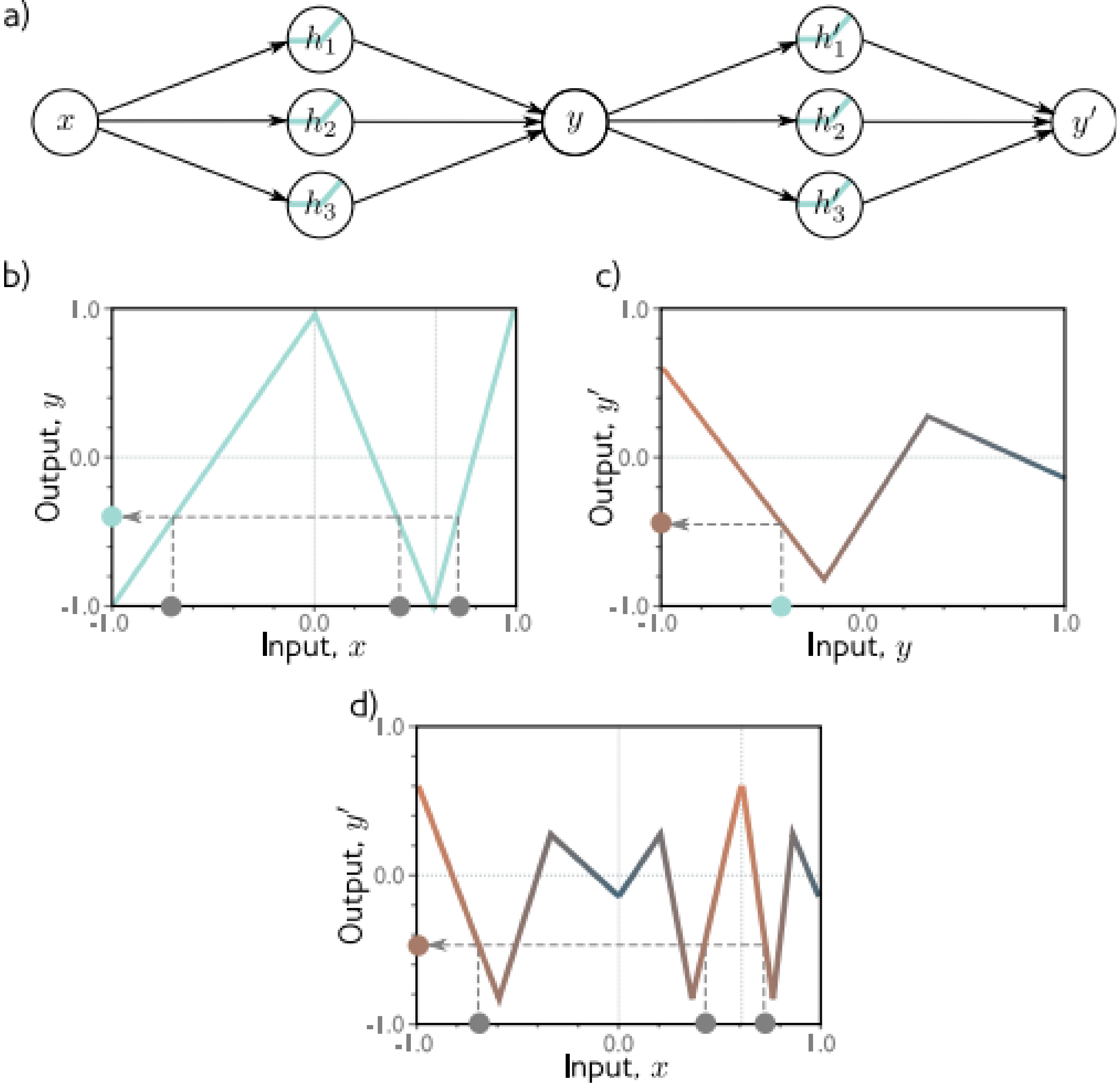
Output, y'
Input, y



a)
x
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h'_1
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b)
Output, y
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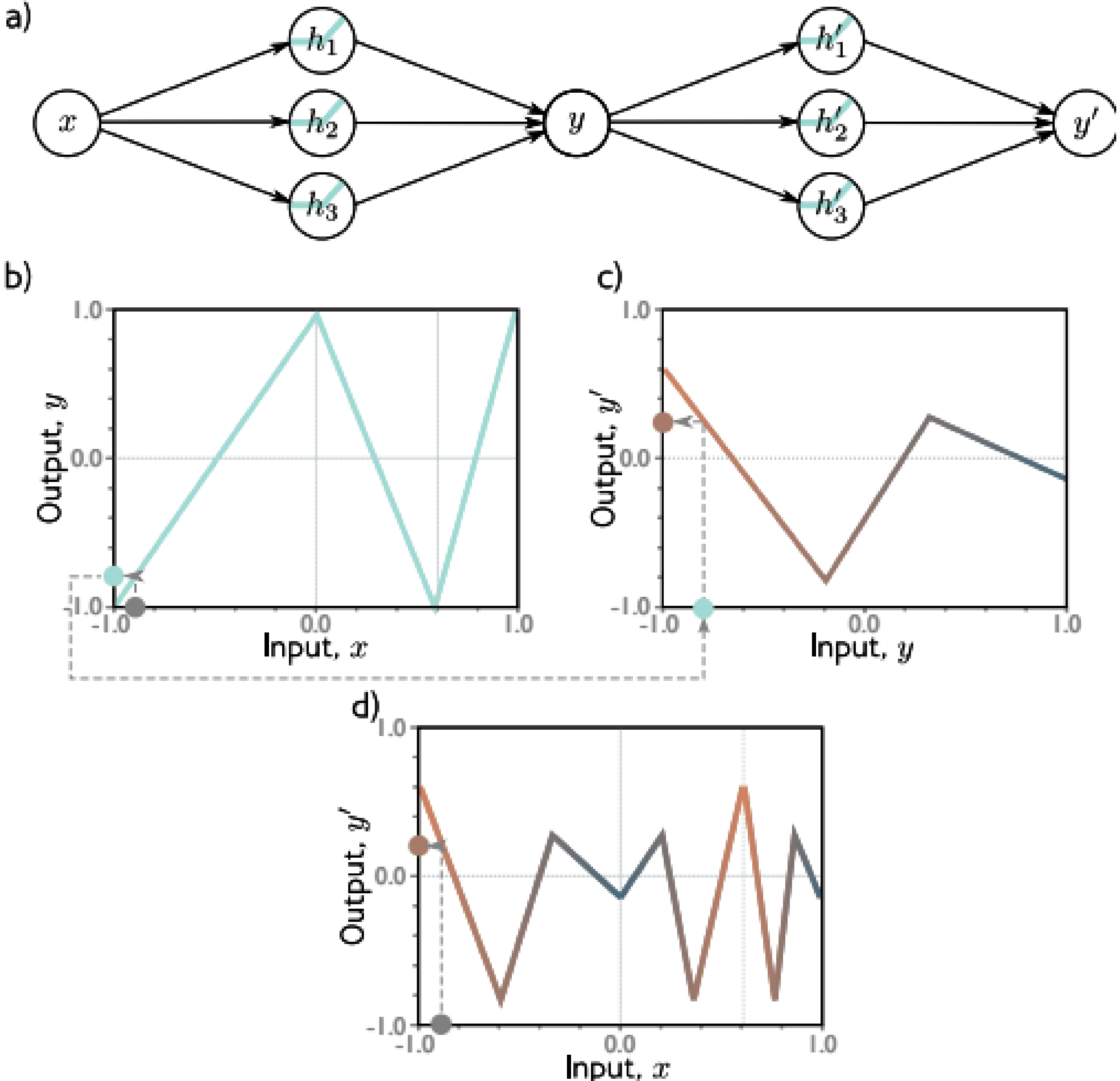
c)
Output, y'
Input, y
d)
Output, y'
Input, x



a)
x
h_1
h_2
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y
h'_1
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y'
b)
Output, y
Input, x
c)
Output, y'
Input, y
d)
Output, y'
Input, x
1.0
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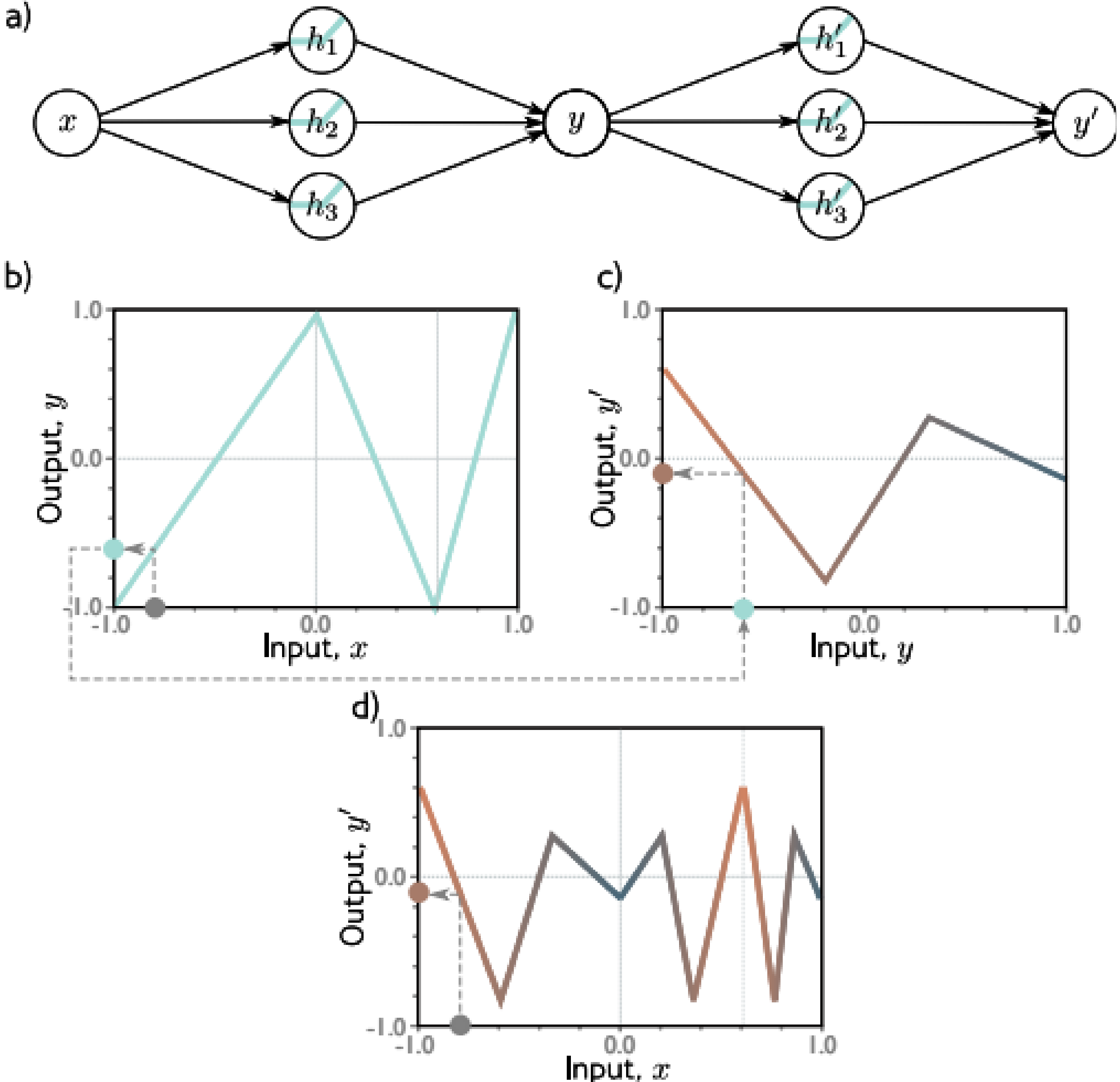

a)
b)
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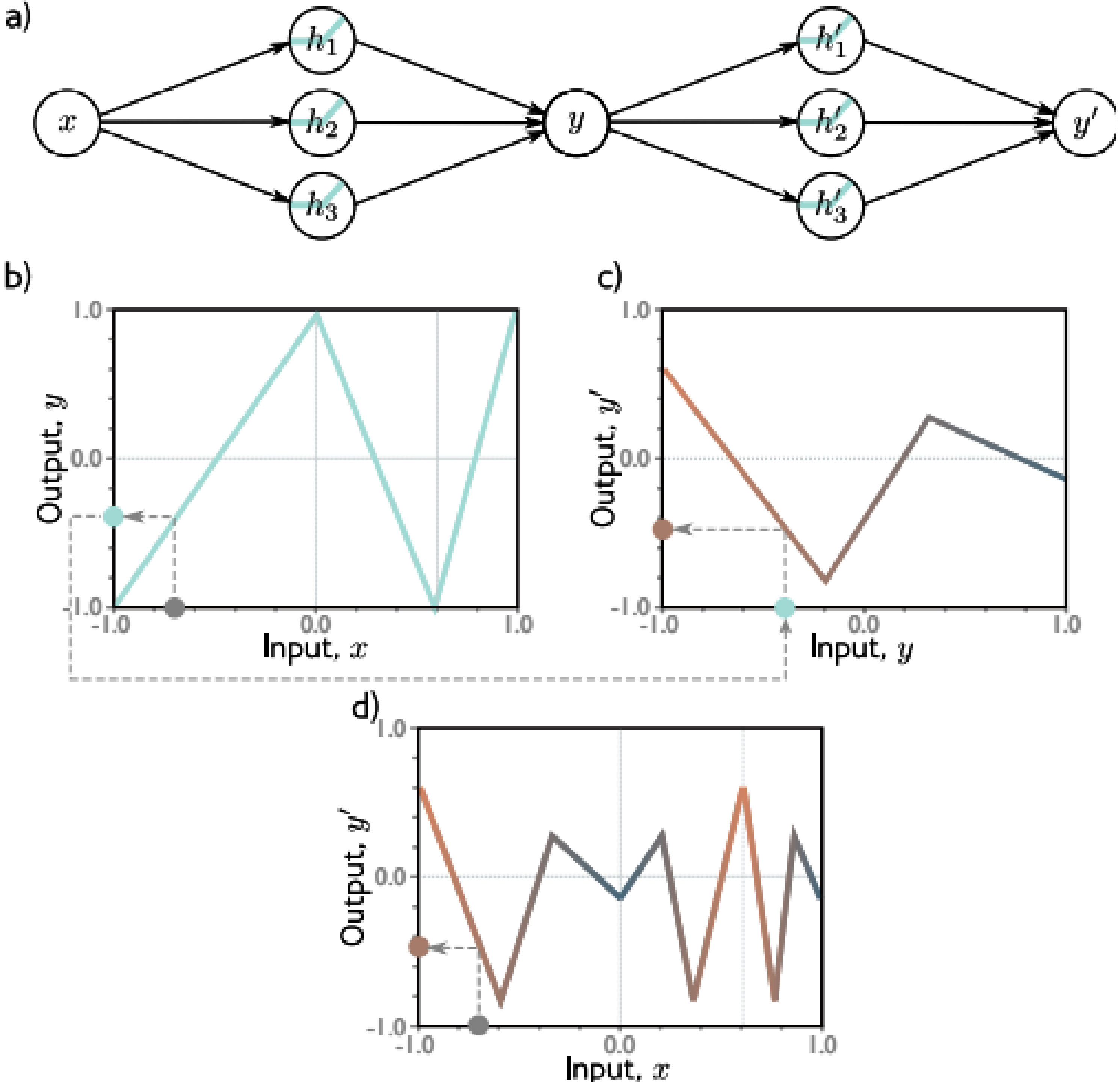
Output, y
Input, x
Output, y'
Input, y



a)
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h'_1
h'_2
h'_3
y'
b)
Output, y
Input, x
1.0
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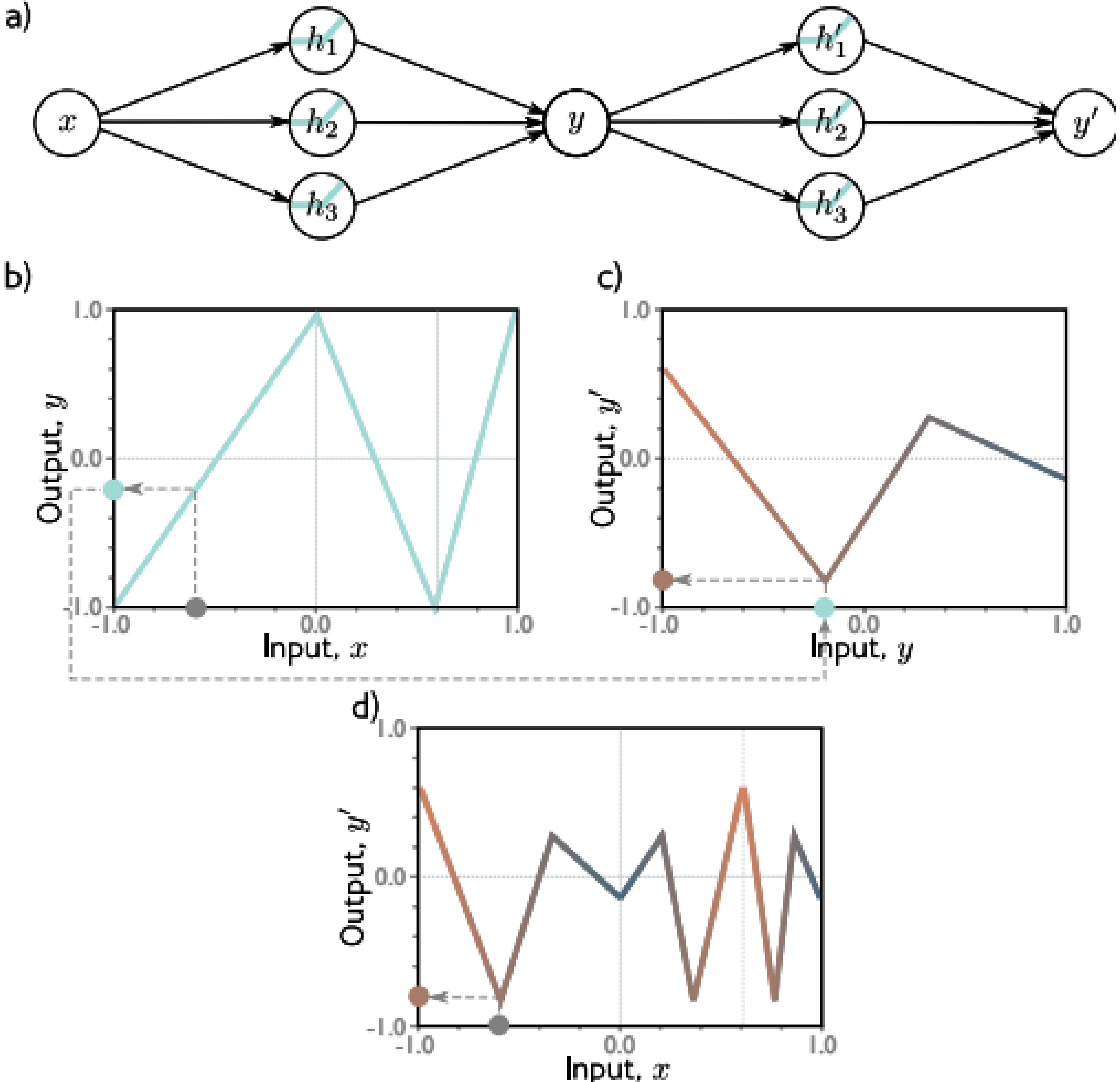
-1.0
c)
Output, y'
Input, y
d)
Output, y'
Input, x


a)
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h_3
y
h'_1
h'_2
h'_3
y'
b)
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-1.0
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0.0
1.0
Output, y
Input, x
c)
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-1.0
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Output, y'
Input, y
d)
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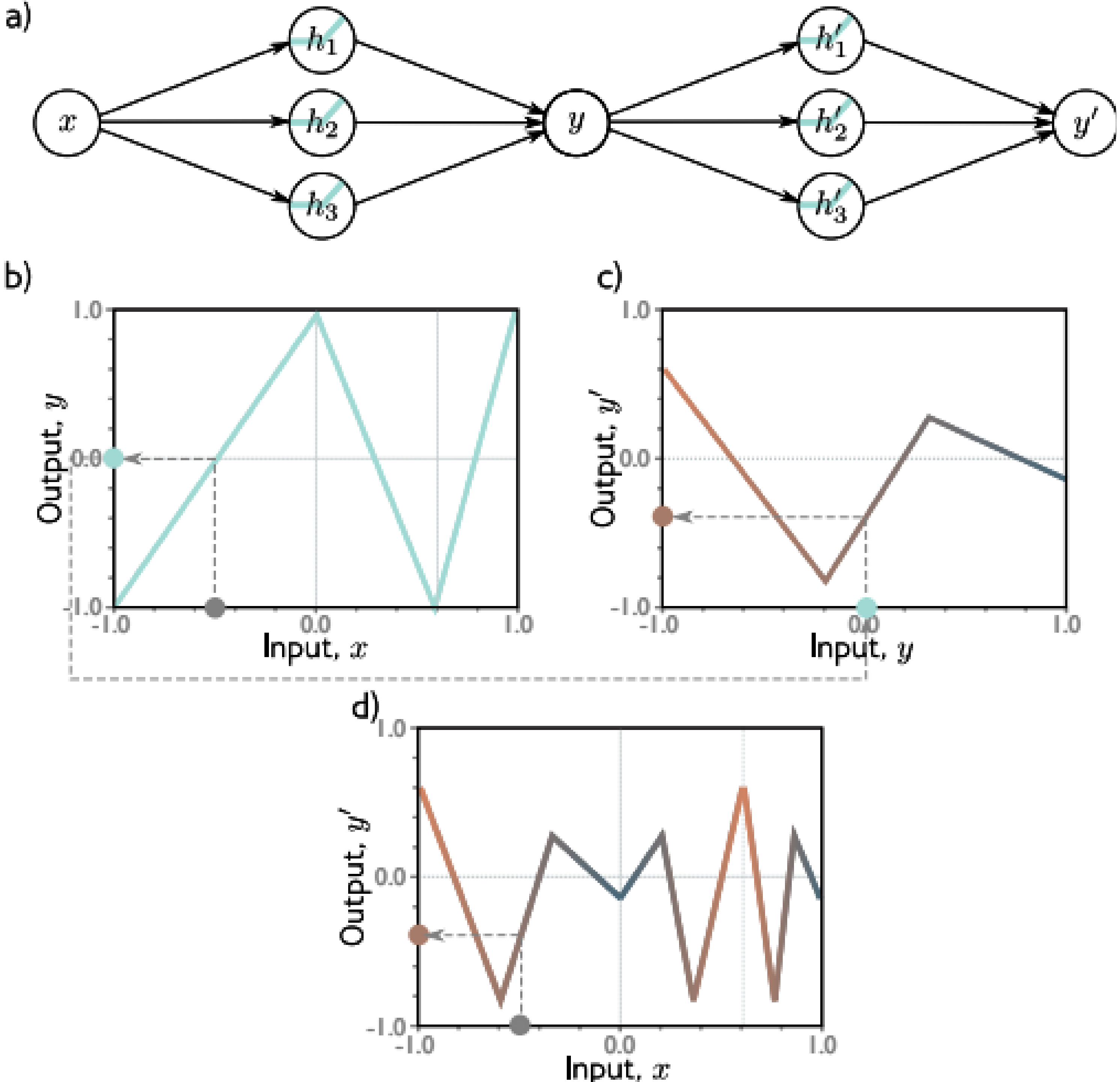
Output, y'
Input, x



a)
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Output, y
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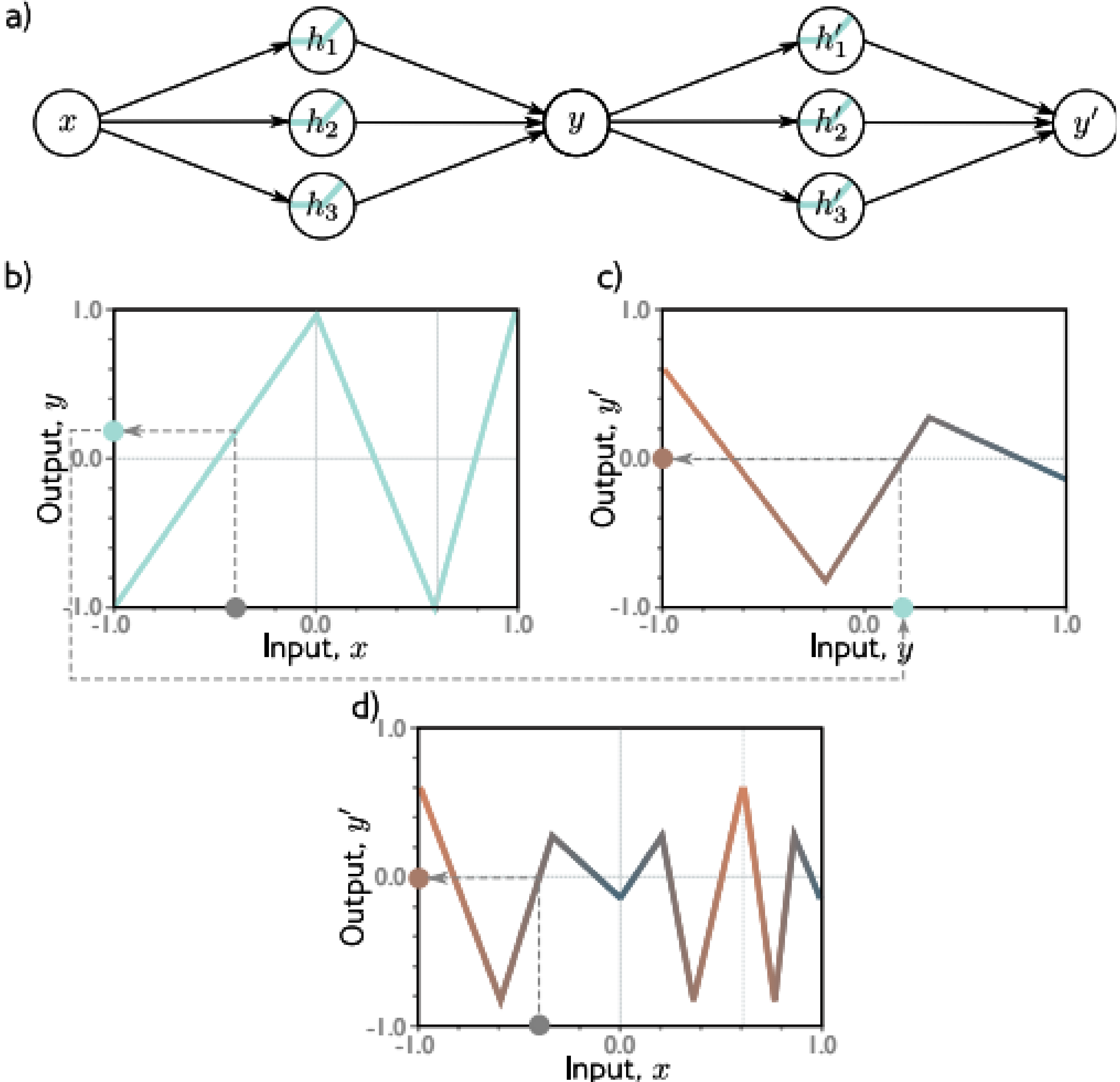

a)
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h'_2
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b)
Output, y
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c)
Output, y'
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d)
Output, y'
Input, x
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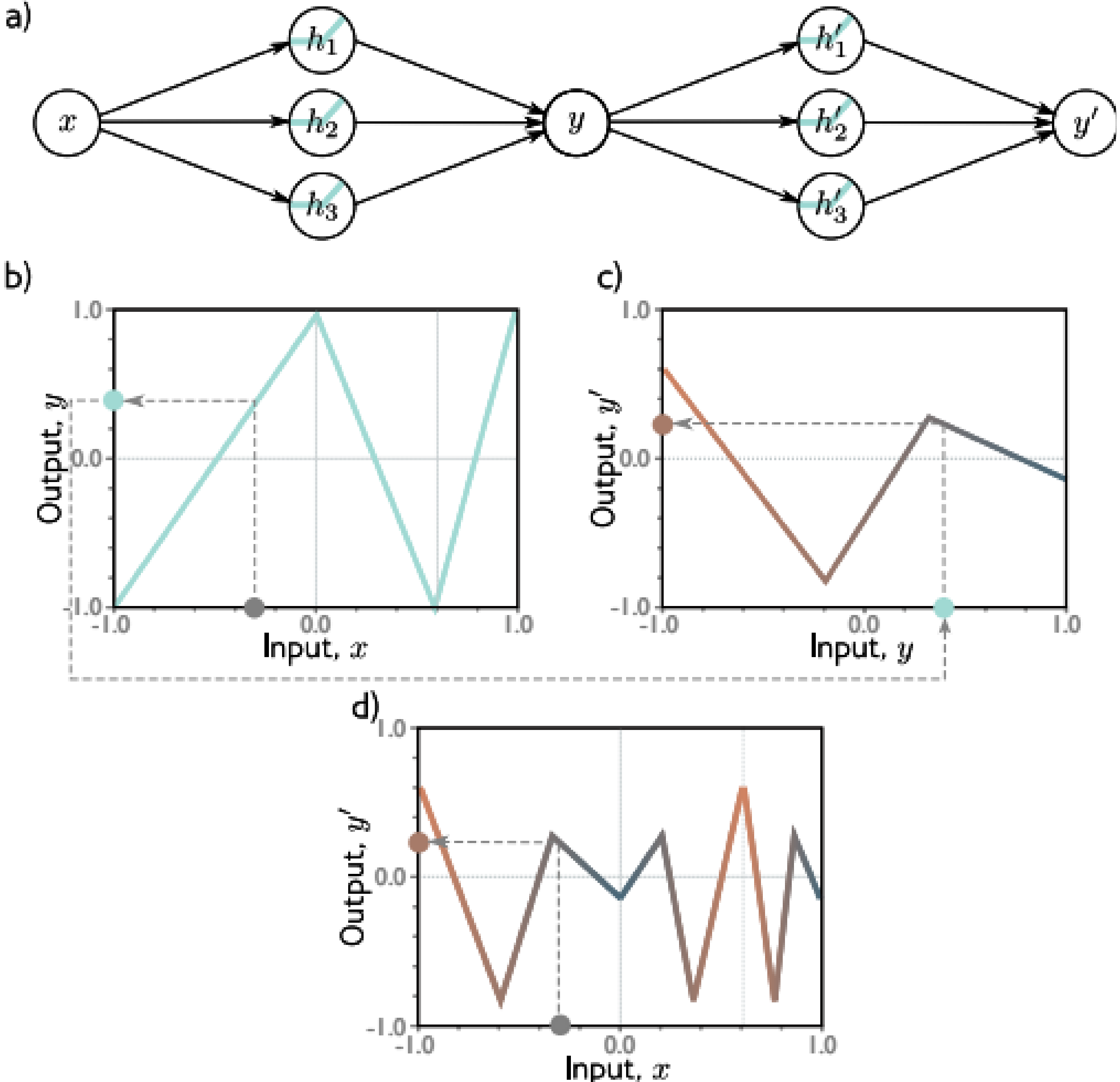

a)
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h'_1
h'_2
h'_3
y'
b)
Output, y
Input, x
c)
Output, y'
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d)
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Input, x
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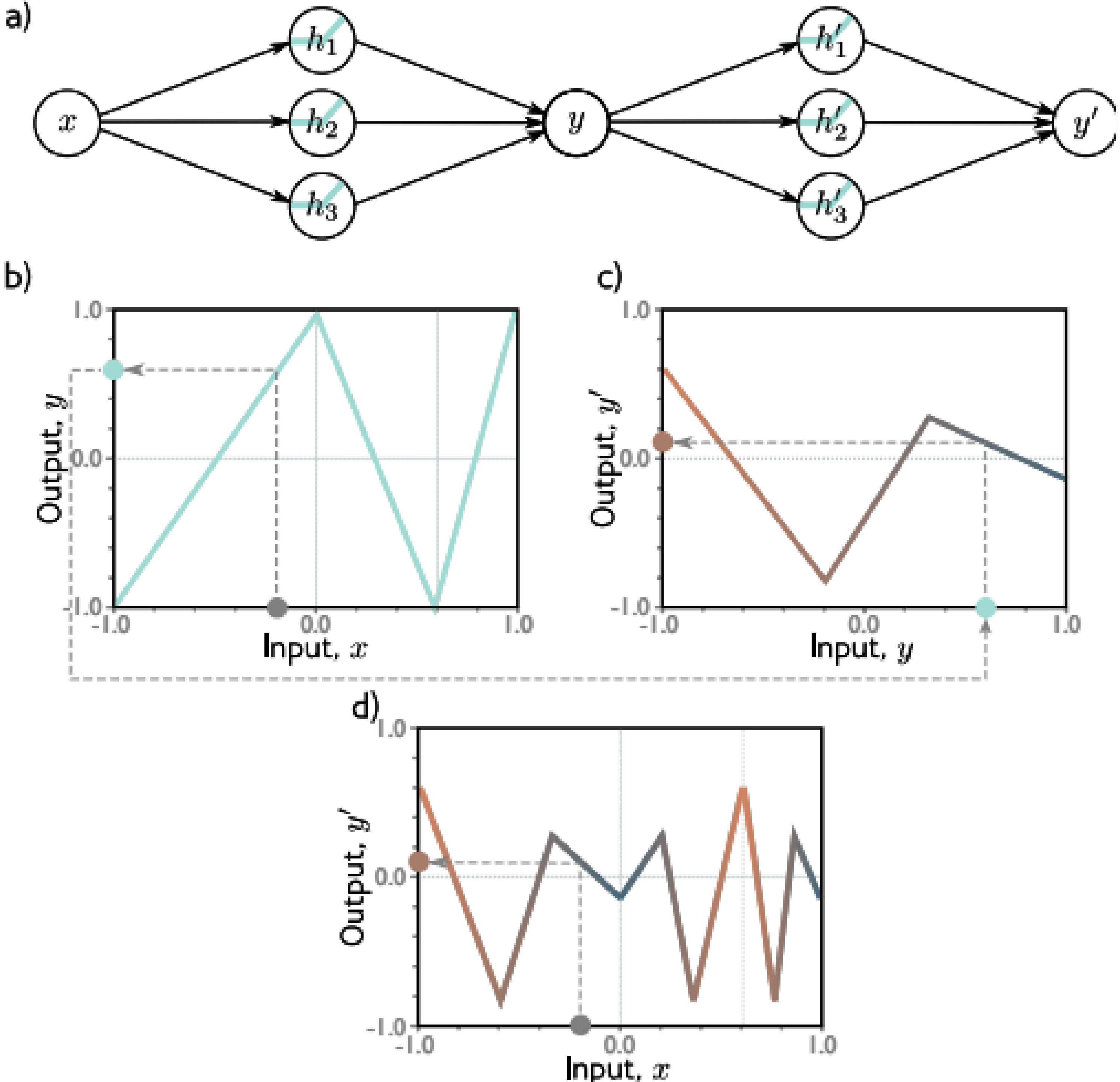

a)
x
h_1
h_2
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y
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h'_2
h'_3
y'
b)
Output, y
Input, x
1.0
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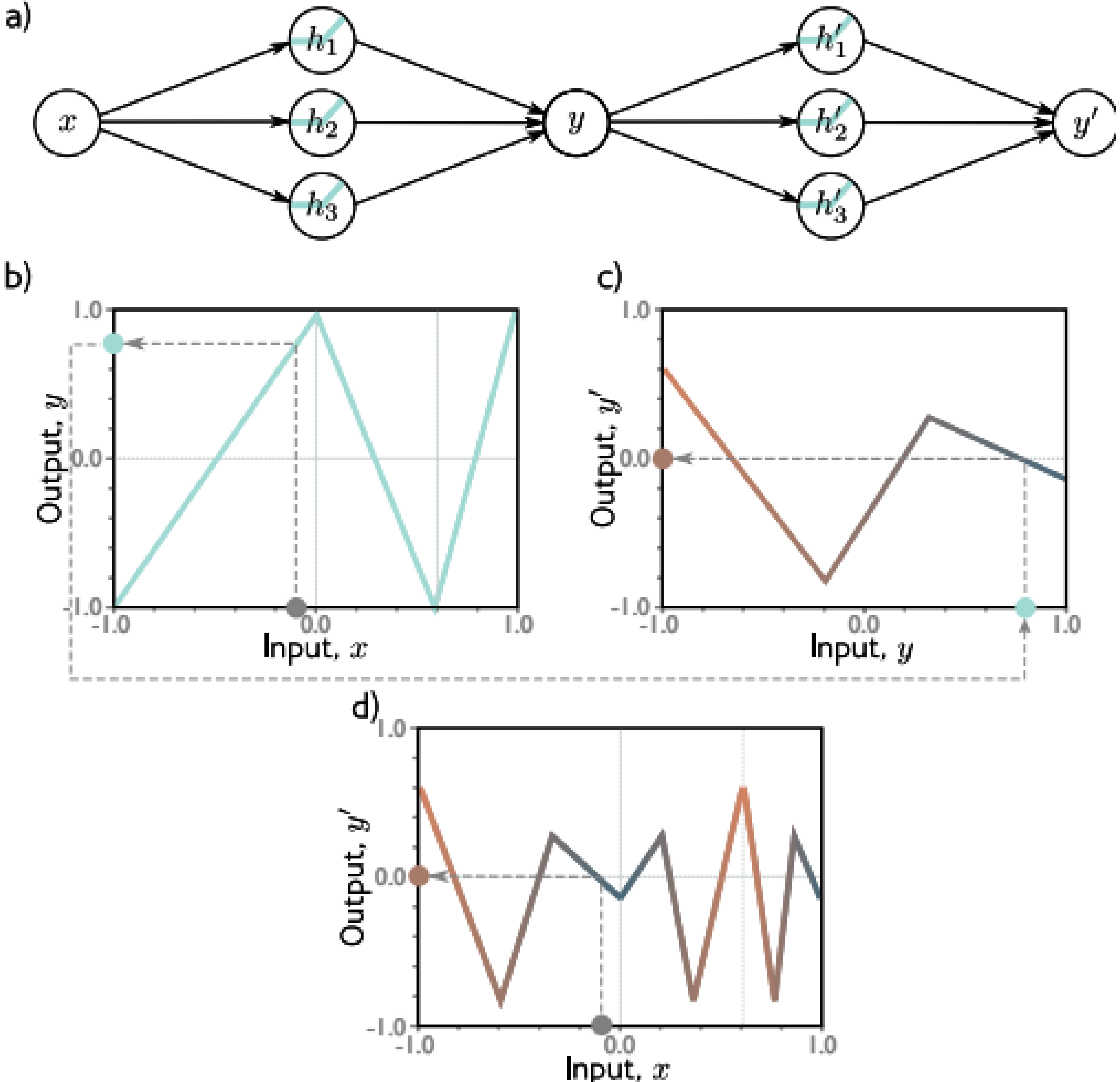
a)
x
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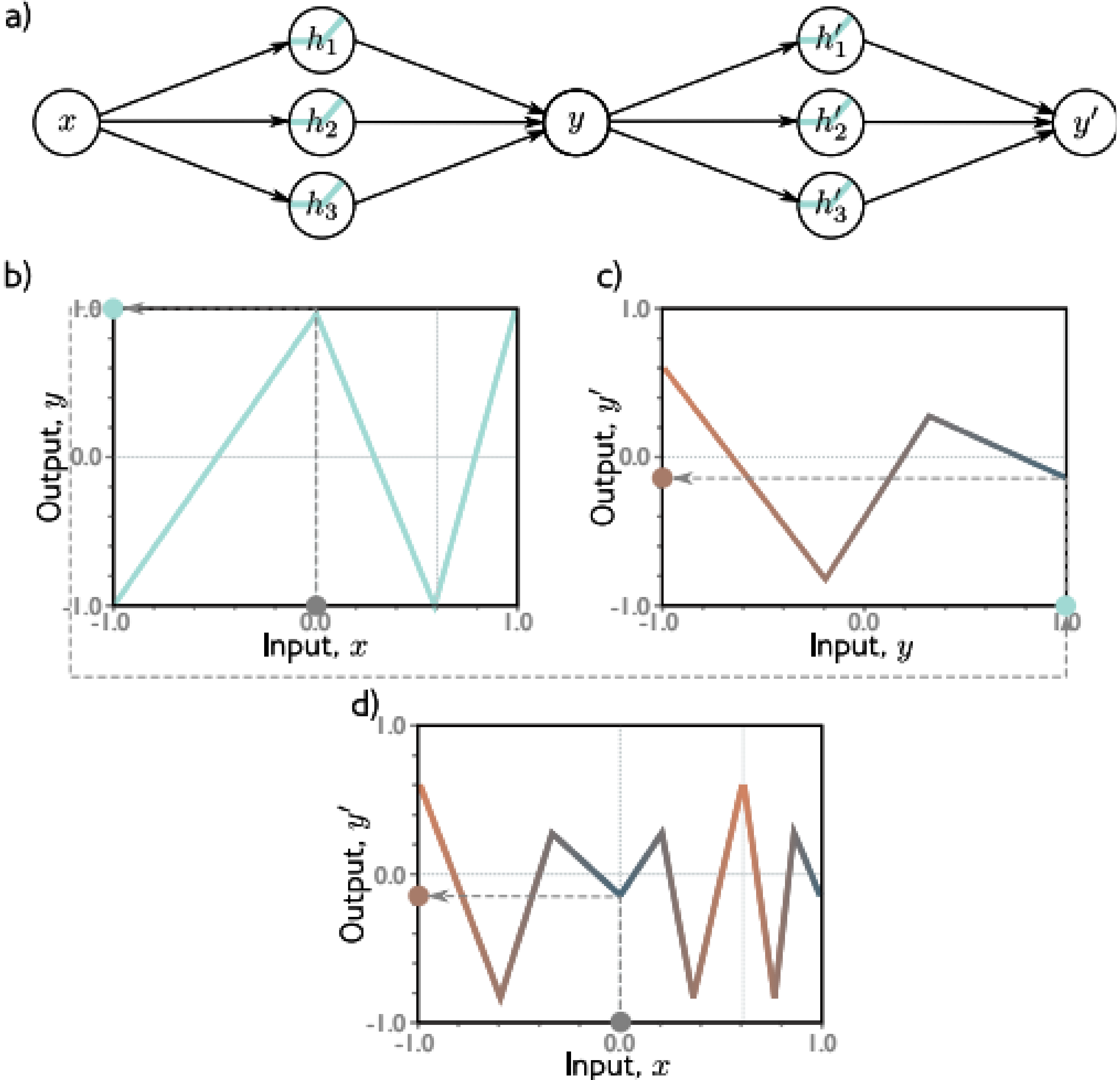

a)
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h'_1
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y'
b)
Output, y
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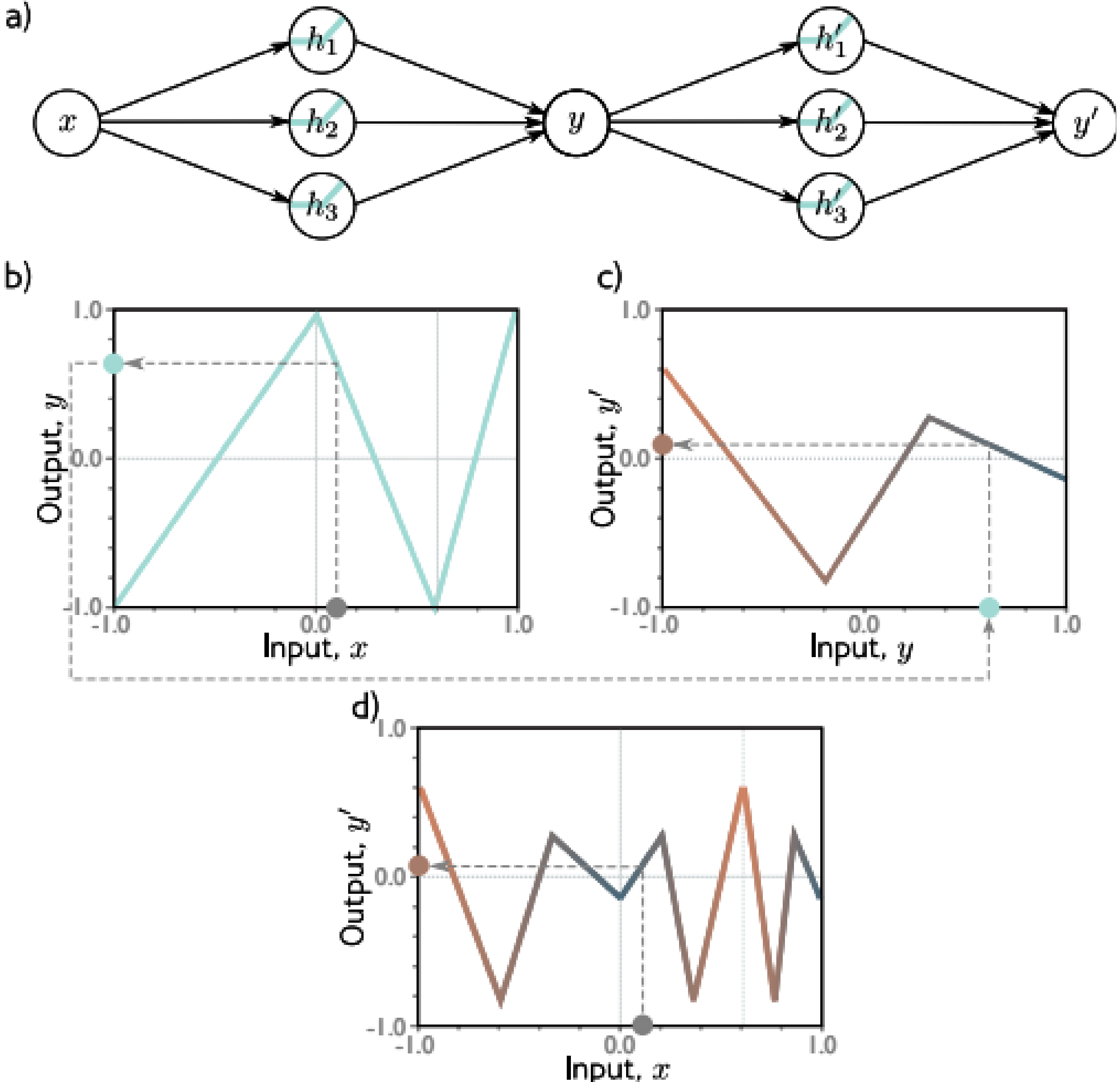
Input, x



a)
x
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h_2
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h'_1
h'_2
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y'
b)
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Output, y
Input, x
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1.0
Output, y'
Input, y
d)
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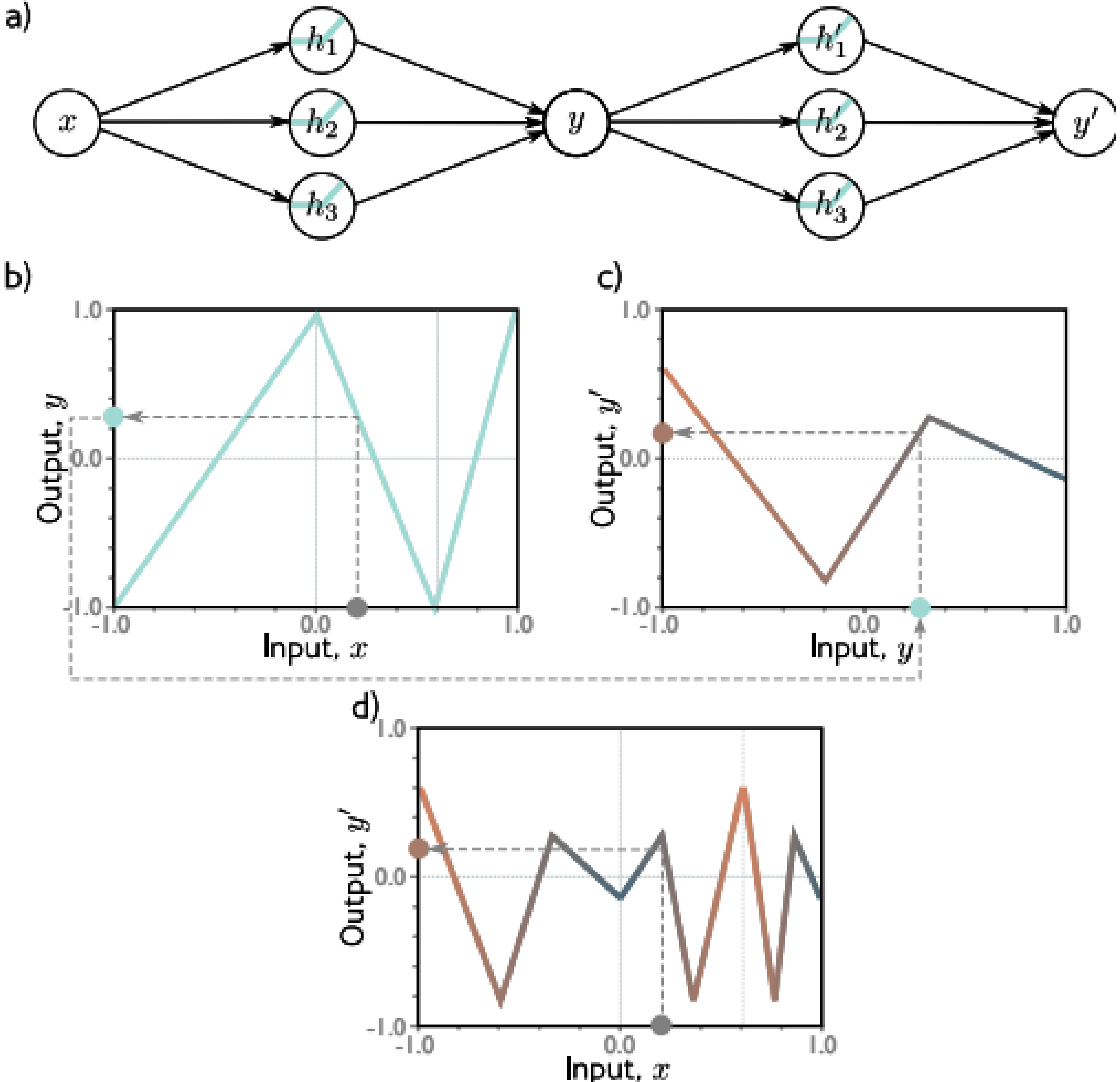
0.0
1.0
Output, y'
Input, x


a)
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y'
b)
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Input, x
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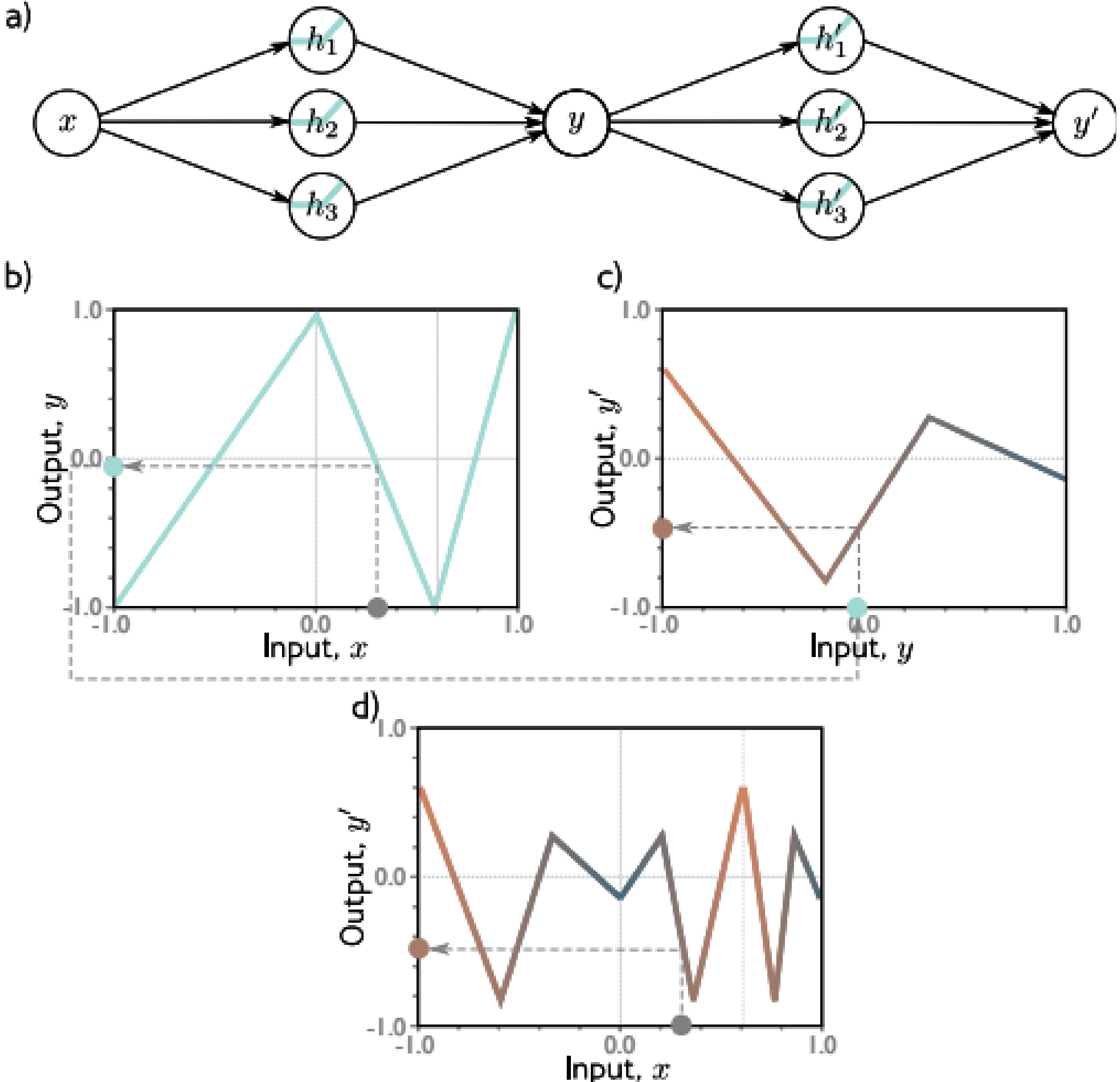
c)
Output, y'
Input, y
d)
Output, y'
Input, x



a)
x
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h_3
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y'
b)
Output, y
Input, x
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Output, y'
Input, y
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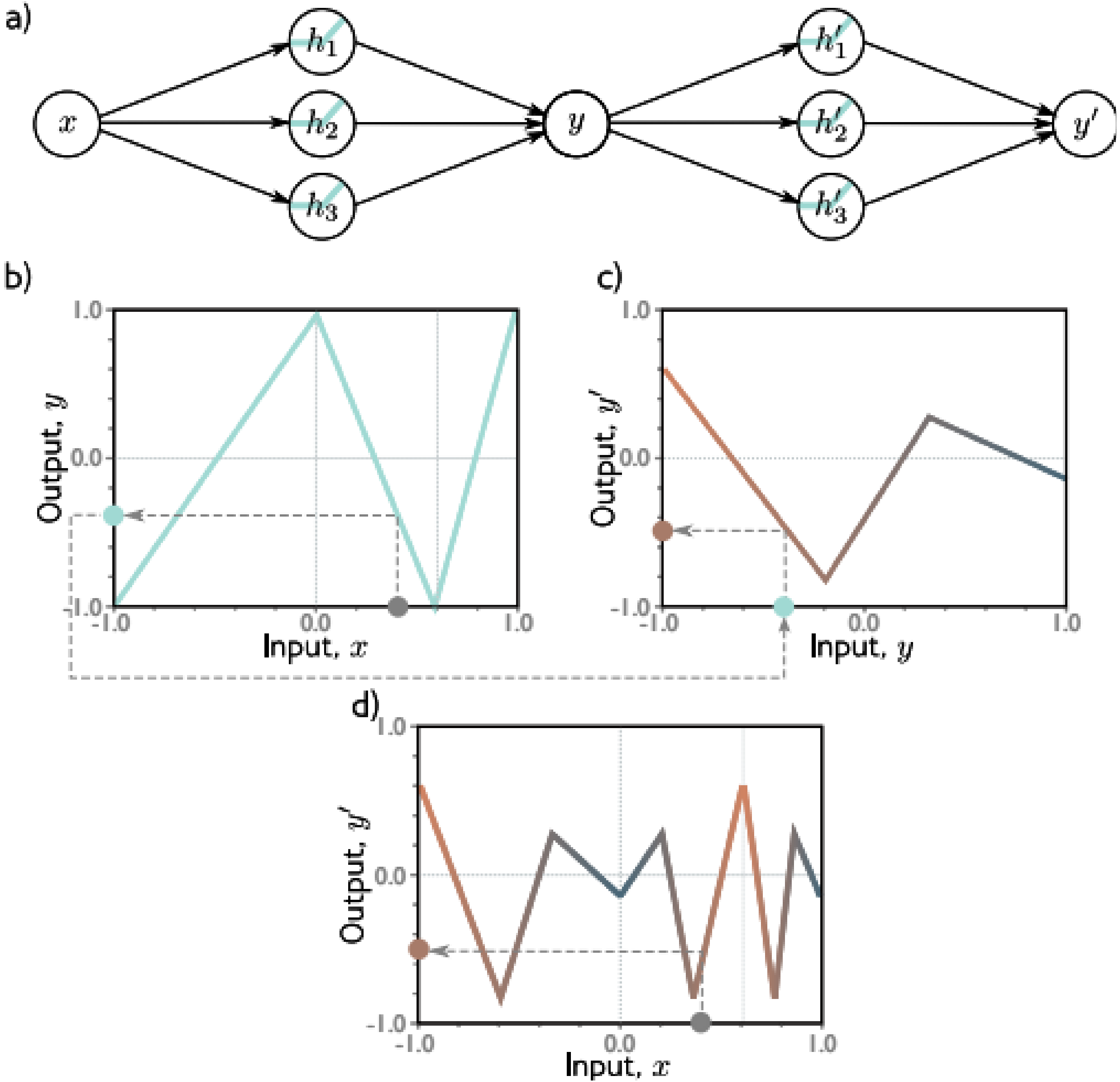
a)
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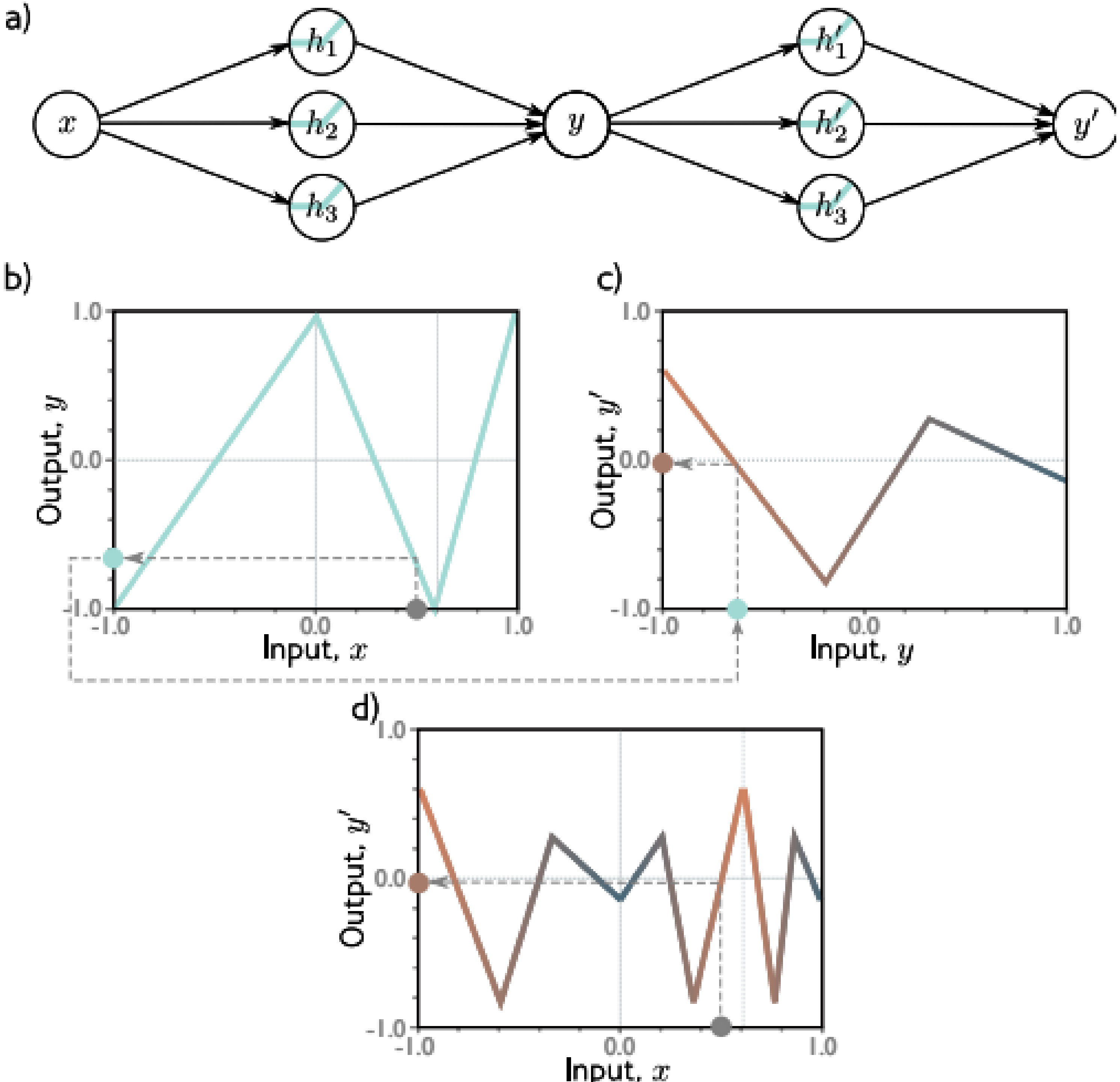

a)
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h_1
h_2
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h'_1
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Output, y
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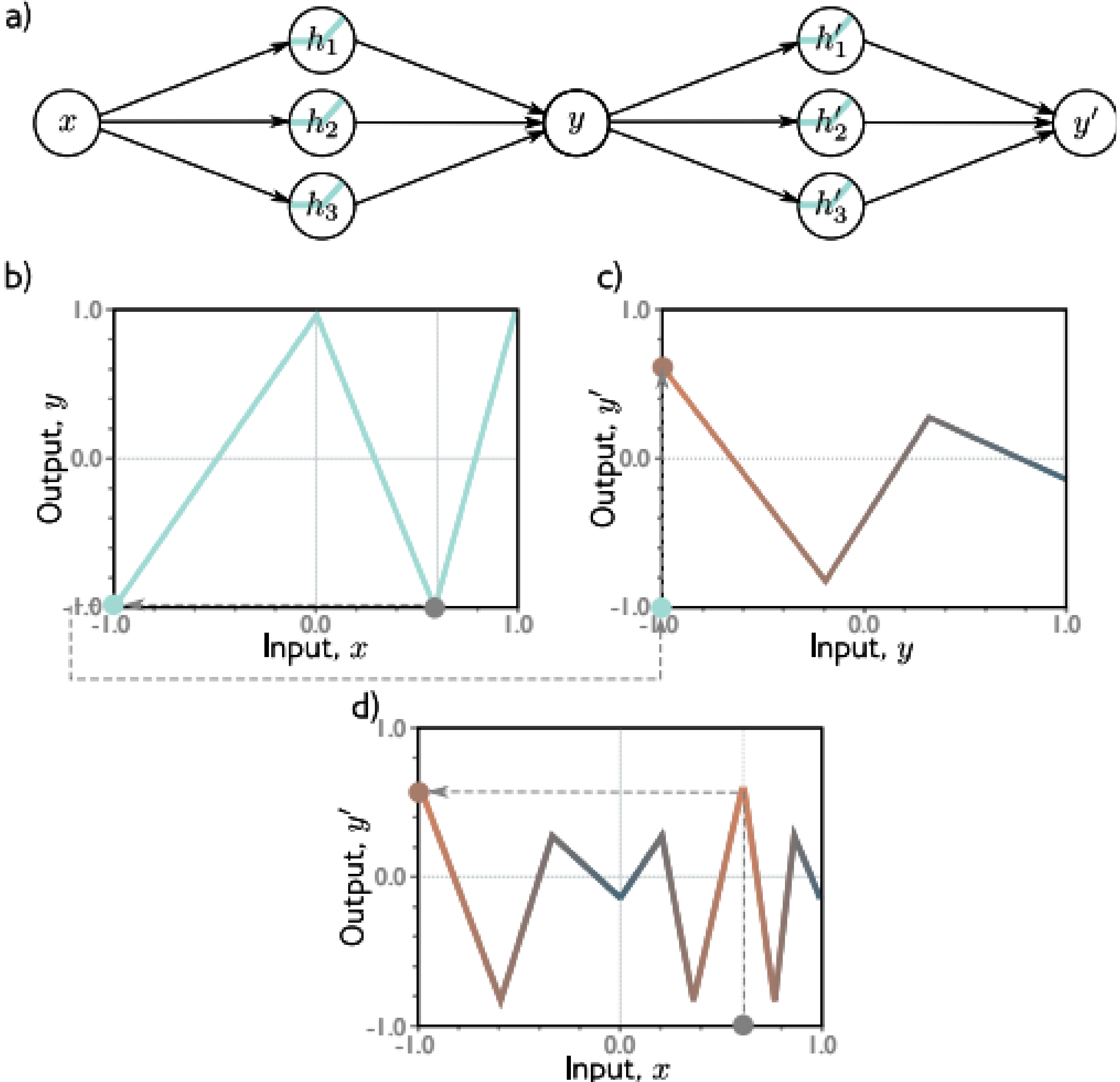

a)
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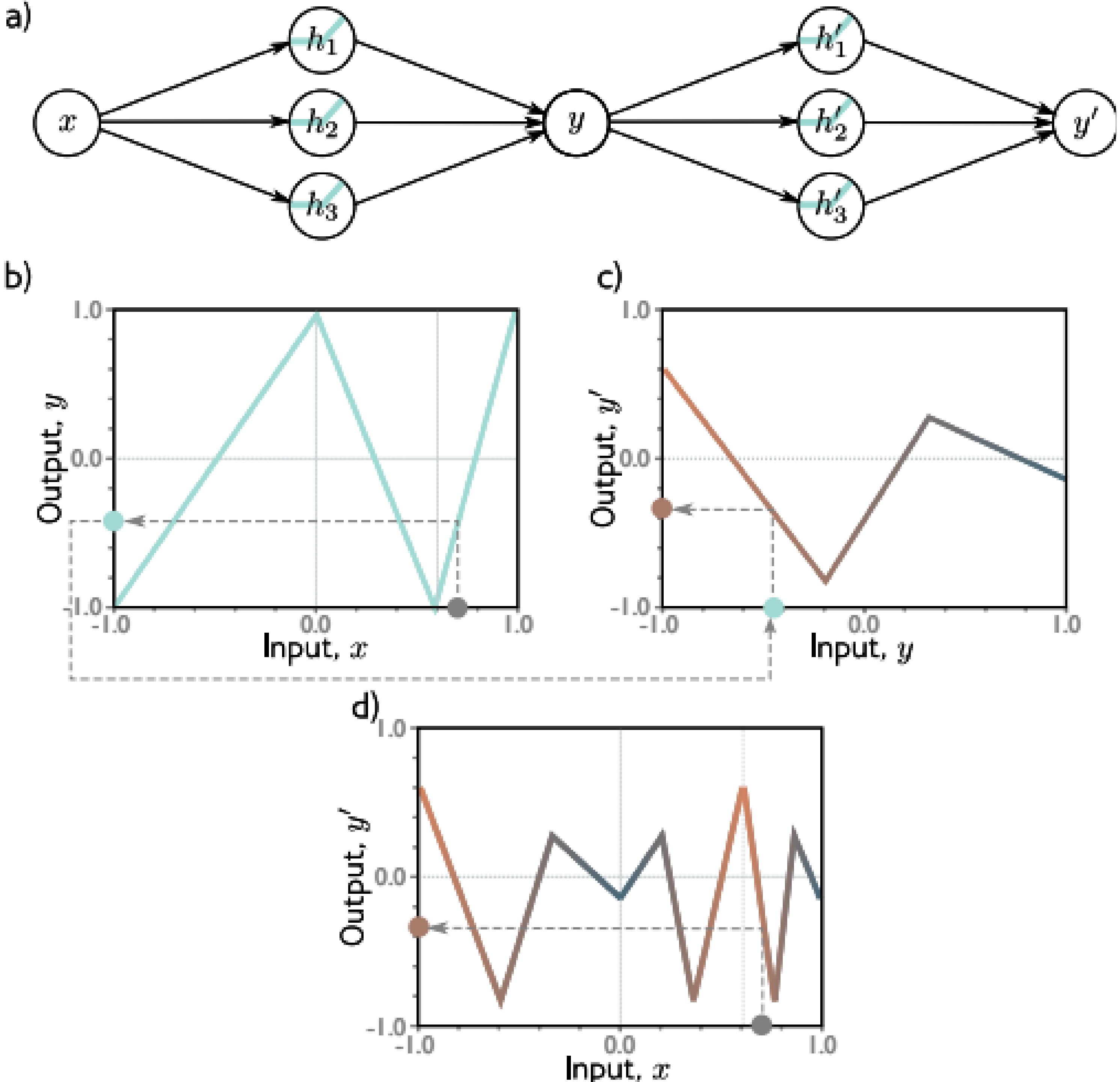
Input, x



a)
x
h_1
h_2
h_3
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h'_1
h'_2
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b)
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Output, y
Input, x
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1.0
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-1.0
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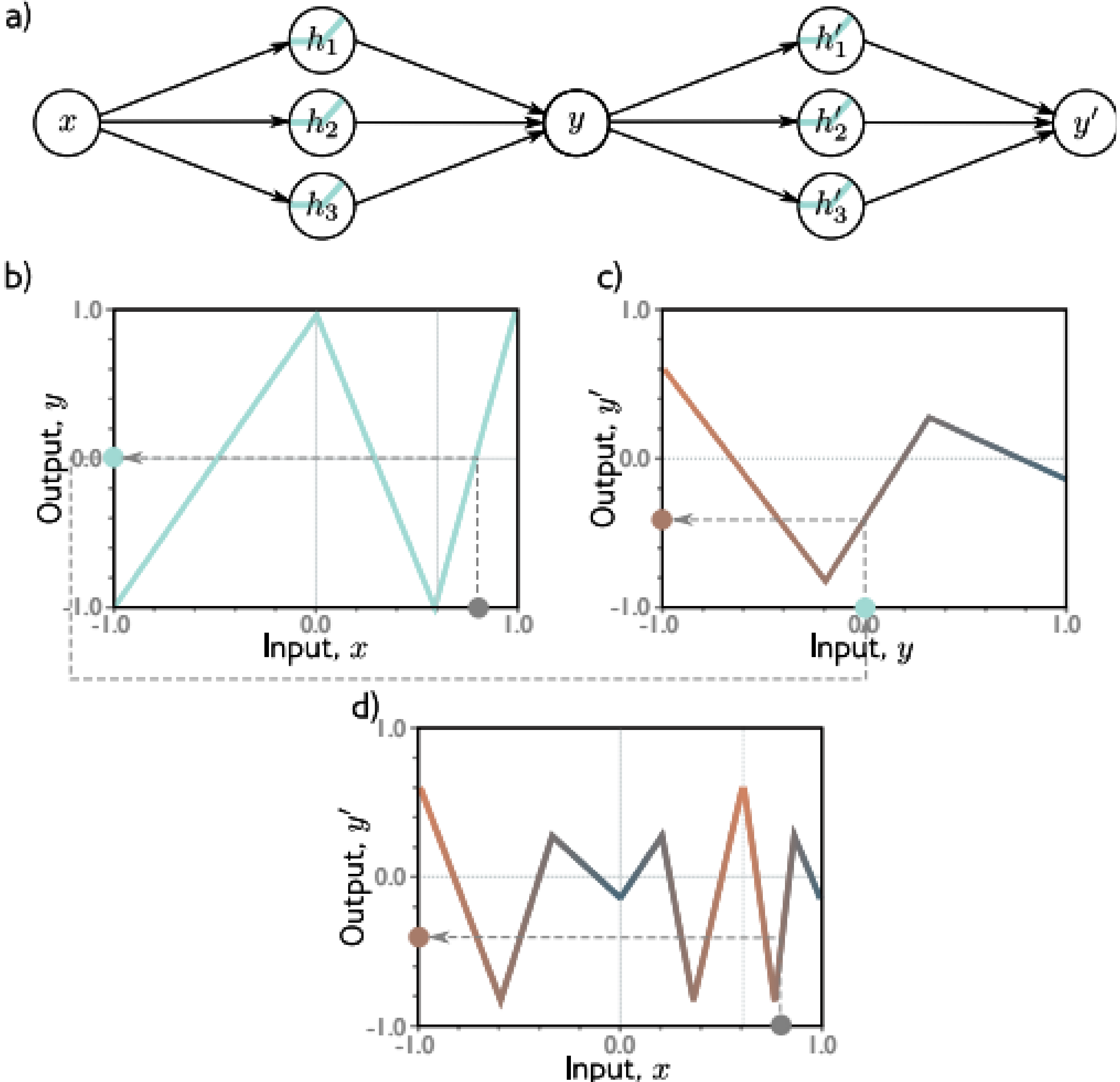
Output, y'
Input, y
d)
1.0
0.0
-1.0
-1.0
0.0
1.0
Output, y'
Input, x



a)
x
h_1
h_2
h_3
y
h'_1
h'_2
h'_3
y'
b)
Output, y
Input, x
1.0
0.0
-1.0
-1.0
0.0
1.0
c)
Output, y'
Input, y
1.0
0.0
-1.0
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Input, x
1.0
0.0
-1.0
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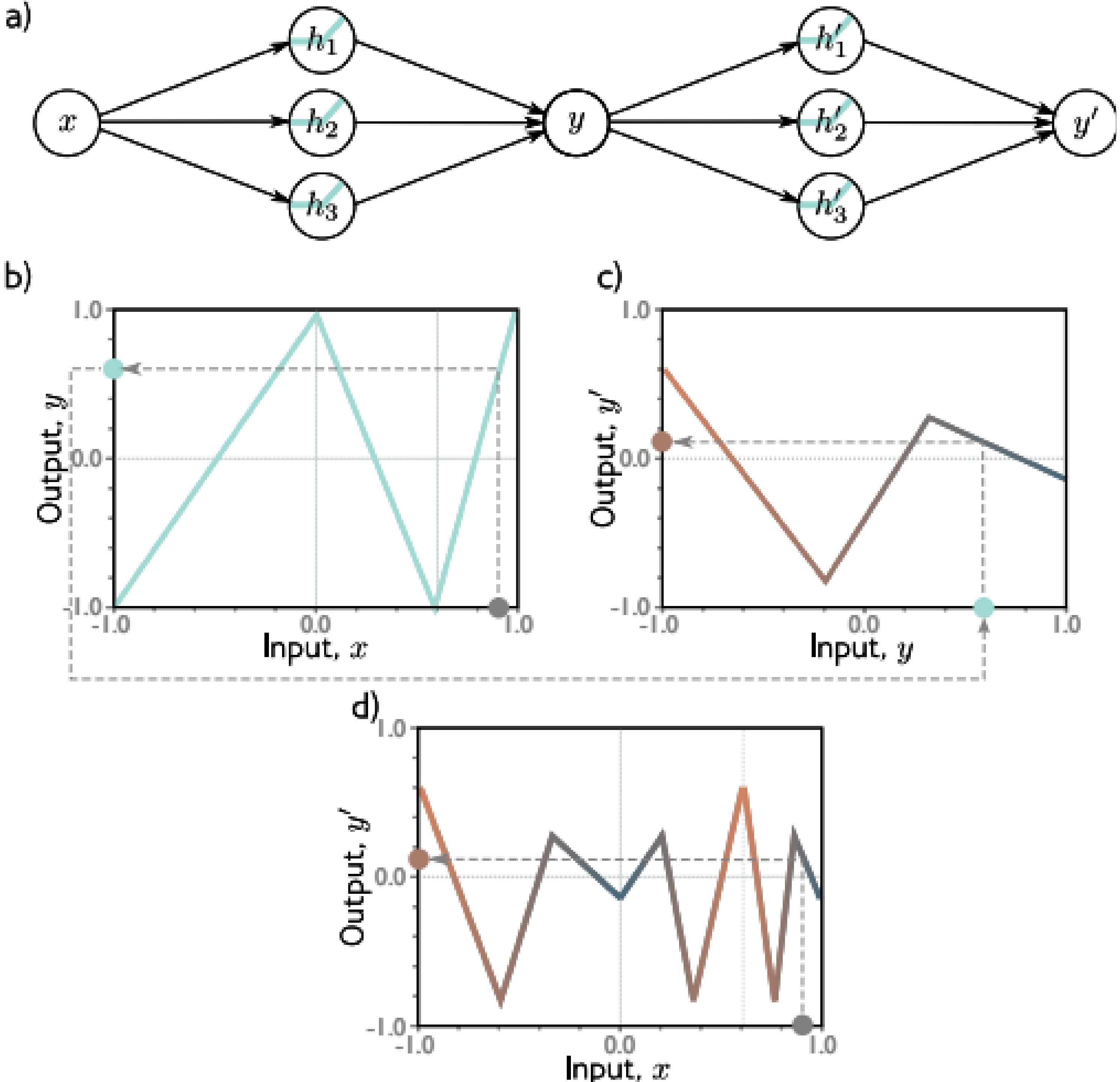
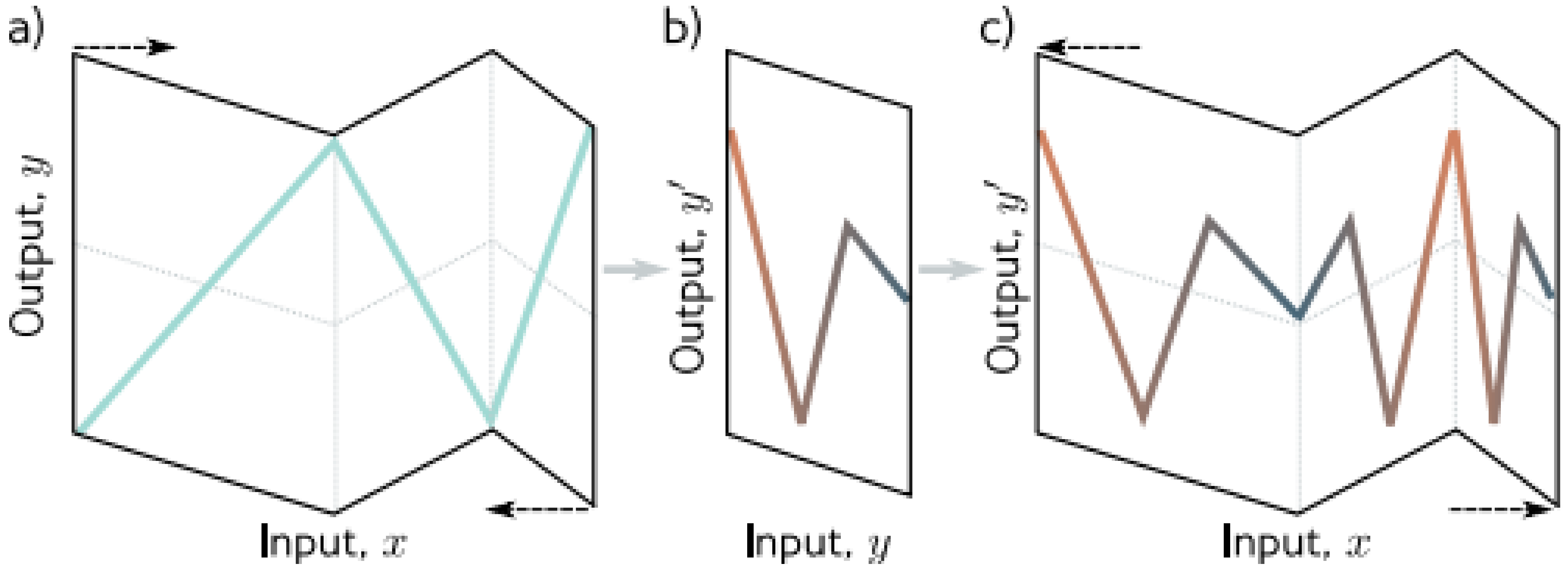
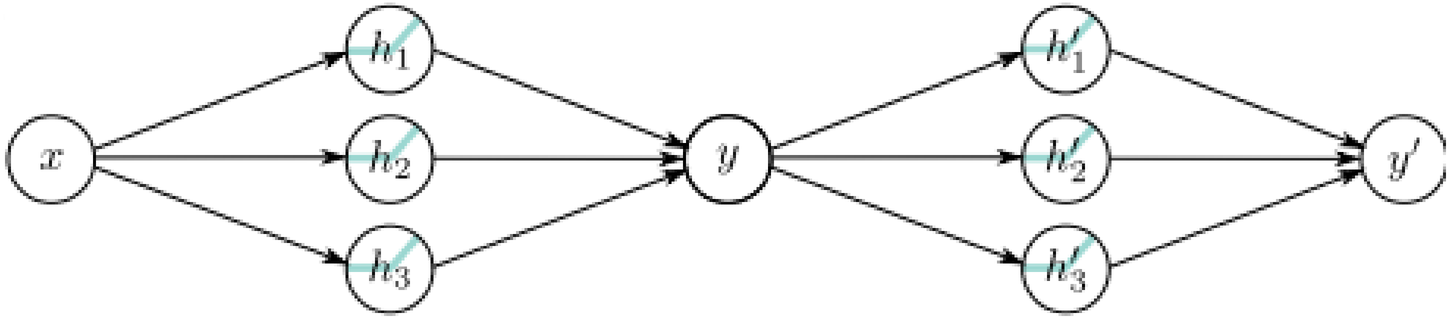


a)
b)
c)
d)
Output, y
Input, x
Output, y'
Input, y

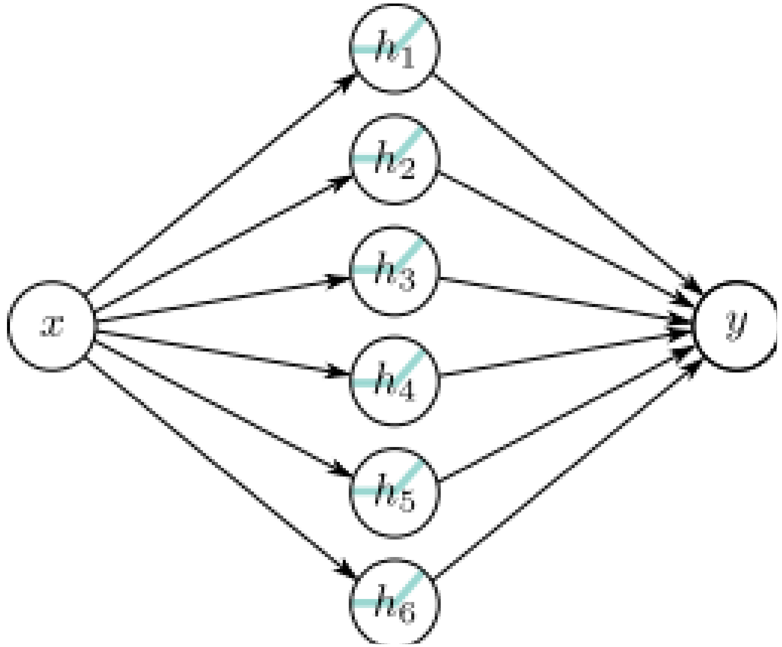
“Folding analogy”



Comparing to shallow with six hidden units

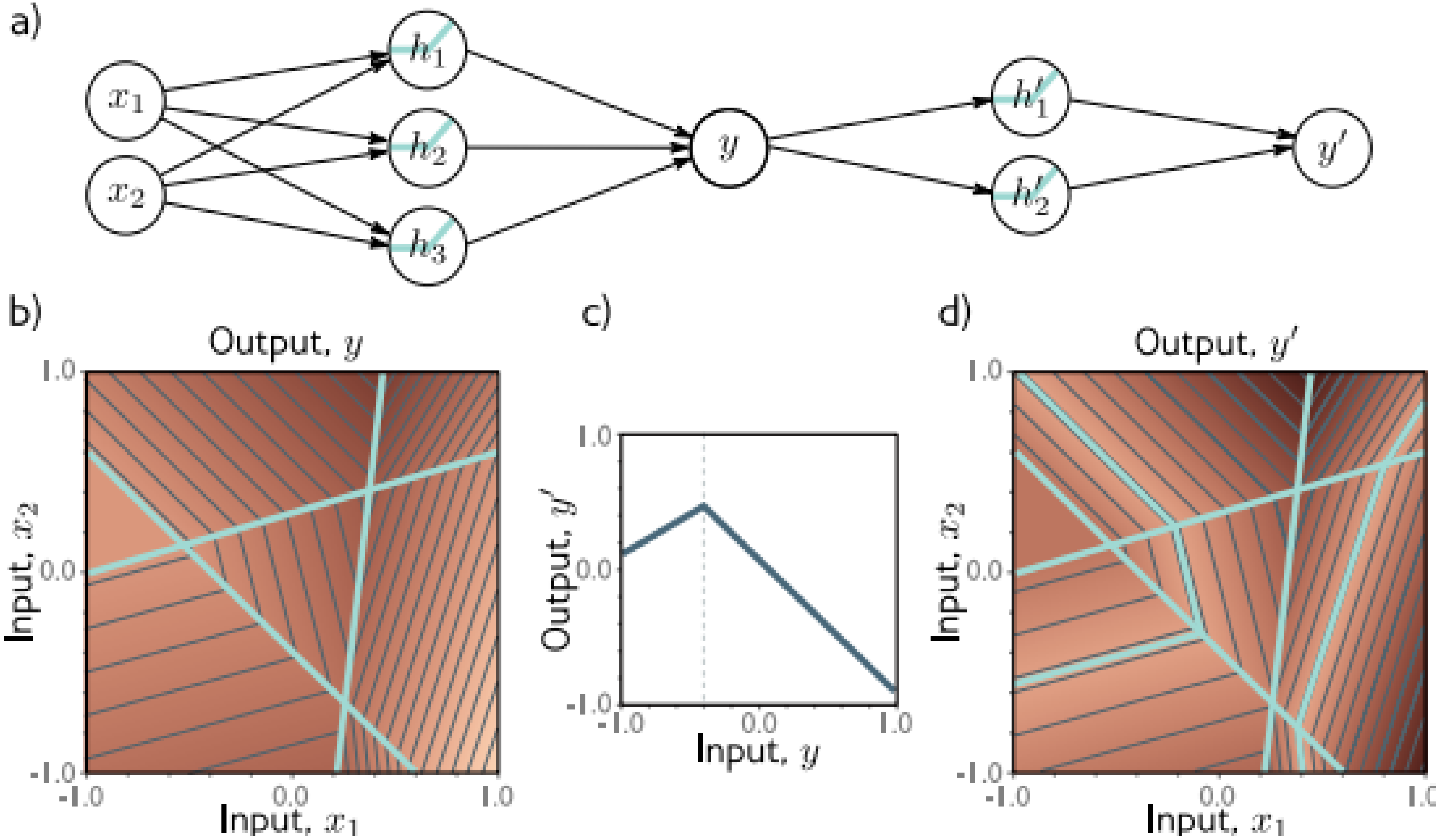


- 20 parameters
- (at least) 9 regions



- 19 parameters
- Max 7 regions

Composing networks in 2D



Deep neural networks

- Composing two networks
- Combining the two networks into one
- Hyperparameters
- Notation change and general case
- Shallow vs. deep networks

Combine two networks into one

Network 1:

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

Network 2:

$$h'_1 = \mathrm{a}[\theta'_{10} + \theta'_{11}y]$$
$$h'_2 = \mathrm{a}[\theta'_{20} + \theta'_{21}y]$$
$$h'_3 = \mathrm{a}[\theta'_{30} + \theta'_{31}y]$$

$$y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$$

Hidden units of second network in terms of first:

$$\begin{aligned}
h'_1 = \quad & \mathrm{a}[\theta'_{10} + \theta'_{11}y] \quad = \quad \mathrm{a}[\theta'_{10} + \theta'_{11}\phi_0 + \theta'_{11}\phi_1 h_1 + \theta'_{11}\phi_2 h_2 + \theta'_{11}\phi_3 h_3] \\
h'_2 = \quad & \mathrm{a}[\theta'_{20} + \theta'_{21}y] \quad = \quad \mathrm{a}[\theta'_{20} + \theta'_{21}\phi_0 + \theta'_{21}\phi_1 h_1 + \theta'_{21}\phi_2 h_2 + \theta'_{21}\phi_3 h_3] \\
h'_3 = \quad & \mathrm{a}[\theta'_{30} + \theta'_{31}y] \quad = \quad \mathrm{a}[\theta'_{30} + \theta'_{31}\phi_0 + \theta'_{31}\phi_1 h_1 + \theta'_{31}\phi_2 h_2 + \theta'_{31}\phi_3 h_3]
\end{aligned}$$

Create new variables

$$
\begin{aligned}
h'_1 = \quad & \mathrm{a}[\theta'_{10} + \theta'_{11} y] & = & \quad \mathrm{a}[\theta'_{10} + \theta'_{11}\phi_0 + \theta'_{11}\phi_1 h_1 + \theta'_{11}\phi_2 h_2 + \theta'_{11}\phi_3 h_3] \\
h'_2 = \quad & \mathrm{a}[\theta'_{20} + \theta'_{21} y] & = & \quad \mathrm{a}[\theta'_{20} + \theta'_{21}\phi_0 + \theta'_{21}\phi_1 h_1 + \theta'_{21}\phi_2 h_2 + \theta'_{21}\phi_3 h_3] \\
h'_3 = \quad & \mathrm{a}[\theta'_{30} + \theta'_{31} y] & = & \quad \mathrm{a}[\theta'_{30} + \theta'_{31}\phi_0 + \theta'_{31}\phi_1 h_1 + \theta'_{31}\phi_2 h_2 + \theta'_{31}\phi_3 h_3]
\end{aligned}
$$

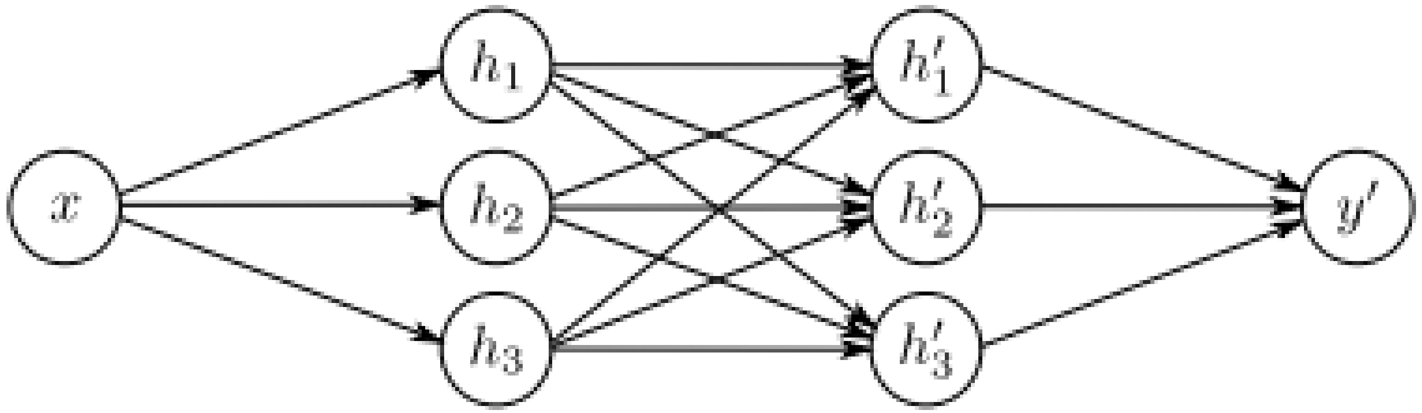
$$
\begin{aligned}
h'_1 &= \mathrm{a}[\psi_{10} + \psi_{11} h_1 + \psi_{12} h_2 + \psi_{13} h_3] \\
h'_2 &= \mathrm{a}[\psi_{20} + \psi_{21} h_1 + \psi_{22} h_2 + \psi_{23} h_3] \\
h'_3 &= \mathrm{a}[\psi_{30} + \psi_{31} h_1 + \psi_{32} h_2 + \psi_{33} h_3]
\end{aligned}
$$

Two-layer network

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$h'_1 = \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3]$$
$$h'_2 = \mathrm{a}[\psi_{20} + \psi_{21}h_2 + \psi_{22}h_2 + \psi_{23}h_3]$$
$$h'_3 = \mathrm{a}[\psi_{30} + \psi_{31}h_2 + \psi_{32}h_2 + \psi_{33}h_3]$$

$$y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$$



Two-layer network as one equation

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$h'_1 = \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3]$$
$$h'_2 = \mathrm{a}[\psi_{20} + \psi_{21}h_1 + \psi_{22}h_2 + \psi_{23}h_3]$$
$$h'_3 = \mathrm{a}[\psi_{30} + \psi_{31}h_1 + \psi_{32}h_2 + \psi_{33}h_3]$$

$$y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$$

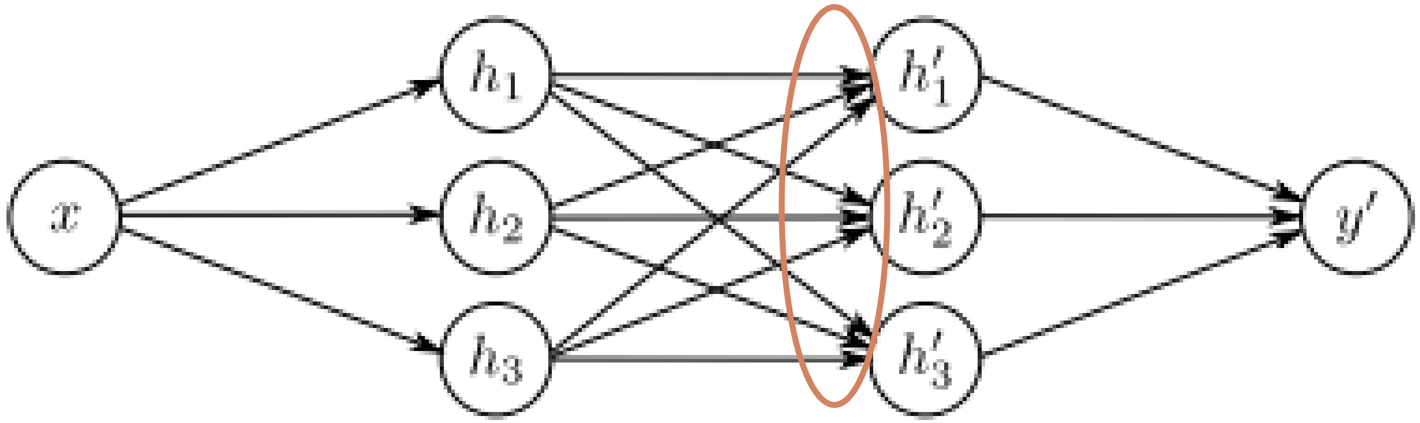
$$\begin{aligned} y' = \phi'_0 &+ \phi'_1 \mathrm{a}\left[\psi_{10} + \psi_{11}\mathrm{a}[\theta_{10} + \theta_{11}x] + \psi_{12}\mathrm{a}[\theta_{20} + \theta_{21}x] + \psi_{13}\mathrm{a}[\theta_{30} + \theta_{31}x]\right] \\ &+ \phi'_2 \mathrm{a}[\psi_{20} + \psi_{21}\mathrm{a}[\theta_{10} + \theta_{11}x] + \psi_{22}\mathrm{a}[\theta_{20} + \theta_{21}x] + \psi_{23}\mathrm{a}[\theta_{30} + \theta_{31}x]] \\ &+ \phi'_3 \mathrm{a}[\psi_{30} + \psi_{31}\mathrm{a}[\theta_{10} + \theta_{11}x] + \psi_{32}\mathrm{a}[\theta_{20} + \theta_{21}x] + \psi_{33}\mathrm{a}[\theta_{30} + \theta_{31}x]] \end{aligned}$$

Networks as composing functions

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$h'_1 = \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3]$$
$$h'_2 = \mathrm{a}[\psi_{20} + \psi_{21}h_1 + \psi_{22}h_2 + \psi_{23}h_3]$$
$$h'_3 = \mathrm{a}[\psi_{30} + \psi_{31}h_1 + \psi_{32}h_2 + \psi_{33}h_3]$$

Consider the pre-activations at the second hidden units
At this point, it’s a one--layer network with three outputs

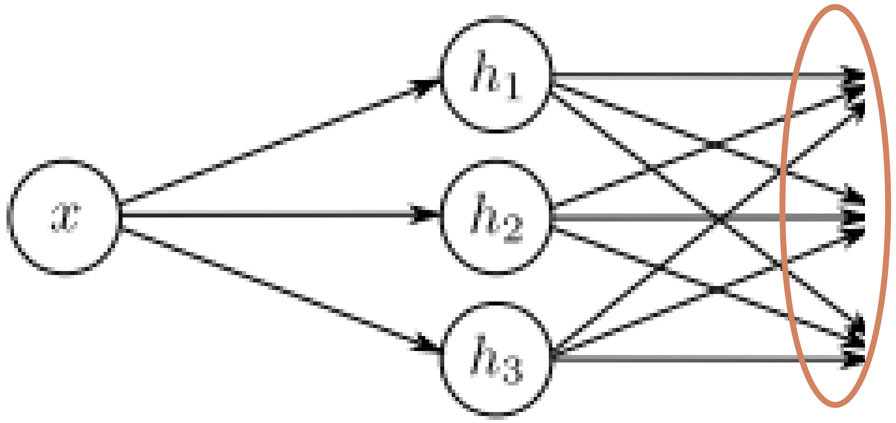


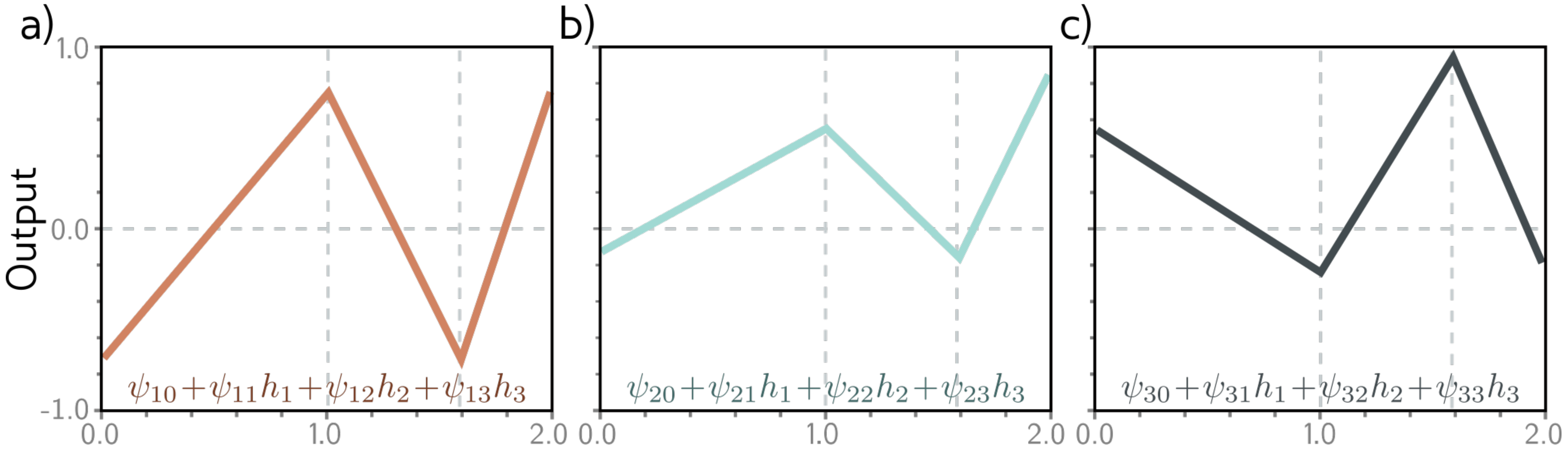
Networks as composing functions

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

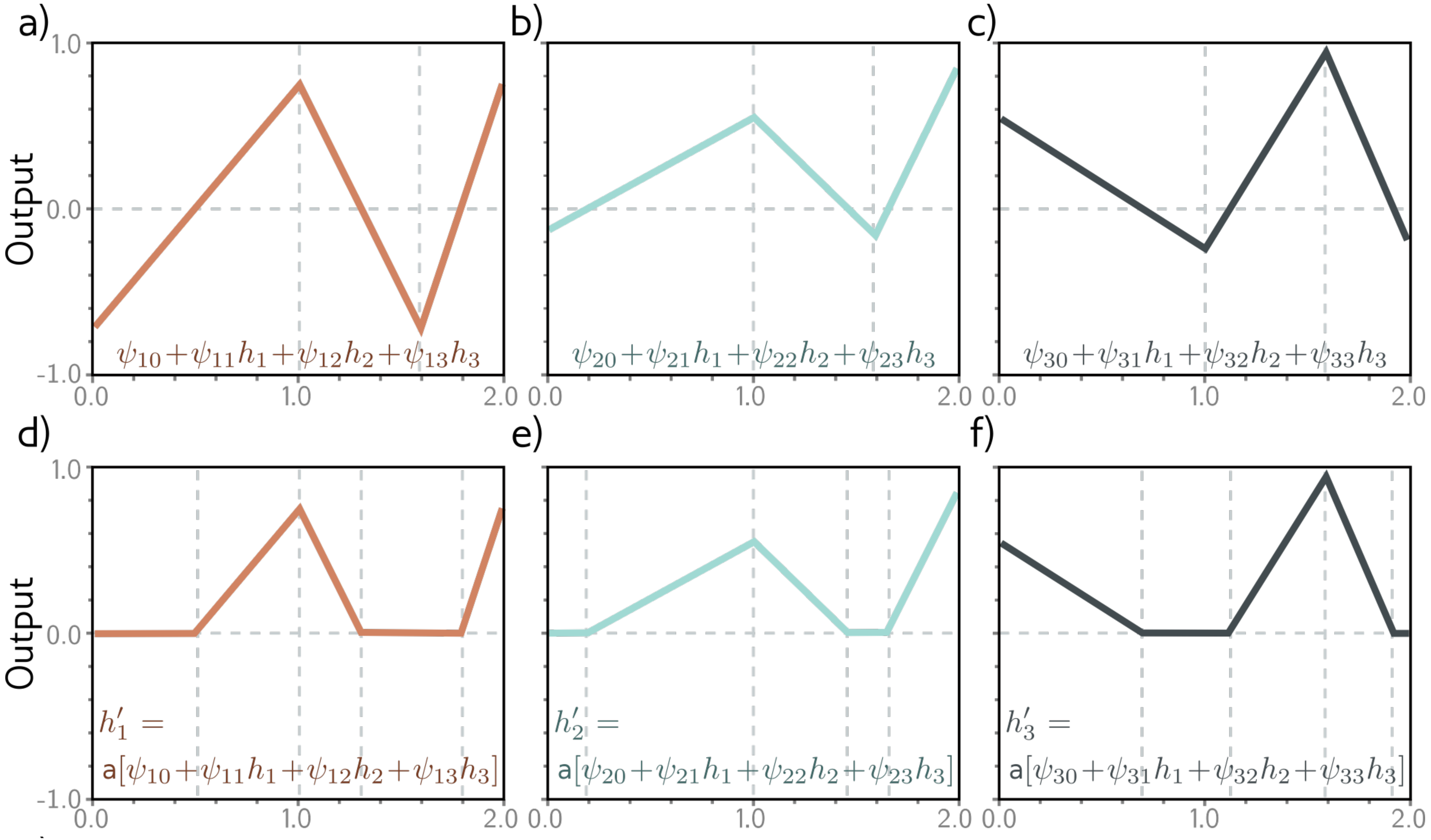
$$h'_1 = \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3]$$
$$h'_2 = \mathrm{a}[\psi_{20} + \psi_{21}h_1 + \psi_{22}h_2 + \psi_{23}h_3]$$
$$h'_3 = \mathrm{a}[\psi_{30} + \psi_{31}h_1 + \psi_{32}h_2 + \psi_{33}h_3]$$

Consider the pre-activations at the second hidden units
At this point, it's a one--layer network with three outputs

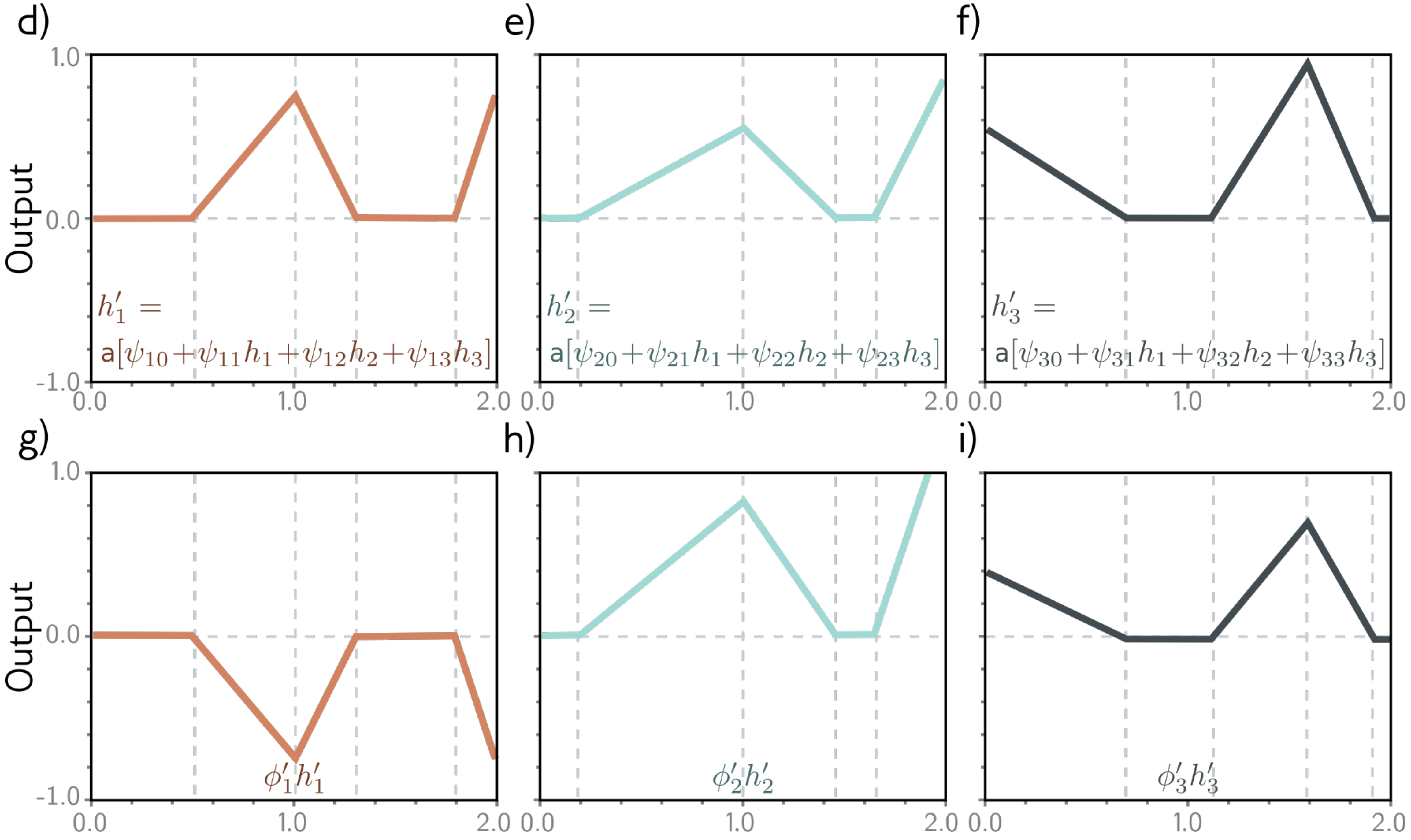


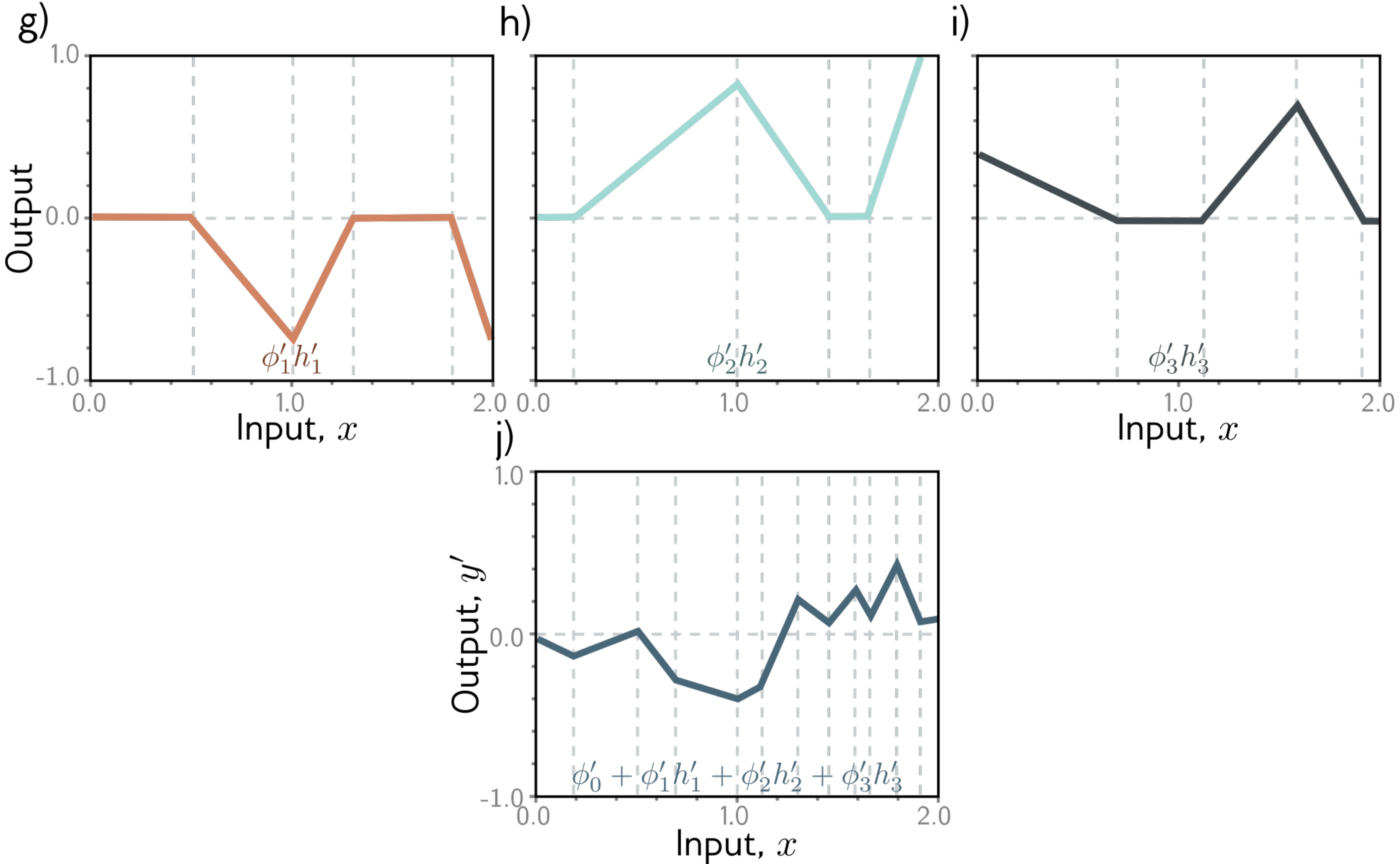


a)
b)
c)
Output
1.0
0.0
-1.0
0.0
1.0
2.0
$\psi_{10}+\psi_{11}h_1+\psi_{12}h_2+\psi_{13}h_3$
$\psi_{20}+\psi_{21}h_1+\psi_{22}h_2+\psi_{23}h_3$
$\psi_{30}+\psi_{31}h_1+\psi_{32}h_2+\psi_{33}h_3$



a)
b)
c)
d)
e)
f)
Output
1.0
0.0
-1.0
2.0
$\psi_{10}+\psi_{11}h_1+\psi_{12}h_2+\psi_{13}h_3$
$\psi_{20}+\psi_{21}h_1+\psi_{22}h_2+\psi_{23}h_3$
$\psi_{30}+\psi_{31}h_1+\psi_{32}h_2+\psi_{33}h_3$
$h'_1 =$
a$[\psi_{10}+\psi_{11}h_1+\psi_{12}h_2+\psi_{13}h_3]$
$h'_2 =$
a$[\psi_{20}+\psi_{21}h_1+\psi_{22}h_2+\psi_{23}h_3]$
$h'_3 =$
a$[\psi_{30}+\psi_{31}h_1+\psi_{32}h_2+\psi_{33}h_3]$


d)
e)
f)
g)
h)
i)
Output
Output
1.0
0.0
-1.0
2.0
$h'_1 =$
$\mathrm{a}[\psi_{10}+\psi_{11}h_1+\psi_{12}h_2+\psi_{13}h_3]$
$h'_2 =$
$\mathrm{a}[\psi_{20}+\psi_{21}h_1+\psi_{22}h_2+\psi_{23}h_3]$
$h'_3 =$
$\mathrm{a}[\psi_{30}+\psi_{31}h_1+\psi_{32}h_2+\psi_{33}h_3]$
$\phi'_1h'_1$
$\phi'_2h'_2$
$\phi'_3h'_3$



g)
h)
i)
j)
1.0
0.0
-1.0
Output
Input, x
2.0
$\phi'_1 h'_1$
$\phi'_2 h'_2$
$\phi'_3 h'_3$
Output, y'
$\phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3$

Deep neural networks

- Composing two networks
- Combining the two networks into one
- Hyperparameters
- Notation change and general case
- Shallow vs. deep networks

Hyperparameters

- K layers = depth of network
- hidden units per layer = width of network

- These are called hyperparameters – chosen before training the network
- Can try retraining with different hyperparameters – hyperparameter optimization or hyperparameter search

Deep neural networks

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Notation change #1

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix} + \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} x\right]$$

Notation change #1

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$\longrightarrow$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix} + \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} x\right]$$

$$h'_1 = \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3]$$
$$h'_2 = \mathrm{a}[\psi_{20} + \psi_{21}h_1 + \psi_{22}h_2 + \psi_{23}h_3]$$
$$h'_3 = \mathrm{a}[\psi_{30} + \psi_{31}h_1 + \psi_{32}h_2 + \psi_{33}h_3]$$

$$\longrightarrow$$

$$\begin{bmatrix} h'_1 \\ h'_2 \\ h'_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \psi_{10} \\ \psi_{20} \\ \psi_{30} \end{bmatrix} + \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}\right]$$

Notation change #1

$$\begin{aligned} h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\ h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\ h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x] \end{aligned} \longrightarrow \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix} + \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} x\right]$$

$$\begin{aligned} h'_1 &= \mathrm{a}[\psi_{10} + \psi_{11}h_1 + \psi_{12}h_2 + \psi_{13}h_3] \\ h'_2 &= \mathrm{a}[\psi_{20} + \psi_{21}h_1 + \psi_{22}h_2 + \psi_{23}h_3] \\ h'_3 &= \mathrm{a}[\psi_{30} + \psi_{31}h_1 + \psi_{32}h_2 + \psi_{33}h_3] \end{aligned} \longrightarrow \begin{bmatrix} h'_1 \\ h'_2 \\ h'_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \psi_{10} \\ \psi_{20} \\ \psi_{30} \end{bmatrix} + \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{32} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}\right]$$

$$y' = \phi'_0 + \phi'_1 h'_1 + \phi'_2 h'_2 + \phi'_3 h'_3 \longrightarrow y' = \phi'_0 + \begin{bmatrix} \phi'_1 & \phi'_2 & \phi'_3 \end{bmatrix} \begin{bmatrix} h'_1 \\ h'_2 \\ h'_3 \end{bmatrix}$$

Notation change #2

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \theta_{10} \\ \theta_{20} \\ \theta_{30} \end{bmatrix} + \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} x\right] \longrightarrow \mathbf{h} = \mathbf{a}\left[\boldsymbol{\theta}_0 + \boldsymbol{\theta} x\right]$$

$$\begin{bmatrix} h'_1 \\ h'_2 \\ h'_3 \end{bmatrix} = \mathbf{a}\left[\begin{bmatrix} \psi_{10} \\ \psi_{20} \\ \psi_{30} \end{bmatrix} + \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{32} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}\right] \longrightarrow \mathbf{h}' = \mathbf{a}\left[\boldsymbol{\psi}_0 + \boldsymbol{\Psi}\mathbf{h}\right]$$

$$y' = \phi'_0 + \begin{bmatrix} \phi'_1 & \phi'_2 & \phi'_3 \end{bmatrix} \begin{bmatrix} h'_1 \\ h'_2 \\ h'_3 \end{bmatrix} \longrightarrow y = \boldsymbol{\phi}'_0 + \boldsymbol{\phi}'\mathbf{h}'$$

Notation change #3

$$\mathbf{h} = \mathbf{a}\left[\boldsymbol{\theta}_0 + \boldsymbol{\theta}x\right] \longrightarrow \mathbf{h}_1 = \mathbf{a}[\boldsymbol{\beta}_0 + \boldsymbol{\Omega}_0\mathbf{x}]$$

$$\mathbf{h}' = \mathbf{a}\left[\boldsymbol{\psi}_0 + \boldsymbol{\Psi}\mathbf{h}\right] \longrightarrow \mathbf{h}_2 = \mathbf{a}[\boldsymbol{\beta}_1 + \boldsymbol{\Omega}_1\mathbf{h}_1]$$

$$y = \phi'_0 + \boldsymbol{\phi}'\mathbf{h}' \longrightarrow \mathbf{y} = \boldsymbol{\beta}_2 + \boldsymbol{\Omega}_2\mathbf{h}_2$$

Notation change #3

Bias vector

Weight matrix

$$\mathbf{h} = \mathbf{a}[\theta_0 + \boldsymbol{\theta} x] \longrightarrow \mathbf{h}_1 = \mathbf{a}[\boldsymbol{\beta}_0 + \boldsymbol{\Omega}_0 \mathbf{x}]$$

$$\mathbf{h}' = \mathbf{a}[\psi_0 + \boldsymbol{\Psi}\mathbf{h}] \longrightarrow \mathbf{h}_2 = \mathbf{a}[\boldsymbol{\beta}_1 + \boldsymbol{\Omega}_1 \mathbf{h}_1]$$

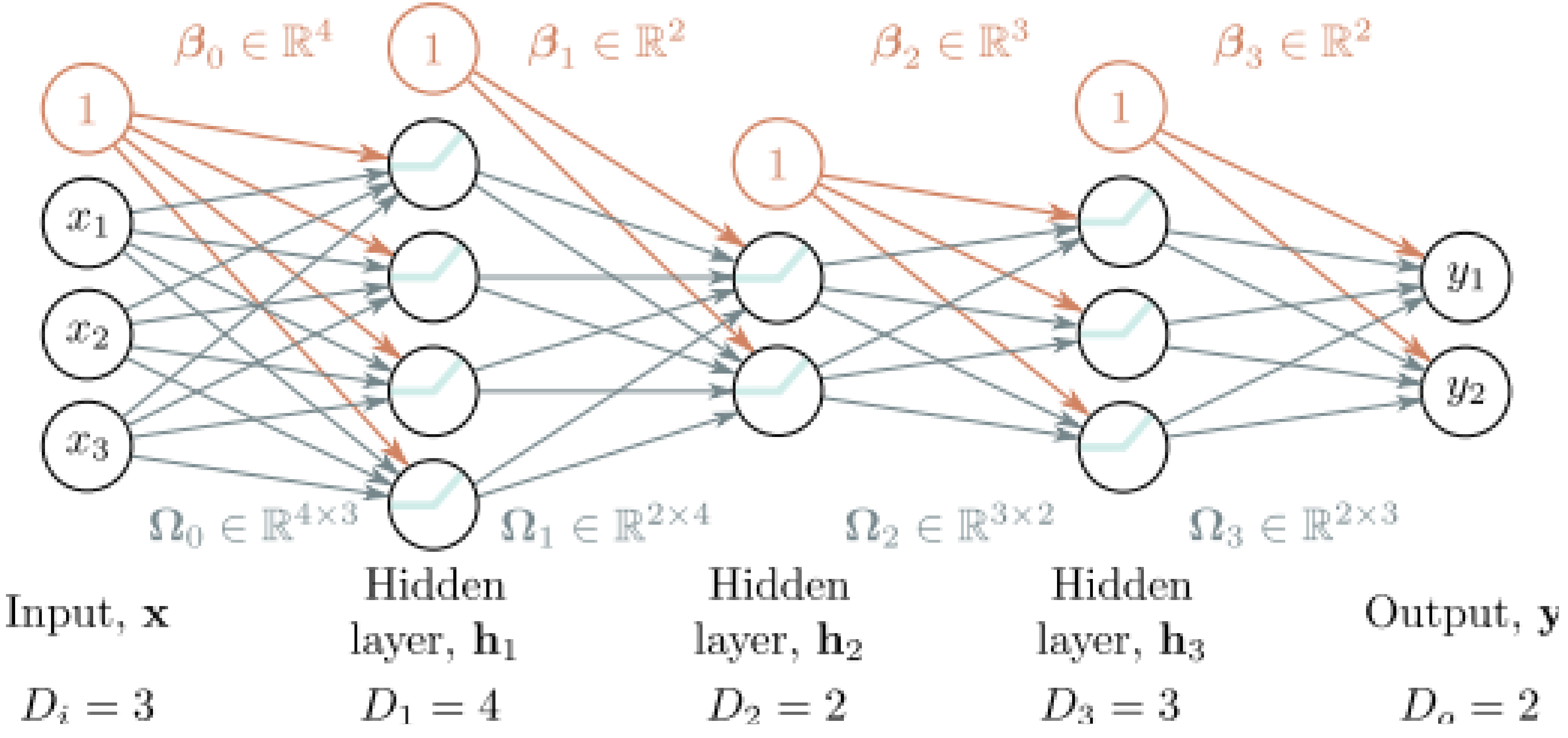
$$y = \phi'_0 + \phi' \mathbf{h}' \longrightarrow \mathbf{y} = \boldsymbol{\beta}_2 + \boldsymbol{\Omega}_2 \mathbf{h}_2$$

General equations for deep network

$$
\begin{aligned}
\mathbf{h}_1 &= \mathbf{a}[\boldsymbol{\beta}_0 + \boldsymbol{\Omega}_0 \mathbf{x}] \\
\mathbf{h}_2 &= \mathbf{a}[\boldsymbol{\beta}_1 + \boldsymbol{\Omega}_1 \mathbf{h}_1] \\
\mathbf{h}_3 &= \mathbf{a}[\boldsymbol{\beta}_2 + \boldsymbol{\Omega}_2 \mathbf{h}_2] \\
&\vdots \\
\mathbf{h}_K &= \mathbf{a}[\boldsymbol{\beta}_{K-1} + \boldsymbol{\Omega}_{K-1} \mathbf{h}_{K-1}] \\
\mathbf{y} &= \boldsymbol{\beta}_K + \boldsymbol{\Omega}_K \mathbf{h}_K,
\end{aligned}
$$

$$
\mathbf{y} = \boldsymbol{\beta}_K + \boldsymbol{\Omega}_K \mathbf{a}\left[\boldsymbol{\beta}_{K-1} + \boldsymbol{\Omega}_{K-1} \mathbf{a}\left[\ldots \boldsymbol{\beta}_2 + \boldsymbol{\Omega}_2 \mathbf{a}\left[\boldsymbol{\beta}_1 + \boldsymbol{\Omega}_1 \mathbf{a}\left[\boldsymbol{\beta}_0 + \boldsymbol{\Omega}_0 \mathbf{x}\right]\right] \ldots\right]\right]
$$

Example



$\beta_0 \in \mathbb{R}^4$
$\beta_1 \in \mathbb{R}^2$
$\beta_2 \in \mathbb{R}^3$
$\beta_3 \in \mathbb{R}^2$
1
x_1
x_2
x_3
y_1
y_2
$\Omega_0 \in \mathbb{R}^{4\times 3}$
$\Omega_1 \in \mathbb{R}^{2\times 4}$
$\Omega_2 \in \mathbb{R}^{3\times 2}$
$\Omega_3 \in \mathbb{R}^{2\times 3}$
Input, $\mathbf{x}$
Hidden layer, $\mathbf{h}_1$
Hidden layer, $\mathbf{h}_2$
Hidden layer, $\mathbf{h}_3$
Output, $\mathbf{y}$
$D_i = 3$
$D_1 = 4$
$D_2 = 2$
$D_3 = 3$
$D_o = 2$

Deep neural networks

- Composing two networks
- Combining the two networks into one
- Hyperparameters
- Notation change and general case
- Shallow vs. deep networks

Shallow vs. deep networks

The best results are created by deep networks with many layers.

- 50-1000 layers for most applications
- Best results in
  - Computer vision
  - Natural language processing
  - Graph neural networks
  - Generative models
  - Reinforcement learning

All use deep networks.
But why?

Shallow vs. deep networks

1. Ability to approximate different functions?

Both obey the universal approximation theorem.

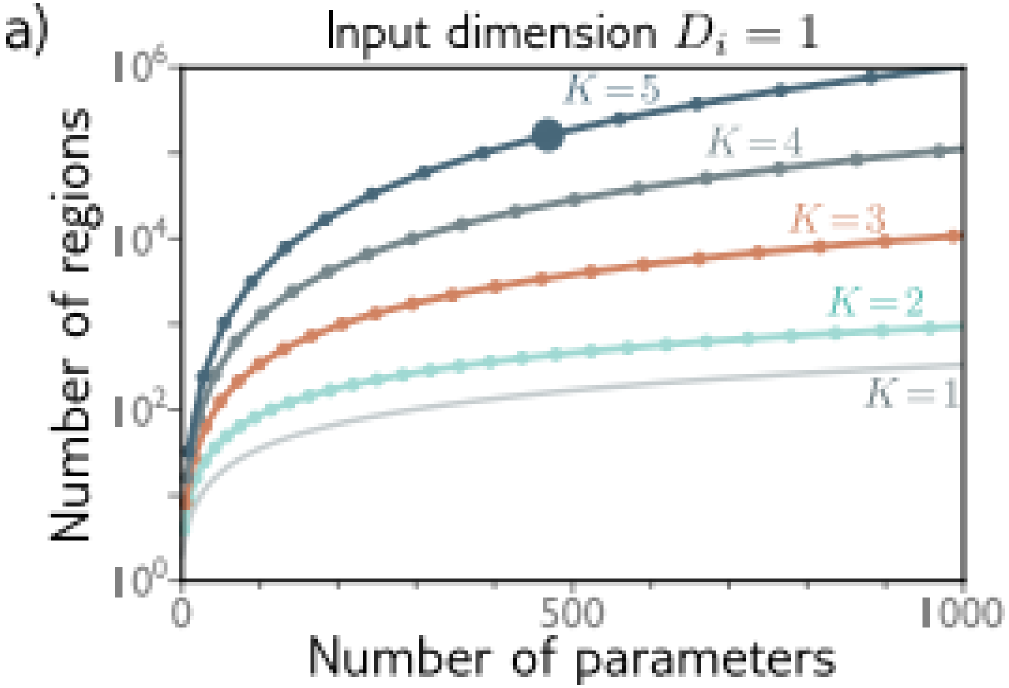
Argument: One layer is enough, and for deep networks could arrange for the other layers to compute the identity function.

Shallow vs. deep networks

2. Number of linear regions per parameter

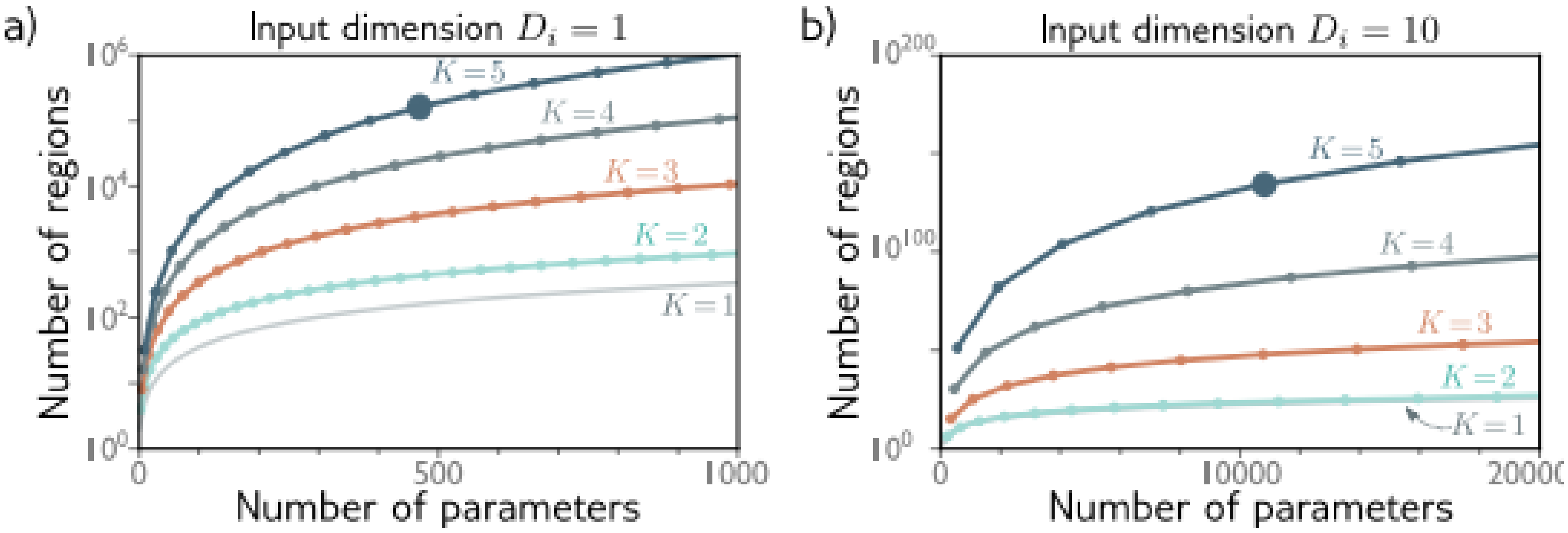
- Deep networks create many more regions per parameters
- But there are dependencies between them
  - Think of folding example
  - Perhaps similar symmetries in real-world functions? Unknown

Number of linear regions per parameter



5 layers
10 hidden units per layer
471 parameters
161,501 linear regions

Number of linear regions per parameter



5 layers
10 hidden units per layer
471 parameters
161,501 linear regions

5 layers
50 hidden units per layer
10,801 parameters
> linear regions

Shallow vs. Deep Networks

3. Depth efficiency

- There are some functions that require a shallow network with exponentially more hidden units than a deep network to achieve an equivalent approximation

- This is known as the depth efficiency of deep networks

- But do the real-world functions we want to approximate have this property? Unknown.

Shallow vs. Deep Networks

4. Large structured networks

- Think about images as input – might be 1M pixels
- Fully connected works not practical
- Answer is to have weights that only operate locally, and share across image
- This leads to convolutional networks
- Gradually integrate information from across the image – needs multiple layers

Shallow vs. Deep Networks

5. Fitting and generalization

- Fitting of deep models seems to be easier up to about 20 layers
- Then needs various tricks to train deeper networks, so (in vanilla form), fitting becomes harder
- Generalization is good in deep networks. Why?

Shallow vs. Deep Networks

5. Fitting and generalization

- Fitting of deep models seems to be easier up to about 20 layers
- Then needs various tricks to train deeper networks, so (in vanilla form), fitting becomes harder
- Generalization is good in deep networks. Why?

Where are we going?

- We have defined families of very flexible networks that map multiple inputs to multiple outputs
- Now we need to train them
  - How to choose loss functions
  - How to find minima of the loss function
  - How to do this in particular for deep networks
- Then we need to test them