Habib University
Course Code: EE 468/CE 468: Mobile Robotics
Course Title: Mobile Robotics
Instructor name: Dr. Basit Memon
Examination: Quiz 2 Exam Date: November 19, 2022
Total Marks: 100 Duration: 50 minutes

Instructions

1. You can refer to course slides, your notes, homework solutions, or any of the books included in the course syllabus. You're not permitted to specifically search for responses to any of the exam questions online. Where appropriate, you can cite the slides and don't have to redo what has already been done.
2. The exam will be administered under HU student code of conduct (see Chapter 3 of `https://habibuniversity.sharepoint.com/sites/Student/code-of-conduct`). You may not talk, discuss, compare, copy from, or consult with any other human on this planet (except me) on questions or your responses during the exam.
3. Make sure that you show your work (intermediate steps). Most of the points for any question will be given based on the followed process.
4. Kindly use a black/blue pen to write your hand-written solutions, so that they are legible.

Questions

Problem 1
85 points

Consider a robot that resides in a circular world consisting of ten different places that are numbered counterclockwise. The robot is unable to sense the number of its present place directly. However, places 0, 3, and 6 contain a distinct landmark, whereas all other places do not. All three of these landmarks look alike. The likelihood that the robot observes the landmark given it is in one of these places is 0.8. For all other places, the likelihood of observing the landmark is 0.4. For each place on the circle compute the probability that the robot is in that place given that the following sequence of actions is carried out deterministically and the following sequence of observations is obtained: The robot detects a landmark, moves 3 grid cells counterclockwise and detects a landmark, and then moves 4 grid cells counterclockwise and finally perceives no landmark.

Solution 1

We can use Markov grid-based localization here. Let x_k be the location of the robot at step k.

$$x_k \in \{0, 1, \cdots, 9\}.$$

Initially, we have no information about the place of the robot. So the prior belief is:

$$\overline{bel}(x_0 = i) = \frac{1}{10}, \quad \forall i \in \{0, 1, \cdots, 9\}.$$

The measurement at step k is denoted by $z_k \in \{0, 1\}$, where $z_k = 1$ indicates that a landmark was detected. We're provided that

$$p(z_k = 1|x_k) = \begin{cases} 0.8 & \text{if } x_k \in \{0, 3, 6\} \\ 0.4 & \text{otherwise.} \end{cases}$$

Correspondingly,

$$p(z_k = 0|x_k) = \begin{cases} 0.2 & \text{if } x_k \in \{0, 3, 6\} \\ 0.6 & \text{otherwise.} \end{cases}$$

Our motion model is deterministic. If u is the number of steps, then our motion model can be described as:

$$p(x_k|x_{k-1}, u_k) = \begin{cases} 1 & \text{if } x_k = x_{k-1} + u_k (\text{mod } 10) \\ 0 & \text{otherwise.} \end{cases}$$

Let's iterate through the measurement and actions sequences step by step. Given that $z_0 = 1$,

$$\begin{aligned} bel(x_0) &= \eta p(z_0|x_0)\overline{bel}(x_0) \\ &= \eta \begin{cases} 0.8/10 & \text{if } x_0 \in \{0, 3, 6\} \\ 0.4/10 & \text{otherwise.} \end{cases} \\ \Rightarrow \eta &= 1/\sum_{x_0} bel(x_0) = 1/0.52 \end{aligned}$$

Next, the robot has moved three steps counter-clockwise. So, our predicted belief at next time step is

$$\begin{aligned} \overline{bel}(x_1) &= \sum_{x_0} p(x_1|x_0, u_1 = 3) bel(x_0) \\ &= bel(x_0 = x_1 - 3) \\ &= \begin{cases} 0.154 & \text{if } x_1 \in \{3, 6, 9\} \\ 0.077 & \text{otherwise.} \end{cases} \end{aligned}$$

A landmark is detected. So, we have

$$\begin{aligned}
bel(x_1) &= \eta p(z_1 = 1|x_1)\,\overline{bel}(x_1) \\
&= \eta \left(\begin{cases} 0.8 & \text{if } x_1 \in \{0,3,6\} \\ 0.4 & \text{otherwise.} \end{cases} \right) \left(\begin{cases} 0.154 & \text{if } x_1 \in \{3,6,9\} \\ 0.077 & \text{otherwise.} \end{cases} \right) \\
&= \eta \begin{cases} 0.123 & \text{if } x_1 \in \{3,6\} \\ 0.062 & \text{if } x_1 \in \{0,9\} \\ 0.031 & \text{otherwise.} \end{cases} \\
\Rightarrow \eta &= 1/\sum_{x_0} bel(x_0) = 1/0.56
\end{aligned}$$

We move four grid cells counterclockwise. So, our predicted belief at next time step is

$$\begin{aligned}
\overline{bel}(x_2) &= \sum_{x_1} p(x_2|x_1, u_1 = 4) bel(x_1) \\
&= bel(x_1 = x_2 - 4) \\
&= \begin{cases} 0.22 & \text{if } x_2 \in \{7,0\} \\ 0.111 & \text{if } x_2 \in \{4,3\} \\ 0.055 & \text{otherwise.} \end{cases}
\end{aligned}$$

A landmark is not detected. So, we have

$$\begin{aligned}
bel(x_2) &= \eta p(z_2 = 0|x_2)\,\overline{bel}(x_2) \\
&= \eta \left(\begin{cases} 0.2 & \text{if } x_2 \in \{0,3,6\} \\ 0.6 & \text{otherwise.} \end{cases} \right) \left(\begin{cases} 0.22 & \text{if } x_2 \in \{7,0\} \\ 0.111 & \text{if } x_2 \in \{4,3\} \\ 0.055 & \text{otherwise.} \end{cases} \right) \\
&= \eta \begin{cases} 0.044 & \text{if } x_2 = 0 \\ 0.022 & \text{if } x_2 = 3 \\ 0.067 & \text{if } x_2 = 4 \\ 0.011 & \text{if } x_2 = 6 \\ 0.132 & \text{if } x_2 = 7 \\ 0.033 & \text{otherwise.} \end{cases} \\
\Rightarrow \eta &= 1/\sum_{x_0} bel(x_0) = 1/0.441
\end{aligned}$$

Consider the polar to Cartesian coordinate transformations: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Assume that r and θ are corrupted by additive Gaussian noise, ϵ_r and ϵ_θ with a zero mean and standard deviations of 0.01 meters and 0.3 rads respectively. It is evident that the vector $\rho = \begin{bmatrix} r & \theta \end{bmatrix}^T$ is normally distributed.

Problem 2
15 points

(a) Is the vector $\alpha = \begin{bmatrix} x & y \end{bmatrix}^T$ also normally distributed? Justify.

(b) Approximate the mean of the vector α, when reference point is $r = 1$ and $\theta = 0$.

(c) Assume that r and θ are uncorrelated. Given the covariance of the vector ρ, approximate the covariance matrix of the vector α.

Solution 2 Using Slide 30-35 from Deck 08,

(a) No, in general α will not be normally distributed as transformation is nonlinear.

(b) Let $\alpha = f(r, \theta)$, where $f = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}$.

$$\begin{aligned} \mu_\alpha &= f(1, 0) \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

(c) The covariance matrix for ρ is

$$\Sigma_\rho = \begin{bmatrix} (0.01)^2 & 0 \\ 0 & (0.3)^2 \end{bmatrix}.$$

The covariance for α is approximately,

$$\Sigma_\alpha = J\Sigma_\rho J^T$$

where

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}_{(r,\theta)=(1,0)} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \Rightarrow \Sigma_\alpha &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (0.01)^2 & 0 \\ 0 & (0.3)^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$