Habib University - City Campus
Instructors: Aeyaz Jamil Keyani
Course: MATH 307 Mathematical Foundations and Reasoning
Examination: Quiz 1 – Spring 2025
Exam Date: Thursday, January 15, 2025
Exam Time: 10:05 – 10:20

Total Marks: 10 Marks | Duration: 15 Minutes

عشرتِ قطرہ ہے دریا میں فنا ہو جانا
درد کا حد سے گزرنا ہے دوا ہو جانا

(مرزا غالب)

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1. Show that if $n \in \mathbb{N}$ and $q \in \mathbb{N}$ such that $q \neq 0$, then there exits $m \in \mathbb{N}$ and $r \in \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

Solution: Let $q \in \mathbb{Z}^+$, we will show by induction that for any $n \in \mathbb{N}$ there exists $m, r \in \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

Base case: $n = 0$

We have that $0 = 0 + 0$, we know that $0 \times q = 0$ and so $0 = 0 \times q + 0$. Here $m = 0$ and $r = 0$, and we have that $0 \leq r < q$ and $n = mq + r$. So the base case holds.

Induction hypothesis: Suppose for $n = k$, there exists $m, r \in \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

Inductive step: We show that for $n = k++$, there exists $m', r' \in \mathbb{N}$ such that $0 \leq r < q$ and $n = k++ = m'q + r'$.

We have that $k{+}{+} = k{+}1$, from inductive hypothesis, $k{+}{+} = k{+}1 = mq{+}r{+}1 = mq{+}(r{+}1)$ (by using associativity). As $0 \leq r < q$ then $r + 1 \leq q$. If $r + 1 < q$ then we are done as we have found $m' = m$ and $r' = r + 1$ that satisfies our conditions. If $r + 1 = q$ then we have that $k{+}{+} = mq + q = (m+1)q = (m+1)q + 0$ (by using distributivity). In this case we have found our $m' = m + 1$ and $r' = 0$ such that $0 \leq r < q$ and $n = k++ = m'q + r'$.

And therefore by the principle of mathematical induction we have that, if $n, q \in \mathbb{N}$ such that $q \neq 0$, then there exits $m, r \in \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

$\square$

2. Show that for sets A, B and X, such that $A \cup B = X$ and $A \cap B = \emptyset$, then $A = X \setminus B$ and $B = X \setminus A$.

Solution: By definition $X \setminus B = \{x \in X | x \notin B\}$, and $A \cup B = \{x | x \in A \vee x \in B\} = X = \{x \in X\}$.

Let $x \in X \setminus B \implies x \in X \wedge x \notin B \implies x \in A \cup B \wedge x \notin B \implies (x \in A \vee x \in B) \wedge x \notin B \implies x \in A \wedge x \notin B$. As $A \cap B = \emptyset$, we have for any object x, $x \notin A \cap B$, therefore there does not exist an object x such that $x \in A$ and $x \in B$. And so an object x is in A iff x is not in B. Therefore $\forall x \in A$, $x \in A \wedge x \notin B$. And so from above we have that $x \in X \setminus B \implies x \in A \wedge x \notin B \implies x \in A$ and so $X \setminus B \subseteq A$.

Conversely, let $x \in A$, then we have that $x \in A \wedge x \notin B$, as $A \cup B = X$, we have $\forall x \in A \cup B,\ x \in X$, and $\forall x \in A,\ x \in A \cup B$, we have that $A \subseteq X$. So we have that $x \in A \implies x \in A \wedge x \notin B \implies x \in X \wedge x \notin B \implies x \in X \setminus B$. So we have that $A \subseteq X \setminus B$. And therefore $A = X \setminus B$.

As set union is commutative, the same argument follows for $B = X \setminus A$ by switching A and B.

$\square$

3. Show that the axiom of replacement implies the axiom of specification.

Solution: Let A be some set and let $P(x)$ be some property pertaining to elements of A. Then the set set $X = \{x \in A | P(x)\}$ can be constructed by axiom of replacement as follow: $X = \{y | \exists x \in A,\ Q(x, y)\}$, where $Q(x, y) : x = y$ and $P(x)$, as we have at most one y for each $x \in A$.

$\square$