Maximum Bipartite Matching

CS-6th

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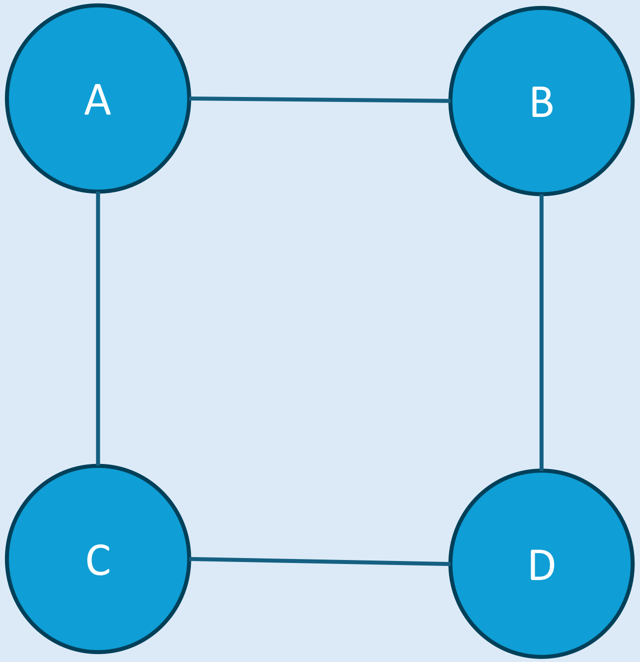
Bipartite Graphs

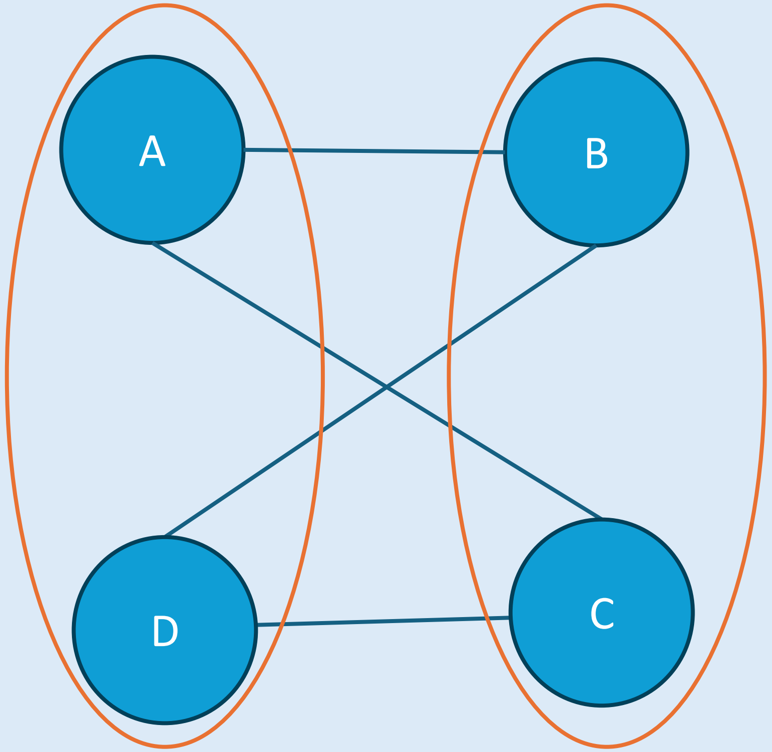
- A simple graph G = (V, E) is bipartite if its vertex set V can be partitioned into two disjoint sets V1 and V2, such that, every edge in the graph connects a vertex v1 in V1 to a vertex v2 in V2 (no edge connects either two vertices in V1 or two vertices in V2). G = (V1, V2, E) is then a bipartite graph.
- We call the pair (V1, V2) a bi-partition of the vertex set V.

Credit: Definition from Dr. Shah Jamal

Example

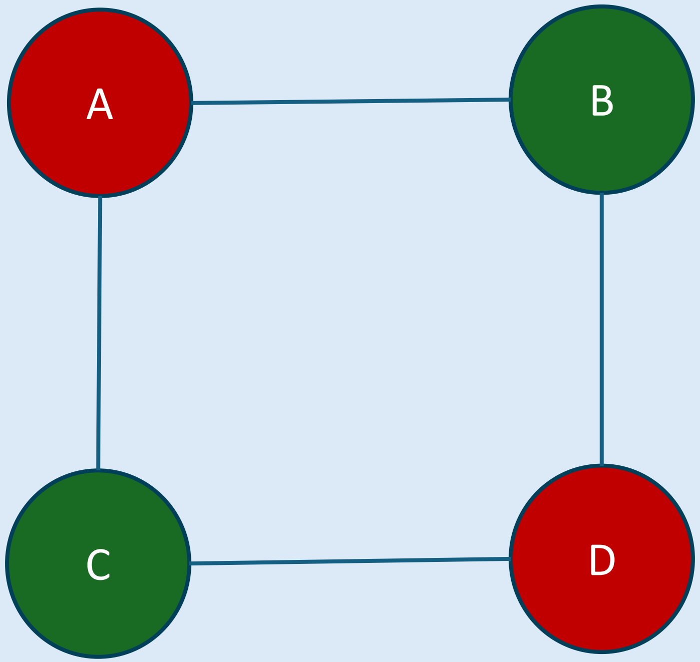
- In this graph, the vertices can be divided into two disjoint sets, {A, D} and {B, C}, such that every edge connects a vertex in one set to a vertex in the other set.



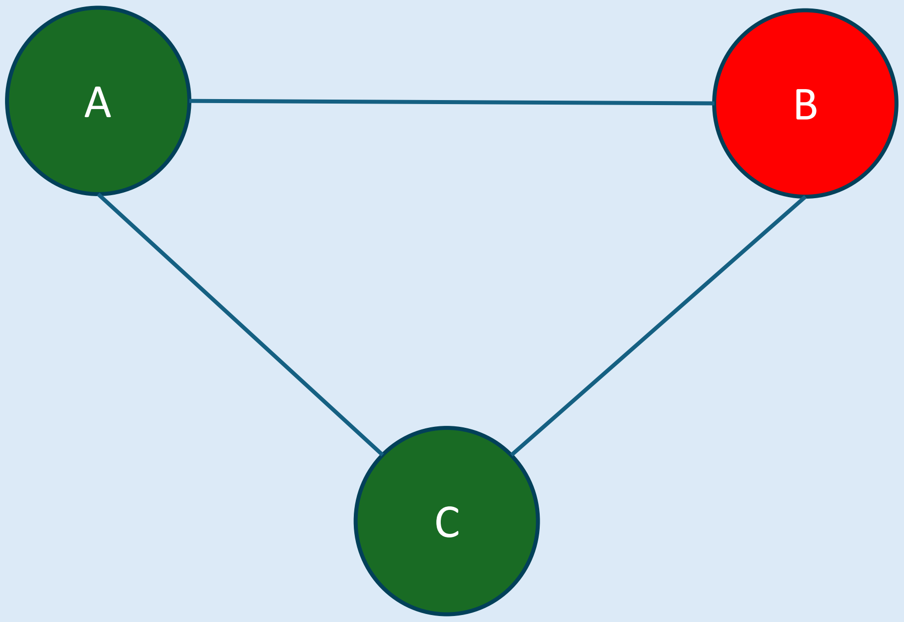


Graph Coloring (biochromatic)

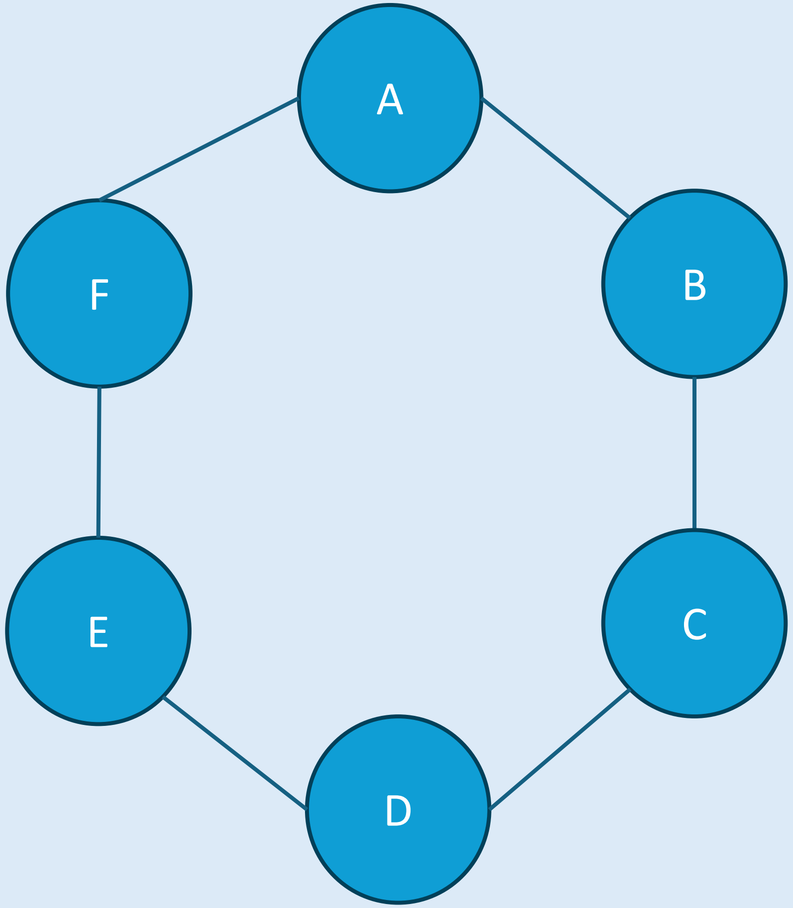
- The problem can be reduced to graph coloring.
- No two adjacent vertices can have same color.



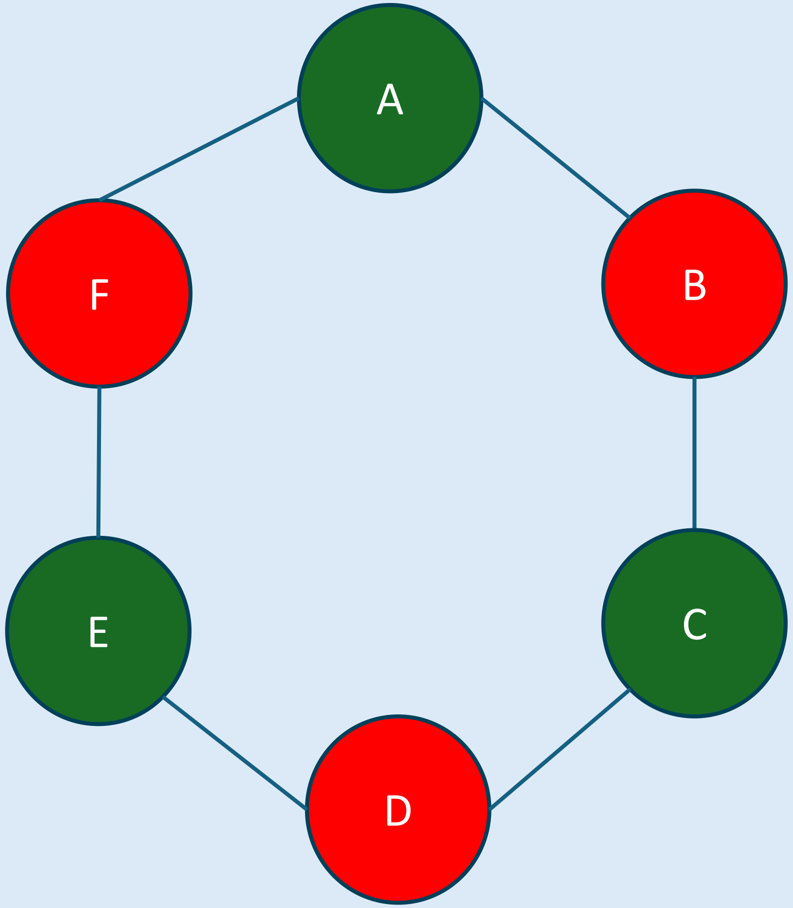
Not a bipartite graph!



Is this a bipartite graph?



Is this a bipartite graph? Yes!



Exercise

- Write down an algorithm that can be used for biochromatic graph coloring.

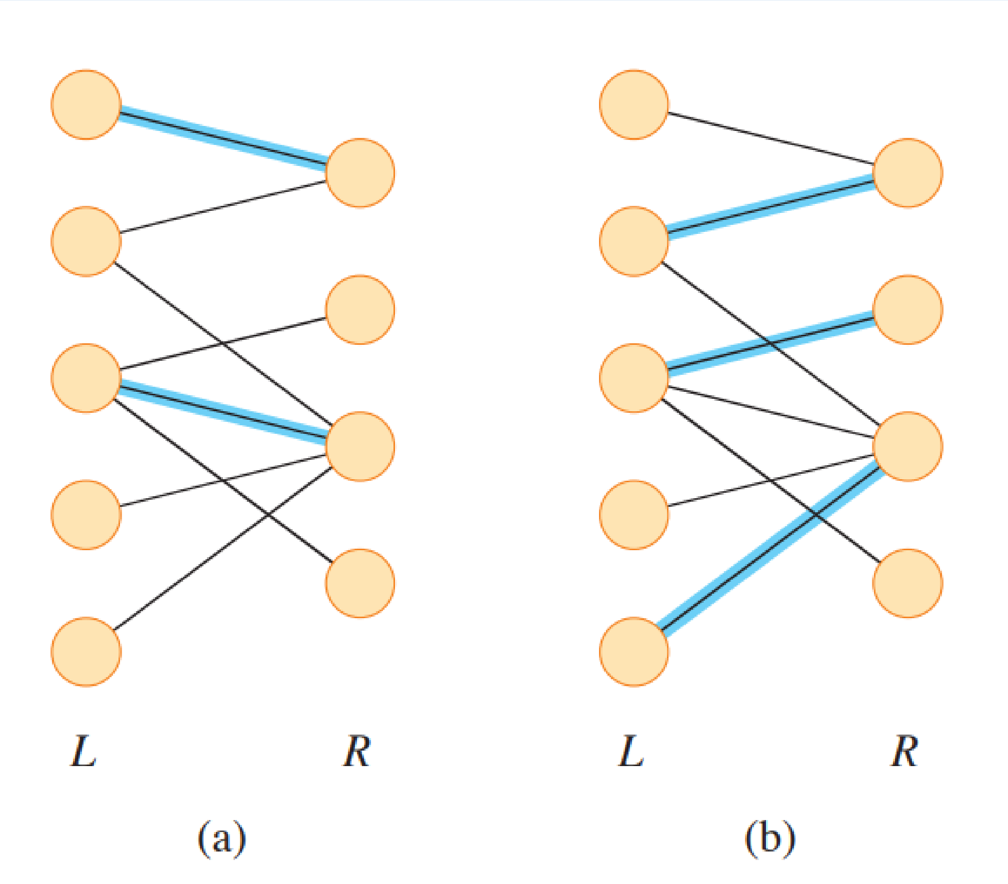
Exercise

- Can a bipartite graph have cycle of odd number?

Maximum Bipartite Matching

Formal Definition:

- Given an undirected graph G=(V,E), a matching is a subset of edges M∈E such that for all vertices v ∈ V , at most one edge of M is incident on v. We say that a vertex v ∈ V is matched by the matching M if some edge in M is incident on v, and otherwise, v is unmatched.
- A maximum matching is a matching of maximum cardinality, that is, a matching M such that for any matching M' , we have |M|>=|M'|.



Cardinality 2

Cardinality 3

Ford-Fulkerson (Max-Bipartite Matching)

- The Ford-Fulkerson method provides a basis for finding a maximum matching in an undirected bipartite graph G=(V,E) in time polynomial in |V| and |E|. The trick is to construct a flow network in which flows correspond to matchings.
- We define the corresponding flow network G=(V,E) for the bipartite graph G as follows:
  - Let the source s and sink t be new vertices not in V , and let V'=V U {s,t}

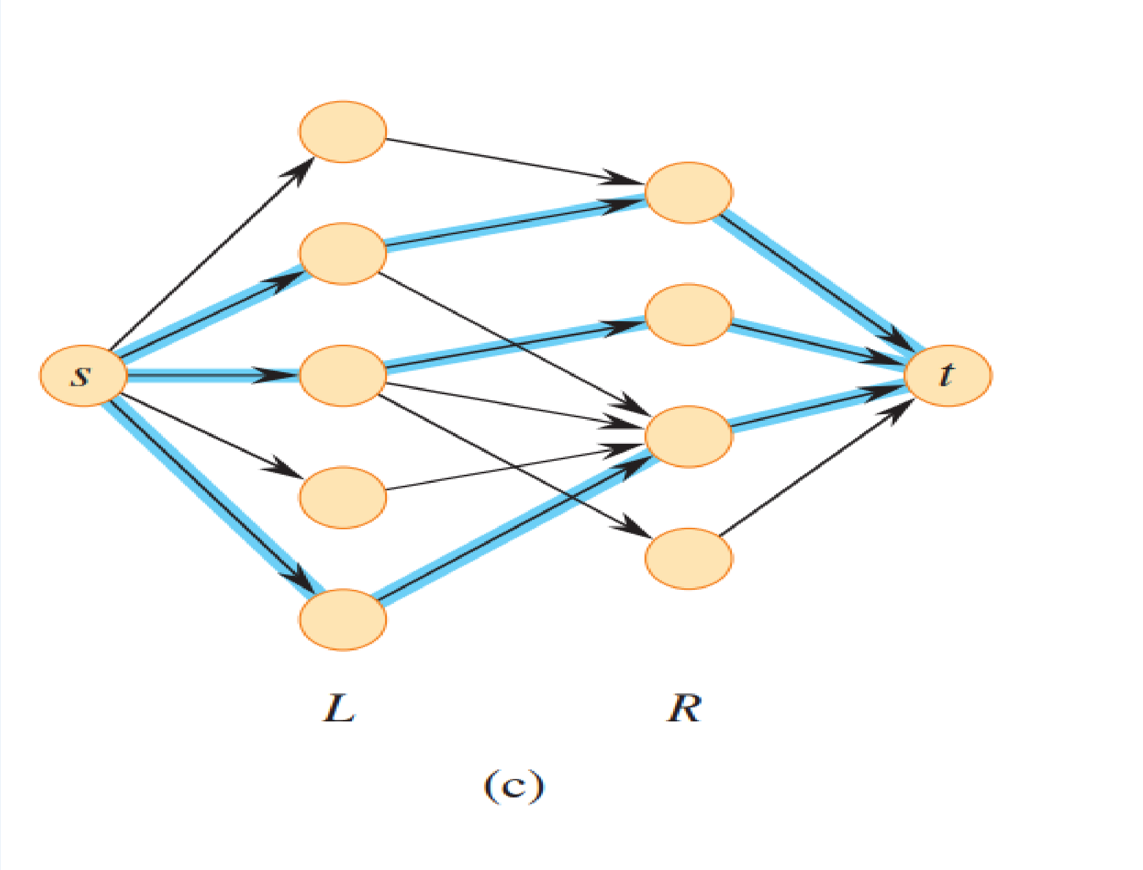
$$
\begin{aligned}
E' = \; & \{(s,u) : u \in L\} \\
& \cup \{(u,v) : u \in L, v \in R, \text{ and } (u,v) \in E\} \\
& \cup \{(v,t) : v \in R\} \; .
\end{aligned}
$$

- To complete the construction, assign unit capacity to each edge in E' .

$$f(u,v) = \begin{cases} 1 & \text{if } (u,v) \in M \ , \\ 0 & \text{if } (u,v) \notin M \ . \end{cases}$$

Mapping to a Flow network

- Max-flow= Max-matching



Time complexity

- Thus, to find a maximum matching in a bipartite undirected graph G, create the flow network G' , run the Ford-Fulkerson method on G' , and convert the integer valued maximum flow found into a maximum matching for G. Since any matching in a bipartite graph has cardinality at most $\min\{|L|,|R|\}=O|V|$, the value of the maximum flow in G' is O(V) Therefore, finding a maximum matching in a bipartite graph takes $O(V.E')=O(VE)$ time, sine $|E'|=O(E)$.