Final Exam (Spring 2023): Solution

Intro to Probability and Statistics - EE 354 / CE 361 / MATH 310

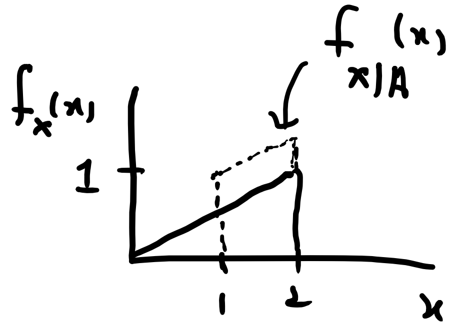
Question 1 - (10 points)

Let X be a random variable with the PDF:

$$f_X(x) = \begin{cases} \frac{x}{2} & if\ 0 \le x \le 2 \\ 0, & otherwise \end{cases}$$

Let A be the event $\{X \ge 1\}$. Let $Y = X^2$. Find the following:

i. $E[X]$ **[2]**
ii. $E[X|A]$ **[3]**
iii. $Var\,[X]$? **[2]**
iv. $Var\,[Y]$? **[3]**



(i)

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^2 x \cdot \frac{x}{2} \cdot dx = \int_0^2 \frac{x^2}{2} \cdot dx = \frac{x^3}{6}\Big|_0^2$$

$$= \frac{8}{6} = 1.33 \text{ — Ans}$$

(ii)

$$P(A) = P(\{X \ge 1\}) = \int_1^{\infty} f_X(x) \cdot dx = \int_1^2 \frac{x}{2} \cdot dx$$

$$= \frac{x^2}{4}\Big|_1^2 = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

$$f_{X|A} = \begin{cases} \frac{f_X(x)}{P(A)} & x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{x}{1.5} & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|A] = \int_{-\infty}^{\infty} x \cdot f_{X|A}(x) = \int_{1}^{2} x \cdot \frac{x}{1.5} \, dx = \int_{1}^{2} \frac{x^2}{1.5} dx$$

$$= \frac{x^3}{4.5} \Big|_{1}^{2} = \frac{8}{4.5} - \frac{1}{4.5}$$

$$E[X|A] = \frac{7}{4.5} = 1.56 \quad \text{— Ans}$$

(iii)

$$Var(X) = E[X^2] - (E[X])^2$$

By Expected Value Rule,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx$$

$$= \int_{0}^{2} x^2 \cdot \frac{x}{2} = \int_{0}^{2} \frac{x^3}{2} \, dx = \frac{x^4}{8} \Big|_{0}^{2}$$

$$= \frac{16}{8} - 0 = 2$$

$$Var(X) = 2 - (1.33)^2 = 0.23 \quad \text{— Ans}$$

(iv)

$$Y = X^2$$

$$Var(Y) = E[Y^2] - (E[Y])^2 \quad \text{— (1)}$$

$$E[Y] = E[X^2] = 2$$

$$E[Y^2] = E[X^4] = \int_{-\infty}^{\infty} x^4 \cdot f_X(x) \, dx$$

$$= \int_0^2 x^4 \cdot \frac{x}{2} \cdot dx = \int_0^2 \frac{x^5}{2} \cdot dx = \frac{x^6}{12}\Big|_0^2$$

$$= \frac{2^6}{12} = \frac{64}{12}$$

$$E[Y^2] = 5.33$$

From (A) $\quad Var(Y) = 5.33 - 2^2$

$$Var(Y) = 1.33 \text{ ——— Ans}$$

Question 2 - (10 points)

Consider a Bernoulli random variable X with parameter $p = 0.1$. We observe X in the presence of additive noise $N \sim U[-2, 2]$. As a result, our observation is random variable Y = X + N. If our observation is 0.8, what is the "most likely" value of X?
(Note: Provide a justification of your answer based on the various theorems/rules/laws/results discussed in this course. Only intuitive argument is not acceptable for this question and will be given no partial credit)

We can approach this using General Version of Bayes' Rule:

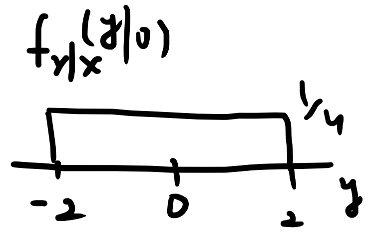
$X \longrightarrow$ $\boxed{\text{Noisy Measurement}}$ $\longrightarrow Y$

Prior Belief:-

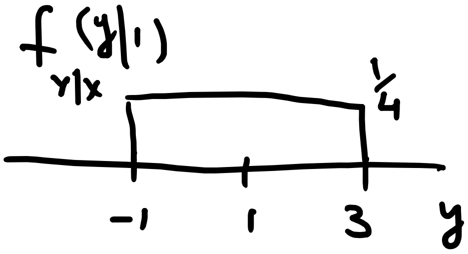
$$p_X(x) = \begin{cases} 0.1 & x = 1 \\ 0.9 & x = 0 \end{cases}$$

Quality of Measurement:-

$$f_{Y|X}(y|0) = \begin{cases} \frac{1}{4} & -2 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$f_{Y|X}(y|1) = \begin{cases} \frac{1}{4} & -1 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



Posterior Belief:-

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{p_X(0) f_{Y|X}(y|0) + p_X(1) f_{Y|X}(y|1)} \quad \text{—(A)}$$

From (A),

$$p_{X|Y}(0|0.8) = \frac{p_X(0)\, f_{Y|X}(0.8|0)}{p_X(0) f_{Y|X}(0.8|0) + p_X(1) f_{Y|X}(0.8|1)}$$

$$= \frac{(0.9)(0.25)}{(0.9)(0.25) + (0.1)(0.25)}$$

$$p_{X|Y}(0|0.8) = 0.9 \quad \text{—(B)}$$

From (A) :-

$$p_{X|Y}(1|0.8) = \frac{p_X(1)\, f_{Y|X}(0.8|1)}{p_X(0) f_{Y|X}(0.8|0) + p_X(1) f_{Y|X}(0.8|1)}$$

$$= \frac{(0.1)(0.25)}{(0.9)(0.25) + (0.1)(0.25)}$$

$$p_{X|Y}(1|0.8) = 0.1 \quad \text{—(C)}$$

Comparing (B) and (C), if the observation is 0.8, the "most likely" value of X is 0.

Question 3 - (10 points)

a) A team-based javelin throw competition involves three athletes from each team. The score of a team is considered to be the maximum of throws achieved by its athletes. Team Pakistan consists of three athletes: Arshad, Akram, and Amjad. All the three athletes can throw between 85m to 95m with all values being equally likely. Throws by each athlete are independent of the other athletes. Let random variable X be the score of Team Pakistan in this competition. What is the PDF of X? **[5]**

b) X and Y are independent continuous random variables that are uniformly distributed with $X \sim U[0,1]$ and $Y \sim U[0,2]$. Find the following:

i. $f_{X,Y}(1.5,1.5)$ **[1]**

ii. $F_{X,Y}(1.5,1.5)$ **[2]**

iii. $E[\,Y|X < 0.5\,]$ **[2]**

(a)

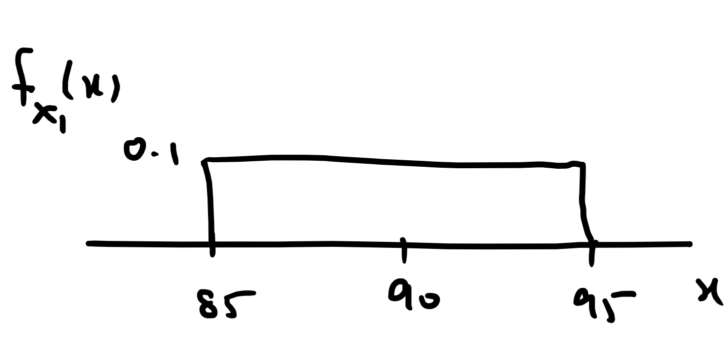
The following strategy can be used:

A. Calculate CDF of X.

B. CDF $\longrightarrow$ PDF

Let X_1, X_2, X_3 be the throws by Arshad, Akram, and Amjad respectively.

$$f_{X_1}(x) = f_{X_2}(x) = f_{X_3}(x) = \begin{cases} 0.1 & 85 \leq x \leq 95 \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = P(X \leq x)$$

$$= P(X_1 \leq x, X_2 \leq x\ X_3 \leq x)$$

Due to Independence:

$$= P(X_1 \le x) \cdot P(X_2 \le x) \cdot P(X_3 \le x)$$

$$F_X(x) = \begin{cases} 0 & x \le 85 \\ [0.1(x-85)][0.1(x-85)][0.1(x-85)] & 85 \le x \le 95 \\ 1 & x \ge 95 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \le 85 \\ (0.1)^3 (x-85)^3 & 85 < x \le 95 \\ 1 & x > 95 \end{cases}$$

Now, for $85 \le x \le 95$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$= \frac{d}{dx}\left[(0.1)^3 (x-85)^3\right]$$

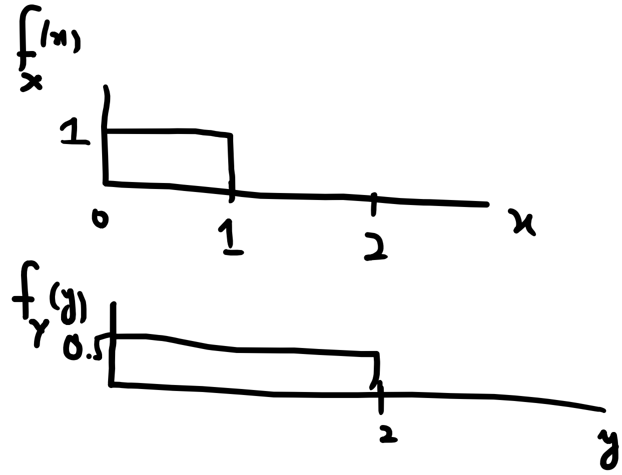
$$= (0.1)^3 \frac{d}{dx}\left[(x-85)^3\right]$$

$$= (0.1)^3\, 3 . (x-85)^2 \frac{d}{dx}(x-85)$$

$$f_X(x) = 3(0.1)^3 (x-85)^2$$

$$\Rightarrow f_X(x) = \begin{cases} 0 & x \le 85 \\ 3(0.1)^3 (x-85)^2 & 85 \le x \le 95 \\ 0 & x \ge 95 \end{cases}$$ — Ans

(b)



(i)

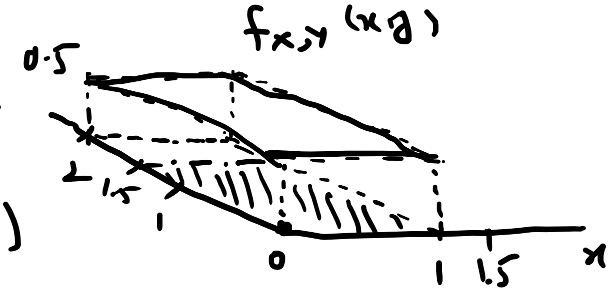
Due to independence,

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_{X,Y}(x,y) = \begin{cases} 0.5 & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y}(1.5, 1.5) = 0 \quad \text{— Ans}$$

(ii)



$$F_{X,Y}(1.5, 1.5) = P(X \leq 1.5, Y \leq 1.5)$$

= Volume under the $f_{X,Y}(x,y)$ over the region $\{X \leq 1.5, Y \leq 1.5\}$

$$= (1.0)(1.5)(0.5)$$

$$= 0.75 \quad \text{— Ans}$$

(iii)

Let A be the event $X < 0.5$

Due to independence of X and Y :-

$$f_{Y|A}(y) = f_Y(y)$$

Now,

$$E[Y|A] = \int_{-\infty}^{\infty} y\, f_{Y|A}(y).dy$$

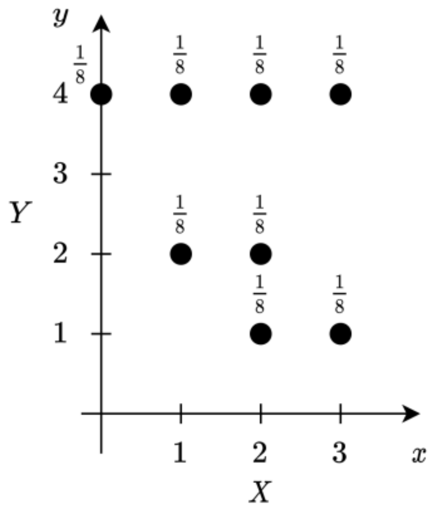
$$= \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = E[Y] = 1 \quad \text{Ans}$$

Question 4 - (10 points)

a) Let X be a Bernoulli random variable with parameter $p = 0.1$. Let Y be an independent Geometric random variable with parameter $p = 0.1$. Find the following:

i. $p_{X,Y}(2,2)$ **[1]**
ii. $p_{X|Y}(2|2)$ **[2]**
iii. $p_{Y|X}(2|2)$ **[2]**

b) Consider the following joint PMF of two random variables X and Y:



i. What is $E[Y|X = 1]$? **[2]**
ii. What is $E[X + Y]$? **[3]**

(a)

$$p_X(x) = \begin{cases} 0.1 & x = 1 \\ 0.9 & x = 0 \end{cases}$$

$$p_Y(y) = \begin{cases} 0.9^{y-1} \cdot 0.1 & y = 1, 2, \dots \\ 0 & y \le 0 \end{cases}$$

(i)

Since X and Y are independent:

$$p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$$

$\because p_X(2) = 0$

$$p_{X,Y}(2,2) = p_X(2) \cdot p_Y(2)$$

$$p_{X,Y}(2,2) = 0 \quad \text{— Ans}$$

(ii)

Since X and Y are independent

$$p_{X|Y}(x|y) = p_X(x)$$

$$p_{X|Y}(2|2) = p_X(2) = 0 \quad \text{—— Ans}$$

(iii)

Since X and Y are independent

$$p_{Y|X}(y|x) = p_Y(y)$$

$$p_{Y|X}(2|2) = p_Y(2)$$

$$= (0.9)^{2-1}(0.1)$$

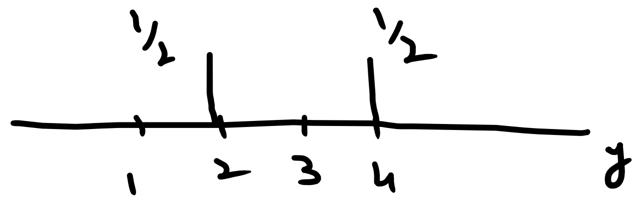
$$= 0.9(0.1)$$

$$p_{Y|X}(2|2) = 0.09 \quad \text{—— Ans}$$

(b)

(i)

$p_{Y|X}(y|1)$



$\Rightarrow \quad E[Y|X=1] = 3$ — Ans

(ii)

Let $Z = X + Y$

Possible (X, Y) pairs with non-zero probability:

(0,4) (1,2) (1,4) (2,1) (2,2) (2,4) (3,1) (3,4)

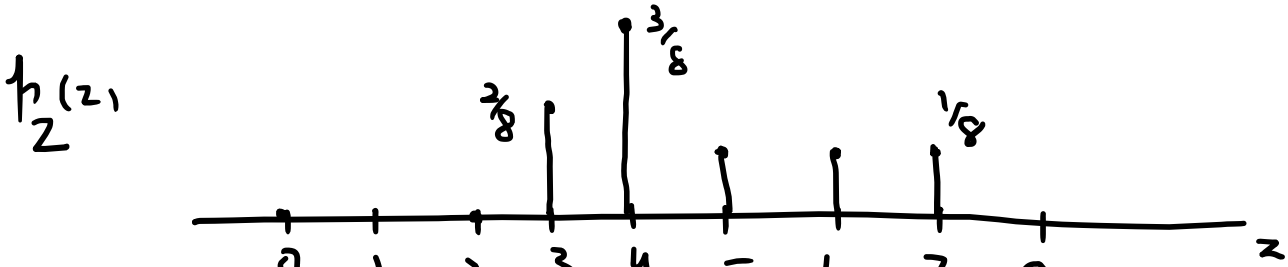
corresponding values of Z:

4 3 5 3 4 6 4 7

Therefore..

$p_Z(0) = 0 \quad p_Z(1) = 0 \quad p_Z(2) = 0 \quad p_Z(3) = \frac{2}{8}$

$p_Z(4) = \frac{3}{8} \quad p_Z(5) = \frac{1}{8} \quad p_Z(6) = \frac{1}{8} \quad p_Z(7) = \frac{1}{8}$



$$E[Z] = \sum z \, p_Z(z)$$

$$= 3\left(\frac{27}{8}\right) + 4\left(\frac{3}{8}\right) + 5\left(\frac{1}{8}\right) + 6\left(\frac{1}{8}\right) + 7\left(\frac{1}{8}\right)$$

$$= \frac{6 + 12 + 5 + 6 + 7}{8}$$

$$E[Z] = \frac{36}{8} = 4.5 \quad \text{— Ans}$$

Question 5 - (10 points)

a) Consider two random variables X & Y:

X = Maximum temperature in Karachi on May 05, 2023 in Celsius (°C)

Y = Maximum temperature in Karachi on May 05, 2023 in Fahrenheit (°F)

Assume that $X \sim N(30,4)$. What is the PDF of Y? **[5]**

(Note: Celsius to Fahrenheit Conversion Formula: $°F = 1.8°C + 32$ *)*

b) Consider the following random variable:

X = Total number of kilometers driven by all the cars in Karachi during one day

Suppose you were told that the number of kilometers travelled by any one car during one day in Karachi is uniformly distributed between 0 and 50 kilometers. Assume that the number of kilometers travelled by each car is independent of the kilometers travelled by any other car. What can you say about the PDF of X? (Justify your answer) **[5]**

(a)

$$Y = 1.8X + 32$$

$$X \sim N(30, 4)$$

If $Y = aX + b$ is a linear function of a Normal RV $X \sim N(\mu, \sigma^2)$, then $Y \sim N(a\mu + b, a^2\sigma^2)$

Therefore,

$$Y \sim N(1.8(30) + 32,\ 1.8^2(4))$$

$$Y \sim N(86, 12.96) \quad \text{— Ans}$$

(b)

X_1 = The number of kilometers driven by Car no. 1 in Karachi in a day

X_2 = " " " " Car no 2 " "

⋮

X_i = The number of kms driven by Car no. i in Karachi in a day

⋮

X_N = The number of kms driven by Car no N in Karachi in a day

So,

$$X = X_1 + X_2 + \cdots + X_N$$

$$X_1 = X_2 = \cdots = X_N \sim U[0, 50]$$

$X_1, X_2, \ldots, X_N$ are independent and identically distributed random variables.

Since N is large, the sum of $X_1, X_2, \ldots, X_N$ will be a Normal random variable (Central Limit Theorem)

$\Rightarrow$ X is a Normal/Gaussian random variable.