Weekly Challenge 12: Closure of Decidable Languages

CS 212 Nature of Computation
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1. Put Together

The *put-together* operation, f, is defined on strings, $u = u_1u_2 \ldots u_m$ and $v = v_1v_2 \ldots v_n$, and extended to languages, L_1 and L_2, over an alphabet, Σ, as follows.

$$f(u, v) = u_1u_2 \ldots u_m v_1 v_2 \ldots v_n$$
$$f(L_1, L_2) = \{f(u, v) \mid u \in L_1, v \in L_2\}.$$

Prove or disprove the following claim.

Claim 1. *The class of decidable languages is closed under the put-together operation.*

Solution: Let L_1 and L_2 be two decidable languages. Let M_1 and M_2 be the Turing Machines that decide L_1 and L_2 respectively. Let L be the language $f(L_1, L_2)$. Then we can construct a Turing Machine M that decides L. For any given string w, the machine M needs to determine if there exists strings u and v such that $f(u, v) = w$.

Then M works as follows:

- As $w = w_1w_2w_3...w_p$ (where $p = m + n$) is a concatenation of u and v, M can try all possible ways of splitting w into two strings u and v as follows; for each i from 0 to p, consider the prefix of w of length i as a potential string u; $u = w_1w_2...w_i$. The remaining part of the string, $w_{i+1}w_{i+2}...w_p$ is considered as a potential string v. [when $i = 0$, $u = \emptyset$ and $v = w$, and when $i = p$, $u = w$ and $v = \emptyset$].
- Then for each potential string u and v, simulate M_1 on u and M_2 on v.
- If M_1 accepts u **and** M_2 accepts v, then M accepts w. If either M_1 or M_2 rejects its respective string, then M rejects w for that particular string.

Since both M_1 and M_2 are deciders, they will halt on all inputs. Therefore, M will also halt on all inputs.

If $w \in f(L_1, L_2)$, then there exists $u \in L_1, v \in L_2$ such that $w = uv$. Machine M will eventually simulate the correct split of w, and both M_1 and M_2 will accept, hence M will accept. If $w \notin f(L_1, L_2)$, then there are no such u and v that can both be accepted by M_1 and M_2 respectively. Thus, for all possible splittings of w, either M_1 or M_2 will reject. Therefore M will also reject. Since M accepts if and only if $w \in f(L_1, L_2)$, and rejects otherwise, M decides $f(L_1, L_2)$. Therefore, $f(L_1, L_2)$ is decidable.

Hence we can conclude that the class of decidable languages is closed under the put-together operation.

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