Algorithms: Design and Analysis - CS 412

Problem Set 03: Asymptotic Analysis

1. Explain why the statement, "The running time of algorithm A is at least $O(n^2)$," is meaningless.

> Since O-notation only provides an upper bound, and not a tight bound, the statement is saying that the running time of algorithm A is at leaast a function whose rate of growth is at most n^2. This is meaningless because the running time of algorithm A could be $O(n^3)$, $O(n^4)$, $O(n^5)$, etc. and the statement would still be true.

2. Prove that the running time of an algorithm is $\Theta(g(n))$ if and only if its worst-case running time is $O(g(n))$ and its best-case running time is $\Omega(g(n))$.

> *Proof.* Refer to Theorem 3.1 in the textbook, and the previous problem set. $\square$

3. Prove that $o(g(n)) \Cap \omega(g(n))$ is the empty set.

> *Proof.* By the definition of $o(g(n))$, for any constant c, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0$.
>
> By the definition of $\omega(g(n))$, for any constant c, there exists a constant $n_1 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_1$.
>
> Let $n_2 = \max(n_0, n_1)$. Then, for all $n \geq n_2$, $0 \leq f(n) < cg(n)$ and $0 \leq cg(n) < f(n)$, which is a contradiction. Therefore, $o(g(n)) \Cap \omega(g(n))$ is the empty set. $\square$

4. Show that $k \ln k = \Theta(n) \implies k = \Theta(n/\ln)$.

Proof. Via symmetry, we have that $f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n))$. So $k \ln k = \Theta(n) \implies n = \Theta(k \ln k)$. Then:

$$\begin{aligned}\ln n &= \Theta(\ln(k \ln k)) \\ &= \Theta(\ln k + \ln \ln k) \\ &= \Theta(\ln k)\end{aligned}$$

Since now we have values of both $\Theta(n)$ and $\Theta(\ln n)$, which by symmetry gives us n and $\ln n$:

$$\begin{aligned}\frac{n}{\ln n} &= \frac{\Theta(k \ln k)}{\Theta(\ln k)} \\ &= \Theta\left(\frac{k \ln k}{\ln k}\right) \\ &= \Theta(k)\end{aligned}$$

Therefore, again, via symmetry, we have that $k \ln k = \Theta(n) \implies k = \Theta(n/\ln)$. □

5. Show that for any real constants a and b, where $b > 0, (n + a)^b = \Theta(n^b)$.

Proof. There exists cosntants c_1, c_2, n_0 such that $\forall n \geq n_0$, $c_1 n^b \leq (n + a)^b \leq c_2 n^b$.

Consider $n \geq |a|$. Then we have:

- Lower Bound: When $n \geq |a|$, $n + a \geq n - |a|$. Since $b > 0$, we can raise both sides to the power of b to get $(n + a)^b \geq (n - |a|)^b$. Then we can choose $c_1 = (1 - \frac{|a|}{n})^b$, and $n_0 = |a|$ to satisfy the lower bound.
- Upper Bound: When $n \geq |a|$, $n + a \leq n + |a|$. Since $b > 0$, we can raise both sides to the power of b to get $(n + a)^b \leq (n + |a|)^b$. Then we can choose $c_2 = (1 + \frac{|a|}{n})^b$, and $n_0 = |a|$ to satisfy the upper bound.

Therefore, we have shown that $(n + a)^b = \Theta(n^b)$. □