Homework 9: Exercise Set 5.2 Solution

Question 07

Which of the following are linear combinations of $\overrightarrow{u} = (0, -2, 2)$ and $\overrightarrow{v} = (1, 3, -1)$?
(a) (2, 2, 2)
(b) (3, 1, 5)
(c) (0, 4, 5)
(d) (0, 0, 0)
Solution: Please take a look at the solution manual for part (a) and (c).
(b) We look for constants a and b such that $a\overrightarrow{u} + b\overrightarrow{v} = (3, 1, 5)$, or $a(0, -2, 2) + b(1, 3, -1) = (3, 1, 5)$ Equating corresponding vector components gives the following system of equations:

$$\begin{aligned} b &= 3 \\ -2a + 3b &= 1 \\ 2a - b &= 5 \end{aligned}$$

From the first equation, we see that $b = 3$. Substituting this value into the remaining equations yields $a = 4$. Thus $(3, 1, 5)$ is a linear combination of $\overrightarrow{u}$ and $\overrightarrow{v}$.
(d) $a\overrightarrow{u} + b\overrightarrow{v} = (0, 0, 0)$, or $a(0, -2, 2) + b(1, 3, -1) = (0, 0, 0)$

$$\begin{aligned} b &= 0 \\ -2a + 3b &= 0 \\ 2a - b &= 0 \end{aligned}$$

Here $a = 0$ and $b = 0$

Question 09

Express the following as linear combinations of $\overrightarrow{p_1} = 2 + x + 4x^2, \overrightarrow{p_2} = 1 - x + 3x^2$ and $\overrightarrow{p_3} = 3 + 2x + 5x^2$
Solution: Please take a look at the solution manual for part (a) and (c).
(b) $6 + 11x + 6x^2$
Solution: Consider a, b, c be the unknowns, so

$$a\overrightarrow{p_1} + b\overrightarrow{p_2} + c\overrightarrow{p_3} = a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 6 + 11x + 6x^2$$

Now solve for a, b, c, it can be written in a matrix form $Ax = b$

$$\begin{bmatrix} 2a & b & 3c \\ a & -b & 2c \\ 4a & 3b & 5c \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix}$$

For more direct requirement because we have determine coefficients a, b, c, it can be written as

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 6 \end{bmatrix}$$

Solving above system, we have $a = 4, b = -5$ and $c = 1$. One can verify by

$$4\overrightarrow{p_1} + (-5)\overrightarrow{p_2} + 1\overrightarrow{p_3} = 4(2+x+4x^2) + (-5)(1-x+3x^2) + 1(3+2x+5x^2) = 6+11x+6x^2$$

(d) $7 + 8x + 9x^2$

Solution: Using similar procedure in the last question. One show the solution is

$$0\overrightarrow{p_1} + (-2)\overrightarrow{p_2} + 3\overrightarrow{p_3} = 0(2+x+4x^2) + (-2)(1-x+3x^2) + 3(3+2x+5x^2) = 7+8x+9x^2$$

Question 15

Find an equation for the plane spanned by the vectors $\overrightarrow{u} = (-1, 1, 1)$ and $\overrightarrow{v} = (3, 4, 4)$.

Solution: Please take a look at the solution manual.

Question 23

Indicate whether each statement is always true or sometimes false. Justify your answer by giving a logical argument or a counterexample.

Solution: Please take a look at the solution manual.

Question 26

(a) Let M_{22} be the vector space of 2×2 matrices. Find four matrices that span $M_{2\times2}$.

Solution: Let $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the four matrices for the vector space $M_{2\times2}$, where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

One can any Matrix $A \in M_{2\times2}$ can be as the linear combination of these matrices.

$$A = aE_{11} + bE_{12} + cE_{13} + dE_{14}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

where a, b, c, d are the coefficients.

(b) In words, describe a set of matrices that spans M_{nn} .

Solution: Let $B' = \{E_{11}, E_{12}, \ldots, E_{nn}\}$ be the n^2 matrices for the span of M_{nn} such that each matrix A in M_{nn} can be written as a linear combination of set B.

$$A = k_{11}E_{11} + k_{12}E_{12} + \ldots + k_{nn}E_{nn}$$

where k_{ii} are the coefficients.