Homework 5 Solutions

Linear Algebra

Question 3:

By inspection, explain why $det(A) = 0$

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

R2 $\implies$ R2 + R1

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 1 & 10 & 6 & 5 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

R3 $\implies$ R3 - R2

$$\begin{bmatrix} -2 & 8 & 1 & 4 \\ 1 & 10 & 6 & 5 \\ 0 & 0 & 0 & 0 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

Since R3 is all zeros, $det(A) = 0$

Question 4

Use Theorem 2.3.3 to determine which of the following matrices are invertible.

Part a

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$$

$$det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 \\ 9 & -1 \end{vmatrix} + 0 \begin{vmatrix} 9 & 4 \\ 8 & -1 \end{vmatrix} - 1 \begin{vmatrix} 9 & -1 \\ 8 & 9 \end{vmatrix}$$

$$det(A) = 1(-35) + 0 - 1(89) = -35 - 89 = -124$$

$det(A) \neq 0$, Hence, the Matrix is invertible

Part b

$$A = \begin{bmatrix} 4 & 2 & 8 \\ -2 & -1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$$

Since C3 is a scalar multiple of C1 (C1 = 2*C3), $det(A) = 0$ hence, matrix is not invertible

Part c

$$A = \begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$$

Since the last column all zeros, $det(A) = 0$ hence, matrix is not invertible

Part d

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{bmatrix}$$

Since the second column all zeros, $det(A) = 0$ hence, matrix is not invertible

Question 5

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

And $det(A) = -7$

Part a

$$det(3A) = 3^3(det(A)) = 27(-7) = -189$$

Part b

$$det(A^{-1}) = \frac{1}{det(A)} = -\frac{1}{7}$$

Part c

$$det(2A^{-1}) = 2^3(det(A^{-1})) = 8\frac{1}{det(A)} = -\frac{8}{7}$$

Part d

$$det((2A)^{-1}) = \frac{1}{det(2A)} = \frac{1}{8 * -7} = -\frac{1}{56}$$

Part e

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

R2 $\iff$ R3

$$A = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$A^T = B = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}^\top = \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$$

Since we swaped rows and took transpose, $det(B) = -(-7) = 7$

Question 6

Without directly evaluating, show that $x = 2$ and $x = 0$ satisfy

$$\begin{vmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 0$$

$x = 0$

$$\begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

R1 $\implies$ R1 - $(\frac{2}{5})$R3

$$\begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

Since the matrix has a row of zeros, its determinant is 0.

$x = 2$

$$\begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

Since R1 is scalar multiple of R2 (R1 = 2*R2), its determinant is 0

Question 7

Without directly evaluating, show that

$$det \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$$

R1 $\implies$ R1 + R2

$$\begin{bmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$$

R1 $\implies$ $\frac{R1}{a+b+c}$

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$$

Since two rows of the matrix are identical (R1 = R3),

$$det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$$

Question 8

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We know that column operations doesn't change the determinant of a matrix
$L.H.S =$

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix}$$

C3 $\implies$ C3 - C1

$$\begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix}$$

C3 $\implies$ C3 - C2

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = R.H.S$$

Quesiton 9

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We know that column operations doesn't change the determinant of a matrix
$L.H.S =$

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

C1 $\implies$ C1 + C2

$$\begin{vmatrix} 2a_1 & a_1-b_1 & c_1 \\ 2a_2 & a_2-b_2 & c_2 \\ 2a_3 & a_3-b_3 & c_3 \end{vmatrix}$$

Taking 2 common from C1

$$2\begin{vmatrix} a_1 & a_1-b_1 & c_1 \\ a_2 & a_2-b_2 & c_2 \\ a_3 & a_3-b_3 & c_3 \end{vmatrix}$$

C2 $\implies$ C2 - C1

$$2\begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix}$$

Taking - common from C2

$$-2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = R.H.S$$

Question 11

$$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We know that column operations doesn't change the determinant of a matrix
$L.H.S =$

$$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix}$$

C2 $\implies$ C2 - t*C1, where t is any integer

$$\begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix}$$

C3 $\implies$ C3 - s*C1, where s is any integer

$$\begin{vmatrix} a_1 & b_1 & c_1 + rb_1 \\ a_2 & b_2 & c_2 + rb_2 \\ a_3 & b_3 & c_3 + rb_3 \end{vmatrix}$$

C3 $\implies$ C3 - r*C2, where r is any integer

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = R.H.S$$

Quesiton 13

Show that

$$\begin{bmatrix} sin^2\alpha & sin^2\beta & sin^2\gamma \\ cos^2\alpha & cos^2\beta & cos^2\beta \\ 1 & 1 & 1 \end{bmatrix}$$

is not invertible for any values of α, β, and γ
R2 $\implies$ R1 + R2

$$\begin{bmatrix} sin^2\alpha & sin^2\beta & sin^2\gamma \\ sin^2\alpha + cos^2\alpha & sin^2\beta + cos^2\beta & sin^2\gamma + cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$$

Since $sin^2\theta + cos^2\theta = 1$

$$\begin{bmatrix} sin^2\alpha & sin^2\beta & sin^2\gamma \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

As two rows are identical (R2 = R3), determinant of the matrix is 0. Hence, the matrix is not invertible for any values of α, β, and γ

Question 16

Let A and B be nxn matrices. Show that if A is invertible, then $det(B) = det(A^{-1}BA)$
As A is invertible, $det(A) \neq 0$ and $\frac{1}{det(A)}$ is defined

$$R.H.S = det(A^{-1}BA)$$

which can be wirtten as

$$= \frac{1}{det(A)} * det(B) * det(A)$$
$$= det(B) = L.H.S$$

Question 21

Let A and B be nxn matrices. You know from earlier work that AB is invertible if A and B are invertible. What can you say about the invertibility of if one or both of the factors are singular? Explain your reasoning.
We know that

$$det(AB) = det(A) * det(B)$$

If one or both of the factors are singular which means either $det(A) = 0$ or $det(B) = 0$ or both.
In any of the cases

$$det(AB) = 0$$

Hence, if one or both of the factors are singular then AB is not invertible