Lecture 9

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$$M_{32} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \cancel{a_{12}} & a_{13} & a_{14} \\ a_{21} & \cancel{a_{22}} & a_{23} & a_{24} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} & \cancel{a_{34}} \\ a_{41} & \cancel{a_{42}} & a_{43} & a_{44} \end{vmatrix}$$

$(-1)^{i+j} M_{ij} = \text{cofactor}$

DEFINITION

If A is a square matrix, then the ***minor of entry*** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the ***cofactor of entry*** a_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Note that the cofactor and the minor of an element a_{ij} differ only in sign; that is, $C_{ij} = \pm M_{ij}$. A quick way to determine whether to use + or − is to use the fact that the sign relating C_{ij} and M_{ij} is in the ith row and jth column of the "checkerboard" array

$(-1)^{2+1} = -1^{3} = -1$

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For example, $C_{11} = M_{11}$, $C_{21} = -M_{21}$, $C_{12} = -M_{12}$, $C_{22} = M_{22}$, and so on.

THEOREM 2.1.1

Expansions by Cofactors

The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \leq i \leq n$ and $1 \leq j \leq n$.

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

EXAMPLE 3 Cofactor Expansion Along the First Column

Let A be the matrix in Example 2. Evaluate $\det(A)$ by cofactor expansion along the first column of A.

Solution

$(-1)^4 = 1$
$(-1)^5 = -1$
$(-1)^6 = 1$

$(1+3)=4$
$(2+3)=5$
$(3+3)=6$

$(+)0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} + (-1)(3)\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 \\ -2 & -4 \end{vmatrix}$

$-3(7) + (-2)(-10)$

$= -21 + 20 = -1$

From 4

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

This agrees with the result obtained in Example 2.

(cofactor matrix)

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Adjoint

$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

adj(A)

DEFINITION

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by $\mathrm{adj}(A)$.

EXAMPLE 6 **Adjoint of a 3×3 Matrix**

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

so the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

THEOREM 2.1.2

Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) \tag{7}$$

THEOREM 2.2.1

Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Proof By Theorem 2.1.1, the determinant of A found by cofactor expansion along the row or column of all zeros is

$$\det(A) = 0 \cdot C_1 + 0 \cdot C_2 + \cdots + 0 \cdot C_n$$

where $C_1, \cdots, C_n$ are the cofactors for that row or column. Hence $\det(A)$ is zero.

THEOREM 2.2.2

Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Proof By Theorem 2.1.1, the determinant of A found by cofactor expansion along its first row is the same as the determinant of A^T found by cofactor expansion along its first column.

CRAMER'S RULE:

WE SHALL DISCUSS ANOTHER METHOD TO FIND THE UNIQUE SOLUTION OF n EQUATIONS IN n UNKNOWNS, PROVIDED DETERMINANT OF THE COEFFICIENT MATRIX $\neq 0$ IN $AX = B$, $B \neq 0$, i.e. LINEAR SYSTEM IS NONHOMOGENEOUS.

SIMPLE CASE

SYSTEM OF TWO EQUATIONS IN TWO UNKNOWNS:

HERE $AX = B \Rightarrow \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad — (1)$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad — (2)$$

FROM (1)

$$x_2 = \frac{b_1 - a_{11}x_1}{a_{12}} \quad — (*)$$

USING (*) IN (2) GIVES THE FOLLOWING:

$$a_{21}x_1 + a_{22}\left(\frac{b_1 - a_{11}x_1}{a_{12}}\right) = b_2$$

$$\Rightarrow a_{12}a_{21}x_1 + a_{22}b_1 - a_{11}a_{22}x_1 = a_{12}b_2$$

$$\Rightarrow x_1(a_{12}a_{21} - a_{11}a_{22}) = a_{12}b_2 - a_{22}b_1$$

$$\Rightarrow x_1 = \frac{a_{12}b_2 - a_{22}b_1}{a_{12}a_{21} - a_{11}a_{22}} = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \Rightarrow$$

$$\boxed{x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}}$$

, SIMILARLY WE CAN OBTAIN

$$\begin{vmatrix} a_{11} & b_1 \\ \dots & \dots \end{vmatrix}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

OR JUST

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|},$$ WHERE

$|A|$ IS THE DETERMINANT OF COEFFICIENT MATRIX.

$$|A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$ IS OBTAINED BY REPLACING

THE ENTRIES IN THE 1ST COLUMN OF $|A|$ BY ENTRIES IN B, ETC. SIMILARLY WE CAN EXTEND THIS METHOD TO THREE EQUATIONS IN THREE UNKNOWNS.

BUT FIRST WE MUST KNOW THE EXPANSION OR METHOD TO EXPAND A DETERMINANT OF A MATRIX OF ORDER $\boxed{3}$.

Seen earlier in this Lecture

EXPANSION BY FIRST ROW:

CONSIDER

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \underbrace{a_{11} (-1)^{1+1} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{M_{11}}}_{C_{11}} + \underbrace{a_{12} (-1)^{1+2} \underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{12}}}_{C_{12}} +$$

$$\begin{vmatrix} a_{31} & a_{33} \end{vmatrix} +$$

$$a_{13}(-1)^{1+3} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{M_{13}}$$

$$= \boxed{\begin{aligned} & a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ & - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ & + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}}$$

CONSIDER

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad -1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad -2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad -3$$

LET US WRITE 1 AND 2 AS

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= c_1 \\ a_{21}x_1 + a_{22}x_2 &= c_2 \end{aligned}\right\} \rightarrow (*)$$

WHERE $\boxed{\begin{aligned} c_1 &= b_1 - a_{13}x_3 \\ c_2 &= b_2 - a_{23}x_3 \end{aligned}}$ AND

SOLUTION OF $(*)$ IS

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$\Rightarrow \boxed{x_1 = \frac{c_1 a_{22} - a_{12} c_2}{a_{11}a_{22} - a_{12}a_{21}}}, \text{ AND}$$

$$\boxed{x_2 = \frac{c_2 a_{11} - c_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}}$$

SUBSTITUTING x_1, x_2 IN 3 AFTER

USING THE VALUES OF $\boxed{C_1}$ AND $\boxed{C_2}$ WE GET THE FOLLOWING VALUE OF $\boxed{x_3}$ AS

$$\boxed{x_3 = \frac{\begin{bmatrix} a_{11}(b_3 a_{22} - a_{32} b_2) \\ - a_{12}(b_3 a_{21} - a_{31} b_2) \\ + b_1(a_{21} a_{32} - a_{31} a_{22}) \end{bmatrix}}{\begin{bmatrix} a_{11}(a_{22} a_{33} - a_{23} a_{32}) \\ - a_{12}(a_{21} a_{33} - a_{31} a_{23}) \\ + a_{13}(a_{32} a_{21} - a_{31} a_{22}) \end{bmatrix}}}$$

CONSIDER THE DENOMINATOR

$$a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{31} a_{23}) + a_{13}(a_{32} a_{21} - a_{31} a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

DETERMINANT OF THE COEFFICIENT MATRIX $= |A| \neq 0$

$+ \; - \; + \; -$
$- \; +$
$+ \; -$

$$(1) \begin{cases} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{cases} \qquad \text{Where } \begin{aligned} c_1 &= b_1 - a_{13}x_3 \\ c_2 &= b_2 - a_{23}x_3 \end{aligned}$$

Also $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ —— (3)

From Crammer's rule on (1), we got $x_1 = \dfrac{c_1 a_{22} - a_{12} c_2}{a_{11}a_{22} - a_{12}a_{21}}$

$x_2 = \dfrac{c_2 a_{11} - c_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$

Putting in (3)

$a_{31}(c_1 a_{22} - a_{12} c_2 \ldots) + a_{32}(c_2 a_{11} - c_1 a_{21} \ldots) + a_{33} x_3 = b_3$

$$\frac{\quad}{a_{11}a_{22} - a_{12}a_{21}}$$

Putting in 3

$$a_{31}\left(\frac{c_1 a_{22} - a_{12} c_2}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{32}\left(\frac{c_2 a_{11} - c_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{33}x_3 = b_3$$

$$\Rightarrow a_{31}\left(\frac{(b_1 - a_{13}x_3)a_{22} - a_{12}(b_2 - a_{23}x_3)}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{32}\left(\frac{(b_2 - a_{23}x_3)a_{11} - (b_1 - a_{13}x_3)a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{33}x_3 = b_3$$

$$\Rightarrow a_{31}\left(\frac{a_{22}b_1 - a_{13}a_{22}x_3 - a_{12}b_2 + a_{12}a_{23}x_3}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{32}\left(\frac{a_{11}b_2 - a_{11}a_{23}x_3 - a_{21}b_1 + a_{13}a_{21}x_3}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{33}x_3 = b_3$$

$$\Rightarrow \left(\frac{a_{22}a_{31}b_1 - a_{13}a_{22}a_{31}x_3 - a_{12}a_{31}b_2 + a_{12}a_{23}a_{31}x_3}{a_{11}a_{22} - a_{12}a_{21}}\right) + \left(\frac{a_{11}a_{32}b_2 - a_{11}a_{23}a_{32}x_3 - a_{21}a_{32}b_1 + a_{13}a_{21}a_{32}x_3}{a_{11}a_{22} - a_{12}a_{21}}\right) + a_{33}x_3 = b_3$$

$$a_{22}a_{31}b_1 - a_{13}a_{22}a_{31}x_3 - a_{12}a_{31}b_2 + a_{12}a_{23}a_{31}x_3 + a_{11}a_{32}b_2 - a_{11}a_{32}a_{23}x_3 - a_{21}a_{32}b_1 + a_{13}a_{21}a_{32}x_3 + (a_{11}a_{22} - a_{12}a_{21})a_{33}x_3 = (a_{11}a_{22} - a_{21}a_{12})b_3$$

$$(-a_{12}a_{22}a_{31} + a_{12}a_{23}a_{31} - a_{11}a_{32}a_{23} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33})x_3 = -a_{22}a_{31}b_1 + a_{12}a_{31}b_2 - a_{11}a_{32}b_2 + a_{21}a_{32}b_1 + a_{11}a_{22}b_3 - a_{21}a_{12}b_3$$

$$\left(a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{21}a_{23}) + a_{13}(a_{32}a_{21} - a_{31}a_{22})\right)x_3 = a_{11}(a_{22}b_3 - a_{32}b_2) - a_{12}(b_3 a_{21} - a_{31}b_2) + b_1(a_{21}a_{32} - a_{22}a_{31})$$

So $x_3 = \dfrac{\quad}{\quad}$

SIMILARLY THE NUMERATOR IS EQUAL TO

$$|A_3| = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

$$\therefore x_3 = \frac{|A_3|}{|A|} = \frac{|A_3|}{|A|}, \quad |A| \neq 0$$

COMPLETE SOLUTION:

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad x_3 = \frac{|A_3|}{|A|}$$

WHERE

$$|A_1| = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$|A_2| = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

Proof of Cramer's rule

$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ with columns C_1, C_2

$C_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore B = (C_1 | C_2)$

$$AB = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 17 & 10 \end{pmatrix}$$

$$AC_1 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

$$AC_2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$\therefore AB = A(C_1|C_2) = (AC_1 | AC_2)$$

Consider the system $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

We can write it as $AX = B \longrightarrow$ Matrix form

Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

We can write it as

$AX = B \longrightarrow$ Matrix form

where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

and A is the coeff. matrix.

If A is invertible, then $X = A^{-1}B$

Note that if A is invertible, then $A^{-1}A = I_n = \begin{array}{c} \begin{matrix} e_1 & e_2 & \cdots & e_n \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \\ n\times n \end{array}$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

So if

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, then $A^{-1}A = A^{-1}\left(C_1 | C_2 | \cdots | C_n\right)$
$\qquad\quad\ \ C_1 \quad C_2 \ \cdots \ C_n$

$A = \left(C_1 | C_2 | \cdots | C_n\right)$

$= \left(A^{-1}C_1 \,|\, A^{-1}C_2 \,|\, \cdots \,|\, A^{-1}C_n\right) = A^{-1}A = I_n$

So $A^{-1}C_1 = e_1 = \underset{n\times 1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$, $A^{-1}C_2 = \underset{n\times 1}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \leftarrow e_2$, $\cdots$, $A^{-1}C_n = \underset{n\times 1}{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \leftarrow e_n$

Now for any k, $1 \le k \le n$, Take the Matrix

$$D_k = \begin{bmatrix} 1 & 0 & \cdots & 0 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & x_2 & & & 0 \\ 0 & 0 & 1 & & 0 & & & \\ & 0 & 0 & \ddots & 0 & & & \\ & & & & 1 & & & \\ & & & 0 & x_k & 0 & & \\ & & & & & 1 & & \\ & & & & & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & x_n & 0 & \cdots & 1 \end{bmatrix}$$

i.e the identity matrix I_n, with the k^{th} Column replaced by

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$(-1)^{k+k}$

recall that
$X = A^{-1}B$

Also
$e_1 = A^{-1}C_1$
$e_2 = A^{-1}C_2$
$\vdots$
$e_n = A^{-1}C_n$

$$\therefore \quad D_k = [e_1 | e_2 | \cdots | e_{k-1} | X | e_{k+1} | \cdots | e_n]$$

$$D_k = [A^{-1}C_1 | A^{-1}C_2 | \cdots | A^{-1}C_{k-1} | A^{-1}B | A^{-1}C_{k+1} | \cdots | A^{-1}C_n]$$

$$\Rightarrow D_k = A^{-1}\underbrace{[C_1 | C_2 | \cdots | C_{k-1} | B | C_{k+1} | \cdots | C_n]}$$

But this matrix is A_k (in Crammer's rule)!

$$\therefore \quad \boxed{D_k = A^{-1}A_k}$$

Next, note that $\underline{\det(D_k) = x_k}$ (Just expand along the row containing x_k. Everything else is zero on that row.)

(*)

recall that
$\det(A^{-1}) = \frac{1}{\det(A)}$

However

$$(\dagger) \quad \det(D_k) = \det(A^{-1}A_k) = \det(A^{-1})\det(A_k) = \frac{\det(A_k)}{\det(A)}$$

From (*) and ($\dagger$), we have $\boxed{x_k = \frac{\det(A_k)}{\det(A)}}$

EXAMPLE 9 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve

$$\begin{aligned} x_1 + + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$\begin{pmatrix} 6 \\ 30 \\ 8 \end{pmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{\underline{11}}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{\underline{11}},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{\underline{11}}$$

Exercise set 2.1
Q 1, 8, 22, 24 (a), 25, 26

Exercise set 2.2
Q 12,13,14,15