CS/CS 316/365 Deep Learning

Activity 6

October 10, 2024

Backpropagation

Activity needs to be handwritten. Submission will be online on canvas only.

- A two-layer network with two hidden units in each layer can be defined as:

$$
\begin{aligned}
y \quad = \quad & \phi_0 + \phi_1 \mathrm{a}\Big[\psi_{01} + \psi_{11}\mathrm{a}[\theta_{01} + \theta_{11}x] + \psi_{21}\mathrm{a}[\theta_{02} + \theta_{12}x]\Big] \\
& + \phi_2 \mathrm{a}\Big[\psi_{02} + \psi_{12}\mathrm{a}[\theta_{01} + \theta_{11}x] + \psi_{22}\mathrm{a}[\theta_{02} + \theta_{12}x]\Big]
\end{aligned}
$$

  where a functions are ReLU functions. Compute the derivatives of the output y with respect to each of the 13 parameters i.e. $\theta s, \phi s, \psi s$, directly (i.e., not using the backpropagation algorithm). The derivative of the ReLU function with respect to its input $\partial a[z]/\partial z$ is the indicator function I[z > 0], which returns one if the argument is greater than zero and zero otherwise.

  Solution:

$$
\begin{aligned}
\frac{\partial y}{\partial \theta_0} &= 1 \\
\frac{\partial y}{\partial \theta_1} &= a[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x]] \\
\frac{\partial y}{\partial \theta_2} &= a[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x]] \\
\frac{\partial y}{\partial \psi_{01}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] \\
\frac{\partial y}{\partial \psi_{11}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] a[\theta_{01} + \theta_{11} x] \\
\frac{\partial y}{\partial \psi_{21}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] a[\theta_{02} + \theta_{12} x] \\
\frac{\partial y}{\partial \psi_{02}} &= \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] \\
\frac{\partial y}{\partial \psi_{12}} &= \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] a[\theta_{01} + \theta_{11} x] \\
\frac{\partial y}{\partial \psi_{22}} &= \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] a[\theta_{02} + \theta_{12} x] \\
\frac{\partial y}{\partial \theta_{01}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] \psi_{11} I[\theta_{01} + \theta_{11} x > 0] + \\
&\quad\ \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] \psi_{12} I[\theta_{01} + \theta_{11} x > 0] \\
\frac{\partial y}{\partial \theta_{11}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] \psi_{11} I[\theta_{01} + \theta_{11} x > 0] + \\
&\quad\ \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] \psi_{12} I[\theta_{01} + \theta_{11} x > 0] \\
\frac{\partial y}{\partial \theta_{02}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] \psi_{21} I[\theta_{02} + \theta_{12} x > 0] + \\
&\quad\ \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] \psi_{22} I[\theta_{02} + \theta_{12} x > 0] \\
\frac{\partial y}{\partial \theta_{12}} &= \phi_1 I[\psi_{01} + \psi_{11} a[\theta_{01} + \theta_{11} x] + \psi_{21} a[\theta_{02} + \theta_{12} x] > 0] \psi_{21} I[\theta_{02} + \theta_{12} x > 0] + \\
&\quad\ \phi_2 I[\psi_{02} + \psi_{12} a[\theta_{01} + \theta_{11} x] + \psi_{22} a[\theta_{02} + \theta_{12} x] > 0] \psi_{22} I[\theta_{02} + \theta_{12} x > 0]
\end{aligned} \tag{1}
$$

- Find an expression for the final term in each of the five chains of derivatives in equation given below.

$$
\begin{aligned}
\frac{\partial \ell_i}{\partial f_2} &= \frac{\partial h_3}{\partial f_2} \left(\frac{\partial f_3}{\partial h_3} \frac{\partial \ell_i}{\partial f_3} \right) \\
\frac{\partial \ell_i}{\partial h_2} &= \frac{\partial f_2}{\partial h_2} \left(\frac{\partial h_3}{\partial f_2} \frac{\partial f_3}{\partial h_3} \frac{\partial \ell_i}{\partial f_3} \right) \\
\frac{\partial \ell_i}{\partial f_1} &= \frac{\partial h_2}{\partial f_1} \left(\frac{\partial f_2}{\partial h_2} \frac{\partial h_3}{\partial f_2} \frac{\partial f_3}{\partial h_3} \frac{\partial \ell_i}{\partial f_3} \right) \\
\frac{\partial \ell_i}{\partial h_1} &= \frac{\partial f_1}{\partial h_1} \left(\frac{\partial h_2}{\partial f_1} \frac{\partial f_2}{\partial h_2} \frac{\partial h_3}{\partial f_2} \frac{\partial f_3}{\partial h_3} \frac{\partial \ell_i}{\partial f_3} \right) \\
\frac{\partial \ell_i}{\partial f_0} &= \frac{\partial h_1}{\partial f_0} \left(\frac{\partial f_1}{\partial h_1} \frac{\partial h_2}{\partial f_1} \frac{\partial f_2}{\partial h_2} \frac{\partial h_3}{\partial f_2} \frac{\partial f_3}{\partial h_3} \frac{\partial \ell_i}{\partial f_3} \right)
\end{aligned}
$$

where forward pass looks like this.

$$
\begin{aligned}
f_0 &= \beta_0 + \omega_0 \cdot x_i \\
h_1 &= \sin[f_0] \\
f_1 &= \beta_1 + \omega_1 \cdot h_1 \\
h_2 &= \exp[f_1] \\
f_2 &= \beta_2 + \omega_2 \cdot h_2 \\
h_3 &= \cos[f_2] \\
f_3 &= \beta_3 + \omega_3 \cdot h_3 \\
\ell_i &= (f_3 - y_i)^2.
\end{aligned}
$$

Solution:

$$\frac{\partial h_3}{\partial f_2} = -\sin[f_2]$$
$$\frac{\partial f_2}{\partial h_2} = \omega_2$$
$$\frac{\partial h_2}{\partial f_1} = \exp[f_1]$$
$$\frac{\partial f_1}{\partial h_1} = \omega_1$$
$$\frac{\partial h_1}{\partial f_0} = \cos[f_0]$$

- Have a look at the expression for derivative of loss with respect to f_2.

$$\frac{\partial \ell_i}{\partial \mathbf{f}_2} = \frac{\partial \mathbf{h}_3}{\partial \mathbf{f}_2} \frac{\partial \mathbf{f}_3}{\partial \mathbf{h}_3} \frac{\partial \ell_i}{\partial \mathbf{f}_3}$$

Here 3 terms on the right have sizes: D_3 x D_3, D_3 x D_f, D_f x D_1. What size are each of the terms in equation of change in loss with respect to f_0 given below?

$$\frac{\partial \ell_i}{\partial \mathbf{f}_0} = \frac{\partial \mathbf{h}_1}{\partial \mathbf{f}_0} \frac{\partial \mathbf{f}_1}{\partial \mathbf{h}_1} \left(\frac{\partial \mathbf{h}_2}{\partial \mathbf{f}_1} \frac{\partial \mathbf{f}_2}{\partial \mathbf{h}_2} \frac{\partial \mathbf{h}_3}{\partial \mathbf{f}_2} \frac{\partial \mathbf{f}_3}{\partial \mathbf{h}_3} \frac{\partial \ell_i}{\partial \mathbf{f}_3} \right)$$

solution:

$$\frac{\partial \ell_i}{\partial \mathbf{f}_0} \in \mathbb{R}^{D_1 \times 1}$$
$$\frac{\partial \mathbf{h}_1}{\partial \mathbf{f}_0} \in \mathbb{R}^{D_1 \times D_1}$$
$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{h}_1} \in \mathbb{R}^{D_1 \times D_2}$$
$$\frac{\partial \mathbf{h}_2}{\partial \mathbf{f}_1} \in \mathbb{R}^{D_2 \times D_2}$$
$$\frac{\partial \mathbf{f}_2}{\partial \mathbf{h}_2} \in \mathbb{R}^{D_2 \times D_3}$$
$$\frac{\partial \mathbf{h}_3}{\partial \mathbf{f}_2} \in \mathbb{R}^{D_3 \times D_3}$$
$$\frac{\partial \mathbf{f}_3}{\partial \mathbf{h}_3} \in \mathbb{R}^{D_3 \times D_f}$$
$$\frac{\partial \ell_i}{\partial \mathbf{f}_3} \in \mathbb{R}^{D_f \times 1}$$