Lecture 15

Tuesday, March 8, 2022 6:45 AM

1) LINEAR ALGEBRA

RESULT:

IF $S = \{ \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \}$ IS A BASIS FOR A VECTOR SPACE $\boxed{V}$, THEN EVERY VECTOR $\underline{v}$ IN $\boxed{V}$ CAN BE EXPRESSED IN THE FORM

$$\underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n$$

IN EXACTLY $\boxed{\text{ONE}}$ WAY.

PROOF:

LET

$$\underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \cdots + c_n \underline{v}_n \quad \text{AND}$$

$$\underline{v} = k_1 \underline{v}_1 + k_2 \underline{v}_2 + \cdots + k_n \underline{v}_n$$

SUBTRACTING THE SECOND EQUATION FROM THE FIRST GIVES

$$\underline{0} = (c_1 - k_1) \underline{v}_1 + (c_2 - k_2) \underline{v}_2 + \cdots + (c_n - k_n) \underline{v}_n$$

THE LINEAR INDEPENDENCE OF VECTORS IN $\{ \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \}$ IMPLIES THAT

$$c_1 - k_1 = 0, \; c_2 - k_2 = 0, \ldots, c_n - k_n = 0$$

$$\Rightarrow \boxed{c_1 = k_1, \; c_2 = k_2, \ldots, c_n = k_n}$$

WHICH COMPLETES THE PROOF.

THEOREM 5.4.2

Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis.

(a) *If a set has more than n vectors, then it is linearly dependent.*

(b) *If a set has fewer than n vectors, then it does not span V.*

Proof (a) Let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of m vectors in V, where $m > n$. We want to show that S' is linearly dependent. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, each $\mathbf{w}_i$ can be expressed as a linear combination of the vectors in S, say

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{n1}\mathbf{v}_n \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{n2}\mathbf{v}_n \\
&\vdots \\
\mathbf{w}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{nm}\mathbf{v}_n
\end{aligned} \tag{6}$$

To show that S' is linearly dependent, we must find scalars $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = \mathbf{0} \tag{7}$$

Using the equations in 6, we can rewrite 7 as

$$\begin{aligned}
&(k_1a_{11} + k_2a_{12} + \cdots + k_ma_{1m})\mathbf{v}_1 \\
&+ (k_1a_{21} + k_2a_{22} + \cdots + k_ma_{2m})\mathbf{v}_2 \\
&\ddots \\
&+ (k_1a_{n1} + k_2a_{n2} + \cdots + k_ma_{nm})\mathbf{v}_n = \mathbf{0}
\end{aligned}$$

Thus, from the linear independence of S, the problem of proving that S' is a linearly dependent set reduces to showing there are scalars $k_1, k_2, \ldots, k_m$, not all zero, that satisfy

$$\begin{aligned}
a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m &= 0 \\
a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m &= 0 \\
\vdots \qquad\qquad & \;\;\vdots \\
a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m &= 0
\end{aligned} \tag{8}$$

But 8 has more unknowns than equations, so the proof is complete since Theorem 1.2.1 guarantees the existence of nontrivial solutions.

■

Proof (b) Let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of m vectors in V, where $m < n$. We want to show that S' does not span V. The proof will be by contradiction: We will show that assuming S' spans V leads to a contradiction of the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

If S' spans V, then every vector in V is a linear combination of the vectors in S'. In particular, each basis vector $\mathbf{v}_i$ is a linear combination of the vectors in S', say

$$\begin{aligned} \mathbf{v}_1 &= a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\ \mathbf{v}_2 &= a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\ &\ \ \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\ \mathbf{v}_n &= a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m \end{aligned} \tag{9}$$

To obtain our contradiction, we will show that there are scalars $k_1, k_2, \ldots, k_n$, not all zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0} \tag{10}$$

But observe that 9 and 10 have the same form as 6 and 7 except that m and n are interchanged and the $\boldsymbol{w}$'s and $\boldsymbol{v}$'s are interchanged. Thus the computations that led to 8 now yield

$$\begin{aligned} a_{11}k_1 + a_{12}k_1 + \cdots + a_{1n}k_n &= 0 \\ a_{21}k_1 + a_{22}k_2 + \cdots + a_{2n}k_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad &\ \ \vdots \\ a_{m1}k_1 + a_{m2}k_2 + \cdots + a_{mn}k_n &= 0 \end{aligned}$$

This linear system has more unknowns than equations and hence has nontrivial solutions by Theorem 1.2.1. ■

It follows from the preceding theorem that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is any basis for a vector space V, then all sets in V that simultaneously span V and are linearly independent must have precisely n vectors. Thus, all bases for V must have the same number of vectors as the arbitrary basis S. This yields the following result, which is one of the most important in linear algebra.

THEOREM 5.4.3

All bases for a finite-dimensional vector space have the same number of vectors.

THEOREM 5.4.4

Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) *If S is a linearly independent set, and if $\boldsymbol{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\boldsymbol{v}$ into S is still linearly independent.*

(b) *If $\boldsymbol{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\boldsymbol{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,*

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$

THEOREM 5.4.5

If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.

Proof Assume that S has exactly n vectors and spans V. To prove that S is a basis, we must show that S is a linearly independent set. But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors. If we remove this vector from S, then it follows from the Plus/Minus Theorem (Theorem 5.4.4*b*) that the remaining set of $n-1$ vectors still spans V. But this is impossible, since it follows from Theorem 5.4.2*b* that no set with fewer than n vectors can span an n-dimensional vector space. Thus S is linearly independent.

Assume that S has exactly n vectors and is a linearly independent set. To prove that S is a basis, we must show that S spans V. But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in span(S). If we insert this vector into S, then it follows from the Plus/Minus Theorem (Theorem 5.4.4*a*) that this set of $n+1$ vectors is still linearly independent. But this is impossible, since it follows from Theorem 5.4.2*a* that no set with more than n vectors in an n-dimensional vector space can be linearly independent. Thus S spans V. ■

THEOREM 5.4.6

Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

(b) *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

Proof (a) If S is a set of vectors that spans V but is not a basis for V, then S is a linearly dependent set. Thus some vector $\mathbf{v}$ in S is expressible as a linear combination of the other vectors in S. By the Plus/Minus Theorem (Theorem 5.4.4*b*), we can remove $\mathbf{v}$ from S, and the resulting set S' will still span V. If S' is linearly independent, then S' is a basis for V, and we are done. If S' is linearly dependent, then we can remove some appropriate vector from S' to produce a set S'' that still spans V. We can continue removing vectors in this way until we finally arrive at a set of vectors in S that is linearly independent and spans V. This subset of S is a basis for V. ■

Proof (b) Suppose that $\dim(V)=n$. If S is a linearly independent set that is not already a basis for V, then S fails to span V, and there is some vector $\mathbf{v}$ in V that is not in span(S). By the Plus/Minus Theorem (Theorem 5.4.4*a*), we can insert $\mathbf{v}$ into S, and the resulting set S' will still be linearly independent. If S' spans V, then S' is a basis for V, and we are finished. If S' does not span V, then we can insert an appropriate vector into S' to produce a set S'' that is still linearly independent. We can continue inserting vectors in this way until we reach a set with n linearly independent vectors in V. This set will be a basis for V by Theorem 5.4.5. ■

THEOREM 5.4.7

If W is a subspace of a finite-dimensional vector space V, then $\dim(W) \leq \dim(V)$; *moreover, if* $\dim(W) = \dim(V)$, *then* $W = V$.

Exercise set 5.4
Q3, Q4, Q6, Q13, Q 15, Q 18, Q 33 [Not included in the syllabus for the midterm, but included in the syllabus for the Final Exam are Q 25 and Q 26]