Lecture 17

Tuesday, March 15, 2022 12:54 AM

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$\boxed{\text{LECTURE 17}}$ $\boxed{\text{LINEAR ALGEBRA}}$

RESULTS:

(1) IF $\underline{v}_1$ IS A NONZERO VECTOR THEN $\{\underline{v}_1\}$ IS ALWAYS INDEPENDENT.

$\because k\underline{v}_1 = \underline{0} \Rightarrow k = 0 \quad \because \underline{v}_1 \neq \underline{0}$

HERE $\{\underline{v}_1\}$ IS A SET CONTAINING ONLY $\boxed{\text{ONE}}$ NONZERO VECTOR.

(2) IF $\underline{0}$ IS A ZERO VECTOR THEN $\{\underline{0}\}$ IS ALWAYS DEPENDENT $\because k\underline{0} = \underline{0}$ IS SATISFIED BY INFINITE $\boxed{\text{NONZERO}}$ VALUES OF $\boxed{k}$.

DEFINITION: THE SOLUTION SPACE OF THE HOMOGENEOUS SYSTEM OF EQUATIONS $A\underline{x} = \underline{0}$, WHICH IS A SUBSPACE OF R^n, IS CALLED THE NULL SPACE OF $\boxed{A}$.

Note: Look back at Lecture 13, Theorem 5.2.2.

EXAMPLE: (DONE LAST TIME) (SLIDE 10 LEC. 16)

NULLSPACE OF

$$A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$ IS GIVEN

BY $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, t IS ANY REAL NUMBER

DEFINITION: THE DIMENSION OF THE NULLSPACE OF $\boxed{A}$ IS CALLED THE NULLITY OF $\boxed{A}$ AND IS DENOTED BY NULLITY(A).

EXAMPLE: IN THE ABOVE EXAMPLE $\boxed{\text{NULLITY}(A) = 1}$ $\because$

THIS IS NUMBER OF ELEMENTS IN THE $\boxed{\text{BASIS}}$ OF NULLSPACE OF $\boxed{A}$ i.e. NUMBER OF ELEMENTS IN $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$ WHICH IS ONE.

IN SIMPLE WORDS THIS IS THE BASIS FOR THE SOLUTION SPACE OF $AX = 0$.

3 **DEFINITION:-** FOR AN $m \times n$ MATRIX

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ THE}$$

VECTORS $\boxed{(\text{in } R^n)}$

$$\underline{r}_1 = (a_{11}, a_{12}, \ldots\ldots, a_{1n})$$
$$\underline{r}_2 = (a_{21}, a_{22}, \ldots\ldots, a_{2n})$$
$$\vdots \qquad\qquad \vdots$$
$$\underline{r}_m = (a_{m1}, a_{m2}, \ldots\ldots, a_{mn})$$

FORMED FROM THE $\boxed{\text{ROWS}}$ OF $\boxed{A}$ ARE CALLED THE ROW VECTORS OF $\boxed{A}$, AND THE VECTORS

$$\underline{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \underline{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots\ldots, \underline{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

(IN R^m) FORMED FROM THE COLUMNS OF $\boxed{A}$ ARE CALLED THE COLUMN VECTORS OF $\boxed{A}$.

DEF.

4 IF A IS AN $m \times n$ MATRIX, THEN THE SUBSPACE OF R^n SPANNED BY THE ROW VECTORS OF A IS CALLED THE ROW SPACE OF A, AND THE SUBSPACE OF R^m SPANNED BY THE COLUMN VECTORS IS CALLED THE COLUMN SPACE OF A.

DEF. THE DIMENSION OF THE ROW SPACE OR COLUMN SPACE OF A MATRIX A IS CALLED THE RANK OF A AND IS DENOTED BY RANK (A).

EXAMPLE: CONSIDER THE FOLLOWING MATRIX

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$$

(a) FIND THE ROW (RANK) OF A?

RANK

ROW (RANK) $\rightarrow$ DIMENSION OF ROW SPACE OR FIND

NOTE: The dimension of the row space and column space are the same and hence Rank will eventually turn out to be unique, but this has to be proved. Otherwise we could have had that Row Rank and Column Rank were different.

NUMBER OF LINEARLY INDEPENDENT ROW VECTORS. SO

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LET $k_1 \underline{r_1} + k_2 \underline{r_2} + k_3 \underline{r_3} = \underline{0}$

$\Rightarrow k_1(2,-1,0,3) +$

$k_2(1,2,5,-1) +$

$k_3(7,-1,5,8) = (0,0,0,0)$

BUT LAST TIME IT WAS PROVED THAT $\boxed{k_1 = 3,\ k_2 = 1,\ k_3 = -1}$

$\therefore$ 3 ROW VECTORS ARE LINEARLY DEPENDENT $\therefore$ ROW (RANK) $\neq 3$.

NOW IGNORE $(7,-1,5,8)$, WHICH IS A LINEAR COMBINATION OF THE FIRST TWO.

CONSIDER FIRST TWO ROW VECTORS S.t.

$$a_1 \underline{r_1} + a_2 \underline{r_2} = \underline{0}$$

$\Rightarrow a_1(2,-1,0,3) + a_2(1,2,5,-1) = (0,0,0,0) \Longrightarrow 2a_1 + a_2 = 0,$

$-a_1 + 2a_2 = 0,\ 5a_2 = 0,\ 3a_1 - a_2 = 0$

$\Rightarrow \boxed{a_1 = a_2 = 0}$ $\because$ FIRST TWO

ARE LINEARLY INDEPENDENT $\therefore$

6 $\boxed{\text{ROW (RANK)}} = \boxed{2}$ OF THE GIVEN MATRIX OR $\boxed{(2,-1,0,3)}$ AND $\boxed{(1,2,5,-1)}$ ARE LINEARLY INDEPENDENT SINCE NONE OF THEM IS A MULTIPLE OF THE OTHER.

NOTE: 1 ROW SPACE OF $\boxed{A}$ IS SPANNED BY $\{(2,-1,0,3), (1,2,5,-1)\}$ WHICH IS A LINEARLY INDEPENDENT SET AND HENCE $\boxed{\text{BASIS}}$ FOR THE ROW SPACE $\therefore$ ROW (RANK) $= \boxed{\text{RANK}(A)} = \boxed{2}$ = DIMENSION OF ROW SPACE OF A = NO. OF LINEARLY INDEPENDENT $\boxed{\text{ROW}}$ VECTORS OF $\boxed{A}$.

2 SIMILARLY WE CAN PROVE THAT COLUMN SPACE OF $\boxed{A}$ IS SPANNED BY $\to \boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}}$ WHICH IS A LINEARLY INDEPENDENT SET OF VECTORS AND HENCE BASIS FOR THE

$\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$

7) COLUMN SPACE

$\therefore$ RANK(A) = COLUMN(RANK) = 2 = DIMENSION OF COLUMN SPACE OF $\boxed{A}$ = No. OF LINEARLY INDEPENDENT COLUMN VECTORS OF $\boxed{A}$.

2 (b) FIND NULLITY (A)

SOLUTION: FOR THIS WE HAVE TO SOLVE $A\underline{x} = \underline{0}$

$$\Rightarrow \underset{3\times 4}{\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \quad (1)$$

AUGMENTED MATRIX IS GIVEN BY

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{array}\right]$$

NOW REDUCE THIS TO REDUCED ECHELON FORM

$$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 2 & -1 & 0 & 3 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

NOTE: IN (1) NO. OF COLUMNS = NO. OF UNKNOWNS

$\boxed{3} \sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 5 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 7R_1$

$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{bmatrix}$ $R_2 \to -\frac{1}{5} R_2$, $R_3 \to -\frac{1}{15} R_3$

$\sim \begin{bmatrix} 1 & 2 & 5 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - R_2$

REDUCED ECHELON FORM $\leftarrow \sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$$\Rightarrow \begin{cases} x_1 + x_3 + x_4 = 0 & \text{— (2)} \\ x_2 + 2x_3 - x_4 = 0 & \text{— (3)} \end{cases}$$

$\to \boxed{2}$ EQUATIONS AND $\boxed{4}$ UNKNOWNS.

LET $x_3 = t$, $x_4 = s$

$\Rightarrow x_1 = -t - s$ from (2)

(3) $\Rightarrow x_2 = -2x_3 + x_4 = x_4 - 2x_3 = s - 2t$

THEREFORE THE COMPLETE SOLUTION IS GIVEN BY

$\boxed{4}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t-s \\ -2t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ BASIS FOR NULLSPACE OF

$A = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\therefore$ $\boxed{\text{NULLITY}}$ OF $\boxed{A = 2}$

= NUMBER OF FREE VARIABLES = DIMENSION OF NULLSPACE OF $\boxed{A}$.

$\boxed{x_3 = t,\ x_4 = s} \longrightarrow$ FREE VARIABLES

$x_1 = -t - s$

$x_2 = -2t + s$

$\longrightarrow$ CORRESPOND TO LEADING 1's IN REDUCED ECHELON FORM.

$\longleftarrow$ LEADING VARIABLES $\longrightarrow$ = RANK(A) = 2

(DIMENSION THEOREM FOR MATRICES)

P. 262 (6th ED.) OR P. 275 (7th ED.)

IF $\boxed{A}$ HAS $\boxed{n}$ COLUMNS THEN

$$\boxed{\text{RANK}(A) + \text{NULLITY}(A) = n}$$

$n \longrightarrow$ NO. OF COLUMNS = TOTAL NUMBER OF VARIABLES

= LEADING + FREE

= RANK(A) + NULLITY(A) = n

THEOREM 5.5.1

*A system of linear equations $Ax = b$ is consistent if and only if **b** is in the column space of A.*

suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It follows from Formula 10 of Section 1.3 that if $c_1, c_2, \ldots, c_n$ denote the column vectors of A, then the product Ax can be expressed as a linear combination of these column vectors with coefficients from ***x***; that is,

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n \tag{1}$$

Thus a linear system, $Ax = b$, of m equations in n unknowns can be written as

$$x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b \tag{2}$$

from which we conclude that $Ax = b$ is consistent if and only if ***b*** is expressible as a linear combination of the column vectors of A or, equivalently, if and only if ***b*** is in the column space of A.

THEOREM 5.5.3

Elementary row operations do not change the nullspace of a matrix.

THEOREM 5.5.5

If A and B are row equivalent matrices, then

(a) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(b) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

The following theorem makes it possible to find bases for the row and column spaces of a matrix in row-echelon form by inspection.

THEOREM 5.5.6

If a matrix R is in row-echelon form, then the row vectors with the leading 1' s (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1' s of the row vectors form a basis for the column space of R.

EXAMPLE 5 **Bases for Row and Column Spaces**

The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row-echelon form. From Theorem 5.5.6, the vectors

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r}_2 &= [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

EXAMPLE 6 **Bases for Row and Column Spaces**

Find bases for the row and column spaces of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution

Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row-echelon form of A. Reducing A to row-echelon form, we obtain (verify)

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 5.5.6, the nonzero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A. These basis vectors are

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4] \\ \mathbf{r}_2 &= [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5] \end{aligned}$$

Keeping in mind that A and R may have different column spaces, we cannot find a basis for the column space of A *directly* from the column vectors of R. However, it follows from Theorem 5.5.5*b* that if we can find a set of column vectors of R that forms a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A.

The first, third, and fifth columns of R contain the leading 1's of the row vectors, so

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R; thus the corresponding column vectors of A—namely,

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

EXAMPLE 8 Basis for the Row Space of a Matrix

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A.

Solution

We will transpose A, thereby converting the row space of A into the column space of A^T; then we will use the method of Example 6 to find a basis for the column space of A^T; and then we will transpose again to convert column vectors back to row vectors. Transposing A yields

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

Reducing this matrix to row-echelon form yields

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns contain the leading 1's, so the corresponding column vectors in A^T form a basis for the column space of A^T; these are

$$c_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad \text{and} \quad c_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$r_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3], \qquad r_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6],$$

and

$$r_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]$$

$$\mathbf{r}_1 = [1 \ \ -2 \ \ 0 \ \ 0 \ \ 3], \qquad \mathbf{r}_2 = [2 \ \ -5 \ \ -3 \ \ -2 \ \ 6],$$

and

$$\mathbf{r}_4 = [2 \ \ 6 \ \ 18 \ \ 8 \ \ 6]$$

for the row space of A.

Exercise set 5.5

Q2 (a) (b), Q6 (a),(c), Q13, Read, understand and explain all the steps in the proof of Theorem 5.5.4