Lecture 28

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Application to Diff Eqns

9.1
APPLICATION TO DIFFERENTIAL EQUATIONS

Many laws of physics, chemistry, biology, engineering, and economics are described in terms of differential equations—that is, equations involving functions and their derivatives. The purpose of this section is to illustrate one way in which linear algebra can be applied to certain systems of differential equations. The scope of this section is narrow, but it illustrates an important area of application of linear algebra.

Terminology

One of the simplest differential equations is

$$y' = ay \tag{1}$$

where $y = f(x)$ is an unknown function to be determined, $y' = dy/dx$ is its derivative, and a is a constant. Like most differential equations, 1 has infinitely many solutions; they are the functions of the form

$$y = ce^{ax} \tag{2}$$

where c is an arbitrary constant. Each function of this form is a solution of $y' = ay$ since

$$y' = cae^{ax} = ay$$

Conversely, every solution of $y' = ay$ must be a function of the form ce^{ax} (Exercise 5), so 2 describes all solutions of $y' = ay$. We call 2 the ***general solution*** of $y' = ay$.

Sometimes the physical problem that generates a differential equation imposes some added conditions that enable us to isolate one ***particular solution*** from the general solution. For example, if we require that the solution of $y' = ay$ satisfy the added condition

$$y(0) = 3 \tag{3}$$

that is, $y = 3$ when $x = 0$, then on substituting these values in the general solution $y = ce^{ax}$ we obtain a value for c—namely, $3 = ce^0 = c$. Thus

$$y = 3e^{ax}$$

is the only solution of $y' = ay$ that satisfies the added condition. A condition such as 3, which specifies the value of the solution at a point, is called an ***initial condition***, and the problem of solving a differential equation subject to an initial condition is called an ***initial-value problem***.

Linear Systems of First-Order Equations

In this section we will be concerned with solving systems of differential equations having the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \vdots\;\; & \quad\;\; \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ y_4' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{aligned} \tag{4}$$

where $y_1 = f_1(x), y_2 = f_2(x), \ldots, y_n = f_n(x)$ are functions to be determined, and the a_{ij}'s are constants. In matrix notation, 4 can be written as

$$\begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or, more briefly,

$$\mathbf{y}' = A\mathbf{y}$$

EXAMPLE 1 Solution of a System with Initial Conditions

(a) Write the following system in matrix form:

$$\begin{aligned} y'_1 &= 3y_1 \\ y'_2 &= -2y_2 \\ y'_3 &= 5y_3 \end{aligned}$$

(b) Solve the system.

(c) Find a solution of the system that satisfies the initial conditions $y_1(0) = 1$, $y_2(0) = 4$, and $y_3(0) = -2$.

Solution (a)

$$\begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{5}$$

or

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \mathbf{y}$$

Solution (b)

Because each equation involves only one unknown function, we can solve the equations individually. From 2, we obtain

$$\begin{aligned} y_1 &= c_1 e^{3x} \\ y_2 &= c_2 e^{-2x} \\ y_3 &= c_3 e^{5x} \end{aligned}$$

or, in matrix notation,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{3x} \\ c_2 e^{-2x} \\ c_3 e^{5x} \end{bmatrix} \tag{6}$$

Solution (c)

From the given initial conditions, we obtain

$$\begin{aligned} 1 &= y_1(0) = c_1 e^0 = c_1 \\ 4 &= y_2(0) = c_2 e^0 = c_2 \\ -2 &= y_3(0) = c_3 e^0 = c_3 \end{aligned}$$

so the solution satisfying the initial conditions is

$$y_1 = e^{3x}, \qquad y_2 = 4e^{-2x}, \qquad y_3 = -2e^{5x}$$

or, in matrix notation,

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{3x} \\ 4e^{-2x} \\ -2e^{5x} \end{bmatrix}$$

◆

The system in the preceding example is easy to solve because each equation involves only one unknown function, and this is the case because the matrix of coefficients for the system in 5 is diagonal. But how do we handle a system $\mathbf{y}' = A\mathbf{y}$ in which the matrix A is not diagonal? The idea is simple: Try to make a substitution for $\boldsymbol{y}$ that will yield a new system with a diagonal coefficient matrix; solve this new simpler system, and then use this solution to determine the solution of the original system.

The kind of substitution we have in mind is

$$\begin{aligned} y_1 &= p_{11}u_1 + p_{12}u_2 + \cdots + p_{1n}u_n \\ y_2 &= p_{21}u_1 + p_{22}u_2 + \cdots + p_{2n}u_n \\ &\vdots \\ y_n &= p_{n1}u_1 + p_{n2}u_2 + \cdots + p_{nn}u_n \end{aligned} \tag{7}$$

or, in matrix notation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \text{or, more briefly,} \qquad \mathbf{y} = P\mathbf{u}$$

In this substitution, the p_{ij}'s are constants to be determined in such a way that the new system involving the unknown functions $u_1, u_2, \ldots, u_n$ has a diagonal coefficient matrix. We leave it for the reader to differentiate each equation in 7 and deduce

$$\mathbf{y}' = P\mathbf{u}'$$

If we make the substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ in the original system

$$\mathbf{y}' = A\mathbf{y}$$

and if we assume P to be invertible, then we obtain

$$P\mathbf{u}' = A(P\mathbf{u})$$

or

$$\mathbf{u}' = (P^{-1}AP)\mathbf{u} \qquad \text{or} \qquad \mathbf{u}' = D\mathbf{u}$$

where $D = P^{-1}AP$. The choice for P is now clear; if we want the new coefficient matrix D to be diagonal, we must choose P

to be a matrix that diagonalizes A.

Solution by Diagonalization

The preceding discussion suggests the following procedure for solving a system $\mathbf{y}' = A\mathbf{y}$ with a diagonalizable coefficient matrix A.

Step 1. Find a matrix P that diagonalizes A.

Step 2. Make the substitutions $\mathbf{y} = P\mathbf{u}$ and $\mathbf{y}' = P\mathbf{u}'$ to obtain a new "diagonal system" $\mathbf{u}' = D\mathbf{u}$, where $D = P^{-1}AP$.

Step 3. Solve $\mathbf{u}' = D\mathbf{u}$.

Step 4. Determine $\boldsymbol{y}$ from the equation $\mathbf{y} = P\mathbf{u}$.

EXAMPLE 2 **Solution Using Diagonalization**

(a) Solve the system

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 4y_1 - 2y_2 \end{aligned}$$

(b) Find the solution that satisfies the initial conditions $y_1(0) = 1, y_2(0) = 6$.

Solution (a)

The coefficient matrix for the system is

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

As discussed in Section 7.2, A will be diagonalized by any matrix P whose columns are linearly independent eigenvectors of A. Since

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$$

the eigenvalues of A are $\lambda = 2, \lambda = -3$. By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if $\boldsymbol{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$—that is, of

$$\begin{bmatrix} \lambda-1 & -1 \\ -4 & \lambda+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\lambda=2$, this system becomes

$$\begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system yields $x_1=t$, $x_2=t$, so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} t \\ t \end{bmatrix}=t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus

$$\mathbf{p}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda=2$. Similarly, the reader can show that

$$\mathbf{p}_2=\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda=-3$. Thus

$$P=\begin{bmatrix} 1 & -\frac{1}{4} \\ 1 & 1 \end{bmatrix}$$

diagonalizes A, and

$$D=P^{-1}AP=\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

Therefore, the substitution

$$\mathbf{y}=P\mathbf{u} \quad \text{and} \quad \mathbf{y}'=P\mathbf{u}'$$

yields the new “diagonal system”

$$\mathbf{u}'=D\mathbf{u}=\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{u} \quad \text{or} \quad \begin{aligned} u_1' &= 2u_1 \\ u_2' &= -3u_2 \end{aligned}$$

From 2 the solution of this system is

$$\begin{aligned} u_1 &= c_1e^{2x} \\ u_2 &= c_2e^{-3x} \end{aligned} \quad \text{or} \quad \mathbf{u}=\begin{bmatrix} c_1e^{2x} \\ c_2e^{-3x} \end{bmatrix}$$

so the equation $\mathbf{y}=P\mathbf{u}$ yields, as the solution for $\mathbf{y}$,

$$\mathbf{y}=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{4} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{2x} \\ c_2e^{-3x} \end{bmatrix}=\begin{bmatrix} c_1e^{2x}-\frac{1}{4}c_2e^{-3x} \\ c_1e^{2x}+c_2e^{-3x} \end{bmatrix}$$

or

$$\begin{aligned} y_1 &= c_1e^{2x}-\tfrac{1}{4}c_2e^{-3x} \\ y_2 &= c_1e^{2x}+c_2e^{-3x} \end{aligned} \tag{8}$$

Solution (b)

If we substitute the given initial conditions in 8, we obtain

$$\begin{aligned} c_1-\tfrac{1}{4}c_2 &= 1 \\ c_1+c_2 &= 6 \end{aligned}$$

Solving this system, we obtain $c_1=2$, $c_2=4$, so from 8, the solution satisfying the initial conditions is

$$y_1 = 2e^{2x} - e^{-3x}$$
$$y_2 = 2e^{2x} + 4e^{-3x}$$

We have assumed in this section that the coefficient matrix of $\mathbf{y}' = A\mathbf{y}$ is diagonalizable. If this is not the case, other methods must be used to solve the system. Such methods are discussed in more advanced texts.

Exercise Set 9.1

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1.

(a) Solve the system

$$\begin{aligned} y_1' &= y_1 + 4y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$$

(b) Find the solution that satisfies the initial conditions $y_1(0) = 0,\ y_2(0) = 0$.

2.

(a) Solve the system

$$\begin{aligned} y_1' &= y_1 + 3y_2 \\ y_2' &= 4y_1 + 5y_2 \end{aligned}$$

(b) Find the solution that satisfies the conditions $y_1(0) = 2,\ y_2'(0) = 1$.

3.

(a) Solve the system

$$\begin{aligned} y_1' &= 4y_1 \phantom{{}+y_2} + y_3 \\ y_2' &= -2y_1 + y_2 \\ y_3' &= -2y_1 \phantom{{}+y_2} + y_3 \end{aligned}$$

(b) Find the solution that satisfies the initial conditions $y_1(0) = -1,\ y_2(0) = 1,\ y_3(0) = 0$.

4. Solve the system