Lecture 18

Thursday, March 17, 2022 12:28 AM

Recall

THEOREM 5.5.4

Elementary row operations do not change the row space of a matrix.

THEOREM 5.5.5

If A and B are row equivalent matrices, then

(a) *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*

(b) *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

Since two matrices are row equivalent if one can be obtained from the other by using EROs, any matrix is row equivalent to its row-echelon forms.

From the above we get

THEOREM 5.6.1

If A is any matrix, then the row space and column space of A have the same dimension.

R.E Form

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof Let R be any row-echelon form of A. It follows from Theorem 5.5.4 that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R)$$

and it follows from Theorem 5.5.5*b* that

$$\dim(\text{column space of } A) = \dim(\text{column space of } R)$$

Thus the proof will be complete if we can show that the row space and column space of R have the same dimension. But the dimension of the row space of R is the number of nonzero rows, and the dimension of the column space of R is the number of columns that contain leading 1's (Theorem 5.5.6). However, the nonzero rows are precisely the rows in which the leading 1's occur, so the number of leading 1's and the number of nonzero rows are the same. This shows that the row space and column space of R have the same dimension. ■

The theorem above justifies using the term "rank" for both the dimension of the row space and the column space!

THEOREM 5.6.2

If A is any matrix, then $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

This is a fairly simple and obvious theorem.

Proof

$$\operatorname{rank}(A) = \dim(\text{row space of } A) = \dim(\text{column space of } A^T) = \operatorname{rank}(A^T)\ .$$

■

We now come to the essential result often called the Dimension Theorem (which was stated last time)

THEOREM 5.6.3

Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n \tag{4}$$

$Ax = 0$

$m \times n$, $n \times 1$, $m \times 1$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Proof Since A has n columns, the homogeneous linear system $Ax = 0$ has n unknowns (variables). These fall into two categories: the leading variables and the free variables. Thus

$$\begin{bmatrix} \text{number of leading} \\ \text{variables} \end{bmatrix} + \begin{bmatrix} \text{number of free} \\ \text{variables} \end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading 1's in the reduced row-echelon form of *A*, and this is the rank of *A*. Thus

$$\operatorname{rank}(A) + \begin{bmatrix} \text{number of free} \\ \text{variables} \end{bmatrix} = n$$

The number of free variables is equal to the nullity of *A*. This is so because the nullity of *A* is the dimension of the solution space of $Ax = 0$, which is the same as the number of parameters in the general solution [see 3, for example], which is the same as the number of free variables. Thus

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

■

Now we see some examples of the reasoning presented above and in the previous lecture in action!

RESULT:
IN $AX=0$ → m → EQUATIONS IN
n → UNKNOWNS ...

Same as above

RESULT:

IN $AX=0 \rightarrow$ $m \rightarrow$ EQUATIONS IN $n \rightarrow$ UNKNOWNS WITH $\boxed{m<n}$, AND IF THERE ARE $\boxed{r}$ NONZERO ROWS IN THE REDUCED ROW-ECHELON FORM OF THE AUGMENTED MATRIX THEN NUMBER OF $\boxed{\text{FREE}}$ VARIABLES ARE $= \boxed{n-r}$. LEADING VARIABLES $= r$.

EXAMPLE: WE PROVED THAT FOR THE FOLLOWING MATRIX

$$\begin{bmatrix} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{bmatrix}$$

REDUCED ROW-ECHELON FORM IS

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

WHICH CONTAINS TWO NON-ZERO ROWS $\Rightarrow \boxed{r=2}$

HERE $m=3<4=n$

$\therefore \quad n-r = 4-2 = 2$, THEREFORE

NUMBER OF FREE VARIABLES

$= \boxed{n-r} = 4-2=2.$

AS WE SAW THAT

$x_3 = t$ $\boxed{x_3=t}$ AND $\boxed{x_4=s}$ WERE FREE VARIABLES AND THEIR NO. = 2

ALSO $\boxed{x_1}$ AND $\boxed{x_2}$ WERE LEADING VARIABLES GIVEN BY

$$\boxed{x_1 = -t-s}$$
$$\boxed{x_2 = s-2t}$$

AND ARE $= \boxed{r} = 2 =$ RANK

WHERE $\boxed{r}$ INDICATES NONZERO ROWS IN THE REDUCED ROW-ECHELON FORM OF THE AUGMENTED MATRIX OF $\boxed{A\underline{X} = \underline{0}}$.

ANOTHER METHOD TO FIND THE BASIS FOR THE ROW SPACE OF A MATRIX.

IN ORDER TO DO THIS WE REFER TO THE FOLLOWING THEOREM:

(5.5.6

THE NONZERO ROW VECTORS IN ANY ROW-ECHELON FORM

THE NONZERO ROW VECTORS IN ANY ROW-ECHELON FORM OF A MATRIX FORM A BASIS FOR THE ROW SPACE OF THAT MATRIX.

EXAMPLE: ∵ THE REDUCED ROW ECHELON FORM OF

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$$ IS GIVEN BY

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ ∴ ACCORDING TO THE ABOVE THEOREM THE TWO NONZERO ROW VECTORS IN R FORM THE BASIS FOR THE ROW SPACE OF A AND R, THEREFORE THE BASIS FOR THE ROW

SPACE OF $\boxed{R}$ AND $\boxed{A}$ IS GIVEN BY

$$\{(1,0,1,1),\ (0,1,2,-1)\}$$

BECAUSE ACCORDING TO THEOREM 5.5.4 (P.251 8th ED.) OR THEOREM (5.5.4) (P.263 7th ED.)

ELEMENTARY ROW OPERATIONS DO NOT CHANGE THE ROW SPACE OF A MATRIX,

$\left\{\begin{array}{l}\text{IF } \boxed{A} \text{ IS AN } m \times n \text{ MATRIX,} \\ \text{AND } B \text{ IS ROW EQUIVALENT TO } A \end{array}\right.$ $\to \boxed{A}$

THEN (i) ROW SPACE IS A SUBSPACE OF $\boxed{R^n}$

(ii) IF THE ROW OPERATION IS A ROW INTERCHANGE, THEN $\boxed{B}$ AND $\boxed{A}$ HAVE SAME ROW VECTORS

(iii) IF THE ROW OPERATION IS MULTIPLICATION OF A ROW BY A NONZERO SCALAR $\boxed{\text{OR}}$ THE ADDITION OF A MULTIPLE OF ONE ROW TO ANOTHER, THEN THE ROW VECTORS

$r'_1, r'_2, \ldots\ldots, r'_m$ OF B ARE LINEAR COMBINATIONS OF $r_1, r_2, \ldots\ldots, r_m$; THUS, THEY LIE IN THE ROW SPACE OF A, $\because$ ROW SPACE (SUBSPACE OF R^n) IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION.

METHOD TO FIND THE BASIS FOR THE COLUMN SPACE OF AN $m \times n$ MATRIX A.

5.5.6

FOR THIS WE REFER TO TH

WHICH STATES THAT

IF A IS AN $m \times n$ MATRIX AND R IS ITS ROW-ECHELON FORM THEN THE COLUMN VECTORS WITH THE LEADING 1'S OF THE ROW VECTORS FORM A BASIS FOR THE COLUMN SPACE OF R, AND THE CORRESPONDING COLUMN VECTORS IN A FORM THE BASIS

FOR THE COLUMN SPACE OF $\boxed{A}$.

EXAMPLE:

FOR $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix}$

THE REDUCED ROW ECHELON FORM IS GIVEN BY

$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_1 + 2C_2 = C_3$

$C_1 - C_2 = C_4$

→ THE FIRST $\boxed{\text{TWO}}$ COLUMN VECTORS IN $\boxed{R}$ WHICH CONTAIN THE LEADING 1'S FORM THE BASIS FOR THE COLUMN SPACE OF $\boxed{R}$ (BUT NOT A) AND THE CORRESPONDING COLUMN VECTORS IN $\boxed{A}$ FORM THE BASIS FOR THE COLUMN SPACE OF $\boxed{A}$ (BUT NOT R).

BECAUSE ELEMENTARY ROW OPERATIONS USUALLY CHANGE THE COLUMN SPACE.

REMARKS:

① **BASIS** FOR THE COLUMN SPACE FOR $\boxed{R} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ AND

② **BASIS** FOR THE COLUMN SPACE FOR $\boxed{A} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}$

③ ELEMENTARY $\boxed{\text{ROW}}$ OPERATIONS CAN CHANGE THE COLUMN SPACE:

LET $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\boxed{3C_1 = C_2}$

$\therefore$ THE COLUMN SPACE OF $\boxed{A}$ CONSISTS OF ALL SCALAR MULTIPLES OF THE FIRST COLUMN VECTOR $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

NOW $A \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \; R_2 \rightarrow R_2 - 2R_1$

AGAIN $\longrightarrow \boxed{3C_1 = C_2}$

$\therefore$ THE COLUMN SPACE OF $B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ CONSISTS OF ALL SCALAR MULTIPLES OF THE FIRST COLUMN VECTOR i.e. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. THIS IS NOT THE SAME AS THE COLUMN SPACE OF A.

REMARK:

IF A IS A MATRIX AND R IS ITS ECHELON FORM THEN THE NUMBER OF NONZERO ROWS OR NUMBER OF THE COLUMN VECTORS THAT CONTAIN THE LEADING 1'S IN R IS THE RANK OF THE MATRIX A. ∴ RANK OF

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 5 & -1 \\ 7 & -1 & 5 & 8 \end{bmatrix} = 2 \quad \because$$

$$\text{IN } R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{TWO NONZERO ROWS}$$

AND NO. OF THE COLUMN VECTORS WITH LEADING 1'S IN R = 2.

ANOTHER DEFINITION OF RANK OF A MATRIX.

RANK (A) = HIGHEST ORDER OF THE NONZERO DETERMINANT OR SUBDETERMINANT OF A.

EXAMPLE:

RANK OF $A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = ?$

$$\text{DET}(A) = 2(2+1) - 1(-1) + 7(-1)$$
$$= 6 + 1 - 7 = 7 - 7 = 0$$

$$\therefore \boxed{\text{RANK}(A) \neq 3}$$

CONSIDER THE SUBDETERMINANT $\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4 + 1 = 5 \neq 0$

$$\therefore \boxed{\text{RANK}(A) = 2}$$

NOTE: LAST TIME WE PROVED THAT NULLITY (A) = 1, AND RANK (A) + NULLITY (A) = 2 + 1 = 3 = NO. OF COLUMNS ACCORDING TO THE DIMENSION THEOREM.

Exercise set 5.6
Q4, Q5, Q6, Q7, Q8, Q15