Midterm Exam A
CS/MATH 113 Discrete Mathematics

Habib University — Spring 2023

11 March, 2023. 1300-1415h.

This exam consists of 4 questions for a total of 40 points on 2 pages. Attempt all problems and submit this sheet with your answer sheet by the end of the exam.

Student ID: ______________________________

Student Name: ______________________________

1. Tautologies (10 points)

Prove or disprove that the following are tautologies.

(a) $(a \implies b) \iff (\neg a \implies \neg b)$

Solution: We prove using a counterexample that this statement is not a tautology.

Proof. Consider $a \equiv T, b \equiv F$.
Then LHS is $T \implies F$ which evaluates to F.
And RHS is $F \implies T$ which evaluates to T. □

(b) $(a \wedge b \wedge c \wedge d) \implies (c \wedge b \wedge a \wedge d)$

Solution: We use the commutative property of conjunction.

Proof. Because of the commutative property, LHS $\equiv$ RHS.
Therefore, LHS $\implies$ RHS. □

2. Logical Equivalence (10 points)

Show that $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$.

Solution: We prove that LHS $\implies$ RHS, and vice versa, through a sequence of inferences.

Proof. <u>Case</u>: LHS $\implies$ RHS

$P(c) \wedge Q(c)$	UI on LHS, c is an arbitrary element	(1)
$P(c)$	simplification on (1)	(2)
$\forall x P(x)$	UG on (2)	(3)
$Q(c)$	simplification on (1)	(4)
$\forall x Q(x)$	UG on (4)	(5)
$\forall x Q(x) \wedge \forall x Q(x)$	conjunction on (3), (5)	

<u>Case</u>: RHS $\implies$ LHS

$P(c) \wedge Q(c)$	UI on RHS, c is an arbitrary element	(1)
$\forall x(P(x) \wedge Q(x))$	UG on (1)	

$\square$

3. Subsets (10 points)

Prove that $\{12n \mid n \in \mathbb{Z}\}$ is a subset of $\{2n \mid n \in \mathbb{Z}\} \cap \{3n \mid n \in \mathbb{Z}\}$.

Solution: Let $A = \{12n \mid n \in \mathbb{Z}\}, B = \{2n \mid n \in \mathbb{Z}\}, C = \{3n \mid n \in \mathbb{Z}\}$.
Then we have to show that $A \subseteq (B \cap C)$.
We do so by showing that $A \subseteq B$ and $A \subseteq C$.

Proof. <u>Case</u>: $A \subseteq B$.
Consider $a \in A$.
Then $a = 12k$ for some $k \in \mathbb{Z}$.
Then $a = 2 \cdot 6k$.
$\therefore a \in B$.

<u>Case</u>: $A \subseteq C$.
Consider $a \in A$.
Then $a = 12k$ for some $k \in \mathbb{Z}$.
Then $a = 3 \cdot 4k$.
$\therefore a \in C$. $\square$

4. Function Properties (10 points)

Given $g : A \to B$ and $f : B \to C$, prove that: if $(f \circ g)$ is onto, then so is f.

Solution: We prove the above by contradiction.

Proof. Assume that $(f \circ g)$ is onto and f is not.
Now, $f \circ g : A \to C$.
For an arbitrary $a \in A$, we have that $(f \circ g)(a) = f(g(a))$.
Because f is not onto, there is an element, $c \in C$ that f does not map to.
That is, $\nexists a \in A \ni f(g(a)) = c$.
$\therefore f \circ g$ is not onto. $\perp$ $\square$

Good luck!