Important facts and Theorems 4

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DEFINITION

A vector w is called a ***linear combination*** of the vectors $v_1, v_2, \ldots, v_r$ if it can be expressed in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

Remark If $r = 1$, then the equation in the preceding definition reduces to $w = k_1 v_1$; that is, w is a linear combination of a single vector v_1 if it is a scalar multiple of v_1.

EXAMPLE 8 Vectors in R^3 Are Linear Combinations of i, j, and k

Every vector $\mathbf{v} = (a, b, c)$ in R^3 is expressible as a linear combination of the standard basis vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

since

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

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EXAMPLE 9 Checking a Linear Combination

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of u and v and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of u and v.

Solution

In order for w to be a linear combination of u and v, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

or

$$(9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Similarly, for $\mathbf{w}'$ to be a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

or

$$(4, -1, 8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equations is inconsistent (verify), so no such scalars k_1 and k_2 exist. Consequently, $\mathbf{w}'$ is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Spanning

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are vectors in a vector space V, then generally some vectors in V may be linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ and others may not. The following theorem shows that if we construct a set W consisting of all those vectors that are expressible as linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, then W forms a subspace of V.

THEOREM 5.2.3

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *are vectors in a vector space V, then*

(a) *The set W of all linear combinations of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *is a subspace of V.*

(b) *W is the smallest subspace of V that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *in the sense that every other subspace of V that contains* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *must contain W.*

Proof (a) To show that W is a subspace of V, we must prove that it is closed under addition and scalar multiplication. There is at least one vector in W—namely $\mathbf{0}$, since $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r$$

and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars. Therefore,

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \cdots + (c_r + k_r)\mathbf{v}_r$$

and, for any scalar *k*,,

$$k\mathbf{u} = (kc_1)\mathbf{v}_1 + (kc_2)\mathbf{v}_2 + \cdots + (kc_r)\mathbf{v}_r$$

Thus $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ and consequently lie in *W*. Therefore, *W* is closed under addition and scalar multiplication.

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Proof (b) Each vector $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ since we can write

$$\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_r$$

Therefore, the subspace *W* contains each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let W' be any other subspace that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Since W' is closed under addition and scalar multiplication, it must contain all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Thus, W' contains each vector of *W*.

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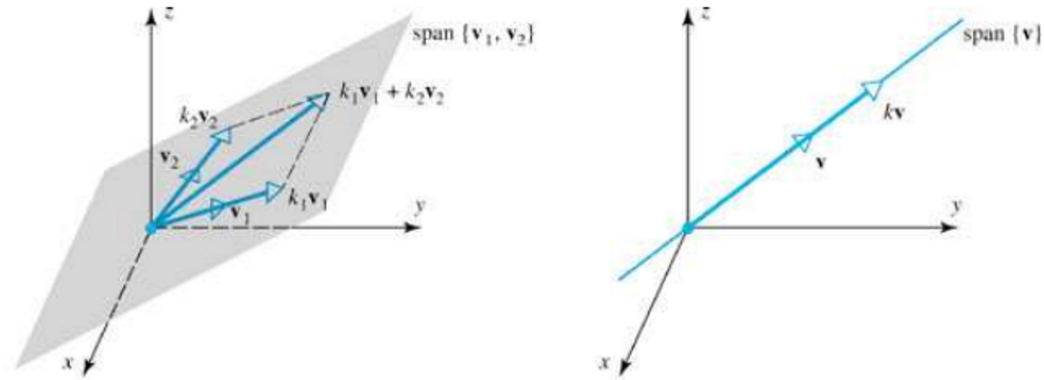
DEFINITION

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the ***space spanned*** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ ***span*** W. To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$, we write

$$W = \text{span}(S) \qquad \text{or} \qquad W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$$

EXAMPLE 10 Spaces Spanned by One or Two Vectors

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are noncollinear vectors in R^3 with their initial points at the origin, then span $\{\mathbf{v}_1, \mathbf{v}_2\}$, which consists of all linear combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$, is the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ (see Figure 5.2.5*a*). Similarly, if $\boldsymbol{v}$ is a nonzero vector in R^2 or R^3, then span $\{\mathbf{v}\}$, which is the set of all scalar multiples $k\mathbf{v}$, is the line determined by $\mathbf{v}$ (see Figure 5.2.5*b*).



(*a*) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(*b*) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

Figure 5.2.5

EXAMPLE 11 Spanning Set for P_n

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n defined in Example 5 since each polynomial $\boldsymbol{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. We can denote this by writing

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$

EXAMPLE 12 Three Vectors That Do Not Span R^3

Determine whether $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

Solution

We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

or

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

or

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

The problem thus reduces to determining whether this system is consistent for all values of b_1, b_2, and b_3. ~~By parts (e) and~~ (g) of Theorem 4.3.4, this system is consistent for all b_1, b_2, and b_3 if and only if the coefficient matrix

Same as Thm 2.3.6

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

has a nonzero determinant. However, $\det(A) = 0$ (verify), so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

Spanning sets are not unique. For example, any two noncollinear vectors that lie in the plane shown in Figure 5.2.5 will span that same plane, and any nonzero vector on the line in that figure will span the same line. We leave the proof of the following useful theorem as an exercise.

THEOREM 5.2.4

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are two sets of vectors in a vector space V, then

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S' and each vector in S' is a linear combination of those in S.

Exercise Set 5.2

Q1, Q2, Q5, Q7, Q9, Q15,Q23, Q26