Lecture 2

Wednesday, January 12, 2022 8:09 PM

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LECTURE NO. 2

1

MATH 221, LINEAR ALGEBRA

RECALL THAT TWO MATRICES ARE EQUAL IF THE CORRESPONDING ENTRIES ARE SAME AND BOTH ARE OF THE SAME SIZE (ORDER) AS WELL.

E.G. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow \boxed{A = B}$

(2) $\therefore$ BY THE DEFINITION OF EQUAL (2 MATRICES THE LINEAR SYSTEM

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad\ \vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

CAN BE WRITTEN AS

$$
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n\\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
=
\overset{m\times 1}{\begin{bmatrix}
b_1\\
b_2\\
\vdots\\
b_m
\end{bmatrix}}
$$

FURTHER THE $\boxed{m \times 1}$ MATRIX ON THE LEFT HAND SIDE CAN BE WRITTEN AS THE $\boxed{\text{PRODUCT}}$ OF $\boxed{\text{TWO}}$ MATRICES GIVEN BY

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n \text{ MATRIX}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}^{n\times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m\times 1}$$

IF WE DESIGNATE THESE MATRICES BY $\boxed{A}$, $\boxed{\underline{X}}$ AND $\boxed{\underline{B}}$, RESPECTIVELY, THE ORIGINAL SYSTEM OF $\boxed{m}$ EQUATIONS IN $\boxed{n}$ UNKNOWNS HAS BEEN REPLACED BY THE <u>SINGLE MATRIX</u> EQUATION

$$A\underline{X} = \underline{B},$$

WHERE $\boxed{A}$ IS CALLED THE <u>COEFFICIENT MATRIX</u>. BUT NOTE THE FOLLOWING:

$$\underset{m\times n}{A} \quad \underset{n\times 1}{\underline{X}} = \underset{m\times 1}{\underline{B}}$$

INSIDE

OUTSIDE

NO. OF COLUMNS OF A = NO. OF ROWS OF $\underline{X}$ = n

P.27 → 27 (8th ED.), P.28 (7th ED.) (MULTIPLICATION OF MATRICES)

TWO MATRICES $\boxed{A}$ AND $\boxed{B}$ CAN BE MULTIPLIED IF THE NUMBER OF COLUMNS OF $\boxed{A}$ ARE EQUAL TO THE NUMBER OF ROWS OF $\boxed{B}$.

$$\underset{m \times r}{A} \quad \underset{r \times n}{B} \quad = \quad \underset{m \times n}{AB}$$

INSIDE

OUTSIDE

BUT HOW TO MULTIPLY? CONSIDER THE FOLLOWING EXAMPLE:

FIND AB, WHERE

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \rightarrow 2\times 3, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \rightarrow 3\times 4$$

SOLUTION: SINCE $\boxed{A}$ IS A $\boxed{2\times 3}$ MATRIX AND $\boxed{B}$ IS A $\boxed{3\times 4}$ MATRIX, THE PRODUCT $\boxed{AB}$ IS $\boxed{2\times 4}$ MATRIX.

5)

AB

$$= \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}_{3\times 4}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \end{bmatrix} = \underset{2\times 4}{C} \text{ (SAY)}$$

TO FIND $\boxed{C_{11}}$ IN $\boxed{AB}$ OR $\boxed{C}$, SINGLE OUT $\boxed{\text{ROW ONE}}$ FROM THE MATRIX $\boxed{A}$ AND $\boxed{\text{COLUMN ONE}}$ FROM THE MATRIX $\boxed{B}$. MULTIPLY THE CORRESPONDING ENTRIES FROM THE $\boxed{\text{ROW}}$ AND $\boxed{\text{COLUMN}}$ TOGETHER AND THEN $\boxed{\text{ADD}}$ UP THE RESULTING PRODUCTS i.e.

$$C_{11} = 1(4) + 2(0) + 4(2) = 12$$

SIMILARLY

$$C_{12} = 1(1) + 2(-1) + 4(7) = 27$$

$$C_{13} = 1(4) + 2(3) + 4(5) = 30$$

$$C_{14} = 1(3) + 2(1) + 4(2) = 13$$

$C_{21} = 8$, $C_{22} = -4$, $C_{23} = 26$, $C_{24} = 12$.

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FINALLY

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix} \rightarrow 2 \times 4$$

<u>DEFINITION</u>: (P. 26)/8th ED., P.27/11th ED.

IF $\boxed{A}$ IS ANY MATRIX AND $\boxed{c}$ IS ANY <u>SCALAR</u>, THEN THE PRODUCT $\boxed{cA}$ IS THE <u>MATRIX</u> OBTAINED BY MULTIPLYING EACH ENTRY OF $\boxed{A}$ BY $\boxed{c}$

<u>EXAMPLE</u>: IF $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}$

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$$

NOTE: TWO MATRICES $\boxed{A}$ AND $\boxed{B}$ COMMUTE IF $\boxed{AB = BA}$ BUT IN GENERAL $AB \neq BA$ (P. 38)/7th ED. (P. 37)/8th ED.

EXAMPLE: LET $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, FIND $\boxed{AB}$ AND $\boxed{BA}$

SOLUTION:

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

WHICH SHOWS THAT

$$\boxed{AB \neq BA}$$

TRY THE FOLLOWING:

IF $\boxed{A}$ AND $\boxed{B}$ ARE TWO SQUARE MATRICES OF SAME SIZE, THEN FIND THE CONDITION SUCH THAT $\boxed{(A+B)^2 = A^2 + B^2 + 2AB}$

NOTE: IN SUCH TYPE OF

QUESTIONS DON'T USE MATRIX ENTRIES. JUST USE MATRIX SYMBOLS.

SOLUTION: $(A+B)^2$

$= (A+B)(A+B) = A^2 + AB + BA + B^2$

IF $\boxed{A}$ AND $\boxed{B}$ COMMUTE THEN

$AB = BA$

$$\boxed{\therefore\ (A+B)^2 = A^2 + 2AB + B^2.}$$

NOTE: IF $\boxed{A}$, $\boxed{B}$ AND $\boxed{C}$ ARE MATRICES SUCH THAT $\boxed{AB}$ AND $\boxed{BC}$ ARE DEFINED THEN $\boxed{A(BC) = (AB)C}$, WHICH IS ALSO CALLED ASSOCIATIVE LAW FOR MULTIPLICATION.

EXAMPLE: IF $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}$,

$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ (P.39)

THEN PROVE THAT

$$\boxed{(AB)C = A(BC)}$$

SOLUTION:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

SIMILARLY

$$A(BC) = A \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

(CHECK)

BASIC RESULTS:

① IF $\boxed{A}$ IS $m \times n$ MATRIX AND $\boxed{D}$ IS A DIAGONAL MATRIX OF OF ORDER $\boxed{m}$ THEN FIND THE MULTIPLICATION RULE FOR $\boxed{DA}$.

SOLUTION: DA

$$= \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & d_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 a_{2n} \\ \vdots & \vdots & & \vdots \\ d_m a_{m1} & d_m a_{m2} & \cdots & d_m a_{mn} \end{bmatrix}$$

10| $\therefore$ $\boxed{DA}$ IS OBTAINED BY MULTIPLYING $\boxed{d_1}$ WITH FIRST ROW OF $\boxed{A}$, $\boxed{d_2}$ WITH SECOND ROW OF $\boxed{A}$, ... AND FINALLY $\boxed{d_m}$ WITH $\boxed{mth}$ ROW OF A.

NOTATION: $\longrightarrow$ DIAGONAL MATRIX OF ORDER n.

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & d_n \end{bmatrix} = \mathrm{diag}(d_1, d_2, \ldots, d_n)$$

② IF $\boxed{A}$ IS $m \times n$ MATRIX AND

$E = \mathrm{diag}(e_1, e_2, \ldots\ldots, e_n)$ THEN

$\boxed{AE}$ IS OBTAINED BY MULTIPLYING THE FIRST COLUMN OF $\boxed{A}$ BY e_1, ..., nth COLUMN OF $\boxed{A}$ BY e_n

$$\because \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} e_1 & \cdots & 0 \\ 0 & e_2 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1 a_{11} & e_2 a_{12} & \cdots & e_n a_{1n} \\ \vdots & \vdots & & \vdots \\ e_1 a_{m1} & e_2 a_{m2} & \cdots & e_n a_{mn} \end{bmatrix}$$

③ IN THE LAST TWO RESULTS WHAT SHOULD BE THE VALUES OF $d_1, d_2, \ldots\ldots, d_m$ AND $e_1, e_2, \ldots\ldots, e_n$ S.t. $AE = A$ AND $DA = A$

ANSWER: ONLY POSSIBLE VALUES ARE $d_1 = d_2 = \cdots\cdots = d_m = 1$ AND $e_1 = e_2 = \cdots\cdots = e_n = 1$, IN BOTH CASES THE DIAGONAL MATRICES E AND D BECOME A SPECIAL MATRIX WHICH IS CALLED THE IDENTITY MATRIX.

NOTATION: $E = I_n$ OR JUST I
$D = I_m$ OR JUST I

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

④ IF A IS A MATRIX OF ORDER n THEN $AI_n = I_n A = A$, THIS MEANS THAT I_n COMMUTES WITH EVERY ·

12] MATRIX WITH WHICH IT MULTIPLIES.

P.42 7th ED.

INVERSE MATRICES: (P.41[illegible]) / 8th ED.

IF $\boxed{Q}$ IS A SQUARE MATRIX AND $\boxed{A}$ IS ALSO A SQUARE MATRIX, (BOTH OF ORDER $\boxed{n}$) THEN $\boxed{Q}$ IS SAID TO BE AN INVERSE OF $\boxed{A}$ IF AND ONLY IF → CONVERSE HOLDS

$$\boxed{QA = AQ = I_n = I}$$

AND IS DENOTED BY $\boxed{A^{-1}}$ i.e. $Q = A^{-1}$

Try this

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -\frac{17}{18} & \frac{8}{18} \\ -\frac{2}{18} & 2/18 \\ 13/18 & -4/18 \end{bmatrix}$$

What do we get?

NOTE: ① THE MATRIX $\boxed{A}$ IS DESCRIBED AS NONSINGULAR OR INVERTIBLE IF AN INVERSE OF $\boxed{A}$ EXISTS, AND SINGULAR IF $\boxed{NO}$ INVERSE OF $\boxed{A}$ EXISTS.

② BOTH $\boxed{Q}$ AND $\boxed{A}$ ARE SQUARE MATRICES OF ORDER $\boxed{n}$ BECAUSE THEY ARE COMMUTING TO GIVE $\boxed{I_n}$.

Exercise Set 1.3
Questions 1 (a) (b) (c) (d) (e) (f), Q7 (read and understand example 7 for this), Q8 (read and understand example 9 for this), Q12, Q13, Q17, Q18 (a)