The Schröder-Bernstein Theorem and Cantor's Theorem

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Schröder-Bernstein Theorem:

Theorem: If there exist injections:

$$f : A \to B \quad \text{and} \quad g : B \to A$$

then there exists a bijection $h : A \to B$, meaning A and B have the same cardinality.

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Why is this useful? - Instead of finding a bijection directly, we can prove two sets have the same size using injections. - This applies to infinite sets, including cases like Hilbert's Hotel.

Hilbert's Hotel and the Theorem

Hilbert's Hotel: The Infinity Paradox

- A hotel has an infinite number of occupied rooms.
- A new guest arrives, and the manager shifts everyone up one room.
- The hotel is still "full" yet accommodates more guests!

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Connecting to Schröder-Bernstein

- Let $A =$ original guests and $B =$ original $+$ new guests.
- There exist injections:

$$f(n) = n + 1, \quad g(n) = 2n - 1$$

- Since there are mutual injections, the theorem guarantees a bijection between A and B.

Step 1: Define Key Sets

- Let $E_0 = A - g(B)$ (elements of A not in the image of g).
- Define recursively:

$$E_{n+1} = g(f(E_n))$$

- Define the union: $E = \bigcup_{n=0}^{\infty} E_n$.

Proof Outline

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Step 2: Define the Function h

$$h(x) = \begin{cases} f(x), & \text{if } x \in E \\ g^{-1}(x), & \text{if } x \notin E \end{cases}$$

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Proof Outline (Continued)

Step 3: Prove h is a Bijection

- **Injectivity:** We must show $h(x) = h(y)$ implies $x = y$.
  - If $x, y \in E$, by injectivity of f, $f(x) = f(y) \implies x = y$.
  - If $x, y \notin E$, by injectivity of g, $g^{-1}(x) = g^{-1}(y) \implies x = y$.
  - If $x \in E$ but $y \notin E$, then $x \in E_n$ for some n. Now $f(x) = g^{-1}(y)$, so $y = gf(x) \in gf[E_n] = E_{n+1}$, a contradiction!

  Therefore h is injective.
- **Surjectivity:** We must show that for each $z \in B$ there is $x \in A$ with $h(x) = z$.
  - Consider $x = g(z)$.
    - If $x \notin E$, then $h(x) = g^{-1}(x) = z$.
    - If $x \in E$, then $x \in E_n$ for some $n > 0$ since $x \in g[B]$. So $x \in gf[E_{n-1}]$, that is, $x = gf(x')$ for some $x' \in E_{n-1}$. Now

$$z = g^{-1}(x) = f(x') = h(x')$$

  Therefore h is surjective, hence bijective as claimed.

Cantor's Theorem

Theorem: For any set S, the power set $P(S)$ (the set of all subsets of S) has a strictly greater cardinality than S itself. That is, there is no surjective function from S to $P(S)$, implying:

$$|P(S)| > |S|.$$

Proof of Cantor's Theorem (Diagonal Argument)

Step 1: Assume a Surjection Exists Assume, for contradiction, that there exists a function $f : S \to P(S)$ that is **surjective**, meaning every subset of S is mapped to by at least one element of S.

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Step 2: Construct the Diagonal Set Define the set:

$$D = \{x \in S \mid x \notin f(x)\}$$

That is, D consists of all elements x in S such that x is **not** in the subset assigned to it by $f(x)$.

Proof of Cantor's Theorem (Continued)

Step 3: Show That D Cannot Be in the Image of f Since f is assumed to be surjective, there must be some element $x^* \in S$ such that:

$$f(x^*) = D.$$

Now, we ask: **Is x^* in D?**

- If $x^* \in D$, then by the definition of D, we must have $x^* \notin f(x^*)$, but since $f(x^*) = D$, this means $x^* \notin D$. This is a contradiction.
- If $x^* \notin D$, then by the definition of D, we must have $x^* \in f(x^*)$, meaning $x^* \in D$. Again, we have a contradiction.

Proof of Cantor's Theorem (Continued)

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- If $x^* \notin D$, then by the definition of D, we must have $x^* \in f(x^*)$, meaning $x^* \in D$. Again, we have a contradiction.

Step 4: Conclude the Contradiction Since D cannot be in the range of f, our assumption that f is surjective must be false. This means that there does not exist a surjection from S to $P(S)$, and therefore:

$$|S| < |P(S)|.$$

This completes the proof. $\square$

Conclusion

Key Takeaways:

- The Schröder-Bernstein Theorem shows that mutual injections imply equal cardinality.
- Hilbert's Hotel paradox is an intuitive example of infinite sets behaving strangely.
- Cantor's Theorem shows that power sets are always strictly larger than the original set.

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- The Schröder-Bernstein Theorem shows that mutual injections imply equal cardinality.
- Hilbert's Hotel paradox is an intuitive example of infinite sets behaving strangely.
- Cantor's Theorem shows that power sets are always strictly larger than the original set.

Final Thought: There are different "sizes" of infinity! **Thank You!** Questions?