Practice Problems (Randomized Algorithms)

CS 6th

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- In HIRE-ASSISTANT, assuming that the candidates are presented in a random order, what is the probability that you hire exactly one time? What is the probability that you hire exactly n times?

- You will hire exactly one time if the best candidate is at first. There are $(n-1)!$ orderings with the best candidate being at first, so the probability that you hire exactly one time is $(n-1)!/n!=1/\mathrm{n}$.
- You will hire exactly n times if the candidates are presented in increasing order. There is only an ordering for this situation, so the probability that you hire exactly n times is $1/n!$.

- Use **indicator random** variables to solve the following problem, which is known as the ***hat-check problem***. Each of n customers gives a hat to a hat-check person at a restaurant. The hat-check person gives the hats back to the customers in a random order. What is the expected number of customers who get back their hat?

Let X be the number of customers who get back their own hat and X_i be the indicator random variable that customer i gets his hat back. The probability that an individual gets his hat back is $\frac{1}{n}$. Thus we have

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

- Let A[1..n] be an array of n distinct numbers. If $i<j$ and A[i]>A[j], then the pair (i,j) is called an ***inversion*** of A. Suppose that the elements of A form a uniform random permutation of $\langle 1,2,\ldots,n \rangle$. Use indicator random variables to compute the expected number of inversions.

Let $X_{i,j}$ for $i < j$ be the indicator of $A[i] > A[j]$. We have that the expected number of inversions

$$
\begin{aligned}
\mathrm{E}\left[\sum_{i<j} X_{i,j}\right] &= \sum_{i<j} E[X_{i,j}] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{A[i] > A[j]\} \\
&= \frac{1}{2} \sum_{i=1}^{n-1} n - i \\
&= \frac{n(n-1)}{2} - \frac{n(n-1)}{4} \\
&= \frac{n(n-1)}{4}.
\end{aligned}
$$

$$\sum_{j=i+1}^{n-1} \frac{1}{2} = \frac{1}{2}(n-i)$$

Sum of first n-1 numbers

$$
\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}\right] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} \\
&= \binom{n}{2} \frac{1}{2} \\
&= \frac{n(n-1)}{2} \cdot \frac{1}{2} \\
&= \frac{n(n-1)}{4}
\end{aligned}
$$

- Write down an algorithm to find the first two largest numbers.

8.2 Finding The Two Largest

The max-two problem is to find the two largest elements from a sequence of n (unique) numbers. For inspiration, we'll go back and look at the naïve algorithm for the problem:

```
function max2(S) = let
   function replace((m1, m2), v) =
      if v ≤ m2 then (m1, m2)
      else if v ≤ m1 then (m1, v)
      else (v, m1)

   val start = if S1 ≥ S2 then (S1, S2) else (S2, S1)

in iter replace start S⟨3, ..., n⟩
end
```

Reference: CMU Randomized Algorithm Notes

$$\textbf{function } Y(e) = \underbrace{1}_{\text{Line } 6} + \underbrace{n-2}_{\text{Line } 3} + \underbrace{\sum_{i=3}^{n} X_i(e)}_{\text{Line } 4}$$

It is common, however, to use the shorthand notation

$$Y = 1 + (n-2) + \sum_{i=3}^{n} X_i$$

Y is total # of comparisons and Xi is an indicator random variable.

where the argument e and function definition is implied.

We are interested in computing the expected value of Y, that is $\mathbf{E}[Y] = \sum_{e\in\Omega} \mathbf{Pr}[e]\, Y(e)$. By linearity of expectation, we have

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}\left[1 + (n-2) + \sum_{i=3}^{n} X_i\right] \\ &= 1 + (n-2) + \sum_{i=3}^{n} \mathbf{E}[X_i]\,. \end{aligned}$$

using a counting argument.) Therefore, the probability that T_i is the largest or the second largest element in $\{T_1, \ldots, T_i\}$ is $\frac{1}{i} + \frac{1}{i} = \frac{2}{i}$, so

$$\mathbf{E}[X_i] = 1 \cdot \tfrac{2}{i} = 2/i.$$

Plugging this into the expression for $\mathbf{E}[Y]$, we get

$$\begin{aligned}
\mathbf{E}[Y] &= 1 + (n-2) + \sum_{i=3}^{n} \mathbf{E}[X_i] \\
&= 1 + (n-2) + \sum_{i=3}^{n} \frac{2}{i} \\
&= 1 + (n-2) + 2\Big(\tfrac{1}{3} + \tfrac{1}{4} + \ldots \tfrac{1}{n}\Big) \\
&= n - 4 + 2\Big(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \tfrac{1}{n}\Big) \\
&= n - 4 + 2H_n,
\end{aligned}$$

where H_n is the n-th Harmonic number. But we know that $H_n \leq 1 + \log_2 n$, so we get $\mathbf{E}[Y] \leq n - 2 + 2\log_2 n$. We could also use the following sledgehammer: