Solution: MidTerm-1 (Spring 2023)

Intro to Probability and Statistics (L3) - EE 354 / CE 361 / MATH 310

Allowed Time: 50 Minutes

Notes:

1. To ensure partial credit, all answers must be supported by proper justification.
2. This is an open-book, open-notes exam. However, you are not allowed to consult internet-based resources, your peers, and classmates.

Question 1 - (12 points)

A basket contains 10 red, 10 white, 5 black, and 5 orange balls. Assume that all balls are equally likely to be chosen. Two balls are chosen from the basket one after the other without replacement. Find the following probabilities:

a) Both the chosen balls are white.

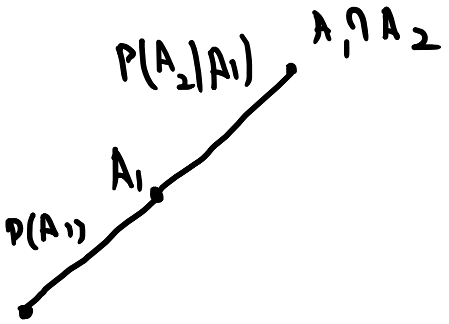
b) The first chosen ball is red and the second chosen ball is black

Solution:

(a)

$A_1 = \{\text{First chosen ball is white}\}$

$A_2 = \{\text{2nd chosen ball is white}\}$



$P(A_1 \cap A_2) = ?$

$P(A_1 \cap A_2) = P(A_1)\, P(A_2|A_1)$ (Multiplication Rule)

$P(A_1) = \frac{10}{30} = \frac{1}{3}$

$P(A_2|A_1) = \frac{9}{29}$

$P(A_1 \cap A_2) = \frac{1}{3}\left(\frac{9}{29}\right)$ — Ans

(b)

$A_1 = \{$ First chosen ball is red $\}$

$A_2 = \{$ 2nd chosen ball is black $\}$

$P(A_1 \cap A_2) = ?$

$P(A_1 \cap A_2) = P(A_1) P(A_2 | A_1)$ (Multiplication Rule)

$P(A_1) = 10/30 = 1/3$

$P(A_2 | A_1) = \frac{5}{29}$

$P(A_1 \cap A_2) = \frac{1}{3} \left(\frac{5}{29} \right)$ — Ans

Question 2 - (12 points)

Imad, an HU student, encounters two traffic lights on his way to campus in the morning. On a good day, the first light is red with a probability 0.1 and green with a probability 0.9. Also, on a good day, the second light is red with a probability 0.25 and green with a probability 0.75. On a bad day, he gets a red or green light with equal probabilities for both lights. With a probability 0.75, he will have a good day, and with a probability 0.25 he will have a bad day. What is the probability that he will have to encounter two red traffic lights next Monday.

<u>Solution</u>:

$G = \{\text{Imad has a good day}\}$

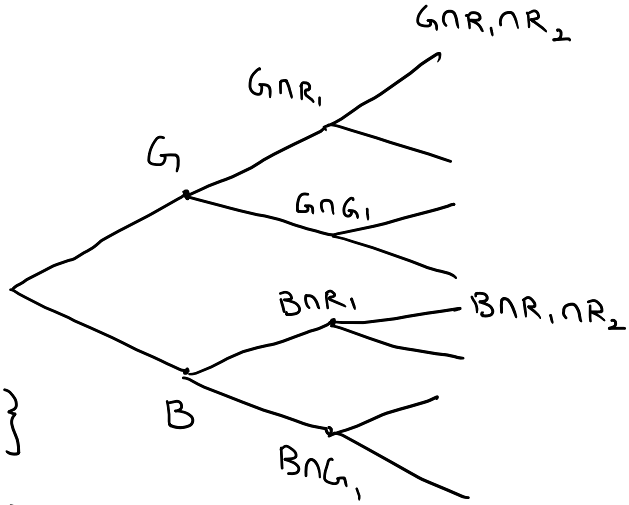
$B = \{\text{Imad has a bad day}\}$

$R_1 = \{\text{1st traffic light is red}\}$

$G_1 = \{\text{1st traffic light is green}\}$

$R_2 = \{\text{2nd traffic light is red}\}$

$G_2 = \{\text{2nd traffic light is green}\}$



$P(G) = 0.75 \qquad P(B) = 0.25$

$P(R_1|G) = 0.1 \qquad P(G_1|G) = 0.9 \qquad P(R_1|B) = 0.5 \qquad P(G_1|B) = 0.5$

$P(R_2|G\cap R_1) = 0.25 \quad P(G_2|G\cap R_1) = 0.75 \quad P(R_2|B\cap R_1) = 0.5 \quad P(G_2|B\cap R_1) = 0.5$

By Total Prob Theorem :-

$$P(R_1\cap R_2) = P(G\cap R_1\cap R_2) + P(B\cap R_1\cap R_2)$$

$$= P(G).\,P(R_1|G)\,P(R_2|G\cap R_1) + P(B)\,P(R_1|B) + P(R_2|B\cap R_1)$$

$$= 0.75(0.1)(0.25) + 0.25(0.5)(0.5)$$

$$= 0.01875 + 0.0625$$

$$P(R_1\cap R_2) = 0.08125 \quad \text{—— Ans}$$

Question 3 - (12 points)

Tomorrow, there is a game between Quetta Gladiators and Lahore Qalandars in Pakistan Super League. Sarfaraz Ahmed is the star wicket-keeper and captain of Quetta Gladiator. Unfortunately, he picked up an injury during his last game. There is only a 40% chance that he plays in the upcoming game against Lahore Qalandars. If he plays, the probability that Quetta win is 0.75. If he does not play, the probability that Quetta win is 0.35 only. Assume that you decided to completely switch off from cricket and did not follow the game. Later, you were told that Quetta Gladiators lost the game, what is the probability that Sarfaraz Ahead did not play the game?

(Note: This is a T20 game that can be decided by a super over. Therefore, the game cannot be tied or drawn.)

Solution:

consider the following events:

$A = \{\text{Sarfaraz Ahmed plays}\}$

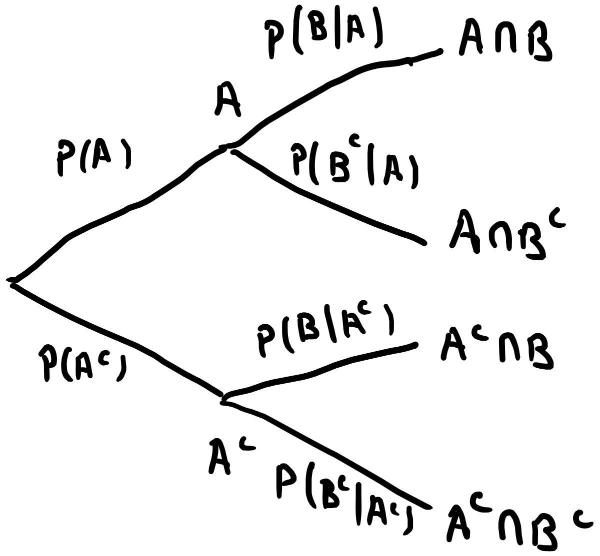
$A^c = \{\text{Sarfaraz Ahmed does not play}\}$

$B = \{\text{QG win}\}$ $\quad B^c = \{\text{QG Lose}\}$

$P(A) = 0.4 \Rightarrow P(A^c) = 0.6$

$P(B|A) = 0.75 \Rightarrow P(B^c|A) = 0.25$

$P(B|A^c) = 0.35 \Rightarrow P(B^c|A^c) = 0.65$



$P(A^c|B^c) = ?$

Using Bayes' Rule: $$P(A^c|B^c) = \frac{P(A^c)\,P(B^c|A^c)}{P(B^c)} \quad \text{—(1)}$$

By Total Probability Theorem:

$$P(B^c) = P(A\cap B^c) + P(A^c \cap B^c)$$

$$= P(A)P(B^c|A) + P(A^c)P(B^c|A^c)$$

$$= (0.4)(0.25) + (0.6)(0.65)$$

$$= 0.49$$

Formula:

$$P(A^c|B^c) = \frac{(0.6)(0.65)}{0.49}$$

$$P(A^c|B^c) = 0.7959 \quad \text{—— Ans}$$

Question 4 - (7 points)

Consider two consecutive rolls of a fair 4-sided die. Are the following events independent?

A = {1st roll results in an even number}

B = {2nd roll results in an odd number}

C = {The result of two rolls is not the same}

Justify your answer.

$$\Omega = \{(1,1),(1,2),(1,3),(1,4)$$
$$(2,1),(2,2),(2,3),(2,4)$$
$$(3,1),(3,2),(3,3),(3,4)$$
$$(4,1),(4,2),(4,3),(4,4)\}$$

$$P(A) = 8/16 = 1/2$$

$$P(B) = 8/16 = 1/2$$

$$P(C) = 12/16 = 3/4$$

$$A \cap B = \{(2,1),(2,3),(4,1),(4,3)\}$$

$$P(A \cap B) = 4/16 = 1/4 = P(A)P(B) \quad \text{—— (1)}$$

$$B \cap C = \{(2,1),(3,1),(3,1)$$
$$(1,3),(2,3),(4,3)\}$$

$$P(B \cap C) = 6/16 = 3/8 = P(B).P(C) \quad \text{—— (2)}$$

$$P(A \cap C) = 6/16 = 3/8 = P(A).P(C) \quad \text{—— (3)}$$

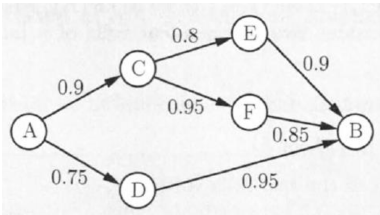
$$A \cap B \cap C = \{(2,1),(2,3),(4,1),(4,3)\}$$

$$P(A \cap B \cap C) = \frac{4}{16} = \frac{1}{4} \neq P(A).P(B).P(C)$$

$\Rightarrow$ A, B, and C are NOT independent. They are only pair wise independent.

Question 5 - (7 points)

For the network shown below, assume that link failures are independent of each other. Each link has been labeled with the probability of that link being up.



What is the probability that the path A → D → B is down?

Solution;

$$P(A \to D \to B \text{ is up}) = p_{AD}\, p_{DB} = 0.75\,(0.95)$$
$$= 0.7125$$

$$P(A \to D \to B \text{ is down}) = 1 - 0.7125 = 0.2875$$