Math Pres 302

Ex 4.3.5 Prove that $2^N \geq N$ for all positive integers N. (Hint: Use Induction)

Solution: If $N = 0$, we have $2^N = 2^0 = 1 \geq 0$. If N is negative, then 2^N is positive, so $2^N \geq N$. So infact, this result holds true for all integers, not just the positive integers.

We first show it holds true for positive integers via induction. Infact, we prove something stronger that $2^N > N$ for all positive integers N.

Base Case: For $N = 1$, we have $2^1 = 2 > 1$.

Now suppose inductively, that $2^N > N$. We want to show that $2^{N+1} > N + 1$.
We have $2^{N+1} = 2 \times 2^N > 2 \times N$ and since $N \geq 1$, we can add N to both sides to show that $2N \geq N + 1$. Thus, we have that $2^{N+1} > 2N \geq N + 1$ which closes the induction.

Alternate Proof:

Let $P(n)$ be $2^n \geq n$. Then, $P(1)$ is true as $2 > 1$. For $P(n+1)$, note $2^{n+1} = 2^n \times 2 \geq n \times 2$ by inductive hypothesis. Then $n \times 2 > n$ (as $n \times 2 - n = n \geq 1 > 0$) as desired.

Ex 4.3.4 Prove proposition 4.3.12. (Hint: induction is not suitable here, instead use proposition 4.3.10)

Proposition 4.3.12 Properties of Exponentiation, II. Let x, y be non-zero rational numbers, and let n, m be integers.

(a) $x^n x^m = x^{n+m}, (x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$

(b) If $x \geq y > 0$, then $x^n \geq y^n > 0$ if n is positive, and $0 < x^n \leq y^n$ if n is negative.

(c) If $x, y > 0$, $n \neq 0$, and $x^n = y^n$, then $x = y$.

(d) We have $|x^n| = |x|^n$

Solution: We will use n, m to denote positive integers, and $-n, -m$ to denote negative integers.

(a) Suppose n, m are both positive. Then the result simply follows from proposition 4.3.10 (a).

Next suppose that $-n, -m$ are negative. Then we have $x^{-n}x^{-m} = \frac{1}{x^n}\frac{1}{x^m} = \frac{1}{x^n x^m} = \frac{1}{x^{n+m}} = x^{-(n+m)} = x^{-n-m}$.

Next suppose that n is positive, while $-m$ is negative. We have two cases, $n-m \geq 0$, or $n-m < 0$.

- $\mathbf{n-m \geq 0}$: Then m is positive, thus by proposition 4.3.10 (a) we have: $x^{n-m}x^m = x^n$. Thus we have $x^n x^{-m} = x^{n-m}x^m x^{-m} = x^{n-m}x^m/x^m = x^{n-m}$
- Suppose instead that $n-m < 0$. Then $-n+m > 0$, and since $n \geq 0$, we can use proposition 4.3.10(a) to get $x^{-n+m}x^n = x^m$. By def of exponentiation, $x^{n-m} = \frac{1}{x^{-n+m}}$. If we multiply both sides by $\frac{1}{x^n}$, we obtain $x^{n-m}\frac{1}{x^n} = \frac{1}{x^{-n+m}}\frac{1}{x^n}$. Simplifying, we get $\frac{1}{x^{-n+m+n}} = \frac{1}{x^m}$. But $x^{-m} = \frac{1}{x^m}$, so we have $x^{n-m}\frac{1}{x^n} = x^{-m}$. So multiplying by x^n on both sides, we get $x^{n-m} = x^n x^{-m}$.

A similar proof follows when $-n$ is negative, and m is positive, and is analogous to the previous cases, which is that $x^{-n}x^m = x^{m-n}$.