Lecture 20

Wednesday, March 30, 2022 12:40 AM

 LINEAR ALGEBRA $\boxed{\text{LECTURE 20}}$ MATH 221

$\boxed{\text{PROBLEM:}}$ GRAM-SCHMIDT PROCESS;

P. 298 (8TH ED.), P. 312 (7TH ED.)

$V \longrightarrow$ INNER PRODUCT SPACE.

GIVEN: $\{\underline{u}_1, \underline{u}_2, \ldots\ldots, \underline{u}_n\}$ BE ANY BASIS FOR V THEN HOW TO PRODUCE AN ORTHOGONAL BASIS $\{\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_n\}$ FOR V?

i.e. $\langle \underline{v}_i, \underline{v}_j \rangle = 0,\ i \neq j,$

$1 \leq j \leq n,\ 1 \leq i \leq n$

WHICH CAN BE NORMALISED TO PRODUCE AN ORTHONORMAL BASIS i.e.

$$\left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \ldots\ldots, \frac{\underline{v}_n}{\|\underline{v}_n\|} \right\}$$

i.e. NORM OF EACH VECTOR $= 1$ (IN ADDITION TO ORTHOGONALITY PROPERTY)

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RECALL THAT FOR EUCLIDEAN INNER PRODUCT (DOT PRODUCT)

$$\text{Proj}_{\underline{a}}\underline{u} = \frac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}$$ AND

VECTOR PROJECTION (COMPONENT) OF $\boxed{\underline{u}}$ PEPENDICULAR TO $\boxed{\underline{a}}$ IS GIVEN BY

$$\boxed{\underline{u} - \text{Proj}_{\underline{a}}\underline{u} = \underline{u} - \frac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}}$$

SIMILARLY IF $\underline{v}_1$ AND $\underline{v}_2$ ARE ORTHOGONAL VECTORS IN AN INNER PRODUCT SPACE $\boxed{V}$ AND $\underline{u} \in V$ SUCH THAT HORIZONTAL ($\overrightarrow{OP}$) PROJECTION OF $\underline{u}$ LIES ALONG $\underline{v}_1$ THEN

$$\boxed{\underline{v}_2 = \underline{u} - \frac{\langle \underline{u}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2}}$$

3

$$\underline{u} = \underline{v}_2 + \overrightarrow{OP}$$

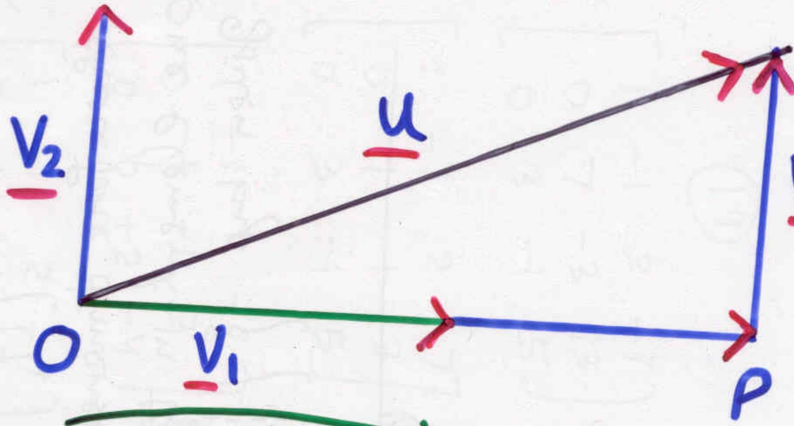
3

PROOF:

3

PROOF:

$\underline{v_2} = ?$



$$\because \quad \underline{u} = \overbrace{k\underline{v_1}}^{\overrightarrow{OP}} + \underline{v_2} = \overrightarrow{OP} + \underline{v_2}$$

$$\because \quad \overrightarrow{OP} = k\underline{v_1}$$

$$\Rightarrow \quad \underline{v_2} = \underline{u} - \overrightarrow{OP} = \underline{u} - k\underline{v_1} \quad \text{—(1)}$$

TAKING INNER PRODUCT ON BOTH SIDES BY $\underline{v_1}$, WE GET

$$\cancelto{0}{\langle \underline{v_2}, \underline{v_1} \rangle} = \langle \underline{u}, \underline{v_1} \rangle - \langle k\underline{v_1}, \underline{v_1} \rangle$$

$$\Rightarrow \langle \underline{u}, \underline{v_1} \rangle = \langle k\underline{v_1}, \underline{v_1} \rangle$$

$$\Rightarrow \langle \underline{u}, \underline{v_1} \rangle = k \langle \underline{v_1}, \underline{v_1} \rangle$$

$$\Rightarrow \boxed{k = \frac{\langle \underline{u}, \underline{v_1} \rangle}{\|\underline{v_1}\|^2}}$$

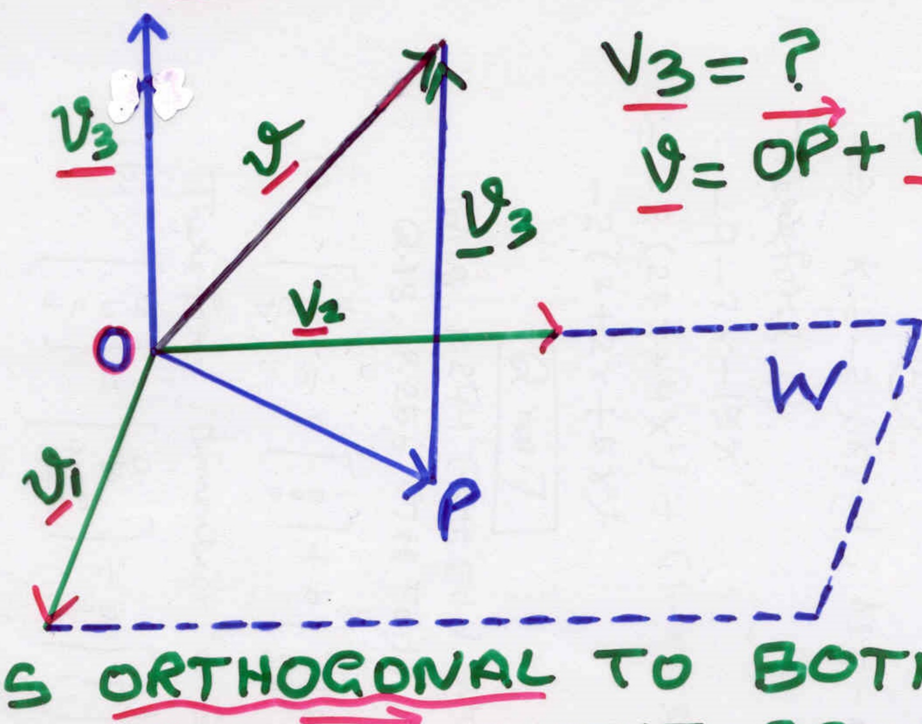
PUTTING IN (1), WE GET

$$\boxed{\underline{v_2} = \underline{u} - \overbrace{\frac{\langle \underline{u}, \underline{v_1} \rangle}{\|\underline{v_1}\|^2}\underline{v_1}}^{\mathrm{Proj}_{\underline{v_1}} \underline{u}}}$$

4 **NOTE:** VECTORS $\underline{e}_1 = (1,0,0)$ AND $\underline{e}_2 = (0,1,0)$ SPAN THE XY-PLANE BECAUSE ANY VECTOR IN THE XY-PLANE CAN BE WRITTEN AS THEIR LINEAR COMBINATION

$$(x,y,0) = x(1,0,0) + y(0,1,0)$$
$$= x\underline{e}_1 + y\,\underline{e}_2$$

SIMILARLY IF $\underline{v}_1$ AND $\underline{v}_2$ ARE ORTHOGONAL VECTORS SPANING A PLANE $\boxed{W}$ AS SHOWN BELOW



$\underline{v}_3 = ?$

$\underline{v} = \vec{OP} + \underline{v}_3$

$\underline{v}_3$ IS ORTHOGONAL TO BOTH $\underline{v}_1$ AND $\underline{v}_2$. $\vec{OP}$ IS THE PROJECTION OR COMPONENT OF $\underline{v}$ IN $\boxed{W}$ $\therefore$ $\vec{OP}$ LIES IN $\boxed{W}$ (SPANNED

TION OR COMPONENT OF $\underline{v}$ IN $\boxed{W}$. $\because$ $\overrightarrow{OP}$ LIES IN $\boxed{W}$ (SPANNED BY $\underline{v_1}$ AND $\underline{v_2}$) THEREFORE

$\overrightarrow{OP}$ IS A LINEAR COMBINATION

5/ OF $\underline{v_1}$ AND $\underline{v_2}$ $\therefore$ $\overrightarrow{OP} = k_1\underline{v_1} + k_2\underline{v_2}$

BUT $\underline{v} = \overrightarrow{OP} + \underline{v_3} \Rightarrow \underline{v_3} = \underline{v} - \overrightarrow{OP}$

$$\Rightarrow \underline{v_3} = \underline{v} - k_1\underline{v_1} - k_2\underline{v_2} \quad \text{— (1)}$$

TO FIND $\boxed{k_1}$ TAKE INNER PRODUCT WITH $\underline{v_1}$

$$\Rightarrow \cancelto{0}{\langle \underline{v_3}, \underline{v_1} \rangle} = \langle \underline{v}, \underline{v_1} \rangle - k_1 \underbrace{\langle \underline{v_1}, \underline{v_1} \rangle}_{\|\underline{v_1}\|^2} - k_2 \cancelto{0}{\langle \underline{v_2}, \underline{v_1} \rangle}$$

$$\Rightarrow k_1 = \frac{\langle \underline{v}, \underline{v_1} \rangle}{\|\underline{v_1}\|^2}$$

SIMILARLY TO FIND $\boxed{k_2}$ TAKE INNER PRODUCT WITH $\underline{v_2}$

$$\therefore (1) \Rightarrow \cancelto{0}{\langle \underline{v_3}, \underline{v_2} \rangle} = \langle \underline{v}, \underline{v_2} \rangle - k_1 \cancelto{0}{\langle \underline{v_1}, \underline{v_2} \rangle} - k_2 \langle \underline{v_2}, \underline{v_2} \rangle$$

$$\Rightarrow k_2 = \frac{\langle \underline{v}, \underline{v_2} \rangle}{\|\underline{v_2}\|^2}$$

, $\therefore$ FROM (1)

$$\|\underline{v}_2\|$$

$$\boxed{\underline{v}_3 = \underline{v} - \frac{\langle \underline{v}, \underline{v}_1 \rangle \underline{v}_1}{\|\underline{v}_1\|^2} - \frac{\langle \underline{v}, \underline{v}_2 \rangle \underline{v}_2}{\|\underline{v}_2\|^2}}$$

TO BE CONTINUED

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ASSIGNMENT NO. 5(a)

Q. no. 1 (a)

CHECK WHETHER $\underline{V}_1 = (1,1,2)$, $\underline{V}_2 = (1,0,1)$, AND $\underline{V}_3 = (2,1,3)$ SPAN THE VECTOR SPACE R^3

Q. no. 1 (b)

ARE THE FOLLOWING TRUE OR FALSE?

(I) A SET THAT CONTAINS THE ZERO VECTOR IS LINEARLY DEPENDENT.

(II) IF W IS A SUBSPACE OF V, THEN $\dim(W) \leq \dim(V)$; MOREOVER, $\dim(W) = \dim(V)$; IF AND ONLY IF $W = V$

(III) EVERY NONZERO FINITE-DIMENSIONAL INNER PRODUCT SPACE HAS AN ORTHONORMAL BASIS.

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(IV) THE PRODUCT $A\underline{X}$ IN A LINEAR SYSTEM IS A LINEAR COMBINATION OF THE COLUMN VECTORS OF $\boxed{A}$.

(V) A SYSTEM OF LINEAR EQUATIONS $A\underline{X}=\underline{b}$ IS CONSISTENT IF AND ONLY IF $\boxed{\underline{b}}$ IS IN THE COLUMN SPACE OF A.

(VI) $A\underline{X}=\underline{b}$ IS CONSISTENT IF AND ONLY IF THE RANK OF THE COEFFICIENT MATRIX $\boxed{A}$ IS THE SAME AS THE RANK OF THE AUGMENTED MATRIX $[A/\underline{b}]$.

Q.no.2

LET $A=\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$

FIND:

(a) ECHELON FORM OF $\boxed{A}$.

(b) BASIS FOR THE COLUMN

(b) BASIS FOR THE COLUMN SPACE OF $\boxed{A}$ BY WATCHING THE

$\boxed{8}$

COLUMN VECTORS IN $\boxed{A}$ WHICH CORRESPOND TO THE COLUMN VECTORS IN ECHELON FORM ((CONTAINING THE LEADING 1'S). $\boxed{8}$

(C) CAN WE ALSO FIND THE BASIS FOR THE ROW SPACE OF $\boxed{A}$ (CONSISTING ENTIRELY OF ROW VECTORS FROM $\boxed{A}$) BY THE METHOD (USED IN (b))? EXPLAIN YOUR ANSWER?

Q.no.3

EXAMPLE (1) P.261 (8th ED.) OR EXAMPLE (1) P.274 (7th ED.) P.261

IS THERE ANY SHORTER METHOD FOR THIS EXAMPLE? SHORTER

Q.no.4 (INNER PRODUCT SPACE)

(a) SEE THE DEFINITION OF WEIGHTED EUCLIDEAN INNER PRODUCT. (P.277 8th ED. OR P.288 7th ED.) P.277

$\overset{\rightarrow P.277}{}$

(P. 277 8th ED. OR P. 298 7th ED.)

(b) LET $\underline{u} = (u_1, u_2)$ AND $\underline{v} = (v_1, v_2)$ BE VECTORS IN R^2. VERIFY THAT THE WEIGHTED EUCLIDEAN INNER PRODUCT

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$$\langle \underline{u}, \underline{v} \rangle = 3u_1v_1 + 2u_2v_2$$

SATISFIES THE FOUR INNER PRODUCT AXIOMS.

$\boxed{\text{Q. no. 5}}$

Question 5

Not Included

(a) IF $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ AND $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ ARE 2×2 MATRICES, THEN PROVE THAT THE FOLLOWING FORMULA DEFINES AN INNER PRODUCT ON M_{22} $(\underline{u} = U,\ \underline{v} = V)$

$$\langle \underline{u}, \underline{v} \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

(b) If $P = a_0 + a_1x + a_2x^2$ AND $Q = b_0 + b_1x + b_2x^2$ ARE ANY TWO VECTORS IN P_2, THEN PROVE THAT THE FOLLOWING FORMULA DEFINES AN INNER PRODUCT ON P_2

$$\langle P, Q \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

$$\langle P, Q \rangle = a_0b_0 + a_1b_1 + a_2b_2$$

Q.no.6

(a) Q.no.7 (P.284 8th ED.) OR

P.297

Question 6

th ED.)

(b) Q.no.17 (P. 285 8th ED.) OR

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NOT INCLUDED

Q.no.17 (P.298 7TH ED.)

(C) Q.no.28 (P.286 8TH ED.) OR

(P.299 7TH ED.)

Q.no.7

FOR ANY INNER PRODUCT SPACE PROVE THAT

(a) $\|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2 = 2\|\underline{u}\|^2 + 2\|\underline{v}\|^2$

(b) $\langle \underline{u}, \underline{v} \rangle = \frac{1}{4}\|\underline{u}+\underline{v}\|^2 - \frac{1}{4}\|\underline{u}-\underline{v}\|^2$

Q.no.8

Q. 17, 20 P.296 8TH ED. OR

Question 8 Not

Include d

Also Exercise Set 6.1
Q17, Q20, Q27, Q28