# Exercise Set 1.5 Solution

## Question 02

Find a row operation that will restore the given elementary matrix to an identity matrix.

(a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Solution: Multiplying Row 1 by 3 and adding to the Row 2.

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} (R_2 + 3 * R_1) \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution: Multiplying Row 3 by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} (\frac{1}{3} * R_3) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution: Row Interchange by $R_1 \leftrightarrows R_4$.

(d) $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: Multiplying Row 3 by 7 and adding to the Row 1.

## Question 03

Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices $E_1, E_2, E_3$, and $E_4$ such that

(a) $E_1A = B$ Solution: If we interchange Rows 1 and 3 of $A$, then we obtain $B$. Therefore, $E_1$ must be the matrix obtained from $I_3$ by interchanging Rows 1 and 3 of $I_3$, i.e.,

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) $E_2B = A$ Solution: If we interchange Rows 1 and 3 of $B$, then we obtain $A$.

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) $E_3A = C$

Solution: If we multiply Row 1 by $-2$ and add to Row 3 in $A$, then we obtain $C$.

$$E_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) $E_4C = A$

Solution: If we multiply Row 1 by 2 and add to Row 3 in $C$, then we obtain $A$.

$$E_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Question 09

Find the inverse of each of the following $4 \times 4$ matrices, where $k_1, k_2, k_3, k_4$ and $k$ are all nonzero.

(a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$

Solution: Using $[A|I] \to [I|A^{-1}]$

$$\frac{1}{k_1} * R_1 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{k_2} * R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{k_3} * R_3 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{1}{k_4} * R_4 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

Solution: Using $[A|I] \rightarrow [I|A^{-1}]$

$$R_1 \leftrightarrows R_4 \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 \leftrightarrows R_3 \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{k_4} * R_1 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{k_3} * R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{k_2} * R_3 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{k_1} * R_4 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 0 & 0 & 0 \end{array}\right]$$

## Question 10

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

(a) Find elementary matrices $E_1$ and $E_2$ such that $E_2E_1A = I$.
Solution: $E_2E_1A = I$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} = I_2$$

(b) Write $A^{-1}$ as a product of two elementary matrices.
Solution: $E_2E_1A = (E_2E_1)A = A^{-1}A = I$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

(c) Write $A$ as a product of two elementary matrices.
Solution: $E_2E_1A = I \longrightarrow A = E_1^{-1}E_2^{-1}I = E_1^{-1}E_2^{-1}$

$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$$

## Question 16

Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

Solution:

$$R_1 \leftrightarrows R_2 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_4 \leftrightarrows R_5 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 - \frac{d}{a}R_2 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 - \frac{e}{h}R_4 \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

One can see the third row is zero but the corresponding row of identity matrix is not zero. Hence it is inconsistent for any values of $a, b, c, d, e, f, g$, and $h$.