Using Packages

SEL Activity 5

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1. Prove that $\mathrm{E}\left[(T-\mu_T)^2\right] = \dfrac{4A^3+2A^2}{N}$.

Solution:

$$\begin{aligned}
\mathrm{E}\left[(T-\mu_T)^2\right] &= \mathrm{var}(T) \\
&= \mathrm{var}\left(\frac{1}{N}\sum_{i=0}^{N} X_i^2\right) \\
&= \frac{1}{N^2}\,\mathrm{var}\left(\sum_{i=0}^{N} X_i^2\right) \\
&= \frac{1}{N^2}\sum_{i=0}^{N}\mathrm{var}\left(X_i^2\right) \\
&= \frac{N}{N^2}\,\mathrm{var}\left(X^2\right) \\
&= \frac{1}{N}\,\mathrm{var}\left(X^2\right)
\end{aligned} \tag{1}$$

Where $X \sim \mathcal{N}(x; A, A)$, therefore the variance of X^2 would be,

$$\begin{aligned}
\mathrm{var}\left(X^2\right) &= \mathrm{E}\left[X^4\right] - \mathrm{E}\left[X^2\right]^2 \\
&= \mathrm{E}\left[X^4\right] - \left(A^2+A\right)^2
\end{aligned}$$

We can use the moment generating function of Guassian Randome Variable to calculate the 4th moment,

$$X \sim \mathcal{N}(\mu, \sigma^2) \implies M_X(0) = \exp\left(t\mu + \frac{1}{2}\sigma^2 t^2\right)$$

We know that,

$$y = e^{f(t)} \Rightarrow y'''' = \left(f''''(0) + 4f'''(0)f'(0) + 3(f''(0))^2 + 6f''(0)(f'(0))^2 + (f'(0))^4\right)e^{f(0)}$$

Let $f(t) = \mu t + \frac{1}{2}\sigma^2 t^2$,

$$
\begin{aligned}
f^{(0)}(t) = \mu t + \frac{1}{2}\sigma^2 t^2 &\implies f^{(0)}(0) = 0 \\
f^{(1)}(t) = \mu + \sigma^2 t &\implies f^{(1)}(0) = \mu \\
f^{(2)}(t) = \sigma^2 &\implies f^{(2)}(0) = \sigma^2 \\
f^{(3)}(t) = 0 &\implies f^{(3)}(0) = 0 \\
f^{(4)}(t) = 0 &\implies f^{(4)}(0) = 0
\end{aligned}
$$

The fourth moment would be,

$$
\begin{aligned}
\mu_4 &= M_X^{(4)}(0) \\
&= \frac{\mathrm{d}^4}{\mathrm{d}\, t^4}\left(\exp\left(t\mu + \frac{1}{2}\sigma^2 t^2\right)\right) \\
&= \left(0 + 4(0)(\mu) + 3\left(\sigma^2\right)^2 + 6\sigma^2(\mu)^2 + (\mu)^4\right) e^0 \\
&= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 \\
&= 3A^2 + 6A^3 + A^4
\end{aligned}
$$

The variance of X^2 would be,

$$
\begin{aligned}
\operatorname{var}\left(X^2\right) &= 3A^2 + 6A^3 + A^4 - \left(A^2 + A\right)^2 \\
&= 3A^2 + 6A^3 + A^4 - A^4 - 2A^3 - A^2 \\
&= 4A^3 + 2A^2
\end{aligned}
$$

Putting the above result in (**??**), we get,

$$
\operatorname{var}(T) = \mathrm{E}\left[(T - \mu_T)^2\right] = \frac{4A^3 + 2A^2}{N}
$$

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2. Prove the following claims,

   1. $g(\mu_T) = A$
   2. $g'(\mu_T) = \frac{1}{2}\left(A + \frac{1}{2}\right)^{-1}$
   3. $g''(\mu_T) = -\frac{1}{4}\left(A + \frac{1}{2}\right)^{-3}$

Solution: We know that, $\mu_T = A^2 + A$ and $g(T) = -\frac{1}{2} + \sqrt{T + \frac{1}{4}}$

$$\therefore g(\mu_T) = -\frac{1}{2} + \left(A^2 + A + \frac{1}{4}\right)^{\frac{1}{2}} = -\frac{1}{2} + A + \frac{1}{2} = A$$

$$\therefore g'(\mu_T) = \left.\frac{\mathrm{d}\, g(T)}{\mathrm{d}\, T}\right|_{T=\mu_T} = \frac{1}{2}\left(A^2 + A + \frac{1}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(A + \frac{1}{2}\right)^{-1}$$

$$\therefore g''(\mu_T) = \left.\frac{\mathrm{d}^2\, g(T)}{\mathrm{d}\, T^2}\right|_{T=\mu_T} = -\frac{1}{4}\left(A^2 + A + \frac{1}{4}\right)^{-\frac{3}{2}} = -\frac{1}{4}\left(A + \frac{1}{2}\right)^{-3}$$

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3. Using (1), (2) and $\mathrm{var}\left(\hat{A}\right) = \sigma^2_{\hat{A}} \approx \sigma^2_T \left[g'\left(\mu_T\right)\right]^2$ to obtain the expression for $\mathrm{E}\left[\hat{A}\right]$

Solution:

$$\begin{aligned} \mathrm{E}\left[\hat{A}\right] &= g(\mu_T) + \frac{1}{2} g''(\mu_T)\, \mathrm{E}\left[(T - \mu_T)^2\right] \\ &= A + \frac{1}{2}\left(-\frac{1}{4}\left(A + \frac{1}{2}\right)^{-3}\right)\left(\frac{4A^3 + 2A^2}{N}\right) \\ &= A - \frac{2A^2}{(2A^2+1)^2} \end{aligned}$$

4. Show that $\operatorname{var}\left(\hat{A}\right) = \dfrac{A^2}{N\left(A+\frac{1}{2}\right)}$.

Solution:

$$
\begin{aligned}
\operatorname{var}\left(\hat{A}\right) = \sigma_{\hat{A}}^2 &\approx \sigma_T^2 \left[g'\left(\mu_T\right)\right]^2 \\
&\approx \frac{4A^3+2A^2}{N}\left[\frac{1}{2}\left(A+\frac{1}{2}\right)^{-1}\right]^2 \\
&\approx 4A^2\frac{A+\frac{1}{2}}{N}\frac{1}{4}\left(A+\frac{1}{2}\right)^{-2} \\
&\approx \frac{A^2}{N\left(A+\frac{1}{2}\right)}
\end{aligned}
$$

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