Assignment No 1

Question 1:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

$A^2 = A \times A$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

1^{st} column of A^2

$$= \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0(0) + 0(2) + 1(1) \\ 2(0) + 1(2) + 0(1) \\ 1(0) + -2(2) + 0(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2\cancel{3} \\ -4 \end{bmatrix}$$

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2^{nd} Column of A^2

$$= \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0(0) + 0(1) + 1(-2) \\ 2(0) + 1(1) + 0(-2) \\ 1(0) - 2(1) + 0(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

3^{rd} Column of A^2

$$= \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0(1) + 0(0) + 1(0) \\ 2(1) + 1(0) + 0(0) \\ 1(1) - 2(0) + 0(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ \cancel{3}2 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(0)-2(2)+0(1) & 1(0)-2(1)+0(-2) & 1(1)-2(0)+0(0) \\ 2(0)+1(2)+2(1) & 2(0)+1(1)+2(-2) & 2(1)+1(0)+2(0) \\ -4(0)-2(2)+1(1) & -4(0)-2(1)+1(-2) & -4(1)-2(0)+1(0) \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -4 & -2 & 1 \\ 4 & -3 & 2\cancel{3} \\ -\cancel{11}3 & -4 & -4 \end{bmatrix}$$

Part (b)

$$A^3 = A^2 + A - 5I$$

$$A^2 + A - 5I = \begin{bmatrix} 1 & -2 & 0 \\ 2\cancel{3} & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ \cancel{5}4 & 2 & 2 \\ -3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -2 & 1 \\ 4 & -3 & 2 \\ -3 & -4 & -4 \end{bmatrix}$$

$$= A^3$$

Part (c)

i) $A^4 = 2A^2 - 4A - 5I$

we have:

$$A^3 = A^2 + A - 5I \quad — ⓘ$$

$$AA^3 = A(A^2 + A - 5I)$$

$$A^4 = A^3 + A^2 - 5AI$$

$$A^4 = A^3 + A^2 - 5A \qquad \because AI = A$$

Using ⓘ

$$A^4 = A^2 + A - 5I + A^2 - 5A$$

$$A^4 = 2A^2 - 4A - 5I$$

ii) $A^{-1} = \frac{1}{5}(I + A - A^2)$

we have:

$$A^3 = A^2 + A - 5I$$

$$A^{-1}A^3 = A^{-1}(A^2 + A - 5I)$$

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$$A^2 = A^{2} + I - 5A^{-1}I \quad \therefore A^{-1}A = I$$

$$A^2 = A + I - 5A^{-1} \quad \therefore A^{-1}I = A^{-1}$$

$$5A^{-1} = A + I - A^2$$

$$A^{-1} = \frac{1}{5}(A + I - A^2)$$

$$A^{-1} = \frac{1}{5}(I + A - A^2)$$

QUESTION 2:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Let's assume that A commutes with P, which means

$$PA = AP$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

By comparision,

$d = 0$ $\qquad$ $f = b$

$g = 0$ $\qquad$ $i = e = a$

$e = a$ $\qquad$ $h = 0$

$h = d = 0$

Putting values in A

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

Hence, the most general matix that Commutes with P is A.

QUESTION 3:

Part (a):

Symmetric matrix means

$$B = B^T$$

For $A + A^T$ ~~be~~ to be symmetric

$$A + A^T = (A + A^T)^T$$

$$= A^T + (A^T)^T \qquad \because (A^T)^T = A$$

$$= A^T + A$$

$$A + A^T = A + A^T$$

Skew Symmetric matrix means

$$B^T = -B$$

For $A - A^T$ to be skew-symmetric

$$-(A - A^T) = (A - A^T)^T$$

$$= A^T - (A^T)^T \qquad \because (A^T)^T = A$$

$$= A^T - A$$

$$= -A + A^T$$

$$-(A - A^T) = -(A - A^T)$$

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Part (b)

AA^T is symmetric

$= (AA^T)^T$ $\quad \therefore (AB)^T = B^T A^T$

$= (A^T)^T A^T$ $\quad \therefore (A^T)^T = A$

$= AA^T$

A^TA is symmetric

$= (A^TA)^T$ $\quad \therefore (AB)^T = B^T A^T$

$= A^T(A^T)^T$ $\quad \therefore (A^T)^T = A$

$= A^T A$

Part (c)

if $A^2 = A$ — (i)

$\cancel{A^{-1}}$ A^{-1} Exists

Pre multiplying A^{-1} in eq (i)

$A^{-1}A^2 = A^{-1}A$

$A^{-1}A \cdot A = A^{-1}A$

$(A^{-1}A)A = A^{-1}A$ $\quad \therefore$ associative property

$IA = I$ $\quad \therefore A^{-1}A = I$

$\boxed{A = I}$ $\quad \therefore IA = A$

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Part (d)

A is invertible

$(A^{-1})^T = (A^T)^{-1}$

Let $X = A^{-1}$

$(X)^T = (A^{-1})^T$

$XA = I$ $\qquad\qquad$ $A^{-1}A = I$

$(XA)^T = I^T$ $\qquad\qquad$ $\cancel{(BA)}$ $(AB)^T = B^T A^T$

$A^T X^T = I$ $\qquad\qquad$ $I^T = I$

Pre multiplying by $(A^T)^{-1}$

$(A^T)^{-1} A^T X^T = (A^T)^{-1} I$ $\qquad\qquad$ $\because (A^T)^{-1} A^T = I$

$I X^T = (A^T)^{-1} I$ $\qquad\qquad$ $\therefore AI = A$

$X^T = (A^T)^{-1}$ $\qquad\qquad$ $\therefore X = A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

Proved.

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QUESTION 5:

$$AX = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$AX = \begin{bmatrix} x & y \\ x+2z & y+2t \end{bmatrix}$$

$$XB = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$XB = \begin{bmatrix} 2x-y & -x+2y \\ 2z-t & -2+2t \end{bmatrix}$$

$$AX = XB$$

$$\begin{bmatrix} x & y \\ x+2z & y+2t \end{bmatrix} = \begin{bmatrix} 2x-y & -x+2y \\ 2z-t & -z+2t \end{bmatrix}$$

$$x = 2x - y \quad \Rightarrow \quad x = y$$

$$y = -x + 2y \quad \Rightarrow \quad x = y$$

$$x + 2z = 2z - t \quad \Rightarrow \quad x = -t$$

$$y + 2t = -z + 2t \quad \Rightarrow \quad \cancel{x = y} \; y = -z = x$$

$$X = \begin{bmatrix} x & x \\ -x & -x \end{bmatrix}$$

Let $x = n$

$$X = n\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

QUESTION 6:

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$A^T = -A$

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

$a = -a$

$d = -b$

$g = -c$

$e = -e$

$h = -f$

$i = -i$

$a = -a$, $e = -e$, $i = -i$

This can ~~oly~~ only be true only if $a, e, i = 0$

Putting values in A:

$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix}$$

For $A^T = -A$, diagonal entries must be zero.

QUESTION 7:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

$$B^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{21} b_{11} + a_{22} b_{21} \\ a_{11} b_{12} + a_{12} b_{22} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{21} b_{11} + a_{22} b_{21} \\ a_{11} b_{12} + a_{12} b_{22} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

→ This result can be extended without the loss of generality.