CS/Math 113 - Problem Set 9

Dead TAs Society
Habib University - Spring 2023

Week 13

Problems

Problem 1. **[Chapter 5.1, Question 5]** Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 \equiv (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer using induction.

Solution:

Let $S(n) = 1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

Base Case: $n = 0$. Then $S(0) = 1^2 = \frac{(1)(1)(3)}{3} = 1$. Hence true for the base case.

Inductive Hypothesis: For the inductive step, assume the inductive hypothesis that $S(k) = 1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$. Assume this is true for $n = k$

Then for $n = k+1$, we have to show that adding one more term $[(2(k+1)+1)^2 \implies (2k+3)^2]$ results in $S(k+1) = \frac{(k+2)(2k+3)(2k+5)}{3}$.

Then from the inductive hypothesis,

$S(k+1) = 1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$

$= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3}$

$= \frac{(2k+3)[(k+1)(2k+1) + 3(2k+3)]}{3}$

$= \frac{(2k+3)(2k^2+9k+10)}{3}$

$= \frac{(2k+3)[(k+2)(2k+5)]}{3} = \frac{(k+2)(2k+3)(2k+5)}{3}$

$\square$

Problem 2. [Chapter 5.1, Question 21] Prove that $2^n > n^2$ if n is an integer greater than 4 using induction.

Solution:

Base Case: $n = 5$. Then $2^5 > 5^2 \implies 32 > 25$. Hence true for the base case.

Inductive Hypothesis: $n = k$. Then $2^k > k^2$ is true.

Then for $n = k + 1$, we have to show that $2^{k+1} > (k + 1)^2$ is true. Then we can show that $2.2^k > k^2 + 2k + 1$

We have that $(k + 1)^2 = k^2 + 2k + 1 < k^2 + 2k + k$ since $k > 4$. Then $(k + 1)^2 < k^2 + 3k$

However, $k^2 + 3k < k^2 + k^2$ as $k > 3$, then $3k < k^2$. Therefore, $k^2 + 3k < 2k^2$. Further, $2k^2 < 2.2^k$ since $k^2 < 2^k$ by the inductive hypothesis, and $2.2^k = 2^{k+1}$.

Therefore, $(k + 1)^2 < 2^{k+1}$.

□

Problem 3. [Chapter 5.1, Question 32] Prove that 3 divides $n^3 + 2n$ whenever n is positive integer using induction.

Solution: Let $P(n) = n^3 + 2n$ and that $3 \mid P(n)$ whenever n is a positive integer. [$3|P(n)$ means 3 divides $P(n)$ and does not leave any remainder]

Base Case: $n = 0$. Then $P(0) = 0$ and $3 \mid 0$. Hence true for the base case.

Inductive Hypothesis: $n = k$. Then $P(k) = k^3 + 2k$ and $3 \mid P(k)$ is true.

Then for the inductive step, we have to show that for $n = k + 1$, and that $3 \mid P(k + 1)$.

$P(k + 1) = (k + 1)^3 + 2(k + 1)$
$= k^3 + 3k^2 + 3k + 1 + 2k + 2$
$= k^3 + 3k^2 + 5k + 3$
$= k^3 + 2k + 3k^2 + 3k + 3$
$= (k^3 + 2k) + 3(k^2 + k + 1)$
$P(k + 1) = (k^3 + 2k) + 3(k^2 + k + 1) = P(k) + 3(k^2 + k + 1)$

We know from the inductive hypothesis that $P(k)$ is divisible by 3, and $3(k^2 + k + 1)$ is definitely divisible by 3. Therefore $3 \mid P(k + 1)$.

□

Problem 4. **[Chapter 5.1, Question 40]** Prove that if $A_1, A_2, \cdots, A_n$ and B are sets, then $(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B \equiv (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B)$ using induction

Solution:

Base Case: $n = 1$, then $A_1 \cup B = A_1 \cup B$ - trivial. And $n = 2$, then $(A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)$ which is the distributive law [nothing to prove - has been already proved].

Inductive Hypothesis: $n = k$, then $(A_1 \cap A_2 \cap \cdots \cap A_k) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B)$ is true.

Then for the inductive step we have to show that for $n = k + 1$, $(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$

Then

$$\begin{aligned}(A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}) \cup B &= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}) \cup B \\ &= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cup B) \cap (A_{k+1} \cup B) \\ &= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B)\end{aligned}$$

The second line follows from the distributive law, and the third line follows from the inductive hypothesis.

$\square$

Problem 5. **[Chapter 5.1, Question 50]** What is wrong with this "proof"?

Proof.

Theorem 1. *For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$*

Basis Step: The formula is true for $n = 1$
Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n + 1)$. By the inductive hypothesis, we have $\sum_{i=1}^{n+1} i = (n + \frac{1}{2})^2/2 + (n + 1) = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n + \frac{3}{2})^2/2 = [(n + 1) + \frac{1}{2}]^2/2$, $\square$

Solution: Consider the base case where $n = 1$. In the above proof, when $n = 1$, the sum yields 1. However, on calculation, $\dfrac{(1 + \frac{1}{2})^2}{2} = \dfrac{9}{4} \times \dfrac{1}{2} = \dfrac{9}{8} \neq 1$. Hence the base case in the proof was wrong.