Weekly Challenge 03: Closure of Regular Languages

CS 212 Nature of Computation
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1. Operation F

Let us define a unary operation, F, on languages as follows.

$F(L) = \{f(w) \mid w \in L\}$ where, given

- $w = w_1 w_2 w_3 \ldots w_n$, and
- each $w_i \in \Sigma$,

$f(w) = w_n w_{n-1} w_{n-2} \ldots w_1$.

Prove or disprove that the class of regular languages is closed under F.

Solution: The above definition of the unary operator F states that for any language L, the operation F reverses each string in L, that is, it reverses the order of the characters in any arbitrary string w. So $F(L)$ consists of all the strings obtained by reversing the strings in a language L.

For any regular langauge L, let $M = (Q, \sum, \delta, q_o, F)$ be the DFA that recognizes L.

Now for the language $F(L)$, we can build an NFA M' as follows:

1. has the same set of states as M,
2. has the same alphabet as M,
3. reverses all the transitions in M,
4. make the start state of M the final state of M', and
5. add a new start state q'_o that has an ε transition to each start state of M, turning the final states of M into normal states of M'

Formally, $M' = (Q', \sum_\varepsilon, \delta', q'_o, q_o)$ where:

1. $Q' = Q \cup \{q'_o\}$ as $q'_o \notin Q$,
2. has the final state the same as the starting state of M,
3. δ' is defined as:
   i $\delta'(q'_o, \varepsilon) = q_f$ where $q_f \in F$
   ii $\delta'(q, a) = p \in M'$ for each transition $\delta(p, a) = q \in M$ and $p, q \in Q, a \in \sum$

From (3.i), we can transition from the starting state q'_o to any of the final states $q_f \in F$, via ε transition thus starting in reverse order. From (3.ii), we can transition from any state q to any state p in M' if there exists a transition from p to q in M.

Hence δ' is δ with the direction of all arcs reversed. So for any path that exists in M from q_o to q_f, there exists a path in M' from q_f to q_o and vice versa.

With this construction of M' we have created an NFA that will process a string w in a language L in reverse order. So if $w \in L$, then $f(w) \in F(L)$ and vice versa.

Thus, M' recognizes $F(L)$ for any arbitrary language L where M recognizes L, therefore, we have shown that $F(L)$ is a regular language, and consequenty, the class of regular languages is closed under the unary operator F.

$\square$