Lecture 14

Sunday, October 24, 2021 5:07 PM

Revision

Linear Independence

How do we check for linear independence?

Process Suppose we are given some vectors $\underline{V_1}, \underline{V_2}, \dots, \underline{V_n}$ in some vector space V. We want to check for linear independence.

STEP1 Immediately Form the following equation:

$$k_1 \underline{V_1} + k_2 \underline{V_2} + \dots + k_n \underline{V_n} = \underline{0}$$

$\underline{0} \longrightarrow$ zero vetor of the vector space

$k_1, k_2, \dots, k_n$: Scalars, i.e real numbers (in $\mathbb{R}$)

STEP2 Plug in the vectors $\underline{V_1}, \underline{V_2}, \dots, \underline{V_n}, \underline{0}$ whatever they are and see what equations we can get for the scalars $k_1, k_2, \dots, k_n$

Example Take $M_{2\times2}$, the vector space of 2X2 matrices. Are the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ linearly independent?

STEP 1 Let $\underline{V_1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\underline{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Form the Eqn. $k_1 \underline{V_1} + k_2 \underline{V_2} = \underline{0}$

STEP 2 $k_1 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

STEP3 Use what we get in step 2 and your normal knowledge of mathematics otherwise to form whatever equations we can for the scalars $k_1, k_2, \dots, k_n$.

THEN i) If we can solve these equations and conclude that

$$k_1 = k_2 = \cdots = k_n = 0 \rightarrow \text{real no. } 0$$

then we know that $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ are linearly independent

ii) If these equations allow more than one possible solution for $k_1, \dots, k_n$ (typically infinite solutions), then $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ ARE NOT linearly independent (i.e they are linearly dependent)

iii) If you cannot tell whether i) or ii) above, then you probably need to look more carefully or recall the relevant mathematics!

Continuing with the example above

$$k_1 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} k_1 + k_2 &= 0 \\ 2k_1 + 3k_2 &= 0 \end{aligned}\right] \rightarrow$$

$k_1 = -k_2$, substitute in eqn. below, $-2k_2 + 3k_2 = 0$

$\Rightarrow k_2 = 0$

$\Rightarrow k_1 = 0$

since we have this as the clear solution and no parameters etc. to confuse the issue with further solutions, $k_1 = k_2 = 0$, Hence we have that

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ **ARE** $\boxed{\text{linearly independent}}$ (in $M_{2\times 2}$)

SPAN

Suppose we have vectors $\underline{v_1}, \underline{v_2}, \underline{v_3}, \ldots, \underline{v_n}$ in a Vector Space V

Then every linear combination of these vectors is also a vector in V

e.g. $2\underline{v_1} + 3\underline{v_2} + \frac{\pi}{\sqrt{3}}\underline{v_n}$, $3\underline{v_3} + 8\underline{v_{n-1}}$ etc.

The general form of the linear combination is

$$\underbrace{a_1}\underline{v_1} + a_2\underline{v_2} + \ldots + a_n\underline{v_n}$$

Scalars, i.e. real nos. (in $\mathbb{R}$)

Note some of the a_i may also be zero. As they are in the examples above.
Or maybe none are zero.

$\boxed{\text{THESE ARE } \underline{\underline{\text{ALL}}} \text{ LINEAR COMBINATIONS.}}$

So, for any vectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ the $\underline{\text{SPAN}}$ of these vectors is the set of all linear combinations of those vectors

$$\therefore \quad \text{SPAN}\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\} = \{\text{All vectors in } V \text{ that are linear Combs. of } \underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}\}$$

Or $\text{SPAN}\{v_1, v_2, \ldots, v_n\} = \{\text{All } v \in V : v = a_1v_1 + a_2v_2 + \ldots + a_nv_n \text{ where } a_i \in \mathbb{R}\}$

Or $\quad SPAN\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\} = \{All\ \underline{v} \in V : \underline{v} = a_1\underline{v_1} + a_2\underline{v_2} + \dots + a_n\underline{v_n} \text{ where } a_i \in \mathbb{R}\}$

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Note two things

1) The SPAN of any set of vectors in a vector space V is itself a vector space (You can check the ten axioms)

Therefore the SPAN is a SUBSPACE of V

2) The SPAN can be the whole of V too!

In this case, we say the set $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ spans V.

BASIS (for a vector space V)

DEFINITION: IF $\boxed{V}$ IS ANY VECTOR SPACE AND $\boxed{S} = \{v_1, v_2, \ldots\ldots, v_n\}$ IS A SET OF VECTORS IN $\boxed{V}$, THEN $\boxed{S}$ IS CALLED A BASIS FOR $\boxed{V}$ IF THE FOLLOWING TWO CONDITIONS HOLD:

(a) $\boxed{S}$ IS LINEARLY INDEPENDENT

(b) $\boxed{S}$ SPANS $\boxed{V}$

LET US CONSIDER SOME EXAMPLES OF $\boxed{\text{SETS}}$ WHICH ARE BASES i.e. THEY ARE LINEARLY INDEPENDENT AS WELL AS SPAN DIFFERENT VECTOR SPACES.

E vs. I for plural.
Common.

EXAMPLES: (OF BASES)

BASES → PLURAL OF BASIS

① $\{e_1, e_2, e_3\}$ IS A $\boxed{\text{BASIS}}$ FOR $\boxed{R^3}$ BECAUSE IT'S LINEARLY INDEPENDENT AS WELL AS SPANS $\boxed{R^3}$.

SIMILARLY

② $\{1, x, x^2, \ldots\ldots, x^n\}$ IS A BASIS FOR $\boxed{P_n}$ AND

③ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

IS THE $\boxed{\text{BASIS}}$ FOR $\boxed{M_{22}}$.

IMPORTANT CONCEPT! ⇒ $\boxed{\text{DIMENSION:}}$

THE $\boxed{\text{DIMENSION}}$ OF A VECTOR SPACE $\boxed{V}$ IS DEFINED TO BE THE $\boxed{\text{NUMBER}}$ OF VECTORS IN A BASIS FOR $\boxed{V}$.

REMARKS: ① DIMENSION OF $R^3 = 3$ SINCE THERE ARE $\boxed{\text{THREE}}$ VECTORS IN $\{e_1, e_2, e_3\}$

② DIMENSION OF $P_n = n+1$

③ DIMENSION OF $M_{22} = 4$

TRY THE FOLLOWING:

CHECK WHETHER

$$\left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

IS A BASIS FOR M_{22}?

HINT: (i) FIRST CHECK THAT THE GIVEN MATRICES ARE LINEARLY INDEPENDENT

WHY? Because that is the first condition for being a basis!

PUT $a_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\left.\begin{aligned} -a_1 + a_2 &= 0 \\ a_1 + a_2 &= 0 \\ a_3 &= 0 \\ a_4 &= 0 \end{aligned}\right]$$

We will see that $a_1 = a_2 = a_3 = a_4 = 0$

AND SEE IF $\boxed{a_1 = a_2 = a_3 = a_4 = 0}$

(ii) TAKE AN ARBITRARY ELEMENT OF M_{22} AS $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ AND CHECK IF IT CAN BE WRITTEN AS A LINEAR COMBINATION OF THE GIVEN MATRICES. FOR THIS

WHY? Because if the set of $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ SPANS M_{22}, then this should be true. And if it is true for any arbitrary element, then it is true for ALL elements! so it does span M_{22}

PUT $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= k_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$+ k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ AND TRY TO FIND $\boxed{k_1, k_2, k_3, k_4}$ IN TERMS OF $\boxed{a, b, c}$ AND $\boxed{d}$.

$\left.\begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{matrix}\right\}$ UNKNOWNS $\qquad \left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\}$ KNOWNS

ANSWER: $\boxed{\text{YES}}$ IT IS A BASIS

$\because$ $a_1 = a_2 = a_3 = a_4 = 0$ AND

$$\boxed{\begin{matrix} k_1 = \dfrac{b-a}{2} \ , & k_2 = \dfrac{a+b}{2} \\ k_3 = c \ , & k_4 = d \end{matrix}}$$

REMARK: A VECTOR SPACE MAY HAVE MORE THAN ONE BASIS. BUT IN ALL THE BASES (PLURAL) THE NUMBER OF ELEMENTS (VECTORS) ARE SAME. AS WE

SAW THAT

$$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

AND $\left\{\begin{bmatrix}-1&1\\0&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

ARE BASES FOR M_{22} AND BOTH CONTAIN $\boxed{4}$ VECTORS = DIMENSION OF M_{22}.

NOTE: ① $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ IS ALSO CALLED STANDARD BASIS FOR $\boxed{M_{22}}$

AND SIMILARLY $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ AND $\{1, x, x^2, \ldots, x^n\}$ ARE STANDARD BASES FOR $\boxed{R^3}$ AND $\boxed{P_n}$ RESPECTIVELY.

② $\left\{\begin{bmatrix}-1&1\\0&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ IS A BASIS

BUT NOT A STANDARD BASIS FOR $\boxed{M_{22}}$.

RESULT: IF $S = \{\underline{V}_1, \underline{V}_2, \ldots, \underline{V}_n\}$ IS A SET OF $\boxed{n}$ VECTORS IN AN n-DIMENSIONAL SPACE $\boxed{V}$, THEN $\boxed{S}$ IS A BASIS FOR $\boxed{V}$ IF EITHER $\boxed{S}$ SPANS $\boxed{V}$ $\boxed{\text{OR}}$ $\boxed{S}$ IS LINEARLY INDEPENDENT.

EXAMPLE: SHOW THAT $\{(-3,7), (5,5)\}$ IS A BASIS FOR R^2 (EASY ONE)

SOLUTION: TWO METHODS

METHOD ①:

SINCE R^2 IS $\boxed{\text{TWO}}$ DIMENSIONAL SPACE WHY?

BECAUSE $\{(1,0), (0,1)\}$ IS THE STANDARD BASIS FOR R^2 WHICH CONTAINS $\boxed{\text{TWO}}$ ELEMENTS, THEREFORE WE

ONLY PROVE THE GIVEN SET TO BE LINEARLY INDEPENDENT

CONSIDER $K_1(-3,7) + K_2(5,5) = (0,0)$

$$\Rightarrow \boxed{\begin{aligned} -3K_1 + 5K_2 &= 0 \\ 7K_1 + 5K_2 &= 0 \end{aligned}}$$

COMPARING BOTH SIDES

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—} (1)$$

$$\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35 \neq 0$$

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} \text{ IS INVERTIBLE}$$

$$(1) \Rightarrow \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \boxed{K_1 = K_2 = 0}$ $\therefore$ LIN. INDEPENDENT AND FINALLY $\{(-3,7), (5,5)\}$ IS A BASIS FOR R^2.

METHOD (2):

SINCE R^2 IS TWO DIMENSIONAL, THEREFORE WE ONLY PROVE THAT THE

GIVEN SET $\{(-3,7),(5,5)\}$

SPANS R^2

CONSIDER

UNKNOWNS

$$(x,y) = k_1(-3,7) + k_2(5,5)$$

$$\Rightarrow \begin{aligned} x &= -3k_1 + 5k_2 \\ y &= 7k_1 + 5k_2 \end{aligned}$$

KNOWNS

$$\Rightarrow \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{—} ①$$

$\therefore \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}$ IS INVERTIBLE

AS $\det \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix} = -15 - 35 = -50 \neq 0$

$$\therefore \begin{bmatrix} -3 & 5 \\ 7 & 5 \end{bmatrix}^{-1} = -\frac{1}{50}\begin{bmatrix} 5 & -5 \\ -7 & -3 \end{bmatrix}$$

$\therefore$

$$① \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = -\frac{1}{50}\begin{bmatrix} 5x - 5y \\ -7x - 3y \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} k_1 &= -\frac{1}{10}(x-y) \\ k_2 &= \frac{1}{50}(7x+3y) \end{aligned}}$$

UNIQUE SOLUTION

$\therefore$ ANY ELEMENT $(x,y) \in R^2$ CAN BE WRITTEN AS A LINEAR COMBINATION OF $(-3,7)$ AND $(5,5)$ $\therefore$ $\{(-3,7),(5,5)\}$ SPANS R^2 $\therefore$ A BASIS FOR R^2.

NOTE: $\{(-3,7),(5,5)\}$ IS A BASIS BUT $\boxed{\text{NOT}}$ A STANDARD BASIS SINCE DIFFERENT FROM $\{(1,0),(0,1)\}$.

TRY THE FOLLOWING:

IF $S = \{v_1, v_2, \ldots, v_n\}$ IS A $\boxed{\text{BASIS}}$ FOR A VECTOR SPACE $\boxed{V}$, THEN EVERY VECTOR v IN $\boxed{V}$ CAN BE EXPRESSED IN THE FORM

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

IN EXACTLY $\boxed{\text{ONE}}$ WAY.

DEFINITION:

HERE $(c_1, c_2, \ldots, c_n)$ IS CALLED THE COORDINATE VECTOR OF v RELATIVE TO $\boxed{S}$.

STUFF COVERED TODAY

1. Defining BASIS and seeing the concept in application
2. Defining DIMENSION of a Vecto Space (no. of elements in a basis)
3. That many different BASES may exist for the same set (typically infinite)

Exercise: Prove This!

but the no. of elements in each set is the same (as the dimension)

4. Coordinates (prove the result).