Matching in Graphs (a refresher)

CS 412 – Week 09

Shah Jamal Alam

Source: *CLRS*

The marriage problem

Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda.

Can each man be matched with a woman he is willing to marry?

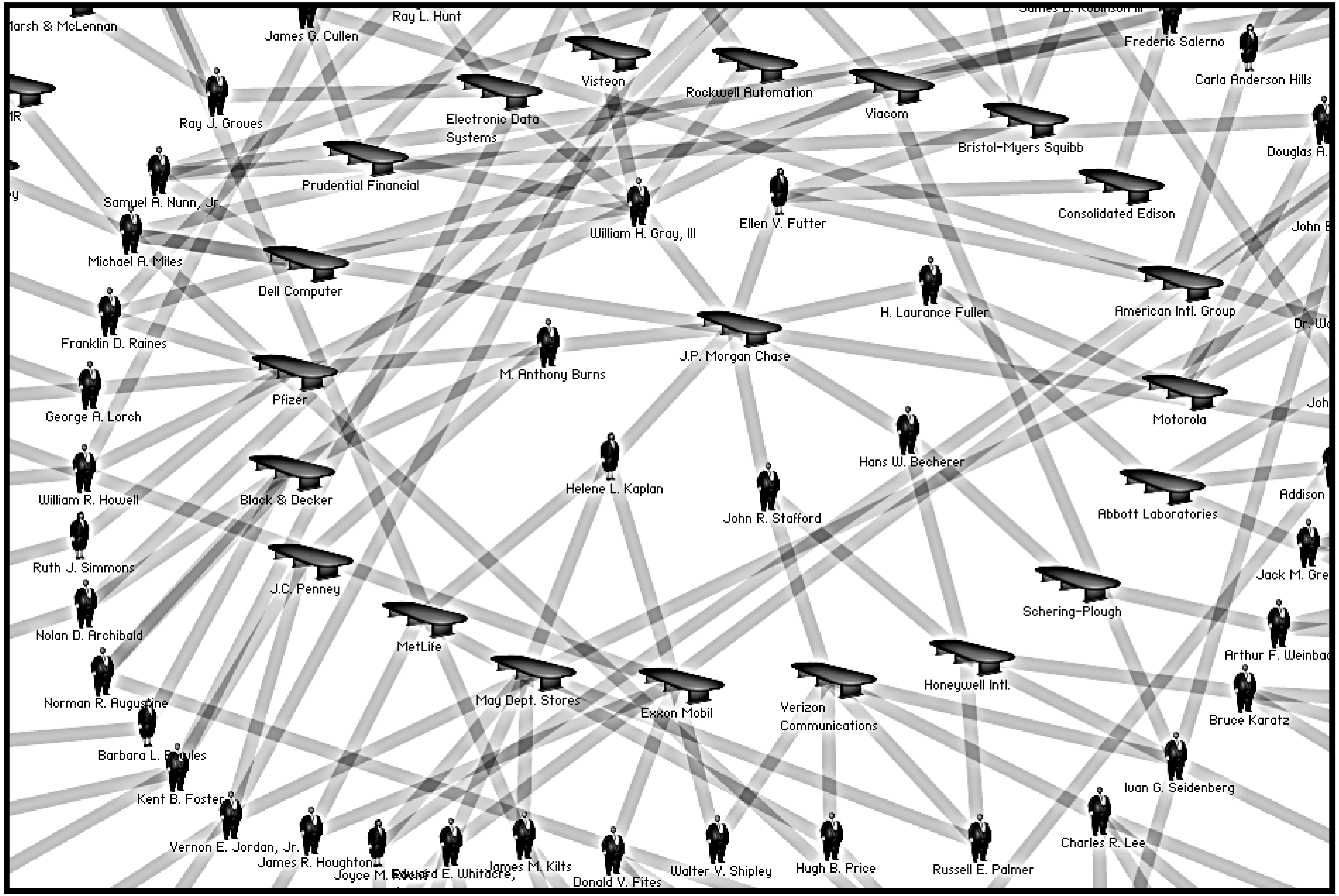
Rosen (2013)

Bipartite Graphs

Definition: A simple graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is bipartite if its vertex set V can be **partitioned** into **two disjoint** sets V_1 and V_2, such that, every edge in the graph connects a vertex v_1 in V_1 to a vertex v_2 in V_2 (no edge connects either two vertices in V_1 or two vertices in V_2). $\mathrm{G} = (\mathrm{V}_1, \mathrm{V}_2, \mathrm{E})$ is then a bipartite graph.

We call the pair $(\mathrm{V}_1, \mathrm{V}_2)$ a **bi-partition** of the vertex set V.

Boards of directors network



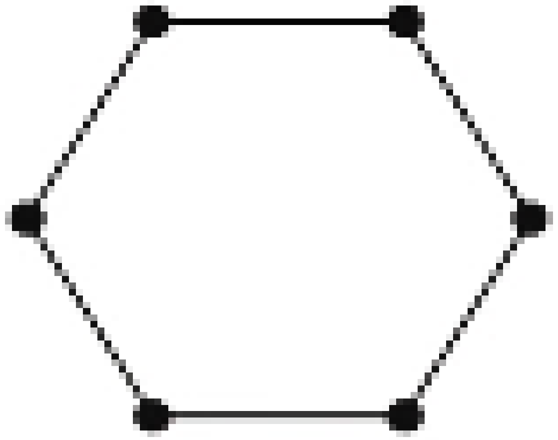
theyrule.net

Non-profits play a central role in the business network (e.g., Carnegie Hall)

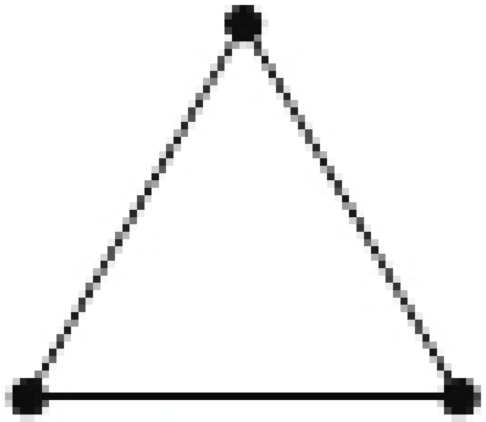
Name	Affiliation
Sandy Weill (Chairman)	CEO, Citigroup
William B. Harrison, Jr.	CEO, JP Morgan Chase
C. Michael Armstrong	CEO, ATT
Harry P. Kamen	CEO, MetLife
Frank Newman	CEO, Bankers Trust
Ronald Perelman	CEO, MacAndrews & Forbes
Joe Roby	Chair Emeritus, CSFB
James D. Wolfensohn	President, World Bank
Rudy Giuliani	Recovering mayor
And ~ 50 others, including: Bill Cosby, Marilyn Horne, Peter Jennings, Yo-Yo Ma, Oscar de la Renta	

Rosen (2013)

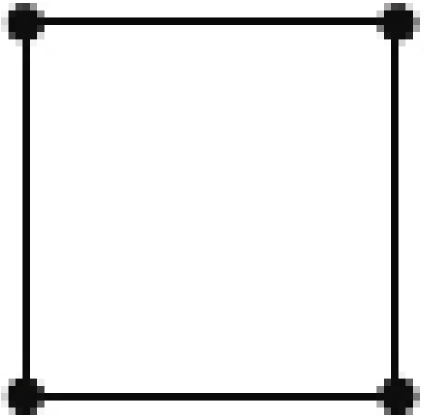
Which of these is a bipartite graph?



C_6



K_3

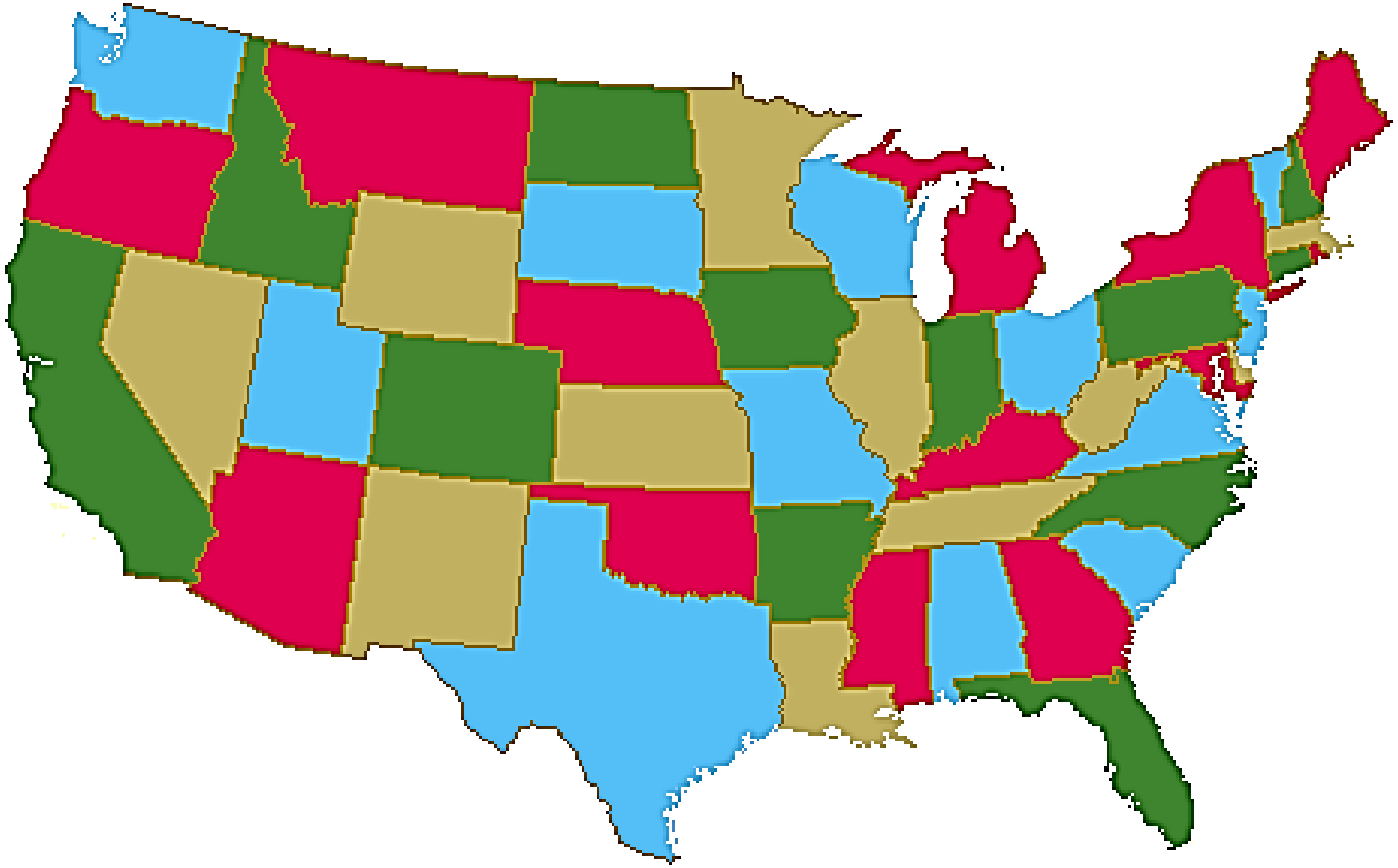


C_4

Rosen (2013)

Bichromatic -> Graph coloring (a more general problem)

Application: coloring a map: limited set of colors, no two adjacent countries should have the same color



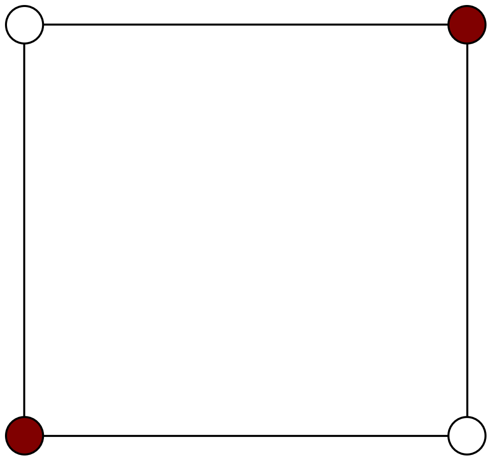
How to determine if a graph is bipartite?

Theorem: A simple graph is bipartite if and only if (iff) it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

bichromatic: vertices can be colored using two colors so that no two vertices of the same color are adjacent

Bipartite Graphs

EG: C_4 is a bichromatic:



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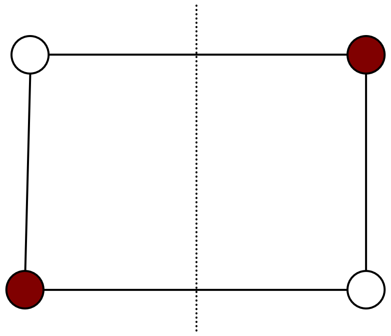

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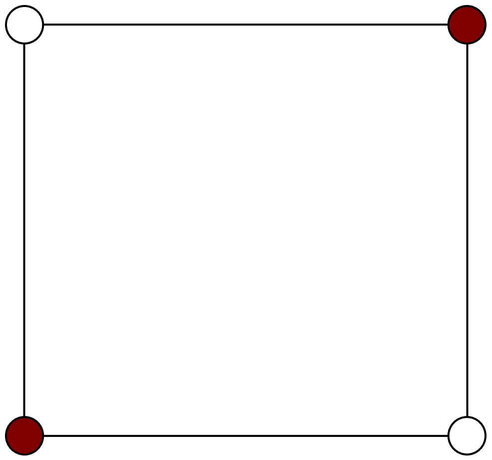

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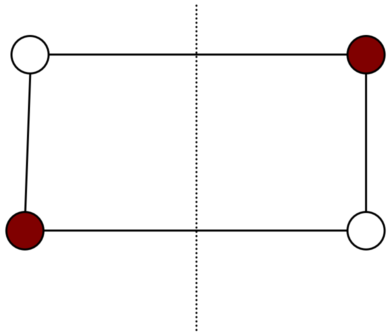

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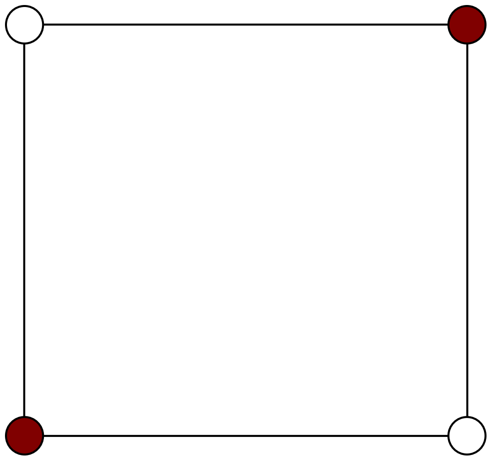

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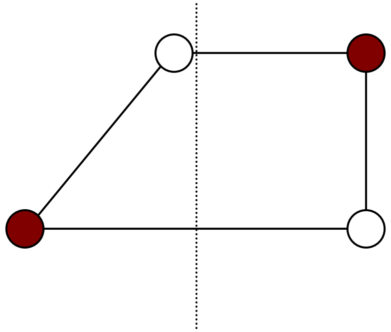

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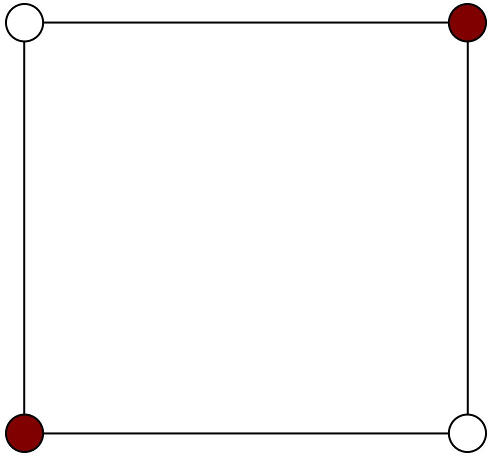

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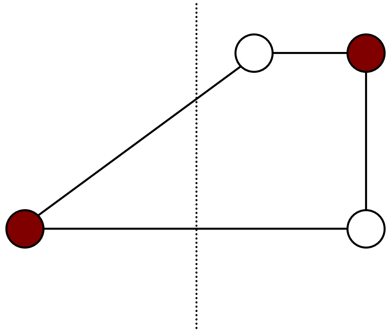

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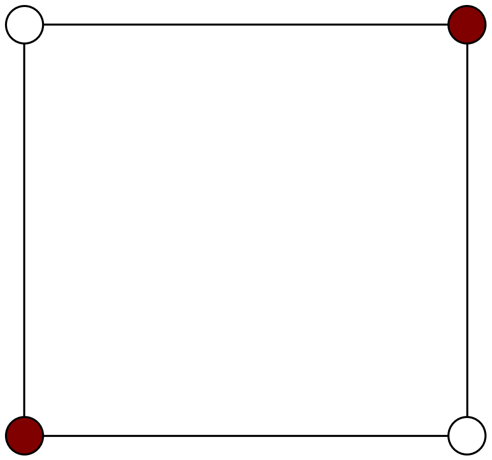

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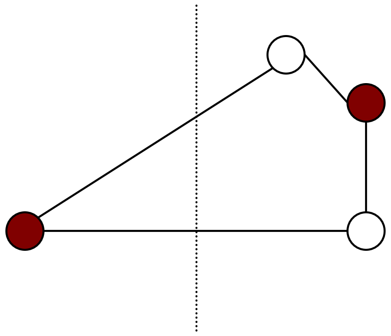

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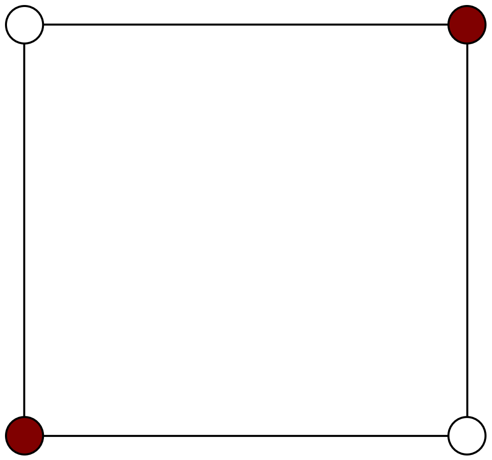

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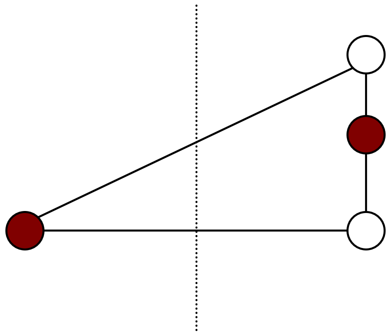

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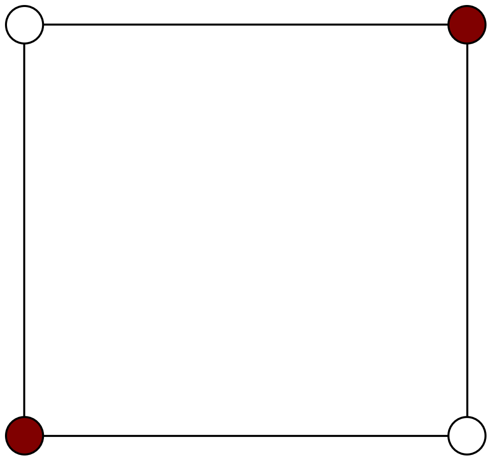

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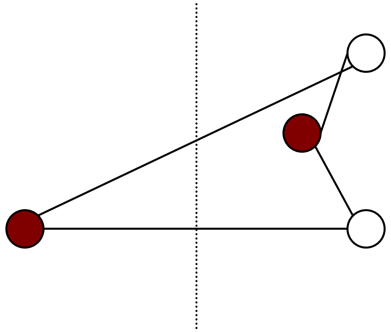

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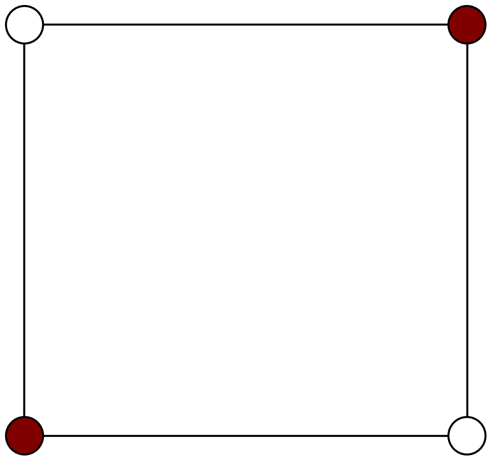

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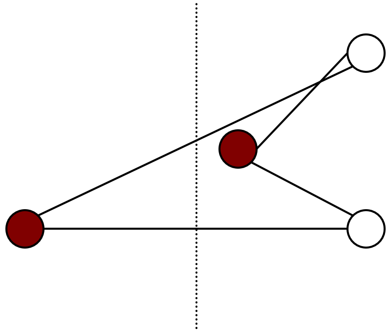

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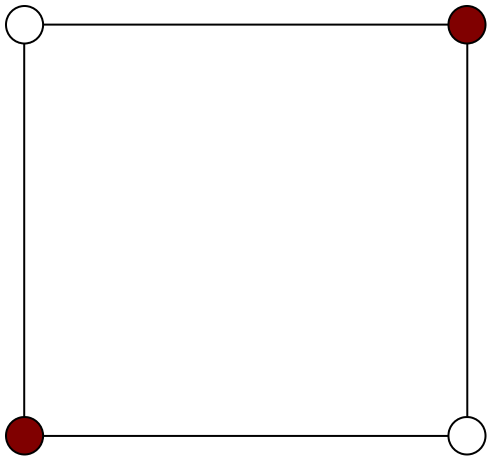

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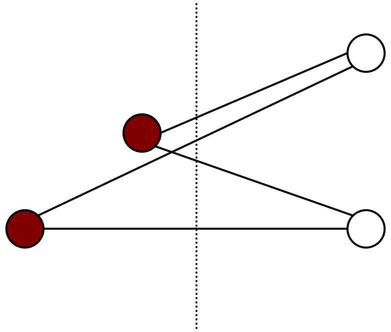

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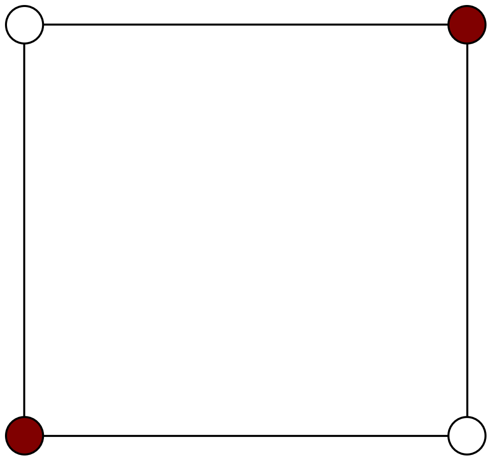

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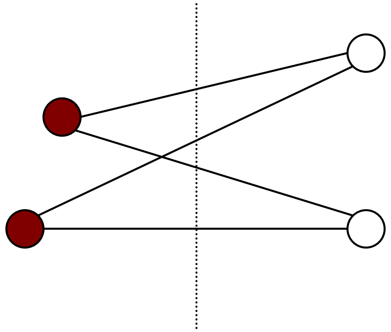

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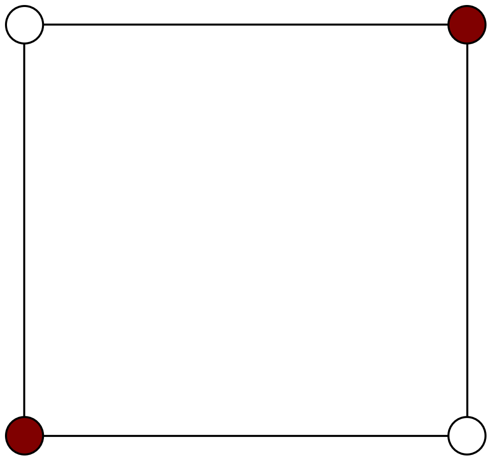

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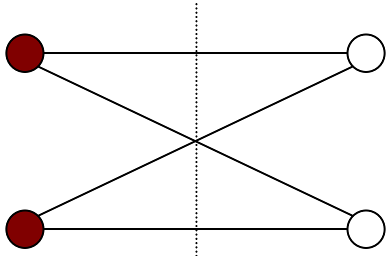


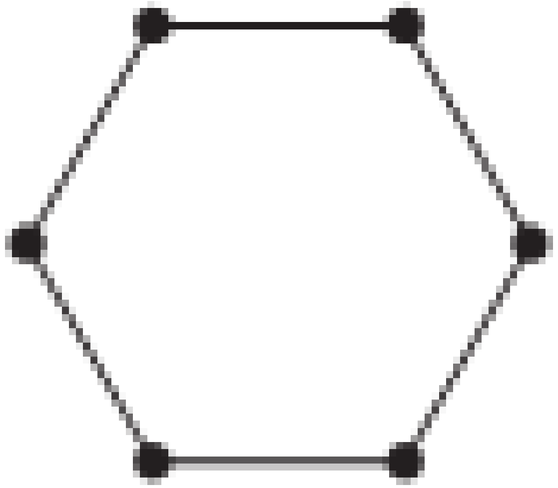
Q: For which n is C_n bipartite?

For which n is C_n bipartite?

C_n is bipartite when n is **even**. For even n, color all odd numbers red and all even numbers green so that vertices are only adjacent to opposite color.

If n is **odd**, C_n is not bipartite. If it were, color 0 red. So 1 must be green, and 2 must be red. This way, all even numbers must be red, including vertex *n*-1. But *n*-1 connects to 0 →←.

Which of these is a bipartite graph?



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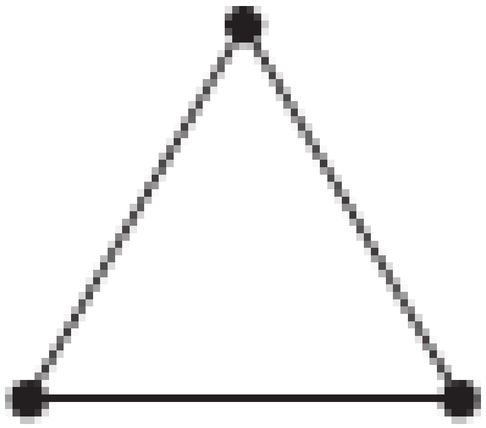

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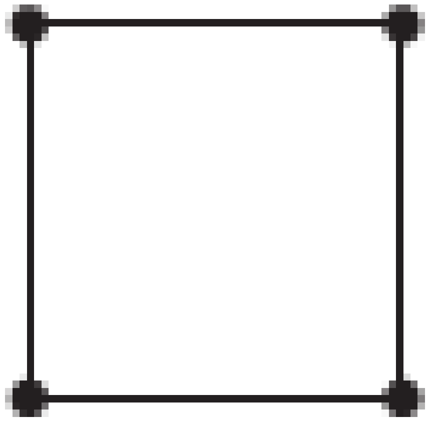

C_4

Definition: A simple graph G is bipartite if and only if (iff) it has no circuits with an odd number of edges.

Rosen et al. (2013)

How to determine if a graph is bipartite?

Theorem: A simple graph is bipartite if and only if (iff) it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

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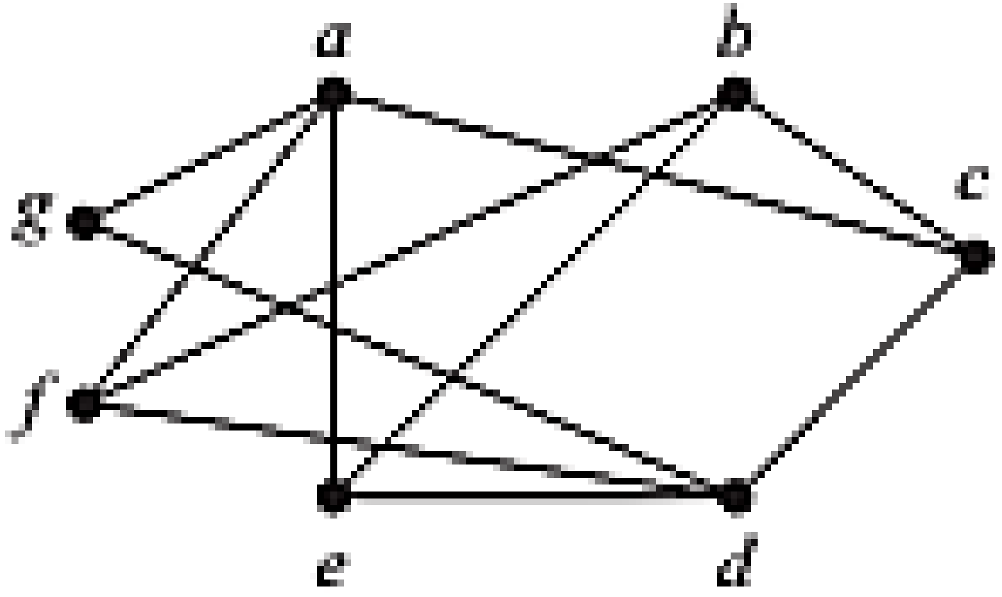
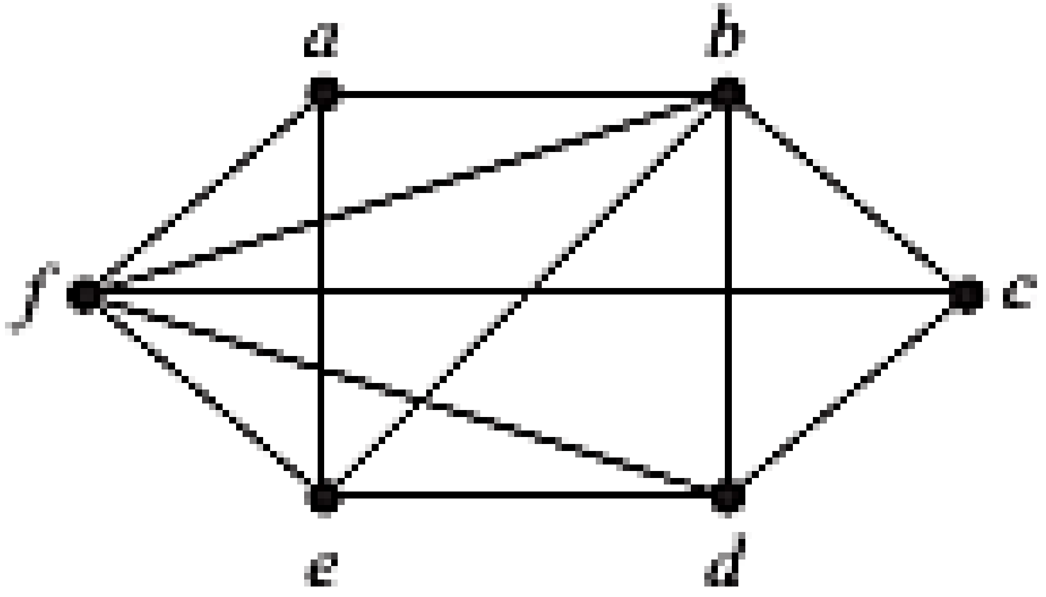


G

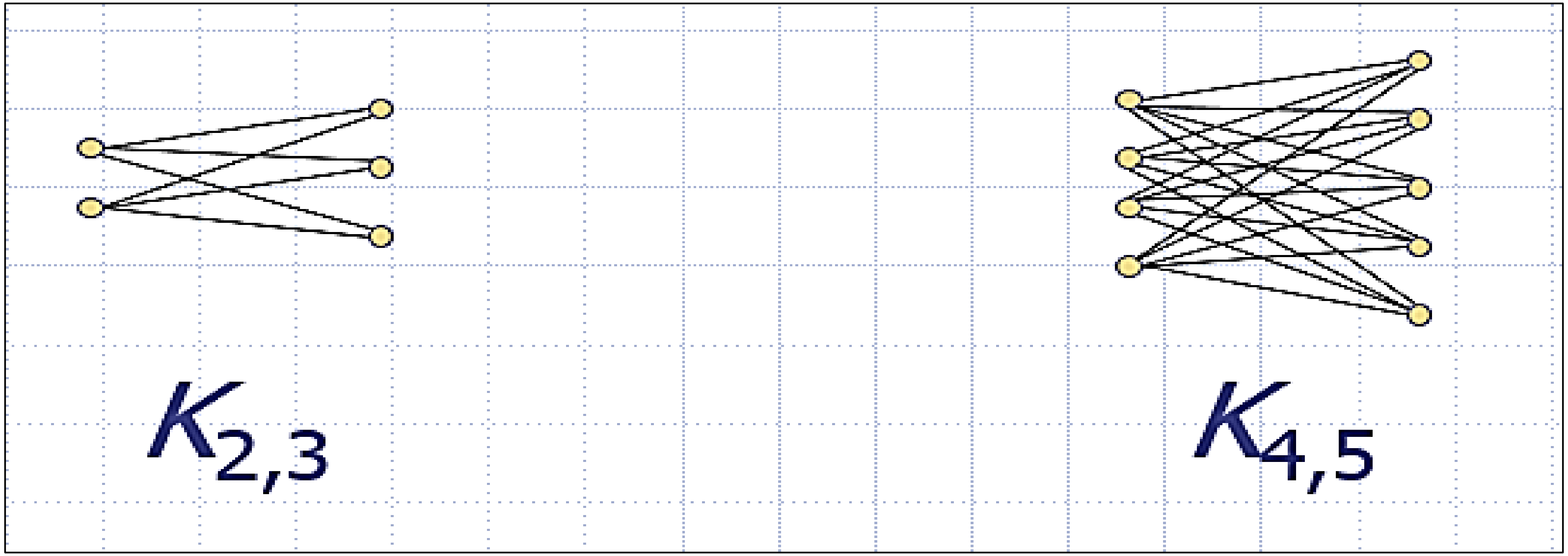


H

Rosen (2013)

Complete Bipartite – $K_{m, n}$

When all possible edges exist in a simple bipartite graph with *m* red vertices and *n* green vertices, the graph is called a ***complete bipartite*** and the notation $K_{m, n}$ is used.



A Job Scheduling problem

Suppose that there are m employees in a company and n different jobs that need to be done, where $m \geq n$. Each employee is trained to do one or more of these n jobs. We would like to assign an employee to each job.

Rosen (2013)

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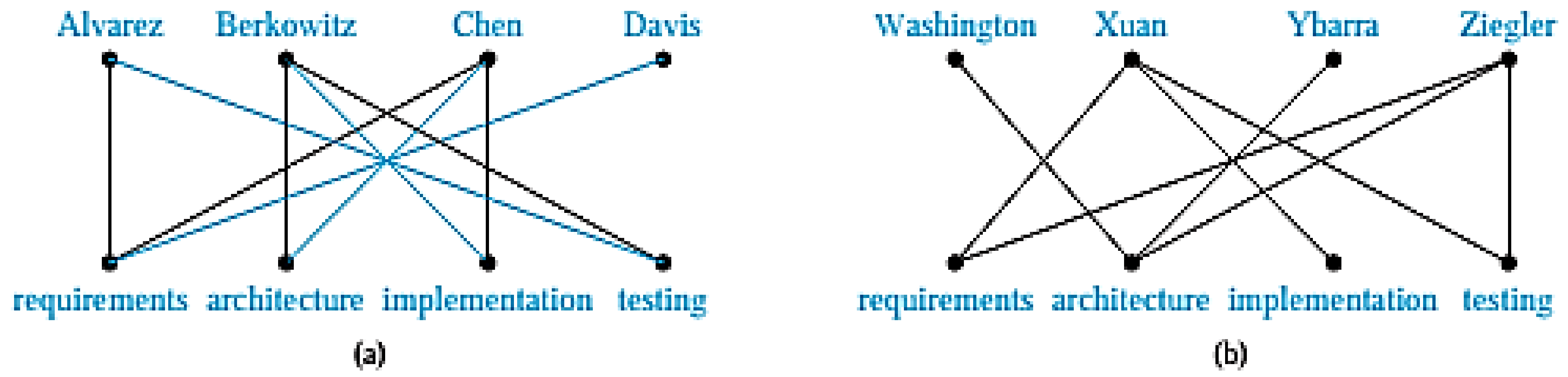


FIGURE 10 **Modeling the Jobs for Which Employees Have Been Trained.**

Rosen (2013)

Matching in graphs

- A subset of edges in a simple graph $G = (V, E)$, such that <u>no two edges are incident with the same vertex</u>.

- In other words, a matching M is a subset of edges such that if (s, t) and (u, v) are two distinct edges then s, t, u and v are distinct vertices in a simple graph.

- A vertex that is the endpoint of an edge of a matching M is said to be **matched** in M; otherwise it is said to be **unmatched.**

Bipartite matching

We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching** from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

HALL'S MARRIAGE THEOREM The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Rosen (2013)

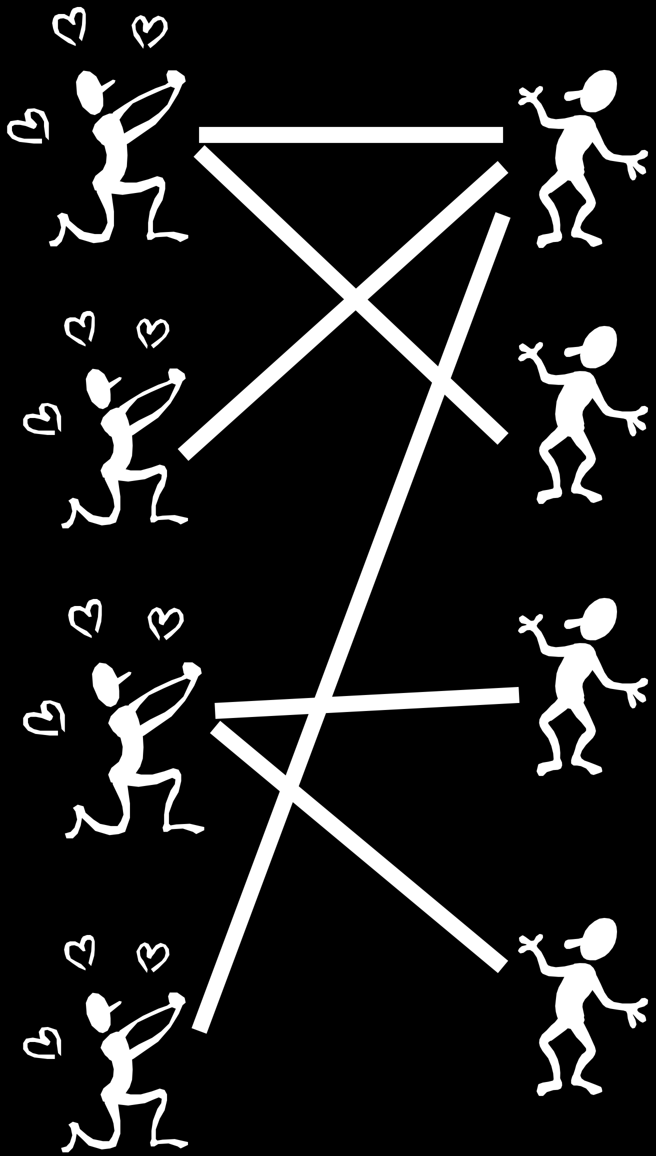
Back to the matching problem

Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda.

Can each man be matched with a woman he is willing to marry?

Rosen (2013)

The Marriage Theorem

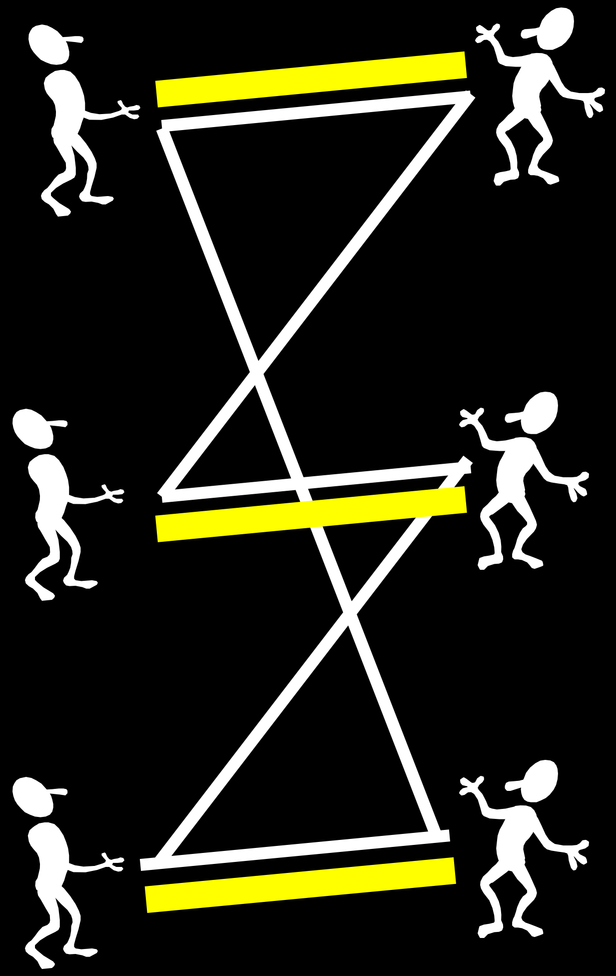


For any subset of (say) k nodes of A there are at least k nodes of B that are connected to at least one of them

The condition fails for this graph

Credits: TBA

Dancing Partners



Credits: TBA

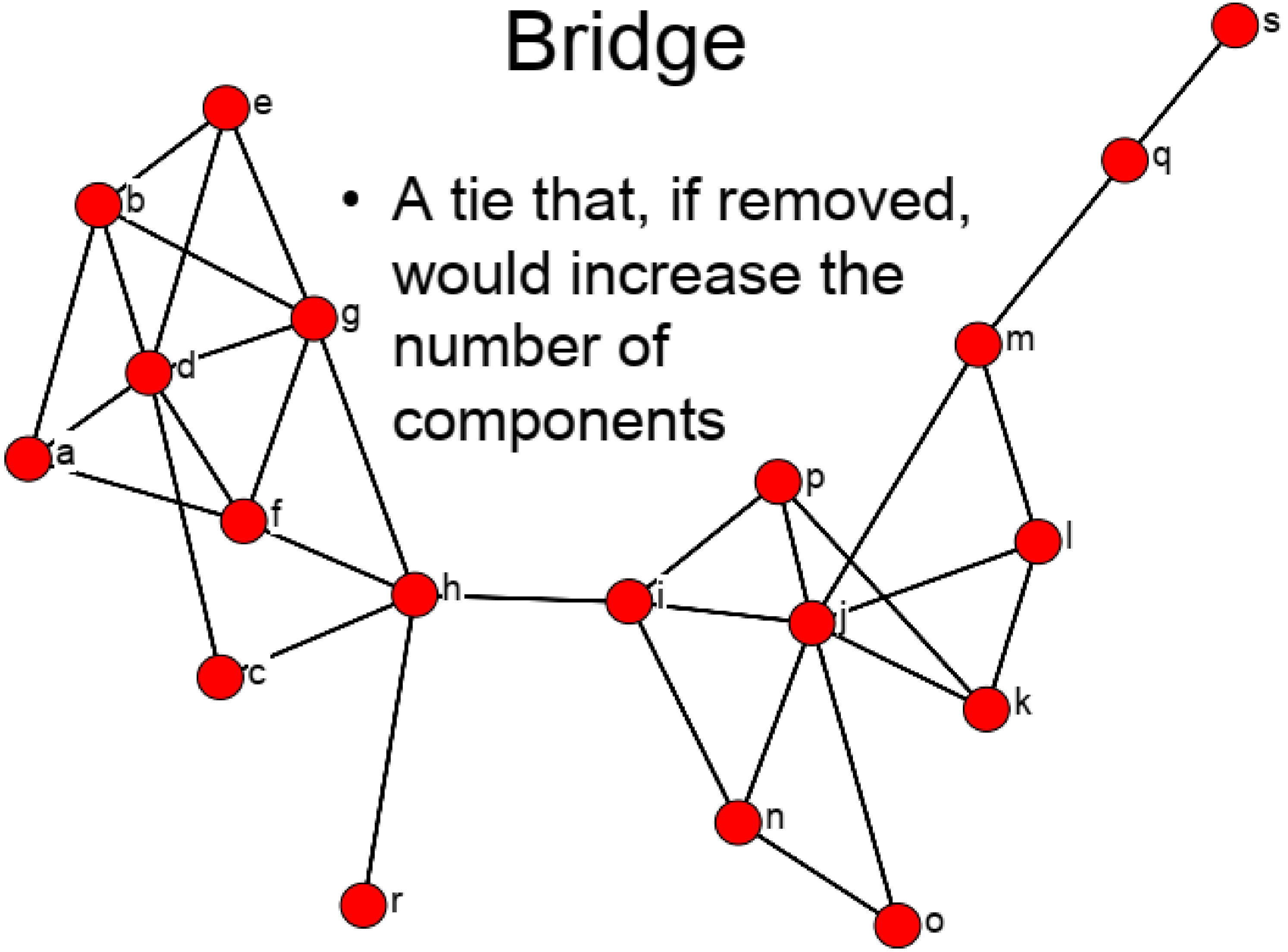
Perfect Matching

Theorem: If every node in a bipartite graph has the same degree $d = 1$, then the graph has a perfect matching.

Note: if degrees are the same then $|A| = |B|$, where A is the set of nodes "on the left" and B is the set of nodes "on the right".

Bridge

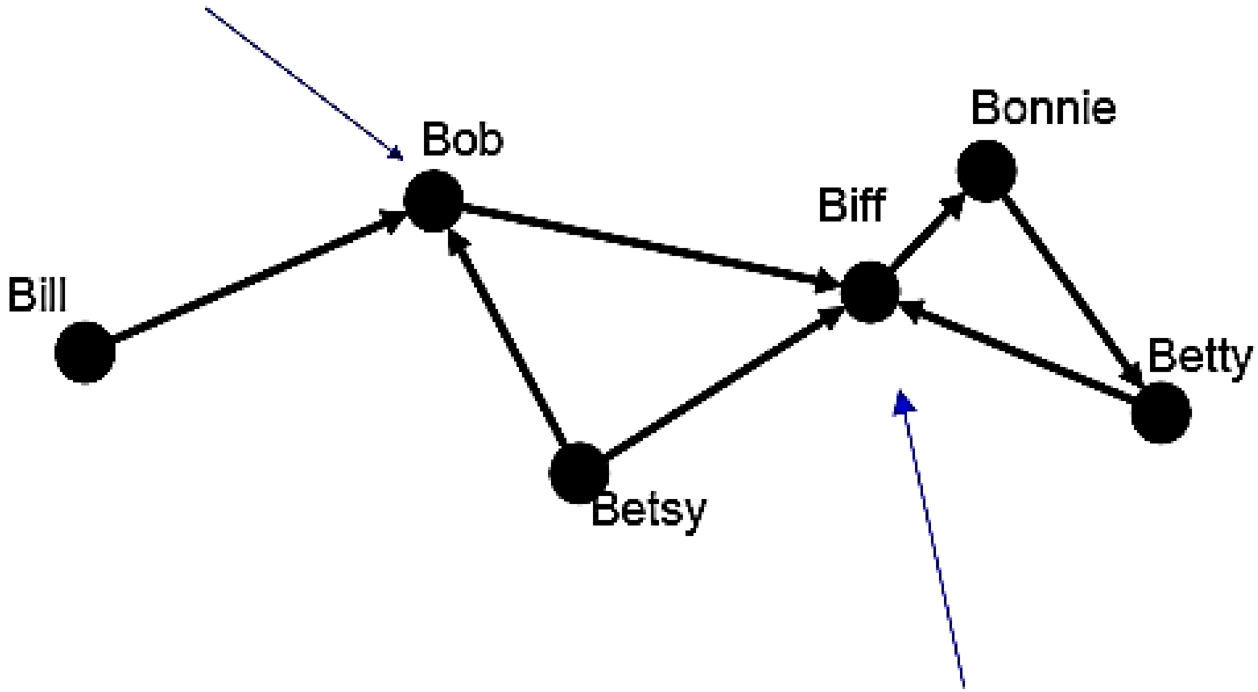
- A tie that, if removed, would increase the number of components



Borgatti (2016)

Cutpoint

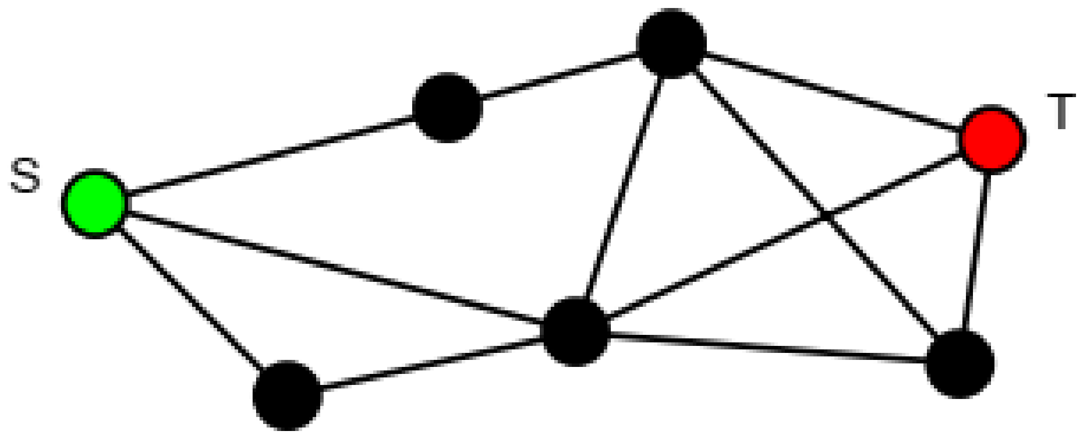
- A node which, if deleted, would increase the number of components



Borgatti (2016)

Maximum Flow

- If ties are pipes with capacity of 1 unit of flow, what is the maximum # of units that can flow from s to t?



Borgatti (2016)