

# Proofs

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## Chapter 3

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### Lemma 3.1.6 (Single choice)

Let  $A$  be a non-empty set. Then there exists an object  $x$  such that  $x \in A$ .

#### Proof

Assume that  $\neg \exists x, x \in A$ , hence,  $x \in A$  is false for all  $x$  (as Definition 3.1.1 ensures the dichotomy of  $x \in A$  and  $x \notin A$ ).

By Axiom 3.2,  $\forall x, x \notin \emptyset$ , hence,  $x \in \emptyset$  is false for all  $x$ .

Therefore, it holds that  $\forall x, x \in A \rightarrow x \in \emptyset$ . By Definition 3.1.4,  $A = \emptyset$ , contradiction as desired.

### Remark 3.1.9

There is only one singleton set for each object  $a$ .

#### Proof

Assume we have two singleton sets  $A, A'$  for object  $a$ . Then,  $x \in A$  iff  $x = a$  by Axiom 3.3, and  $x \in A'$  iff  $x = a$  by Axiom 3.3. Hence by transitivity  $x \in A \rightarrow x \in A'$ . By Definition 3.1.4,  $(x \in A \rightarrow x \in A') \rightarrow A = A'$ , as desired.

Given any two objects  $a$  and  $b$ , there is only one pair set formed by  $a$  and  $b$

#### Proof

Assume there exists two pair sets  $P, P'$  for objects  $a, b$ . Then,  $x \in P$  iff  $x = a \vee x = b$  by Axiom 3.3, and  $x \in P'$  iff  $x = a \vee x = b$  by Axiom 3.3. The rest of the proof follows from proof for "There is only one singleton set for each object  $a$ ."

$\{a, a\} = \{a\}$  (aka pair set of  $a, a$  is  $\{a\}$ )

#### Proof

Denote  $A = \{a, a\}$ ,  $A' = \{a\}$ , then,  $x \in A$  iff  $x = a \vee x = a$  by Axiom 3.3. aka,  $x \in A$  iff  $x = a$ . Similarly,  $x \in A'$  iff  $x = a$  by Definition 3.1.1. Similar reasoning (based on transitivity) follows.

Hence, Axiom 3.3 is redundant, in the way that if pair set exists, we can construct singleton sets from pair sets. We can also construct pair set from singleton set and Axiom 3.4.

### Exercise 3.1.1

Show that the definition of equality in Definition 3.1.4 is reflexive, symmetric, and transitive.

Proof:

By definition of equality:  $\forall x, x \in A \Leftrightarrow x \in B$

Note that  $\Leftrightarrow$  is reflexive, symmetric and transitive, hence set equality is also reflexive, symmetric and transitive.

### Exercise 3.1.2

Show  $\emptyset, \{\emptyset\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}$  are distinct.

Proof:

We need to show:

- $\emptyset$  is distinct from all other sets
- $\{\emptyset\}$  is distinct from  $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}$
- $\{\{\emptyset\}\}$  is distinct from  $\{\emptyset, \{\emptyset\}\}$

Part 1:  $\emptyset$  is distinct from all other sets

By Axiom 3.3:

- $\emptyset \in \{\emptyset, \{\emptyset, \{\emptyset\}\}$
- $\{\emptyset\} \in \{\{\emptyset\}\}$

By Lemma 3.1.6, All 3 other sets are non-empty, as desired.

Part 2:  $\{\emptyset\}$  is distinct from  $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

Assume  $\{\emptyset\} = \{\{\emptyset\}\}$ , then by Definition 3.1.4,  $\emptyset \in \{\emptyset\} \Leftrightarrow \emptyset \in \{\{\emptyset\}\}$ . However,  $\emptyset \in \{\emptyset\}$  is true while  $\emptyset \in \{\{\emptyset\}\}$  is false (By Axiom 3.3), contradiction as desired.

Showing  $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$  is similar.

Part 3:  $\{\{\emptyset\}\}$  is distinct from  $\{\emptyset, \{\emptyset\}\}$

Similar to Part 2.

### Remark 3.1.12

Show Pairwise Union (as defined in Axiom 3.4) satisfies substitution axiom.

Aka:

- If  $A, B, A'$  are sets, and  $A = A'$ , then  $A \cup B$  is equal to  $A' \cup B$
- If  $B'$  is a set which is equal to  $B$ , then  $A \cup B$  is equal to  $A \cup B'$

Proof

WLOG we prove the first case. (Proof for second case is similar)

We have:

$$x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$$

$$x \in A' \cup B \Leftrightarrow (x \in A' \vee x \in B)$$

Since  $A = A'$ ,  $x \in A \Leftrightarrow x \in A'$  (By Definition 3.1.4). Hence:

$$x \in A' \cup B \Leftrightarrow (x \in A \vee x \in B)$$

Therefore, by transitivity,  $x \in A \cup B \Leftrightarrow x \in A' \cup B$ . By definition 3.1.4,  $A \cup B = A' \cup B$  as desired.

### Lemma 3.1.13

- If  $a$  and  $b$  are objects, then  $\{a, b\} = \{a\} \cup \{b\}$ .
- If  $A, B, C$  are sets, then the union operation
  - is commutative (i.e.,  $A \cup B = B \cup A$ )
  - is associative (i.e.,  $(A \cup B) \cup C = A \cup (B \cup C)$ ).
- Also, we have  $A \cup \emptyset = \emptyset \cup A = A$ .

Proof

This is actually 4 Lemmas.

Part 1: Pair set can be constructed from singleton set & Pairwise Union

Proof

We have  $x \in \{a, b\} \Leftrightarrow (x = a) \vee (x = b)$  (By Axiom 3.3)

We have  $x \in \{a\} \cup \{b\} \Leftrightarrow (x = a) \vee (x = b)$  (By Axiom 3.4)

Hence by transitivity,  $x \in \{a, b\} \Leftrightarrow x \in \{a\} \cup \{b\}$ . By definition 3.1.4,  $\{a, b\} = \{a\} \cup \{b\}$  as desired.

Part 2: Pairwise Union is commutative

Proof

By Axiom 3.4:

- $x \in A \cup B \rightarrow x \in A \vee x \in B$
- $x \in B \cup A \rightarrow x \in B \vee x \in A$

By propositional logic,  $(x \in A \vee x \in B) \rightarrow (x \in B \vee x \in A)$ . Hence, by transitivity  $x \in A \cup B \rightarrow x \in B \cup A$ . By Definition 3.1.4  $x \in A \cup B = x \in B \cup A$  as desired.

Part 3: Pairwise Union is associative

By Axiom 3.4:

- $x \in A \cup (B \cup C) \rightarrow x \in A \vee x \in (B \cup C)$
- $x \in (A \cup B) \cup C \rightarrow x \in (A \cup B) \vee x \in C$

Hence, note  $x \in A \cup B \rightarrow x \in A \vee x \in B$ , therefore since  $\rightarrow$  is analogous to equality in propositional logic, we can employ substitution axiom:

- $x \in (A \cup B) \cup C \rightarrow x \in A \vee x \in B \vee x \in C$

Similarly, we can show:

- $x \in A \cup (B \cup C) \rightarrow x \in A \vee x \in B \vee x \in C$

By transitivity,  $x \in A \cup (B \cup C) \rightarrow x \in (A \cup B) \cup C$ , by definition 3.1.4  $A \cup (B \cup C) = (A \cup B) \cup C$  as desired.

Part 4:  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$

$A \cup \emptyset = \emptyset \cup A$  follows from commutativity. Remains to show that:

- $A \cup A = A$
- $A \cup \emptyset = A$

For  $A \cup A = A$ , note (by Axiom 3.4):

$$x \in A \cup A \rightarrow x \in A \vee x \in A$$

And since  $x \in A \vee x \in A = x \in A$  (by propositional logic), we use substitution axiom.

$$x \in A \cup A \rightarrow x \in A$$

Similarly by Axiom 3.4:

$$x \in A \rightarrow x \in A$$

Rest follows from transitivity and Definition 3.1.4.

For  $A \cup \emptyset = A$ , by Axiom 3.4:

$$x \in A \cup \emptyset \rightarrow x \in A \vee x \in \emptyset$$

Since  $x \in \emptyset$  is always false (By Axiom 3.2), above can be simplified using propositional logic.

$$x \in A \cup \emptyset \iff x \in A$$

Note:

$$x \in A \cup A \iff x \in A$$

Rest follows from transitivity and Definition 3.1.4.

### Proposition 3.1.18 (Sets are partially ordered by set inclusion)

Let  $A, B, C$  be sets. If  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ . If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$ . Finally, if  $A \not\subseteq B$  and  $B \not\subseteq C$  then  $A \not\subseteq C$ .

Proof

Part 1: If  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$

By definition 3.1.15:

- $x \in A \implies x \in B$
- $x \in B \implies x \in C$

We need to show:

- $x \in A \implies x \in C$

Above can be shown with propositional logic. Can either use propositional logic laws or enumerate through all  $2^3$  states.

Part 2: If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$

By definition 3.1.15, we have:

- $x \in A \implies x \in B$
- $x \in B \implies x \in A$

We need to show (from definition 3.1.4):

- $x \in A \iff x \in B$

Similarly we can use propositional logic to perform the deduction.

Part 3: if  $A \not\subseteq B$  and  $B \not\subseteq C$  then  $A \not\subseteq C$

Ah now I need first order logic there...

- $\forall x, x \in A \implies x \in B$
- $\forall x, x \in B \implies x \in C$

- $\sim \forall x, x \in A \rightarrow x \in B$
- $\sim \forall x, x \in B \rightarrow x \in C$

Need to show:

- $\forall x, x \in A \rightarrow x \in C$
- $\sim \forall x, x \in A \rightarrow x \in C$

Part 1:  $\forall x, x \in A \rightarrow x \in C$

We can combine first 2 statements to form:

- $\forall x, (x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in C)$

Above, by propositional logic, simplifies to:

$\forall x, x \in A \rightarrow x \in C$

As desired.

Part 2:  $\sim \forall x, x \in A \rightarrow x \in C$  (aka  $A \neq C$ )

Assume  $A = C$ , then all element of  $C$  is in  $A$  (Definition 3.1.4). However, since  $B \not\subseteq C$ , some element  $x \in C$  is not in  $B$  (Follows from simple contradiction from Definition 3.1.4 and Definition 3.1.15). There are two cases:  $x \in A$  and  $x \notin A$  (Definition 3.1.1)

Case 1:  $x \in A$

Then,  $x \in B$  as  $A \subseteq B$  (Definition 3.1.15). Contradiction as  $x \notin B$ .

Case 2:  $x \notin A$

Then, since  $x \in C$ , by definition 3.1.4 it must hold that  $x \in A$ . Contradiction.

Hence, in neither case,  $A = C$ , as desired.

## Proposition 3.1.28 (Sets form a boolean algebra)

Let  $A, B, C$  be sets, and let  $X$  be a set containing  $A, B, C$  as subsets.

- (a) (Minimal element) We have  $A \cup \emptyset = A$  and  $A \cap \emptyset = \emptyset$ .
- (b) (Maximal element) We have  $A \cup X = X$  and  $A \cap X = A$ .
- (c) (Identity) We have  $A \cap A = A$  and  $A \cup A = A$ .
- (d) (Commutativity) We have  $A \cup B = B \cup A$  and  $A \cap B = B \cap A$ .
- (e) (Associativity) We have  $(A \cup B) \cup C = A \cup (B \cup C)$  and  $(A \cap B) \cap C = A \cap (B \cap C)$ .
- (f) (Distributivity) We have  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  and  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
- (g) (Partition) We have  $A \cup (X \setminus A) = X$  and  $A \cap (X \setminus A) = \emptyset$

- (h) (de Morgan laws) we have  $\neg(A \cup B) = (\neg A) \cap (\neg B)$  and  $\neg(A \cap B) = (\neg A) \cup (\neg B)$ .

Proof

a)

Follows from Lemma 3.1.13

b)

Part 1

By Axiom 3.4:

$$x \in A \cup X \iff x \in A \vee x \in X$$

$$\text{By Definition 3.1.15 } x \in A \implies x \in X$$

Therefore:

$$x \in A \cup X \iff x \in X$$

Hence by definition of equality  $A \cup X = X$  as desired.

Part 2

By definition 3.1.23:

$$x \in A \cap X \iff x \in A \wedge x \in X$$

$$\text{By Definition 3.1.15 } x \in A \implies x \in X$$

Hence, if  $x \in A$ , then  $x \in A \wedge x \in X$  is true, if  $x \notin A$ , then  $x \in A \wedge x \in X$  is false. Therefore:

$$x \in A \cap X \iff x \in A$$

As desired.

c)

Part 1

Follows from Lemma 3.1.13

Part 2

By Definition of intersection:

$$x \in A \cap A \iff x \in A \wedge x \in A$$

By propositional logic,  $x \in A \text{ and } x \in A$  simplifies to  $x \in A$ , hence:

$$x \in A \cap A \Leftrightarrow x \in A$$

As desired.

d)

Part 1:

Follow from Lemma 3.1.13

Part 2:

By Definition of intersection:

$$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$$

$$x \in B \cap A \Leftrightarrow x \in B \text{ and } x \in A$$

Since  $\text{and}$  is commutative:

$$x \in B \cap A \Leftrightarrow x \in A \text{ and } x \in B$$

Hence, by transitivity:

$$x \in A \cap B \Leftrightarrow x \in B \cap A$$

As desired.

e)

Part 1:

Follow from Lemma 3.1.13

Part 2:

Similar to proof for Lemma 3.1.13, hence omitted

f)

Again we can reduce the problem to propositional logic.

Part 1

$$x \in A \cap (B \cup C) \Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$$

$$x \in (A \cup B) \cap (A \cup C) \Leftrightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$$

We perform further expansion:



$$\forall x \in (A \cup B) \cap (A \cup C) \implies (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$$

We remain to show that  $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \implies (x \in B \vee x \in C)$ , which follows from propositional logic, as desired.

Part 2:

Similar to part 1.

g) h)

We similarly:

- Apply the definition
- Use propositional logic
- Transitivity + Definition of set equality

### Exercise 3.1.5

Let  $A, B$  be sets. Show that the three statements  $A \subseteq B$ ,  $A \cup B = B$ ,  $A \cap B = A$  are logically equivalent (any one of them implies the other two).

Proof:

We have  $A \subseteq B$  if  $x \in A \implies x \in B$ .

We have  $A \cup B = B$  if  $x \in A \vee x \in B \implies x \in B$

We have  $A \cap B = A$  if  $x \in A \wedge x \in B \implies x \in A$

We can use propositional logic to prove that the above 3 are equivalent. Simply enumerate all 4 configurations in truth table.

### Exercise 3.1.7

- Let  $A, B, C$  be sets. Show that  $A \cap B \subseteq A$  and  $A \cap B \subseteq B$ .
- Furthermore, show that  $C \subseteq A$  and  $C \subseteq B$  if and only if  $C \subseteq A \cap B$ .
- In a similar spirit, show that  $A \subseteq A \cup B$  and  $B \subseteq A \cup B$ , and furthermore that  $A \subseteq C$  and  $B \subseteq C$  if and only if  $A \cup B \subseteq C$ .

Proof

Part 1

WLOG consider the first equation. We need to show  $x \in A$  from  $x \in A \cap B$ . This is trivially true as  $x \in A \cap B$  is true implies that  $x \in A$ .

## Part 2 & 3

Similar reasoning follows.

### Exercise 3.1.11. Show that the axiom of replacement implies the axiom of specification

To construct  $\{x \in A : P(x)\}$ , we construct  $P(x, y)$  such that it is true iff  $P(x)$  is true and  $y = x$ .  $P(x, y)$  is permissible as there's at most one  $y$  for each  $x$  such that  $P(x, y)$  is true (with the one  $y$  being  $y = x$ ).

Hence, via Axiom of specification we construct the set:

$$\{y : P(x, y) \wedge x \in A\}$$

We need to prove that the set is equal to  $\{x \in A : P(x)\}$ .

Note, an element  $y' \in \{y : P(x, y) \wedge x \in A\}$  if and only if  $P(x, y') \wedge (x \in A)$ , which is  $P(x) \wedge (y' = x) \wedge (x \in A)$ .

Similarly, an element  $y' \in \{x \in A : P(x)\}$  iff  $P(y') \wedge (y' \in A)$ .

Note,  $P(x) \wedge (y' = x)$  iff  $P(y')$  by axiom of substitution. Hence,  $P(x) \wedge (y' = x) \wedge (x \in A)$  iff  $P(y') \wedge (y' \in A)$

Rest follows from transitivity and definition for set equivalence.

### Contradiction via introduction of Axiom 3.8

We can construct:

$$\Omega = \{x : x \text{ is a set} \wedge x \notin x\}$$

Now,  $\Omega \in \Omega$  leads to contradiction (due to  $x \notin x$  constraint.). Hence, it must hold that  $\Omega \notin \Omega$ . However, since  $\Omega$  is a set and  $\Omega \notin \Omega$ ,  $\Omega$  satisfies the condition to be in  $\Omega$  (aka  $\Omega \in \Omega$ ), contradiction as desired (as neither  $\Omega \in \Omega$  nor  $\Omega \notin \Omega$ )

Hence Axiom 3.8 cannot be introduced as an axiom.

### Exercise 3.2.1

Show that the universal specification axiom, Axiom 3.8, if assumed to be true, would imply Axioms 3.2, 3.3, 3.4, 3.5, and 3.6. (If we assume that all natural numbers are objects, we also obtain Axiom 3.7.) Thus, this axiom, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as “naive set theory”).

Unfortunately, as we have seen, Axiom 3.8 is “too good to be true”!

Proof

Axiom 3.2

Let  $P(x)$  always return false.

Axiom 3.3

Let  $P(x)$  be  $x = a$  (for singleton set)

Let  $P(x)$  be  $x = a \vee x = b$  (for pair set)

Axiom 3.4

Let  $P(x)$  be  $x \in A \vee x \in B$

Axiom 3.5

Let  $P'(x)$  be  $x \in A \wedge P(x)$ .

Axiom 3.6

Let  $P(y)$  be  $P(x, y)$  ;  $\text{is true for some } x \in A$ .

Axiom 3.7

Under the assumption that all natural number are objects:

Let  $P(x)$  be  $x$  is a natural number.

## Exercise 3.2.2

Part 1:

Assume  $A$  exists, and  $A \in A$ , then by Axiom 3.3,  $\{A\}$  exists. Hence, by Axiom 3.9,  $\{A\}$  must contain an element that is either not a set, or a set that is disjoint from  $\{A\}$ .

By Axiom 3.3, only  $A \in \{A\}$ , hence it must hold that  $A$  must be disjoint from  $\{A\}$ . (aka  $A \cap A' = \emptyset$ )

However,  $A \in A$  and  $A \in \{A\}$ , indicating that  $A \in A \cap \{A\}$  (By definition of intersection). Contradiction as desired.

Part 2:

Equivalent of showing that :

- $A \in B \wedge B \in A$  never happens
- All 3 others can happen
  - Existence by example is sufficient.

- e.g.  $\{\{1,2\}, \{1\}\}$ ,  $\{\{1\}, \{2\}\}$ ,  $(\{1\}, \{1,2\})$

By Axiom 3.3, we construct the pair set  $\{A, B\}$ . Hence, by Axiom 3.9,  $\{A, B\}$  must contain an element that is either not a set or a set that is disjoint from  $\{A, B\}$ .

The element must be either be  $A$  or  $B$  (By Axiom 3.3). The element cannot be  $A$ , as:

$$B \in \{A, B\} \cap A$$

Contradiction.

The element cannot be  $B$ , as:

$$A \in \{A, B\} \cap B$$

Contradiction.

Hence in either case we reach contradiction, as desired.

### Exercise 3.2.3

Show (assuming the other axioms of set theory) that the universal specification axiom, Axiom 3.8, is equivalent to an axiom postulating the existence of a “universal set”  $\Omega$  consisting of all objects (i.e., for all objects  $x$ , we have  $x \in \Omega$ ). In other words, if Axiom 3.8 is true, then a universal set exists, and conversely, if a universal set exists, then Axiom 3.8 is true. (This may explain why Axiom 3.8 is called the axiom of universal specification). Note that if a universal set  $\Omega$  existed, then we would have  $\Omega \in \Omega$  by Axiom 3.1, contradicting Exercise 3.2.2. Thus the axiom of foundation specifically rules out the axiom of universal specification.

Proof

Converse:

If  $\Omega$  exists, then we can apply Axiom 3.5 to  $x \in \Omega$  to obtain axiom of universal specification.

Other direction:

Use Axiom 3.8 with property  $P(x): x \text{ ; is an object}$ .

### Lemma 3.3.12 (Composition is associative)

Part 1: Show that  $f \circ (g \circ h)$ ,  $(f \circ g) \circ h$  have the same domain & range.

Reason: they may have different domain & range, and they are equal in the intersection of domain & range, which is a false positive.

Proof

$g \circ h$  maps from  $X$  to  $Z$  (by definition of composition), hence  $f \circ (g \circ h)$  maps from  $X \rightarrow W$  (by definition of composition)

Similarly  $(f \circ g) \circ h$  also maps from  $X \rightarrow W$ , as desired.

Part 2: Show that  $f \circ (g \circ h) = (f \circ g) \circ h$

Proof

Via definition of composition,  $f \circ (g \circ h) = f(g(h(x)))$ . Similarly,  $(f \circ g) \circ h = f(g(h(x)))$ . Hence, by transitivity of equality  $f \circ (g \circ h) = (f \circ g) \circ h$  as desired.

### Corollary of Definition 3.3.1

Show that Function obeys axiom of substitution.

Proof:

Assumption: Property satisfies Axiom of substitution.

Need to show  $x = x' \rightarrow f(x) = f(x')$

Note,  $P(x, f(x)), P(x', f(x'))$  are true, by definition of function.

Hence, by Axiom of substitution (on property),  $P(x, f(x)), P(x, f(x'))$  are both true. By definition of function, there's exactly one  $y$  such that  $P(x, y)$  is true. Hence, it must hold that  $y = f(x) = f(x')$  (otherwise there will be at least two different  $y$ , being  $f(x)$  and  $f(x')$ ). As desired.

### Exercise 3.3.1

Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric, and transitive.

Reflexivity:

Say we have  $f: X \rightarrow Y$ , then, by definition of equality, we need to show:

$\forall x, f(x) = f(x)$

Note  $f(x)$  has a single output  $y$  (By definition of function). Since  $f$  also satisfies Axiom of substitution,  $f(x) = f(x)$  as desired.

Symmetry:

Need to show if  $f = g$ , then  $g = f$ .

Say if  $f = g$ , then by definition of equality, it must hold that:

- $f, g$  has same domain & range.
- $\forall x, f(x) = g(x)$

The condition for  $g = f$  are (by definition of equality):

- $g, f$  has same domain & range.
- $\forall x, g(x) = f(x)$

Which are the same as:

- $f, g$  has same domain & range.
  - By symmetry of set equality
- $\forall x, f(x) = g(x)$ 
  - By symmetry of object equality.

As desired.

Transitivity:

Need to show if  $f = g$  and  $g = h$ , then  $f = h$ .

Aka, need to show:

- $f, h$  has same domain & range.
- $\forall x, f(x) = h(x)$

Since  $f, g$  and  $g, h$  has same domain and range, it must hold that  $f, h$  has same domain & range, by transitivity of set equality.

Also, since  $\forall x, f(x) = g(x) \wedge g(x) = h(x)$ , it holds that  $f(x) = h(x)$  as desired.

Composition obeys the axiom of substitution.

Proof

Need to show  $x = x' \rightarrow g(f(x)) = g(f(x'))$

Since function satisfies axiom of substitution, we have  $f(x) = f(x')$ . For convenience let  $y = f(x)$ ,  $y' = f(x')$ ,  $y = y'$ . Hence we remain to show that  $g(y) = g(y')$ , which is true by axiom of substitution, as desired.

### Exercise 3.3.2

Part 1:  $f, g$  injective implies  $g(f(x))$  is injective

We directly show  $x \neq x' \rightarrow g(f(x)) \neq g(f(x'))$ .

Since  $f$  is injective,  $f(x) \neq f(x')$  (by definition of injectivity). Similarly it must hold that  $g(f(x)) \neq g(f(x'))$  as desired.

Part 2:  $f, g$  surjective implies  $g(f(x))$  is surjective

Say  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , then  $g(f(x)): X \rightarrow Z$ .

Need to show  $\forall z \in Z, \exists x, g(f(x)) = z$ .

Direct Proof:

Since  $g$  is surjective, by definition of surjectivity,  $\exists y, g(y) = z$ . Hence, we need to show  $\exists x, f(x) = y$ . By definition of surjectivity,  $x$  must exist as desired.

### Exercise 3.3.3

When is the empty function  $f: \emptyset \rightarrow Y$  injective? surjective? bijective?

Injectivity:

Always satisfied. Assume injectivity is not satisfied, then there exists  $x \neq x', f(x) = f(x')$ , requiring that  $x, x' \in \emptyset$ , contradiction as desired.

Surjectivity:

Only when  $Y = \emptyset$ . If we have  $y \in Y$ , then surjectivity is not satisfied as there exists no  $x$  such that  $f(x) = y$  (necessity). In addition if  $Y = \emptyset$ , surjectivity is vacuously true (sufficiency).

Bijjective:

$Y = \emptyset$ , being the necessary and sufficient condition for surjectivity.

### Exercise 3.3.4

Part 1

We have:

- $\forall x \in X, g(f(x)) = g(\hat{f}(x))$
- $g$  is injective

Need to show:

$\forall x \in X, f(x) = \hat{f}(x)$

Proof

By injectivity of  $g$ , it holds that  $\forall x \in X, f(x) = \hat{f}(x)$ , as desired. (as  $g(f(x)) = g(\hat{f}(x)) \rightarrow f(x) = \hat{f}(x)$ ).

If  $g$  is not injective, then it is false. e.g. we can have  $g$  be constant function. Hence, regardless of  $f$ ,  $g(f(x)) = g(\hat{f}(x))$  holds.

Part 2

We have:

- $\forall x \in X, g(f(x)) = \hat{g}(f(x))$
- $f$  is surjective

Need to show:

$$\forall y \in Y, g(y) = \hat{g}(y)$$

Proof

Choose an arbitrary  $y$ . It must hold that  $y = f(x)$  (by surjectivity of  $f$ ). Hence,  $g(y) = g(f(x))$ ,  $\hat{g}(y) = \hat{g}(f(x))$ . Since  $g(f(x)) = \hat{g}(f(x))$ ,  $g(y) = \hat{g}(y)$  by transitivity as desired.

If  $f$  is not surjective, then it is false. As, there must be  $y$  not in range of  $f(x)$ , and  $g(y)$  may not equal to  $\hat{g}(y)$  at such points.

### Exercise 3.3.5

Part 1

Since  $g(f(x))$  is injective, it holds that for  $x, x' \in X$ ,  $x \neq x' \rightarrow g(f(x)) \neq g(f(x'))$ .

Assume  $f$  is not injective, then there exists  $x, x' \in X$  s.t.  $x \neq x'$ ,  $f(x) = f(x')$ . Then, there exists  $x \neq x'$ ,  $g(f(x)) = g(f(x'))$ . Contradiction as desired.

Injectivity of  $g$  is not necessary, by example,  $g(x): \{1,2\} \rightarrow \{1\}$ , being constant function that maps everything to  $1$ .

And  $f: \{1,2\} \rightarrow \{1\}$ , mapping only  $1$  to  $1$ . The composition is injective, yet  $g$  is not injective.

Part 2

Since  $g(f(x))$  is surjective, it holds that  $\forall z \in Z, \exists y, z = g(y)$ .

Assume  $g$  is not surjective, then there exists

### Exercise 3.3.6

Part 1:

Recall the definition of  $f^{-1}(y)$ , it is implicitly defined as:

$$\forall y \in Y, \forall x \in X, f^{-1}(y) = x \iff f(x) = y$$

(Aka  $f(x) = y$  is the desired property)

We proceed with two steps:



- Show that the property is permissible
  - If the property is not permissible, then we can prove everything via absurdum, by treating something that is not a function as a function.
- Show that  $f^{-1}(f(x)) = x$

Permissibility:

We need to show there's exactly one  $x$  s.t.  $f(x) = y$ .

Note, there's at least one  $x$  s.t.  $f(x) = y$  by surjectivity. Assume, there exists more than one  $x$ , say  $x \neq x'$ ,  $f(x) = y$  and  $f(x') = y$ , then  $f$  is not injective, contradiction. Hence there's exactly one  $x$  such that  $f(x) = y$ , as desired.

Remark: We have also shown the converse. If  $f$  is not invertible (aka the property is not permissible), then there's either no  $x$  or more than one  $x$  s.t.  $f(x) = y$ , which implies that  $f$  is not bijective. Hence bijectivity = invertibility.

Show that  $f^{-1}(f(x)) = x$ :

We first show that  $f(x)$  is in the domain of  $f^{-1}$ , aka  $f(x) \in Y$ .

- The motivation is again, to ensure no absurdity happens

Since  $f: X \rightarrow Y$ , it follows that  $f(x) \in Y$  by definition of function, as desired.

Hence, say we have  $f^{-1}(f(x))$ . Then denote  $y = f(x)$ .  $f^{-1}(y)$  therefore evaluates to the only object  $x'$  such that  $y = f(x')$ . Since  $f$  is injective,  $f(x) = f(x') \rightarrow x = x'$ . Hence  $f^{-1}(f(x)) = x' = x$  as desired.

Part 2:

We need to show:

- $f^{-1}(y)$  is in the domain of  $f$  (aka  $f^{-1}(y) \in X$ )
- $f(f^{-1}(y)) = y$

For the first statement, note  $f^{-1}: Y \rightarrow X$  by definition, as desired.

For the second statement, denote  $f^{-1}(y) = x$ , then, by definition of inverse function, it must hold that  $f(x) = y$ . Hence, we can use Axiom of substitution (substitute  $f^{-1}(y) = x$  into  $f(x) = y$ ) to conclude  $f(f^{-1}(y)) = y$ , as desired.

Part 3:

We first show that  $f^{-1}$  is invertible.

In Part 1, it is shown that invertibility = bijectivity, hence it remains to show that  $f^{-1}$  is bijective.

Surjectivity:

Need to show  $\exists y \in Y$  s.t.  $\forall x \in X, f^{-1}(y) = x$ .

Note, for a given  $x$ , there exists  $f(x) = y$ . Which, by definition of inverse function implies  $f^{-1}(y) = x$ , as desired.

Injectivity:

Need to show  $y \neq y' \rightarrow f^{-1}(y) \neq f^{-1}(y')$

Denote  $x = f^{-1}(y)$  and  $x' = f^{-1}(y')$ . Then, assume  $x = x'$ , it must hold that  $f(x) = f(x')$ . Since, by definition of inverse function,  $f(x) = y$ ,  $f(x') = y'$ , we have  $y = y' \wedge y \neq y'$ , contradiction as desired.

Then, note, by Part 1:

$$f(f^{-1}(f^{-1}(y))) = y$$

By Part 2:

$$f(f^{-1}(y)) = y$$

Since  $f^{-1}(y)$  is surjective, by Exercise 3.3.4:

$$f = (f^{-1})^{-1}$$

As desired.

### Exercise 3.3.7

Part 1: Show  $g \circ f$  is bijective

Consequence of Exercise 3.3.2.

Part 2: Show  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

By definition of inverse function:

$$\forall z \in Z, \forall x \in X, (g \circ f)^{-1}(z) = x \Leftrightarrow g(f(x)) = z$$

By definition of inverse function,  $g(f(x)) = z \Leftrightarrow f(x) = g^{-1}(z) \Leftrightarrow x = f^{-1}(g^{-1}(z))$ .

Hence:

$$\forall z \in Z, \forall x \in X, (g \circ f)^{-1}(z) = x \Leftrightarrow x = f^{-1}(g^{-1}(z))$$

By universal instantiation:

$$\forall z \in Z, (g \circ f)^{-1}(z) = x \Leftrightarrow x = f^{-1}(g^{-1}(z))$$

Hence, by transitivity of equality:

$$\forall z \in Z, (g \circ f)^{-1}(z) = f^{-1}(g^{-1}(z))$$

Rest follows from definition of function equality, as desired.

### Exercise 3.3.8

a)

Note both sides is a mapping from  $X$  to  $Z$  (By definition of composition). Need to show for any arbitrary input  $x \in X$ , both sides are equal (by definition of function equality.)

Since  $X \subseteq Y \subseteq Z$ , it holds that  $X \subseteq Z$  by proposition 3.1.18. Hence, by definition of inclusion map,  $RHS = x$ .

For  $LHS$ , note  $\iota_{X \rightarrow Y}(x) = x$  if  $x \in X$ , and  $\iota_{Y \rightarrow Z}(x) = x$  if  $x \in Y$ . Note  $x \in X \rightarrow x \in Y$  by definition of subset.

Hence:

$$LHS = \iota_{Y \rightarrow Z}(\iota_{X \rightarrow Y}(x)) = \iota_{Y \rightarrow Z}(x) = x = RHS$$

As desired.

b)

Need to show  $f = f \circ \iota_{A \rightarrow A}$  and  $f = \iota_{B \rightarrow B} \circ f$ .

Similar to  $a$ , say we are given input  $x$ , we show  $LHS = RHS$ .

Part 1:

Note  $f(\iota_{A \rightarrow A}(x)) = f(x)$ , as  $x \in A$ , and by definition of inclusion map  $\iota_{A \rightarrow A}(x) = x$  if  $x \in A$ , as desired.

Part 2:

Note  $\iota_{B \rightarrow B}(f(x)) = f(x)$ . As,  $f: A \rightarrow B$ , hence  $f(x) \in B$ , rest follows from definition of inclusion map.

c)

Since  $f$  is bijective,  $f^{-1}$  exists. Easy to verify both sides maps from  $B \rightarrow B$ .

We remains to show  $LHS = RHS$  for any arbitrary input  $y \in B$ .

For  $RHS$ , since  $y \in B$ , by definition of inclusion map  $RHS = y$ .

For  $LHS$ , by Exercise 3.3.6,  $LHS = y$  as desired.

d)

We know  $h: X \cup Y \rightarrow Z$ , hence by definition of function, we can define  $h$  via finding a permissible property  $P(t, z)$ .

We let  $P(t, z)$  be:

- If  $t \in X$ ,  $z = f(t)$
- If  $t \in Y$ ,  $z = g(t)$

We proceed to show that the property is permissible. Aka, for every  $t \in X \cup Y$ , there's exactly one  $z$  such that  $P(t, z)$  is true.

Given  $t \in X \cup Y$ , by definition of pairwise union it must hold that  $t \in X$  or  $t \in Y$ . By definition of disjoint set it must hold that  $t \in X$  and  $t \in Y$  cannot be mutually true.

Hence, it must hold that exactly one of  $t \in X$  and  $t \in Y$  is true. WLOG assume  $t \in X$ , then there exists exactly one  $z$ , being  $f(t)$ , for which  $P$  is true, as desired.  $f(t)$  is also well defined as  $f: X \rightarrow Z$  and  $t \in X$ .

After defining  $h$ , we proceed to show that  $h$  does satisfy the desired properties.

- $h \circ \iota_X = f$
- $h \circ \iota_Y = g$

WLOG we prove  $h \circ \iota_X = f$ . First, easy to verify both sides maps  $X \rightarrow Z$ . Then,  $RHS = f(t)$ ,

$LHS = h \circ \iota_X(t) = h(t)$ ;  $\iota_X(t) = t$ ;  $t \in X$  so  $h(t) = f(t)$ .

As desired.

Challenge: Construct Image of sets with Axiom of specification

Choose  $Y$  as the set,  $\exists x, y = f(x)$  as the filtering property.

### Exercise 3.4.1

We call the forward image  $S$  and inverse image  $S'$ . By definition of image:

$S = \{f^{-1}(y) : y \in V\}$   $S' = \{x \in X : f(x) \in V\}$

We proceed with 2 steps:

- Show  $S \subseteq S'$
- Show  $S' \subseteq S$

Then we use Prop 3.1.18 to conclude  $S = S'$

Part 1

Need to show  $\forall x, x \in S \rightarrow x \in S'$  (by definition of subset)

Proof

Assume  $x \in S$ , then it holds that  $x = f^{-1}(y)$  for some  $y \in V$ . Since  $V \subseteq Y$ ,  $y \in V \rightarrow y \in Y$ . Hence  $f^{-1}(y)$  is well defined and it holds that  $x \in X$  (as  $f^{-1}: Y \rightarrow X$ ). It also holds that  $y = f(x)$  (by definition of inverse function). As  $y \in V$ ,  $f(x) \in V$ , as desired.

Part 2:

Need to show  $\forall x, x \in S' \rightarrow x \in S$  (by definition of subset)

Proof:

Assume  $x \in S'$ , then it holds that  $x \in X$  and  $f(x) \in V$ . We need to show that there exists  $y$  s.t.  $x = f^{-1}(y)$  and  $y \in V$ . We claim that  $y = f(x)$  satisfies both conditions.

Since  $y = f(x)$ , then by definition of inverse function  $x = f^{-1}(y)$ , as desired.

Since  $f(x) = y$ ,  $f(x) \in V \rightarrow y \in V$  as desired.

## Exercise 3.4.2

Part 1:

Claim:  $S \subseteq f^{-1}(f(S))$

Note: it cannot hold that  $S = f^{-1}(f(S))$  or  $S \supseteq f^{-1}(f(S))$  in general, consider  $f$  being constant function.

Proof:

Assume  $x \in S$ . Then,  $f(x) \in f(S)$ , by definition of image. Then, since  $f(x) \in f(S)$ ,  $x \in f^{-1}(f(S))$  by definition of inverse image, as desired.

Part 2:

Claim:  $f(f^{-1}(U)) \subseteq U$

Note: Example on page 58 shows it cannot hold that  $f(f^{-1}(U)) = U$  or  $f(f^{-1}(U)) \supseteq U$  in general.

Proof:

Assume  $y \in f(f^{-1}(U))$ . There must exist  $x \in f^{-1}(U)$  such that  $f(x) = y$  (by definition of image). Since  $x \in f^{-1}(U)$ , it must hold that  $f(x) \in U$ . Since  $f(x) = y$ , it holds that  $y = f(x) \in U$ , as desired.

### Exercise 3.4.3

#### Part 1

Need to show  $y \in f(A \cap B) \rightarrow y \in f(A) \text{ and } y \in f(B)$

By definition of image  $y \in f(A \cap B)$  only if  $\exists x \in A \cap B, f(x) = y$ . Hence, it must hold that  $x \in A$  and  $x \in B$ . Therefore by definition of image  $y \in f(A)$  and  $y \in f(B)$  as desired.

The converse is not true. Let  $f$  be constant function, and  $A, B$  be disjoint sets.  $f(A) \cap f(B)$  is not empty, yet  $f(A \cap B)$  is empty.

#### Part 2

Need to show  $y \in f(A) \text{ and } y \notin f(B) \rightarrow y \in f(A \setminus B)$

By definition of image,  $\exists x \in A, f(x) = y$ , and  $\forall x \in B, f(x) \neq y$ . Hence, it must hold that  $\exists x \in A \setminus B, f(x) = y$ .

#### Proof

We know  $\exists x \in A, f(x) = y$ . Assume  $x \in B$ . Then,  $y \in f(B)$  by definition of image, contradiction. Hence if  $x$  exists, it must hold that  $x \in A$  and  $x \notin B$ , as desired.

Therefore, since  $\exists x \in A \setminus B, f(x) = y$ , it follows from definition of image that  $y \in f(A \setminus B)$ , as desired.

The converse is not true, using same construction from Part 1.  $f(A \setminus B)$  is nonempty, and  $y \notin f(B)$  is false, as  $f(A) = f(B) = \{\text{constant}\}$ .

#### Part 3

Need to show:

- $y \in f(A) \text{ or } y \in f(B) \rightarrow y \in f(A \cup B)$
- $y \in f(A \cup B) \rightarrow y \in f(A) \text{ or } y \in f(B)$

Proof for the first statement

WLOG  $y \in A$ . Then,  $\exists x \in A, f(x) = y$ . Since  $A \subseteq A \cup B$  (by Exercise 3.1.7), we have  $\exists x \in A \cup B, f(x) = y$ . By definition of image  $y \in f(A \cup B)$ , as desired.

Proof for the second statement

Since  $y \in f(A \cup B)$ ,  $\exists x \in A \cup B, y = f(x)$ . By definition of pairwise union it must hold that  $x \in A$  or  $x \in B$  (or both).

WLOG assume  $x \in A$ , then  $\exists x \in A, y = f(x)$ , by definition of image,  $y \in f(A)$ , hence  $y \in f(A) \cup f(B)$ , as desired.

### Exercise 3.4.4

Part 1

Need to show  $\forall x, x \in f^{-1}(U \cup V) \iff x \in f^{-1}(U) \cup f^{-1}(V)$  (By definition of set equality).

Note,  $x \in f^{-1}(U \cup V)$  iff  $f(x) \in U \cup V$  (by definition of inverse image), and  $x \in f^{-1}(U) \cup f^{-1}(V)$  iff  $x \in f^{-1}(U) \vee x \in f^{-1}(V)$  (by definition of pairwise union), which is equivalent to  $f(x) \in U \vee f(x) \in V$  (by definition of inverse image). Hence, the predicate is equivalent to:

$\forall x, x \in f^{-1}(U \cup V) \iff f(x) \in U \cup V$

The above is trivially true by definition of pairwise union, as desired.

Part 2, 3

Similar reasoning.

### Exercise 3.4.5

Surjectivity (if direction)

Need to show:

- $y \in f(f^{-1}(S)) \implies y \in S$
- $y \in S \implies y \in f(f^{-1}(S))$

Part 1:

$y \in f(f^{-1}(S))$  implies  $\exists x \in f^{-1}(S), f(x) = y$  (by definition of image). Hence,  $\exists y' \in S, f(x) = y'$  (By definition of inverse image). Note  $y = f(x)$ , hence  $y = y' \in S$ , as desired.

Part 2:

(Sufficiency) Assume  $y \in S$ , then  $y \in Y$  as  $S \subseteq Y$ . Since  $y \in Y$ ,  $\exists x, f(x) = y$  by surjectivity. By definition of reverse image, it holds that the same  $x$  is in  $f^{-1}(S)$ . By definition of image,  $\exists y' \in f(f^{-1}(S)), f(x) = y'$ . Hence,  $y = y' \in f(f^{-1}(S))$ , as desired.

Surjectivity (only if direction)

Proof by contradiction

Assume we have  $f$  non-surjective, which means  $\exists y \in Y, \forall x \in X, f(x) \neq y$ . Denote that  $y$  as  $y'$ .

Since  $y' \in Y$ ,  $\{y'\} \subseteq Y$ , hence it must hold that:

$$\{y'\} = f(f^{-1}(\{y'\}))$$

However,  $f^{-1}(\{y'\}) = \emptyset$ , as  $\forall x \in X, f(x) \neq y'$ . Hence,  $f(f^{-1}(\{y'\})) = f(\emptyset) = \emptyset \neq \text{LHS}$ , contradiction as desired.

Injectivity (if direction)

- $x \in f^{-1}(f(S)) \rightarrow x \in S$
- $x \in S \rightarrow x \in f^{-1}(f(S))$

Part 1:

Similarly to surjectivity part 1

Part 2:

(Sufficiency) Since  $x \in S$ ,  $x \in X$  as  $S \subseteq X$ . Hence,  $\exists y \in f(S), f(x) = y$  (By definition of image). By definition of reverse image,  $\exists x' \in f^{-1}(f(S)), f(x') = y$ . By injectivity,  $f(x') = y \wedge f(x) = y \rightarrow x = x'$ . Hence  $x = x' \in f^{-1}(f(S))$ , as desired.

Injectivity (only of direction)

Assume  $f$  is not injective, then  $\exists x_1, x_2 \in X, (x_1 \neq x_2) \wedge (f(x_1) = f(x_2))$

Consider the set  $S = \{x_1\}$ . Then, it must hold that

$$S = f^{-1}(f(S))$$

However, note  $x_2 \notin S$ , yet  $x_2 \in f^{-1}(f(S))$ . Contradiction as desired.

## Exercise 3.4.6

Intuition: the set hinted by Tao contains all possible partitions for  $X$ , being  $(f^{-1}(0), f^{-1}(1))$  pairs.

Denote the set formed by applying the replacement Axiom on  $\{0, 1\}^X$  as  $P$ , aka:

$$P = \{Y: Y = f^{-1}(\{1\}) ; \text{for some } f \in \{0, 1\}^X\}$$

We claim that  $2^X = P$ , via showing that:

- $Y \in 2^X \rightarrow Y \in P$
- $Y \in P \rightarrow Y \in 2^X$



## Part 1

$Y \in 2^X$  iff  $Y \subseteq X$ , by definition of power set. Consider the following function  $f: X \rightarrow \{0, 1\}$  s.t.  $P(x,y)$  being:

- $x \in Y$  and  $y = 1$
- $x \notin Y$  and  $y = 0$

Above property is permissible as:

- for any input  $x$ , there's a corresponding output
  - As either  $x \in Y$  or  $x \notin Y$ , at least one of the clauses are true
- Fixing  $x$ , There's exactly one  $y$  that that  $P(x,y)$  is true.
  - WLOG if  $x \in Y$ , then  $y = 1$  is the only candidate.

Since  $f: X \rightarrow \{0, 1\}$ ,  $f \in \{0, 1\}^X$ . Take  $Y' = f^{-1}(\{1\})$ , by definition of  $P$ ,  $Y' \in P$ . We remain to show that  $Y = Y'$ , hence, we show:

- $x \in Y \rightarrow x \in Y'$
- $x \in Y' \rightarrow x \in Y$

First statement:

Given  $x \in Y$ , note  $Y \subseteq X$ , hence  $x \in X$ , which means  $x$  is a valid input for  $f$ . Note  $f(x) = 1$  (as  $x \in Y$ ). Hence  $f(x) \in \{1\}$ , therefore by definition of inverse image,  $x \in f^{-1}(\{1\})$ , as desired.

Second statement:

Since  $x \in f^{-1}(\{1\})$ ,  $f(x) = 1$ , by definition of inverse image. Hence, assume  $x \notin Y$ , then  $f(x) = 0 \neq 1$ , contradiction as desired.

## Part 2

For  $Y \in P$ , it holds that there's  $f: X \rightarrow \{0, 1\}$  s.t.  $Y = f^{-1}(\{1\})$ . We need to show  $Y \in 2^X$ , which is equivalent to showing  $Y \subseteq X$ .

Given  $x \in Y$ , by definition of inverse image, it holds that  $x \in X$  and  $f(x) = 1$ . Hence,  $x \in Y \rightarrow x \in X$ , as desired.

## Exercise 3.4.7

By Exercise 3.4.6, the followings are sets:

$$A = \{X: X \subseteq X\} \quad B = \{Y: Y \subseteq Y\}$$

For each  $Y \in B$ ,  $Y^X$  is a set. Hence, for each  $X \subseteq X$ , denote  $A_X = Y^X$ . Then, we construct:

$$B_{\{Y\}} = \bigcup_{X' \in X} A_{\{X'\}}$$

We claim:

$$C = \bigcup_{Y' \in B} B_{\{Y'\}}$$

Is the desired set.

Proof:

Part 1: Show element in  $C$  are partial functions

Say given an element  $f \in C$ , then by definition of union,  $\exists Y' \in B, f \in B_{\{Y'\}}$ . Since  $B_{\{Y'\}} = \bigcup_{X' \in X} A_{\{X'\}}$ ,  $\exists X' \in A, f \in A_{\{X'\}}$ . Since  $A_{\{X'\}} = Y'^{\{X'\}}$ , it holds that  $f: X' \rightarrow Y'$ , with  $X' \in X, Y' \in Y$ , as desired.

Part 2: Show a given partial function is in  $C$

Given  $f: X'' \rightarrow Y''$  s.t.  $X'' \subseteq X, Y'' \subseteq Y$ . Note  $f \in C$  iff  $\exists Y' \in B, f \in B_{\{Y'\}}$ . We claim  $Y' = Y''$  is permissible. Aka, we need to verify:

- $Y'' \in B$
- $f \in B_{\{Y''\}}$

The first statement is true by definition of  $B$ . For the second statement, note  $f \in B_{\{Y''\}}$  iff  $\exists X'' \in A, f \in A_{\{X''\}}$ , by definition of  $B_{\{Y''\}}$ . We let  $X' = X''$ . Then, we need to verify:

- $X'' \in A$
- $f \in A_{\{X''\}}$

The first one is true, by definition of  $A$ . For the second one, note  $A_{\{X''\}} = Y''^{\{X''\}}$ , hence by definition of power set  $f \in A_{\{X''\}}$ , as desired.

### Exercise 3.4.8

Given two sets  $A, B$ , we wish to construct a set  $A \cup B$  s.t.:

$$x \in A \cup B \iff x \in A \vee x \in B$$

Proof

By Axiom 3.1,  $A, B$  are objects. Hence, by Axiom 3.4,  $\{A, B\}$  is a set. By Axiom 3.11, there exists a set  $U$  s.t.

$$\forall x \in U \iff x \in S ; \text{for some } S \in \{A, B\}$$

We claim  $U = A \cup B$ .

Part 1:  $x \in A \cup B \implies x \in U$

Given an element  $x \in A \cup B$ , then it must hold that  $x \in A$  or  $x \in B$  (or both).

WLOG assume  $x \in A$ . then since  $A \in S$ , the property  $\exists S \in \{A, B\}, x \in S$  is satisfied, hence  $x \in S$  as desired.

Part 2:  $x \in U \rightarrow x \in A \cup B$

Given  $x \in U$ , it holds  $\exists S \in \{A, B\}, x \in S$ . WLOG let  $S = A$ , then  $x \in A$ , hence  $x \in A \cup B$  as desired.

### Exercise 3.4.9

For sake of simplicity denote two sets as LHS and RHS.

WLOG we show  $LHS \subseteq RHS$ .

Assume  $x \in LHS$ , then it holds that:

- $x \in A_\alpha$  for all  $\alpha \in I$

Hence, since  $\beta' \in I$ , it must hold that:

- $x \in A_{\beta'}$

Hence. since:

- $x \in A_{\beta'}$
- $x \in A_\alpha$  for all  $\alpha \in I$

$x \in RHS$ , as desired.

### Exercise 3.4.10

Part 1

Need to show:  $(\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in I \cup J} A_\alpha$

Proof

Assume  $x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$ , then, by definition of pairwise union, it must either hold that:

- $x \in \bigcup_{\alpha \in I} A_\alpha$
- $x \in \bigcup_{\alpha \in J} A_\alpha$

WLOG assume  $x \in \bigcup_{\alpha \in I} A_\alpha$ , then, by definition of union,  $\exists \beta \in I, x \in A_\beta$ . Hence,  $\exists \beta \in I \cup J, x \in A_\beta$  (as  $I \subseteq I \cup J$  by Exercise 3.1.7). Therefore  $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$  as desired.

Note: if  $\bigcup_{\alpha \in I} A_\alpha$  is empty, above reasoning still applies to  $x \in \bigcup_{\alpha \in J} A_\alpha$ , if both are empty, then  $LHS \subseteq RHS$  is vacuously true as desired.

For the other direction, assume  $x \in \bigcup_{\alpha \in I \cup J} A_\alpha$ , then it must hold that  $\exists \beta \in I \cup J, x \in A_\beta$ . Hence, it must hold that either:

- $\exists \beta \in I, x \in A_\beta$
- $\exists \beta \in J, x \in A_\beta$

(By definition of Pairwise Union)

WLOG say  $\exists \beta \in I, x \in A_\beta$ , then  $x \in \bigcup_{\alpha \in I} A_\alpha$  by definition of union. Hence,  $x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$  by definition of pairwise union, as desired.

## Part 2

Need to show  $(\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in I \cup J} A_\alpha$

(If  $I, J$  are non-empty)

## Proof

Given  $x \in (\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha)$ , it must hold that:

- $x \in (\bigcap_{\alpha \in I} A_\alpha)$
- $x \in (\bigcap_{\alpha \in J} A_\alpha)$

By definition of intersection.

Hence, it must hold that:

- $\forall a \in I, x \in A_a$
- $\forall a \in J, x \in A_a$

(By (3.4))

Which can be rewritten as:

- $\forall a, a \in I \rightarrow x \in A_a$
- $\forall a, a \in J \rightarrow x \in A_a$

Combining both:  $\forall a, (a \in I \rightarrow x \in A_a) \wedge (a \in J \rightarrow x \in A_a) \iff \forall a, (a \in I \vee a \in J) \rightarrow x \in A_a \iff \forall a, a \in I \cup J \rightarrow x \in A_a$

Therefore  $x \in \bigcap_{\alpha \in I \cup J} A_\alpha$ , as desired.

Other direction:

Given  $x \in \bigcap_{\alpha \in I \cup J} A_\alpha$ , it must hold that:

- $\forall a \in I \cup J, x \in A_\alpha$

Hence, both of:

- $\forall a \in I, x \in A_\alpha$
- $\forall a \in J, x \in A_\alpha$

holds (via previous reasoning)

Therefore

- $x \in (\bigcap_{\alpha \in I} A_\alpha)$
- $x \in (\bigcap_{\alpha \in J} A_\alpha)$

holds

As desired.

Note: intersection on family of sets is undefined if either of  $I, J$  are empty. Hence, important to check if the index set is non-empty when applying intersection lemmas.

For instance, (3.4) is vacuously true for any  $S$ , if the index set is empty.

This shows the importance of ZFC, as by applying ZFC to construct intersection on family set if becomes obvious that non-emptiness of index is necessary.

Side Note: when taking out an element from a set, need to consider what if the set is empty.

## Exercise 3.4.11

Part 1

Show:

$$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$$

Proof

We show:

- $LHS \subseteq RHS$
- $RHS \subseteq LHS$

For  $LHS \subseteq RHS$ , note  $x \in X \setminus \bigcup_{\alpha \in I} A_\alpha$  iff  $x \in X$  and  $x \notin \bigcup_{\alpha \in I} A_\alpha$ . Hence, we have:

- $x \in X$
- $\nexists a \in I, x \in A_\alpha$ 
  - which, by first-order logic, equates to  $\forall a \in I, x \notin A_\alpha$

Hence, it holds that:

- $\forall a \in I, x \in X \wedge x \notin A_\alpha$

Which becomes  $\forall a \in I, x \in X \setminus A_\alpha$  as desired.

For  $RHS \subseteq LHS$ , note given  $x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ , we have:

$$\forall a \in I, x \in X \setminus A_\alpha \iff \forall a \in I, x \in X \wedge x \notin A_\alpha \iff \forall a \in I, x \notin A_\alpha$$

Hence,  $x \notin \bigcup_{\alpha \in I} A_\alpha$ . Remains to show  $x \in X$ . Assume  $x \in X$ , then  $\forall a \in I, x \in X \wedge x \notin A_\alpha$  is either false, or vacuously true (which requires  $I$  being empty). Either case is a contradiction, as desired.

- Note: indeed cannot deduce  $x \in X$  directly from  $\forall a \in I, x \notin A_\alpha$ , but rather can only deduce  $x \in X \vee I = \emptyset$ . Need to be careful doing Predicate Logic when the manipulation is outside of what I had seen in CS121.

Part 2:

Show:  $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

Need to show that:

- $LHS \subseteq RHS$
- $RHS \subseteq LHS$

Part 1

Given  $x \in X \setminus \bigcap_{\alpha \in I} A_\alpha$ , it must hold that:

- $x \in X$
- $x \notin \bigcap_{\alpha \in I} A_\alpha$ 
  - Which equates to  $\exists \alpha \in I, x \notin A_\alpha$

Hence:

$$\exists \alpha \in I, x \in X \wedge x \notin A_\alpha \iff \exists \alpha \in I, x \in X \setminus A_\alpha$$

As desired.

Part 2

For  $x \in \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

We have:

$\exists \alpha \in I, x \in X \setminus A_\alpha \iff \exists \alpha \in I, x \in X \text{ and } x \notin A_\alpha$

Assume  $x \notin X$ , then  $\exists \alpha \in I, x \in X \text{ and } x \notin A_\alpha$  is false, contradiction.  
Hence  $x \in X$ .

By weakening  $\exists \alpha \in I, x \in X \text{ and } x \notin A_\alpha$ , we deduce:

$\exists \alpha \in I, x \notin A_\alpha$

As desired.

## Lemma 3.5.12

We use strong induction.

- Mainly to have  $P(1)$  as base case instead of  $P(0)$

Base case  $P(1)$ :

Need to show there exists a 1-tuple  $(x_1)$  s.t.  $x_1 \in X_1$ , for no-empty set  $X_1$ .

Note by Lemma 3.1.6,  $x_1$  exists, hence,  $(x_1)$  exists as desired.

- Exact construction of  $(x_1)$  from  $X_1$  to be done in future exercises

Inductive Step:

Suppose  $P(n)$  is true, then a  $n$ -tuple  $(x_i)_{1 \leq i \leq n}$  exists such that  $x_i \in X_i$  for all  $1 \leq i \leq n$ .  
Hence, for  $(x_i)_{1 \leq i \leq n+1}$ , we simply take  $(x_i)_{1 \leq i \leq n}$ , and take an element  $e \in X_{n+1}$  to append it ( $e$  exists due to Lemma 3.1.6), as desired.

## Exercise 3.5.1

Part 1: Show that  $(x, y) = \{\{x\}, \{x, y\}\}$  satisfies:

$(x, y) = (x', y') \iff (x = x' \text{ and } y = y')$

Proof:

We show:

$(x, y) = (x', y') \implies (x = x' \text{ and } y = y') \iff (x = x' \text{ and } y = y') \implies (x, y) = (x', y')$

First statement

Assume  $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ . Then, assume  $x \neq x'$ , then  $\{x\} \neq \{x'\}$  (as  $x \in \{x\}$ , yet  $x \notin \{x'\}$ ). Then,  $\{x\} \in \{\{x\}, \{x, y\}\}$ , yet  $\{x\} \notin \{\{x'\}, \{x', y'\}\}$  (Assume  $\{x\} \in \{\{x'\}, \{x', y'\}\}$ , then  $\{x\} = \{x', y'\}$ , however  $y' \notin \{x\}$ , contradiction).

Similarly, assume  $y \neq y'$ , then,  $\{x, y\} \neq \{x', y'\}$ , As,  $y \in \{x, y\}$ , and  $y \notin \{x', y'\}$ , contradiction. (Unless  $y = x'$ , however in this case,  $y = x' = x$ , and similarly we conclude  $y' = x' = x$ , hence  $y = y'$ , contradiction as desired).

## Second Statement

Assume  $x = x', y = y'$ . Then, note:

- $\{x\} \in \{\{x'\}, \{x', y'\}\}$
- $\{x, y\} \in \{\{x'\}, \{x', y'\}\}$
- $\{x'\} \in \{\{x\}, \{x, y\}\}$
- $\{x', y'\} \in \{\{x\}, \{x, y\}\}$

Rest follows from Definition of set equality, as desired.

## Exercise 3.5.1 (Additional Challenge)

Part 1: Show that  $(x, y) = \{x, \{x, y\}\}$  satisfies:

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y')$$

Proof:

We show:

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y') \iff (x = x' \text{ and } y = y') \iff (x, y) = (x', y')$$

First statement:

Assume  $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$ . Then, assume  $x \neq x'$ . Then,  $x \in \{x, \{x, y\}\}$ , yet  $x \notin \{x', \{x', y'\}\}$ .

As,  $x \in \{x', \{x', y'\}\}$  iff  $x = x'$  or  $x = \{x', y'\}$ .  $x = x'$  is false. Hence, only  $x = \{x', y'\}$  is possible. Therefore we have:

$$\{\{x', y'\}, \{x, y\}\} = \{x', \{x', y'\}\}$$

Hence, it holds that  $x' = \{x, y\}$  or  $x' = \{x', y'\}$ . Assume  $x' = \{x', y'\}$ , then  $x' = x$ , contradiction. Hence, must hold that  $x' = \{x, y\}$ .

Therefore, we have:

- $x' = \{x, y\}$
- $x = \{x', y'\}$



Hence  $x' \in x$ ,  $x \in x'$ . Contradiction by Exercise 3.2.2.

Assume  $y \neq y'$ , then,  $\{x, y\} \neq \{x', y'\}$ . (As  $y \in \{x, y\}$ , yet  $y \notin \{x', y'\}$ ), contradiction. Or, we may have  $y \in \{x', y'\}$ , if  $y = x'$ . However, we can then deduce  $y' = x$ , and hence  $y' = x' = x = y$ , contradiction as desired.

## Second Statement

Assume  $x = x'$ ,  $y = y'$ , then:

- $x \in \{x', \{x', y'\}\}$
- $\{x, y\} \in \{x', \{x', y'\}\}$
- $x' \in \{x, \{x, y\}\}$
- $\{x', y'\} \in \{x, \{x, y\}\}$

Rest follows from Definition of set equality, as desired.

## Exercise 3.5.2

### Part 1

Need to show:

$$(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$$

Given LHS, RHS are two surjective functions with domain  $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ , and  $\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = y(i)$ .

Step 1: Verify that two functions have same domain and range.

$x, y$  have same domain, by definition of  $n$ -tuple. Assume  $x$  has range  $X$ , and  $y$  has range  $X'$ . We proceed to show  $X = X'$ .

Assume there exists  $e \in X$ . Then, by surjectivity of  $x$ ,  $\exists i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = e$ . Since  $\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = y(i)$ , it holds that:

$$\exists i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, y(i) = e$$

Hence  $e \in X \rightarrow e \in X'$ . Similarly we show  $e \in X' \rightarrow e \in X$ , hence  $X = X'$ , as desired.

With domain & range shown to be identical, by definition of function equality  $x, y$  are equal iff:

$$\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = y(i)$$

Which is a priori, as desired.

### Part 2

Need to show:

$$\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = y(i)$$

Given  $x = y$ ,  $x, y$  surjective and  $x, y$  has domain  $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ .

Proof: by definition of function equality, it must hold that:

$$\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i) = y(i)$$

As desired.

Part 3

Need to show:

$$\prod_{1 \leq i \leq n} X_i = \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

is a set

For which  $(X_i)_{1 \leq i \leq n}$  is an ordered  $n$ -tuple of sets.

Proof

Lemma:  $\prod_{1 \leq i \leq n} X_i$  contains only functions with domain  $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ , and range being subset of  $\bigcup_{1 \leq i \leq n} X_i$ .

Proof: assume for some  $x \in \prod_{1 \leq i \leq n} X_i$ , range of  $x$  (denote it  $R$ ) is not subset of  $\bigcup_{1 \leq i \leq n} X_i$ . Then,  $\exists y \in R, y \notin \bigcup_{1 \leq i \leq n} X_i$ . By surjectivity of  $x$ , it holds  $\exists i' \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, x(i') = y$ . However,  $y = x(i') \in X_{i'}$  by definition of  $\prod_{1 \leq i \leq n} X_i$ , contradiction.

Hence, we construct the following set:

$$\{x : x \text{ is a partial function from } i' \in \{i \in \mathbb{N} : 1 \leq i \leq n\} \text{ to } \bigcup_{1 \leq i \leq n} X_i\}$$

The above set exists by Exercise 3.4.7. Then, we apply axiom of specification, with the property  $P(x)$  being:

- $x(i) \in X_i \text{ for all } 1 \leq i \leq n \text{ and } x \text{ is surjective} \text{ and } x \text{ has domain } \{i \in \mathbb{N} : 1 \leq i \leq n\}$ .

As desired.

### Exercise 3.5.3

Note ordered pair is a 2-tuple.

Definition: Two ordered  $n$ -tuples  $(x_i)_{1 \leq i \leq n}$  and  $(y_i)_{1 \leq i \leq n}$  are said to be equal iff  $x_i = y_i$  for all  $1 \leq i \leq n$ .

Reflexivity ( $A = A$ )

Given  $n$ -tuple  $A = (x_i)_{1 \leq i \leq n}$ , it holds that  $x_i = x_i$  for all  $1 \leq i \leq n$  (via Reflexivity of  $x_i$ ), as desired.

Symmetry ( $A = B \Leftrightarrow B = A$ )

Say  $A = (x_i)_{1 \leq i \leq n}$ ,  $B = (y_i)_{1 \leq i \leq n}$ ,  $A = B$  implies  $x_i = y_i$  for all  $1 \leq i \leq n$ , hence  $y_i = x_i$  for all  $1 \leq i \leq n$ , implying  $B = A$  as desired.

Similarly, we deduce  $B = A \Rightarrow A = B$ , as desired.

Transitivity ( $A = B \wedge (B = C) \Rightarrow A = C$ )

Say  $A = (x_i)_{1 \leq i \leq n}$ ,  $B = (y_i)_{1 \leq i \leq n}$ ,  $C = (z_i)_{1 \leq i \leq n}$ . Then,  $A = B$  implies  $x_i = y_i$  for all  $1 \leq i \leq n$ ,  $B = C$  implies  $y_i = z_i$  for all  $1 \leq i \leq n$ , hence  $x_i = z_i$  for all  $1 \leq i \leq n$  as desired.

### Exercise 3.5.4

Part 1:  $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Assume  $x \in A \times (B \cup C)$ , then  $x_1 \in A$ ,  $x_2 \in B \cup C$ .

WLOG  $x_2 \in B$ , then  $(x_1, x_2) \in (A \times B)$ , hence  $(x_1, x_2) \in (A \times B) \cup (A \times C)$ , as desired.

Assume  $x \in (A \times B) \cup (A \times C)$ , then either:

- $x_1 \in A \wedge x_2 \in B$
- $x_1 \in A \wedge x_2 \in C$

WLOG assume  $x_1 \in A \wedge x_2 \in B$ , then  $x_1 \in A$ ,  $x_2 \in B$  implying  $x_2 \in B \cup C$  as desired.

Part 2:  $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Assume  $x \in A \times (B \cap C)$ , then  $x_1 \in A$ ,  $x_2 \in B \cap C$ .

Hence,  $x_2 \in B \cap C$  implies  $x_2 \in B$ , therefore  $x_1 \in A$ ,  $x_2 \in B$  implies  $x \in (A \times B)$ .

Similarly we show  $x \in (A \times C)$ , hence  $x \in (A \times B) \cap (A \times C)$ , as desired.

Assume  $x \in (A \times B) \cap (A \times C)$ , then:

- $x_1 \in A$
- $x_2 \in B$

- $x_2 \in C$

Hence,  $x_2 \in B \cap C$ . Therefore  $x \in A \times (B \cap C)$  as desired.

Part 3:  $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Assume  $x \in A \times (B \setminus C)$ , then  $x_1 \in A$ ,  $x_2 \in B$ ,  $x_2 \notin C$ .

From  $x_1 \in A$ ,  $x_2 \in B$ ,  $x \in (A \times B)$ .

From  $x_2 \notin C$ ,  $x \notin (A \times C)$ .

Hence,  $x \in (A \times B) \setminus (A \times C)$  as desired.

Assume  $x \in (A \times B) \setminus (A \times C)$ , then:

- $x_1 \in A$  and  $x_2 \in B$
- $\neg (x_1 \in A \text{ and } x_2 \in C)$ 
  - Which, by de Morgan  $(x_1 \notin A \text{ or } x_2 \notin C)$

Hence, we have  $x_1 \in A$ ,  $x_2 \in B$ ,  $x_2 \notin C$  as desired.

### Exercise 3.5.5

Need to show:  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Assume  $x \in (A \times B) \cap (C \times D)$ , then  $x \in (A \times B)$  and  $x \in (C \times D)$ . Hence, it holds that:

- $x_1 \in A$
- $x_1 \in C$
- $x_2 \in B$
- $x_2 \in D$

Therefore  $x_1 \in A$  and  $x_1 \in C$ , hence  $x_1 \in A \cap C$ . Similarly  $x_2 \in B \cap D$ , as desired.

Assume  $x \in (A \cap C) \times (B \cap D)$ , then:

- $x_1 \in A$  and  $x_1 \in C$
- $x_2 \in B$  and  $x_2 \in D$

Hence:

- $x_1 \in A$  and  $x_2 \in B$
- $x_1 \in C$  and  $x_2 \in D$

Therefore  $x \in (A \times B)$  and  $x \in (C \times D)$ , hence  $x \in (A \times B) \cap (C \times D)$ , as desired.

Is it true that  $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ ?

False, consider  $A, B, C, D = \{a\}, \{b\}, \{c\}, \{d\}$ .

Is it true that  $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$ ?

False, consider  $A, B, C, D = \{a\}, \{b\}, \{a\}, \{d\}$ .

### Exercise 3.5.6

Recall definition:

$$\prod_{1 \leq i \leq n} X_i = \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

If direction:

For  $(x_i)_{1 \leq i \leq 2} \in A \times B$ , it must hold that  $x_1 \in A$  and  $x_2 \in B$  (by definition of Cartesian product).

Hence, since  $A \subseteq C$ ,  $B \subseteq D$ , it holds that  $x_1 \in C$  and  $x_2 \in D$ , hence  $(x_i)_{1 \leq i \leq 2} \in C \times D$  as desired.

Only if direction:

Assume  $A \not\subseteq C$ , then  $\exists x_1 \in A, x_1 \notin C$ . Since  $B$  is nonempty, we can pick an arbitrary  $x_2 \in B$ , s.t.  $(x_1, x_2) \in A \times B$ . Since  $A \times B \subseteq C \times D$ ,  $(x_1, x_2) \in C \times D$ , hence  $x_1 \subseteq C$  (by definition of Cartesian product), contradiction.

Similarly we show  $B \subseteq D$  is necessary.

Removal of non-empty constraint:

Only if direction no longer holds. If direction still holds.

### Exercise 3.5.7

Part 1: Showing  $h$  exists

Define  $h: Z \rightarrow X \times Y$  as:

$$h(z) = (f(z), g(z))$$

Aka, the property  $P(z, (x, y))$  for  $h$  being  $(x, y) = (f(z), g(z))$

We proceed to show that  $h$  is a function. Note  $f(z) \in X$ ,  $g(z) \in Y$ , hence  $(f(z), g(z)) \in X \times Y$ .

Then, assume there exists  $x', y'$  s.t.  $P(z, (x, y)), P(z, (x', y'))$  are true, then it must hold that  $(x, y) = (x', y) = (f(z), g(z))$ , as desired. Hence  $h$  is a function.

Then, note:

$$\begin{aligned} \pi_{X \times Y \rightarrow X} \circ h(z) &= \pi_{X \times Y \rightarrow X}((f(z), g(z))) && \text{by definition of } h \\ &= f(z) && \text{by definition of } \pi \end{aligned}$$

As desired.

Similarly we show  $\pi_{X \times Y \rightarrow Y} \circ h(z) = g(z)$

Part 2: Showing  $h$  is unique

Assume there exists  $h' \neq h$  that also satisfies the property. Therefore,  $\exists z \in Z, h(z) \neq h'(z)$ . Call that  $z = z'$ . Denote  $h(z') = (f(z), g(z))$ ,  $h'(z') = (a, b)$ .

WLOG assume  $f(z') \neq a$ , then:

$$\pi_{X \times Y \rightarrow X} \circ h'(z') = \pi_{X \times Y \rightarrow X}((a, b)) = a \neq f(z') = \pi_{X \times Y \rightarrow X} \circ h(z')$$

Contradiction, as desired. Similarly  $g(z') \neq b$  also yields contradiction, hence it must be the case that  $h(z') = h'(z')$ , implying  $h = h'$ .

## Exercise 3.5.8

Recall definition:

$$\prod_{1 \leq i \leq n} X_i = \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}.$$

If direction:

Note  $\exists i \in 1 \leq i \leq n, X_i = \emptyset$ . Denote the  $i$  as  $i'$ . Then, assume  $\prod_{1 \leq i \leq n} X_i$  is non-empty, we have  $x \in \prod_{1 \leq i \leq n} X_i$  (by single choice), such that  $x_i \in X_i$  for all  $1 \leq i \leq n$ .

However,  $x_i \in X_i$  for all  $1 \leq i \leq n$  is always false. (As otherwise we have  $x_{i'} \in X_{i'}$ , yet  $X_{i'}$  is empty, contradiction). Contradiction as desired.

Only if:

Lemma 3.5.12

## Exercise 3.5.9

Show that:

$$\left( \bigcup_{\alpha \in I} A_\alpha \right) \cap \left( \bigcup_{\beta \in J} B_\beta \right) = \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$$

Proof

Assume  $x \in (\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta)$ , then it holds that:

- $x \in \bigcup_{\alpha \in I} A_\alpha$
- $x \in \bigcup_{\beta \in J} B_\beta$

Hence, it must hold that:

- $\exists a \in I, x \in A_\alpha$
- $\exists b \in J, x \in B_\beta$

Hence, by definition of ordered pair:

$\exists (a, b) \in I \times J, x \in A_\alpha \text{ \& } x \in B_\beta$   $\exists (a, b) \in I \times J, x \in A_\alpha \cap B_\beta$   $x \in \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$

As desired.

Assume  $x \in \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta)$ , then:

$\exists (a, b) \in I \times J, x \in A_\alpha \cap B_\beta$   $\exists (a, b) \in I \times J, x \in A_\alpha \text{ \& } x \in B_\beta$

By definition of ordered pair:

- $\exists a \in I, x \in A_\alpha \text{ \& } x \in B_\beta$
- $\exists b \in J, x \in A_\alpha \text{ \& } x \in B_\beta$

Hence:

- $\exists a \in I, x \in A_\alpha$
- $\exists b \in J, x \in B_\beta$

Hence:

- $x \in \bigcup_{\alpha \in I} A_\alpha$
- $x \in \bigcup_{\beta \in J} B_\beta$

Therefore:

$x \in (\bigcup_{\alpha \in I} A_\alpha) \cap (\bigcup_{\beta \in J} B_\beta)$

As desired.

## Exercise 3.5.10

Part 1: show that two functions  $f, g: X \rightarrow Y$  are equal if and only if they have the same graph.

If direction:

Assume  $f = g$ , then the graph of  $f$  is:

$$\{(x, f(x)) : x \in X\}$$

The graph of  $g$  is:

$$\{(x, g(x)) : x \in X\}$$

WLOG given  $(x', y') \in \{(x, f(x)) : x \in X\}$ , it holds that  $(x', y') \in \{(x, g(x)) : x \in X\}$ . As, by Axiom of Replacement,  $(x, y) \in \{(x, f(x)) : x \in X\}$  iff  $\exists x'' \in X, (x', y') = (x'', g(x''))$ . We simply let  $x'' = x'$ , as desired (we can easily verify that  $x'' = x'$  works, steps omitted).

Only if direction:

Assume:

$$\{(x, f(x)) : x \in X\} = \{(x, g(x)) : x \in X\}$$

Then, assume  $f \neq g$ , then there exists  $x' \in X, f(x') \neq g(x')$ . Then,  $(x', f(x')) \in \{(x, f(x)) : x \in X\}$ , yet  $(x', f(x')) \notin \{(x, g(x)) : x \in X\}$ , contradiction.

As, if  $(x', f(x')) \in \{(x, g(x)) : x \in X\}$ , it must hold that, by Axiom of replacement,  $\exists x'' \in X, (x', f(x')) = (x'', g(x''))$ . Hence,  $x' = x''$  and  $f(x') = g(x'') = g(x')$ , contradiction.

Part 2

Show  $f$  exists

Denote the property  $P(x, y)$  as  $(x, y) \in G$ , then we claim that:

- $f: X \rightarrow Y$  with property  $P(x, y)$  is a function
- $f$  has graph  $G$

Note  $(x, y) \in G$  is permissible if for all  $x \in X$ , there exists exactly one  $y$  such that  $(x, y) \in G$  is true. As, there is at least one  $y$  by taking an element from  $\{y \in Y : (x, y) \in G\}$  (as the set is non-empty). Assume there more than one  $y$  (call them  $y, y'$ ) s.t. the property is true, then  $\{y \in Y : (x, y) \in G\}$  has two elements  $y, y'$ , contradiction.

Say  $f$  has graph  $G' = \{(x, f(x)) : x \in X\}$ , then we show  $G = G'$ .

Assume  $(x', y') \in G$ , then it holds that  $y' = f(x')$  (As  $P(x', y')$  is true). Let  $x = x'$ , then by Axiom of Replacement  $(x', f(x')) \in G'$ . Hence, since  $(x', y') = (x', f(x'))$ ,  $(x', y') \in G'$  as desired.

Assume  $(x', y') \in G'$ , then it holds that  $y' = f(x')$ , by definition of graph. Consequently  $P(x', y')$  is true, hence  $(x', y') \in G$ , as desired.

Show  $f$  is unique



Assume there exists  $f' \neq f$  s.t.  $f'$  also has graph  $G$ .

Then  $\exists x' \in X, f(x') \neq f'(x')$ . Then, by definition of graph, it must hold that  $(x', f(x')) \in G$  and  $(x', f'(x')) \in G$ . Hence,  $f(x') \in \{y \in Y: (x, y) \in G\}$ , and  $f'(x') \in \{y \in Y: (x, y) \in G\}$ , yet  $f'(x') \neq f(x')$ , implying that  $\{y \in Y: (x, y) \in G\}$  has two elements, contradiction.

### Exercise 3.5.11

Consider  $X \times Y$ , then, by Lemma 3.4.9, there exists a set  $P$  that contains all subsets of  $X \times Y$ . Then, we apply Axiom of Specification to  $P$ :

$$P' = \{G \in X \times Y: \forall x \in X, \{y \in Y, (x, y) \in G\} \text{ has exactly one element}\}$$

Via Exercise 3.5.10, there exists exactly one function  $f: X \rightarrow Y$  for each  $G \in P'$  such that  $G$  is graph of  $f$ . Hence, we apply Axiom of Replacement to  $P'$ , to form:

$$F = \{f: X \rightarrow Y \text{ has graph } G: G \in P'\}$$

We claim that  $F$  is the desired set, aka the set of all functions  $f: X \rightarrow Y$ .

Proof:

Denote the set of all functions  $f: X \rightarrow Y$  as  $F'$ .

Take  $f \in F$ ,  $f: X \rightarrow Y$ , hence  $f \in F'$  as desired.

Take  $f \in F'$ , then  $f$  has a graph  $G$ . Hence we need to show  $G \in P'$ , aka, we need to show:

- $G \in X \times Y$
- $\forall x \in X, \{y \in Y, (x, y) \in G\} \text{ has exactly one element}$

Take  $(x, y) \in G$ , then, by definition of graph,  $x \in X$  and  $y = f(x) \in Y$ , hence  $(x, y) \in X \times Y$  as desired.

Assume  $\forall x \in X, \{y \in Y, (x, y) \in G\} \text{ has exactly one element}$  is false, then for some  $x' \in X$ ,  $\{y \in Y, (x', y) \in G\}$  is either empty or has more than one elements. Note  $f(x') \in \{y \in Y, (x', y) \in G\}$ , hence  $\{y \in Y, (x', y) \in G\}$  is non-empty. Similarly if  $y, y' \in \{y \in Y, (x', y) \in G\}$ ,  $y \neq y'$ , then by definition of graph  $y = f(x')$ ,  $y' = f(x')$ , yet  $y \neq y'$ , contradiction.

Hence  $f \in F \rightarrow f \in F'$  and  $f \in F' \rightarrow f \in F$ , indicating that  $F = F'$  by definition of set equality, as desired.

### Exercise 3.5.12

Let  $P(N)$  denote: a function  $\phi_N: \{n \in \mathbb{N}, n \leq N\} \rightarrow \mathbb{N}$  s.t.  $\phi_N(0) = c$  and  $\forall n < N, \phi_N(n+1) = f(n, \phi_N(n))$  exists.

Remark:  $P(N)$  only states a function with certain properties exist.  $P(N)$  does not state there's only one function with such properties exists (unless uniqueness is proven, there may be two, three or more functions with such properties).

Remark 2:  $\phi_N$  is merely a mock variable, as with  $\xi_N$ .

For base case  $P(0)$ , we construct  $\phi_0: \{0\} \rightarrow \mathbb{N}$  s.t.  $\phi_0(0) = c$ , as desired.

For  $P(N)$ , by  $P(N)$  there exists function  $\phi_N: \{n \in \mathbb{N}, n \leq N\} \rightarrow \mathbb{N}$  s.t.  $\phi_N(0) = c$  and  $\forall n < N, \phi_N(n) = f(n, \phi_N(n))$ .

We claim the following construction defines a permissible  $\phi_{N+1}$ :

$$\phi_{N+1}(n) = \begin{cases} \phi_N(n) & \text{if } n < N+1 \\ f(n, \phi_N(n)) & \text{if } n = N+1 \end{cases}$$

We need to verify that the construction defines a function from  $\{n \in \mathbb{N}, n \leq N+1\} \rightarrow \mathbb{N}$ . Pick  $n \in \{n \in \mathbb{N}, n \leq N+1\}$ , by Definition 2.2.11, either  $n < N+1$  or  $n = N+1$ . We proceed by cases. If  $n < N+1$ , then by construction,  $\phi_{N+1}(n) = \phi_N(n)$ . Hence  $\phi_{N+1}(n)$  is assigned once, and  $a_{N+1}(n) = a_N(n) \in \mathbb{N}$ . Case  $n = N+1$  follows similarly. Hence, by Definition 3.3.1,  $\phi_{N+1}$ , as by construction, is a function from  $\{n \in \mathbb{N}, n \leq N+1\} \rightarrow \mathbb{N}$ .

To prove  $P(N+1)$ , we remain to show that:

$$\phi_{N+1}(0) = c \quad \forall n < N+1, \phi_{N+1}(n) = f(n, \phi_{N+1}(n))$$

To show  $\phi_{N+1}(0) = c$ , we first argue by contradiction that  $0 < N+1$ . Assume  $0 < N+1$  is false, then by Proposition 2.2.13, we have  $0 \geq N+1$ , which, by Definition 2.2.11, implies  $0 = d + (N+1)$  for some  $d \in \mathbb{N}$ . Since  $0 = d + (N+1)$ , by Corollary 2.2.9,  $0 = N+1$ . However, by Axiom 3.3,  $\forall n \in \mathbb{N}, 0 \neq n$ , contradiction.

For  $\forall n < N+1, \phi_{N+1}(n) = f(n, \phi_N(n))$ . By Proposition 2.2.12,  $n < N+1$  iff  $n \leq N$ . Hence either  $n = N$  or  $n < N$ . We proceed by cases. If  $n = N$ , then  $a_{N+1}(n) = f(n, a_N(n))$  by construction. If  $n < N$ , then we claim  $n < N+1$ , as, by Definition 2.2.12 f), it holds that  $n + d = N$  for some positive  $d$ , therefore  $(n + d) \leq N$ , therefore  $(n + d) + d = N + d = N+1$  by Definition 2.2.1, therefore  $n < N+1$  by Definition 2.2.12 f). Hence, by construction  $\phi_{N+1}(n) = \phi_N(n) = f(n, \phi_N(n)) = f(n, \phi_{N+1}(n))$ , as desired.

Then, we argue for all  $N \in \mathbb{N}$ , only one function is specified by  $P(N)$ . We argue by induction. For each  $N \in \mathbb{N}$ , by  $P(N)$  we take function  $\phi_N: \{n \in \mathbb{N}, n \leq N\} \rightarrow \mathbb{N}$  s.t.  $\phi_N(0) = c$  and  $\forall n < N, \phi_N(n) = f(n, \phi_N(n))$ , and  $\xi_N: \{n \in \mathbb{N}, n \leq$

$N \rightarrow N$  s.t.  $\xi_N(0) = c$  and  $\forall n < N, \xi_N(n+1) = f(n, \xi_N(n))$ . We then denote  $P'(n)$  as  $n \leq N \rightarrow \phi_N(n) = \xi_N(n)$ .

For base case  $P'(0)$ , by  $P(N)$  we have  $\phi_N(0) = \xi_N(0) = c$ , as desired.

For  $P'(n+1)$ , by  $P(N)$  we have  $\phi_N(n+1) = f(n, \phi_N(n))$ ,  $\xi_N(n+1) = f(n, \xi_N(n))$ . Since, by  $P'(n)$ , we have  $\phi_N(n) = \xi_N(n)$ , we then have  $f(n, \phi_N(n)) = f(n, \xi_N(n))$ , therefore  $\phi_N(n+1) = f(n, \xi_N(n)) = \xi_N(n+1)$ , as desired.

Since only one function is specified by each  $P(N)$ , for each  $N \in \mathbb{N}$ , we can define  $a_N$  as the function specified by  $P(N)$ . Then  $a(0) = a_0(0) = c$ . To verify  $a(n+1) = f(n, a(n))$ , we first verify that  $a_{N+1}(n) = a_N(n)$ . Recall, during construction of some function specified by  $P(n+1)$  (and referred to it as  $\phi_{N+1}$ ), we used some function specified by  $P(n)$  (and referred to it as  $\phi_N$ ), and set  $\phi_{N+1}(n) = \phi_N(n)$  if  $n < N+1$ . Since two functions specified by  $P(N)$ ,  $P(N+1)$  are unique,  $\phi_{N+1}(n) = \phi_N(n)$  implies  $a_{N+1}(n) = a_N(n)$ , hence:

$$\begin{aligned} a(n+1) &= a_{N+1}(n+1) = f(n, a_{N+1}(n)) = f(n, a_N(n)) = f(n, a(n)) \end{aligned}$$

As desired.

We remains to show  $a$  is unique, aka  $a$  is the only function that satisfies the conditions  $a(0) = c$ ,  $a(n+1) = f(n, a(n))$ .

Assume there exists  $b$  that satisfies the conditions. We proceed to show that  $\forall n \in \mathbb{N}, a(n) = b(n)$ , aka  $a = b$ .

We let  $P''(n)$  be  $a(n) = b(n)$ . Then,  $a(0) = b(0) = c$  by definition, hence the base case is satisfied.

For  $P''(n+1)$ , note  $a(n+1) = f(n, a(n))$ ,  $b(n+1) = f(n, b(n))$ . Hence since  $a(n) = b(n)$  by  $P''(n)$ , it holds that  $f(n, a(n)) = f(n, b(n))$ , hence  $a(n+1) = b(n+1)$  as desired.

Therefore  $a$  is unique.

Remark 3: Additional Challenge is not merely a challenge. In fact, the above proof is unsound. Above proof used order, which is defined using addition, which is then defined using recursive definition. Hence the above proof is circular.

Additional Challenge : TODO

### Exercise 3.5.13

We first show that  $f$  exists, using Exercise 3.5.12, then we show that  $f$  has the desired property:

- $f$  needs to be bijective
- $f(n) = n' \iff f(n'') = n''$

Existence: we define  $f: \mathbb{N} \rightarrow \mathbb{N}'$  as:

- $f(0) = 0'$
- $f(n'') = f(n)''$

Above uniquely defines a function by Exercise 3.5.12, as desired.

For  $f(n) = n' \iff f(n'') = n''$ , note  $f(n) = n'$  implies  $f(n'') = f(n)' = f(n)'' = n''$ , as desired. And,  $f(n'') = n''$  implies  $f(n)'' = n''$ , which, by Axiom 2.4, implies that  $f(n) = n'$ , as desired. Using this, we can also inductively show that  $f(n) = n'$ .

We proceed to prove that  $f$  is bijective. For surjectivity, let the inductive hypothesis  $P(n)$  be  $\exists x \in \mathbb{N}, f(x) = n'$ . Then, for  $P(0')$ , we let  $x = 0$  as desired. For  $P(n'')$ , note by inductive hypothesis we have  $f(c) = n'$ . Hence  $f(c'') = n''$ , as desired.

For injectivity, assume  $f(n) = f(m)$ . Then, we have  $f(n) = n'$ ,  $f(m) = m'$ , hence  $f(n) = f(m) \implies n' = m'$ , as desired.

## Why? on page 70

### There's no bijection between an empty set and a non-empty set

Say  $f: \emptyset \rightarrow Y$  to some non-empty set  $Y$ . Then, by single choice we take  $y \in Y$ . However,  $\nexists x \in \emptyset, f(x) = y$  (otherwise  $x \in \emptyset$ , contradiction). Hence,  $f$  is not surjective.

### Show $g$ is a bijection

A similar example is shown in Why? on Page 160.

## Exercise 3.6.1

Reflexivity:

Consider the mapping  $f: X \rightarrow X$  s.t.  $f(x) = x$ . Easy to confirm that it is a bijection.

Symmetry:

If  $X$  has same cardinality with  $Y$ , then there exists a bijective function  $f: X \rightarrow Y$ . Hence, by Exercise 3.3.6, there exists bijective  $f^{-1}: Y \rightarrow X$ , hence  $Y$  has same cardinality with  $X$  as desired.

Transitivity:

If  $X$  has same cardinality with  $Y$ , there exists bijective  $f: X \rightarrow Y$ . If  $Y$  has same cardinality with  $Z$ , then there exists bijective  $g: Y \rightarrow Z$ . By Exercise 3.3.7,  $f \circ g: X \rightarrow Z$  exists & is bijective, as desired.

### Exercise 3.6.2

Note  $\{i \in \mathbb{N}; 1 \leq i \leq 0\}$  is empty, as  $i \leq 0$  implies  $i + 1 = 0$ , which then by Corollary 2.2.9 implies that  $i = 0$ , yet  $1 + a = i$  for some  $a$ , leading to  $1 = 0$ , yet  $1 \neq 0$ , hence  $0 = 0$ , contradicting Axiom 2.3.

Only If direction:

Assume  $X$  is non-empty, yet  $X$  has cardinality  $0$ , then we have bijection  $f: \emptyset \rightarrow X$ . However,  $f$  is not surjective, as given  $x \in X$ , there exists no  $e \in \emptyset$  s.t.  $f(e) = x$ . Hence  $f$  is not bijective, contradiction.

If direction:

If  $X$  is empty, then, obvious bijection  $f: \emptyset \rightarrow \emptyset$  exists, as desired.

### Exercise 3.6.3

We induct on  $n$ . Vacuously true for  $n = 0$ . For  $P(n)$ , consider  $f: \{i \in \mathbb{N}, 1 \leq i \leq n\} \rightarrow \mathbb{N}$ , consider  $g: \{i \in \mathbb{N}, 1 \leq i \leq n\} \rightarrow \mathbb{N}$  s.t.  $\forall i, 1 \leq i \leq n, g(i) = f(i)$ , then by inductive hypothesis  $\exists M' \in \mathbb{N}, \forall i, 1 \leq i \leq n, g(i) = f(i) \leq M'$ .

Therefore, we let  $M = \max(M', f(n))$ , then:

$$\forall i, 1 \leq i \leq n, f(i) \leq M' \leq \max(M', f(n))$$

$$f(n) \leq \max(M', f(n))$$

Combined:

$$\forall i, 1 \leq i \leq n, f(i) \leq \max(M', f(n))$$

As desired.

### Exercise 3.6.4

(a) Let  $\#(X) = n$ . Then, there exists a bijective function  $f: X \rightarrow \{i \in \mathbb{N}, 1 \leq i \leq n\}$

We define  $g: X \cup \{x\} \rightarrow \{i \in \mathbb{N}, 1 \leq i \leq n + 1\}$  as:

- If  $e \in X$ ,  $g(e) = f(e)$
- Else  $g(e) = n + 1$

$g$  is surjective, as, for  $y \in \{i \in \mathbb{N}, 1 \leq i \leq n + 1\}$ , it must either hold that  $y \leq n$  or  $y = n + 1$ , by rules of order. If  $y \leq n$ , then  $\exists e \in X, f(e) = y$ , by surjectivity of  $f$ . Hence,  $\exists e \in X, f(e) = g(e) = y$ , therefore  $\exists e \in X \cup \{x\}, g(e) = y$ , as desired.

If  $y = n + 1$ , then  $g(x) = n + 1$  (as  $x \notin X$ ), as desired.

$g$  is injective. as, say  $g(a) = g(b)$ , then either  $a, b \in X$  or some of  $a, b \notin X$ .

If  $a, b \in X$ , then  $g(a) = g(b)$  implies  $f(a) = f(b)$ , hence  $a = b$  by injectivity of  $f$ .

Else, WLOG assume  $a \in X$  and  $b \notin X$ , then  $g(b) = n + 1, g(a) = f(a)$ . However  $f(a) \in \{i \in \mathbb{N}, 1 \leq i \leq n\}$ ,  $g(b) \notin \{i \in \mathbb{N}, 1 \leq i \leq n\}$ . Hence,  $f(a) \neq g(b)$ , contradiction.

Therefore, it must hold that both  $a, b \notin X$ . However,  $a, b \in X \cup \{x\}$ , hence  $a = b = x$ , as desired.

(b)

Note  $Y - X \subseteq Y$ , hence by (c):

$$\begin{aligned} \#(Y - X) &\leq \#(Y) - \#(X) + \#(Y - X) \leq \#(X) + \#(Y) - \#(X) \end{aligned}$$

We remain to show that  $\#(X \cup Y) = \#(X) + \#(Y - X)$ . Note there exists bijection  $f: X \rightarrow \{i \in \mathbb{N}: 1 \leq i \leq \#(X)\}$ ,  $g: Y - X \rightarrow \{i \in \mathbb{N}: 1 \leq i \leq \#(Y - X)\}$ .

And, we construct  $h: X \cup Y \rightarrow \{i \in \mathbb{N}: 1 \leq i \leq \#(X) + \#(Y - X)\}$ , for which:

- For  $e \in X, h(e) = f(e)$
- Else,  $h(e) = g(e) + \#(X)$

We proceed to show that  $h$  is bijective.

For injectivity, consider  $h(e) = h(e')$ , then if  $e \in X, e' \notin X$ , then  $h(e) \leq \#(X), h(e') = h(e) > \#(X)$ , violating trichotomy of order. If  $e \in X, e' \in X$ , then  $h(e) = h(e')$  implies  $f(e) = f(e')$ , which implies  $e = e'$  by injectivity of  $f$ .

Similarly, if  $e, e' \notin X, h(e) = h(e')$  implies  $g(e) + \#(X) = g(e') + \#(X)$ , which implies  $g(e) = g(e')$ , which implies  $e = e'$  by injectivity of  $g$ , as desired.

For surjectivity, consider  $i' \in \{i \in \mathbb{N}: 1 \leq i \leq \#(X) + \#(Y - X)\}$ , we either have  $i' \leq \#(X)$  or  $i' > \#(X)$ .

For the first case, note  $\exists e \in X, f(e) = i'$ , since  $f(e) = g(e)$ , the same  $e$  yields  $g(e) = i'$  as desired.

For the second case, note  $\exists e \notin X, g(e) = i' - \#(X)$ . (As  $\#(X) + \#(Y - X) \geq i'$  implies  $\#(Y - X) \geq i' - \#(X)$  which implies  $i' - \#(X) \in \{i \in \mathbb{N}: 1 \leq i \leq \#(Y - X)\}$ ), hence  $\exists e \notin X, g(e) + \#(X) = i'$ , which implies  $\exists e \notin X, h(e) = i'$  as desired.

Note: Above used subtraction, which will be defined later. And introduction here should not cause circular reasoning.

If  $X, Y$  disjoint, then  $Y - X = Y$ , hence  $\#(X \cup Y) = \#(X) + \#(Y - X) = \#(X) + \#(Y)$  as desired.

(c)

Note, if  $Y = X$ , then we have  $\#(Y) = \#(X)$ , hence  $\#(Y) \leq \#(X)$  as desired.

For case  $Y \neq X$ ,  $Y \subset X$ . We induct on  $\#(X)$ . When  $\#(X) = 0$ , by Exercise 3.6.2,  $X = \emptyset$ . For  $Y \subset X$ , either  $Y = \emptyset$  or  $Y \neq \emptyset$ .  $Y = \emptyset$  implies  $X = Y$ , hence not under the case  $Y \neq X$ .  $Y \neq \emptyset$  implies  $y \in Y$  (by single choice), yet  $Y \subset X$  implies  $y \in X$ , contradiction. Hence,  $\#(X) = 0$  is vacuously true.

For  $\#(X) = 1$ , by Exercise 3.6.2,  $X$  is non-empty, hence by single choice there exists  $x \in X$ . We claim  $X = \{x\}$ . Assume there exists  $x' \in X$ ,  $x' \neq x$ , then we can define bijection  $f: X \rightarrow \{1, 2\}$ ,  $f(x) = 1$ ,  $f(x') = 2$ . Hence  $\#(X) = 2$ , contradiction.

Assume  $Y$  is non-empty, then we claim  $Y = X = \{x\}$ . As, assume  $Y$  contains 2 elements  $y, y'$ , then  $y \in Y$ ,  $y' \in Y$  as  $Y \subset X$ , hence  $y = y' = x$ , as desired. Hence, this does not fall under the case  $Y \neq X$ .

Assume  $Y$  is empty, then  $Y \subset X$ , as desired.

For  $P(n)$ , we take  $x \in X$ ,  $x \notin Y$  (which exists as  $X \neq Y$ ). By Lemma 3.6.9  $\#(X - \{x\}) = n$ . Hence, if  $X - \{x\} \neq Y$ , it must then hold that  $X - \{x\} \subset Y$ , therefore  $\#(Y) < \#(X - \{x\})$ , by inductive hypothesis. Hence  $\#(Y) < \#(X - \{x\}) < n = \#(X)$  as desired.

If  $X - \{x\} = Y$ , then  $\#(Y) = \#(X - \{x\}) = n < n = \#(X)$ , closing the induction.

(d)

Let  $\#(X) = n$ , we induct on  $\#(X)$ .

For  $n = 0$ , by Exercise 3.6.2,  $X = \emptyset$ , hence  $f(X) = \emptyset$ . Therefore  $\#(X) = \#(f(X)) = 0$ , as desired.

For  $P(n)$ , we take an element  $x \in X$  and consider  $X - \{x\}$ . Note by Lemma 3.6.9,  $\#(X - \{x\}) = n$ , hence  $\#(f(X - \{x\})) \leq \#(X - \{x\})$  by inductive hypothesis.

Then, we perform case analysis. If there exist  $x' \in X$ ,  $x \neq x'$  s.t.  $f(x) = f(x')$ , then  $f(X - \{x\}) = f(X)$ . As, given  $y \in f(X)$ , it must hold that  $y = f(c)$  for some  $c \in X$ . If  $c \in X - \{x\}$ , then  $y \in f(X - \{x\})$ . If  $c \notin X - \{x\}$ , since  $c \in X$  it must hold that  $c = x$ . Hence  $y = f(c) = f(x) \notin f(X - \{x\})$  as desired. Similarly for  $y \in f(X - \{x\})$ , we can show that  $y \in f(X)$ , as desired.

Therefore,  $f(X - \{x\}) = f(X)$ , hence  $\#(f(X - \{x\})) = \#(f(X))$ , consequently  $\#(f(X)) \leq \#(X - \{x\})$ , note  $\#(X - \{x\}) = n$ , hence  $\#(f(X)) \leq n < n + 1 = \#(X)$  as desired.



If there doesn't exist  $x' \in X$ ,  $x \neq x'$  s.t.  $f(x) = f(x')$ , then easy to verify  $f(X - \{x\}) = f(X) - \{f(x)\}$ . Since  $f(x) \in f(X)$ ,  $\#(f(X - \{x\})) = \#(f(X) - \{f(x)\}) = \#(f(X)) - 1$ . Hence:

$$\begin{aligned} \#(f(X - \{x\})) &\leq \#(X - \{x\}) \leq \#(f(X)) - 1 \leq n \leq \#(f(X)) \leq n \end{aligned}$$

As desired.

For  $f$  one-to-one, we can either modify the above induction proof or directly consider function  $g: X \rightarrow f(X)$  s.t.  $g(x) = f(x)$ , and show  $g$  is bijective.

(e)

Since  $X, Y$  are finite sets, denote  $\#(X) = n$ ,  $\#(Y) = m$ . WLOG assume  $n = 0$ , then  $RHS = 0$ , and  $LHS = 0$  by Exercise 3.5.8, as desired.

Consider the case  $n, m \neq 0$ . There exists bijections:

$$f: \{X \rightarrow \{i \in \mathbb{N}, 1 \leq i \leq n\}\} \rightarrow \{Y \rightarrow \{i \in \mathbb{N}, 1 \leq i \leq m\}\}$$

Define  $h: \{X \times Y \rightarrow \{i \in \mathbb{N}, 1 \leq i \leq nm\}\}$  as:

$$h(x, y) = (g(y) - 1)n + f(x)$$

Easy to verify that  $h$  does have desired domain & range.

We claim that  $h$  is bijective. For injectivity, assume we have  $h(x, y) = h(x', y')$ , then  $(g(y) - 1)n + f(x) = (g(y') - 1)n + f(x')$ , then  $(g(y) - 1)n + f(x) = (g(y') - 1)n + f(x')$ , then  $g(y)n + f(x) = g(y')n + f(x')$ .

We induct on  $n$  to show that  $g(y)n + f(x) = g(y')n + f(x')$  implies  $f(x) = f(x')$ ,  $g(y) = g(y')$ . We prove a Lemma:

- Given  $n, q \in \mathbb{N}$ ,  $n, q \geq 1$ , there exists unique  $r \in \mathbb{N}$ ,  $1 \leq r \leq q$ ,  $m \in \mathbb{N}$  s.t.  $n = mq + r$ .

Proof:

For existence, note there exists  $m$ ,  $0 \leq r - 1 < q$  by applying Proposition 2.3.9 to  $n - 1 = mq + (r - 1)$ .

For uniqueness, Induct on  $n$ .

For  $n = 1$ ,  $1 = mq + r$ . Hence, it must hold that  $r = 1$ , then  $mq = 0$ . Since  $q \geq 1$ , then by Lemma 2.3.3,  $m = 0$  must hold, as desired.

For  $P(n)$ , we have  $n = mq + r$ . Assume we also have  $n = m'q + r'$ , s.t.  $r \neq r'$ , then contradiction as  $m'q + r = mq + r' \rightarrow r = r'$ . Similarly, cannot have



$m \neq m' \wedge r = r'$ . Therefore must have  $m \neq m' \wedge r \neq r'$

Then, if  $r, r' > 1$ , we have  $n = mq + (r - 1) = m'q + (r' - 1)$ , contradicting  $P(n)$ .

If  $r = 1 \wedge r' \geq 1$ , then  $n = mq, n = (m - 1)q + q$ . (Note: we can take  $m - 1$  as if  $m = 0$ ,  $n = mq = 0$ , violating  $n \geq 1$ ). Also  $n = m'q + (r' - 1)$ . Note since  $r' \geq q$ , it must hold that  $r' - 1 < q$ , hence  $(r' - 1) \neq q$ , contradicting  $P(n)$ . This closes the induction.

Hence, let  $L = g(y)n + f(x) = g(y')n + f(x')$ . It immediately follows that  $g(y) = g(y'), f(x) = f(x')$  via the Lemma, as desired.

For surjectivity, given  $L \in \{i \in \mathbb{N} : 1 \leq i \leq nm\}$ , there exists  $u, t$  s.t.  $L = un + t$  by Lemma, hence  $L = ((u + 1) - 1)n + t$ . We take  $a = g^{-1}(u + 1), b = f^{-1}(t)$ . Since  $1 \leq t \leq n$ ,  $t$  is in domain of  $f^{-1}$ . Similarly, assume  $u + 1 > m$ , then  $L \geq mn + t > mn$ . Contradiction. Hence  $u + 1$  is in domain of  $g^{-1}$ . Then:

$$\begin{aligned} h(a, b) &= h(g^{-1}(u + 1), f^{-1}(t)) \mid = (g(g^{-1}(u + 1)) - 1)n + f(f^{-1}(t)) \mid = ((u + 1) - 1)n + t \mid = L \end{aligned}$$

As desired.

Note: For injectivity, induction can be circumvented, by bounding  $f(x) - f(x')$  and show that it lies between multiple of  $n$ . Surjectivity can also be shown by Euclidean Algorithm + bound estimation. This offers an alternative way of doing the problem.

(f)

Let  $\#(X) = n$ , We induct on  $n$ .

When  $n = 0$ , we have  $\#(Y)^{\#(X)} = 1$  and  $X = \emptyset$  (By 3.6.2), and there exists 1 function  $f: X \rightarrow \emptyset$ , as desired.

For  $P(n)$ , consider taking  $x \in X$ , and hence  $Y^{X - \{x\}} = \#(Y)^{n-1}$  by inductive hypothesis.

We proceed to show that there exists a bijection between  $Y^{X - \{x\}} \times Y^{\{x\}}$  and  $Y^X$ . Let  $g: Y^{X - \{x\}} \times Y^{\{x\}} \rightarrow Y^X$  be:

- $g(h, i) = f$  s.t.  $\forall x' \in X - \{x\}, f(x') = h(x')$ , else  $f(x) = i(x)$

We proceed to prove that  $g$  is injective. For  $g(h, i), g(h', i')$ , it must hold that:

- $\forall x' \in X - \{x\}, f(x') = h(x')$
- $\forall x' \in X - \{x\}, f(x') = h'(x')$

Hence,  $\forall x' \in X - \{x\}, h(x') = h'(x')$ , therefore  $h = h'$ . Similarly,  $i(x) = i'(x)$ , as desired.

For surjectivity, given a function  $f \in Y^X$ , we define:

- $h: X - \{x\} \rightarrow Y$  s.t.  $\forall x' \in X - \{x\}, h(x') = f(x')$
- $i: \{x\} \rightarrow Y$  s.t.  $i(x) = f(x)$ .

Hence,  $h \in Y^{X - \{x\}}, i \in Y^{\{x\}}$ . Easy to verify  $g(h, i) = f$ , as desired.

Then, we show that  $\#(Y^{\{x\}}) = \#(Y)$ . We similarly establish construct  $h: Y^{\{x\}} \rightarrow Y$  s.t.  $h(f) = f(x)$ . Then,  $h$  is injective, as, given  $f(x) = f'(x)$ , it must hold that  $f = f'$  by definition of equality.  $h$  is surjective, as, given  $y \in Y$ , we can construct  $f(x) = y$  as desired.

Hence:

$$\begin{aligned} \#(Y^X) &= \#(Y^{X - \{x\}}) \times \#(Y^{\{x\}}) \quad \text{By e)} \\ &= \#(Y)^n \times \#(Y) \quad \text{By inductive hypothesis} \\ &= \#(Y)^{n+1} \end{aligned}$$

As desired.

### Exercise 3.6.5

We consider  $f: A \times B \rightarrow B \times A$  s.t.  $f(a, b) = (b, a)$ .

$f$  is injective as, given  $f(a, b) = f(c, d)$ , then  $(b, a) = (d, c)$ , hence  $b = d$  and  $a = c$ , therefore  $(a, b) = (c, d)$  as desired.

$f$  is surjective, as, given  $(a, b) \in B \times A$ , we have  $f(b, a) = (a, b)$  as desired.

Alternate proof of Lemma 2.3.2:

For  $m, n \in \mathbb{N}$ , there exists two sets  $M, N$  with cardinality  $m, n$ . By Proposition 3.6.14 e),  $\#(M \times N) = m \times n$ ,  $\#(N \times M) = n \times m$ . Yet, by Exercise 3.6.5,  $\#(M \times N) = \#(N \times M)$ , hence  $m \times n = n \times m$ , as desired.

### Exercise 3.6.6

Note:

$$f \in (A^B)^C \Leftrightarrow f: C \rightarrow A^B$$

$$g \in A^{B \times C} \Leftrightarrow g: B \times C \rightarrow A$$

Consider  $h: (A^B)^C \rightarrow A^{B \times C}$  s.t.  $\forall (b, c) \in B \times C, h(f)(b, c) = f(c)(b)$ . We claim that  $h$  is a bijection.

$h$  is injective as say we have  $h(f) = h(f')$ , then  $\forall (b, c) \in B \times C, f(c)(b) = f'(c)(b)$ . Denote  $f(c) = g, f'(c) = g'$ , then  $\forall (b, c) \in B \times C, g(b) = g'(b)$ . Hence  $\forall b \in B, g(b) = g'(b)$ , hence  $g = g'$ . Therefore  $\forall c \in C, f(c) = f'(c)$ , hence  $f = f'$  as desired.

$h$  is surjective as, given  $g: B \times C \rightarrow A$ , we define  $f: C \rightarrow A^B$  as  $f(c) = g(b, c)$ . With  $c$  fixed,  $g(b, c) \in A^B$  as desired.

Given  $a, b, c \in \mathbb{N}$ , construct  $A, B, C$  with cardinality  $a, b, c$ . Then,  $\#((A \cup B)^C) = (a+b)^c$ ,  $\#(A^B \times C) = a^b \times c$  by Proposition 3.6.14. Hence,  $(a+b)^c = a^b \times c$  as desired.

For  $a^b \times a^c = a^{b+c}$ , consider sets  $A, B, C$  with cardinality  $a, b, c$  and  $B, C$  disjoint. Then, consider bijection  $f: A^B \cup A^C \rightarrow A^{B \cup C}$ , for which  $f(h) = (f|_B, f|_C)$ , with  $\forall b \in B, f(b) = h(b), \forall c \in C, g(c) = h(c)$ . Rest similar to the above proof.

### Exercise 3.6.7

Only if

$f: A \rightarrow B$  is an injection. Denote  $g: A \rightarrow f(A)$  s.t.  $\forall a \in A, g(a) = f(a)$ . Then,  $g$  is a bijection. As,  $g$  is injective, given  $g(a) = g(a')$ , then  $f(a) = f(a')$ , hence  $a = a'$  by injectivity of  $f$ .  $g$  is surjective, as, for all  $b \in f(A)$ ,  $\exists a \in A$  s.t.  $f(a) = b$ , by definition of image.

Hence,  $\#(A) = \#(f(A))$ . Note  $f(A) \subseteq B$ , hence by 3.6.14 c),  $\#(A) = \#(f(A)) \leq \#(B)$ .

If

Given  $\#(A) \leq \#(B)$ , then there exists:

- $f: A \rightarrow \{i \in \mathbb{N} : 1 \leq i \leq \#(A)\}$
- $g: \{i \in \mathbb{N} : 1 \leq i \leq \#(B)\} \rightarrow B$

We construct  $g': \{i \in \mathbb{N} : 1 \leq i \leq \#(A)\} \rightarrow B$  s.t.  $\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq \#(A)\}, g'(i) = g(i)$ .

Then, we claim that  $h = g \circ f, h: A \rightarrow B$  is an injection. First, note domain & range of  $g, f$  satisfies conditions in Definition 3.3.10, hence  $h: A \rightarrow B$  is a function. Then, given  $h(a) = h(a')$ , we deduce  $g(f(a)) = g(f(a'))$ , which by injectivity of  $g$  we deduce  $f(a) = f(a')$ , hence  $a = a'$  by injectivity of  $f$  as desired.

### Exercise 3.6.8

Consider  $h: A \rightarrow f(A)$ . Then, similar to Exercise 3.6.7 reasoning,  $h$  is bijective. Hence, consider  $h^{-1}: f(A) \rightarrow A$ , then  $h^{-1}$  is bijective, hence surjective. Finally consider  $g: B \rightarrow A$  s.t.:

- If  $b \in f(A)$ ,  $g(b) = h^{-1}(b)$
- Else we choose  $a \in A$ , and let  $g(b) = a$

Note: we can take  $a \in A$ , as if  $A = \emptyset$ , injectivity of  $f$  implies  $B = \emptyset$ , for which  $g: \emptyset \rightarrow \emptyset$  exist as trivial bijection.

Easy to verify  $g$  has desired domain & range.

For surjectivity, take  $a \in A$ , then  $g(h(a)) = h^{-1}(h(a)) = a$  as desired.

### Exercise 3.6.9

In Exercise 3.6.4 b), it is shown that  $\#(X \cup Y) = \#(Y) + \#(X - Y)$ .

We proceed to show that  $\#(X \cap Y) = \#(X) - \#(X - Y)$ . Note  $\#(X \cap Y) + \#(X - Y) = \#((X \cap Y) \cup (X - Y)) = \#(X)$  (by Exercise 3.6.4 b, as  $X - Y$ ,  $X \cap Y$  disjoint), hence  $\#(X \cap Y) = \#(X) - \#(X - Y)$ .

Add  $\#(X \cap Y) = \#(X) - \#(X - Y)$  and  $\#(X \cup Y) = \#(Y) + \#(X - Y)$  yields the result.

### Exercise 3.6.10

Assume  $\forall i \in \{1, \dots, n\}, \#(A_i) \leq 1$ .

Then, by 3.6.14 b):

$$\#\bigcup_{i \in \{1, \dots, n\}} A_i = \sum_{i \in \{1, \dots, n\}} \#(A_i) \leq \sum_{i \in \{1, \dots, n\}} 1 = n$$

Contradiction. Hence  $\exists i \in \{1, \dots, n\}, \#(A_i) > 1$ , which by rules of order,  $\exists i \in \{1, \dots, n\}, \#(A_i) \geq 2$ , as desired.

### Exercise 4.1.1

Reflexivity:

Given  $a \in \mathbb{N}$ , obviously  $a + b = a + b$ , hence  $a \leq b$ .

Symmetry:

If  $a \leq b$ , then  $a + d = c + b$ , hence by Reflexivity of Natural Number equality,  $c + b = a + d$ , hence  $c \leq d$  as desired.

### Exercise 4.1.2

If  $a \leq b'$ , then  $a + b' = a' + b$ . Hence  $b' + a = b + a'$ , therefore  $(b' \leq a') = (b \leq a)$ , hence  $-(a \leq b) = -(a' \leq b')$  as desired.

### Exercise 4.1.3

$$\begin{aligned} \text{Let } a = x - y \quad & (-1) \times (x - y) = (0 - 1) \times (x - y) = (0 - 1) \times x + (0 - 1) \times (-y) = 0 - x + y = y - x = -a \end{aligned}$$

As desired.

### Exercise 4.1.4

For  $x + y = y + x$ , let  $x = a$  and  $y = c$ , then  $x + y = (a + c)$  and  $y + x = (c + a)$ .

Note  $(a + c) + (d + b) = (c + a) + (b + d)$ , hence  $x + y = y + x$  as desired.

For  $(x + y) + z = x + (y + z)$ , let  $x = a$ ,  $y = c$ ,  $z = e$ , then:

- $(x + y) + z = (a + c) + e$  and  $(b + d) + f$
- $x + (y + z) = a + (c + e)$  and  $b + (d + f)$

Hence, we remain to show  $(a + c) + e + b + (d + f) = a + (c + e) + (b + d) + f$ , which is true by natural number addition rules as desired.

For  $x + 0 = 0 + x = x$ , let  $x = a$  and  $y = 0$ , then  $x + 0 = (a + 0)$  and  $0 + x = (0 + a) = a$  as desired. For  $x + 0 = 0 + x$ , it follows from  $x + y = y + x$ .

For  $x + (-x) = (-x) + x = 0$ , similarly  $x + (-x) = (-x) + x$  follows from  $x + y = y + x$ . Remains to show  $x + (-x) = 0$ . Let  $x = a$  and  $y = b$ , then  $x + (-x) = a + b$  and  $a + b = (a + b)$  and  $(b + a)$ . We claim  $(a + b) + (b + a) = 0$ , as  $a + b + 0 = 0 + b + a$ , as desired.

For  $xy = yx$ , let  $x = a$  and  $y = c$ , then  $xy = (ac + bd)$  and  $yx = (ca + db)$ . Rest follows from reflexivity.

For  $x1 = 1x = x$ , note  $x1 = 1x$  by  $xy = yx$ . Remains to show that  $x1 = x$ . Let  $x = a$  and  $y = b$ , then  $x1 = (a + b)$  and  $(1 + 0) = (a1 + b0) = a + b$  as desired.

For  $x(y + z) = xy + xz$ , let  $x = a$ ,  $y = c$ ,  $z = e$ , then  $x(y + z) = (a + b)(c + e)$  and  $xy + xz = (ac + bd) + (ae + bf)$ .

$xy + xz = (ac + bd) + (ae + bf)$  and  $(af + be) = ((ac + bd) + (ae + bf))$  and  $(ad + bc) + (af + be)$ . Note  $(ac + bd) + (ae + bf) = a(c + e) + b(d + f)$  and  $(ad + bc) + (af + be) = a(d + f) + b(c + e)$ , rest follows from reflexivity.

For  $(y + z)x = yx + zx$ , note  $(y + z)x = x(y + z)$  by  $xy = yx$ . Then,  $x(y + z) = xy + xz$ . Finally  $xy + xz = yx + zx$  by  $xy = yx$ , as desired.

## Exercise 4.1.5

Let  $x = (a + b)$ ,  $y = (c + d)$ , then  $xy = (ac + bd) + (ad + bc) = 0$ . Hence  $ac + bd + 0 = (ad + bc) + 0$  implying  $ac + bd = ad + bc$ . Then, we need to prove either  $a = b$  or  $c = d$ .

Note:

$$\begin{aligned} ac + bd &= ad + bc \mid ac + bd - bd - ad = ad + bc - bd - ad \quad \text{by substitution} \\ ac - ad &= bc - bd \mid a(c - d) = b(c - d) \mid a(c - d) - b(c - d) = 0 \mid (a - b)(c - d) = 0 \end{aligned}$$

WLOG assume  $a > b$ ,  $c > d$ , then By Lemma 4.1.5,  $(a - b)$ ,  $(c - d)$  equates to natural numbers. Hence, by Proposition 4.1.8 either  $a = b$  or  $c = d$  as desired.

If  $a - b < 0$  or  $c - d < 0$ , then we can simply multiply both sides by  $-1$  and the same argument holds.

### Exercise 4.1.6

If  $ac = bc$ , then  $ac - bc = 0$ , hence  $c(a - b) = 0$ . Therefore, by Proposition 4.1.8, either  $c = 0$  or  $a - b = 0$ . Since  $c \neq 0$ , then  $a - b = 0$  must hold, therefore  $a = b$  as desired.

Alternate proof:

By Corollary 2.3.7,  $c$  is either positive, or negation of positive number.

WLOG say  $c$  is positive (if  $c$  is negative, we multiply both sides by  $-1$ ). Then, assume  $a, b \in \mathbb{N}$ , then the conclusion follows from Corollary 2.3.7. Assume  $a, b \notin \mathbb{N}$ , then we multiply both sides by  $-1$  to yield  $-a = -b$ , which then we can deduce  $a = b$  as desired. Assume  $a \in \mathbb{N}$ ,  $b \notin \mathbb{N}$ , then we can show  $ac \in \mathbb{N}$ ,  $bc = -(-b)c \notin \mathbb{N}$ , contradiction hence the case is vacuous, as desired.

### Exercise 4.1.7

a)

Only if:

Assume  $a > b$ , then  $a = b + c$  for some positive  $c$ . Hence:

$$\begin{aligned} a &= b + c \mid a - b &= b + c - b \mid a - b &= c \end{aligned}$$

As desired.

If:

Assume  $a - b = c$  for some positive  $c$ , then:

$$a - b &= c \mid a - b + b &= c + b \mid a &= b + c \mid a > b \end{aligned} \quad \text{As desired.}$$

b)

$a + c - (b + c) = a + c - b - c = a - b$ . Since  $a > b$ ,  $a + c - (b + c) = a - b$  is a positive number, hence  $a + c > b + c$  as desired.

c)

$ac - bc = (a - b)c$ . Since  $a - b$ ,  $c$  are positive,  $(a - b)c$  is positive, hence  $ac > bc$  as desired.

d)

$-b - (-a) = -b + a = a - b$ . Since  $a > b$ ,  $-b - (-a) = a - b$  is positive, hence  $-b > -a$  as desired.

e)

Note  $a = b + e$ ,  $b - f = c$  for some positive  $e, f$ , hence:

$a - c = (b + e) - (b - f) = e + f > 0$ , hence  $a > c$  as desired.

f)

Equivalent of showing exactly one of  $a - b > 0$ ,  $a - b$  positive,  $a - b$  negative is true, which follows from Trichotomy of integers.

### Exercise 4.1.8

Let  $P(n)$  denote  $n \geq 0$ . Then,  $P(0)$  is true as  $0 \geq 0$ .

For  $P(n+1)$ , note  $n+1 > n \geq 0$  (by  $P(n)$ ), as desired.

However,  $P(-1)$ , aka  $-1 \geq 0$  is false.

### Exercise 4.2.1

Reflexivity:

Let  $x = a // b$ , then  $x = x \Leftrightarrow ab = ab$ . Since  $ab = ab$  always true  $x = x$  as desired.

Symmetry:

Let  $x = a // b$ ,  $y = c // d$ . Then  $x = y$  implies  $ad = bc$ , which implies  $bc = ad$  hence  $y = x$  as desired.

Transitivity:

Let  $x = a // b$ ,  $y = c // d$ ,  $z = e // f$ . Assume  $x = y$ ,  $y = z$ , then  $ad = bc$ ,  $cf = ed$ . Hence  $bcd = bed$ , hence  $adf = bed$ . Since  $d \neq 0$ ,  $af = be$  by Corollary 4.1.9, hence  $x = z$  as desired.

### Exercise 4.2.2

For addition, say we have  $x = x'$ ,  $x = a // b$ ,  $y = c // d$ ,  $x' = a' // b'$ , then:

$$\begin{aligned} x + y &= (ad + bc) // bd \\ x' + y &= (a'd + b'c) // b'd \end{aligned}$$

Now, we remain to show that  $(ad + bc)b'd = (a'd + b'c)bd$ . Note  $ab' = a'b$ , then  $(ad + bc)b'd = ab'dd + bb'cd = a'bdd + bb'cd = (a'd + b'c)bd$  as desired.

Similarly, say  $x = a // b$ ,  $y = c // d$ ,  $y' = y$ ,  $y' = c' // d'$ , then:

$$\begin{aligned} x + y &= (ad + bc) // bd \\ x + y' &= (ad' + bc') // bd' \end{aligned}$$

Then, remain to show  $(ad + bc)bd' = (ad' + bc')bd$ . Note  $cd' = c'd$ , then  $(ad + bc)bd' = add' + cd'bb = add' + c'dbb = (ad' + c'b)bd$  as desired.

For product:

$$\begin{aligned} x * y &= ac // bd \\ x' * y &= a'c // b'd \end{aligned}$$

Note  $ab' = a'b$ , then  $ac b'd = a'cbd$  as desired.

Similarly:

$$\begin{aligned} x * y &= ac // bd \\ x * y' &= ac' // bd' \end{aligned}$$

Note  $cd' = c'd$ , then  $acbd' = ac'bd$  as desired.

For negation, say we have  $x = a // b$ ,  $x' = x$ ,  $x' = a' // b'$ , then  $-x = (-a) // b$ ,  $-x' = (-a') // b'$ . Remains to show  $-ab' = -a'b$ . Note  $ab' = a'b$ , then  $-ab' = -a'b$  as desired.

### Exercise 4.2.3

Let  $x = a // b$ ,  $y = c // d$ ,  $z = e // f$ .

For  $x + y = y + x$ , note  $x + y = (ad + bc) // bd$ ,  $y + x = (da + cb) // bd = (ad + bc) // bd$  as desired.

For  $x + 0 = 0 + x = x$ , note  $x + 0 = 0 + x$  by  $x + y = y + x$ . Remains to show  $x + 0 = 0$ . Note  $x + 0 = (a1 + b0) // b = a // b = x$  as desired.

For  $x + (-x) = (-x) + x = 0$ , note  $x + (-x) = (-x) + x$  by  $x + y = y + x$ , hence remains to show  $x + (-x) = 0$ . Note  $x + (-x) = (ab + b(-a)) // bb = 0 // bb$ . We claim  $0 // bb = 0 // 1$ , as  $0 \times 1 = 0 \times bb = 0$ , as desired.

For  $xy = yx$ , note  $xy = ac // bd$ ,  $yx = ca // db = ac // bd$ , rest follows by reflexivity.

For  $(xy)z = x(yz)$ , note  $(xy)z = (ac)e // (bd)f$ ,  $x(yz) = a(ce) // b(df) = (ac)e // (bd)f$  as desired.

For  $x1 = 1x = x$ , note  $x1 = 1x$  follows from  $xy = yx$ . For  $x1 = x$ , note  $x1 = a1 // b1 = a // b = x$  as desired.

For  $x(y + z) = xy + xz$ , note  $x(y + z) = (a // b)(cf + de // df) = a(cf + de) // bdf$

$$xy + xz = ac // bd + ae // bf = (acbf + aebd) // bdbf$$

Then, need to show  $a(cf + de)bdbf = (acbf + aebd)bdf$ , which, since  $b, d, f \neq 0$ , is equivalent to  $a(cf + de) = (acf + aed)$ . which is true by distributive law of integer, as desired.

For  $(y + z)x = yx + zx$ , note  $(y + z)x = x(y + z)$  by  $xy = yx$ , then  $x(y + z) = xy + xz = yx + zx$  as desired.



For  $x^{-1} = x^{-1} x = 1$ , note  $x^{-1} = x^{-1} x$  by  $xy = yx$ . Remains to show  $x x^{-1} = 1$ . Note  $x^{-1} = ab // ba$ . Since  $ab1 = ba1$ ,  $x x^{-1} = 1$  as desired.

## Exercise 4.2.4

Let  $x = a / b$ . Then, assume  $b$  positive. (If  $b$  is not positive, easy to prove  $(-a) / (-b) = a / b$ ), with  $(-b)$  positive.

Then, we either have  $a < 0$ ,  $a = 0$ ,  $a > 0$  by Lemma 4.1.11 f).

If  $a < 0$ , then by definition  $x$  is negative. If  $a = 0$ , then by definition  $x = 0$ . If  $a > 0$ , then by definition  $x$  is positive. Hence at least one of (a) (b) (c) is true.

Then, we show (a), (b) cannot happen at the same time. Assume  $a / b = 0 / 1$ , then  $a1 = b0$ , hence  $a = 0$ , contradiction, Similarly (b) (c) cannot happen at the same time.

For (a), (c), assume  $x = a / b = c / d$  for  $a, b, d > 0$ ,  $c < 0$ , then  $ad = bc$  holds. Note  $ad$  is positive by Lemma 4.1.11 c), yet  $bc$  is negative, contradiction as desired.

## Exercise 4.2.5

a) Equivalent to exactly one of  $x - y = 0$ ,  $x - y$  positive,  $x - y$  negative is true, which follows from Lemma 4.2.7.

b) One have  $x < y$  if  $x - y$  is negative. We claim  $y - x$  is positive. Say  $x = a/b$ ,  $y = c/d$  for some  $b, d \in \mathbb{Z}$ ,  $b, d > 0$ , then  $x - y = (ad - bc) / bd$ ,  $y - x = (bc - ad) / bd = -(ad - bc) / bd$ , hence by definition of negation  $x - y = -(y - x)$ . Hence  $y - x$  negative by Definition 4.2.6, therefore  $y > x$ .

The other direction can be proved similarly.

c) If  $x < y$  and  $y < z$ , then  $y - x$ ,  $z - y$  are both positive. Hence,  $(y - x) + (z - y) = z - x$  is positive, therefore  $x < z$  as desired.

Note: easy to prove sum of two positive numbers are positive. Assume we have  $x, y > 0$ ,  $x = a/b$ ,  $y = c/d$ ,  $a, b, c, d > 0$ , then  $x + y = (ad + bc) / bd$ . Note  $bd$  is positive as  $b > 0$ , by 4.1.11 c)  $bd > 0$ , and  $ad + bc$  is positive as say we have  $m, n > 0$ , then  $m + n > 0 + n$  by 4.1.11 b), and since  $0 + n = n > 0$ ,  $m + n > 0$  by 4.1.11 e) as desired.

d)  $x < y$  implies  $y - x$  is positive. Hence  $y + z - (x + z) = y - x$  is positive as desired.

e)  $yz - xz = (y - x)z$ . Since  $y > x$ ,  $y - x$  is positive, we call it  $d$ . We remains to prove that product of two positive rational is positive. Say  $m = a / b$ ,  $n = c / d$ , for some  $a, b, c, d > 0$ , then  $mn = ac / bd$ . Hence, we remain to show  $ac, bd$  are positive, given  $a, b, c, d$  positive, It is already shown in c), as desired.

## Exercise 4.2.6

$xz - yz = (x - y)z = 1(x - y)z = (-1)(-1)(x - y)z = (y - x)(-z)$ . Since  $y - x, -z$  are positive, their product is positive via similar reasoning in Exercise 4.2.9 c).

### Exercise 4.3.1

(a) By trichotomy of rational number, we either have  $x$  positive,  $x$  negative or  $x = 0$ .

Assume  $x$  positive, then by definition  $|x| = x$ , which is positive as desired.

Assume  $x$  negative, then by definition  $|x| = -x$ . Assume  $x = a / b$  for some  $a < 0, b > 0$ , then  $-x = (-a) / b$  with  $-a, b$  positive hence  $-x$  is positive.

Assume  $x = 0$ , then by definition  $x = 0 \geq 0$  as desired,

(b) By trichotomy of rational number, we break into cases.

If  $x = 0$ ,  $|x + y| = |y| = 0 + |y| = |x| + |y| \leq |x| + |y|$  as desired. Similar reasoning for  $y = 0$ .

If  $x, y$  are positive, then  $|x + y| = x + y = |x| + |y| \leq |x| + |y|$  as desired.

If  $x, y$  are negative, then  $|x + y| = -(x + y) = (-x) + (-y) = |x| + |y| \leq |x| + |y|$  as desired.

If  $x$  positive and  $y$  negative, then we either have  $x + y$  positive,  $x + y$  negative, or  $x + y = 0$ . If  $x + y$  is positive, then  $|x + y| = x + y, |x| + |y| = x - y, |x| + |y| - |x + y| = x - y - (x + y) = -2y > 0$  as desired.

If  $x + y = 0$ , then  $|x + y| = 0, |x| + |y| \geq 0 = |x + y|$  as desired.

If  $x + y < 0$ , then  $|x + y| = -(x + y), |x| + |y| = x - y, |x| + |y| - |x + y| = x - y + (x + y) = 2x > 0$  as desired.

c)

If direction:

If  $y \geq |x|$ , assume  $x \geq 0$ , then  $y \geq x \geq 0$ . Note since  $y \geq 0, -y \leq 0$  by Exercise 4.2.6, hence  $y \geq x \geq 0 \geq -y$  as desired.

Assume  $x < 0$ , then  $-x > 0$  by Exercise 4.2.6, hence  $y \geq -x > 0$ , therefore  $-y \leq x < 0$  by Exercise 4.2.6. Note since  $y > 0, -y \leq x < 0 < y$  as desired.

d)

Assume  $x, y$  both positive, then  $|xy| = xy = |x||y|$  as desired. Assume  $x, y$  both negative, then  $|xy| = (-x)(-y) = |x||y|$  as desired. Assume  $x = 0$ , then  $|xy| = |0| = 0 = 0|y| = |x||y|$  as desired. Assume  $x$  positive and  $y$  negative then  $|xy| = x(-y) = |x||y|$  as desired.

e)

$|x - y| \geq 0$  by a). Also, by a),  $|x - y| = 0$  iff  $x - y = 0$  iff  $x = y$  as desired.

f)

$|x - y| = 1|x - y| = |-1||x - y| = |(-1)(x - y)| = |y - x|$  as desired.

g):

Need to show  $|x - z| \leq |x - y| + |y - z|$ . Let  $x - y = a$ ,  $y - z = b$ , then  $x - z = x - y + y - z = a + b$ . Hence, since by b)  $|a + b| \leq |a| + |b|$ ,  $|x - z| \leq |x - y| + |y - z|$  by substitution as desired.

## Exercise 4.3.2

a)

If  $x = y$ , then  $d(y, x) = 0$ . Hence  $\epsilon > 0 = d(y, x)$  as desired.

For the converse, WLOG assume  $x < y$ , then  $y - x = d$  for some positive  $d$ . Then,  $d(x, y) = d$ . We choose  $\epsilon = d / 2$ , then we claim  $d > d / 2$ . Say  $d = a / b$  for some  $a, b > 0$ , then  $d / 2 = (a / b) \times (1 / 2) = a / 2b$ . One can easily verify  $d = 2a / 2b$ , hence  $d - d / 2 = (2a - a) / 2b = a / 2b$ , which is positive, as desired.

Hence,  $d > d/2$  implies  $d(y, x) > d/2$ , hence  $x$  is not  $\epsilon$ -close to  $y$  for  $\epsilon = d / 2$ , contradiction as desired.

b)

If  $|x - y| \leq \epsilon$ , then  $|y - x| = |x - y| = \epsilon$  as desired.

c)

Given  $|x - y| \leq \epsilon$ ,  $|y - z| \leq \delta$ , then  $|x - y| + |y - z| \leq \epsilon + |y - z| \leq \epsilon + \delta$ . Note by 4.3.3 f),  $|x - z| \leq |x - y| + |y - z| \leq \epsilon + |y - z| \leq \epsilon + \delta$  as desired.

d)

Note: 
$$d(x + z, y + w) = |x + z - (y + w)| = |(x - y) + (z - w)| \leq |x - y| + |z - w|$$
 *by 4.3.3 (b)*  $\leq \epsilon + \delta$

As desired.

Similarly:

$$d(x - z, y - w) = |x - z - (y - w)| = |(x - y) + (-1)(z - w)| \leq |x - y| + |z - w|$$
 *by 4.3.3 (b), (d)*  $\leq \epsilon + \delta$

As desired.

e)

$|x - y| \leq \epsilon < \epsilon'$  as desired.

f)

WLOG assume  $y \leq w \leq z$ . Then  $y - x \leq w - x \leq z - x$ . Hence we either have  $w < x$ ,  $w \geq x$ . Assume  $w < x$ , then  $x - w > 0$ , note  $x - y \geq x - w > 0$ , hence  $|x - y| \geq |x - w|$  implying  $\epsilon \geq d(w, x)$  as desired.

Assume  $w \geq x$ , then  $w - x \geq 0$ , hence  $0 \leq w - x \leq z - x$ , therefore  $|w - x| \leq |z - x|$  implying  $d(w, x) \leq \epsilon$  as desired.

g)

Note  $\epsilon > 0$  non-zero required as Definition 4.3.4 mandates that  $\epsilon > 0$ .

$$\begin{aligned} d(xz, yz) &= |xz - yz| = |z(x - y)| = |z||x - y| \quad \text{By 4.3.3 (d)} \\ &\leq |z|\epsilon \end{aligned}$$

As desired.

### Exercise 4.3.3

a)

Let  $P(n)$  be  $x^n x^m = x^{n+m}$ .

Then,  $P(0)$  denote  $x^0 x^m = x^m$ , which is true as  $x^0 = 1$  by definition 4.3.9.

For  $P(n + 1)$ , note  $x^{n+1} x^m = x^n x x^m = x^n x^{m+1}$  by  $P(n)$ , then  $x^{n+1} x^m = x^{n+m+1} = x^{(n+1)+m}$  by definition 4.3.9 as desired.

Similarly, let  $P(m)$  denote  $(x^n)^m = x^{nm}$ , then  $P(0)$  denote  $(x^n)^0 = x^{n0}$ , which is true as both side evaluates to  $1$ .

For  $P(m + 1)$ , note  $(x^n)^{m+1} = (x^n)^m x^n = x^{nm} x^n$  by  $P(m)$ , which, by  $x^n x^m = x^{n+m}$  equals to  $x^{nm+n} = x^{n(m+1)}$  as desired.

Let  $P(n)$  denote  $(xy)^n = x^n y^n$ , then  $P(0)$  is true as both sides equates to  $1$ . For  $P(n + 1)$ , note  $(xy)^{n+1} = (xy)^n xy = x^n y^n xy$  by inductive hypothesis, which then, by Definition 4.3.9, equals to  $x^{n+1} y^{n+1}$  as desired.

b)

If direction:

Note  $x^n = x^{n-1}x$ . Note  $n - 1 \in \mathbb{N}$  as each positive natural number has a natural predecessor (Lemma 2.2.10). Hence, since  $x = 0$ ,  $x^{n-1}x = x^{n-1}0 = 0$  as desired.

Only if direction:

We prove the contraposition, let  $P(n)$  be  $x \neq 0 \rightarrow x^n \neq 0$ . Then,  $P(1)$  is true as  $x^1 = x^0 x = x \neq 0$ . For  $P(n + 1)$ , note  $x^{n+1} = x^n x$ . Since  $x \neq 0$  and  $x^n \neq 0$  by  $P(n)$ , then their product  $x^n x \neq 0$  as desired.

c)

Note if  $y = 0$ , then  $y^n = 0 \geq 0$  as desired. For  $y > 0$ , Let  $P(n)$  denote  $y > 0 \rightarrow y^n > 0$ . Then,  $P(0)$  is true as  $y^0 = 1 > 0$ . For  $P(n+1)$ , note if  $y > 0$ , then  $y^{n+1} = y^n y > y^n 0 = 0$  as desired.

For  $x^n \geq y^n$  (given  $x \geq y \geq 0$ ), let  $P(n)$  denote  $x^n \geq y^n$ . Then,  $P(0)$  is true as  $x \geq y$ . For  $P(n + 1)$ , note

$$\begin{aligned} x^{n+1} &= x^n x \geq x^n y \quad \text{As } x^n \geq 0, x > y \geq y^n y \quad \text{By } P(n) \\ &= y^{n+1} \quad \text{By Definition 4.3.9} \end{aligned}$$

As desired.

d)

Assume  $x \geq 0$ , then  $|x^n| = x^n = |x|^n$  as desired. Assume  $x < 0$ , then  $|x^n| = |(-1)^n (-x)^n| = |(-1)^n| |(-x)^n| = |(-x)^n| = (-x)^n = |x|^n$  as desired.

We remain to show  $|(-1)^n| = 1$ . If  $n = 0$ , then  $|(-1)^0| = 1$  as desired. For  $P(n+1)$ , note  $|(-1)^{n+1}| = |(-1)^n (-1)| = |(-1)^n| | -1| = 1 \times 1 = 1$  as desired.

## Exercise 4.3.4

a)

We start with proving  $x^n x^m = x^{n+m}$ . We proceed by cases. Note if any of  $n, m = 0$ , then the case is trivially true. Hence, we remain to show two cases. One being  $n, m$  both being negative, another being  $n$  being positive and  $m$  being negative.

### Case 1 $n, m$ both being negative

We first show for any rational  $a, b \neq 0$ ,  $a^{-1} b^{-1} = (ab)^{-1}$ . Observe,  $a^{-1}$  as exponent is consistent with  $a^{-1}$  as inverse, as  $a^{-1} = 1/a = \text{Inverse of } a$ .

Then  $(ab)^{-1} ab = 1$  by Proposition 4.2.4, then  $(ab)^{-1} a b a^{-1} b^{-1} = a^{-1} b^{-1}$ , then,  $(ab)^{-1} a b a^{-1} b^{-1} = (ab)^{-1} (a a^{-1}) (b b^{-1}) = (ab)^{-1}$  by Proposition 4.2.4 as desired.

Then:

$$\begin{aligned} x^n x^m &= (x^{-n})^{-1} \times (x^{-m})^{-1} \quad \text{By 4.3.11} \\ &= (x^{-n} x^{-m})^{-1} \quad \text{As } a^{-1} b^{-1} = (ab)^{-1} \\ &= (x^{-(n+m)})^{-1} \quad \text{By 4.3.10 a)} \\ &= x^{n+m} \end{aligned}$$

$\text{\textit{By 4.3.11}} \end{align} \quad \square$

As desired.

## Case 2: $n$ is positive and $m$ is negative

Assume  $n$  is positive,  $m$  is negative. Then, let  $m' = -m$  be a positive integer.

We propose Lemma 1:  $x^a / x^b = x^{a-b}$  if  $a, b, a-b \in \mathbb{N}$ . To prove it:

$$\begin{align} x^a &= x^{a-b} x^b && \text{\textit{By 4.3.10 a)}} \\ x^a / x^b &= x^{a-b} && \end{align} \quad \square$$

As desired.

We propose Lemma 2:  $1/x^n = x^{-n}$  for any  $n \in \mathbb{Z}$ .

Assume  $n \geq 0$ , then  $1/x^n = 1 / (1 / x^{-n})$  for some negative  $-n$  by Definition 4.3.11. Easy to verify  $1 / (1 / a) = a$  for  $a \in \mathbb{Q}$ , hence  $1 / x^n = 1 / (1 / x^{-n}) = x^{-n}$ , as desired.

We remain to show  $x^n x^{-m'} = x^{n-m'}$ . Observe:

$$\begin{align} x^n x^{-m'} &= x^n \times (1 / x^{m'}) && \text{\textit{By 4.3.11}} \\ &= x^n / x^{m'} && \text{\textit{By Definition of division}} \end{align} \quad \square$$

Hence, if  $n - m' \geq 0$ , result directly follows from Lemma 1. Otherwise:

$$\begin{align} x^n / x^{m'} &= 1 / (x^{m'} / x^n) && \text{\textit{By Lemma 1}} \\ &= 1 / x^{m' - n} && \text{\textit{By Lemma 2}} \end{align} \quad \square$$

As desired.

We proceed to prove  $(x^n)^m = x^{nm}$ . We again proceed by cases. Again, if any of  $n, m = 0$ , result is trivial. Hence, remains the case  $n, m < 0$ , the case  $n > 0, m < 0$  and the case  $n < 0, m > 0$ .

## Case 1: $n, m < 0$

We remain to show  $(x^{-n'})^{m'} = x^{(-n')(-m')}$  for  $n' = -n, m' = -m, n', m' \in \mathbb{N}$ .

We propose Lemma 3:  $1/a^n = (1/a)^n$  for  $n \in \mathbb{N}$ .  $P(0)$  is true as  $1/a^0 = (1/a)^0 = 1$ . For  $P(n+1)$ :

$$\begin{align} \frac{1}{a^{n+1}} &= \frac{1}{a a^n} && \text{\textit{By Definition 4.3.9}} \\ &= \frac{1}{a} \frac{1}{a^n} && \text{\textit{By } P(n)} \\ &= \frac{1}{a} (1/a)^n && \text{\textit{By Definition 4.3.9}} \end{align} \quad \square$$

As desired.

$$\begin{align} (x^{-n'})^{m'} &= (1/x^{n'})^{m'} && \text{\textit{By Definition 4.3.11}} \\ &= \frac{1}{(1/x^{n'})^{m'}} && \text{\textit{By Definition 4.3.11}} \\ &= \frac{1}{1 / (x^{n'})^{m'}} \end{align}$$

$\text{\textit{By Lemma 3}} \mid \&= \frac{1}{x^{n'm}} \text{\textit{By 4.3.10 (a)}} \mid \&= x^{n'm} \mid \&= x^{(-n)(-m)}$   
 $\text{\textit{end{align}}} \text{\textit{\$}}$

As desired.

## Case 2: $n > 0, m < 0$

We let  $m' = -m \in \mathbb{N}$ , then:

$\text{\textit{\$}} \text{\textit{\begin{align}}} (x^n)^{-m} \&= 1 / (x^n)^{m'} \text{\textit{By Definition 4.3.11}} \mid \&= 1 / x^{nm'} \text{\textit{By 4.3.10 (a)}} \mid \&= x^{n(-m')} \text{\textit{By Lemma 2}} \text{\textit{end{align}}} \text{\textit{\$}}$

As desired.

## Case 2: $n < 0, m > 0$

Let  $n' = -n \in \mathbb{N}$ , then:

$\text{\textit{\$}} \text{\textit{\begin{align}}} (x^{-n})^m \&= (1 / x^{n'})^m \text{\textit{By Definition 4.3.11}} \mid \&= 1 / (x^{n'})^m \text{\textit{By Lemma 3}} \mid \&= 1 / x^{n'm} \text{\textit{By 4.3.10 (a)}} \mid \&= x^{(-n')m} \text{\textit{By Lemma 2}} \text{\textit{end{align}}} \text{\textit{\$}}$

As desired.

For  $(xy)^n = x^n y^n$ , consider  $n$  being negative, and let  $n' = -n \in \mathbb{N}$ :

$\text{\textit{\$}} \text{\textit{\begin{align}}} (xy)^{-n'} \&= 1 / (xy)^{n'} \text{\textit{By Definition 4.3.11}} \mid \&= 1 / (x^{n'} y^{n'}) \text{\textit{By 4.3.10 (a)}} \mid \&= (1 / x^{n'})(1 / y^{n'}) \text{\textit{By Definition 4.2.2}} \mid \&= x^{-n'} y^{-n'} \text{\textit{By Definition 4.3.11}} \text{\textit{end{align}}} \text{\textit{\$}}$

As desired.

b)

We show if  $x \geq y > 0$  and  $n$  is negative,  $0 < x^n \leq y^n$ .

Let  $n' = -n \in \mathbb{N}$ ,  $x = a/b$ ,  $y = c/d$ ,  $a, b, c, d \in \mathbb{N}$ . Easy to verify  $x^{-n'} = b^{n'} / a^{n'}$ . By 4.3.10 c),  $b^{n'} > 0$ ,  $a^{n'} > 0$ , then  $x^{-n'}$  is positive by Definition 4.2.6, hence  $x^{-n'} > 0$  by 4.2.8, proving  $0 < x^n$  part of the inequality.

For  $x^n \leq y^n$ , aka  $x^{-n'} \leq y^{-n'}$ , note:

$\text{\textit{\$}} \text{\textit{\begin{align}}} y^{n'} \&\leq x^{n'} \text{\textit{By 4.3.10 (c)}} \mid 1 \&\leq \frac{x^{n'}}{y^{n'}} \text{\textit{By 4.2.9 (e) and Lemma 1}} \mid x^{-n'} \&\leq y^{-n'} \text{\textit{By 4.2.9 (e) and Lemma 1}} \text{\textit{end{align}}} \text{\textit{\$}}$

As desired.

c)

We start with Lemma 4:  $a^n = 1 \iff a = 1$ , for  $a \in \mathbb{Q}$ ,  $n \in \mathbb{Z} - \{0\}$ .

We proceed by cases. If  $n$  is positive, then we can induct on  $n$ . Let  $n = 1$ , then  $a^1 = 1$  implies  $a = 1$  as desired. For  $P(n+1)$ , note  $a^{n+1} = 1$  implies  $a \cdot a^n = 1$  which by  $P(n)$  implies  $a = 1$  as desired.

If  $n$  is negative, let  $n' = -n \in \mathbb{N}$ , then:

$$\begin{aligned} a^{-n} &= 1 \mid a^{-n} a^{n'} = a^{n'} \mid 1 = a^{n'} \mid 1 = a \end{aligned}$$

As desired.

Hence:

$$\begin{aligned} x^n &= y^n \mid x^n (\frac{1}{y})^n = 1 \quad \text{By Lemma 3} \mid (x/y)^n = 1 \quad \text{By 4.3.12 (a)} \mid x/y = 1 \quad \text{By Lemma 4} \mid x = y \end{aligned}$$

As desired.

d)

We start with Lemma 5:  $|1/a| = 1/|a|$ ,  $a \neq 0$ .

We proceed with two cases,  $a$  is positive and  $a$  is negative.

Assume  $a$  is positive, then easy to verify  $1/a$  is also positive, hence  $|1/a| = 1/a = 1/|a|$  as desired.

Assume  $a$  is negative, then easy to verify  $1/a$  is negative, hence  $|1/a| = 1/(-a) = 1/|a|$  as desired.

Assume  $n$  is negative, then let  $n' = -n \in \mathbb{N}$ , then:

$$\begin{aligned} |x^{-n}| &= \left| \frac{1}{x^{n'}} \right| \quad \text{By Definition 4.3.11} \mid = \frac{1}{|x^{n'}|} \quad \text{By Lemma 5} \mid = \frac{1}{|x|^{n'}} \quad \text{By 4.3.10 (d)} \mid = |x|^{-n} \quad \text{By Definition 4.3.11} \end{aligned}$$

As desired.

### Exercise 4.3.5

Let  $P(n)$  be  $2^n \geq n$ . Then,  $P(1)$  is true as  $2 > 1$ . For  $P(n+1)$ , note  $2^{n+1} = 2^n \times 2 \geq n \times 2 > n$  (as  $n \times 2 - n = n \geq 1 > 0$ ) as desired.

### Proposition 4.4.5

Let  $\epsilon > 0$ ,  $\epsilon \in \mathbb{Q}$ . Then, for contradiction assume no such  $x$  exists. Then, the statement  $x^2 < 2 < (x + \epsilon)^2$  is false for all nonnegative  $x \in \mathbb{Q}$ . This implies whenever  $x^2 < 2$ ,  $(x + \epsilon)^2 \leq 2$ . (Otherwise  $x^2 < 2 < (x + \epsilon)^2$  would be true).



By Proposition 4.4.4, since  $x + \epsilon \in \mathbb{Q}$ , it cannot hold that  $(x + \epsilon)^2 = 2$ , hence it must hold that  $(x + \epsilon)^2 < 2$ .

We set  $x = 0$ . Since  $x^2 < 2$  and  $x$  is non-negative, it holds that  $(0 + \epsilon)^2 < 2$ , hence  $\epsilon^2 < 2$ .

Similarly, we set  $x = \epsilon$ , then since  $\epsilon^2 < 2$ , it holds that  $(\epsilon + \epsilon)^2 \leq 2$ , aka  $(2\epsilon)^2 < 2$ .

We then claim  $(n\epsilon)^2 < 2$  for  $n \in \mathbb{N}$ .  $P(0)$  is already proven. For  $P(n + 1)$ , note  $(n\epsilon)^2 < 2$  by inductive hypothesis, hence set  $x = n\epsilon$ , then  $(n\epsilon + \epsilon)^2 < 2$  implying  $((n + 1)\epsilon)^2 < 2$  as desired.

However, by proposition 4.4.1, there exists a natural number  $n > 2/\epsilon$ , as  $2/\epsilon$  is a rational number. Hence, it holds that, for that particular  $n$ ,  $n\epsilon > 2$ , hence  $(n\epsilon)^2 > 4$ . However  $(n\epsilon)^2 < 2 < 4$ , contradiction as desired.

My comment: basically the proof bounds  $n\epsilon$  and constructs a large enough  $n$  to show that  $n\epsilon$  cannot be bounded to create contradiction. Proposition 4.4.1 is probably not strictly needed, and other estimation would suffice in the construction of sufficiently large  $n$ .

### Exercise 4.4.1

Consider  $x \geq 0$ , then  $x = a/b$  for some  $a, b \in \mathbb{N}$ .

$$\begin{aligned} n \leq x &< n + 1 \mid n \leq a/b < n + 1 \mid bn \leq a < b(n + 1) \end{aligned}$$

We claim there exists  $n \in \mathbb{N}$  s.t. the above inequality is true. Note  $bn \leq a$  iff  $bn + c = a$  for some  $c \in \mathbb{N}$ , then by proposition 2.3.9 there exists such  $n, c \in \mathbb{N}$  s.t.  $bn + c = a$  and furthermore  $c < b$ .

We claim that the found  $n$  satisfies  $a < b(n + 1)$ . Note  $b(n + 1) - a = b(n + 1) - (bn + c) = b - c > 0$  as desired.

For  $x < 0$ , let  $x' = -x$ , then there exists  $n' \in \mathbb{N}$  s.t.  $n' \leq x' < n' + 1$ , aka  $-n' - 1 < x \leq -n'$ .

Assume  $x \neq n'$ , then  $x < n'$ . We let  $n = -n' - 1$ , then  $x < -n' = n + 1$ , and  $-n' - 1 = n + 1 - 1 = n$ , hence  $-n' - 1 < x$  implies  $n < x$ , which implies  $n \leq x$  as desired.

Assume  $x = n'$ , then  $x \in \mathbb{Z}$ . We simply let  $n = x$  as desired.

For uniqueness, say we have  $n \leq x < n + 1$ ,  $m \leq x < m + 1$ . WLOG say  $n < m$ . Then  $n < m \leq x < n + 1 < m + 1$ . Hence  $n < m < n + 1$ . Hence  $m \in \mathbb{Z}$  is between two consecutive integers, contradiction.

We can prove that  $n < m < n + 1$  does indeed lead to contradiction. Since  $n < m$ ,  $m - n = d$  for some positive integer  $d$ . Similarly  $n + 1 - m = d'$  for some positive  $d'$ . Note  $n + 1 - m = -(m - n) + 1 = -d + 1 = d'$ , hence  $d' + d - 1 = 0$ . Since  $d - 1$  is natural by Lemma 2.2.10, by Corollary

2.2.9,  $d' = d - 1 = 0$ , hence  $d = 1$ . However,  $d' + (d - 1) = (d' - 1) + d$ , hence similarly  $d = 0$ , contradiction as desired.

## Exercise 4.4.2

a)

We claim  $\forall n \in \mathbb{N}, a_n \geq k$  for all  $k \in \mathbb{N}$ .

We induct on  $k$ . Base case  $P(0)$  is true as  $\forall n \in \mathbb{N}, a_n \in \mathbb{N}$ , and a natural number is either equal or greater than  $0$  as desired.

For  $P(k + 1)$ , note:

$$\begin{aligned} \forall n \in \mathbb{N}, a_n > a_{n+1} \geq k &\quad \text{By } P(k) \\ \forall n \in \mathbb{N}, a_n &\geq k + 1 \end{aligned}$$

As desired.

Hence, we claim  $a_1 \geq a_0$ . Yet  $a_0 > a_1$ , contradiction as desired.

b)

Principle of infinite descent doesn't work if the sequence can take integer values. As, integer is not bounded below. Positive rational value also doesn't suffice. Consider the sequence  $a_0 = 1, a_{n+1} = a_n / 2$ . Easy to prove for any arbitrary  $n$ ,  $a_n > 0$ .

## Exercise 4.4.3

A natural number  $p$  is either even or odd as say  $p = 2k + r$ , then there exists  $k \in \mathbb{N}, 0 \leq r < 2$  s.t.  $p = 2k + r$ . Since  $0 \leq r < 2$ ,  $r = 0, 1$  as desired.

A natural number  $p$  cannot be both even and odd, as, say  $p = 2a = 2b + 1$  for some  $a, b \in \mathbb{N}$ , then  $2a = 2b + 1, 2(a - b) = 1, a - b = 1/2$ . However, integer subtraction is closed, yet  $1/2$  is not an integer, contradiction as desired.

We can show  $1/2$  is not an integer, as, assume  $1/2$  is an integer, then there exists  $a/1 = 1/2$ , then  $1 = 2a$  for some integer  $a$ . Easy to verify via induction  $\forall a \in \mathbb{N}, 2a \neq 1$ , and similarly  $\forall a \in \mathbb{N}, -1 \neq 2a$  by trichotomy of order as desired.

If  $p$  is odd, then  $p = 2k + 1, p^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2k' + 1$  as desired.

Since  $p^2 = 2q^2$ , then assume  $q = p$ , then  $p^2 = 2p^2$  implying  $pp = 0$  hence  $p = 0$ , contradiction as  $p$  is positive integer.

Similarly assume  $q > p$ , then  $q^2 > pq$  and  $pq > p^2$ , hence  $q^2 > p^2, p^2 = 2q^2 > 2p^2$  implying  $p^2 > 2p^2$ , hence  $0 > p^2$ , hence  $0 = p^2 + d$  for some  $d \in \mathbb{N}$ , hence  $p^2 = 0$ ,

hence  $\$p = 0\$, contradiction as desired.$

### Exercise 5.1.1

Choose  $\epsilon = 1$ , then for some  $N$ ,  $|a_j - a_k| \leq 1$  for all  $j, k \geq N$ .

Then,  $a_{1}^{N-1}$  is bounded by some  $M'$  Lemma 5.1.14. We claim  $a_N^{\infty}$  is also bounded. Pick  $a_k$  s.t.  $k \geq N$ , then  $|a_k - a_N| \leq 1$ . Then, note  $|a_k| - |a_N| \leq |a_k - a_N|$ , then  $|a_k| \leq |a_k - a_N| + |a_N| \leq 1 + |a_N|$  as desired.

Hence, pick  $M = M' + 1 + |a_N|$  as desired.

Proof for  $|x| - |y| \leq |x - y|$ :

Note  $x = (x - y) + y$ , hence  $|x| \leq |x - y| + |y|$  by triangle inequality. Therefore  $|x| - |y| \leq |x - y|$  as desired.

### Exercise 5.2.1

Note, for any  $\epsilon > 0$ , there exists  $N, N'$  s.t.

$$\begin{aligned} \forall i, j \geq N, |a_i - a_j| \leq \epsilon \quad \forall i \geq N', |b_i - b_j| \leq \epsilon' \end{aligned}$$

Then, pick  $N'' = \max(N, N')$ , it holds that:

$$\begin{aligned} \forall i, j \geq N'', |b_i - b_j| &\leq |b_i - a_i| + |a_i - b_j| \quad \text{By triangle inequality} \\ \forall i, j \geq N'', |b_i - b_j| &\leq \epsilon' + |a_i - b_j| \quad \text{As } |b_i - a_i| \leq \epsilon' \quad \forall i, j \geq N'', \\ |b_i - b_j| &\leq \epsilon' + |a_i - a_j| + |a_j - b_j| \quad \text{By triangle inequality} \\ \forall i, j \geq N'', |b_i - b_j| &\leq \epsilon' + \epsilon + \epsilon' \quad \forall i, j \geq N'', |b_i - b_j| \leq 2\epsilon' + \epsilon \end{aligned}$$

Hence, we simply let  $\epsilon' = \epsilon = \epsilon'' / 3$ , then it holds that:

$$\forall i, j \geq N'', |b_i - b_j| \leq \epsilon''$$

Hence  $(a_n)_{n=1}^{\infty}$  is Cauchy Sequence implies  $(b_n)_{n=1}^{\infty}$  is Cauchy Sequence. Easy to verify equivalent sequence is reflexive, symmetric and transitive, hence the other direction follows from symmetry as desired.

### Exercise 5.2.2

Assume  $a$  is bounded by some  $M$ , then:

$$\begin{aligned} \forall i \geq N, |a_i - b_i| \leq \epsilon \quad \text{As } a, b \text{ eventually } 1 \text{ close} \\ \forall i \geq N, |b_i| - |a_i| \leq |a_i - b_i| \leq 1 \quad \text{By reverse triangle inequality} \\ \forall i \geq N, |b_i| - |a_i| \leq 1 \quad \forall i \geq N, |b_i| \leq 1 + |a_i| \quad \forall i \geq N, |b_i| \leq 1 + M \\ \text{As } a \text{ is bounded by some } M \end{aligned}$$

By Lemma 5.1.14,  $(b_n)_{n=1}^{N-1}$  is bounded by some  $M'$ , hence  $(b_n)_{n=1}^{\infty}$  is bounded by  $\max(M, 1 + M')$  as desired.

The other direction follows similarly.

### Exercise 5.3.1

Reflexivity:

Note  $\forall n \geq 1, |a_n - a_n| = 0 \leq \epsilon$  for any  $\epsilon > 0$  as desired.

Symmetry:

If  $\forall n \geq N, |a_n - b_n| \leq \epsilon$ , then  $\forall n \geq N, |b_n - a_n| = |a_n - b_n| \leq \epsilon$  as desired.

Transitivity:

Say  $\forall n \geq N, |a_n - b_n| \leq \epsilon$ ,  $\forall n \geq M, |b_n - c_n| \leq \delta$ , then  $\forall n \geq \max(N, M), |a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq \epsilon + \delta$ . Remains to set  $\epsilon = \delta = \epsilon'/2$ , then  $\forall n \geq \max(N, M), |a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq \epsilon + \delta = \epsilon'$ . as desired.

### Exercise 5.3.2

Show  $xy$  is a real number:

By proposition 4.3.7 h), say  $\forall i, j \geq N, |a_i - a_j| \leq \epsilon$ ,  $\forall i, j \geq M, |b_i - b_j| \leq \delta$ , then:

$$\forall i, j \geq \max(N, M), |a_i b_i - a_j b_j| \leq \epsilon |b_i| + \delta |a_i| + \epsilon \delta$$

Note by Lemma 5.1.15,  $|a_i| \leq A, |b_i| \leq B$  for some  $A, B \in \mathbb{Q}$ . Hence:

$$\forall i, j \geq \max(N, M), |a_i b_i - a_j b_j| \leq \epsilon B + \delta A + \epsilon \delta$$

We wish that:

$$\forall i, j \geq N', |a_i b_i - a_j b_j| \leq \epsilon'$$

Hence, we let  $N' = \max(N, M), \epsilon = \epsilon'/(3B), \delta = 2\epsilon' / 3(A + \epsilon)$ , then:

$$\begin{aligned} & \forall i, j \geq \max(N, M), |a_i b_i - a_j b_j| \leq \epsilon B + \delta A + \epsilon \delta \\ & \forall i, j \geq N', |a_i b_i - a_j b_j| \leq \frac{\epsilon'}{3B} B + \frac{2\epsilon'}{3(A + \epsilon)} (A + \epsilon) + \epsilon \delta \\ & \forall i, j \geq N', |a_i b_i - a_j b_j| \leq \frac{\epsilon'}{3} + \frac{2\epsilon'}{3} = \epsilon' \end{aligned}$$

As desired.

Show  $x = x' \rightarrow xy = x'y$

Need to show:

$$\forall n \geq N, |a_n b_n - a_n' b_n| \leq \epsilon$$

Note  $\forall n \geq N, |a_n - a_n'| \leq \epsilon$  as  $x = x'$ .

Hence, let  $N' = N, \epsilon = \epsilon' / B$  for which  $B$  is bounded by  $B$  (exists by Lemma 5.1.15), then:

$$\begin{aligned} |a_n b_n - a_n' b_n| &= |b_n| |a_n - a_n'| \leq B \epsilon' / B = \epsilon' \end{aligned}$$

As desired.

### Exercise 5.3.3

If direction

It holds that  $|a - b| \leq \epsilon$  for any  $\epsilon \in \mathbb{Q}, \epsilon > 0$ .

Assume  $a \neq b$ , then  $|a - b| \neq 0$ . Easy to verify  $|a - b| > |a - b| / 2$ . Hence, it cannot hold that  $|a - b| \leq |a - b| / 2$  by trichotomy of order, contradiction as desired.

Only if direction

Since  $a = b$ ,  $|a - b| = 0 \leq \epsilon$  for any  $\epsilon > 0$  as desired.

### Exercise 5.3.4

Since  $(b_n)_{n=0}^{\infty}$  is equivalent to  $(a_n)_{n=0}^{\infty}$ ,  $(b_n)_{n=0}^{\infty}$  is eventually  $\epsilon$ -close to  $(a_n)_{n=0}^{\infty}$  for some  $\epsilon > 0$ . Rest follows from Exercise 5.2.2.

### Exercise 5.3.5

Need to show there exists  $N$  s.t.  $\forall n \geq N, |1/n - 0| \leq \epsilon$  for any  $\epsilon > 0$ .

Note  $|1/n - 0| = 1/n \leq 1/N$ , and,  $1/N \leq \epsilon \Leftrightarrow 1/\epsilon \leq N$ . By Proposition 4.4.1 such  $N$  exists as desired.

### Exercise 5.4.1

Say  $x$  is negative, then  $x = (a_n)_{n=1}^{\infty}$  for some  $(a_n)_{n=1}^{\infty}$  negatively bounded. Easy to verify  $-x = -(a_n)_{n=1}^{\infty}$  is positively bounded. Other direction follows similarly.

Say  $x = \lim_{n \rightarrow \infty} (a_n)_{n=1}^{\infty}, y = \lim_{n \rightarrow \infty} (b_n)_{n=1}^{\infty}$ , with both sequence positively bounded away from zero. Then

it holds that  $a_n \geq A$ ,  $b_n \geq B$  for some  $A, B > 0$ ,  $A, B \in \mathbb{Q}$ . Hence  $a_n + b_n \geq A + B > 0$  as desired. Similar reasoning for  $xy$  being positive.

Other Part: Note we either have  $x = 0$  or  $x \neq 0$ . Say  $x = 0$  aka  $x = \lim_{n \rightarrow \infty} (0)_{n=1}^{\infty}$ . Easy to verify  $x$  is neither positive nor negative, hence the trichotomy.

Say  $x \neq 0$ , then we claim  $x$  is either positive or negative. By Lemma 5.3.14  $x = \lim_{n \rightarrow \infty} (a_n)_{n=1}^{\infty}$  for which  $\forall n$ ,  $|a_n| \geq c$  for some rational  $c > 0$ .

As  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence, it holds that  $\exists N$ ,  $\forall n \geq N$ ,  $|a_n - a_N| \leq \epsilon$  for all  $\epsilon > 0$ .

Since  $a_N \neq 0$  (as  $|a_N| > c$ ), either  $a_N < 0$  or  $a_N > 0$ . WLOG assume  $a_N < 0$ , then  $a_N = |a_N| \geq c$ . Then WLOG assume  $a_n - a_N \geq 0$ , we then have  $a_n - a_N \leq \epsilon$ . Setting  $\epsilon = c/2$  yields  $a_n - a_N \leq c/2$ . Since  $|a_N| \geq c$  and  $a_N < 0$ ,  $a_N \leq -c$ , hence:

$$\begin{aligned} a_n - a_N &\leq c/2 \implies a_n \leq c/2 + a_N \leq c/2 - c \leq -c/2 \end{aligned}$$

Hence  $\exists N$ ,  $\forall n \geq N$ ,  $a_n \leq -c/2$ . Hence, we define  $(b_n)_{n=1}^{\infty}$  s.t.  $\forall n < N$ ,  $b_n = -1$ ,  $\forall n \geq N$ ,  $b_n = a_n$ . Easy to verify  $|b_n| \leq \min(-1, -c/2)$  (aka  $(b_n)_{n=1}^{\infty}$  is negatively bounded from  $0$ ),  $(b_n)_{n=1}^{\infty}$  is Cauchy and  $(a_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty}$  as desired.

## Exercise 5.4.2

Note by Proposition 5.3.11, Proposition 4.1.6 holds for  $\mathbb{R}$ .

a)

Consider  $x - y$ , then we either have  $x - y > 0$ ,  $x - y = 0$ ,  $x - y < 0$ , hence we either have  $x > y$ ,  $x = y$ ,  $x < y$  as desired.

b)

One has  $x < y$  iff  $x - y$  is negative. Note  $y - x = -(x - y)$ . By Proposition 5.4.4,  $y - x$  positive, hence  $y > x$  as desired.

Other direction follows similarly.

c)

If  $x < y$ , then  $y - x$  is positive, similarly  $z - y$  is positive. Hence  $(y - x) + (z - y) = z - x$  is positive (by Proposition 5.4.4), hence  $x < z$  as desired.

d)

If  $x < y$ , then  $y - x$  is positive. Note  $(y + z) - (x + z) = y - x$ , which is also positive, hence  $y + z > x + z$  as desired.

e)

Proven in the text.

### Exercise 5.4.3

Uniqueness follows similar reasoning from Exercise 4.4.1.

For existence, we either have  $x < 0$ ,  $x = 0$ ,  $x > 0$ .

Consider the case  $x > 0$ , then we argue by contradiction. Assume  $\neg (\forall x \in \mathbb{R}^+, \exists N \in \mathbb{Z}, N \leq x < N + 1)$ , then  $\exists x \in \mathbb{R}^+, \forall N \in \mathbb{Z}, x < N \vee x \geq N + 1$ . We claim for that particular  $x$ ,  $x \geq n$  for all  $n \in \mathbb{N}$ . We prove by induction.

For  $P(0)$ ,  $x > 0$  hence  $x \geq 0$  as desired.

For  $P(n+1)$ , note we have  $x < n \vee x \geq n + 1$ . Easy to verify  $x < n$ ,  $x \geq n + 1$  are mutually exclusive. Note by  $P(n)$ ,  $x \geq n$ . Hence  $x < n$  cannot hold, therefore  $x \geq n + 1$  as desired.

However, there exists  $N \in \mathbb{Z}^+$  s.t.  $x \leq N$  by Proposition 5.4.12. Hence we have  $x \geq N + 1$  (by  $P(N + 1)$ ) and  $x \leq N$ , contradiction.

(Above proof similar to in-text proof for Proposition 4.4.5 and the inductive proof for principle of infinite descent).

For the case  $x = 0$ ,  $0 \leq 0 < 1$  as desired.

For the case  $x < 0$ , same reasoning in Exercise 4.4.1 solution applies.

Note: A direct proof using Proposition 4.4.1 and Corollary 5.4.10 would also suffice. Albeit I had enough suffering in this section already and I will not bother with it. The indirect proof is just much more elegant.

### Exercise 5.4.4

Easy to show for all  $N \in \mathbb{N}$ ,  $1/N > 0$ .

Remains to show  $\exists N \in \mathbb{N}, x > 1/N$  for any  $x \in \mathbb{R}^+$ . Note  $x > 1/N \iff N > 1/x \iff N_1 > 1/x$ . Since  $1, 1/x$  are positive real numbers,  $N$  exists by Corollary 5.4.13 as desired.

### Exercise 5.4.5

Say  $q = a/b$ , then:



$$\begin{aligned} x < a/b < y &\wedge bx < a < by \wedge 0 < a - bx < b(y - x) \end{aligned}$$

Now, note by Exercise 5.4.4, there exists  $y - x > 1/N$  for some  $N \in \mathbb{Z}^+$ . We make leap of faith and claim  $b = N$ , then we remains to find  $a \in \mathbb{Z}$ . Note  $y - x > 1/b$  implies  $b(y - x) > 1$ . Hence  $a - bx \leq 1$  implies  $a - bx < b(y - x)$

$$0 < a - bx \leq 1 \wedge bx < a \leq bx + 1 \end{aligned}$$

We proceed to prove in general  $\exists a \in \mathbb{Z}, r < a \leq r + 1$  for all  $r \in \mathbb{R}$ . Note by Exercise 5.4.3, there exists  $n \in \mathbb{Z}$  s.t.  $n \leq r < n + 1$ . We claim  $a = n + 1$  is suitable, as  $n \leq r \rightarrow n + 1 \leq r + 1 \rightarrow a \leq r + 1$ , and  $r < n + 1 \rightarrow r < a$ , as desired.

My comment: Too many leap of faith required for this question...

## Exercise 5.4.6

WLOG we only prove  $<$  case as  $\leq$  case is similar

If direction

We either have  $x - y \geq 0$  or  $x - y < 0$ .

WLOG  $x - y \geq 0$ , then  $|x - y| = x - y$ . Hence  $x - y < \epsilon$  implies  $x < y + \epsilon$ . Note  $x - y \geq 0$  implies  $y \leq x$ , hence  $y - \epsilon < y \leq x < y + \epsilon$  as desired.

Only if direction

Note  $y - \epsilon < x < y + \epsilon$  iff  $-\epsilon < x - y < \epsilon$ . We argue by contradiction. Assume  $|x - y| \geq \epsilon$ , then either  $x - y \geq \epsilon$  or  $-\epsilon \geq x - y$ . First one contradict  $x - y < \epsilon$  and second one contradict  $-\epsilon < x - y$  as desired.

## Exercise 5.4.7

First statement

If direction

Say  $x \leq y$ , then  $x \leq y < y + \epsilon$  hence  $x \leq y + \epsilon$  as desired.

Only if direction

Assume  $x > y$ , then  $x - y = c$  for some  $c > 0$ . Then, we claim  $x > y + c/2$ , as  $x - (y + c/2) = c - c/2 = c/2 > 0$ . Hence let  $\epsilon = c/2$ , it cannot hold that  $x \leq y + \epsilon$  as desired.

Second Statement

If direction

If  $x = y$  then  $x - y = 0$ . Hence  $|x - y| = x - y = 0 < \epsilon \leq \epsilon$  as desired.



Only if direction

Say we have  $|x - y| \leq \epsilon$ , then by Exercise 5.4.6 it holds that  $y - \epsilon \leq x \leq y + \epsilon$ . Hence,  $x \leq y + \epsilon$  for any  $\epsilon > 0$  implies  $x \leq y$  by the first statement.  $y - \epsilon \leq x \rightarrow y \leq y + \epsilon$  for any  $\epsilon > 0$  implies  $y \leq x$ . Hence,  $x \leq y$  and  $y \leq x$  implies  $x = y$  as desired.

### Exercise 5.4.8

We prove the contraposition. Assume  $\lim_{n \rightarrow \infty} a_n > x$ , then by Proposition 5.4.14 there exists  $q \in \mathbb{Q}$  s.t.  $\lim_{n \rightarrow \infty} a_n > q > x$ . Hence,  $\lim_{n \rightarrow \infty} a_n > \lim_{n \rightarrow \infty} q$ . Therefore  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} q$ , hence by Corollary 5.4.10  $\exists n \geq 1, a_n > q > x$ . Hence  $\nexists n \geq 1, a_n \leq x$  as desired.

### Exercise 5.5.1

Assume  $-M \neq \inf(-E)$ , then, either:

- $-M$  is not a lower bound
- There exists  $-M' > -M$  which is a lower bound

Assume  $-M$  is not a lower bound, then  $\exists -x \in -E, -x < M$ . Hence it holds that  $x \geq M$ . Note  $-x \in -E \rightarrow x \in E$ . Hence  $x \geq M$  means  $M \neq \sup(E)$ , contradiction.

Assume  $-M' > -M$  exists which is a lower bound, then  $\forall -x \in -E, -x \geq -M'$ . Hence  $\forall x \in E, -x \geq -M'$ , hence  $\forall x \in E, x \leq M'$ . Therefore  $M' < M$  is also an upper bound for  $E$ , contradiction.

Note: it may seem trivial that  $-x \in -E \rightarrow x \in E$ , however it can still be proven.  $x \in E \rightarrow -x \in -E$  follows from Axiom of substitution.  $-x \in -E \rightarrow x \in E$  as for every  $-x \in -E$  there exists  $x' \in E$  s.t.  $-x' = -x$  by Axiom of substitution, hence  $x = x' \in E$  as desired.

### Exercise 5.5.2

Similar to Exercise 5.4.3 Solution.

Say  $\nexists L < m \leq k, m/n$  is upper bound and  $(m - 1) / n$  is not upper bound, then  $\forall L < m \leq k, m/n$  is not upper bound or  $(m - 1) / n$  is upper bound.

Now we induct on  $m$  and claim  $L \leq m \leq k \rightarrow m/n$  is not upper bound.

For base case  $m = L$ ,  $L / n$  is not upper bound as desired.

For  $P(m + 1)$ , note if  $m + 1 \leq k$ , then either  $(m + 1) / n$  is not upper bound or  $m/n$  is upper bound. Since  $m/n$  is not upper bound by  $P(m)$ ,  $(m + 1) / n$  is not upper bound has to hold as

desired, closing the induction.

Hence, since  $L \leq k \leq k$ , it holds that  $k/n$  is not an upper bound, contradiction.

### Exercise 5.5.3

Since  $(m-1)/n$  is not upper bound, there exists element  $x \in E$  s.t.  $x > (m-1)/E$ . Similarly there exists  $x' \in E$ ,  $x' > (m'-1)/E$ . Let  $X = \max(x, x')$ , easy to verify  $X \in E$ ,  $X > (m-1)/E$ ,  $X > (m'-1)/E$ .

WLOG say  $m > m'$ , then:

$$\begin{aligned} m/n &\geq X > (m-1)/n \quad m'/n \geq X > (m'-1)/n \quad m/n > m'/n \geq x > (m-1)/n > \\ &(m'-1)/n \end{aligned}$$

Hence we have  $m'/n > (m-1)/n$  and  $m/n > (m'-1)/n$ , therefore  $m' > m-1$ ,  $m > m'-1$ , hence by Proposition 2.2.12 e),  $m' \geq m$ ,  $m \geq m'$ , therefore  $m = m'$ , contradiction as desired.

### Exercise 5.5.4

By Corollary 5.4.13, for any  $\epsilon$  there exists  $M \in \mathbb{Z}^+$  s.t.  $1 < \epsilon M$ , hence there exists  $M$  s.t.  $1/M < \epsilon$ .

Therefore,  $|q_n - q_{n'}| \leq 1/M < \epsilon$  as desired.

To prove that  $\forall M \geq 1, |q_M - S| \leq 1/M$ , note:

$$\begin{aligned} \forall M \geq 1, \forall n \geq M, |q_n - q_M| &\leq 1/M \quad \text{By Given Property} \\ \forall M \geq 1, \forall n \geq M, q_M - 1/M &\leq q_n \leq q_M + 1/M \quad \text{By Exercise 5.4.6} \end{aligned}$$

Now define  $q'_n = q_n$  if  $n \geq M$ ,  $q'_M$  if  $n < M$ , easy to verify  $(q'_n)^{\infty}_{n=1}$  is Cauchy and  $\lim_{n \rightarrow \infty} (q'_n)^{\infty}_{n=1} = \lim_{n \rightarrow \infty} (q_n)^{\infty}_{n=1}$ .

Note  $\forall n \geq 1, q_M - 1/M \leq q'_n \leq q_M + 1/M$ , hence by Exercise 5.4.8,  $q_M - 1/M \leq S \leq q_M + 1/M$ , hence  $|q_M - S| \leq 1/M$  follows from Exercise 5.4.6.

### Exercise 5.5.5

We start by proving a stronger version of Proposition 5.5.14. Given  $x, y \in \mathbb{R}$ ,  $x < y$ , we can always find  $r \in \mathbb{Q}$ ,  $r \neq 0$  s.t.  $x < r < y$ .

If  $x, y > 0$  or  $x, y < 0$ , then apply Proposition 5.5.14, easy to verify  $r > 0$  or  $r < 0$  correspondingly.

WLOG if  $x \leq 0, y > 0$ , apply Proposition 5.5.14 to find  $y/2 < r < y$ , as desired.

Denote  $x = \sqrt{2}$  as the real number such that  $x > 0$ ,  $x^2 = 2$ . It exists by Proposition 5.5.12.

Then, by stronger version of Proposition 5.4.14, there exists  $r \in \mathbb{Q}$  s.t.  $x / \sqrt{2} < r < y / \sqrt{2}$ . Hence,  $x < \sqrt{2}r < y$ . We claim that  $\sqrt{2}r$  is irrational. Assume  $\sqrt{2}r$  is rational, then  $\sqrt{2}r / r = \sqrt{2}$  (we can divide as  $r \neq 0$ ), hence  $\sqrt{2}$  is rational as division is closed, contradicting Proposition 4.4.4.

## Exercise 5.6.1

a)

We denote  $E: \{y' \in \mathbb{R}, y' \geq 0, (y')^n \leq x\}$ . Hence  $y = x^{1/n} = \sup(E)$

We argue by contradiction. Assume  $y^n \neq x$ , then either  $y^n < x$  or  $y^n > x$ .

Assume  $y^n < x$ , then we claim we can find  $0 < \epsilon < 1$  s.t.  $(y + \epsilon)^n \leq x$ , hence  $y + \epsilon \in E$ , yet  $y + \epsilon > \sup(E)$ , contradiction.

For construction of  $\epsilon$ , we establish estimation for  $(y + \epsilon)^n$ :

$$\begin{aligned} (y + \epsilon)^n &= \sum_{j=0}^n \binom{n}{j} y^j \epsilon^{n-j} && \text{Binomial Theorem} \\ &= y^n + \sum_{j=0}^{n-1} \binom{n}{j} y^j \epsilon^{n-j} && \\ &= y^n + \sum_{j=0}^{n-1} \binom{n}{j} y^j \epsilon^{n-j-1} \epsilon && \\ &\leq y^n + \sum_{j=0}^{n-1} \binom{n}{j} y^j \epsilon^{n-j-1} && \text{As } \epsilon \leq 1, \epsilon^i \leq 1 \text{ for } i \in \mathbb{N} \\ &= y^n + k_{n,y} \epsilon && \text{By factoring out } \epsilon \end{aligned}$$

With  $k_{n,y} \in \mathbb{R}^+$  depending on  $n, y$ . We remain to show that we can find  $0 < \epsilon < 1$  such that  $y^n + k_{n,y} \epsilon \leq x$ . Since  $y^n < x$ , it is always possible, as desired. (e.g. we can let  $\epsilon = \min(1/2, (x - y^n)/k_{n,y})$ , easy to verify  $0 < \epsilon < 1$  and  $y^n + k_{n,y} \epsilon \leq x$ ).

Note: we take  $\epsilon$  as  $x - y^n$  may exceed  $k_{n,y}$  causing the obvious choice  $\epsilon = (x - y^n)/k_{n,y}$  to fail as  $\epsilon > 1$ .

Similarly, assume  $y^n > x$ , we claim we can find  $0 < \epsilon < 1$  s.t.  $(y - \epsilon)^n > x$ .

Hence,  $y - \epsilon$  is a lower bound of  $E$ , yet  $y - \epsilon < \sup(E)$ , contradiction.

For  $y - \epsilon$  being lower bound: assume  $\exists y' \in E, y - \epsilon < y'$ , then  $(y - \epsilon)^n \leq (y')^n \leq x$ , yet  $(y - \epsilon)^n > x$ , contradiction.

To find  $0 < \epsilon < 1$ , we establish estimation for  $(y - \epsilon)^n$ :

$$\begin{aligned} (y - \epsilon)^n &= \sum_{j=0}^n \binom{n}{j} y^j (-\epsilon)^{n-j} && \text{Binomial Theorem} \\ &= y^n + \sum_{j=0}^{n-1} \binom{n}{j} y^j (-\epsilon)^{n-j} && \\ &= y^n + \sum_{\substack{j=0 \\ \text{odd}}}^{n-1} \binom{n}{j} y^j (-\epsilon)^{n-j} && \text{Discard Positive Terms} \\ &= y^n + \sum_{\substack{j=0 \\ \text{odd}}}^{n-1} \binom{n}{j} y^j \epsilon^{n-j-1} \epsilon && \\ &\geq y^n + \sum_{\substack{j=0 \\ \text{odd}}}^{n-1} \binom{n}{j} y^j \epsilon^{n-j-1} && \text{As } \epsilon \leq 1, -\epsilon^i \geq -1 \text{ for } i \in \mathbb{N} \end{aligned}$$

$$\sum_{n-j \text{ odd}, j=0}^{n-1} \binom{n}{j} y^j \leq y^n - k_{\{n, y\}} \epsilon$$

*By factoring out  $\epsilon$*

Remains to show there exists  $0 < \epsilon < 1$  s.t.  $y^n - k_{\{n, y\}} \epsilon \geq x$ . Since  $y^n > x$ , we similarly take  $\epsilon = \min(1/2, (y^n - x)/k_{\{n, y\}})$  as desired.

Note: Part a) basically shows  $(x^{1/n})^n = x$ , aka cancellation.

b)

Note since  $y^n = x$ ,  $((y^n)^{1/n})^n = (x^{1/n})^n$ . By a),  $((y^n)^{1/n})^n = y^n$ , hence we have  $y^n = (x^{1/n})^n$ , with  $n \in \mathbb{Z}^+$ . By Proposition 5.6.3 & Proposition 4.3.12 c),  $y = x^{1/n}$  as desired.

Note: Part b) basically shows  $(y^n)^{1/n} = y$ , aka cancellation in slightly different order.

c)

I assume it actually means  $x > 0 \Rightarrow x^{1/n} > 0$ .

Note by definition  $x^{1/n} = \sup\{y \in \mathbb{R}, 0 \leq y \text{ and } y^n \leq x\}$ . Since  $0 \in \{y \in \mathbb{R}, 0 \leq y \text{ and } y^n \leq x\}$ , it holds that  $x^{1/n} \geq 0$ . Remains to show  $x^{1/n} \neq 0$ . Assume  $x^{1/n} = 0$ , then  $(x^{1/n})^n = 0$  as  $n \neq 0$ . By Lemma 5.6.6 a),  $(x^{1/n})^n = x$ , hence  $x = 0$ , contradicting  $x > 0$  as desired.

d)

By Proposition 4.3.10 c),  $x^{1/n} > y^{1/n}$  iff  $(x^{1/n})^n > (y^{1/n})^n$ . By a),  $(x^{1/n})^n = x$ ,  $(y^{1/n})^n = y$ . Hence  $(x^{1/n})^n > (y^{1/n})^n$  iff  $x > y$  as desired.

e)

For case  $x > 1$ :

$$\begin{aligned} x > 1 & \Rightarrow x^{k+1} > x^k & \text{By 5.4.7 \& 4.2.9 e)} \\ (x^{1/k})^{k(k+1)} & > (x^{1/(k+1)})^{k(k+1)} & \text{By a), 5.6.3, 4.3.10 a)} \\ x^{1/k} & > x^{1/(k+1)} & \text{By 5.6.3 \& 4.3.10 c)} \end{aligned}$$

Remain follows from simple induction.

Case  $x < 1$  is similar.

$$\begin{aligned} xy & = xy & \mid ((xy)^{1/n})^n & = (x^{1/n})^n (y^{1/n})^n & \text{By a)} \\ & = (x^{1/n} y^{1/n})^n & \text{By 5.6.3 \& 4.3.10 a)} & \mid (xy)^{1/n} & = x^{1/n} y^{1/n} & \text{By 5.6.3 \& 4.3.10 c)} \end{aligned}$$

g)

$$\begin{aligned} x &= x \wedge ((x^{1/n})^{1/m})^{nm} = (x^{1/nm})^{nm} \quad \text{By a), 5.6.3, 4.3.10 a)} \\ (x^{1/n})^{1/m} &= x^{1/nm} \quad \text{By 5.6.3 \& 4.3.10 c)} \end{aligned}$$

## Exercise 5.6.2

Let  $q = a/b$ ,  $r = c/d$ ,  $a, b, c, d \in \mathbb{Z}^+$

$x^q = (x^{1/a})^b$  by Definition 5.6.7. Since  $x > 0$ ,  $x^{1/a} > 0$  by 5.6.6 d), and  $(x^{1/a})^b > 0$  by 4.3.10 c) as desired.

b)

$$\begin{aligned} (x^{1/bd})^{ad+bc} &= (x^{1/bd})^{ad} (x^{1/bd})^{bc} \quad \text{By 4.3.10 a)} \\ x^{a/b+c/d} &= x^{a/b} x^{c/d} \quad \text{By 5.6.7} \end{aligned}$$

$$\begin{aligned} x^{a/b} &= x^{ad/bd} \quad \text{By 5.6.8} \\ ((x^{a/b})^{1/d})^d &= (x^{a/bd})^d \quad \text{By 5.6.6 a), 5.6.7} \\ (x^{a/b})^{1/d} &= x^{a/bd} \quad \text{By 4.3.10 c)} \\ ((x^{a/b})^{1/d})^c &= (x^{a/bd})^c \\ (\lim_{n \rightarrow \infty} (x^{a/b})^{1/d})^c &= ((x^{1/bd})^a)^c \quad \text{By 5.6.7} \\ (x^q)^r &= (x^{1/bd})^{ac} \quad \text{By 4.3.10 a)} \\ (x^q)^r &= x^{ac/bd} \quad \text{By 5.6.7} \\ (x^q)^r &= x^{qr} \end{aligned}$$

c)

$$\begin{aligned} x^{-q} &= x^{-a/b} = (x^{1/b})^{-a} = 1/(x^{1/b})^a = 1/x^{a/b} = 1/x^q \end{aligned}$$

d)

Note  $x^{a/b} > y^{a/b}$  iff  $(x^{a/b})^b > (y^{a/b})^b$  by Proposition 4.3.10 c) iff  $x^a > y^a$  by 5.6.9 b) iff  $x > y$  by Proposition 4.3.10 c) as desired.

e)

Let  $q = a/b$ ,  $r = c/d$ ,  $a, b, c, d \in \mathbb{Z}$ ,  $c, d > 0$ . Then  $q > r$  iff  $q - r > 0$  iff  $ad - bc > 0$  iff  $ad > bc$ .

WLOG say  $x > 1$  then  $x^q = x^{ad/bd} > x^{bc/bd} = x^r$  iff  $ad > bc$  by 5.6.6 e) as desired.

## Exercise 5.6.3

Assume  $x = 0$ , easy to verify  $LHS = RHS = 0$ .

Assume  $x > 0$ , then by Lemma 5.6.9 b),  $RHS = x$ .  $LHS = x$  by definition of absolute value as desired.

Assume  $x < 0$ , then  $x = -y$  for some  $y > 0$ . Remains to show  $|-y| = ((-y)^2)^{1/2}$ . Note  $LHS = y$  by definition. For RHS,  $(-y)^2 = y^2$ , and  $(y^2)^{1/2} = y$  by 5.6.9 b) as desired.

## Exercise 6.1.1

We fix  $n$  and induct on  $m$ .

For  $m = n + 1$ , clearly  $a_m = a_{n+1} > a_n$  as desired.

For  $m+1$ , note  $a_{m+1} > a_m$ , and  $a_m > a_n$  by inductive hypothesis as desired.

### Exercise 6.1.2

$(a_n)_{n=m}^{\infty}$  converges to  $L$  iff  $(a_n)_{n=m}^{\infty}$  is eventually  $\epsilon$ -close to  $L$  for any  $\epsilon > 0$ , which happens iff  $(a_n)_{n=N}^{\infty}$  is  $\epsilon$ -close to  $L$  for some  $N \geq m$  any  $\epsilon > 0$ , which happens iff  $\forall \epsilon > 0, \exists N \geq M, \forall n \geq N, |a_n - L| \leq \epsilon$  as desired.

Every iff is ensured by Definition 6.1.5.

### Exercise 6.1.3

Only if:

Since  $(a_n)_{n=m}^{\infty}$  is convergent to  $c$ ,  $\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |a_n - c| \leq \epsilon$ .

Since  $m' \geq m$ , it holds that  $N \geq m'$ , hence  $\forall \epsilon > 0, \exists N \geq m', \forall n \geq N, |a_n - c| \leq \epsilon$ . Therefore  $(a_n)_{n=m'}^{\infty}$  is convergent as desired.

If

We have  $\forall \epsilon > 0, \exists N' \geq m', \forall n \geq N', |a_n - c| \leq \epsilon$ . We choose  $N = \max(N, m)$ , easy to verify  $N \geq m$  and  $N \geq N'$ . Since  $N \geq N'$ ,  $\forall n \geq N, |a_n - c| \leq \epsilon$  implies  $\forall n \geq N', |a_n - c| \leq \epsilon$  as desired.

### Exercise 6.1.4

If

We have  $\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |a_{n+k} - c| \leq \epsilon$ .

Hence, we have  $\forall \epsilon > 0, \exists N + k \geq m, \forall n, n + k \geq N + k, |a_{n+k} - c| \leq \epsilon$ . Let  $N' = N + k$ , we claim  $\forall \epsilon > 0, \exists N' \geq m, \forall n' \geq N', |a_{n'} - c| \leq \epsilon$ . Say given  $n' \geq N'$ , then  $n' = n + k$  for some  $n \geq N' - k = N$ , hence  $|a_{n'} - c| = |a_{n+k} - c| \leq \epsilon$  as desired.

Only if:

We have  $\forall \epsilon > 0, \exists N' \geq m, \forall n' \geq N', |a_{n'} - c| \leq \epsilon$ .

We need to show  $\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |a_{n+k} - c| \leq \epsilon$ .



Similarly, we let  $N = N'$ . Note  $n + k \geq N = N'$ . Hence let  $n' = n + k$ ,  $|a_{n+k} - c| = |a_{n'} - c| \leq \epsilon$  as desired.

Alternative proof (informal):

Note we have  $(a_{n+k})_{n=m}^\infty$  term by term equal to  $(a_n)_{n=m+k}^\infty$ . Hence result follows from Exercise 6.1.3.

Albeit, the problem being... need to show term by term equal yields the same limit. The concept seems to be obvious, as both intuitively represent the same sequence, but proving it may be another story.

### Exercise 6.1.5

We have  $\forall \epsilon' > 0, \exists N' \geq m, \forall n \geq N', |a_n - c| \leq \epsilon'$

We need to show  $\forall \epsilon > 0, \exists N \geq m, \forall i, j \geq N, |a_i - a_j| \leq \epsilon$ .

Let  $\epsilon' = \epsilon / 2$ , then  $\exists N' \geq m, \forall n \geq N', |a_n - c| \leq \epsilon / 2$ . Choose  $N = N'$ , then  $|a_i - a_j| \leq |a_i - L| + |L - a_j| = \epsilon / 2 + \epsilon / 2 = \epsilon$  as desired.

### Exercise 6.1.6

Let  $(a_n)_{n=m}^\infty$  be a Cauchy Sequence of rationals and let  $L = \lim_{n \rightarrow \infty} a_n$ . Then:

$\forall \epsilon > 0, \exists N \geq m, \forall i, j \geq m, |a_i - a_j| \leq \epsilon$

We argue by contradiction. Assume  $(a_n)_{n=m}^\infty$  does not converge to  $L$ , then:

$\exists \epsilon > 0, \forall M \geq m, \exists i' \geq M, |a_{i'} - L| > \epsilon$

Hence, choosing that  $\epsilon$ , and noting  $(a_n)_{n=m}^\infty$  is  $\epsilon/2$  stable, then:

$\exists N \geq m, \forall i, j \geq N, |a_i - a_j| \leq \epsilon/2$

Since  $i' \geq n$ , then  $\forall j \geq N, |a_{i'} - a_j| \leq \epsilon/2$ . By Exercise 5.4.6,  $\forall j \geq N, |a_{i'} - \epsilon/2| \leq |a_j| \leq |a_{i'}| + \epsilon/2$ .

WLOG say  $a_{i'} - L \geq 0$ , then  $a_{i'} - L = |a_{i'} - L| > \epsilon$ . Note  $a_j \geq a_{i'} - \epsilon/2$ . Hence  $a_{i'} - L + a_j > \epsilon + a_{i'} - \epsilon/2$ , which simplifies to  $\forall j \geq N, a_j > L + \epsilon/2$

We construct sequence  $(b_n)_{n=1}^\infty$  s.t.  $\forall j < M, b_j = L + \epsilon/2, \forall j \geq M, b_j = a_j$ . Then it holds that  $\forall j \geq 1, b_j \geq L + \epsilon/2$ . By Exercise 5.4.3,  $L = \lim_{n \rightarrow \infty} b_j \geq L + \epsilon/2$ . Hence  $L \geq L + \epsilon/2 \rightarrow 0 \geq \epsilon/2$ , contradiction as desired.

Comment: Basically we show that if  $(a_n)_{n=m}^\infty$  doesn't converge to  $L$ , then its formal limit is not  $L$  either. The tricky part is usage of Exercise 5.4.6 and how cases should be split. Very tricky to see the sign of  $a_i - L$  matters.

### Exercise 6.1.7

Assume a sequence of rational numbers  $(a_n)_{n=m}^\infty$  is bounded by a real number  $M$ . Then, by proposition 5.4.14 there exists  $r \in \mathbb{Q}$ ,  $M < r < M + 1$ . Hence  $(a_n)_{n=m}^\infty$  is bounded by  $r$  as desired.

### Exercise 6.1.8

a)

Need to show:

$$\forall \epsilon > 0, \exists M \geq m, \forall n \geq M, |a_n + b_n - (x + y)| \leq \epsilon$$

Note we have  $\exists A \geq M, \forall n \geq M, |a_n - x| \leq \epsilon/2$ ,  $\exists B \geq M, \forall n \geq M, |b_n - y| \leq \epsilon/2$ . Hence, choose  $M = \max(A, B)$ , then  $\forall n \geq M, |a_n + b_n - (x + y)| \leq |a_n - x| + |b_n - y| = \epsilon$  as desired.

b)

One can use 4.3.7 h). We present an alternative approach.

We need to prove:  $\forall \epsilon > 0, \exists M \geq m, \forall n \geq M, |a_n b_n - xy| \leq \epsilon$

Fix  $\epsilon$ . By Corollary 6.1.17,  $\forall n, |a_n| \leq U$  for some  $U \in \mathbb{R}^+$ . Then, since  $(a_n)_{n=m}^\infty$  converges to  $x$ , it holds that:

$$\exists A \geq m, \forall n \geq M, |a_n - x| \leq \epsilon/(2U)$$

We then proceed by cases. If  $y \neq 0$ , then:

$$\exists B \geq m, \forall n \geq M, |b_n - y| \leq \epsilon/(2|y|)$$

Hence, choose  $M = \max(A, B)$ , it holds that, for all  $n \geq M$ :

$$\begin{aligned} |a_n b_n - xy| &= |a_n b_n + a_n y - a_n y - xy| \leq |a_n(b_n - y) + y(a_n - x)| \leq \\ &|a_n||b_n - y| + |y||a_n - x| \leq U|b_n - y| + |y||a_n - x| \leq U \epsilon/(2U) + |y| \epsilon/(2|y|) \leq \epsilon \end{aligned}$$

If  $y = 0$ , then, since  $(b_n)_{n=m}^\infty$  converges to  $0$ , it holds that:

$$\exists B \geq m, \forall n \geq M, |b_n| \leq \epsilon/U$$

Therefore:



$$\begin{aligned} |a_n b_n - xy| &= |a_n b_n| \leq |a_n| |b_n| \leq U |b_n| \leq U \epsilon / (U) \leq \epsilon \end{aligned}$$

As desired.

c)

Follows from b) by defining  $b_n = c$  for all  $n$ .

d)

$$\begin{aligned} \lim_{n \rightarrow \infty} (a_n - b_n) &= \lim_{n \rightarrow \infty} (a_n + (-b_n)) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} (-b_n) \quad \text{(By a)} \\ &= \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n \quad \text{(By c)} \end{aligned}$$

e)

Note we have:

$$\forall \epsilon' > 0, \exists N' \geq m, \forall n \geq N', |b_n - y| \leq \epsilon'$$

Then, note by Lemma 5.3.15,  $b_n^{-1}$  is a Cauchy Sequence, by Proposition 6.1.15, a Cauchy Sequence is convergent, by Corollary 6.1.17, a convergent sequence is bounded, hence,  $|b_n^{-1}| \leq U$  for some  $U$ .

We need to show:

$$\forall \epsilon > 0, \exists N \geq M, |1/b_n - 1/y| \leq \epsilon$$

Hence, choose  $\epsilon' = \epsilon y / M$ ,  $N = N'$ , it holds that, for  $n \geq N$ :

$$\begin{aligned} |1/b_n - 1/y| &= \frac{1}{b_n y} |y - b_n| \leq \frac{M}{y} y \epsilon' / M = \epsilon \end{aligned}$$

As desired.

f)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} a_n b_n^{-1} = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n^{-1} \quad \text{(By b)} \\ &= \lim_{n \rightarrow \infty} a_n (\lim_{n \rightarrow \infty} b_n)^{-1} \quad \text{(By c)} \end{aligned}$$

As desired.

g)

WLOG assume  $x \leq y$ , then:

$$\exists A \geq m, \forall n \geq A, |a_n - x| \leq \frac{y - x}{2}$$

$$\exists B \geq m, \forall n \geq B, |b_n - y| \leq \frac{y - x}{2}$$

Hence, for all  $n \geq \max(A, B)$ :

$$a_n \leq x + \frac{y - x}{2} = \frac{x + y}{2} = y - \frac{y - x}{2} \leq b_n$$

Hence  $\forall n \geq \max(A, B), a_n \leq b_n$ .

Since  $(b_n)_{n=m}^{\infty}$  is convergent, it holds:

$$\begin{aligned} \forall \epsilon > 0, \exists N' \geq m, \forall n \geq N', |b_n - y| \leq \epsilon \\ \end{aligned}$$

Choose  $N = \max(N', A, B)$ , it holds that:

$$\begin{aligned} \forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |b_n - y| \leq \epsilon \quad \forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |\max(a_n, b_n) - \max(x, y)| \leq \epsilon \\ \end{aligned}$$

As desired.

h)

$$\begin{aligned} \text{We exploit duality of } \max, \min. \quad \lim_{n \rightarrow \infty} \min(a_n, b_n) &= \lim_{n \rightarrow \infty} -\max(-a_n, -b_n) = -\lim_{n \rightarrow \infty} \max(-a_n, -b_n) \quad \text{(By c)} \\ &= -\max(\lim_{n \rightarrow \infty} -a_n, \lim_{n \rightarrow \infty} -b_n) \quad \text{(By g)} \\ &= \min(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n) \quad \end{aligned}$$

## Exercise 6.1.9

We can have  $a_n = b_n = 1/n$ ,  $a_n/b_n = 1$ . Hence  $LHS = 1$ , yet  $RHS$  is undefined as  $\lim_{n \rightarrow \infty} b_n = 0$ .

## Exercise 6.1.10

Only if direction is trivial as  $\epsilon \in \mathbb{Q} \rightarrow \epsilon \in \mathbb{R}$ .

For if direction, given  $\epsilon \in \mathbb{R}$ , then there exists  $0 < \epsilon' < \epsilon$ , s.t.  $\epsilon' \in \mathbb{Q}$  by Proposition 5.4.14. Hence,  $\exists N' \geq 0, \forall n \geq N', |a_n - b_n| \leq \epsilon'$ . We let  $N = N'$ , then  $\forall \epsilon \in \mathbb{R}^+, \exists N \geq 0, \forall n \geq N, |a_n - b_n| \leq \epsilon' < \epsilon$  as desired.

## Exercise 6.2.1

Reflexivity:

Note  $\mathbb{R}^* = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

Hence, since  $-\infty$ ,  $\infty$  are distinct from all elements in  $\mathbb{R}$  by Definition 6.2.1, it holds that a number  $x \in \mathbb{R}^*$  is either:

- $x \in \mathbb{R}$
- $x = \infty$
- $x = -\infty$

$x \in \mathbb{R}$  case follows from Definition 6.2.3 a) and analogue of reflexivity in  $\mathbb{R}$ .

For  $x = \infty$ ,  $x \leq x$  follows from Definition 6.2.3 b).  $x = -\infty$  case follows from Definition 6.2.3 c), as desired.

Trichotomy:

When  $x, y \in \mathbb{R}$ , trichotomy follows from analogue in  $\mathbb{R}$  (Proposition 5.4.7).

When  $x \in \mathbb{R}$ ,  $y = \infty$ ,  $x = y$  is false by Definition 6.2.1.  $x < y$  is true as  $x \leq y$  (By Definition 6.2.3 b)) and  $x \neq y$ .  $x > y$  is false as  $x \geq y$  (aka  $y \leq x$ ) is false. (We can verify neither of 6.2.3 a), b), c) are satisfied).

7 cases remain, and the verification is similar to above cases hence omitted.

Transitivity:

If  $x, y, z \in \mathbb{R}$ , follows from analogue in  $\mathbb{R}$ .

If  $x \in \mathbb{R}^* - \mathbb{R}$ ,  $y, z \in \mathbb{R}$ , easy to verify  $x = -\infty$ , hence since  $z \in \mathbb{R}$ ,  $x < z$  as desired.

Other cases follow similarly. There may be a proof not involving 8 cases (prob via building some Lemmas first) but I am too average to figure it out.

Negation reverses order:

If  $x, y \in \mathbb{R}$ , follows from analogue in  $\mathbb{R}$ .

If  $x \in \mathbb{R}^* - \mathbb{R}$ ,  $y \in \mathbb{R}$ , easy to verify  $x = -\infty$ . Hence  $-x = \infty$ ,  $-y \leq \infty = -x$  as desired.

Other case follow similarly.

## Exercise 6.2.2

If  $E \subseteq \mathbb{R}$ , then the 3 claims follow from Definition 5.5.10 and Exercise 5.5.1.

If  $\infty \in E$ , then by Definition 6.2.6,  $\sup(E) = \infty$ ,  $\inf(E) = -\sup(-E)$ . Then, if  $-\infty \in E$ , then  $\infty \in -E$  hence  $\inf(E) = -\infty$ .

Hence, a) is true by Definition 6.2.3.

For b), since  $-\infty \in E$ , then  $-\infty \leq M$ . Hence  $\sup(E) = -\infty \leq M$  as desired.

For c), note  $-\infty \in E$ , hence  $-\infty \geq M$ . Using similar reasoning  $\inf(E) = -\infty \geq M$  as desired.

If  $-\infty \notin E$ , consider  $\inf(E - \{-\infty\})$ . Since  $E - \{-\infty\} \subseteq \mathbb{R}$ , it follows from Exercise 5.5.1 that  $\inf(E - \{-\infty\})$  is the greater lower bound of  $E - \{-\infty\}$ . Note  $\inf(E - \{-\infty\}) \leq -\infty$ , and  $\inf(E) = \inf(E - \{-\infty\})$ , claim a), c) follows.

Why  $\inf(E) = \inf(E - \{-\infty\})$ ?

Note  $\inf(E) = -\sup(-E)$ . Remains to show  $\sup(-E) = \sup(-E - \{-\infty\})$ . It follows from Definition 6.2.6 c) as desired.

$-\infty \notin E$  case follows similarly.

### Exercise 6.3.1

For  $\inf(a_n)^{\infty}_{n=m} = 0$ . Note, by Proposition 6.2.5 b), we either have  $\inf(a_n)^{\infty}_{n=m} < 0$ ,  $\inf(a_n)^{\infty}_{n=m} = 0$ ,  $\inf(a_n)^{\infty}_{n=m} > 0$ .

If  $\inf(a_n)^{\infty}_{n=m} < 0$ , then note  $0$  is a lower bound. (As  $\forall n \geq 1, 1/n > 0$ ). Yet by 6.2.11 c) we have  $\inf(a_n)^{\infty}_{n=m} \geq 0$ , contradiction.

If  $\inf(a_n)^{\infty}_{n=m} > 0$ , then denote it as  $\epsilon$ . By Archimedes easy to verify we can find sufficiently large  $N \in \mathbb{Z}^+$  s.t.  $1/N < \epsilon$ . Hence  $\inf(a_n)^{\infty}_{n=m}$  is not a lower bound, contradicting Exercise 5.5.1.

Hence it must hold  $\inf(a_n)^{\infty}_{n=m} = 0$ .

For  $\sup(a_n)^{\infty}_{n=m} = 1$ , easy to verify  $1$  is an upper bound. (As  $\forall n \in \mathbb{Z}^+, 1/n \leq 1$ ). Hence, remains to show it is the least upper bound by Definition 5.5.10. Assume  $1$  is not least upper bound, then by Theorem 6.2.11 b),  $1 > \sup(a_n)^{\infty}_{n=m}$ . However since  $1 = a_1$ ,  $a_1 > \sup(a_n)^{\infty}_{n=m}$ , contradicting Definition 5.5.10 as desired.

### Exercise 6.3.2

$\forall n \geq m, a_n \leq \sup(a_n)^{\infty}_{n=m}$  follows from 6.2.11 a) (via letting  $E = (a_n)^{\infty}_{n=m}$ )

$x \leq M$  for any upper bound  $M$  follows from 6.2.11 b)

For third claim, we argue by contradiction. Assume  $\exists y < x, \forall n \geq m, y \geq a_n$ , then  $y$  is a lower bound. Hence  $y \geq x$  by 6.2.11 b). However,  $y < x$ , contradiction as desired.

### Exercise 6.3.3

Note for all  $\epsilon > 0$ , there exists  $N \geq m$  s.t.:

$$\sup(a_n)^{\infty}_{n=m} - \epsilon \leq a_N \leq \sup(a_n)^{\infty}_{n=m} + \epsilon$$

The first inequality follows from 6.3.6 3rd claim as  $\sup(a_n)^{\infty}_{n=m} - \epsilon < \sup(a_n)^{\infty}_{n=m}$ . The second inequality follows from same claim as  $a_n \leq \sup(a_n)^{\infty}_{n=m} \leq \sup(a_n)^{\infty}_{n=m} + \epsilon$ .

Note for all  $n \geq N$ ,  $a_n \geq a_N$ , hence:

$$\sup(a_n)^{\infty}_{n=m} - \epsilon \leq a_N \leq a_n \leq \sup(a_n)^{\infty}_{n=m} + \epsilon$$

Therefore, by Exercise 5.4.6, it holds that:

$$\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |a_n - \sup(a_n)^{\infty}_{n=m}| \leq \epsilon$$

As desired.

## Exercise 6.3.4

Explain: When  $x > 1$ , the sequence is no longer bounded above.

Show: Assume  $\lim_{n \rightarrow \infty} x^n = L$ . Then since  $x > 1$ ,  $1/x < 1$ . Hence,  $\lim_{n \rightarrow \infty} (1/x)^n = 0$ .

However,  $\lim_{n \rightarrow \infty} x^n (1/x)^n = 1$ , yet  $\lim_{n \rightarrow \infty} x^n \lim_{n \rightarrow \infty} (1/x)^n = L \times 0 = 0 \neq 1$ , contradiction as desired.

Example 1.2.3 flaw: Change of variable (As in Exercise 6.1.4) applies only to convergent series. In fact Example 1.2.3 serves as a valid proof by contradiction for proving divergence of the series.

## Exercise 6.4.1

We have:

$$\forall \epsilon > 0, \exists M \geq m, \forall n \geq M, |a_n - L| \leq \epsilon$$

We need to show:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - L| \leq \epsilon$$

Fix  $\epsilon$ , we choose  $n = \max(N, M)$ , then  $|a_n - L| \leq \epsilon$  as desired.

## Exercise 6.4.2

### Limit Point

Exercise 6.1.3

Only if direction:

Recall definition of Limit Point, we have:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - x| \leq \epsilon$$

Note since  $m' \geq m$ ,  $\forall N \geq m$  implies  $\forall N \geq m'$ , hence:

$$\forall \epsilon > 0, \forall N \geq m', \exists n \geq N, |a_n - x| \leq \epsilon$$

Which, by definition of limit point implies  $x$  is a limit point of  $(a_n)_{n=m'}^\infty$ , as desired.

If direction:

We have:

$$\forall \epsilon > 0, \forall N \geq m', \exists n \geq N, |a_n - x| \leq \epsilon \quad \text{(1)}$$

Hence, by instantiating  $N = m'$ :

$$\forall \epsilon > 0, \exists n' \geq m', |a_{n'} - x| \leq \epsilon \quad \text{(2)}$$

We need to show:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - x| \leq \epsilon \quad \text{(3)}$$

Fix  $\epsilon$ , we split  $\forall N \geq m$  into cases, and show in each case there exists a permissible  $n$ .

For  $m' > N$ , we can merely choose  $n = n'$  with  $n'$  from (2) (using the same  $\epsilon$  in (2)). We need to test  $n' \geq N$  and  $|a_{n'} - x| \leq \epsilon$ . Since  $n' \geq m'$  and  $m' > N$ ,  $n' \geq N$ . By (2),  $|a_{n'} - x| \leq \epsilon$ . Hence  $n'$  satisfies the conditions.

For  $N \geq m'$ , we can choose the same  $n$  in (1). Easy to confirm the conditions are satisfied by such choice.

Hence, in both case  $\exists n \geq N, |a_n - x| \leq \epsilon$ , and the reasoning holds for any  $\epsilon > 0$ , as desired.

#### Exercise 6.1.4

Only if direction:

By Definition of Limit Point, we have:

$$\forall \epsilon > 0, \forall N' \geq m, \exists n' \geq N, |a_{n'} - x| \leq \epsilon \quad \text{(1)}$$

We need to show:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_{n+k} - x| \leq \epsilon \quad \text{(2)}$$

Fix  $\epsilon, N$ , we need to show a permissible  $n$  exists via (1).

We instantiate (1) via setting  $N' = N + k$ , then  $\exists n' > N, |a_{n'} - x| \leq \epsilon$ . We claim  $n = n' - k$  is permissible.

We first verify  $n \geq N$ . Note  $n' \geq N + k$ , hence  $n' - k \geq N$ . Therefore  $n = n' - k \geq N$  as desired.

Then,  $|a_{n+k} - x| = |a_{n'} - x| \leq \epsilon$ , as desired.

If direction:

We have:

$\forall \epsilon > 0, \forall N \geq m + k, \exists n' \geq N', |a_{n'} - x| \leq \epsilon \quad \text{(1)}$

We need to show:

$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - x| \leq \epsilon$

Fix  $\epsilon$ . We split into case  $N \geq m + k$  and  $N < m + k$ , and show in each case a permissible  $n$  exists.

Case  $N \geq m + k$  is shown by (1), by instantiating  $N' = N$  and take  $n'$  as  $n$ . Easy to verify it is permissible.

Remains to consider case  $N < m + k$ . In that case, note by instantiating  $N' = m + k$  in (1), we deduce  $\exists n' \geq m + k, |a_{n'} - x| \leq \epsilon$ . We show  $n'$  is permissible, as since  $n' \geq m + k$  and  $m + k > N$ , it holds that  $n' \geq N$ . And,  $|a_{n'} - x| \leq \epsilon$ , as desired.

### $\limsup_{n \rightarrow \infty} a_n$ case

#### Exercise 6.1.3

Need to show  $\inf(\sup(a_n)_{n=N})_{N=m} = \inf(\sup(a_n)_{n=N})_{N=m}$  for which  $m' \geq m$ .

We first show  $\sup(a_n)_{n=N} \geq \sup(a_n)_{n=N+1}$

First note by Proposition 6.3.6,  $\forall n \geq N, \sup(a_n)_{n=N} \geq a_n$ . Hence,  $\forall n \geq N + 1, \sup(a_n)_{n=N} \geq a_n$ . Therefore  $\sup(a_n)_{n=N}$  is an upper bound for  $(a_n)_{n=N+1}$ . Rest follows by Proposition 6.3.6 as desired.

Then we perform induction on  $m'$ . For ease of notation denote  $I(m') = \inf(\sup(a_n)_{n=N})_{N=m'}$ ,  $b_N = \sup(a_n)_{n=N}$ . Note  $b_N \geq b_{N+1}$ . Hence, the inductive hypothesis can be expressed as  $P(m')$ :  $I(m) = I(m')$ .

Base case  $P(m)$  is trivially true. Assume true for  $P(m')$ . By Proposition 6.3.6, we have  $\forall N \geq m', I(m') \leq b_N$ . Hence it holds that  $\forall N \geq m' + 1, I(m') \leq b_N$ . Hence  $I(m')$  is a lower bound of  $(b_N)_{N=m'+1}$ .

Similarly,  $\forall N \geq m' + 1, l(m' + 1) \leq b_N$ . And, noting  $l(m'+1) \leq b_{m'+1} \leq b_{m'}$ , we have  $\forall N \geq m', l(m' + 1) \leq b_N$ . Hence  $l(m'+1)$  is lower bound of  $\{b_N\}_{N=m}^\infty$ .

Therefore, by Proposition 6.3.6, we have  $l(m') \geq l(m'+1), l(m'+1) \geq l(m')$ . Hence  $l(m') = l(m'+1)$ . And, by inductive hypothesis  $l(m'+1) = l(m) = l(m)$  as desired.

#### Exercise 6.1.4

We start with a Lemma:

$$\{\sup(a_n)_{n=N+k}^\infty : N \geq m\} = \{\sup(a_n)_{n=N'}^\infty : N' \geq m+k\}$$

Proof:

Say given  $\sup(a_n)_{n=N+k}^\infty$  for some  $N \geq m$ , then  $N+k \geq m+k$ , hence letting  $N' = n+k$ , by Axiom of Substitution  $\sup(a_n)_{n=N+k}^\infty = \sup(a_n)_{n=N'}^\infty \in \text{RHS}$  as desired.

Similarly, given  $\sup(a_n)_{n=N'}^\infty$  for some  $N' \geq m+k$ , then  $N' - k \geq m$ . Hence letting  $N = N' - k$ ,  $\sup(a_n)_{n=N'}^\infty = \sup(a_n)_{n=N'+k-k}^\infty = \sup(a_n)_{n=N+k}^\infty \in \text{LHS}$  as desired.

Hence:

$$\begin{aligned} \inf(\sup(a_n)_{n=N+k}^\infty)_{N=m}^\infty &= \inf(\sup(a_n)_{n=N}^\infty)_{N=m+k}^\infty \\ &\quad \text{By Lemma} \\ &= \inf(\sup(a_n)_{n=N}^\infty)_{N=m}^\infty \\ &\quad \text{By Exercise 6.1.3} \end{aligned}$$

As desired.

#### $\liminf_{n \rightarrow \infty}$ case

Similar to  $\limsup_{n \rightarrow \infty}$  case hence omitted.

### Exercise 6.4.3

c)

We first show  $\inf(a_n)_{n=m}^\infty \leq L^-$ .

By definition of  $L^-$  and  $a_N^-$ , above is equivalent to  $\inf(a_n)_{n=m}^\infty \leq \sup(\inf(a_n)_{n=N}^\infty)_{N=m}^\infty$ .

Since  $\inf(a_n)_{n=m}^\infty \in \{\inf(a_n)_{n=N}^\infty : N \geq m\}$  (can show via setting  $N = m$  on the right side and result follows from Axiom of Substitution), by Proposition 6.3.6 we have  $\inf(a_n)_{n=m}^\infty \leq \sup(\inf(a_n)_{n=N}^\infty)_{N=m}^\infty$  as desired.

Can similarly show  $L^+ \leq \sup(a_n)_{n=m}^\infty$



Remains to show  $L^- \leq L^+$ . By definition of  $L^-$ ,  $L^+$ , the inequality is equivalent to  $\sup(a_n)_{n=N}^\infty \leq \inf(a_n)_{n=N}^\infty$ . We argue by contradiction. Assume  $\sup(a_n)_{n=N}^\infty > \inf(a_n)_{n=N}^\infty$ , then, by Proposition 6.3.6:

$$\exists N_0 \geq m, \sup(a_n)_{n=N_0}^\infty \geq a_{N_0}^- > \inf(a_n)_{n=N_0}^\infty$$

Taking  $a_{N_0}^- > \inf(a_n)_{n=N_0}^\infty$ , we apply Proposition 6.3.6 again yielding:

$$\exists N_1 \geq m, a_{N_0}^- > a_{N_1}^+ \geq \inf(a_n)_{n=N_1}^\infty$$

We claim  $a_{N_0}^- > a_{N_1}^+$  is the desired contradiction. Let  $M = \max(N_0, N_1)$ , then by Proposition 6.3.6,  $\sup(a_n)_{n=M}^\infty \geq a_M \geq \inf(a_n)_{n=M}^\infty$ , hence  $\sup(a_n)_{n=M}^\infty \geq \inf(a_n)_{n=M}^\infty$ , contradicting  $\inf(a_n)_{n=M}^\infty > \sup(a_n)_{n=M}^\infty$  as desired.

d)

We argue by contradiction. To show  $L^- \leq c$  for any sequence, assume  $L^- > c$  for some sequence  $(a_n)_{n=N}^\infty$ . Then, by Definition 6.4.6, we have  $\sup(a_n)_{n=N}^\infty > c$ . By Proposition 6.3.6, we have:

$$\exists N_0 \geq m, \sup(a_n)_{n=N_0}^\infty \geq a_{N_0}^- > c$$

Hence we have  $a_{N_0}^- > c$  for some  $N_0 \geq m$ . By Definition 6.4.6, we have  $\inf(a_n)_{n=N_0}^\infty > c$ . Since by Proposition 6.3.6, we have  $\forall n \geq N_0, a_n \geq \inf(a_n)_{n=N_0}^\infty$  and  $\inf(a_n)_{n=N_0}^\infty > c$ , we have  $\forall n \geq N_0, a_n > c$ .

Since  $c$  is a limit point of  $(a_n)_{n=N}^\infty$ , by Definition 6.4.1, we have:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - c| \leq \epsilon$$

Set  $\epsilon = (\inf(a_n)_{n=N_0}^\infty - c) / 2$ ,  $N = N_0$ , we have  $\exists n \geq N_0, |a_n - c| \leq (\inf(a_n)_{n=N_0}^\infty - c) / 2$ , hence:

$$\begin{aligned} a_n - c &\leq (\inf(a_n)_{n=N_0}^\infty - c) / 2 \text{ as } n \geq N_0 \Rightarrow \\ a_n &> c \mid a_n + a_n - c \leq \inf(a_n)_{n=N_0}^\infty \mid a_n < \inf(a_n)_{n=N_0}^\infty \text{ as } \\ a_n &> c \end{aligned}$$

It was shown that  $\forall n \geq N_0, a_n \geq \inf(a_n)_{n=N_0}^\infty$ , since  $n \geq N_0$ , we have  $a_n \geq \inf(a_n)_{n=N_0}^\infty$ , contradicting  $a_n < \inf(a_n)_{n=N_0}^\infty$  as desired.

e)

By Definition 6.4.1, Showing  $L^-$  is a limit point is equivalent of showing:

$$\forall \epsilon > 0, \forall N \geq m, \exists n \geq N, |a_n - L^-| \leq \epsilon$$

Fix  $\epsilon$ ,  $N$ . Then, by Definition 6.4.6,  $L^- = \sup(a_{N'}^-)^{\infty}_{N' = m}$ . By Exercise 6.4.2, it equals to  $\sup(a_{N'}^-)^{\infty}_{N' = N}$ .

Hence, by Proposition 6.3.6:

$$\begin{aligned} &\exists N_0 \geq N, L^- \geq a_{N_0}^- > L^- - \epsilon / 2 \mid \exists N_0 \geq N, 0 \leq a_{N_0}^- - L^- < \epsilon / 2 \mid \exists N_0 \geq N, 0 \leq L^- - a_{N_0}^- < \epsilon / 2 \mid \\ &\exists N_0 \geq N, |L^- - a_{N_0}^-| < \epsilon / 2 \end{aligned}$$

Then, by Proposition 6.3.6 again:

$$\begin{aligned} &\exists n_0 \geq N_0, a_{N_0}^- \leq a_{n_0} < a_{N_0}^- + \epsilon / 2 \mid \exists n_0 \geq N_0, 0 \leq a_{n_0} - a_{N_0} < \epsilon / 2 \mid \exists n_0 \geq N_0, |a_{n_0} - a_{N_0}| < \epsilon / 2 \end{aligned}$$

We claim  $n = n_0$  is permissible. First,  $n_0 \geq N_0$ , and  $N_0 \geq N$ , hence  $n_0 \geq N$ . Then:

$$\begin{aligned} &|a_{n_0} - L^-| \leq |a_{n_0} - a_{N_0}^-| + |a_{N_0}^- - L^-| \text{ \textit{By Triangle Inequality}} \\ &\leq \epsilon / 2 + \epsilon / 2 = \epsilon \end{aligned}$$

As desired.

f)

**If  $(a_n)_{n=m}^\infty$  converges to  $c$ , then we must have  $L^+ = L^- = c$**

For simplicity we only show  $L^+ = c$ .

Since  $(a_n)_{n=m}^\infty$  converges to  $c$ , by Definition 6.1.5, we have:

$$\forall \epsilon > 0, \exists N_{\epsilon} > m, \forall n \geq N_{\epsilon}, |a_n - c| \leq \epsilon$$

By Exercise 5.4.6, the above is equivalent to:

$$\forall \epsilon > 0, \exists N_{\epsilon} > m, \forall n \geq N_{\epsilon}, c - \epsilon \leq a_n \leq c + \epsilon$$

Hence, taking the right side of the inequality:

$$\forall \epsilon > 0, \exists N_{\epsilon} > m, \forall n \geq N_{\epsilon}, a_n \leq c + \epsilon$$

Therefore, by Definition 5.5.1,  $c + \epsilon$  is an upper bound for  $(a_n)_{n=N_{\epsilon}}^\infty$ .

Hence, by Proposition 6.3.6, we have  $\sup(a_n)_{n=N_{\epsilon}}^\infty \leq c + \epsilon$ . Since, by Definition 6.4.6,  $\sup(a_n)_{n=N_{\epsilon}}^\infty = a^{+}_{N_{\epsilon}}$ , and  $L^+ = \inf(a^{+}_{N})_{N=m}^\infty$ , we have, by Proposition 6.3.6:

$$\forall \epsilon > 0, \exists N_{\epsilon} > m, L^+ = \inf(a^{+}_{N})_{N=m}^\infty \leq a^{+}_{N_{\epsilon}} \leq c + \epsilon$$

$$\forall \epsilon > 0, \exists N_{\epsilon} > m, L^+ \leq c + \epsilon$$

$$\forall \epsilon > 0, L^+ \leq c + \epsilon$$

We claim  $L^+ \leq c$ . We argue by contradiction. Assume  $L^+ > c$ , then choose  $\epsilon = (L^+ - c)/2$ , we deduce:

$$\begin{aligned} L^- &\leq c + (L^+ - c)/2 \leq 2L^- \leq 2c + L^+ - c \leq L^- \leq c \end{aligned}$$

Which contradicts  $L^+ > c$  as desired.

Then, we deduce  $c \leq L^+$ .

Similar to how we established  $\sup(a_n)_{n = N_{\epsilon}} \leq c + \epsilon$ , we have  $c - \epsilon \leq \inf(a_n)_{n = N_{\epsilon}}$ . Then, by Definition 6.4.2,  $\inf(a_{N^+})_{N = m} = \inf(a_{N^+})_{N = N_{\epsilon}}$ , hence  $L^+ = \inf(a_{N^+})_{N = N_{\epsilon}}$ . Then, by Proposition 6.3.6,  $\forall N \geq N_{\epsilon}, a_N \leq a^{+N}$ . Hence, by Lemma 6.4.13,  $\inf(a_n)_{n = N_{\epsilon}} \leq \inf(a^{+N})_{N = N_{\epsilon}}$ , therefore:

$$\forall \epsilon > 0, c - \epsilon \leq \inf(a_n)_{n = N_{\epsilon}} \leq \inf(a^{+N})_{N = N_{\epsilon}} = L^+$$

$$\forall \epsilon > 0, c - \epsilon \leq L^+$$

Using similar argument for showing  $L^+ \leq c$ , we can show above predicate implies  $c \leq L^+$ . Hence  $c \leq L^+ \leq c$ , implying  $L^+ = c$  as desired

**If  $L^+ = L^- = c$ , then  $(a_n)_{n = m}$  converges to  $c$**

By Exercise 6.1.2,  $(a_n)_{n = m}$  converges to  $c$  iff:

$$\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, |a_n - c| \leq \epsilon$$

By Exercise 5.4.6, above is true iff:

$$\forall \epsilon > 0, \exists N \geq m, \forall n \geq N, c - \epsilon \leq a_n \leq c + \epsilon$$

Fix  $\epsilon$ . Since  $c + \epsilon = L^+ + \epsilon > L^+$ , by Proposition 6.4.12 a):

$$\exists N_0 \geq m, \forall n \geq N_0, a_n < c + \epsilon$$

Similarly:

$$\exists N_1 \geq m, \forall n \geq N_1, c - \epsilon < a_n$$

Hence, we claim  $N = \max(N_0, N_1)$  is permissible. Note  $N \geq m$  as  $N \geq N_0$  and  $N_0 \geq m$ . And, since  $n \geq N$  implies  $n \geq N_0$  and  $n \geq N_1$ :

$$\forall n \geq N, c - \epsilon < a_n < c + \epsilon$$

$\forall n \geq N, c - \epsilon \leq a_n \leq c + \epsilon$

As desired.

### Exercise 6.4.4

For  $\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$ . We argue by contradiction. Assume  $\sup(a_n)_{n=m}^\infty > \sup(b_n)_{n=m}^\infty$ . Then by Proposition 6.3.4,  $\exists n_0 > 0$ ,  $\sup(a_n)_{n=m}^\infty \geq a_{n_0} > \sup(b_n)_{n=m}^\infty$ . Hence we have  $a_{n_0} > \sup(b_n)_{n=m}^\infty$ . However, since  $\forall n \geq m, a_n \leq b_n$ , we have  $a_{n_0} \leq b_{n_0}$ . We also have  $b_{n_0} \leq \sup(b_n)_{n=m}^\infty$  by Proposition 6.3.6. Hence we have  $a_{n_0} \leq \sup(b_n)_{n=m}^\infty$ , contradicting  $a_{n_0} > \sup(b_n)_{n=m}^\infty$  as desired.

Similarly we can show  $\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$ .

For  $\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n$ , by Definition 6.4.6 the inequality is equivalent to  $\inf(a_{N^+})_{n=m}^\infty \leq \inf(b_{N^+})_{n=m}^\infty$ . Note since  $\forall n \geq m, a_n \leq b_n$ , by 6.4.4 1) we have  $\forall N \geq m, a_{N^+} \leq b_{N^+}$ . Hence by 6.4.4 2) we have  $\inf(a_{N^+})_{n=m}^\infty \leq \inf(b_{N^+})_{n=m}^\infty$  as desired.

Similarly we can show  $\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n$ , as desired.

### Exercise 6.4.5

By Proposition 6.4.12 f), since  $L = \lim_{n \rightarrow \infty} a_n$ ,  $L = \limsup_{n \rightarrow \infty} a_n$ . Similarly  $L = \limsup_{n \rightarrow \infty} c_n$ .

Since  $a_n \leq b_n \leq c_n$ , By Lemma 6.4.13 f), we have:

$$L = \limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n \leq \limsup_{n \rightarrow \infty} c_n = L$$

$$\limsup_{n \rightarrow \infty} b_n = L$$

Similarly we can conclude  $\liminf_{n \rightarrow \infty} b_n = L$ . Hence by Lemma 6.4.13 f),  $\lim_{n \rightarrow \infty} b_n = L$  as desired.

### Exercise 6.4.6

Let  $a_n = 1 - 1/n$ ,  $b_n = 1$ . It is the desired example as  $a_n < b_n$  and easy to verify  $\limsup_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n = 1$ .

It does not contradict Lemma 6.4.13 as Lemma 6.4.13 only claims  $\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n$ , which still holds as  $1 \leq 1$ .

## Exercise 6.4.7

If direction:

Note  $\lim_{n \rightarrow \infty} -|a_n| = \lim_{n \rightarrow \infty} |a_n| = 0$ , and  $-|a_n| \leq a_n \leq |a_n|$ . By Corollary 6.4.14,  $\lim_{n \rightarrow \infty} a_n = 0$  as desired.

Only if direction

Note  $\min(-a_n, a_n) \leq |a_n| \leq \max(-a_n, a_n)$ .

By 6.1.19 g) h),  $\lim_{n \rightarrow \infty} \min(-a_n, a_n) = \lim_{n \rightarrow \infty} \max(-a_n, a_n) = 0$ . Hence  $\lim_{n \rightarrow \infty} a_n = 0$  by Corollary 6.4.14 as desired.

If  $0$  is replaced with some other number (say  $1$ ), clearly false. Note  $\lim_{n \rightarrow \infty} -1 \neq \lim_{n \rightarrow \infty} |-1|$ .

## Exercise 6.4.8

We proceed to prove by cases. Either  $(a_n)_{n=m}^{\infty}$  has finite upper bound or  $(a_n)_{n=m}^{\infty}$  does not have finite upper bound.

If  $(a_n)_{n=m}^{\infty}$  has finite upper bound, then we show  $\limsup_{n \rightarrow \infty} a_n$  is finite. Let  $M$  be the finite upper bound of  $(a_n)_{n=m}^{\infty}$ . Define  $(b_n)_{n=m}^{\infty}$  as constant sequence with each term being  $M$ . Hence,  $\lim_{n \rightarrow \infty} b_n = M$ . Therefore, by Proposition 6.4.12 f),  $\limsup_{n \rightarrow \infty} b_n = M$ . By definition of upper bound, it holds that  $\forall n \geq m, a_n \leq b_n$ . Hence, by Lemma 6.4.13,  $\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n = M$ . Hence  $\limsup_{n \rightarrow \infty} a_n$  is finite.

Therefore, by Proposition 6.4.12 e),  $\limsup_{n \rightarrow \infty} a_n$  is a limit point as under Definition 6.4.1. Hence, by Proposition 6.4.12 f), for all limit point  $c$  as under Definition 6.4.1,  $c \leq \limsup_{n \rightarrow \infty} a_n$ . There does not exist other limit points as  $(a_n)_{n=m}^{\infty}$  has finite upper bound, hence for all limit point  $c$ ,  $c \leq \limsup_{n \rightarrow \infty} a_n$  as desired.

If  $(a_n)_{n=m}^{\infty}$  has no finite upper bound, then we show  $\limsup_{n \rightarrow \infty} a_n = \infty$ . By Definition 6.4.6, it is equivalent of showing  $\limsup_{n \rightarrow \infty} a_n = \inf(a_{N^+})_{N=m}^{\infty}$ . We start by showing  $\forall N \geq m, a_{N^+} = \infty$ .

By Definition 5.5.10, it is equivalent of showing  $(a_n)_{n=N}^{\infty}$  has no finite upper bound for all  $N \geq m$ . We use induction. For base case  $N = m$ ,  $(a_n)_{n=m}^{\infty}$  has no finite upper bound due to the case we are under. For  $P(N + 1)$ , we prove by contraposition and show  $\neg P(N + 1) \rightarrow \neg P(N)$ . Assume  $P(N + 1)$  has upper bound  $M$ , then  $\forall n \geq N + 1, a_n \leq M$ . Since  $M \leq \max(M, a_N)$ , we deduce  $\forall n \geq N + 1, a_n \leq \max(M, a_N)$ . Note  $a_N \leq \max(M, a_N)$ , hence  $\forall n \geq N, a_n \leq \max(M, a_N)$ , therefore  $\max(M, a_N)$  is a finite upper bound for  $(a_n)_{n=N}^{\infty}$ , hence  $\neg P(N)$ , as desired.

Alternative method of showing  $(a_n)^{\infty}_{n=N}$  for all  $N \geq m$  has no finite upper bound: We argue by contradiction. Assume for some  $N \geq m$ ,  $(a_n)^{\infty}_{n=N}$  has finite upper bound  $M$ , then by Lemma 5.1.14,  $a_m, a_{m+1}, \dots, a_N$  is bounded by some finite  $M'$ . Hence,  $(a_n)^{\infty}_{n=m}$  is bounded by  $\max(M, M')$ , contradicting  $(a_n)^{\infty}_{n=m}$  has no finite upper bound, as desired.

Since  $\forall N \geq m, a_N^+ = \infty$ , we deduce  $\forall N \geq m, a_N^+ \geq \infty$ . Hence  $\infty$  is a lower bound for  $a_N^+$ . Since  $\infty$  is a lower bound for  $a_N^+$ , it holds  $\inf(a_N^+)^{\infty}_{N=m} \geq \infty$  by Proposition 6.2.11 c), hence  $\inf(a_N^+)^{\infty}_{N=m} = \infty$ , therefore  $\limsup_{n \rightarrow \infty} a_n = \inf(a_N^+)^{\infty}_{N=m} = \infty$ .

Since  $\limsup_{n \rightarrow \infty} a_n = \infty$  and  $\infty$  is a limit point,  $\limsup_{n \rightarrow \infty} a_n$  is a limit point. Furthermore, for any given limit point  $c \in \mathbb{R}^*$ , it holds that  $c \leq \infty = \limsup_{n \rightarrow \infty} a_n$  as desired.

### Exercise 6.4.9

Consider  $1, -1, 0, 2, -2, 0, 3, -3, 0, \dots$

TODO: prove it has exactly 3 limit points.

### Exercise 6.4.10

Since each  $b_m$  is limit point of  $(a_n)^{\infty}_{n=N}$ , we have, for any  $m \geq M$ :

$$\forall \epsilon_0 > 0, \forall N_0 \geq N, \exists n_0 \geq N_0, |a_{n_0} - b_m| \leq \epsilon_0 \quad \text{(1)}$$

Since  $c$  is limit point of  $b_n$ , we also have:

$$\forall \epsilon_1 > 0, \forall M_0 \geq M, \exists m_0 \geq M_0, |b_{m_0} - c| \leq \epsilon_1 \quad \text{(2)}$$

By Definition of limit point, showing  $c$  is a limit point of  $(a_n)^{\infty}_{n=N}$  is equivalent of showing:

$$\forall \epsilon > 0, \forall N' \geq N, \exists n \geq N', |a_n - c| \leq \epsilon \quad \text{(3)}$$

We fix  $\epsilon, N'$ . Then, we instantiate (2), setting  $\epsilon_1 = \epsilon / 2, M_0 = M$ , yielding:

$$\exists m_0 \geq M, |b_{m_0} - c| \leq \epsilon / 2 \quad \text{(4)}$$

Then, we instantiate (1), setting  $m = m_0, \epsilon_0 = \epsilon / 2, N_0 = N'$ , then:

$$\exists n_0 \geq N', |a_{n_0} - b_{m_0}| \leq \epsilon / 2 \quad \text{(5)}$$

We claim, in (3), setting  $n = n_0$  is permissible. Note  $n_0 \geq N'$ , remains to check  $|a_{n_0} - c| \leq \epsilon$ .



$$\begin{aligned} |a_{n_0} - c| &\leq |a_{n_0} - b_{m_0}| + |b_{m_0} - c| && \text{By Triangle Inequality} \\ &\leq \epsilon / 2 + \epsilon / 2 && \text{By (4), (5)} \\ &\leq \epsilon \end{aligned}$$

As desired.

## Exercise 6.5.1

$$\begin{aligned} \lim_{n \rightarrow \infty} (1/n)^q &= \lim_{n \rightarrow \infty} (1/n)^{a/b} = \\ &= \lim_{n \rightarrow \infty} ((1/n)^{1/b})^a && \text{By Definition 5.6.7} \\ &= (\lim_{n \rightarrow \infty} (1/n)^{1/b})^a && \text{By Theorem 6.1.19 b) and Induction} \\ &= 0^a && \text{As } a > 0 \end{aligned}$$

Remark: The fact that  $a, b \in \mathbb{Z}^+$  is essential. Definition 6.5.7 only applies if  $a \in \mathbb{Z}$ ,  $b \in \mathbb{Z}^+$ . Theorem 6.1.19 b) and induction applies only because  $a \in \mathbb{Z}^+$ .  $0^a$  is well defined because  $a > 0$  ( $0^{-1}$  would be undefined).

## Exercise 6.5.2

**$\lim_{n \rightarrow \infty} x^n = 0$  when  $|x| < 1$**

By Corollary 6.4.17,  $\lim_{n \rightarrow \infty} x^n = 0$  iff  $\lim_{n \rightarrow \infty} |x^n| = 0$ . Hence:

$$\begin{aligned} \lim_{n \rightarrow \infty} |x^n| &= \lim_{n \rightarrow \infty} |x|^n && \text{by 4.3.10 d)} \\ &= 0 && \text{by 6.3.10} \end{aligned}$$

Remark: Proposition 4.3.10 d) can be used since  $n \in \mathbb{N}$ . If  $n \in \mathbb{R}$  things are more tricky. Proposition 6.3.10 can be used as  $0 < |x| < 1$ . Note we may have  $|x| = 0$ , but in that case the limit is still trivially  $0$ , as desired.

**$\lim_{n \rightarrow \infty} x^n = 1$  when  $x = 1$**

Since  $x^n = 1^n = 1$ , by Definition 6.1.5 it follows that  $\lim_{n \rightarrow \infty} x^n = \lim_{n \rightarrow \infty} 1 = 1$  as desired.

**$\lim_{n \rightarrow \infty} x^n$  diverges when  $x = -1$  or when  $|x| > 1$**

When  $x = -1$ , then both  $-1, 1$  are limit points (can show via Definition 6.4.1), hence by Proposition 6.4.12 d) it holds that  $L^- \leq -1$ ,  $L^+ \geq 1$ , hence  $L^- \neq L^+$ , implying  $\lim_{n \rightarrow \infty} x^n$  diverges by Proposition 6.4.12 f).

(Or we can manually evaluate  $L^-$ ,  $L^+$  and show they are not equal).

When  $|x| > 1$ , note similar reasoning for Exercise 6.3.4 still applies. Assume  $\lim_{n \rightarrow \infty} x^n = L$ , then  $\lim_{n \rightarrow \infty} x^n \lim_{n \rightarrow \infty} (1/x)^n = 1$ . However, since  $|x| > 1$ ,  $|1/x| < 1$ , hence  $\lim_{n \rightarrow \infty} (1/x)^n = 0$ , hence we have  $L \cdot 0 = 1$ , contradiction.

## Exercise 6.5.3

We argue by cases. Assume  $x = 1$ , then  $x^{1/n} = 1$ , hence  $\lim_{n \rightarrow \infty} x^{1/n} = 1$  as desired.

Assume  $x > 1$ , then by Lemma 5.6.9 d),  $x^{1/n} > 1^{1/n}$ , hence  $x^{1/n} > 1$ . Hence,  $1$  is a lower bound for  $(x^{1/n})_{n=1}^{\infty}$ . Since  $x > 1$ , by Lemma 5.6.6 e),  $x^{1/n}$  is a decreasing function of  $n$ . Hence, by Proposition 6.3.8,  $(x^{1/n})_{n=1}^{\infty}$  converges, and  $\lim_{n \rightarrow \infty} x^{1/n} = \inf(x^{1/n})_{n=1}^{\infty} \geq 1$ .

We then show  $\inf(x^{1/n})_{n=1}^{\infty} > 1$  causes contradiction. Assume  $\inf(x^{1/n})_{n=1}^{\infty} = 1 + d$  for some  $d \in \mathbb{R}^+$ , then  $\forall n \geq 1, (1 + d) \leq x^{1/n}$ . By 4.3.12 c),  $\forall n \geq 1, (1 + d)^n \leq x^n$ . Via Binomial Theorem, easy to show  $(1 + d)^n \geq nd^2$ . By Corollary 5.4.13, there exists  $n' \in \mathbb{Z}^+$ ,  $n' d^2 > x$ . Hence  $(1 + d)^{n'} \geq n' d^2 > x$ , which is the desired contradiction.

Hence,  $\lim_{n \rightarrow \infty} x^{1/n} = \inf(x^{1/n})_{n=1}^{\infty} = 1$ , as desired.

Case  $0 < x < 1$  remains. Via similar application of Lemma 5.6.9 d), 5.6.6 e), Proposition 6.3.8, we can show  $\lim_{n \rightarrow \infty} x^{1/n}$  exists. Then, by Theorem 6.1.19 b);

$$\lim_{n \rightarrow \infty} x^{1/n} \lim_{n \rightarrow \infty} (1/x)^{1/n} = \lim_{n \rightarrow \infty} x^{1/n} (1/x)^{1/n}$$

Easy to show  $RHS = 1$ . Then, since  $0 < x < 1$ ,  $1/x > 1$ , hence  $\lim_{n \rightarrow \infty} (1/x)^{1/n} = 1$ , therefore we have  $\lim_{n \rightarrow \infty} x^{1/n} \times 1 = 1$ , hence  $\lim_{n \rightarrow \infty} x^{1/n} = 1$  as desired.

Remark: The estimation  $(1 + d)^n \geq nd^2$  cannot be proven easily by induction. However, one may still avoid usage of Binomial Theorem by using estimation  $(1 + d)^n \geq 1 + dn$  instead. This estimation, also called Bernoulli's inequality can be easily proven with induction.

## Exercise 6.6.1

Show  $(a_n)_{n=0}^{\infty}$  is subsequence of  $(a_n)_{n=0}^{\infty}$

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be  $f(n) = n$ . Then,  $f$  is strictly increasing. Since  $\forall n \in \mathbb{N}, a_n = a_{f(n)}$ ,  $(a_n)_{n=0}^{\infty}$  is a subsequence of  $(a_n)_{n=0}^{\infty}$  by Definition 6.6.1.

Show  $(c_n)_{n=0}^{\infty}$  is subsequence of  $(a_n)_{n=0}^{\infty}$

Since  $(c_n)_{n=0}^{\infty}$  is subsequence of  $(b_n)_{n=0}^{\infty}$ , by Definition 6.6.1 there exists  $g: \mathbb{N} \rightarrow \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}, c_n = b_{g(n)}$ . Since  $(b_n)_{n=0}^{\infty}$  is subsequence of  $(a_n)_{n=0}^{\infty}$ , by Definition 6.6.1 there exists  $f: \mathbb{N} \rightarrow \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}, b_n = a_{f(n)}$ .



We now claim  $\forall n \in \mathbb{N}, c_n = a_{f(g(n))}$ . Since  $\forall n \in \mathbb{N}, b_n = a_{f(n)}$  and  $g(n) \in \mathbb{N}$ , it holds that  $\forall n \in \mathbb{N}, b_{g(n)} = a_{f(g(n))}$ . Since  $\forall n \in \mathbb{N}, c_n = b_{g(n)}$ , it holds that  $\forall n \in \mathbb{N}, c_n = a_{f(g(n))}$ , as desired.

Then, note by Definition 3.3.10,  $f \circ g: \mathbb{N} \rightarrow \mathbb{N}$ . Hence  $f \circ g$  satisfies the domain & range requirement by Definition 6.6.1. Remains to show  $\forall n \in \mathbb{N}, f(g(n+1)) \geq f(g(n))$ . Since  $g$  is strictly increasing,  $g(n+1) > g(n)$ . Since  $f$  is strictly increasing, easy to show via induction that  $a \geq b \rightarrow f(a) \geq f(b)$ , hence  $f(g(n+1)) \geq f(g(n))$  as desired.

## Exercise 6.6.2

Consider  $a: 1,2,1,2,1,2,\dots$  and  $b: 2,1,2,1,2,1,\dots$

Note  $b_n = a_{n+1}, a_n = b_{n+1}$ .

## Exercise 6.6.3

We define  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as:

$$f(j, k) = \min\{n \in \mathbb{N} : |a_n| \geq j + 1; n > k\}$$

For which  $\min(E)$  is implicitly defined as a natural number s.t.:

- $\min(E) \in E$
- $\forall e \in E, \min(E) \leq e$

We claim that if  $E$  is a non-empty subset of  $\mathbb{N}$ , then:

- $\min(E)$  exists
- $\min(E)$  is unique.

For existence of  $\min(E)$ , we argue by contradiction. Assume  $\min(E)$  does not exist for a set  $E$ . Then, we use strong induction: let  $P(n)$  denote  $n \notin E$ . Then base case  $P(0)$  is true. As, if  $0 \in E$ ,  $\forall e \in E, 0 \leq e$  (as  $e \in E$  implies  $e \in \mathbb{N}$ ), hence  $\min(E) = 0$  is permissible, contradiction.

For  $P(n+1)$ , note  $0, 1, 2, \dots, n \notin E$  by inductive hypothesis. Hence, assume  $n+1 \in E$ , then  $\forall e \in E, n+1 \leq e$  (otherwise  $\exists e \in E, n+1 > e$ , hence  $e \leq n$ , contradicting  $0, 1, 2, \dots, n \notin E$ ). Hence  $\min(E) = n+1$  is permissible, contradiction, thus closing the induction.

Since  $E$  is a non-empty subset of  $\mathbb{N}$ , by Lemma 3.1.16 we can take  $c \in E$ , and since  $E \subseteq \mathbb{N}$ , it holds that  $c \in \mathbb{N}$ . Since  $c \in \mathbb{N}$ ,  $c \notin E$  by  $P(c)$ , contradiction as desired.

We proceed to show  $f$  is well defined, which is equivalent to showing  $\min\{n \in \mathbb{N} : |a_n| \geq j + 1; n > k\}$  exists & is unique, which is equivalent to showing  $\{n \in \mathbb{N} : |a_n| \geq j + 1; n > k\}$  is subset of

$\mathbb{N}$  & non-empty.

For  $\{n \in \mathbb{N} : |a_n| \geq j; n > k\} \subseteq \mathbb{N}$ , it follows from Axiom of specification. For  $\{n \in \mathbb{N} : |a_n| \geq j + 1; n > k\} \neq \emptyset$ . We argue by contradiction. Assume the contrary, then  $\forall n > k, |a_n| < j + 1$ . Yet, by Lemma 5.1.14,  $\forall n \leq k, |a_n| \leq M$  for some finite  $M$ . Hence,  $(a_n)_{n=0}^\infty$  is bounded by  $\max(M, j + 1)$ , contradiction. Hence  $f$  is a function.

We recursively define  $n: \mathbb{N} \rightarrow \mathbb{N}$  as:

$$\begin{aligned} n(0) &= \{n \in \mathbb{N} : |a_n| \geq 0\} \setminus n(j + 1) = f(j, n(j)) \end{aligned}$$

(This does uniquely define a function by Exercise 3.5.12)

Then, easy to verify  $n(j + 1) > n(j)$ . Hence, by Definition 6.6.1 we can define subsequence  $(b_j)_{j=0}^\infty$  of  $(a_n)_{n=0}^\infty$  via:

$$b_j = a_{f(j)}$$

We claim  $(b_j)_{j=0}^\infty$  is the desired subsequence, aka:

$$\lim_{j \rightarrow \infty} \frac{1}{b_j} = 0$$

By definition of  $f(j)$ , it holds that  $|a_{f(j)}| \geq j + 1$ , hence:

$$\begin{aligned} \lim_{j \rightarrow \infty} \left| \frac{1}{b_j} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{a_{f(j)}} \right| \leq \lim_{j \rightarrow \infty} \frac{1}{j + 1} \quad \text{By 6.4.14} = \lim_{j \rightarrow \infty} \frac{1}{j} \\ &= 0 \quad \text{By 6.1.4} \end{aligned}$$

Hence, by Corollary 6.4.17,  $\lim_{j \rightarrow \infty} 1/b_j = 0$ , as desired.

## Exercise 6.6.4

### (b) implies (a)

By Lemma 6.6.4,  $(a_n)_{n=0}^\infty$  is a subsequence of  $(a_n)_{n=0}^\infty$ . Hence, every subsequence of  $(a_n)_{n=0}^\infty$  converges to  $L$  implies  $(a_n)_{n=0}^\infty$  converges to  $L$ .

### (a) implies (b)

Since  $(a_n)_{n=0}^\infty$  converges to  $L$ , by Definition 6.1.5:

$$\forall \epsilon' > 0, \exists N' \geq 0, \forall n' \geq N', |a_{n'} - L| \leq \epsilon' \quad \text{(1)}$$

Denote a subsequence of  $(a_n)_{n=0}^\infty$  as  $(b_n)_{n=0}^\infty$  s.t.  $\forall n \geq 0, b_n = a_{f(n)}$  for some strictly increasing function  $f: \mathbb{N} \rightarrow \mathbb{N}$ . We need to show  $(b_n)_{n=0}^\infty$  converges to  $L$ , which, By Definition 6.1.5, is equivalent to:

$$\forall \epsilon > 0, \exists N \geq 0, \forall n \geq N, |b_n - L| \leq \epsilon \quad \text{(2)}$$

Since  $\forall n \geq 0, b_n = a_{f(n)}$ , and  $n \geq N \geq 0$  in (2), we conclude  $|b_n - L| \leq \epsilon$  iff  $|a_{f(n)} - L| \leq \epsilon$ , hence (2) is equivalent to:

$$\forall \epsilon > 0, \exists N \geq 0, \forall n \geq N, |a_{f(n)} - L| \leq \epsilon \quad \text{(3)}$$

Fix  $\epsilon$ , then by instantiating (1) with  $\epsilon' = \epsilon$ , we deduce:

$$\exists N' \geq 0, \forall n' \geq N', |a_{n'} - L| \leq \epsilon \quad \text{(4)}$$

Then, we claim  $\exists N'' \in \mathbb{N}, f(N'') > N'$ . We argue by contradiction. Assume  $\forall N \in \mathbb{N}, f(N) \leq N'$ . Then, define  $(c_N)_{N=0}^\infty$  s.t.  $\forall N \in \mathbb{N}, c_N = N' - f(N)$ . Then, since  $\forall N \in \mathbb{N}, f(N) < f(N+1)$ ,  $\forall N \in \mathbb{N}, c_N > c_{N+1}$ . Yet,  $\forall N \in \mathbb{N}, c_N \in \mathbb{N}$ . Thus contradiction by Exercise 4.4.2 (a).

We claim (3) is true, as  $N = N''$  is permissible. Easy to verify  $N'' \geq 0$ . Remains to show  $\forall n \geq N'', |a_{f(n)} - L| \leq \epsilon$ . Since  $f$  is strictly increasing, via easy induction  $\forall n \geq N'', f(n) \geq f(N'')$ . Since  $f(N'') > N'$ ,  $\forall n \geq N'', f(n) > N'$ . Therefore by (4),  $|a_{f(n)} - L| \leq \epsilon$  as desired.

## Exercise 6.6.5

### (a) implies (b)

We borrow definition of  $\min$  from Exercise 6.6.3

Then, we define  $n: \mathbb{N} \rightarrow \mathbb{N}$  via:

$$\begin{aligned} n(0) &= 0 \mid n(j+1) = f(j, n(j)) \end{aligned}$$

For which  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  is defined as:

$$f(j, k) = \min\{n > k : |a_n - L| \leq \frac{1}{j+1}\}$$

Similar to Exercise 6.6.3 solution, to ensure that  $n: \mathbb{N} \rightarrow \mathbb{N}$  is well defined, we merely need to show  $\{n > k : |a_n - L| \leq \frac{1}{j+1}\}$  is non-empty.

Since  $(a_n)_{n=0}^\infty$  has limit point  $L$ , by Definition 6.4.1 it holds that:

$$\forall \epsilon > 0, \forall N \geq 0, \exists n \geq N, |a_n - L| \leq \epsilon$$

Instantiate the above with  $\epsilon = 1/(j+1)$ ,  $N = k+1$  yields:

$$\exists n \geq k+1, |a_n - L| \leq \frac{1}{j+1}$$

$$\exists n > k, |a_n - L| \leq \frac{1}{j+1}$$

Hence  $\{n > k : |a_n - L| \leq \frac{1}{j+1}\}$  is non-empty.

Easy to verify  $n(j+1) > n(j)$ , hence by Definition 6.6.1, we can define subsequence  $(b_j)_{j=0}^\infty$  of  $(a_n)_{n=0}^\infty$  via:

$$b_j = a_{f(j)}$$

We claim  $(b_j)_{j=0}^\infty$  is the desired subsequence, aka  $(b_j)_{j=0}^\infty$  converges to  $L$ . By Definition 6.1.5, it is equivalent to showing:

$$\forall \epsilon > 0, \exists N \geq 0, \forall n \geq N, |b_n - L| \leq \epsilon \quad \text{(1)}$$

Observe,  $|a_{f(n)} - L| \leq 1/n$ . Hence,  $|b_n - L| \leq 1/n$ . Fixing  $\epsilon$ , we can choose  $N$  s.t.  $1/N \leq \epsilon$ . Such  $N$  exists as by Corollary 5.4.13,  $\exists N \in \mathbb{Z}^+, N \epsilon \geq 1$ . Divide  $\epsilon$  by both sides yields  $1/N \leq \epsilon$ . (Or we can note  $1/N \leq \epsilon \iff 1/\epsilon \leq N$  and use Proposition 5.4.12).

We claim the same  $N$  is permissible for (1), as, for all  $n \geq N$ :

$$\begin{aligned} |b_n - L| &\leq 1/n \leq 1/N \leq \epsilon \quad \text{As } n \geq N \leq \epsilon \end{aligned}$$

As desired.

### (b) implies (a)

We prove a stronger statement. Call the subsequence  $(b_n)_{n=0}^\infty$ , for which  $\forall n \in \mathbb{N}, b_n = a_{f(n)}$  for a strictly increasing function  $f: \mathbb{N} \rightarrow \mathbb{N}$ . Say  $L$  is a limit point of  $(b_n)_{n=0}^\infty$ , then, by Definition 6.4.1:

$$\forall \epsilon' > 0, \forall N' \geq 0, \exists n' \geq N', |b_{n'} - L| \leq \epsilon' \quad \text{(1)}$$

We want to show  $(a_n)_{n=0}^\infty$  has limit point  $L$ , which, by Definition 6.4.1, is equivalent to:

$$\forall \epsilon > 0, \forall N \geq 0, \exists n \geq N, |a_n - L| \leq \epsilon \quad \text{(2)}$$

Fix  $\epsilon, N$ . By solution of Exercise 6.6.4,  $\exists N_0 \in \mathbb{N}, f(N_0) \geq N$ . We instantiate (1) with  $\epsilon' = \epsilon, N' = N_0$ , then:

$$\exists n' \geq N_0, |b_{n'} - L| \leq \epsilon \quad \exists n' \geq N_0, |a_{f(n')} - L| \leq \epsilon$$

Then,  $n = f(n')$  is permissible, as  $f(n') \geq f(N_0) \geq N$  and  $|a_{f(n')} - L| \leq \epsilon$ , as desired.

## Exercise 6.7.1

Say we have  $(q_n)_{n=1}^\infty$  of rationals converging to  $q$ , and  $(r_n)_{n=1}^\infty$  of rationals converging to  $r$ .

a)

$$\begin{aligned} x^q &= \lim_{n \rightarrow \infty} x^{q_n} \mid \& \geq \lim_{n \rightarrow \infty} 0 \& \text{As } \\ x^{q_n} &> 0 \text{ by 5.6.9 a), then 6.4.13} \end{aligned}$$

Hence  $x^q \geq 0$ . Assume  $x^q = 0$ , then  $(x^q)^{1/q} = 0^{1/q} = 0$ . By b),  $(x^q)^{1/q} = x$ , hence  $x = 0$ , contradicting  $x > 0$ . Hence  $x^q > 0$  as desired.

**b)**

We propose Lemma 1. Given  $a \in \mathbb{N}$ ,  $q \in \mathbb{R}$ , and a sequence  $(q_n)_{n=m}^{\infty}$  converging to  $q$ , it holds that:

$$(x^q)^a = x^{qa}$$

Proof:

$$\begin{aligned} (x^q)^a &= (\lim_{n \rightarrow \infty} x^{q_n})^a \& \text{By 6.7.2} \& \\ &= \lim_{n \rightarrow \infty} (x^{q_n})^a \& \text{By 6.1.19 b) \& Induction} \& \\ &= \lim_{n \rightarrow \infty} x^{q_n a} \& \text{By 5.6.9 b)} \& \\ &= x^{qa} \& \text{By 6.7.2 as } q_n \text{ converges to } q \end{aligned}$$

As desired.

We propose Lemma 2. Given  $a \in \mathbb{Z}$ ,  $q \in \mathbb{R}$ , and a sequence  $(q_n)_{n=m}^{\infty}$  converging to  $q$ , it holds that:

$$(x^q)^a = x^{qa}$$

Proof:

Case  $a \geq 0$  is covered by Lemma 1. For case  $a < 1$ , let  $-a' = a$  s.t.  $-a' > 0$ , then:

$$\begin{aligned} (x^q)^a &= (x^q)^{-a'} \& \\ &= (\lim_{n \rightarrow \infty} x^{q_n})^{-a'} \& \text{By 6.7.2} \& \\ &= ((\lim_{n \rightarrow \infty} x^{q_n})^{-1})^{a'} \& \text{By 5.6.9 b)} \& \\ &= (\lim_{n \rightarrow \infty} (x^{q_n})^{-1})^{a'} \& \text{By 6.1.19 e)} \& \\ &= \lim_{n \rightarrow \infty} (x^{q_n})^{-1})^{a'} \& \text{By 6.1.19 b) \& Induction} \& \\ &= \lim_{n \rightarrow \infty} x^{q_n(-a')} \& \text{By 5.6.9 b)} \& \\ &= \lim_{n \rightarrow \infty} x^{q_n a} \& \\ &= x^{qa} \& \text{By 6.7.2 as } q_n \text{ converges to } q \end{aligned}$$

For 6.1.19 e), it can be used as since  $x > 0$ ,  $x^{q_n} > 0$  by 5.6.9 d).

We propose Lemma 3. Given  $b \in \mathbb{Z} - \{0\}$ ,  $q \in \mathbb{R}$ , and a sequence  $(q_n)_{n=m}^{\infty}$  converging to  $q$ , it holds that:

$$(x^q)^{1/b} = x^{q(1/b)}$$

Proof:

By Proposition 5.6.3 & Proposition 4.3.12, if  $((x^q)^{1/b})^b = (x^{q(1/b)})^b$ , then  $(x^q)^{1/b} = x^{q(1/b)}$ . We proceed to show  $((x^q)^{1/b})^b = (x^{q(1/b)})^b$ . For LHS:

$$((x^q)^{1/b})^b = x^q \quad \text{By 5.6.6}$$

$$(x^{q(1/b)})^b = x^{q(1/b)b} = x^q \quad \text{By Lemma 2}$$

Hence  $LHS = RHS = x^q$ , as desired.

We propose Lemma 4: Given  $q \in \mathbb{Q}$ ,  $x \in \mathbb{R}$ , and a sequence  $(q_n)_{n=m}^\infty$  converging to  $q$ , it holds that:

$$(x^q)^p = x^{qp}$$

Proof:

Let  $p = a/b$  for some  $a, b \in \mathbb{Z}$ . Then:

$$(x^q)^p = (x^q)^{a/b} = ((x^q)^a)^{1/b} \quad \text{By 5.6.9 b)} = (x^{qa})^{1/b} \quad \text{By Lemma 2} = x^{qa(1/b)} \quad \text{By Lemma 3} = x^{qp}$$

As desired.

$$(x^q)^r = \lim_{n \rightarrow \infty} (x^q)^{r_n} \quad \text{By 6.7.2} = \lim_{n \rightarrow \infty} x^{qr_n} \quad \text{By Lemma 4} = x^{qr} \quad \text{By 6.7.2 as } qr_n \text{ converges to } qr$$

As desired.

**c)**

$$x^{-q} = (x^q)^{-1} \quad \text{By b)} = \frac{1}{x^q} \quad \text{By 5.6.9 c)}$$

As desired.

**d)**

If direction:

If  $x > y$ , then  $\forall n \geq 1, x^{q_n} > y^{q_n}$  by 5.6.9 d). Hence,  $\lim_{n \rightarrow \infty} x^{q_n} \geq \lim_{n \rightarrow \infty} y^{q_n}$  (By Lemma 6.4.13), hence  $x^q \geq y^q$ . Assume  $x^q = y^q$ , then by b),  $(x^q)^{1/q} = x, (y^q)^{1/q} = y$ , hence  $x = y$ , contradiction. Hence  $x^q > y^q$  as desired

Only if direction:

To show  $x^q > y^q \Rightarrow x > y$ . We prove the contrapositive  $x \leq y \Rightarrow x^q \leq y^q$ .

Since  $x \leq y$ ,  $\forall n \geq 1, x^{q_n} \leq y^{q_n}$  by 5.6.9 d). Hence,  $\lim_{n \rightarrow \infty} x^{q_n} \leq \lim_{n \rightarrow \infty} y^{q_n}$ , hence  $x^q \leq y^q$  as desired.

e)

Say  $x > 1$ .

If direction:

If  $q > r$ , then  $\forall n \geq 1, x^{q_n} > x^{r_n}$  (by 5.6.9 e). Hence  $\lim_{n \rightarrow \infty} x^{q_n} \geq \lim_{n \rightarrow \infty} x^{r_n}$  (By Lemma 6.4.13). However, since  $q \neq r$ , by 6.7.1,  $\lim_{n \rightarrow \infty} x^{q_n} \neq \lim_{n \rightarrow \infty} x^{r_n}$ . Hence,  $\lim_{n \rightarrow \infty} x^{q_n} > \lim_{n \rightarrow \infty} x^{r_n}$ . Therefore  $x^q > x^r$  as desired.

Only if direction:

We prove the contrapositive, aka  $q \leq r \Rightarrow x^q \leq x^r$ .

If  $q \leq r$ , then  $\forall n \geq 1, x^{q_n} \leq x^{r_n}$  (by 5.6.9 e). Hence  $\lim_{n \rightarrow \infty} x^{q_n} \leq \lim_{n \rightarrow \infty} x^{r_n}$  (By Lemma 6.4.13). Hence  $x^q \leq x^r$  as desired.

## Why? on Page 160

$g$  (when restricted to  $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ ) is a bijection from  $\{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow X - \{x\}$

For sake of simplicity call  $g$  with restricted domain  $g'$ . We argue by contradiction.

Assume  $g'$  is not injective, then:

$$\exists i, j, g'(i) \neq g'(j)$$

Since  $\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, g'(i) = g(i)$ , it holds that:

$$\exists i, j, g(i) \neq g(j)$$

Hence  $g$  is not injective. Contradiction.

We can similarly show that  $g'$  is surjective, completing the proof.

$\tilde{h}$  is a bijection from  $\{i \in \mathbb{N} : 1 \leq i \leq n\} \rightarrow X - \{x\}$

For injectivity, we argue by contradiction. Assume  $\tilde{h}$  is not injective, it holds that:

$$\exists p, q, p \neq q \wedge \tilde{h}(p) = \tilde{h}(q)$$

Recall definition of  $h$ :

$$\tilde{h}(i) = \begin{cases} h(i) & \text{when } i < j \\ h(i + 1) & \text{when } i \geq j \end{cases}$$

Hence, we either have  $\tilde{h}(p) = h(p)$ ,  $\tilde{h}(p) = h(p + 1)$ ,  $\tilde{h}(q) = h(q)$ ,  $\tilde{h}(q) = h(q + 1)$ .



Case 1:  $\tilde{h}(p) = h(p)$ ,  $\tilde{h}(q) = h(q)$

Hence  $p \neq q$  yet  $h(p) = h(q)$ . Therefore  $h$  is not injective, contradiction.

Case 2:  $\tilde{h}(p) = h(p + 1)$ ,  $\tilde{h}(q) = h(q + 1)$

Similarly  $p + 1 \neq q + 1$  yet  $h(p + 1) = h(q + 1)$ , contradiction.

Case 3:  $\tilde{h}(p) = h(p)$ ,  $\tilde{h}(q) = h(q + 1)$

Assume  $p \neq q + 1$ , then  $h(p) = h(q + 1)$ , contradiction.

Assume  $p = q + 1$ , then since  $\tilde{h}(q) = h(q + 1)$ ,  $q \geq j$ . Hence,  $p = q + 1 \geq j$ . Hence  $\tilde{h}(p) = h(p + 1)$ . Rest follows from Case 2.

For surjectivity. Since  $h$  is surjective, then:

$$\forall e \in X, \exists i' \in \{i \in \mathbb{N} : 1 \leq i \leq n+1\}, h(i') = e \quad \text{(1)}$$

To show  $\tilde{h}$  is surjective is equivalent to showing:

$$\forall e \in X - \{x\}, \exists i \in \{i \in \mathbb{N} : 1 \leq i \leq n+1\}, h(i) = e \quad \text{(2)}$$

Fix  $e$ , then by (1):

$$\exists i' \in \{i \in \mathbb{N} : 1 \leq i \leq n+1\}, h(i') = e \quad \text{(3)}$$

Hence, if  $i' < j$ , then note  $\tilde{h}(i') = h(i') = e$ . Hence setting  $i = i'$  is permissible in (2).

If  $i' \geq j$ , then  $\tilde{h}(i' + 1) = h(i') = e$ . Hence setting  $i = i' + 1$  is permissible in (2), as desired.

## Why? On page 163

### Why is $h$ a bijection?

Similar to the last Why?, some grunt work suffices to show that  $h$  is injective and surjective.

### Why does Proposition 7.1.11(c) give us what we want?

By Proposition 7.1.11 c), we have:

$$\sum_{(y,x) \in Y \times X} f(x,y) = \sum_{(x,y) \in X \times Y} f(h(x,y))$$

Since  $h(x,y) = (y,x)$ , we have:

$$\sum_{(x,y) \in X \times Y} f(h(x,y)) = \sum_{(y,x) \in Y \times X} f(y,x)$$

As desired.



## Exercise 7.1.1

a)

We can either induct on  $n$  or induct on  $p$ . (or maybe induct on  $m$ ). For sake of simplicity we induct on  $p$ .

Denote  $P(p)$  as: for any given integer  $m$ ,  $n$  s.t.  $m \leq n < p$ :

$$\sum_{i=m}^n a_i + \sum_{i=n+1}^p a_i = \sum_{i=m}^p a_i$$

When  $p \leq m$ , the inductive hypothesis is trivially true. Hence, consider the base case  $P(m+1)$ , we need to show:

$$\sum_{i=m}^{n+1} a_i + \sum_{i=n+1}^{m+1} a_i = \sum_{i=m}^p a_i$$

Since  $m \leq n < m+1$ , it holds that  $n = m$ . Hence:

$$\begin{aligned} \sum_{i=m}^n a_i + \sum_{i=n+1}^{m+1} a_i &= \sum_{i=m}^m a_i + \sum_{i=m+1}^{m+1} a_i \\ &= a_m + a_{m+1} \quad \text{\textit{By Definition 7.1.1}} \\ &= \sum_{i=m}^{m+1} a_i \quad \text{\textit{By Definition 7.1.1}} \end{aligned}$$

As desired.

For  $P(p+1)$ :

$$\begin{aligned} \sum_{i=m}^n a_i + \sum_{i=n+1}^{p+1} a_i &= (\sum_{i=m}^n a_i + \sum_{i=n+1}^p a_i) + a_{p+1} \quad \text{\textit{By Definition 7.1.1}} \\ &= \sum_{i=m}^p a_i + a_{p+1} \quad \text{\textit{By } } P(p) \\ &= \sum_{i=m}^{p+1} a_i \quad \text{\textit{By Definition 7.1.1}} \end{aligned}$$

As desired.

b)

We induct on  $k$ . Let  $P(n)$  denote:

$$\sum_{i=m}^n a_i = \sum_{j=m+k}^{n+k} a_{j-k}$$

For base case  $P(m)$ , by Definition 7.1.1:

$$\sum_{i=m}^m a_i = a_m = a_{m+k-k} = \sum_{j=m+k}^{m+k} a_{j-k}$$

As desired.

For  $P(n+1)$  given  $P(n)$ :

$$\begin{aligned} \sum_{j=m+k}^{n+k+1} a_{j-k} &= a_{n+1} + \sum_{j=m+k}^{n+k} a_{j-k} \\ &\quad \text{\textit{By Definition 7.1.1}} \\ &= a_{n+1} + \sum_{i=m}^n a_i \quad \text{\textit{By } } P(n) \\ &= \sum_{i=m}^{n+1} a_i \quad \text{\textit{By Definition 7.1.1}} \end{aligned}$$

As desired.

c)

We induct on  $n$ :

For base case  $n = m$ , easy to verify  $LHS = RHS = a_m + b_m$ .

For  $P(n + 1)$  given  $P(n)$ :

$$\begin{aligned} \sum_{i=m}^{n+1} (a_i + b_i) &= a_{n+1} + b_{n+1} + \sum_{i=m}^n (a_i + b_i) \\ &= a_{n+1} + b_{n+1} + \sum_{i=m}^n a_i + \sum_{i=m}^n b_i \\ &= (a_{n+1} + \sum_{i=m}^n a_i) + (b_{n+1} + \sum_{i=m}^n b_i) \\ &= \sum_{i=m}^{n+1} a_i + \sum_{i=m}^{n+1} b_i \end{aligned}$$

As desired.

d)

We induct on  $n$ :

For base case  $n = m$ , easy to verify  $LHS = RHS = c a_m$ .

For  $P(n + 1)$  given  $P(n)$ :

$$\begin{aligned} \sum_{i=m}^{n+1} c a_i &= c a_{n+1} + \sum_{i=m}^n c a_i \\ &= c a_{n+1} + c \sum_{i=m}^n a_i \\ &= c (a_{n+1} + \sum_{i=m}^n a_i) \\ &= c \sum_{i=m}^{n+1} a_i \end{aligned}$$

e)

We induct on  $n$ :

For base case  $n = m$ ,  $|a_m| \leq |a_m|$  as desired.

For  $P(n + 1)$  given  $P(n)$ :

$$\begin{aligned} \left| \sum_{i=m}^{n+1} a_i \right| &= |a_{n+1} + \sum_{i=m}^n a_i| \\ &\leq |a_{n+1}| + \left| \sum_{i=m}^n a_i \right| \quad \text{By Triangle Inequality} \\ &\leq |a_{n+1}| + \sum_{i=m}^n |a_i| \quad \text{By } P(n) \\ &= \sum_{i=m}^{n+1} |a_i| \end{aligned}$$

As desired.

f)

Follows from simple induction on  $n$ .

## Exercise 7.1.2

a)

By Definition 7.1.6, we choose a bijection  $g: \emptyset \rightarrow X$ , then:

$$\sum_{x \in X} f(x) = \sum_{i=1}^0 f(g(i))$$

Since  $0 < 1$ , by Definition 7.1.1 RHS equals to 0 as desired.

b)

Choose bijection  $g: \{1\} \rightarrow X$  s.t.  $g(1) = x_0$ , then:

$$\sum_{x \in X} f(x) = \sum_{i=1}^1 f(g(i)) = f(x_0)$$

As desired.

c)

Denote  $|X| = n$ , and denote bijection  $k: \{1 \leq i \leq n\} \rightarrow Y$ . Observe  $g \circ k$  is a bijection from  $\{1 \leq i \leq n\} \rightarrow X$ . Hence:

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{i=1}^n f(g(k(i))) = \sum_{y \in Y} f(g(y)) \end{aligned}$$

As desired.

d)

By Lemma 7.1.14 b):

$$\sum_{i=n}^m a_i = \sum_{i=1}^{m-(n-1)} a_{i+(n-1)}$$

We denote  $g: \{1 \leq i \leq m-(n-1)\} \rightarrow X$ , s.t.  $g(i) = i + (n-1)$ . Observe  $g$  is bijective.

Hence:

$$\sum_{i=n}^m a_i = \sum_{i=1}^{m-(n-1)} a_{i+(n-1)} = \sum_{i \in X} a_i$$

As desired.

e)

Denote  $|X| = n$ ,  $|Y| = m$ . Then since  $X \cap Y = \emptyset$ ,  $|X \cup Y| = n + m$ .

Then denote bijections  $h: \{1 \leq i \leq n\} \rightarrow X$ ,  $j: \{1 \leq i \leq m\} \rightarrow Y$ ,  $k: \{1 \leq i \leq n+m\} \rightarrow X \cup Y$ .

We define  $k$  as:

$$k(i) = \begin{cases} h(i) & \text{if } i \leq n \\ j(i-n) & \text{Otherwise} \end{cases}$$

Observe, under this definition,  $k$  is indeed bijective.

Hence:

$$\begin{aligned} \sum_{z \in X \cup Y} f(x) &= \sum_{i=1}^{n+m} f(k(i)) \mid &= \sum_{i=1}^n f(k(i)) + \sum_{i=n+1}^{n+m} f(k(i)) \mid &= \sum_{i=1}^n f(h(i)) + \sum_{i=n+1}^{n+m} f(j(i-n)) \mid &= \\ &= \sum_{i=1}^n f(h(i)) + \sum_{i=1}^m f(j(i)) \mid &= \sum_{x \in X} f(x) + \sum_{y \in Y} f(y) \end{aligned}$$

As desired.

f)

Let  $n = |X|$ . Choose a bijection  $h: \{1 \leq i \leq n\} \rightarrow X$ , then:

$$\begin{aligned} \sum_{x \in X} f(x) + g(x) &= \sum_{i=1}^n f(h(i)) + g(h(i)) \mid &= \sum_{i=1}^n f(h(i)) + \sum_{i=1}^n g(h(i)) \mid &= \sum_{x \in X} f(x) + \sum_{x \in X} g(x) \end{aligned}$$

As desired.

g)

Similarly to f)

h)

Similar to f)

## Exercise 7.1.3

### Define Finite Product

Let  $m, n$  be integers, and let  $(a_i)_{i=m}^n$  be a finite sequence of real numbers, assigning a real number  $a_i$  to each integer  $i$  between  $m$  and  $n$  inclusive (i.e.,  $m \leq i \leq n$ ). Then we define the finite product  $\prod_{i=m}^n a_i$  by the recursive formula  $\prod_{i=m}^n a_i = 1$  whenever  $n < m$

$$\prod_{i=m}^{n+1} a_i = (\prod_{i=m}^n a_i) \times a_{n+1} \text{ whenever } n \geq m - 1$$

### Which of the above result still holds?

Remark: some may require minor modification (e.g. replace addition sign with multiplication sign)

- 7.1.4 a), b), c)
- 7.1.4 d)
  - Scaling factor on RHS needs major modification
- 7.1.4 e)
  - Seems to become an equality instead
- 7.1.4 f)
  - If we restrict  $a_i, b_i \geq 0$ .
  - Otherwise... may need to check number of negative elements, determine if it is odd/even then make a case by case rule.

- 7.1.11 a)
  - RHS No longer \$0\$, should be \$1\$
- 7.1.11 b), c), d), e) f)
- 7.1.11 g)
  - Similar to 7.1.4 d), major change required
- 7.1.11 h)
  - Similar to 7.1.4 f), same constraint required
- 7.1.11 i)
  - Becomes equality

## Exercise 7.1.4

We induct on \$n\$:

For base case \$n = 0\$, \$LHS = (x + y)^0 = 1\$, \$RHS = \frac{0!}{0!(0-0)!}x^0y^{n-0} = 1\$, as desired.

For inductive case:

$$\begin{aligned}
 (x + y)^{n+1} &= (x + y)(x + y)^n \quad \&= (x + y) \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n-j} \\
 &= (x \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n-j}) + (y \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n-j}) \\
 &= \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} + \sum_{j=0}^n \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \\
 &= x^{n+1} + \sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} + y^{n+1} + \sum_{j=1}^n \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \\
 &= x^{n+1} + \sum_{j=1}^n \frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} + y^{n+1} \\
 &= x^{n+1} + \sum_{j=1}^n \left( \frac{n!}{(j-1)!(n+1-j)!} + \frac{n!}{j!(n-j)!} \right) x^j y^{n+1-j} + y^{n+1} \\
 &= x^{n+1} + \sum_{j=1}^n \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} + y^{n+1} = \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}
 \end{aligned}$$

As desired.

Remark:

The identity:

$$\frac{n!}{(j-1)!(n+1-j)!} + \frac{n!}{j!(n-j)!} = \frac{(n+1)!}{j!(n+1-j)!}$$

Can be proven as follows:

$$\begin{aligned}
 (n+1)! &= (n+1)! \cdot n! \cdot (n+1) \quad \&= (n+1)! \cdot n!(j+n+1-j) \quad \&= (n+1)! \cdot n!j + n!(n+1-j) \\
 &= (n+1)! \cdot n! \left( \frac{j}{j} (n+1-j) \right) + n! \left( \frac{n+1-j}{1} (n+1-j) \right) \quad \&= \frac{(n+1)!}{j} (n+1-j) \\
 &\quad + \frac{n!}{(j-1)!(n+1-j)!} + \frac{n!}{j!(n-j)!} \quad \&= \frac{(n+1)!}{j!(n+1-j)!} \quad \text{end{align}}
 \end{aligned}$$

With the division in the second last step can be done as \$j!(n+1-j)! \neq 0\$ for any \$n, j\$.

- We check division by \$0\$ as it is illegal. We also need to check multiplication by \$0\$ as we otherwise we can deduce \$3 = 1\$ via \$3 \cdot 0 = 1 \cdot 0\$. Cancellation law requires the

cancelled out part to be  $\neq 0$ .

As desired.

Remark 2: Above derivation may seem obscured at first. But, consider:

$$\begin{aligned} x(x+y)^2 &= xy^2 + 2x^2y + x^3 \\ y(x+y)^2 &= y^3 + 2xy^2 + x^2y \end{aligned}$$

Observe the two expressions shares common terms in the middle ( $xy^2$ ,  $x^2y$ ).

The scary looking algebra is merely the exploitation of such pattern.

## Exercise 7.1.5

Denote  $n = |X|$ . We induct on  $n$ .

If  $n = 0$ , then  $LHS = RHS = 0$  as desired.

For  $P(n+1)$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{x \in X} a_n(x) &= \lim_{n \rightarrow \infty} (a_n(x_0) + \sum_{x \in X - \{x_0\}} a_n(x)) \quad \text{\textit{By 7.1.11 e}} \\ &= \lim_{n \rightarrow \infty} a_n(x_0) + \lim_{n \rightarrow \infty} \sum_{x \in X - \{x_0\}} a_n(x) \quad \text{\textit{By 6.1.19 a}} \\ &= \lim_{n \rightarrow \infty} a_n(x_0) + \sum_{x \in X - \{x_0\}} \lim_{n \rightarrow \infty} a_n(x) \quad \text{\textit{By 3.6.9 \& P(n)}} \\ &= \sum_{x \in X} \lim_{n \rightarrow \infty} a_n(x) \quad \text{\textit{By 7.1.11 e}} \end{aligned}$$

As desired.

## Exercise 7.2.1

The series is divergent by alternating series test.

Example 1.2.2 used Proposition 7.2.14 c) d), for which, both requires the series to be convergent, yet the series is divergent.

## Exercise 7.2.2

By Theorem 6.4.18,  $\sum_{n=m}^{\infty} a_n$  converges iff  $(\sum_{n=m}^N a_n)_{N=m}^{\infty}$  is a Cauchy Sequence.

We remains to show:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q a_n \right| \leq \epsilon$$

iff  $(\sum_{n=m}^N a_n)_{N=m}^{\infty}$  being a Cauchy Sequence.

If:

Note that case  $p > q$  is trivial. Hence we consider the case  $p \leq q$ .

Since  $(\sum_{n=m}^N a_n)^{\infty}_{N=m}$  is a Cauchy Sequence, then:

$$\forall \epsilon' > 0, \exists N' \geq m, \forall p', q' \geq N', \left| \sum_{n=m}^{q'} a_n - \sum_{n=m}^{p'} a_n \right| \leq \epsilon'$$

Observe, since  $p \leq q$ :

$$\sum_{n=m}^{q'} a_n - \sum_{n=m}^{p'} a_n = \sum_{n=p'+1}^{q'} a_n$$

Fix  $\epsilon$ , we then can choose  $\epsilon' = \epsilon$ , then:

$$\exists N' \geq m, \forall p', q' \geq N', \left| \sum_{n=m}^{q'} a_n - \sum_{n=m}^{p'} a_n \right| \leq \epsilon$$

We then choose  $N = N' + 1$ . It is permissible as, for any  $p, q \geq N$ , we have  $q, p - 1 \geq N'$ , hence:

$$\begin{aligned} \left| \sum_{n=p}^q a_n \right| &= \sum_{n=(p-1)+1}^q a_n \leq \epsilon \end{aligned}$$

As desired

Only if:

We have:

$$\forall \epsilon' > 0, \exists N' \geq m, \forall p', q' \geq N', \left| \sum_{n=p'}^{q'} a_n \right| \leq \epsilon'$$

We need to show:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=m}^q a_n - \sum_{n=m}^p a_n \right| \leq \epsilon$$

Fix  $\epsilon$ . We let  $\epsilon' = \epsilon$ , then:

$$\exists N' \geq m, \forall p', q' \geq N', \left| \sum_{n=p'}^{q'} a_n \right| \leq \epsilon$$

Then, we claim  $N = N'$  is feasible. As, for any  $p, q \geq N$ :

$$\begin{aligned} \left| \sum_{n=m}^q a_n - \sum_{n=m}^p a_n \right| &= \left| \sum_{n=p+1}^q a_n \right| \leq \epsilon \end{aligned}$$

## Exercise 7.2.3

By Proposition 7.2.5, we have:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q a_n \right| \leq \epsilon$$

Hence, setting  $p = q$  yields:

$$\forall \epsilon > 0, \exists N \geq m, \forall p \geq N, \left| \sum_{n=p}^p a_n \right| \leq \epsilon$$

Hence:

$$\forall \epsilon > 0, \exists N \geq m, \forall p \geq N, |a_p| \leq \epsilon$$

Therefore, by definition of convergence,  $(a_n)_{n=m}^{\infty}$  converges to  $0$ , as desired.

## Exercise 7.2.4

Part 1: If the series is absolutely convergent, it is conditionally convergent.

Since the series is absolutely convergent, Proposition 7.2.5, it holds that:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q |a_n| \right| \leq \epsilon$$

Observe, by triangle inequality:

$$\left| \sum_{n=p}^q a_n \right| \leq \sum_{n=p}^q |a_n| = \left| \sum_{n=p}^q |a_n| \right| \leq \epsilon$$

Hence:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q a_n \right| \leq \epsilon$$

Conditional convergence follows by Proposition 7.2.5, as desired.

Part 2: Triangle inequality holds for absolutely convergent infinite series

Follows from Corollary 5.4.10

## Exercise 7.2.5

a)

Follows from 6.1.19 a)

b)

Follows from 6.1.19 c)

c)



Say  $(\sum_{n=m+k}^{\infty} a_n)$  is convergent. By 7.1.4 a)

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n$$

Then, since both  $\sum_{n=m}^{m+k-1} a_n$ ,  $\sum_{n=m+k}^{\infty} a_n$  are convergent, by Theorem 6.1.19 a),  $\sum_{n=m}^{\infty} a_n$  also converges and:

$$\lim_{N \rightarrow \infty} \sum_{n=m}^N a_n = \lim_{N \rightarrow \infty} \sum_{n=m}^{m+k-1} a_n + \lim_{N \rightarrow \infty} \sum_{n=m+k}^N a_n = \sum_{n=m}^{m+k-1} a_n + \lim_{N \rightarrow \infty} \sum_{n=m+k}^{\infty} a_n$$

As desired.

Similarly, for the case of  $(\sum_{n=m}^{\infty} a_n)$  being convergent, we can use Theorem 6.1.19 d).

d)

Observe  $\sum_{n=m}^N a_n = \sum_{n=m+k}^{N+k} a_{n-k}$ .

Hence, if the sequence  $(\sum_{n=m}^N a_n)$  converges, the sequence  $(\sum_{n=m+k}^{N+k} a_{n-k})$  converges to the same value. Then, observe:

$$\sum_{n=m+k}^{N+k} a_{n-k} = \sum_{n=m+k}^{N+k} a_{n-k} - \sum_{n=N+1}^{N+k} a_{n-k}$$

We then show  $(\sum_{n=N+1}^{N+k} a_{n-k})$  is convergent. By Exercise 6.1.3 and Lemma 7.1.4 b) equivalent of showing  $(\sum_{n=N-k+1}^N a_n)$  converges. We claim it converges to 0 by contradiction. Since  $(\sum_{n=m}^N a_n)$  converges, by Proposition 7.2.5:

$$\forall \epsilon > 0, \exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q a_n \right| \leq \epsilon$$

Since  $(\sum_{n=N-k+1}^N a_n)$  does not converge to 0, then:

$$\exists \epsilon > 0, \forall L \geq m+k-1, \exists N_0 \geq L, \left| \sum_{n=N_0-k+1}^{N_0} a_n \right| > \epsilon$$

Pick this  $\epsilon$ , then by (1):

$$\exists N \geq m, \forall p, q \geq N, \left| \sum_{n=p}^q a_n \right| \leq \epsilon$$

Take  $N$  from (3), set  $L = \max(m+k-1, N)$  in (2) yields:

$$\exists N_0 \geq L, \left| \sum_{n=N_0-k+1}^{N_0} a_n - L \right| > \epsilon$$

Yet, since  $N_0 - k + 1 \geq L \geq m+k-1$ ,  $N_0 \geq N$ . In addition,  $N_0 - k + 1 \geq L \geq N$ . Hence, setting  $p = N_0 - k + 1$ ,  $q = N_0$  in (3) yields:

$$\left| \sum_{n=N_0-k+1}^{N_0} a_n - L \right| \leq \epsilon$$

Contradicting (4), as desired.

Recall:

$$\sum_{n=m+k}^N a_{n-k} = \sum_{n=m+k}^{N+k} a_{n-k} - \sum_{n=N+1}^{N+k} a_{n-k}$$

Since  $\sum_{n=N+1}^{N+k} a_{n-k}$  converges to 0,  $\sum_{n=m+k}^{N+k} a_{n-k}$  converges to  $\sum_{n=m}^{\infty} a_n$ , by Theorem 6.1.19 d),  $\sum_{n=m+k}^N a_{n-k}$  converges to  $\sum_{n=m}^{\infty} a_n - 0 = \sum_{n=m}^{\infty} a_n$ , as desired.

The converse can be shown similarly.

## Exercise 7.2.6

We start with Lemma:

$$\sum_{n=0}^N (a_n - a_{n+1}) = a_0 - a_{N+1}$$

Proof: We induct on  $N$ . For  $N = 0$ :

$$\sum_{n=0}^0 (a_n - a_{n+1}) = a_0 - a_1$$

As desired.

For  $P(n+1)$  given  $P(n)$ :

$$\begin{aligned} \sum_{n=0}^{N+1} (a_n - a_{n+1}) &= a_{N+1} - a_{N+2} + \sum_{n=0}^N (a_n - a_{n+1}) \\ &= a_{N+1} - a_{N+2} + a_0 - a_{N+1} \quad \text{By } P(n) \\ &= a_0 - a_{N+2} \end{aligned}$$

This closes the induction.

Hence, showing the infinite sum converges is equivalent to showing whether  $(a_0 - a_{N+2})^{\infty}_{N=0}$  converges. Since both  $(a_0)^{\infty}_{N=0}$ ,  $(a_{N+2})^{\infty}_{N=0}$  converges, by Theorem 6.1.19 d)  $(a_0 - a_{N+2})^{\infty}_{N=0}$  converges to  $a_0 - 0 = a_0$ , as desired.

**How does the proposition change if we assume that  $a_n$  does not converge to zero, but instead converges to some other real number  $L$ ?**

With the same derivation, easy to show the infinite series converges to  $a_0 - L$ .

## Exercise 7.3.1

Observe, since  $b_n \geq |a_n|$ , by 7.1.4 f):

$$\left| \sum_{n=m}^N a_n \right| \leq \sum_{n=m}^N |a_n| \leq \sum_{n=m}^N b_n$$

Since  $\sum_{n=m}^N b_n$  is convergent, then by Proposition 7.1.3,  $\forall N \geq m, \sum_{n=m}^N b_n \leq M$

Hence:

$$\sum_{n=m}^N |a_n| \leq \sum_{n=m}^N b_n \leq M$$

Since  $\sum_{n=m}^N |a_n| \leq M$ , by Proposition 7.1.3,  $\sum_{n=m}^\infty |a_n|$  is convergent, hence  $\sum_{n=m}^\infty a_n$  is absolutely convergent.

Furthermore, by Corollary 5.4.10:

$$\sum_{n=m}^\infty |a_n| \leq \sum_{n=m}^\infty b_n \leq M$$

Remaining part of the inequality follows from Proposition 7.2.9, as desired.

## Exercise 7.3.2

**If  $|x| \geq 1$ , then the series  $\sum_{n=0}^\infty x^n$  is divergent**

Since  $|x| \geq 1$ , by 4.3.12 d),  $|x|^n \geq 1$ . Hence,  $|x^n| \geq 1$ . We claim  $\lim_{n \rightarrow \infty} x^n \neq 0$ . Assume  $\lim_{n \rightarrow \infty} x^n = 0$ , then by Corollary 6.4.17,  $\lim_{n \rightarrow \infty} |x^n| = 0$ . However, since  $|x^n| \geq 1$ , by Corollary 5.4.10,  $\lim_{n \rightarrow \infty} |x^n| \geq 1$ , contradiction.

Since  $\lim_{n \rightarrow \infty} x^n \neq 0$ , by Corollary 7.2.6,  $\sum_{n=0}^\infty x^n$  diverges, as desired.

## Second Part

Lemma:

$$\sum_{n=m}^N x^n = \frac{1 - x^{N+1}}{1 - x}$$

Proof:

We induct on  $N$ . If  $N = m$ , then  $LHS = RHS = 1$ , as desired.

For  $P(N + 1)$  given  $P(N)$ :

$$\begin{aligned} \sum_{n=m}^{N+1} x^n &= \sum_{n=m}^N x^n + x^{N+1} \quad \&= \frac{1 - x^{N+1}}{1 - x} + x^{N+1} \quad \&= \frac{1 - x^{N+1} + (1 - x)x^{N+1}}{1 - x} \quad \&= \frac{1 - x^{N+2}}{1 - x} \end{aligned}$$

As desired.

Then, we remain to show:

$$\lim_{N \rightarrow \infty} \frac{1 - x^{N+1}}{1 - x}$$

exists and equals to  $\frac{1}{1-x}$ :

Proof:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1-x^{N+1}}{1-x} &= \lim_{N \rightarrow \infty} \frac{1}{1-x} - \lim_{N \rightarrow \infty} \frac{x^{N+1}}{1-x} \quad \text{(By 6.1.19 a)} \\ &= \frac{1}{1-x} - 0 \quad \text{(By 6.1.19 c)} \\ &= \frac{1}{1-x} \quad \text{As } |x| < 1 \end{aligned}$$

One can verify the conditions for 6.1.19 a), c) are met, as desired.

### Exercise 7.3.3

We argue by contradiction. Assume  $a_N \neq 0$  for some  $N \in \mathbb{N}$ . Then, by 7.2.14 c):

$$\begin{aligned} \sum_{n=0}^N |a_n| + \sum_{n=N+1}^{\infty} |a_n| &= \sum_{n=0}^{\infty} |a_n| - \sum_{n=0}^{N-1} |a_n| + |a_N| + \sum_{n=N+1}^{\infty} |a_n| \\ &= \sum_{n=0}^{\infty} |a_n| - \sum_{n=0}^{N-1} |a_n| + |a_N| + \sum_{n=N+1}^{\infty} |a_n| \\ &= 0 - \sum_{n=0}^{N-1} |a_n| + |a_N| + \sum_{n=N+1}^{\infty} |a_n| = -|a_N| \end{aligned}$$

Hence, since  $a_N \neq 0$ ,  $|a_N| > 0$ , therefore  $\sum_{n=0}^{N-1} |a_n| + \sum_{n=N+1}^{\infty} |a_n| < 0$ .

By triangle inequality,  $\sum_{n=0}^{N-1} |a_n| \geq 0$ . By Corollary 7.3.2,  $\sum_{n=N+1}^{\infty} |a_n| \geq 0$ .  $\sum_{n=0}^{N-1} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \geq 0$ , contradicting  $\sum_{n=0}^{N-1} |a_n| + \sum_{n=N+1}^{\infty} |a_n| < 0$ , as desired.

Remark: One can also use a direct proof via directly evaluating  $|a_N|$  for any  $N \in \mathbb{N}$  and show  $|a_N| = 0$ , in a manner similar to the presented proof.

### Exercise 7.4.1

Since  $f(n+1) > f(n)$ ,  $f$  is injective via simple induction. Hence, restricting the domain of  $f$  to  $\{m \in \mathbb{N} : m \leq M\} \rightarrow \{f(m) : m \leq M\}$ ,  $f$  is bijective.

Therefore, by Proposition 7.1.11 c):

$$\sum_{m=0}^M |a_{f(m)}| = \sum_{n \in \{f(m) : m \leq M\}} |a_n|$$

Observe  $\{f(m) : m \leq M\}$  has a least upper bound. Hence, by 5.4.12, it is bounded above by some integer  $N$ . Since  $f: \mathbb{N} \rightarrow \mathbb{N}$ ,  $\{f(m) : m \leq M\}$  is bounded below by  $0$ . Hence,  $\{f(m) : m \leq M\} \subseteq \{n \in \mathbb{N} : 0 \leq n \leq N\}$ . Hence, by Proposition 7.1.11 e):

$$\begin{aligned} \sum_{n \in \{n \in \mathbb{N} : 0 \leq n \leq N\}} |a_n| &= \sum_{n \in \{f(m) : m \leq M\}} |a_n| + \sum_{n \in \{n \in \mathbb{N} : 0 \leq n \leq N\} - \{f(m) : m \leq M\}} |a_n| \\ &\geq \sum_{n \in \{f(m) : m \leq M\}} |a_n| \end{aligned}$$

Since  $\sum_{n \in \{n \in \mathbb{N} : 0 \leq n \leq N\}} |a_n| = \sum_{n=0}^N |a_n|$ , by Proposition 7.3.1:

$$\sum_{n \in \{n \in \mathbb{N}, 0 \leq n \leq N\}} |a_n| \leq \sum_{n=0}^{\infty} |a_n|$$

Hence, for any given  $M \in \mathbb{N}$ :

$$\sum_{m=0}^M |a_{f(m)}| = \sum_{n \in \{f(m): m \leq M\}} |a_n| \leq \sum_{n \in \{n \in \mathbb{N}, 0 \leq n \leq N\}} |a_n| \leq \sum_{n=0}^{\infty} |a_n|$$

Therefore,  $\sum_{m=0}^M |a_{f(m)}|$  is bounded above, by Proposition 7.3.1,  $\sum_{m=0}^{\infty} |a_{f(m)}|$  is convergent as desired.

Remark: Proposition 7.1.11 c)'s bijectivity requirement is insidious. Very easy to miss.

## Exercise 7.5.1

Let  $L = \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$ . Then, since  $(c_n)_{n=m}^{\infty}$  is always positive,  $\frac{c_{n+1}}{c_n}$  is always positive, hence by Lemma 6.4.13:

$$\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

Since  $LHS = 0$ , it holds that:

$$0 \leq \lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}$$

Hence  $0 \leq L$ . If  $0 = L$ , there's nothing to prove. As, by Lemma 5.6.9 a), since  $c_n$  is positive real,  $(c_n)^{1/n}$  is positive real, hence by Lemma 6.4.13:

$$\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} (c_n)^{1/n}$$

And  $LHS = L$ , as desired.

If  $0 < L$ , Let  $\epsilon > 0$  s.t.  $L - \epsilon > 0$ , (i.e.  $0 < \epsilon < L$ ), by Proposition 6.4.12 c), there exists  $N \geq m$  s.t.  $L - \epsilon \leq \frac{c_{n+1}}{c_n}$  for all  $n \geq N$ , aka  $c_n(L - \epsilon) \leq c_{n+1}$

Hence, we claim, for all  $n \geq N$ :

$$c_{N(L-\epsilon)^{n-N}} \leq c_n$$

Proof: we induct on  $n$ . For  $P(n)$ ,  $c_{N(L-\epsilon)^{N-N}} \leq c_N$  as desired.

For  $P(n+1)$ :

$$\begin{aligned} c_{n+1} &\geq c_{N(L-\epsilon)} \geq c_{N(L-\epsilon)^{n-N}(L-\epsilon)} \quad \text{By } P(n) \\ &= c_{N(L-\epsilon)^{(n+1)-N}} \end{aligned}$$

Closing the induction.

Hence, let  $A = c_{N(L-\epsilon)^{-N}}$ , we have:

$$\begin{aligned} A(L-\varepsilon)^n &\leq c_n A^{1/n}(L-\varepsilon) \leq c_n^{1/n} && \text{(By 5.6.9 e)} \\ \liminf_{n \rightarrow \infty} A^{1/n}(L-\varepsilon) &\leq \liminf_{n \rightarrow \infty} c_n^{1/n} && \text{(By 6.4.13)} \end{aligned}$$

To evaluate  $\lim_{n \rightarrow \infty} A^{1/n}(L-\varepsilon)$ , we claim  $\lim_{n \rightarrow \infty} A^{1/n}(L-\varepsilon)$  exists and evaluate  $\lim_{n \rightarrow \infty} A^{1/n}(L-\varepsilon)$  instead. First note that since  $L - \varepsilon > 0$ ,  $(L - \varepsilon)^{-N} > 0$  by Proposition 6.7.3 and Lemma 5.6.9 a). consequently since  $c_N > 0$ ,  $A = c_N(L - \varepsilon)^{-N} > 0$ , therefore by Lemma 6.5.4 a):

$$\lim_{n \rightarrow \infty} A^{1/n} = 1$$

Hence:

$$\lim_{n \rightarrow \infty} A^{1/n}(L-\varepsilon) = (L-\varepsilon) \lim_{n \rightarrow \infty} A^{1/n} \quad \text{(By 6.1.19 c))} = (L-\varepsilon)$$

Since  $\liminf_{n \rightarrow \infty} A^{1/n}(L-\varepsilon) \leq \liminf_{n \rightarrow \infty} c_n^{1/n}$ , we have:

$$\forall 0 < \varepsilon < L, \quad L - \varepsilon \leq \liminf_{n \rightarrow \infty} c_n^{1/n}$$

To show  $L \leq \liminf_{n \rightarrow \infty} c_n^{1/n}$ , we argue by contradiction. Assume  $L > \liminf_{n \rightarrow \infty} c_n^{1/n}$ , then  $L - (L - \liminf_{n \rightarrow \infty} c_n^{1/n})/2 > \liminf_{n \rightarrow \infty} c_n^{1/n}$ . However,  $0 < (L - \liminf_{n \rightarrow \infty} c_n^{1/n})/2 < L$ . Hence, let  $\varepsilon = (L - \liminf_{n \rightarrow \infty} c_n^{1/n})/2$ , we have  $L - \varepsilon - \liminf_{n \rightarrow \infty} c_n^{1/n} > 0$ , contradiction, as desired.

## Exercise 7.5.2

If  $x = 0$ , then the series trivially converges to 0.

If  $x \neq 0$ , then every term is non-negative. Hence, by Corollary 7.5.3:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1)^q x^{n+1}}{n^q x^n} &= \\ |x| \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^q &\leq \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^q \\ &\text{(Since } |x| < 1 \text{)} \end{aligned}$$

If  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^q = 1$ , then  $\lim_{n \rightarrow \infty} \frac{(n+1)^q x^{n+1}}{n^q x^n} < 1$ , hence the series is absolute convergent. We remain to show  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^q = 1$ . Observe, we have  $N \leq q < N+1$  for some integers  $N$  by Exercise 5.4.3. By Lemma 5.6.9 e), we have:

$$\left(\frac{n+1}{n}\right)^N \leq \left(\frac{n+1}{n}\right)^q \leq \left(\frac{n+1}{n}\right)^{N+1}$$

Via Proposition 7.5.4 & induction, easy to show  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^N = 1$  for any  $N \in \mathbb{N}$ , then via Theorem 6.1.19 e) easy to show  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{N+1} = 1$  for any  $N \in \mathbb{N}$ , hence:

Consequently, by Corollary 6.4.14,  $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^q = 1$ , as desired.

$\lim_{n \rightarrow \infty} n^q x^n = 0$  follows from Corollary 7.2.6, as desired.

### Exercise 7.5.3

Use Corollary 7.3.7:

Divergent series:  $\sum_{n=1}^{\infty} 1/n$ .

Convergent series:  $\sum_{n=1}^{\infty} 1/n^2$ .

Easy to verify the ratio for both is 1. And hence by Lemma 7.5.2,  $\alpha$  for root test is also 1, as desired.

### Exercise 8.1.1

**There exists a proper subset  $Y \subsetneq X$  of  $X$  which has the same cardinality as  $X$  implies  $X$  is infinite**

We prove the contrapositive. Assume  $X$  is finite, then by Proposition 3.6.14 c), for all proper subset  $Y \subsetneq X$ ,  $|Y| < |X|$ , hence  $|Y| \neq |X|$ , as desired.

**$X$  is infinite implies there exists a proper subset  $Y \subsetneq X$  of  $X$  which has the same cardinality as  $X$**

We first establish that  $X$  is infinite implies  $X$  has a countable subset  $C$ .

Remark: This may seem as an arbitrary move. However, since it is more tractable to show a countable subset  $C$  has proper subset  $C - \{c\}$  with the same cardinality as  $C$ , taking a countable subset may help to reduce a less tractable problem to a more tractable problem.

Then for every natural number  $n$ , define  $A_n: \{S \subseteq X: |S| = n\}$ .  $A_n$  is a valid set by Lemma 3.4.9 and Axiom of Specification.

Remark 2: Similar to our proof for Exercise 3.5.12, to build an infinite set, one way is to establish series of finite approximations and then combine them.

We proceed to prove that  $A_n: \{S \subseteq X: |S| = n\}$  is non-empty for all  $n \in \mathbb{N}$ . Assume there exists  $n$  s.t.  $A_n$  is empty. Then, by Proposition 8.1.4, there exists minimum  $n$  s.t.  $A_n$  is empty. Call it  $n_0$ . We proceed by cases. If  $n_0 = 0$ , then  $\emptyset \in \{S \subseteq X: |S| = n\}$ , contradiction. If  $n_0 > 0$ , then  $A_{n_0 - 1}$  is non-empty. Take  $a_{n_0 - 1} \in A_{n_0 - 1}$ , we claim  $X - a_{n_0 - 1}$  is non-empty. (Assume  $X - a_{n_0 - 1}$  is empty, then  $X \subseteq a_{n_0 - 1}$ , hence by Proposition 3.6.14 d),  $X$  is finite, contradiction.) Hence, take  $x \in X - a_{n_0 - 1}$ , easy to verify  $a_{n_0 - 1} \cup \{x\} \in A_{n_0}$ , contradiction.

Therefore, by the Axiom of Choice, there exists  $f: \mathbb{N} \rightarrow \bigcup_{n \in \mathbb{N}} A_n$  such that  $\forall n \in \mathbb{N}, f(n) \in A_n$ .



Then, we let  $C = \bigcup_{n \in \mathbb{N}} f(n)$ . Easy to verify  $C \subseteq X$ . Remains to show  $C$  is countable. We proceed to show  $C$  is at most countable.

We prove a more general statement, we claim:

$$\bigcup_{n \in \mathbb{N}} Q_n$$

is at most countable if each  $Q_n$  is at most countable. (Lemma 1)

Remark 3: Say we have some at most countable sets  $Q_0, Q_1, \dots$ , then a natural way to represent them is to write them out as a 2D array.

$f(n)$	$Q_0$	$Q_1$	$Q_2$	$Q_3$
$f(0)$	$a$			
$f(1)$		$b$		
$f(2)$	$b$			$a$
$f(3)$	$b$		$a$	$c$

Then, a natural idea is map each element in  $\bigcup_{n \in \mathbb{N}} f(n)$  to  $\mathbb{N} \times \mathbb{N}$  with each element's coordinate in the 2D array.

However, the coordinates are not unique (e.g.  $a$  is at  $(0, 0)$ ,  $(3, 2)$ ,  $(2, 3)$ ) at the same time. Hence we instead aim for an injection by taking the coordinate with the smallest  $y$  axis.

Formalizing the idea from Remark 3, since  $n \in \mathbb{N}$ ,  $Q_n$  is at most countable, hence by Exercise 8.1.6, there exists injection from  $Q_n \rightarrow \mathbb{N}$ . Hence, the set

$$T_n = \{t \mid t \text{ is an injection with domain } Q_n \text{ and range } \mathbb{N}\}$$

is non-empty. Hence, by Axiom of choice, for each  $n \in \mathbb{N}$  we have a sequence  $(t_n)_{n \in \mathbb{N}}$  s.t.  $t_n: Q_n \rightarrow \mathbb{N}$  is an injection.

We define  $u: \bigcup_{n \in \mathbb{N}} Q_n \rightarrow \mathbb{N} \times \mathbb{N}$  as:

$$u(x) = (t_{n_x}(x), n_x)$$

For which  $n_x = \min\{n \in \mathbb{N} : x \in Q_n\}$ , which exists by Proposition 8.1.4.

We then claim  $u$  is an injection. Take  $x, x' \in \bigcup_{n \in \mathbb{N}} Q_n$  s.t.  $x \neq x'$ . Then, we claim  $(t_{n_x}(x), n_x) \neq (t_{n_{x'}}(x'), n_{x'})$ . We proceed by cases. If  $n_{x'} \neq n_x$ , then  $(t_{n_x}(x), n_x) \neq (t_{n_{x'}}(x'), n_{x'})$ , as desired. If  $n_{x'} = n_x$ , then  $t_{n_x}(x) = t_{n_{x'}}(x')$ . Since  $t_{n_{x'}}$  is an injection and  $x \neq x'$ ,  $t_{n_x}(x) = t_{n_{x'}}(x') = t_{n_{x'}}(x)$  as desired.



Since, by Corollary 8.1.13,  $\mathbb{N} \times \mathbb{N}$  is countable, there exists bijection  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . Hence, by Exercise 3.3.2,  $f \circ u: \bigcup_{n \in \mathbb{N}} Q_n \rightarrow \mathbb{N}$  is injective. By Exercise 8.1.6,  $\bigcup_{n \in \mathbb{N}} Q_n$  is at most countable, as desired.

Since each  $f(n)$  is at most countable, by Lemma 1,  $C = \bigcup_{n \in \mathbb{N}} f(n)$  is at most countable. We then argue  $C$  is not finite. Assume  $C$  is finite, then since  $f(|C| + 1) \subseteq C$ , by Proposition 3.6.14 c),  $|f(|C| + 1)| \leq |C|$ . However  $|f(|C| + 1)| = |C| + 1 > |C|$ , contradiction. Since  $C$  is at most countable and  $C$  is not finite,  $C$  is countable.

Since  $C$  is countable, take  $c \in C$ . By Corollary 8.1.7,  $C - \{c\}$  is at most countable. We argue  $C - \{c\}$  is countable. Assume  $C - \{c\}$  is finite, then By Proposition 3.6.14 b),  $C = (C - \{c\}) \cup \{c\}$  is finite, contradiction.

Since  $C, C - \{c\}$  are countable, there exists bijection  $g: C \rightarrow C - \{c\}$ .

We then define  $h: X \rightarrow X - \{c\}$  as:

$$h(x) = \begin{cases} g(x) & \text{if } x \in C \\ x & \text{Otherwise} \end{cases}$$

We claim  $g$  is a bijection.

We argue  $g$  is injective by contradiction. If  $\exists a, b \in X, a \neq b \text{ and } h(a) = h(b)$ , then either  $a, b \in C, a \in C \text{ and } b \notin C, a, b \notin C$ . We proceed by cases. Assume  $a, b \in C$ , then  $\exists a, b \in C, a \neq b \text{ and } g(a) = g(b)$ , hence  $g$  is not injective, contradiction. Assume  $a \in C \text{ and } b \notin C$ , then  $g(a) \in C - \{c\}$  and  $g(b) = b \notin C$ . Since  $g(a) \in C - \{c\}, g(a) \in C$ , yet  $g(a) = g(b) \notin C$ , contradiction. Assume  $a, b \notin C$ , then  $h(a) = a, h(b) = b$ , hence  $h(a) = h(b)$  implies  $a = b$ , contradiction.

We argue  $g$  is surjective by cases, for all  $y \in C - \{c\}$ , we have  $h(g^{-1}(y)) = y$ . For all  $y \notin C - \{c\}$ , we have  $h(y) = y$ . Hence in either case there exist  $x \in X$  s.t.  $h(x) = y$ .

Since  $h: X \rightarrow X - \{c\}$  is a bijection,  $|X| = |X - \{c\}|$ . Since  $|X| = |X - \{c\}|$  and  $X - \{c\} \subsetneq X$ ,  $Y = X - \{c\}$  is permissible, as desired.

## Exercise 8.1.2

A inductive proof is presented in the solution for Exercise 6.6.3.

Instead, we present a proof by infinite descent.

Assume the contrapositive:

$$\forall n \in X, \exists m \in X, n > m \quad \text{(1)}$$

Then take an element  $e_0 \in X$ . By (1),  $\exists e_1 \in X, e_0 > e_1$ . Hence, we can recursively construct a sequence  $(e_n)_{n=0}^{\infty}$  s.t.  $e_n > e_{n+1}$ , yet  $\forall n \in \mathbb{N}, e_n \in \mathbb{N}$ , contradiction by infinite descent.

## Exact Construction of $(e_n)_{n=0}^\infty$

Since  $\forall n \in X, \exists m \in X, n > m$ , for every  $x \in X$ , the set  $A_x = \{y \in X: y < x\}$  is non-empty. Therefore we define the choice function  $g: X \rightarrow \bigcup_{x \in X} A_x$  s.t.  $\forall x \in X, g(x) \in A_x$ . Therefore  $\forall x \in X, g(x) < x$ . Then, we take  $c \in X$ . Finally by Exercise 3.5.12, there exists sequence  $(e_n)_{n=0}^\infty$  s.t.

$$e_0 = c \quad \forall n \in \mathbb{N}, e_{n+1} = f(n, e_n)$$

for which  $f(n, e_n) = g(e_n)$ .

Since  $e_{n+1} = f(n, e_n) = g(e_n)$ , it holds that  $\forall n \in \mathbb{N}, e_{n+1} < e_n$ . Since  $g$  has range  $\bigcup_{x \in X} A_x \subseteq X$ , easy to verify  $\forall n \in \mathbb{N}, e_n \in X$ , as desired.

Remark: it turns out to demonstrate existence of a sequence for which we make arbitrary choice at each step (e.g. multiple permissible  $e_{n+1}$  given a single  $e_n$ ), there's a special Axiom called [Axiom of Dependent Choice](#). And, its proof from Axiom of Choice is identical to the above reasoning.

**Does the well-ordering principle work if we replace the natural numbers by the integers? What if we replace the natural numbers by the positive rationals?**

The well-ordering principle doesn't work if we replace the natural numbers by the integers. For example, we can have  $\{-1, -2, -3, -4, \dots\}$ , which does not have a minimum element.

The well-ordering principle doesn't apply to sets of positive rational. For example, consider the set  $\{1/n: n \in \mathbb{Z}^+\}$ . Given any positive rational  $q$ , we can find  $1/n < q$ , hence the set has no minimal element.

## Exercise 8.1.3

**Since  $X$  is infinite, the set  $\{x \in X: x \neq a_m \text{ for all } m < n\}$  is infinite**

We argue by contradiction. Assume  $\{x \in X: x \neq a_m \text{ for all } m < n\}$  is finite for some  $n$ . Easy to verify  $X = \{x \in X: P(x)\} \cup \{x \in X: \neg P(x)\}$  for any property  $P(x)$ . Let  $P(x)$  be  $x \neq a_m \text{ for all } m < n$ , then:

$$X = \{x \in X: x \neq a_m \text{ for all } m < n\} \cup \{x \in X: x = a_m \text{ for some } m < n\}$$

We claim  $\{x \in X: x = a_m \text{ for some } m < n\}$  is finite.

We induct on  $n$ . Let  $P(n)$  denote  $\{x \in X: x = a_m \text{ for some } m < n\}$  is finite. Then for  $P(0)$ ,  $\{x \in X: x = a_m \text{ for some } m < 0\}$  is empty, as desired.

For  $P(n+1)$ , observe  $\{x \in X: x = a_m \text{ for some } m < n+1\} = \{x \in X: x = a_m \text{ for some } m < n\} \cup \{x \in X: x = a_n\}$ . Easy to verify  $\{x \in X: x = a_n\}$  is finite. And,  $\{x \in X: x = a_m \text{ for some } m < n\}$  is finite by  $P(n)$ , hence their union is finite by 3.6.14 b), as desired.

Hence,  $X = \{x \in X : x \neq a_m \text{ for all } m < n\} \cup \{x \in X : x = a_m \text{ for some } m < n\}$  is finite by finite by 3.6.14 b), yet  $X$  is infinite, which is the desired contradiction.

**One can show(?) that  $a_n$  is an increasing sequence**

We show  $\forall n \in \mathbb{N}, a_n < a_{n+1}$ .

Recall:

$$a_n = \min\{x \in X : x \neq a_m \text{ for all } m < n\}$$

$$a_{n+1} = \min\{x \in X : x \neq a_m \text{ for all } m < n+1\}$$

To show  $a_n < a_{n+1}$ , it is sufficient to show  $a_n \neq a_{n+1}$  and  $a_n \leq a_{n+1}$ .

We have  $a_n \neq a_{n+1}$ , as, since  $a_{n+1} \in \{x \in X : x \neq a_m \text{ for all } m < n+1\}$  and  $n < n+1$ ,  $a_n \neq a_{n+1}$ .

We have  $a_n \leq a_{n+1}$ , as, observe  $a_{n+1} \in \{x \in X : x \neq a_m \text{ for all } m < n\}$  (as  $a_{n+1} \in X$  and  $a_{n+1} \neq a_m \text{ for all } m < n+1$  implies  $a_{n+1} \neq a_m \text{ for all } m < n$ ). Then, since  $a_n$  is the minimal element of  $\{x \in X : x \neq a_m \text{ for all } m < n\}$ , it holds that  $a_n \leq a_{n+1}$ , as desired.

**in particular that(?)  $a_n \neq a_m$  for all  $n \neq m$ .**

We had shown similar result before (an increasing function is injective). WLOG assume  $m > n$ , then we can induct on  $m$  to prove  $a_n < a_m$  for all  $m > n$ , hence  $a_n \neq a_m$  for all  $m > n$ .

**We have(?)  $a_n \in X$  for each natural number  $n$**

Recall definition of  $a_n$ :

$$a_n = \min\{x \in X : x \neq a_m \text{ for all } m < n\}$$

Hence,  $a_n \in \{x \in X : x \neq a_m \text{ for all } m < n\}$ , hence  $a_n \in X$ , as desired.

**This implies(?) that  $x$  is an element of the set  $\{x \in X : x \neq a_m \text{ for all } m < n\}$  for all  $n$**

Since we have  $x \in X$ , and we have  $\forall m \in \mathbb{N}, x \neq a_m$ . Hence we have  $\forall m < n, x \neq a_m$ , hence  $x \in \{x \in X : x \neq a_m \text{ for all } m < n\}$ , as desired.

**Since  $a_n$  is an increasing sequence, we have  $a_n \geq n$  (?)**

We use induction. Let  $P(n): a_n \geq n$ . Then for  $P(0)$ , since  $a_0 \in \mathbb{N}$ ,  $a_0 \geq 0$  as desired.

For  $P(n+1)$ :

$$\begin{aligned} a_{n+1} &> a_n \mid \& \geq a_n + 1 \& \geq n + 1 \end{aligned}$$

As desired.

Remark: Basically the gist of this part of proof is observing  $a_n$  is not bounded above. By showing  $\forall n \geq N, x \geq a_n$ , we can find sufficiently large  $n$  to cause contradiction.

### But we then must have(?)

We have:

$$m = \min\{n \in \mathbb{N} : g(n) \neq f(n)\}$$

Hence, we have  $g(m) \neq f(m) = a_m$  and  $\forall t < m, g(t) \neq f(m)$  (otherwise  $m$  is not the minimal element of  $\{n \in \mathbb{N} : g(n) \neq f(n)\}$ ).

Therefore, since  $\forall t < m, g(t) \neq f(m)$ , we have  $\forall t < m, g(m) \neq a_t$ . Since  $g: \mathbb{N} \rightarrow X$ ,  $g(m) \in X$ . Therefore  $g(m) \in \{x \in X : x \neq a_t \text{ for all } t < m\}$ .

Then, we argue by contradiction. Assume  $g(m) \neq \min\{x \in X : x \neq a_t \text{ for all } t < m\}$ , then we have  $g(m) > \min\{x \in X : x \neq a_t \text{ for all } t < m\}$ . Since  $g$  is increasing, we have:

$$g(m-1) \leq \min\{x \in X : x \neq a_t \text{ for all } t < m\} < g(m)$$

However, since  $g(m-1) = f(m-1) = a_{m-1}$ , and  $\min\{x \in X : x \neq a_t \text{ for all } t < m\} = a_m > a_{m-1}$ , we can change  $\leq$  to  $<$ :

$$g(m-1) < a_m < g(m)$$

This contradicts the surjectivity of  $g$ . As, for all  $n \leq m-1$ , since  $g$  is increasing, we have  $g(n) \leq g(m-1) < a_m$ , hence  $g(n) \neq a_m$ . Similarly, for all  $n \geq m$ ,  $g(n) \neq a_m$ . Hence  $g$  is not surjective, as desired.

### Exercise 8.1.4

We define:

$$A = \{n \in \mathbb{N} : f(m) \neq f(n) \text{ for all } 0 \leq m < n\}$$

Define  $g: A \rightarrow f(\mathbb{N})$  s.t.  $\forall n \in A, g(n) = f(n)$ . We claim  $g$  is bijective.

We first show  $g$  is injective by contradiction. Assume there exists  $a, b$  s.t.  $a \neq b, g(a) = g(b)$ . WLOG assume  $a < b$ . Then, since  $\forall 0 \leq m \leq b, f(m) \neq f(a)$ , and  $a < b$ , it holds that  $f(a) \neq f(b)$ . However since  $g(a) = g(b)$ ,  $f(a) = f(b)$ , contradiction.

We then show  $g$  is surjective by contradiction. Assume  $g$  is not surjective, then  $\exists y \in f(\mathbb{N}), \forall a \in A, g(a) \neq y$ . However, since  $y \in f(\mathbb{N}), \exists n \in \mathbb{N}, f(n) = y$ . Hence  $\forall a \in A, g(a) \neq f(n)$ . We proceed by cases. Assume  $n \in A$ , then  $g(n) = f(n)$ , contradicting  $\forall a \in A, g(a) \neq f(n)$ . Assume  $n \notin A$ , then  $f(m) = f(n)$  for some  $0 \leq m < n$ .

By Proposition 8.1.4, we let  $m$  be the minimum element of  $\{m \in \mathbb{N} : 0 \leq m < n : f(m) = f(n)\}$ . Then, we claim  $m \in A$ . Assume the contrary, then  $f(m') = f(m)$  for some  $0 \leq m' < m$ . Hence  $f(m') = f(n)$  and  $0 \leq m' < n$ , therefore  $m' \in \{m \in \mathbb{N} : 0 \leq m < n : f(m) = f(n)\}$ , yet  $m' < m$ , contradiction. Since  $m \in A$ , we have  $f(m) = g(m)$ . Since  $f(m) = f(n)$ , we have  $g(m) = f(n)$ , contradicting  $\forall a \in A, g(a) \neq f(n)$ , as desired.

Since  $g: A \rightarrow f(\mathbb{N})$  is a bijection,  $f(\mathbb{N})$  has the same cardinality as  $A$ . Since  $A \subseteq \mathbb{N}$ ,  $A$  is at most countable. Since  $|A| = |f(\mathbb{N})|$ ,  $f(\mathbb{N})$  is at most countable, as desired.

## Exercise 8.1.5

Since  $X$  is countable, then there exists bijection  $g: \mathbb{N} \rightarrow X$ . Hence  $f \circ g: \mathbb{N} \rightarrow Y$ . Therefore by Proposition 8.1.8,  $f \circ g(\mathbb{N})$  is at most countable. Since  $g(\mathbb{N}) = X$ ,  $f(X)$  is at most countable, as desired.

## Exercise 8.1.6

### If direction

Let  $g: A \rightarrow f(A)$  s.t.  $\forall a \in A, g(a) = f(a)$ . Easy to verify  $g$  is bijective. Hence,  $|A| = |f(A)|$ . Since  $f(A) \subseteq \mathbb{N}$ ,  $f(A)$  is at most countable by Corollary 8.1.6, hence  $A$  is at most countable.

### Only if direction

If  $A$  is at most countable, by Corollary 8.1.6 there's a bijective map  $g: A \rightarrow Y$ , for some  $Y \subseteq \mathbb{N}$ . Hence, let  $f: A \rightarrow \mathbb{N}$  s.t.  $\forall a \in A, f(a) = g(a)$ , easy to verify  $f$  is injective, as desired.

## Exercise 8.1.7

We have bijection  $f: \mathbb{N} \rightarrow X, g: \mathbb{N} \rightarrow Y$ . We define  $h: \mathbb{N} \rightarrow X \cup Y$  by:

$$\begin{aligned} h(2n) &= f(n) \text{ and } h(2n+1) = g(n) \end{aligned}$$

Above does define a function  $\mathbb{N} \rightarrow X \cup Y$ , as by Exercise 4.4.3, for any natural number  $k$ , either  $k = 2n$  or  $k = 2n + 1$  for some  $n \in \mathbb{N}$  (but not both). Hence for any  $k \in \mathbb{N}$ ,  $h(k)$  is assigned with exactly one value, hence  $h(k)$  has domain  $\mathbb{N}$ . And, we either have  $h(k) = f(n)$  or  $h(k) = g(n)$ , in either case  $h(k) \in X \cup Y$ , hence  $h$  has range  $X \cup Y$ .

We claim  $h(\mathbb{N}) = X \cup Y$ . We first show  $h(\mathbb{N}) \subseteq X \cup Y$ . Since  $h(\mathbb{N}) = \{h(k) : k \in \mathbb{N}\}$ , and  $h$  has range  $X \cup Y$ , it holds that  $h(\mathbb{N}) \subseteq X \cup Y$ . We then show  $X \cup Y \subseteq h(\mathbb{N})$ . Take an element  $e \in X \cup Y$ . We have two overlapping cases of  $e \in X$  or  $e \in Y$ . WLOG  $e \in X$ , then we have  $f(n) = e$  for some  $n \in \mathbb{N}$ . Hence, we have  $h(2n) = e$ , hence  $e \in X \cup Y \rightarrow e \in h(\mathbb{N})$ , as desired.

Therefore, by Corollary 8.1.9, since  $\mathbb{N}$  is countable,  $\mathcal{P}(\mathbb{N})$  is countable. Since  $\mathcal{P}(\mathbb{N}) = X \cup Y$ ,  $X \cup Y$  is countable, as desired.

### Exercise 8.1.8

By Corollary 8.1.13,  $\mathbb{N} \times \mathbb{N}$  is countable. Hence, remains to show  $\mathbb{N} \times \mathbb{N}$  is bijective to  $X \times Y$ . Since  $X, Y$  are countable, there exists bijection  $f: X \rightarrow \mathbb{N}$ ,  $g: Y \rightarrow \mathbb{N}$ . We claim  $h: X \times Y \rightarrow \mathbb{N} \times \mathbb{N}$  s.t.  $h(x, y) = (f(x), g(y))$  is a bijection, by showing  $h$  is injective and surjective.

$h$  is injective, as say we have  $x, x', y, y'$  s.t.  $x \neq x', y \neq y', x \in X, y \in Y$ , then  $f(x) \neq f(x'), g(y) \neq g(y')$ , hence  $(f(x), g(y)) \neq (f(x'), g(y'))$ , as desired.

$h$  is surjective, as, for any  $p, q \in \mathbb{N}$ , we always have  $(p, q) = h(f^{-1}(p), g^{-1}(q))$ .

Hence,  $h$  is bijective, therefore  $|X \times Y| = |\mathbb{N} \times \mathbb{N}|$ . Since  $|\mathbb{N} \times \mathbb{N}|$  is countable,  $|X \times Y|$  is countable, as desired.

### Exercise 8.1.9

Proved in Exercise 8.1.1 Solution

### Exercise 8.1.10

#### Overview

We first construct bijections:  $g: \mathbb{N} \rightarrow \mathbb{N}^+ + \{0\}$ . Then, easy to construct bijection  $h: \mathbb{N}^- \rightarrow \mathbb{N}^-$  from  $g$ . Then we can construct bijection  $i: \mathbb{Z} \rightarrow \mathbb{Q}$  via:

$$i(n) = \begin{cases} g(n) & \text{if } n \geq 0 \\ h(n) & \text{if } n < 0 \end{cases}$$

Easy to verify if  $g: \mathbb{N}^+ \rightarrow \mathbb{N}^+$ ,  $h: \mathbb{N}^- \rightarrow \mathbb{N}^-$  are indeed bijections, then  $i: \mathbb{Z} \rightarrow \mathbb{Q}$ , as defined above, is a bijection.

Then, since  $\mathbb{Z}$  is countable, there exists bijection  $j: \mathbb{N} \rightarrow \mathbb{Z}$  (easy to explicitly construct either). Easy to verify  $i \circ j: \mathbb{N} \rightarrow \mathbb{Q}$  is a bijection. And, if we can explicitly define  $g$ , we can explicitly define  $i \circ j$ .

#### Explicitly define $g$

We construct the following sequence:

$$\begin{aligned} a_0 &= 0, a_1 = 1 \\ a_{2n} &= a_n \\ a_{2n+1} &= a_n + a_{n+1} \\ g(n) &= \frac{a_n}{a_{n+1}} \end{aligned}$$

With induction, one can show above recursive definition does uniquely define a sequence.



Then, we define  $g$  as:

$$g(n) = \frac{a_n}{a_{n+1}}$$

We then show  $g$ , as defined above, is bijective.

We first show  $g$  is surjective. It is sufficient to show, for all  $a, b \in \mathbb{N} \times (\mathbb{N} - \{0\})$  relatively prime,  $\exists n \in \mathbb{N}, g(n) = a/b$ . We argue by contradiction, assume for some relatively prime  $a, b$  pairs,  $\forall n \in \mathbb{N}, g(n) \neq a/b$ . Then, denote the set containing such  $a, b$  pairs as  $C$ .

We claim, we can take  $p/q \in C$ , for which  $p/q$  has the smallest denominator out of all rationals in  $C$ , and the smallest numerator out of all rationals with the smallest denominator in  $C$ . (Existence of such  $p/q$  may seem obvious, but it is not true for certain sets, say if  $C = \mathbb{Z} \times \mathbb{Z}$ , hence justification is needed).

By Axiom of Replacement, we replace every  $(a, b) \in C$  with  $b$  and form a new set  $C_b$ . Then, by Proposition 8.1.4, we take  $q = \min(C_b)$ . We then use Axiom of Specification to construct  $\{(a, b) \in C : b = q\}$ . We then use Axiom of replacement to replace  $(a, b) \in \{(a, b) \in C : b = q\}$  with  $b$  and form a new set  $C_{\{b\}}$ . We use Proposition 8.1.4 again to take  $p = \min(C_{\{b\}})$ , as desired.

Hence, we either have  $p = q$ ,  $p < q$ ,  $p > q$ . If  $p = q$ , then since  $p, q$  are relatively prime,  $p = q$  implies  $p = q = 1$ . Hence  $g(1) = 1$ , as desired.

If  $p < q$ , then we claim  $p/(q - p) \in C$ . Since  $p, q$  are relatively prime, hence  $\gcd(p, q) = 1$ , hence  $\gcd(p, q - p) = 1$ , therefore  $p, q - p$  are relatively prime. We then argue by contradiction.

Assume  $p/(q - p) \notin C$ , then  $\exists n \in \mathbb{N}, g(n) = p/(q - p)$ .

We then propose Lemma 1:  $a_n, a_{n+1}$  are relatively prime for all  $n \in \mathbb{Z}^+$ . We induct on  $n$ . Let  $P(n)$  denote  $a_n, a_{n+1}$  are relatively prime.

For  $P(1)$ , easy to compute  $a_1 = a_2 = 1$ , hence  $a_1, a_2$  are relatively prime, as desired.

For  $P(n)$ , if  $n$  is even, then  $a_n = a_{n'}$ ,  $a_{n+1} = a_{n'} + a_{n'+1}$  for some  $n' < n$ . Hence,  $\gcd(a_n, a_{n+1}) = \gcd(a_{n'}, a_{n'} + a_{n'+1}) = \gcd(a_{n'}, a_{n'+1}) = 1$  as desired.

Similar reasoning follows if  $n$  is odd, closing the induction.

Hence, by Lemma 1, since  $p, q - p$  are relatively prime and  $\exists n \in \mathbb{N}, g(n) = p/(q - p)$ , it holds that for the same  $n$ ,  $p = a_n$ ,  $q - p = a_{n+1}$ . Hence,  $g(2n) = a_{2n}/a_{2n+1} = a_n/(a_n + a_{n+1}) = p/(p + q - p) = p/q$ . Hence  $p/q \notin C$ , contradiction.

Hence  $p/(q - p) \in C$ . However,  $p/q$  has the smallest denominator out of all rationals in  $C$ , yet  $p/(q - p)$  has smaller denominator, contradiction. Hence  $p, q \in C$ . Similarly  $p > q$  leads to a contradiction. Hence  $g$  is surjective.

To show  $g$  is injective. We argue by contradiction. If  $g$  is not injective, then the set  $C = \{a, b \in \mathbb{N} \times \mathbb{N} - \{0\} : \gcd(a, b) = 1 : \exists n, n' \in \mathbb{N}, n \neq n' \text{ and } g(n) = g(n') = a/b\}$  is non-empty.

Choose  $p/q \in C$  s.t.  $p/q$  has the smallest denominator in  $C$ , and has the smallest numerator among those with the smallest denominator.

We proceed by cases. Either  $p = q$ ,  $p < q$ ,  $p > q$ .

If  $p = q$ , we must have  $p = q = 1$ . However, we can inductively show that  $\forall n \geq 2, g(n) \neq 1$ . Hence  $1/1 \notin C$ , contradiction.

If  $p < q$ , by Lemma  $a_n = a_{n'} = p$ ,  $a_{n+1} = a_{n'+1} = q$ .

We start with Lemma 2: If  $a_n < a_{n+1}$ ,  $n$  is even. We argue by contradiction. Assume  $n$  is odd, then  $a_n = a_{2k+1}$ ,  $a_{n+1} = a_{2(k+1)}$ . Yet,  $a_{2(k+1)} = a_{k+1}$ ,  $a_{2k+1} = a_k + a_{k+1}$ . Hence,  $a_{2(k+1)} \leq a_{2k+1}$ , hence  $a_{n+1} \leq a_n$ , contradiction.

Therefore, since  $p < q$ ,  $n, n'$  are both even. Hence, easy to verify  $g(n/2) = g(n'/2) = p/(q-p)$ . Yet,  $\gcd(p, q-p) = 1$ , hence  $p/(q-p) \in C$ , yet  $p/(q-p)$  has smaller denominator than  $p/q \in C$ , contradiction, as desired.

Case  $p > q$  follows similarly. Hence  $g$  is injective.

## The second equality follows easily from Proposition 7.4.3 (Page 189)

We need to show:

$$\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n, m) = \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(m, n)$$

By Corollary 8.1.13,  $\mathbb{N} \times \mathbb{N}$  is countable, hence there exists bijection  $g: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ , therefore, by Definition 8.2.1:

$$\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n, m) = \sum_{i=0}^{\infty} f(g(i))$$

Define  $s: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  s.t.  $s(m, n) = (n, m)$ , then:

$$\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(m, n) = \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(s(n, m))$$

Hence:

$$\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(m, n) = \sum_{j=0}^{\infty} f(s(g(j)))$$

Observe:

$$\sum_{j=0}^{\infty} f(s(g(j))) = \sum_{j=0}^{\infty} f(g(g^{-1}s(g(j))))$$

Easy to verify  $g(g^{-1}s(g(j)))$  is a bijection from  $\mathbb{N} \rightarrow \mathbb{N}$ , hence, by Proposition 7.4.3:

$$\sum_{j=0}^{\infty} f(g(i)) = \sum_{j=0}^{\infty} f(g(g^{-1}s(g(j))))$$



Therefore:

$$\begin{aligned} \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n, m) &= \sum_{i=0}^{\infty} f(g(i)) = \sum_{j=0}^{\infty} f(g(g^{-1}s(g(j)))) = \sum_{j=0}^{\infty} f(s(g(j))) = \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(m, n) \end{aligned}$$

As desired.

## Exercise 8.2.1

According to Erratum,  $X$  is countable instead of at most countable.

### Only If direction

Since  $X$  is countable, there exists bijection  $g: \mathbb{N} \rightarrow X$ . Hence, by Definition 8.2.1:

$$\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} f(g(n))$$

And furthermore  $\sum_{n=0}^{\infty} |f(g(n))|$  is convergent.

For a given finite subset  $A \subseteq X$ . If  $A = \emptyset$ ,  $\sum_{x \in A} f(x) = 0$ , hence  $\sum_{x \in A} |f(x)| < \infty$ .

If  $A \neq \emptyset$ , consider the set  $\{g^{-1}(a) : a \in A\}$ . Since  $A$  is finite,  $\{g^{-1}(a) : a \in A\}$  is finite. Since  $\{g^{-1}(a) : a \in A\}$  is finite and  $\{g^{-1}(a) : a \in A\} \subseteq \mathbb{N}$ , by Exercise 3.6.3, it holds that  $\exists M \in \mathbb{N}$ ,  $\forall a \in A$ ,  $g^{-1}(a) \leq M$ . Then, denote  $S = \{g(n) : 0 \leq n \leq M\}$ . We claim  $A \subseteq S$ . We argue by contradiction. Assume  $a \notin S$ , then  $\forall 0 \leq n \leq M$ ,  $g(n) \neq a$ . Since  $g: \mathbb{N} \rightarrow X$  is surjective, and  $\forall 0 \leq n \leq M$ ,  $g(n) \neq a$ , there must exist  $g(n') = a$  for some  $n' > M$ . Hence,  $g^{-1}(a) = n' > M$ , contradicting  $\forall a \in A$ ,  $g^{-1}(a) \leq M$ .

Therefore, by Proposition 7.1.11 e):

$$\sum_{x \in A} |f(x)| + \sum_{x \in S - A} |f(x)| = \sum_{x \in S} |f(x)|$$

$$\sum_{x \in A} |f(x)| \leq \sum_{x \in S} |f(x)|$$

Easy to show  $\sum_{x \in S} |f(x)| = \sum_{n=0}^M |f(g(n))|$ . Since  $\sum_{n=0}^{\infty} |f(g(n))|$  is convergent and  $(\sum_{n=0}^M |f(g(n))|)_{M=0}^{\infty}$  is an increasing sequence, by Proposition 6.3.8, it holds that  $\sum_{n=0}^{\infty} |f(g(n))|$  is supremum of  $(\sum_{n=0}^M |f(g(n))|)_{M=0}^{\infty}$ , therefore:

$$\sum_{n=0}^M |f(g(n))| \leq \sum_{n=0}^{\infty} |f(g(n))|$$

Hence:

$$\begin{aligned} \sum_{x \in A} |f(x)| &\leq \sum_{x \in S} |f(x)| = \sum_{n=0}^M |f(g(n))| \leq \sum_{n=0}^{\infty} |f(g(n))| = \sum_{x \in X} |f(x)| \end{aligned}$$

Hence  $\sum_{x \in X} |f(x)|$  is an upper bound for  $\{\sum_{x \in X} |f(x)| : A \subseteq X, A \text{ finite}\}$ , therefore:

$$\sup\{\sum_{x \in X} |f(x)| : A \subseteq X, A \text{ finite}\} \leq \sum_{x \in X} |f(x)| < \infty$$

As desired.

### If direction

Since  $X$  is countable, there exists bijection  $g: \mathbb{N} \rightarrow X$ . Hence, by Definition 8.2.1:

$$\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} f(g(n))$$

We need to show  $\sum_{x \in X} f(x)$  is absolutely convergent, which is equivalent to showing  $\sum_{n=0}^{\infty} f(g(n))$  is absolutely convergent, which is equivalent to showing  $\sum_{n=0}^{\infty} |f(g(n))|$  is convergent.

Observe  $\sum_{n=0}^{\infty} |f(g(n))|$  is a series of non-negative numbers. Hence, by Proposition 7.3.1, if we can find a finite upper bound  $M$  for  $(\sum_{n=0}^M |f(g(n))|)^{\infty}_{M=0}$ ,  $\sum_{n=0}^{\infty} |f(g(n))|$  is convergent.

We claim  $\sup\{\sum_{x \in X} |f(x)| : A \subseteq X, A \text{ finite}\}$  is the desired finite upper bound. Easy to verify it is finite. For it being an upper bound, we claim, for any  $M \in \mathbb{N}$ ,

$$\sum_{n=0}^M |f(g(n))| = \sum_{a \in A} |f(a)|$$

For some set  $A \subseteq X$ .

To prove the claim, we restrict the domain of  $g$  to  $\{0 \leq i \leq M\} \rightarrow \{g(n) : 0 \leq i \leq M\}$ . Easy to prove  $g$  is a bijection. Furthermore,  $\{g(n) : 0 \leq i \leq M\} \subseteq X$ . Hence set  $A = \{g(n) : 0 \leq i \leq M\}$  yields the equality.

Therefore, for any  $M \in \mathbb{N}$ , it holds that:

$$\sum_{n=0}^M |f(g(n))| = \sum_{a \in A} |f(a)| \leq \sup\{\sum_{x \in X} |f(x)| : A \subseteq X, A \text{ finite}\}$$

Hence, by Proposition 7.3.1,  $\sum_{n=0}^{\infty} |f(g(n))|$  converges, as desired.

### Exercise 8.2.2

Define  $M$  as:

$$M = \sup\{\sum_{x \in A} |f(x)| : A \subseteq X, A \text{ finite}\}$$

We show the set  $\{x \in X : |f(x)| > 1/n\}$  are finite with cardinality at most  $Mn$  for every  $n \in \mathbb{Z}^+$ . We argue by contradiction. Assume  $\{x \in X : |f(x)| > 1/n\}$  is infinite, then by our solution of Exercise 8.1.1, we can find a countably infinite subset  $C \subseteq \{x \in X : |f(x)| > 1/n\}$ . Since

$\sum_{x \in X} |f(x)|$  is absolutely convergent and  $C \subseteq X$ ,  $\sum_{x \in C} |f(x)|$  is absolutely convergent by Lemma 8.2.3. Therefore, by Definition 8.2.1, it holds that:

$$\sum_{x \in C} |f(x)| = \sum_{n=0}^{\infty} |f(g(n))|$$

for some bijection  $g: \mathbb{N} \rightarrow C$ .

Since  $g(n) \in C$ ,  $\forall n \in \mathbb{N}$ ,  $f(g(n)) > 1/n$ . Therefore, by Corollary 7.2.6 and Lemma 7.5.2 (or By Corollary 7.3.2),  $\sum_{x \in C} |f(x)| = \sum_{n=0}^{\infty} |f(g(n))|$  is divergent. Contradiction.

Hence  $\{x \in X : |f(x)| > 1/n\}$  is finite. Denote  $u = |\{x \in X : |f(x)| > 1/n\}|$ . Hence there exists bijection  $g: \{1 \leq i \leq u\} \rightarrow \{x \in X : |f(x)| > 1/n\}$

We then show  $u \leq Mn$ .

Observe:

$$\begin{aligned} M &\geq \sum_{x \in X : |f(x)| > 1/n} |f(x)| = \sum_{i=1}^u |f(g(i))| \geq u \frac{1}{n} \end{aligned}$$

Hence  $u \leq Mn$ , as desired.

Therefore, by Exercise 8.1.9,  $\bigcup_{n \in \mathbb{Z}^+} \{x \in X : |f(x)| > 1/n\}$  is at most countable. We then claim  $\{x \in X : f(x) \neq 0\} = \bigcup_{n \in \mathbb{Z}^+} \{x \in X : |f(x)| > 1/n\}$ . If  $x \in \bigcup_{n \in \mathbb{Z}^+} \{x \in X : |f(x)| > 1/n\}$ , then  $|f(x)| > 1/n$  implies  $f(x) \neq 0$ , hence  $x \in \{x \in X : f(x) \neq 0\}$ . If  $x \in \{x \in X : f(x) \neq 0\}$ , then  $|f(x)| > 0$ , therefore  $\exists n \in \mathbb{Z}^+$ ,  $|f(x)| > 1/n$ , as desired. Hence  $\{x \in X : f(x) \neq 0\}$  is at most countable.

## Exercise 8.2.3

(a)

If  $X$  is countable, then, consider bijection  $h: \mathbb{N} \rightarrow X$ :

$$\begin{aligned} \sum_{x \in X} f(x) + \sum_{x \in X} g(x) &= \sum_{n=0}^{\infty} f(h(n)) + \sum_{n=0}^{\infty} g(h(n)) = \sum_{n=0}^{\infty} (f(h(n)) + g(h(n))) \quad \text{\textit{By 7.2.14 a)}} \\ &= \sum_{x \in X} (f(x) + g(x)) \end{aligned}$$

As desired.

If  $X$  is uncountable, we first show  $\sum_{x \in X} |f(x) + g(x)|$  is absolutely convergent.

Observe, for any finite  $A \subseteq X$ , it holds that:

$$\begin{aligned} \sum_{x \in A} |f(x) + g(x)| &\leq \sum_{x \in A} |f(x)| + \sum_{x \in A} |g(x)| \leq \sup\{\sum_{x \in A} |f(x)|, A \subseteq X, A \text{ finite}\} + \sup\{\sum_{x \in A} |g(x)|, A \subseteq X, A \text{ finite}\} \\ &< \infty \end{aligned}$$

Hence  $\sum_{x \in X} |f(x) + g(x)|$  is absolutely convergent by Definition 8.2.4.

Then, it suffices to prove:

$$\sum_{x \in X: f(x) \neq 0} f(x) + \sum_{x \in X: g(x) \neq 0} g(x) = \sum_{x \in X: f(x) + g(x) \neq 0} (f(x) + g(x))$$

We decompose  $f(x) + g(x) \neq 0$  into mutually exclusive cases. The row means  $f(x) = 0$

	$f(x)=0$	$f(x) \neq 0$
$g(x)=0$	x	✓
$g(x) \neq 0$	✓	$f(x) + g(x) \neq 0$

Hence, we can break  $\{x \in X: f(x) + g(x) \neq 0\}$  into three pairwise disjoint components:

$$\begin{aligned} \{x \in X: f(x) + g(x) \neq 0\} &= \{x \in X: f(x) = 0, g(x) \neq 0\} \sqcup \{x \in X: f(x) \neq 0, g(x) = 0\} \sqcup \{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0\} \end{aligned}$$

Then, by Proposition 8.2.6 c), we can break  $\sum_{x \in X: f(x) + g(x) \neq 0} f(x) + g(x)$  into three absolutely convergent sums:

$$\begin{aligned} \sum_{x \in X: f(x) + g(x) \neq 0} f(x) + g(x) &= \sum_{x \in X: f(x) = 0, g(x) \neq 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x) \neq 0, g(x) = 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) \end{aligned}$$

We proceed to simplify each components (with Proposition 8.2.6 c):

$$\begin{aligned} \sum_{x \in X: f(x) = 0, g(x) \neq 0} f(x) + g(x) &= \sum_{x \in X: f(x) = 0, g(x) \neq 0} g(x) \sqcup \sum_{x \in X: g(x) \neq 0} g(x) - \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x) \end{aligned}$$

Similarly:

$$\sum_{x \in X: f(x) \neq 0, g(x) = 0} f(x) + g(x) = \sum_{x \in X: f(x) \neq 0} f(x) - \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x)$$

Then:

$$\begin{aligned} \sum_{x \in X: f(x) + g(x) \neq 0} f(x) + g(x) &= \sum_{x \in X: f(x) = 0, g(x) \neq 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x) \neq 0, g(x) = 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) \\ &= \sum_{x \in X: g(x) \neq 0} g(x) - \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x) \sqcup \sum_{x \in X: f(x) \neq 0} f(x) - \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x) \\ &\sqcup \sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x) \neq 0} f(x) + \sum_{x \in X: g(x) \neq 0} g(x) \\ &\sqcup \sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) \sqcup \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x) + \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x) \end{aligned}$$

We remains to prove:

$$\sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) - (\sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x) + \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x)) = 0$$

Aka:

$$\sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + g(x) = \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x) + \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x)$$

Observe, by countable version of 8.2.6 a):

$$\sum_{x \in X: f(x), g(x) \neq 0} f(x) + g(x) = \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} f(x) + \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x)$$

Hence, remains to show:

$$\sum_{x \in X: f(x), g(x) \neq 0} f(x) + g(x) = \sum_{x \in X: f(x), g(x) \neq 0, f(x) + g(x) \neq 0} f(x) + \sum_{x \in X: f(x) \neq 0, g(x) \neq 0} g(x)$$

We prove a more general statement. Say  $\sum_{x \in X} f(x)$  is absolutely convergent, then  $\sum_{x \in X: f(x) \neq 0} f(x)$  is absolutely convergent, and:

$$\sum_{x \in X} f(x) = \sum_{x \in X: f(x) \neq 0} f(x)$$

The above statement is true by definition if  $X$  is uncountable. Hence, remains to prove above statement is true if  $X$  is countable.

Observe:

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{x \in X: f(x) \neq 0} f(x) + \sum_{x \in X: f(x) = 0} f(x) \\ &\text{\textit{By 8.2.6 c)}} \quad = \sum_{x \in X: f(x) \neq 0} f(x) \quad \text{\textit{As }} \sum_{x \in X: f(x) = 0} f(x) = 0 \\ \end{aligned}$$

Remark: Above also shows if  $\sum_{x \in X: f(x) \neq 0} f(x)$  is absolutely convergent, then  $\sum_{x \in X} f(x)$  is absolutely convergent and is equal to  $\sum_{x \in X: f(x) \neq 0} f(x)$ .

As desired.

**(b)**

If  $X$  is countable, proof is similar to countable case of (a).

If  $X$  is uncountable. We first show  $\sum_{x \in X} cf(x)$  is absolutely convergent.

Observe, for any finite set  $A \subseteq X$ :

$$\begin{aligned} \sum_{x \in A} |cf(x)| &= \sum_{x \in A} c|f(x)| = c \sum_{x \in A} |f(x)| \quad \text{\textit{By 7.1.11 g)}} \\ &\leq c \sup \{ \sum_{x \in A} |f(x)|, A \subseteq X, A \text{ finite} \} < \infty \end{aligned}$$

Hence  $\sum_{x \in X} cf(x)$  is absolutely convergent by Definition 8.2.4

Then:

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{x \in X: f(x) \neq 0} f(x) = \sum_{x \in X: f(x) \neq 0} cf(x) \\ &= \sum_{x \in X} cf(x) \end{aligned}$$

As desired.

(c)

We proceed by cases. If  $X$  is finite, then  $X_1, X_2$  are finite. Result then follows from 7.1.11 e).

If  $X$  is countable, then, either  $X_1$  is finite and  $X_2$  is countable or  $X_1, X_2$  are both countable.

If  $X_1$  is finite and  $X_2$  is countable, then, despite absolute convergent is defined for infinite series, we extend the definition for finite series and claim all finite series are absolutely convergent. Therefore  $\sum_{x \in X_1} f(x)$  is absolutely convergent. To show  $\sum_{x \in X_2} f(x)$  is absolutely convergent, note, since  $X_2 \subseteq X$ , it follows that for all finite set  $A \subseteq X_2$ , it holds that  $A \subseteq X$ , hence:

$$\sum_{x \in A} |f(x)| < \sup(\sum_{x \in A} |f(x)|, A \subseteq X, A \text{ finite}) < \infty$$

Hence  $\sum_{x \in X_2} f(x)$  is absolutely convergent by Lemma 8.2.3.

Observe, there exists bijection  $g: \{1 \leq i \leq |X_1|\} \rightarrow X_1$  and  $h: \mathbb{N} \rightarrow X_2$ . Then, we construct bijection  $t: \mathbb{N} \rightarrow X$  as following.

$$t(n) = \begin{cases} g(n + 1) & \text{if } n < |X_1| \\ h(n - |X_1|) & \text{Otherwise} \end{cases}$$

Easy to prove  $t: \mathbb{N} \rightarrow X$  is a bijection. Therefore:

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{n=0}^{\infty} f(t(n)) = \sum_{n=0}^{|X_1|-1} f(t(n)) + \\ &= \sum_{n=|X_1|}^{\infty} f(t(n)) \end{aligned} \quad \text{By 7.2.14 c)}$$

We remains to show  $\sum_{n=0}^{|X_1|-1} f(t(n)) = \sum_{x \in X_1} f(x)$  and  $\sum_{n=|X_1|}^{\infty} f(t(n)) = \sum_{x \in X_2} f(x)$ .

$$\begin{aligned} \sum_{n=0}^{|X_1|-1} f(t(n)) &= \sum_{n=1}^{|X_1|} f(t(n-1)) \quad \text{By 7.1.4 b)} \\ &= \sum_{n=1}^{|X_1|} f(g(n)) = \sum_{x \in X_1} f(x) \end{aligned}$$

$$\begin{aligned} \sum_{n=|X_1|}^{\infty} f(t(n)) &= \sum_{n=|X_1|}^{\infty} f(h(n - |X_1|)) = \sum_{n=0}^{\infty} f(h(n)) \\ &= \sum_{x \in X_2} f(x) \end{aligned} \quad \text{By 7.2.14 d)}$$

As desired.

If both  $X_1, X_2$  are countable. Via similar argument,  $\sum_{x \in X_1} f(x), \sum_{x \in X_2} f(x)$  are absolutely convergent.



Let  $g: \mathbb{N} \rightarrow X_1$ ,  $h: \mathbb{N} \rightarrow X_2$  be bijections. Then, similar to Exercise 8.1.7, define  $t: \mathbb{N} \rightarrow X_1 \cup X_2$  as follows:

$$t(2n) = g(n) \text{ and } t(2n+1) = h(n)$$

Since  $X_1 \cap X_2 = \emptyset$ , easy to verify  $t$  is a bijection.

Then, we claim, for all  $N \in \mathbb{N}$ :

$$\sum_{n=0}^{2N+1} f(t(n)) = \sum_{n=0}^N f(g(n)) + \sum_{n=0}^N f(h(n))$$

We prove via induction. For  $P(0)$ :

$$\begin{aligned} \sum_{n=0}^1 f(t(n)) &= f(t(0)) + f(t(1)) = f(g(0)) + f(h(0)) \\ &= \sum_{n=0}^0 f(g(n)) + \sum_{n=0}^0 f(h(n)) \end{aligned}$$

As desired.

For  $P(n+1)$ :

$$\begin{aligned} \sum_{n=0}^{2(N+1)+1} f(t(n)) &= \sum_{n=0}^{2(N+1)} f(t(n)) + f(t(2N+2)) + f(t(2N+3)) \\ &= \sum_{n=0}^N f(g(n)) + \sum_{n=0}^N f(h(n)) + g(N+1) + h(N+1) \\ &= \sum_{n=0}^{N+1} f(g(n)) + \sum_{n=0}^{N+1} f(h(n)) \end{aligned}$$

As desired.

Define  $(S_N)_{N=0}^\infty = (\sum_{n=0}^N f(t(n)))_{N=0}^\infty$ , it holds that  $(S_{2N+1})_{N=0}^\infty$  is a subsequence of  $(S_N)_{N=0}^\infty$ . By 6.1.19 a),  $(S_{2N+1})_{N=0}^\infty$  converges to  $\sum_{n=0}^\infty f(g(n)) + \sum_{n=0}^\infty f(h(n)) = \sum_{x \in X_1} f(x) + \sum_{x \in X_2} f(x)$ . Therefore, by Proposition 6.6.5,  $(S_N)_{N=0}^\infty$  converges to  $\sum_{x \in X_1} f(x) + \sum_{x \in X_2} f(x)$ , as desired.

For the converse with at most countable  $X_1, X_2$ , we merely need to show  $\sum_{x \in X_1 \cup X_2} f(x)$  is absolutely convergent. Then, the equality follows the same reasoning as above.

To show  $\sum_{x \in X_1 \cup X_2} f(x)$  is absolutely convergent, given a finite set  $A \subseteq X_1 \cup X_2$ , observe  $A = (X_1 \cap A) \cup (X_2 \cap A)$ . Therefore:

$$\begin{aligned} \sum_{x \in A} |f(x)| &= \sum_{x \in X_1 \cap A} |f(x)| + \sum_{x \in X_2 \cap A} |f(x)| \\ &\leq \sup_{x \in A} |f(x)|: A \subseteq X: A \text{ finite} + \sup_{x \in A} |f(x)|: A \subseteq X: A \text{ finite} \\ &< \infty \end{aligned}$$

Hence  $\sum_{x \in X_1 \cup X_2} f(x)$  is absolutely convergent by Lemma 8.2.3.

For case with uncountable  $X, X_1, X_2$ , observe, since  $X = X_1 \cup X_2$

$$\{x \in X: f(x) \neq 0\} = \{x \in X_1: f(x) \neq 0\} \cup \{x \in X_2: f(x) \neq 0\}$$

Hence, the uncountable case follows from the countable case.

(d)

Assume  $X, Y$  are countable. We first show  $\sum_{y \in Y} f(\phi(y))$  is absolutely convergent. Given finite  $B \subseteq Y$ , it holds that:

$$\begin{aligned} \sum_{y \in B} |f(\phi(y))| &= \sum_{x \in \phi(B)} |f(x)| \quad (\text{By 7.1.11 c}) \leq \\ &\sup \{ \sum_{x \in A} |f(x)|, A \subseteq X, A \text{ finite} \} < \infty \end{aligned}$$

Hence  $\sum_{y \in Y} f(\phi(y))$  is absolutely convergent by Lemma 8.2.3.

Denote bijection  $h: \mathbb{N} \rightarrow X$ . Observe  $\phi^{-1} \circ h: \mathbb{N} \rightarrow Y$  is a bijection. Hence:

$$\begin{aligned} \sum_{x \in X} f(x) &= \sum_{n=0}^{\infty} f(h(n)) = \sum_{n=0}^{\infty} \\ &f(\phi(\phi^{-1}(h(n)))) = \sum_{y \in Y} f(\phi(y)) \end{aligned}$$

As desired.

If  $X, Y$  are uncountable. Then suffices to  $\phi: \{y \in Y: f(\phi(y)) \neq 0\} \rightarrow \{x \in X: f(x) \neq 0\}$  remains a bijection.

$\phi$  is surjective. As, given  $x \in \{x \in X: f(x) \neq 0\}$ , it holds that  $\phi(\phi^{-1}(x)) = x$ . Furthermore,  $\phi^{-1}(x) \in \{y \in Y: f(\phi(y)) \neq 0\}$ , as  $\phi^{-1}(x) \in Y$ , and since  $f(x) \neq 0$ ,  $f(\phi(y)) = f(\phi(\phi^{-1}(x))) = f(x) \neq 0$ , as desired.

$\phi$  is injective. As, assume  $\phi: \{y \in Y: f(\phi(y)) \neq 0\} \rightarrow \{x \in X: f(x) \neq 0\}$  is not injective, then  $\exists y, y' \in \{y \in Y: f(\phi(y)) \neq 0\}, y \neq y' \wedge \phi(y) = \phi(y')$ . Since  $\phi(y), \phi(y') \in \{x \in X: f(x) \neq 0\}$ , it holds that  $\phi(y), \phi(y') \in X$ . Similarly  $y, y' \in Y$ . Therefore  $\phi: Y \rightarrow X$  is not injective, contradiction.

## Exercise 8.2.4

We first define conditional convergence for summation on infinite sets. Analogous to Definition 8.2.1, we say a series  $\sum_{x \in X} f(x)$  is conditionally convergent iff for some bijection  $g: \mathbb{N} \rightarrow X$ , the sum  $\sum_{n=0}^{\infty} f(g(n))$  is conditionally convergent.

We then argue by contradiction. Assume not both of the series  $\sum_{n \in A^+} a_n, \sum_{n \in A^+} a_n$  are conditionally convergent, then, there exists two overlapping cases:

- $\sum_{n \in A^+} a_n$  is conditionally convergent
- $\sum_{n \in A^-} a_n$  is conditionally convergent

### Case 1: $\sum_{n \in A^+} a_n$ is conditionally convergent

Define  $(b_n)_{n=0}^{\infty}$  s.t.

$$b_n = \begin{cases} a_n & \text{if } n \in A^+ \\ 0 & \text{Otherwise} \end{cases}$$



Recall, by Solution of Exercise 8.2.3 a), if  $\sum_{x \in X: f(x) \neq 0} f(x)$  is absolutely convergent, then  $\sum_{x \in X} f(x)$  is absolutely convergent and is equal to  $\sum_{x \in X: f(x) \neq 0} f(x)$ . Hence:

$$\sum_{n \in A^+} a_n = \sum_{n \in \mathbb{N}: b_n \neq 0} b_n = \sum_{n \in \mathbb{N}} b_n$$

With  $\sum_{n \in \mathbb{N}} b_n$  being absolutely convergent.

Since  $\sum_{n \in \mathbb{N}} a_n, \sum_{n \in \mathbb{N}} b_n$  are convergent, by Proposition 7.2.14 a),  $\sum_{n \in \mathbb{N}} (b_n - a_n)$  is convergent. Since  $\forall n \in \mathbb{N}, b_n - a_n \geq 0$ , hence  $\sum_{n \in \mathbb{N}} (b_n - a_n)$  is absolutely convergent. Hence by Proposition 8.2.6 a),  $\sum_{n \in \mathbb{N}} -a_n = \sum_{n \in \mathbb{N}} (b_n - a_n) - \sum_{n \in \mathbb{N}} b_n$  is absolutely convergent. Hence by Proposition 8.2.6 c),  $\sum_{n \in \mathbb{N}} a_n$  is absolutely convergent, contradiction.

## Case 2: $\sum_{n \in A^+} a_n$ is conditionally convergent

Similar to Case 1

## Exercise 8.2.5

### $A^+$ and $A^-$ are infinite

We proceed to show  $A^+, A^-$  are countable. Since  $A^+, A^- \subseteq \mathbb{N}$ , by Corollary 8.1.6,  $A^+, A^-$  are at most countable.

Assume  $A^+, A^-$  are both finite, then by Proposition 3.6.14 b),  $\mathbb{N} = A^+ \cup A^-$  is finite, contradiction.

Assume  $A^+$  is countable and  $A^-$  is finite. Then, let  $M = \max(A^-)$ . (Easy to verify we can take  $\max$  on a finite set by inducting on the cardinality of the set). By Proposition 7.2.14 c), since  $\sum_{n=0}^{\infty} a_n$  is conditionally convergent,  $\sum_{n=M+1}^{\infty} a_n$  is conditionally convergent. Since  $M = \max(A^-)$ , it holds that  $\forall n > M, a_n \geq 0$ , hence  $\sum_{n=M+1}^{\infty} a_n = \sum_{n=M+1}^{\infty} |a_n|$  is conditionally convergent. Hence, by Proposition 7.2.14 c),  $\sum_{n=0}^{\infty} |a_n|$  is conditionally convergent, hence  $\sum_{n=0}^{\infty} a_n$  is absolutely convergent, contradiction.

Similarly it cannot hold that  $A^-$  is countable and  $A^+$  is finite, as desired.

### The sums $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ both fail to be absolutely convergent

We argue by contradiction. WLOG assume  $\sum_{m=0}^{\infty} a_{f_+(m)}$  is absolutely convergent, then  $\sum_{n \in A^+} a_n$  is absolutely convergent, contradiction.

### The map $j \mapsto n_j$ is injective

We argue by contradiction. Assume  $\exists j, j' \in \mathbb{N}, j \neq j' \text{ and } n_j = n_{j'}$ .

Assume  $n_j \in \mathbb{A}^+$ ,  $n_{j'} \in \mathbb{A}^-$ , then it holds that  $n_j \geq 0$ ,  $n_{j'} < 0$ , hence  $n_j \neq n_{j'}$ , contradiction.

Assume  $n_j \in \mathbb{A}^+$ ,  $n_{j'} \in \mathbb{A}^+$ . WLOG say  $j' < j$ . Then  $n_j = \min\{n \in \mathbb{A}^+ : n \neq n_i \text{ for all } i < j\}$ . Hence, since  $j' < j$ , it holds that  $n_j \neq n_{j'}$ , contradiction.

Case  $j \in \mathbb{A}^-$ ,  $j' \in \mathbb{A}^-$  follows similarly, as desired.

### Case I occurs an infinite number of times, and Case II also occurs an infinite number of times

We first translate "Case I occurs an infinite number of times" into mathematical language. We define the number of times Case I appears as  $\{j \in \mathbb{N} : \sum_{0 \leq i < j} a_{n_i} < L\}$ . (And vice versa for Case 2)

We prove Case I occurs an infinite number of times. We argue by contradiction. Assume Case I occurs finitely many times. Then, the set  $\{j \in \mathbb{N} : \sum_{0 \leq i < j} a_{n_i} < L\}$  is finite. Therefore,  $\max\{j \in \mathbb{N} : \sum_{0 \leq i < j} a_{n_i} < L\}$  exists.

We then let  $M = \max\{j \in \mathbb{N} : \sum_{0 \leq i < j} a_{n_i} < L\}$ . Then, for all  $j > M$ , it holds that  $\sum_{0 \leq i < j} a_{n_i} \geq L$ . Therefore

$$\begin{aligned} \sum_{0 \leq i < j} a_{n_i} &\geq L = \sum_{0 \leq i < M} a_{n_i} + \sum_{M+1 \leq i < j} a_{n_i} \\ &\geq L = \sum_{M+1 \leq i < j} a_{n_i} + \sum_{0 \leq i < M} a_{n_i} \\ &\geq L - \sum_{0 \leq i < M} a_{n_i} + \sum_{M+1 \leq i < j} a_{n_i} \end{aligned} \quad \text{As } \sum_{0 \leq i < M} a_{n_i} < L$$

Yet since for all  $j > M$ ,  $n_j \in \mathbb{A}^+$ , hence for all  $j > M$ ,  $a_{n_j} < 0$ , therefore  $\sum_{M+1 \leq i < j} a_{n_i} < 0$ , contradiction.

### The map $j \mapsto n_j$ is surjective

We argue by contradiction. Assume  $j \mapsto n_j$  is surjective. then, by Proposition 8.1.4, there exists a minimum  $N$  s.t.  $\forall j \in \mathbb{N}$ ,  $n_j \neq N$ .

WLOG say  $N \in \mathbb{A}^+$ . We then argue

$$\forall j \in \mathbb{N}, \sum_{0 \leq i < j} a_{n_i} < L, n_j < N$$

We argue by contradiction. Assume  $\exists j \in \mathbb{N}$ ,  $\sum_{0 \leq i < j} a_{n_i} < L$ ,  $n_j \geq N$ , then since  $\forall j \in \mathbb{N}$ ,  $n_j \neq N$ , it holds that  $n_j > N$  and  $N \in \{n \in \mathbb{A}^+ : n \neq n_i \text{ for all } i < j\}$ . Since  $n_j = \min\{n \in \mathbb{A}^+ : n \neq n_i \text{ for all } i < j\}$ , it holds that  $n_j \leq N$ . Yet  $n_j > N$ , contradiction.

Then, consider the set  $\{n_j : j \in \mathbb{N}, \sum_{0 \leq i < j} a_{n_i} < L\}$ , easy to verify  $\{n_j : j \in \mathbb{N}, \sum_{0 \leq i < j} a_{n_i} < L\} \subseteq \{n \in \mathbb{N} : n < N\}$ . Hence, by Proposition 3.6.14 c),  $\{n_j : j \in \mathbb{N}, \sum_{0 \leq i < j} a_{n_i} < L\}$  is finite.

However, since Case I occurs infinitely many times, it holds that  $\{j \in \mathbb{N} : \sum_{0 \leq i < j} a_{n_i} < L\}$  is infinite. Since  $j \mapsto n_j$  is injective, it holds that the image  $\{n_j : j \in \mathbb{N}, \sum_{0 \leq i < j} a_{n_i} < L\}$

$\{j \mid a_{n_j} < L\}$  is infinite. Contradiction.

**We have  $\lim_{j \rightarrow \infty} a_{n_j} = 0$**

Since  $\sum_{n=0}^{\infty} a_n$  converges, by Corollary 7.2.6,  $\lim_{n \rightarrow \infty} a_n = 0$ . Easy to verify  $j \mapsto n_j$  is increasing. Hence,  $(a_{n_j})_{j \in \mathbb{N}}$  is a subsequence of  $(a_n)_{n \in \mathbb{N}}$ , therefore  $\lim_{j \rightarrow \infty} a_{n_j} = 0$  by Proposition 6.6.5.

**We have  $\lim_{j \rightarrow \infty} \sum_{0 \leq i < j} a_{n_i} = L$**

Denote  $S_j = \sum_{0 \leq i < j} a_{n_i}$ .

We claim:  $\forall \epsilon > 0, \exists K \in \mathbb{N}, S_K < L + \epsilon \quad \text{(1)}$

To prove the claim, since  $\lim_{j \rightarrow \infty} a_{n_j} = 0$ , we have:

$\forall \epsilon' > 0, \exists J' \geq 0, \forall j \geq J', |a_{n_j}| \leq \epsilon' \quad \text{(2)}$

We then argue, for any  $J \in \mathbb{N}$ , there exists  $j \geq J$  s.t.  $S_j < L$

$\forall J \in \mathbb{N}, \exists j \geq J, S_j < L \quad \text{(3)}$

We argue by contradiction. Assume  $\exists J \in \mathbb{N}, \forall j \geq J, S_j \geq L$ . Then,  $\{j \in \mathbb{N} : S_j < L\}$  is finite, hence Case I occurs finitely many times, contradiction.

Instantiate (1) with  $\epsilon' = \epsilon$ , we have  $\exists J' \geq 0, \forall j \geq J', |a_{n_j}| \leq \epsilon$ . Instantiate (3) with  $J = J'$ , we have  $\exists j' \geq J', S_{j'} < L$ . Since  $S_{j'} < L$ , it holds that  $S_{j'} < L + \epsilon$ . Hence  $K = j'$  is permissible, proving (0). Furthermore, since  $\forall j \geq J', |a_{n_j}| \leq \epsilon$  and  $K = j' \geq J'$ , it holds that  $\forall j \geq K, |a_{n_j}| \leq \epsilon$ .

We then claim:

$\forall j \geq K, S_j < L + \epsilon$

We argue by induction. Let  $P(j)$  denote  $S_j < L + \epsilon$ , then the base case  $P(j)$  is trivially true.

For  $P(j+1)$ , we proceed by cases. Either  $S_j < L$  or  $L \leq S_j < L + \epsilon$ :

If  $S_j < L$ , then:

$S_j + a_{n_{j+1}} < L + a_{n_{j+1}}$

$S_{j+1} < L + a_{n_{j+1}} \quad \text{Since } S_j + a_{n_{j+1}} = S_{j+1}$

$S_{j+1} < L + \epsilon \quad \text{Since } L + a_{n_{j+1}} \leq L + |a_{n_{j+1}}| \leq L + \epsilon$

As desired.

If  $L \leq S_j < L + \epsilon$ , then by Case II,  $a_{n_j}$  is negative, therefore  $S_{j+1} = S_j + a_{n_j} < S_j < L + \epsilon$  as desired.

Since  $\forall \epsilon > 0, \exists K \in \mathbb{N}, S_K < L + \epsilon$  and  $\forall j \geq K, S_j < L + \epsilon$ , it holds that

$$\forall \epsilon > 0, \exists K \in \mathbb{N}, \forall j \geq K, S_j < L + \epsilon \quad \text{(3)}$$

Via similar reasoning, one can show:

$$\forall \epsilon > 0, \exists K' \in \mathbb{N}, \forall j \geq K, L - \epsilon \leq S_j \quad \text{(4)}$$

By (3) and (4):

$$\forall \epsilon > 0, \forall j \geq \max(K, K'), |S_j - L| \leq \epsilon$$

Hence  $\lim_{j \rightarrow \infty} \sum_{0 \leq i < j} a_{n_i} = L$ , as desired.

## Exercise 8.2.6

Remark: the question need to be amended. Check Errata for detail.

### ► Informal Idea

We aim to make minimal tweak to Exercise 8.2.5 Solution, while incorporating the informal idea.

We define  $(n_j)_{n=0}^{\infty}$  recursively as follows. Denote  $\text{cnt}(j) = |\{i < j: n_i \in A^-\}|$  (Informally, number of negative elements added), then:

$$n_j = \begin{cases} \min\{n \in A^+: n \neq n_i \text{ for all } i < j\} & \text{if } \sum_{0 \leq i < j} n_i < \text{cnt}(j) \\ \min\{n \in A^-: n \neq n_i \text{ for all } i < j\} & \text{if } \sum_{0 \leq i < j} n_i \geq \text{cnt}(j) \end{cases}$$

### $j \mapsto n_j$ is injective

Follows from similar reasoning in Exercise 8.2.5

### Case I, II appears infinitely many times

Denote  $S_j = \sum_{0 \leq i < j} n_i$ . We argue by contradiction.

Assume Case I appears finitely many times, then take  $M = \max\{j \in \mathbb{N}: n_j \in A^+\}$ . Then, it holds that  $\forall n > M, S_n \geq \text{cnt}(n)$ . However, easy to verify  $\forall n > M, S_M \geq S_n$  and  $\text{cnt}(n) = \text{cnt}(M) + n - M$  (e.g. by induction), Hence,  $\forall n > M$ , we have

$$S_M > S_n \geq \text{cnt}(n) = \text{cnt}(M) + n - M$$

$$S_M > \text{cnt}(M) + n - M$$

We can always choose sufficiently large  $n$  s.t.  $S_M \leq \operatorname{cnt}(M) + n - M$  (e.g. by Proposition 5.4.12) Contradiction.

Assume Case II appears finitely many times, then take  $M = \max\{j \in \mathbb{N} : n_j \in A^+\}$ . Then, it holds that  $\forall n > M, S_n < \operatorname{cnt}(n)$ . Furthermore,  $\operatorname{cnt}(n) = \operatorname{cnt}(m)$ . Hence  $(S_n)_{n=M+1}^\infty$  is bounded above.

Since  $\forall j \geq M, a_{\{n_j\}} \in A^+$ ,  $(S_n)_{n=M+1}^\infty$  is increasing. Therefore by Proposition 6.3.8,  $(S_n)_{n=M+1}^\infty$  converges. Hence, by Definition 7.2.2,  $\sum_{i=0}^\infty a_{\{n_j\}}$  converges. Hence, by Proposition 7.2.14 c),  $\sum_{i=M+1}^\infty a_{\{n_j\}}$  converges. Since  $\forall j > M, a_{\{n_j\}} \in A^+$ , it holds that  $\sum_{i=M+1}^\infty a_{\{n_j\}}$  converges absolutely.

We then claim  $A^+ - \{n_j : j > M\}$  is a finite set. We first prove  $A^+ = \{n_j : j \in \mathbb{N}, n_j \in A^+\}$ . Easy to verify  $\{n_j : j \in \mathbb{N}, n_j \in A^+\} \subseteq A^+$ . For  $A^+ \subseteq \{n_j : j \in \mathbb{N}, n_j \in A^+\}$ . We argue by contradiction. Suppose for some  $p \in A^+$ ,  $\forall j \in \mathbb{N}, n_j \neq p$ . Then, it holds that, for all  $j \in \mathbb{N}$  s.t.  $n_j = \min\{n \in A^+ : n \neq n_i \text{ for all } i < j\}$ ,  $p \geq n_j$ .

However, observe for each  $n_j \in A^+$ ,  $n_j$  forms a strictly increasing sequence of natural number.

(Assume the contrary, then define  $\operatorname{suc}(n) = \min\{j > n : n_j \in A^+\}$ . We can then find minimum  $M$  s.t.  $n_M, n_{\operatorname{suc}(M)} \in A^+$  and  $n_M \geq n_{\operatorname{suc}(n)}$ . However, then it holds that  $n_{\operatorname{suc}(n)} \in \{n \in A^+ : n \neq n_i \text{ for all } i < N\}$ . Hence  $n_M > n_{\operatorname{suc}(n)}$ . Therefore  $n_M = n_{\operatorname{suc}(n)}$ . However  $n_M \notin \{n \in A^+ : n \neq n_i \text{ for all } i < \operatorname{suc}(M)\}$ , hence  $n_M \neq n_{\operatorname{suc}(n)}$ , contradiction.)

Since Case I happens infinitely many times. the sequence is strictly increasing. Hence, we can always find  $j$  s.t.  $n_j \in A^+$  is arbitrary large. Therefore  $\exists j, p < n_j$ , contradiction. Therefore  $A^+ = \{n_j : j \in \mathbb{N}, n_j \in A^+\}$ .

Hence  $A^+ - \{n_j : j > M\} = \{n_j : j \in \mathbb{N}, n_j \in A^+\} - \{n_j : j > M\} = \{n_j : j \leq M, n_j \in A^+\}$ , which is a finite set, as desired.

Therefore, since:

- $A^+ = \{n_j : j > M\} \cup (A^+ - \{n_j : j > M\})$
- $\{n_j : j > M\} \cap (A^+ - \{n_j : j > M\}) = \emptyset$
- $\sum_{n \in \{n_j : j > M\}} a_n = \sum_{i=M+1}^\infty a_{\{n_j\}}$  is absolutely convergent
- $(A^+ - \{n_j : j > M\})$  is finite

By Proposition 8.2.6 c),  $\sum_{n \in A^+} a_n$  is absolutely convergent, contradiction. Therefore, Case II appears infinitely many times.

## $\$j \mapsto n_j\$ is surjective$

We had already shown  $A^+ = \{n_j: j \in \mathbb{N}, n_j \in A^+\}$ . Hence  $A^+ \subseteq n(\mathbb{N})$ . We can similarly show  $A^- \subseteq n(\mathbb{N})$ . Hence  $\$j \mapsto n_j\$ is surjective, as desired.$

$$\lim_{j \rightarrow \infty} a_{n_j} = 0$$

Follows from similar reasoning in Exercise 8.2.5

$$\liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_{f(n)} = \limsup_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_{f(n)} = \infty$$

We first claim:

$$\forall \epsilon > 0, \exists J \geq 0, \forall j \geq J, \operatorname{cnt}(j) - \epsilon \leq S(j) \leq \operatorname{cnt}(j) + \epsilon$$

To prove the claim, since  $\lim_{j \rightarrow \infty} a_{n_j} = 0$ , we have:

$$\forall \epsilon' > 0, \exists N \geq 0, \forall j \geq N, |a_{n_j}| \leq \epsilon' \quad (1)$$

Instantiate (1) with  $\epsilon' = \epsilon$ . We can find  $J > N$  s.t. Case II happens, aka  $S_J \geq \operatorname{cnt}(J)$  (Otherwise, similar to the reasoning in Exercise 8.2.5, Case II happens finitely, contradiction).

We claim such  $J$  is permissible in (0). We argue by induction on  $j$ . For  $P(J)$ , it holds that  $\operatorname{cnt}(J) - \epsilon \leq \operatorname{cnt}(J) \leq S(J)$ , as desired.

For  $P(j+1)$ , assume  $j+1$  is under Case I. First, observe  $j+1$  is under Case I,  $\operatorname{cnt}(j) = \operatorname{cnt}(j+1)$ . Then we proceed by cases.

If  $j$  is under Case I, then

$$\begin{aligned} S_{j+1} &= S_j + a_{n_j} \geq S_j \quad \text{As } j \text{ is under Case I, } a_{n_j} \geq 0 \\ &\geq \operatorname{cnt}(j) - \epsilon \geq \operatorname{cnt}(j+1) - \epsilon \quad \text{By } P(j) \\ &\geq \operatorname{cnt}(j+1) - \epsilon \quad \text{As } j+1 \text{ is under Case I} \end{aligned}$$

If  $j$  is under Case II, then  $S_N \geq \operatorname{cnt}(j)$ , hence:

$$\begin{aligned} S_{j+1} &= S_j + a_{n_j} \leq S_j - |a_{n_j}| \quad \text{As } j \text{ is under Case II, } a_{n_j} < 0 \\ &\geq \operatorname{cnt}(j) - |a_{n_j}| \quad \text{As } j \text{ is under Case II, } S_j \geq \operatorname{cnt}(j) \\ &\geq \operatorname{cnt}(j) - \epsilon \geq \operatorname{cnt}(j+1) - \epsilon \quad \text{As } |a_{n_j}| \leq \epsilon \\ &\geq \operatorname{cnt}(j+1) - \epsilon \quad \text{As } j+1 \text{ is under Case I} \end{aligned}$$

Assume  $j+1$  is under Case II, then:

$$\begin{aligned} S_{j+1} &\geq \operatorname{cnt}(j+1) \quad \text{As } j+1 \text{ is under Case II} \\ &\geq \operatorname{cnt}(j+1) - \epsilon \end{aligned}$$



As desired.

Therefore, instantiate  $\epsilon = 1$ , we have, for all  $j > J$ :

$$S_j \geq \operatorname{cnt}(j) - 1$$

Since Case 2 happens infinitely many times, easy to verify  $\limsup_{n \rightarrow \infty} \operatorname{cnt}(j) - 1 = \liminf_{n \rightarrow \infty} \operatorname{cnt}(j) - 1 = \infty$ . Therefore, by Corollary 6.4.14:

$$\liminf_{n \rightarrow \infty} S_n \geq \liminf_{n \rightarrow \infty} \operatorname{cnt}(n) - 1 = \infty$$

$$\limsup_{n \rightarrow \infty} S_n \geq \limsup_{n \rightarrow \infty} \operatorname{cnt}(n) - 1 = \infty$$

As desired.

### Exercise 8.3.1

### Exercise 8.4.1

#### Show that the Axiom of Choice implies Proposition 8.4.7

Consider the set  $Y_x = \{y \in Y : P(x, y) \text{ is true}\}$ . Then, by Axiom of Choice (using  $X$  as index set), there exists a function  $g: X \rightarrow \bigcup_{x \in X} Y_x$  s.t. for each  $x \in X$ ,  $g(x) \in Y_x$ . Observe,  $g(x) \in Y_x$  iff  $P(x, g(x))$  is true, hence  $g: X \rightarrow \bigcup_{x \in X} Y_x$  has the property s.t. for each  $x \in X$ ,  $P(x, g(x))$  is true. Easy to verify  $\bigcup_{x \in X} Y_x \subseteq Y$ . Hence we define  $f: X \rightarrow Y$  as  $\forall x \in X, f(x) = g(x)$ . Since  $\forall x \in X, P(x, g(x))$  is true and  $\forall x \in X, f(x) = g(x)$ ,  $\forall x \in X, P(x, f(x))$  is true, as desired.

#### Show that Proposition 8.4.7 implies the Axiom of Choice

Let  $I$  be a set and for each  $\alpha \in I$  let  $X_\alpha$  be a non-empty set. Then, define the property  $P(\alpha, x)$  pertaining to object  $\alpha \in I$  and  $x \in \bigcup_{\beta \in I} X_\beta$ , with the property being  $x \in X_\alpha$ . Pick  $x' \in X_{\alpha'}$  (by Lemma 3.1.6), easy to verify  $x' \in \bigcup_{\beta \in I} X_\beta$ . Hence  $P(\alpha, x')$  is true. Therefore there is at least one  $x \in \bigcup_{\beta \in I} X_\beta$  such that  $P(\alpha, x)$  is true.

Then, by Proposition 8.4.7, there exists a function  $f: I \rightarrow \bigcup_{\beta \in I} X_\beta$  s.t.  $\forall \alpha \in I, P(\alpha, f(\alpha))$  is true. Hence  $\forall \alpha \in I, f(\alpha) \in X_\alpha$  is true, as desired.

### Exercise 8.4.2

Using the axiom of choice, show that there exists a set  $Y$  such that  $|Y \cap X_\alpha| = 1$  for all  $\alpha \in I$

By Axiom of Choice, there exists  $f: I \rightarrow \bigcup_{\beta \in I} X_{\beta}$  s.t.  $\forall \alpha \in I, f(\alpha) \in X_{\alpha}$ . We claim  $Y = \{f(\alpha) : \alpha \in I\}$  is permissible. Observe  $f(\alpha) \in Y \cap X_{\alpha}$ . Hence  $|Y \cap X_{\alpha}| \geq 1$  by 3.6.14 c). Assume  $|Y \cap X_{\alpha}| > 1$ , then there exists  $e \in Y \cap X_{\alpha}, e \neq f(\alpha)$ . Since  $e \in Y$ , it holds that  $e = f(\beta)$  for some  $\beta \in I - \{\alpha\}$ . Therefore,  $e \in X_{\beta}$ . Since  $e \in X_{\alpha}$  and  $e \in X_{\beta}$ , it holds that  $e \in X_{\alpha} \cup X_{\beta}$ , hence by single choice  $X_{\alpha} \cap X_{\beta} \neq \emptyset$ , however  $X_{\alpha} \cap X_{\beta} = \emptyset$ , contradiction. Hence  $|Y \cap X_{\alpha}| = 1$

**Show that if the above statement was true for an arbitrary choice of sets  $I$  and non-empty disjoint sets  $X_{\alpha}$ , then the axiom of choice is true**

Given an index set  $I$ , we need to show there exists  $f: I \rightarrow \bigcup_{\beta \in I} X_{\beta}$  s.t.  $\forall \alpha \in I, f(\alpha) \in X_{\alpha}$ . Consider the sets  $M_{\alpha} = \{(\alpha, x) : x \in X_{\alpha}\}$ . Easy to verify  $\forall \alpha, \beta \in I, \alpha \neq \beta \rightarrow M_{\alpha} \cap M_{\beta} = \emptyset$ . Hence, there exists a set  $Y$  s.t.  $\forall \alpha \in I, |Y \cap M_{\alpha}| = 1$ .

We then define  $f: I \rightarrow \bigcup_{\beta \in I} X_{\beta}$  implicitly with the property  $P(\alpha, x)$  as  $(\alpha, x) \in Y \cap M_{\alpha}$ . The property is permissible, as, given  $\alpha$ , since  $|Y \cap M_{\alpha}| = 1$ , there exists  $(\alpha, x) \in Y \cap M_{\alpha}$ . Furthermore, assume  $(\alpha, x') \in Y \cap M_{\alpha}$  for some  $x' \neq x$ , then  $|Y \cap M_{\alpha}| > 1$ , contradiction.

We claim  $\forall \alpha \in I, f(\alpha) \in X_{\alpha}$ . Since  $(\alpha, f(\alpha)) \in Y \cap M_{\alpha}$ , it holds that  $(\alpha, f(\alpha)) \in M_{\alpha}$ , hence  $f(\alpha) \in X_{\alpha}$ , as desired. Therefore  $f$  is the desired choice function.

### Exercise 8.4.3

Errata: Tao stated that  $f: A \rightarrow B$  must not only be an injection, but  $g \circ f : A \rightarrow A$  is the identity map

For some reason the error remain uncorrected in the 3rd edition.

**Using the axiom of choice, show that there then exists an injection  $f : A \rightarrow B$  s.t.  $g \circ f : A \rightarrow A$  is the identity map**

Denote  $X_{\alpha} = g^{-1}(\{\alpha\})$ . Since  $g: B \rightarrow A$  is surjective,  $g^{-1}(\{\alpha\})$  is non-empty for all  $\alpha \in A$ . Then, by Axiom of Choice there exists  $f: A \rightarrow \bigcup_{\alpha \in A} X_{\alpha}$  s.t.  $\forall \alpha \in A, f(\alpha) \in X_{\alpha}$ . Easy to verify  $\bigcup_{\alpha \in A} X_{\alpha} \subseteq B$ . Hence we can extend the range of  $f$  s.t.  $f: A \rightarrow B$ .

We claim  $f$  is the desired injection. Given any  $\alpha, \alpha' \in A$  s.t.  $\alpha \neq \alpha'$ , since  $\forall \alpha \in A, f(\alpha) \in X_{\alpha}$ , it holds that  $f(\alpha) \in g^{-1}(\{\alpha\})$  and  $f(\alpha') \in g^{-1}(\{\alpha'\})$ . Therefore  $g(f(\alpha)) = \alpha, g(f(\alpha')) = \alpha'$ . (and  $g \circ f$  is the identity map). We then argue by contradiction. Assume  $f(\alpha) = f(\alpha')$ , then  $g(f(\alpha)) =$



$g(f(\alpha))$  implying  $\alpha = \alpha'$ , contradiction. Hence given any  $\alpha, \alpha' \in A$ , it holds that  $f(\alpha) \neq f(\alpha')$ , as desired.

**Show that if the above statement is true for arbitrary sets  $A, B$  and surjections  $g : B \rightarrow A$ , then the axiom of choice is true**

As shown in Exercise 8.4.2, if, for any arbitrary choice of sets  $I$  and non-empty disjoint sets  $X_\alpha$ , there exists a set  $Y$  s.t.  $|Y \cap X_\alpha| = 1$  for all  $\alpha \in I$ , then the Axiom of Choice is true.

Hence, suffices to show such  $Y$  exists given for any arbitrary sets  $A, B$  and surjections  $g : B \rightarrow A$ , there exists an injection  $f : A \rightarrow B$ . We define the function  $g : \bigcup_{\alpha \in I} X_\alpha \rightarrow I$  with the property  $P(x, \alpha)$  being  $x \in X_\alpha$ .  $P(x, \alpha)$  is permissible as for each  $x \in \bigcup_{\alpha \in I} X_\alpha$ , there exists  $\alpha \in I$  s.t.  $x \in X_\alpha$ . Furthermore, there's exactly one  $\alpha \in I$  s.t.  $x \in X_\alpha$ , as, assume  $\exists \alpha, \alpha' \in I$  s.t.  $\alpha \neq \alpha'$  and  $x \in X_\alpha, x \in X_{\alpha'}$ , then  $X_\alpha \cap X_{\alpha'} = \emptyset$ , contradiction.

Then, there exists injection  $f : I \rightarrow \bigcup_{\alpha \in I} X_\alpha$  s.t.  $g \circ f : I \rightarrow I$  is identity map. We claim  $Y = \{f(\alpha) : \alpha \in I\}$  is permissible. We first show  $|Y \cap X_\alpha| \neq 0$  for any given  $\alpha \in I$ . Observe  $g(f(\alpha)) = \alpha$ , hence, by definition of  $g$ ,  $f(\alpha) \in X_\alpha$ . Therefore  $\{f(\alpha)\} \subseteq Y \cap X_\alpha$  hence  $|Y \cap X_\alpha| \neq 0$ , as desired.

We then show  $|Y \cap X_\alpha| \leq 1$  by contradiction. Assume  $|Y \cap X_\alpha| > 1$ , then there exists  $e \neq f(\alpha)$  s.t.  $e \in Y \cap X_\alpha$ . Since  $e \in Y$ ,  $\exists \beta \in I, f(\beta) = e$  by definition of  $Y$ . Furthermore since  $f(\beta) = e \neq f(\alpha)$ , it holds that  $\beta \neq \alpha$ . Since  $g(f(\beta)) = \beta$ ,  $f(\beta) \in X_\beta$  by definition of  $g$ . However, since  $e \in Y \cap X_\alpha$ ,  $f(\beta) = e \in X_\alpha$ . Hence  $f(\beta) \in X_\alpha \cap X_\beta$ , hence  $X_\alpha, X_\beta$  not disjoint, contradiction. Therefore  $|Y \cap X_\alpha| = 1$ , as desired.