HABIB ID:

NAME:

LINEAR ALGEBRA

SPRING 2023

QUIZ 12 L1

Max Marks: 10

Time: 12 minutes

Q. 1 State and prove the Dimension Theorem for matrices.

Q. 2 Hence or otherwise, if A is an m x n matrix, what is the largest possible value for its rank and the smallest possible value for its nullity?

Dimension Theorem for Matrices

If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These fall into two categories: the leading variables and the free variables. Thus

$$\begin{bmatrix}\text{number of leading} \\ \text{variables}\end{bmatrix} + \begin{bmatrix}\text{number of free} \\ \text{variables}\end{bmatrix} = n$$

But the number of leading variables is the same as the number of leading 1's in the reduced row-echelon form of A, and this is the rank of A. Thus

$$\operatorname{rank}(A) + \begin{bmatrix}\text{number of free} \\ \text{variables}\end{bmatrix} = n$$

The number of free variables is equal to the nullity of A. This is so because the nullity of A is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$, which is the same as the number of parameters in the general solution [see 3, for example], which is the same as the number of free variables. Thus

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

Max Value of Rank & Min Value of Nullity

$\operatorname{Rank}(A) \leq \min(m, n)$ & $\operatorname{Nullity}(A) \geq n - \min(m, n)$.