# Lecture 16

Thursday, March 10, 2022 4:33 AM

## PREVIOUS RESULTS:

(1) $\{(1,0),(0,1)\}$ IS STANDARD BASIS FOR $R^2$.

(2) $\{(-3,7),(5,5)\}$ IS A BASIS BUT $\boxed{\text{NOT}}$ A STANDARD BASIS FOR $R^2$

RECALL: $\boxed{(x,y) = x(1,0) + y(0,1)}$

AND
$$(x,y) = \frac{(y-x)}{10}(-3,7) + \frac{(7x+3y)}{50}\overset{(5,5)}{\uparrow}$$

$$\Rightarrow \boxed{(x,y) = \left(\frac{y-x}{10}\right)(-3,7) + \frac{(7x+3y)}{50}(5,5)}$$

## HOMOGENEOUS LINEAR SYSTEM

P.16 8TH ED.
P.17 7TH ED.

FOR $A\underset{\sim}{X} = \underset{\sim}{0}$, EXACTLY $\boxed{\text{ONE}}$ OF THE FOLLOWING IS TRUE:

(1) THE SYSTEM HAS $\boxed{\text{ONLY}}$ THE TRIVIAL SOLUTION (ZERO SOLUTION) IF $\boxed{A}$ IS INVERTIBLE

(2) SYSTEM HAS INFINITELY MANY SOLUTIONS (NONTRIVIAL) IN ADDITION TO THE TRIVIAL SOLUTION. i.e. $\boxed{A}$ IS SINGULAR OR UNKNOWNS ARE $\boxed{\text{MORE}}$ THAN EQUATIONS.

IF $S = \{\underline{V_1}, \underline{V_2}, \ldots\ldots, \underline{V_r}\}$ IS A NON-EMPTY SET OF VECTORS, THEN IN VECTOR EQUATION

$$K_1\underline{V_1} + K_2\underline{V_2} + \cdots\cdots + K_r\underline{V_r} = \underline{0}$$

IF ANY $\boxed{\text{ONE}}$ OF THE SCALARS $\boxed{K_i \neq 0}$, $1 \leq i \leq r$, THEN $\boxed{S}$ IS LINEARLY DEPENDENT AND ALL THE VECTORS $\underline{V_i}$, $1 \leq i \leq r$, ARE LINEARLY DEPENDENT VECTORS. (P.222 8th ED.)
(P.232 7th ED.)

EXAMPLE: CHECK WHETHER $\{(2,2),(1,1)\}$ IS INDEPENDENT OR DEPENDENT?

SOLUTION: LET

$$k_1(2,2) + k_2(1,1) = (0,0) \quad \text{①}$$

$$\Rightarrow \quad 2k_1 + k_2 = 0$$

$$2k_1 + k_2 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \because \det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 0$$

$\therefore$ NONTRIVIAL (NONZERO) SOLUTIONS EXIST, $\therefore$ GIVEN VECTORS ARE LINEARLY DEPENDENT.

NOTE: IN $2k_1 + k_2 = 0$

LET $\boxed{k_1 = t}$, $\boxed{k_2 = -2t}$,

IF $\boxed{t=1}$, $k_1 = 1$, $k_2 = -2$

$\therefore$ WE GET $(2,2) - 2(1,1) = (0,0)$

$\Rightarrow \boxed{(2,2) = 2(1,1)}$ FROM ①

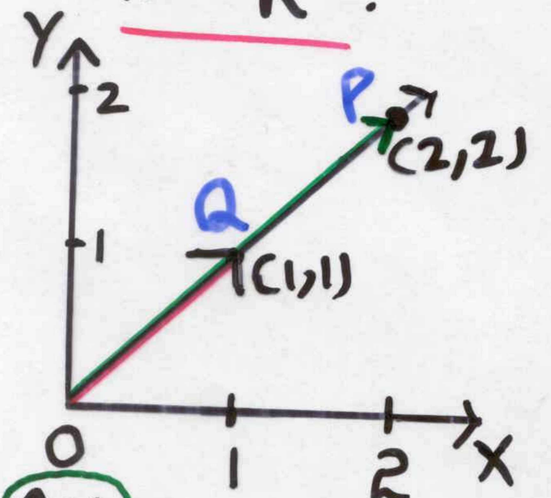
NOTE: RECALL THAT IF $A\underline{x} = \underline{0}$ REPRESENTS A HOMOGENEOUS SYSTEM OF EQUATIONS THEN $\boxed{\text{INFINITELY}}$ MANY NONTRIVIAL (NONZERO) SOLUTIONS EXIST IF $\boxed{A}$ IS SINGULAR i.e. $\boxed{\det(A)=0}$ OR $\boxed{A^{-1}}$ DOESN'T EXIST.

RESULTS:

① $\{(1,1),(2,2)\}$ IS LINEARLY DEPENDENT AND

② $\{(5,5),(-3,7)\}$ IS LINEARLY $\boxed{\text{INDEPENDENT}}$ (PROVED LAST TIME)

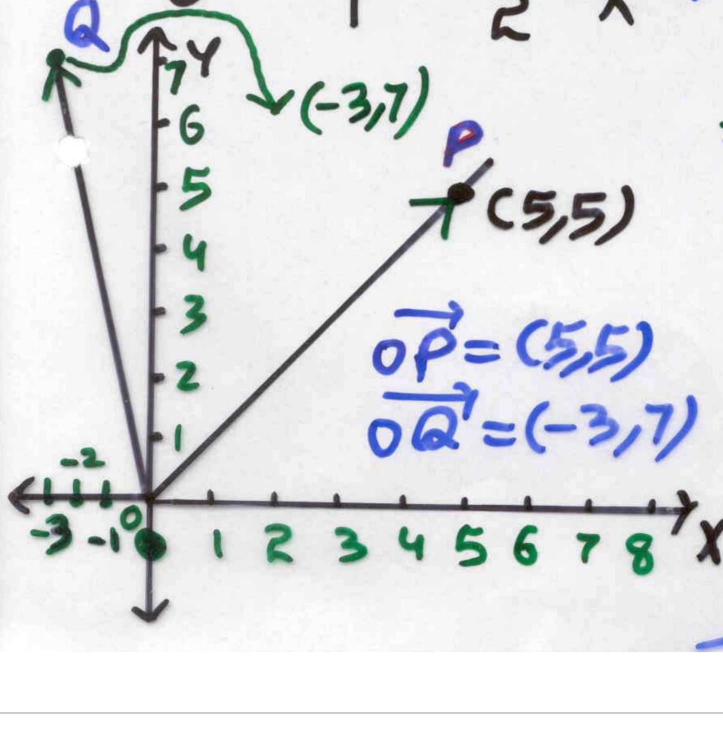
GEOMETRIC INTERPRETATION IN $R^2$:




IN $R^2$ $\boxed{\text{TWO}}$ VECTORS ARE LINEARLY DEPENDENT IF THEY LIE ON THE SAME LINE.

$\overrightarrow{OQ} = (1,1)$

$\overrightarrow{OP} = (2,2)$




IN $R^2$ $\boxed{\text{TWO}}$ VECTORS ARE LINEARLY INDEPENDENT IF THEY $\boxed{\text{DO NOT}}$ LIE ON THE SAME LINE.

NOTE: ALSO FOR ANY VECTOR SPACE $V$, THE SET $\{v_1, v_2\}$ IS DEPENDENT IF $v_1, v_2$ ARE SCALAR MULTIPLES OF EACHOTHER AND $\{v_1, v_2\}$ IS INDEPENDENT IF AND ONLY IF NEITHER VECTOR IS A SCALAR MULTIPLE OF EACH OTHER.

AS WE SAW THAT

$\{(1,1), (2,2)\}$ IS DEPENDENT AND $(2,2) = 2(1,1)$

i.e. $(1,1), (2,2)$ ARE SCALAR MULTIPLES OF EACHOTHER.

BUT

$\{(-3,7), (5,5)\}$ IS INDEPENDENT AND NONE OF $(-3,7), (5,5)$ IS A SCALAR MULTIPLE OF THE OTHER.

HERE $\{v_1, v_2\} \longrightarrow$ A SET CONTAINING EXACTLY TWO VECTORS

EXAMPLE: P. 229 (6th ED.)
P. 232 (7th ED.)

LET $\underline{v_1} = (2,-1,0,3)$, $\underline{v_2} = (1,2,5,-1)$ AND $\underline{v_3} = (7,-1,5,8)$ THEN

(a) CHECK WHETHER $\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ IS LINEARLY DEPENDENT OR INDEPENDENT.

SOLUTION: LET $k_1\underline{v_1} + k_2\underline{v_2} + k_3\underline{v_3} = \underline{0}$, $\Rightarrow k_1(2,-1,0,3) + k_2(1,2,5,-1) + k_3(7,-1,5,8) = (0,0,0,0)$ — (1)

COMPARING BOTH SIDES

$$2k_1 + k_2 + 7k_3 = 0 \rightarrow (2)$$

$$-k_1 + 2k_2 - k_3 = 0 \rightarrow (3)$$

$$k_2 + k_3 = 0 \rightarrow (4)$$

$$3k_1 - k_2 + 8k_3 = 0 \rightarrow (5)$$

WHICH IS A HOMOGENEOUS SYSTEM OF (4) EQUATIONS WITH THREE UNKNOWNS. SO THERE ARE TWO POSSIBILITIES

TO SOLVE THIS SYSTEM, SOLVE ②, ③, ④ AND CHECK IF THE SOLUTION SATISFIES ⑤ OR ONLY SOLVE ②, ③, ④ $\because$ ⑤ CAN BE OBTAINED BY SUBTRACTING ③ FROM ②

$$\because \quad \left.\begin{aligned} 2k_1 + k_2 + 7k_3 &= 0 \rightarrow (2) \\ \underset{+}{\mp} k_1 \underset{-}{\pm} 2k_2 \underset{+}{\mp} k_3 &= 0 \rightarrow (3) \end{aligned}\right\} \text{SUB-TRACT-ING}$$

$$3k_1 - k_2 + 8k_3 = 0 \rightarrow (5)$$

NOW IN MATRIX NOTATION, WE HAVE

$$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - (6)$$

BUT $\det \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 0$ (CHECK)

$\therefore$ NONTRIVIAL SOLUTIONS EXIST $\Rightarrow$ THE GIVEN VECTORS ARE LINEARLY DEPENDENT $\because$ ②, ③, ④ ARE SATISFIED BY NONZERO VALUES OF $k_1, k_2$ AND $k_3$.

SINCE THE SYSTEM IS HOMOGENEOUS AND THE COEFFICIENT MATRIX IS $\boxed{\text{NOT}}$ INVERTIBLE i.e. $\boxed{A^{-1}}$ DOESN'T EXIST THEREFORE INFINITE NONTRIVIAL (NONZERO) SOLUTIONS EXIST FOR

$A\underline{X} = \underline{0}$ WHERE

$$A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

(b) PROVE THAT $\underline{V_3}$ CAN BE WRITTEN AS THE LINEAR COMBINATION OF $\underline{V_2}$ AND $\underline{V_1}$.

SOLUTION: HINT: FOR THIS WE HAVE TO SOLVE THE ABOVE SYSTEM i.e. $A\underline{X} = \underline{0}$ FOR $\boxed{k_1, k_2 \text{ AND } k_3}$, i.e.

SOLUTION OF $\begin{cases} 2k_1 + k_2 + 7k_3 = 0 & \text{—①} \\ -k_1 + 2k_2 - k_3 = 0 & \text{—②} \\ k_2 + k_3 = 0 & \text{—③} \end{cases}$

TO FIND THE

LET $\boxed{k_2 = t}$ ③ $\Rightarrow$ $\boxed{k_3 = -k_2 = -t}$

② $\Rightarrow$ $k_1 = 2k_2 - k_3 = 2t - (-t) = 3t = k_1$

$\therefore \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ ALSO SATISFIES ①

FOR $\boxed{t = 1, \ k_1 = 3, k_2 = 1, k_3 = -1}$

$k_1\underline{v_1} + k_2\underline{v_2} + k_3\underline{v_3}$

$= 3(2,-1,0,3) + (1,2,5,-1)$

$-(7,-1,5,8) = (6,-3,0,9) +$

$(-6,3,0,-9) = (0,0,0,0)$

$\Rightarrow 3\underline{v_1} + \underline{v_2} - \underline{v_3} = \underline{0}$

OR $\boxed{\underline{v_3} = \underline{v_2} + 3\underline{v_1}}$

RESULT:

A SET S WITH TWO OR MORE VECTORS IS

(a) LINEARLY DEPENDENT IF AND ONLY IF AT LEAST $\boxed{\text{ONE}}$ OF THE VECTORS IN $\boxed{S}$ IS EXPRESSIBLE AS A LINEAR COMBINATION OF THE OTHER VECTORS IN $\boxed{S}$.

E.G. IN THE LAST EXAMPLE $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ IS DEPENDENT AND $\underline{v}_3 = \underline{v}_2 + 3\underline{v}_1$ i.e. $\underline{v}_3$ IS A LINEAR COMBINATION OF $\underline{v}_1$ AND $\underline{v}_2$.

(b) LINEARLY INDEPENDENT IF AND ONLY IF $\boxed{\text{NO}}$ VECTOR IN $\boxed{S}$ IS EXPRESSIBLE AS A LINEAR COMBINATION OF THE OTHER VECTORS IN $\boxed{S}$.

E.G. $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ IS LINEARLY INDEPENDENT $\because$ NONE OF $\underline{e}_1$, $\underline{e}_2, \underline{e}_3$ IS A LINEAR COMBINATION OF THE OTHER TWO.

LET $\underline{e}_3 = k_1 \underline{e}_1 + k_2 \underline{e}_2$

$$\Rightarrow \underline{e}_3 = k_1 \underline{e}_1 + k_2 \underline{e}_2$$

$$\Rightarrow (0,0,1) = k_1(1,0,0) + k_2(0,1,0)$$

$$\Rightarrow (0,0,1) = (k_1, k_2, 0)$$

WHICH IS $\boxed{\text{NOT}}$ POSSIBLE

$\therefore$ $\underline{e}_3$ IS NOT EXPRESSIBLE AS A LINEAR COMBINATION OF $\underline{e}_1$ AND $\underline{e}_2$.