Linear Algebra
Homework 1

Ali Muhammad Asad

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.3 : Matrices and Matrix Operations

4. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix},\ C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},\ D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix},\ E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Using these matrices, compute the following:
(a) $2A^T + C$ (b) $D^T - E^T$ (c) $(D-E)^T$ (d) $B^T + 5C^T$

(e) $\frac{1}{2}C^T - \frac{1}{4}A$ (f) $B - B^T$ (g) $2E^T - 3D^T$ (h) $(2E^T - 3D^T)^T$

Solution:

(a) $= 2\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ 0 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(b) $= \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(c) $= \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right)^T = \left(\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

(d) Cannot be computed since the matrix B^T and C^T have different orders, so addition is not defined.

(e) $= \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{2} \\ \frac{9}{4} & 0 \\ \frac{3}{4} & \frac{9}{4} \end{bmatrix}$

(f) $= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$(g) = 2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}$$

$$(h) = \left(2\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}\right)^T = \left(\begin{bmatrix} 9 & 1 & -1 \\ -13 & 2 & -4 \\ 0 & 1 & -6 \end{bmatrix}\right)^T = \begin{bmatrix} 9 & -13 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -6 \end{bmatrix}$$

Ex Set 1.4 : Inverses; Rules of Matrix Arithmetic

11. Find the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Solution: Determinant of the matrix $= (\cos\theta \times \cos\theta) - (-\sin\theta \times \sin\theta)$
Det $= \cos^2\theta + \sin^2\theta \implies$ Det $= 1$
Adjoint $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Inverse $= \frac{\text{Adjoint}}{\text{Determinant}} \implies$ Inverse $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

13. Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11}a_{22}\cdots a_{nn} \neq 0$. Show that A is invertible, and find its inverse.

Solution: $A = \text{diag}(a_{11}, a_{22}, ..., a_{nn})$.
For A to be invertible, $\det(A) \neq 0$.
The deterinant of any diagonal matrix is the product of its daigonals. Then $\det(A) = a_{11}a_{22}...a_{nn} \neq 0$ since $a_{11}a_{22}...a_{nn} \neq 0$. Therefore, the inverse of A exists.

For A^{-1} to exist, $AA^{-1} = I$ and $A^{-1}A = I$ must be true by the definition.

Let $A^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & ... & \alpha_{nn} \end{bmatrix}$

Then $AA^{-1} = I \implies \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & ... & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}$

$\implies \begin{bmatrix} a_{11}\alpha_{11} & a_{11}\alpha_{12} & ... & a_{11}\alpha_{1n} \\ a_{22}\alpha_{21} & a_{22}\alpha_{22} & ... & a_{22}\alpha_{2n} \\ \vdots & \vdots & & \vdots \\ a_{nn}\alpha_{n1} & a_{nn}\alpha_{n2} & ... & a_{nn}\alpha_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}$

By comparison, $a_{11}\alpha_{11} = 1 \implies \alpha_{11} = \frac{1}{a_{11}}$,
$a_{11}\alpha_{12} = 0 \implies \alpha_{12} = 0$ as $a_{11} \neq 0$. The same argument holds for $\alpha_{12}...\alpha_{1n}$. Similarly, $a_{22} \neq 0 \therefore \alpha_{22} = \frac{1}{a_{22}}$ however $\alpha_{21} = 0$ and $\alpha_{23}...\alpha_{2n} = 0$.

Then we can conclude that $\alpha_{ij} = \frac{1}{a_{ij}}\ \forall i = j \mid i, j \in \mathbb{N}$ where $i \leq n$ and $j \leq n$ and $\alpha_{ij} = 0\ \forall i \neq j \mid i, j \in \mathbb{N}$ where $i \leq n$ and $j \neq n$.

Then $A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & ... & 0 \\ 0 & \frac{1}{a_{22}} & ... & 0 \\ 0 & 0 & ... & \frac{1}{a_{nn}} \end{bmatrix}$

15. (a) Show that a matrix with a row of zeroes cannot have an inverse.

(b) Show that a matrix with a column of zeroes cannot have an inverse.

Solution:

(a) Let $A = \begin{bmatrix} 0 & 0 & ... & 0 \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$. Let B be the inverse of A. Then $AB = I$ should hold true. However, we previously proved that if a matrix A has a row of zeroes, then AB also has a row of zeroes. Hence, we know that $AB \neq I$ since AB has a row of zeroes. So we can conclude that A does not have an inverse if A has a row of zeroes.

(b) Let $A = \begin{bmatrix} 0 & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{m2} & ... & a_{nn} \end{bmatrix}$ where A has a column of zeroes.

Let $B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix}$ be the inverse of A, then $AB = I$ and $BA = I$ should hold true.

Then $BA = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix} \begin{bmatrix} 0 & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & ... & a_{nn} \end{bmatrix}$

$= \begin{bmatrix} b_{11}0 + b_{12}0 + ... + b_{1n}0 & b_{11}a_{12} + b_{12}a_{22} + ... + b_{1n}a_{n2} & ... & b_{11}a_{1n} + b_{12}a_{2n} + ... + b_{1n}a_{nn} \\ b_{21}0 + b_{22}0 + ... + b_{2n}0 & b_{21}a_{12} + b_{22}a_{22} + ... + b_{2n}a_{n2} & ... & b_{21}a_{1n} + b_{22}a_{2n} + ... + b_{2n}a_{nn} \\ \vdots & \vdots & & \vdots \\ b_{n1}0 + b_{n2}0 + ... + b_{nn}0 & b_{n1}a_{12} + b_{n2}a_{22} + ... + b_{nn}a_{n2} & ... & b_{n1}a_{1n} + b_{n2}a_{2n} + ... + b_{nn}a_{nn} \end{bmatrix}$

$= \begin{bmatrix} 0 & b_{11}a_{12} + b_{12}a_{22} + ... + b_{1n}a_{n2} & ... & b_{11}a_{1n} + b_{12}a_{2n} + ... + b_{1n}a_{nn} \\ 0 & b_{21}a_{12} + b_{22}a_{22} + ... + b_{2n}a_{n2} & ... & b_{21}a_{1n} + b_{22}a_{2n} + ... + b_{2n}a_{nn} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n1}a_{12} + b_{n2}a_{22} + ... + b_{nn}a_{n2} & ... & b_{n1}a_{1n} + b_{n2}a_{2n} + ... + b_{nn}a_{nn} \end{bmatrix}.$

Then BA also has a column of zeroes which implies $BA \neq I$. Hence if A has a column of zeroes, then BA also has a column of zeroes, therefore, A does not have an inverse.

16. Is the sum of two invertible matrices necessarily invertible? [is not, why?]

Solution: No, the sum of two invertible matrices may not necessarily be invertible. Consider two invertible matrices A and B. It is quite possible that the $A + B$ may produce a matrix with a row of zeroes, or a colummn of zeroes, in which case the resultant matrix is no longer invertible.
Consider, $A = I$ which is invertible [the identity matrix is invertible], and $B = -I$, which is also invertible [multiplying an invertible matrix with a non-zero constant, the resultant matrix is also invertible]. Then $A+B = 0$ that is, the sum of A and B produces a 0 matrix which is not invertible. Hence proved.

17. Let A and B be square matrices such that $AB = 0$. Show that if A is invertible, then B = 0.

Solution: If A is invertible, then $A^{-1}A = I$ is true. Then premultiplying A^{-1} on both sides, we get $A^{-1}AB = A^{-1}0 \implies IB = 0$ as a matrix times the 0 matrix is equal to the zero matrix. Then $IB = 0 \implies B = 0$ as $IB = B$. Hence shown that $B = 0$ if A is invertible and $AB = 0$. [If A is invertible, then $A \neq 0$ as 0 is not invertible]

29. (a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 (from the book) do not contradict one another.

Solution:

(a) If A is invertible, and $AB = AC$ then $B = C$.
If A is invertible, then $A^{-1}A = I$ holds true. Then premultiplying by A^{-1} we get $A^{-1}AB = A^{-1}AC \implies IB = IC \implies B = C$ as $IA = A$.

(b) They donot contradict one another as A was not invertible in Example 3. So when A is not invertible, then the operation $A^{-1}A$ does not hold as it does not exist. Therefore, A cannot be cancelled from both sides.
Since A is invertible over here, it can be cancelled, therefore, it does not contradict the example.