Linear Algebra Spring 23
Exercise 6 Solutions

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.6 : Further Results on Systems of Equations and Invertibility

Question 16 Find the conditions that the $b's$ must satisfy for the system to be consistent.
$6x_1 - 4x_2 = b_1$
$3x_1 - 2x_2 = b_2$

Solution: $\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \quad \mathrm{R}_1 - 2\mathrm{R}_2 \left[\begin{array}{cc|c} 0 & 0 & b_1 - b_2 \\ 3 & -2 & b_2 \end{array}\right]$

We can see that the first row gets a row of zeroes, then it must be consistent for $b_1 = 2b_2$

Question 23 Let $Ax = 0$ be a homogenous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k x = 0$ also has only the trivial solution.

Solution:

Since $Ax = 0$ has only the trivial solution, then theorem 1.6.4 guarantees that A is invertible. Then by Theorem 1.4.8 (b), A^k is also invertible. Then $(A^k)^{-1} = (A^{-1})^k$

We can note that $\underbrace{A^{-1}A^{-1}A^{-1}\cdots A^{-1}}_{\text{k factors}}\underbrace{AAA\cdots A}_{\text{k factors}} = I$

Since A^k is invertible, therefore, by theorem 1.6.4 we can conclude that $A^k x = 0$ also only has the trivial solution.

Question 24 Let $Ax = 0$ be a homogenous system of n linear equations, in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $Ax = 0$ has the trivial solution if and only if $(QA)x = 0$ has just the trivial solution.

Solution: Let $Ax = 0$ hold true. Then $Q(Ax) = Q0 \implies (QA)x = 0 \because$ Associative Law
Now let $(QA)x = 0$ hold true, [as shown above].
Then $Q^{-1}(QA)x = Q^{-1}0 \implies (Q^{-1}Q)Ax = Q^{-1}0 \because$Associative Law
$\implies IAx = 0 \implies Ax = 0$

Hence proved.

Question 25 Let $Ax = b$ be any consistent system of linear equations, and let x_1 be a fixed solution. Show that every solution to the system can be written in the form $x = x_1 + x_0$ where x_0 is a solution $Ax = 0$. Show that every matrix of this form is a solution.

Solution: Suppose that x_1 is a fixed matrix which satisfies the equation $Ax_1 = b$. Further let x be any matrix whatsoever which satisfies the equation $Ax = b$. We must show that there is a matrix x_0 which satisfies both of the equations $x = x_1 + x_0$ and $Ax_0 = 0$. Show that every matrix of this form is a solution.

Then the first equation implies that $x_0 = X - X_1$.
This candidate for x_0 will satisfy the second equation because $Ax_0 = A(x - x_1) = Ax - Ax_1 = b - b = 0$

We must also show that if both $Ax_1 = b$ and $Ax_0 = 0$, then $A(x_1 + x_0) = b$.
But $A(x_1 + x_0) = Ax_1 + Ax_0 = b + 0 = b$

Question 26 Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$

Use part (a) to prove part (b)

Solution: For $B = A^{-1}$ [from (a)].
Given that $AB = I$, and $BA = I$.
Hence $AB = BA = I \implies B = A^{-1}$ for $AB = I$.
Another thing can be seen is $A^{-1}A = AA^{-1} = I$