Linear Algebra
Homework 1

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.1 Intro to Systems of Linear Equations

1. Which of the following are linear equations in x_1, x_2 and x_3?

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$

(b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$

(d) $x_1^{-2} + x_2 + 8x_3 = 4$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

Solution:

(a) Is a linear equation $\rightarrow$ obvious.
(b) Is not a linear equation because of the term x_1x_3
(c) Is a linear equation
(d) Is not a linear equation because of the term x_1^{-2}
(e) Is not a linear equation because of the term $x_1^{3/5}$
(f) Is a linear equation

2. Given that k is a constant, which of the following are linear equations?

(a) $x_1 - x_2 + x_3 = sin(k)$

(b) $kx_1 - \frac{1}{k}x_2 = 9$

(c) $2^k x_1 + 7x_2 - x_3 = 0$

Solution:

(a) Is linear - k is a constant $\implies sin(k)$ is a constant.
(b) Is linear. [Assuming that $k \neq 0$]
(c) Is linear - k is a constant $\implies 2^k$ is a constant.

3. Find the solution set of each of the following linear equations.

(a) $7x - 5y = 3$

(b) $3x_1 - 5x_2 + 4x_3 = 7$

(c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

(d) $3v - 8w + 2x - y + 4z = 0$

Solution:

(a) $7x - 5y = 3 \implies 7x = 3 + 5y \implies x = \frac{3+5y}{7}$
So our solution set becomes $\{(x, y)\} = \{(\frac{3+5y}{7}, y) : y \in \mathbb{R}\}$

(b) $3x_1 - 5x_2 + 4x_3 = 7 \implies 3x_1 = 7 + 5x_2 - 4x_3 \implies x_1 = \frac{7+5x_2-4x_3}{3}$
So our solution set becomes $\{(x_1, x_2, x_3)\} = \{(\frac{7+5x_2-4x_3}{3}, x_2, x_3) : x_2, x_3 \in \mathbb{R}\}$

(c) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1 \implies x_2 = \frac{1+8x_1+5x_3-6x_4}{2}$
Our solution set becomes $\{(x_1, x_2, x_3, x_4)\} = \{(x_1, \frac{1+8x_1+5x_3-6x_4}{2}, x_3, x_4) : x_1, x_3, x_4 \in \mathbb{R}\}$

(d) $3v - 8w + 2x - y + 4z = 0 \implies y = 3v - 8w + 2x + 4z$
Our solution set becomes $\{(v, w, x, y, z)\} = \{(v, w, x, 3v - 8w + 2x + 4z) : v, w, x, z \in \mathbb{R}\}$

4. Find the augmented matrix for each of the following systems of linear equations.

(a)
$$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 \phantom{{}-x_2} + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 \phantom{{}+x_3} - x_4 + x_5 &= 1 \\ 3x_2 + x_3 \phantom{{}+7x_4} - x_5 &= 2 \\ x_3 + 7x_4 \phantom{{}-x_5} &= 1 \end{aligned}$$

(d)
$$\begin{aligned} x_1 \phantom{{}+x_2+x_3} &= 1 \\ x_2 \phantom{{}+x_3} &= 2 \\ x_3 &= 3 \end{aligned}$$

Solution:

(a) $\begin{bmatrix} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & -1 & 1 & 1 \\ 0 & 3 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 7 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

5. Find a system of linear equations corresponding to the augmented matrix.

(c) $\begin{bmatrix} 7 & 2 & 1 & 3 & -5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$

Solution:

$7x_1 + 2x_2 + x_3 + 3x_4 = -5$
$x_1 + 2x_2 + 4x_3 \qquad = 1$

7. The curve $y = ax^2 + bx + c$ passes through the points $(x_1, y_1), (x_2, y_2),$ and (x_3, y_3). Show that the coefficients $a, b,$ and $, c$ are a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

Solution:

By substituting the points in the equation we get a system of linear equations as follows:
$ax_1^2 + bx_1 + c = y_1$
$ax_2^2 + bx_2 + c = y_2$
$ax_3^2 + bx_3 + c = y_3$
We can substitute the values as the values are a solution to the equation, as the equation passes through those points.
Then by taking out the augmented matrix from the system of linear equations, we get the augmented matrix $\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$ as required. Hence shown that the coefficients $a, b,$ and $, c$ are a solution.

8. Consider the system of equations
$x + y + 2z = a$
$x \qquad + z = b$
$2x + y + 3z = c$
Show that for this system to be consistent, the constants $a, b,$ and $, c$ must satisfy $c = a + b$

Solution: If we add the first and second equation, we get $x + x + y + 2z + z = a + b \implies 2x + y + 3z = a + b$
Comparing this equation with the third equation $2x + y + 3z = c$ we get $c = a + b$. Hence shown that for the system of equations to be consistent, the constants must satisfy $c = a+b$.

9. Show that if the linear equations $x_1 + kx_2 = c$ and $x_1 + lx_2 = d$ have the same solution set, then the equations are identical.

Solution:

For the first equation, $x_1 = c - kx_2$. Taking an arbitrary constant $x_2 = t$, our equation becomes $x_1 = c - kt, x_2 = t$ and our solution set becomes $\{(c - kt, t) : t \in \mathbb{R}\}$

For the second equation, $x_1 = d - lx_2$. Taking an arbitrary constant $x_2 = s$, our equation becomes $x_1 = d - ls$ and our solution set becomes $\{(d - ls, s) : s \in \mathbb{R}\}$

On the assumption that both equations have the same solution set, then

$$c - kt = d - ls \tag{1}$$

$$t = s \tag{2}$$

Using (2) and substituting in (1),
$\implies c - ks = d - ls$
$\implies c - d = ks - ls$
$\implies c - d = s(k - l)$ (3)

On the assumption that the solution set is equal, (3) has to hold true $\forall s : s \in \mathbb{R}$ which is only possible in the condition that $c = d$ and $k = l$ as $c = d \iff k = l$.
If $c = d$, then $0 = s(k - l)$. So either $s = 0$, or $k = l$. $\forall s : s \neq 0, k = l$.
If $k = l$, then $c - d = s(0) \implies c = d$.
$\therefore c = d \iff k = l$

Hence shown that if they have the same solution set, then the equations are equal.