Linear Algebra
Homework 3 part i

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.3: Matrices and Matrix Operations

18.
(a) Show that if A has a row of zeroes, and B is any matrix for which AB is defined, then AB also has a row of zeroes.
(b) Find a similar result involving a column of zeroes.

Solution:
(a) Let A be an $m \times n$ matrix. Then for AB to be defined, B is an $n \times p$ matrix.
We claim that the ith row of AB is a row of zeroes. Then by definition of multiplication of matrices, an entry c_{ij} in the ith row of AB should be:
$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.
Since the ith row of A is zero, we have $a_{i1} = a_{i2} = ... = a_{in} = 0$
$\implies c_{ij} = 0b_{1j} + 0b_{2k} + ... + 0b_{nj} = \sum_{k=1}^{n} 0b_{kj} = 0$.
Hence the ith row of AB is a row of zeroes.

(b) In general if $A = [a_{ij}]$ is an $n \times p$ matrix and $B = [b_{ij}]$ is an $m \times n$ matrix. We claim that the jth column of BA is a column of zeroes, then the jth column of A must be a column of zeroes.

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{np} \end{bmatrix}$$

Then the jth column of the matrix can be represented in the form of a column vector:

$$j\text{th} = \begin{bmatrix} b_{11}a_{1j} + b_{12}a_{2j} + ...b_{1n}a_{nj} \\ b_{21}a_{1j} + b_{22}a_{2j} + ...b_{2n}a_{nj} \\ \vdots \\ b_{m1}a_{1j} + b_{m2}a_{2j} + ...b_{mn}a_{nj} \end{bmatrix}$$

Let the jth row of A be a column of zero, so we have $a_{1j} = a_{2j} = ... = a_{nj} = 0$.
Therefore, our jth column of the matrix as represented by a column vector becomes

$$\begin{bmatrix} b_{11}0 + b_{12}0 + ...b_{1n}0 \\ b_{21}0 + b_{22}0 + ...b_{2n}0 \\ \vdots \\ b_{m1}0 + b_{m2}0 + ...b_{mn}0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence the jth column of the matrix BA is a column of zeroes if the jth column of A is a column of zeroes.

19. Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, then $k = 0$ or $A = 0$.

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

If $kA = 0$, then every element of kA is zero.

Then $kA = k \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = 0 \implies ka_{ij} = 0 \; [\forall i, j \geq 0 \mid i \leq m, j \leq n]$

This is only possible if and only if $k = 0$, in which case all entries of A are multiplied by 0 and we get the zero matrix as required.
Or if all entries of A are zero, that is $a_{ij} = 0 \; [\forall i, j \geq 0 \mid i \leq m, j \leq n]$
Therefore, for $kA = 0$ to be true, either $k = 0$ or $A = 0$

25. Prove: If A and B are $n \times n$ matrices, then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

Solution: By definition, the ***trace of*** A, denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. Therefore, trace is only defined for square matrices.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$

Then $\text{tr}(A) = a_{11} + a_{22} + ... + a_{nn}$ and $\text{tr}(B) = b_{11} + b_{22} + ... + b_{nn}$.
Then $\text{tr}(A) + \text{tr}(B) = a_{11} + a_{22} + ... + a_{nn} + b_{11} + b_{22} + ... + b_{nn}$.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nn} + b_{nn} \end{bmatrix}$$

Then $\text{tr}(A+B) = a_{11} + b_{11} + a_{22} + b_{22} + ... + a_{nn} + b_{nn}$ which is the same as $\text{tr}(A) + \text{tr}(B)$.
Hence proved.