Linear Algebra
Homework 1

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.3 Matrices and Matrix Operations

1.

A	B	C	D	E
(4 x 5)	(4 x 5)	(5 x 2)	(4 x 2)	(5 x 4)

Determine which of the following matrix operations are defined. For those that are defined, give the size of the resulting matrix.

(a) BA
(b) $AC + D$
(c) $AE + B$
(d) $AB + B$
(e) $E(A + B)$
(f) $E(AC)$

Solution:

(a) Not defined since B has 5 columns but A has 4 rows so columns $\neq$ rows.
(b) Is defined as AC is defined and will result in a (4 x 2) matrix which has the same order as D and can be added.
(c) Not defined. AE will result in a matrix of order (4 x 4) which is not the same as B which has order (4 x 5). So addition is not possible so not defined.
(d) Not defined as AB is not defined as A has 5 columns while B has 4 rows.
(e) Is defined and will result in a matrix of order (5 x 5) as $A + B$ will give a matrix of order (4 x 5) which when is post multiplied by E will give matrix of order (5 x 5).
(f) Is defined. AC will give a (4 x 2) matrix which when post multiplied by E will give a (5 x 2) matrix.

7. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

Use the method of Example 7 [given in the book] to find
(a) the first row of AB
(b) the third row of AB
(c) the second column of AB
(d) the first column of BA
(e) the third row of AA
(f) the third column of AA

Solution:

(a) The first row of AB can be obtained by

$$\begin{bmatrix} 3 & -2 & 7 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 3(6) - 2(0) + 7(7) & 3(-2) - 2(1) + 7(7) & 3(4) - 2(3) + 7(5) \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$$

(b) the third row of AB can be obtained by

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 0(6) + 4(0) + 9(7) & 0(-2) + 4(1) + 9(7) & 0(4) + 4(3) + 9(5) \end{bmatrix}$$

$$\begin{bmatrix} 63 & 67 & 57 \end{bmatrix}$$

(c) the second column of AB can be obtained by

$$\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}$$

(d) the first column of BA can be obtained by

$$\begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$$

(e) the third row of AA

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 24 & 56 & 97 \end{bmatrix}$$

(f) the third column of AA

$$\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

8. Let A and B be matrices from Q7. Use method of Example 9 [from the book] to
(a) express each column matrix of AB as a linear combination of the column matrices of A
(b) express each column matrix of BA as a linear combination of the column matrices of B

Solution: $AB = \begin{bmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{bmatrix}$ and $BA = \begin{bmatrix} 6 & -6 & 70 \\ 6 & 17 & 31 \\ 63 & 41 & 122 \end{bmatrix}$

(a) $\begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$

$\begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 7 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$

$\begin{bmatrix} 41 \\ 59 \\ 57 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix} = 3 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} -6 \\ 17 \\ 41 \end{bmatrix} = -2 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 70 \\ 31 \\ 122 \end{bmatrix} = 7 \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} + 9 \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

12. (a) Show that if AB and BA are both defined, then AB and BA are square matrices.
(b) Show that if A is an mxn matrix and $A(BA)$ is defined, then B is an nxm matrix.

Solution:

(a) Let A be an mxn matrix, and let B be an kxl matrix. Then for AB to be defined, the number of columns of A have to be equal to the number of rows of B so $n = k$.

Similarly, for BA to be defined, number of columns of B have to be equal to the number of rows of A so $m = l$. Then A is an mxn matrix and B is an nxm matrix.

Then AB is an mxm matrix and BA is an nxn matrix which are both square matrices. Hence shown.

(b) Let A be an mxn matrix. If $A(BA)$ is defined, then the number of columns of A has to be equal to the number of rows of BA, then BA has n number of rows.
For BA to be defined, the number of columns of B has to be equal to the number of rows

of A. Hence, B has m columns. To produce matrix BA with n number of rows, B must have n number of rows.
Hence shown that B is an nxm matrix.

13. In each part, find matrices A, x, and b that express the given system of linear equations as a single matrix equation $Ax = b$.
(a)

$$\begin{array}{llll} 2x_1 & -3x_2 & +5x_3 & = 7 \\ 9x_1 & -x_2 & +x_3 & = -1 \\ x_1 & +5x_2 & +4x_3 & = 0 \end{array}$$

(b)

$$\begin{array}{llllll} 4x_1 & & -3x_3 & +x_4 = 1 & \\ 5x_1 & +x_2 & & -8x_4 = 3 & \\ 2x_1 & -5x_2 & +9x_3 & -x_4 & = 0 \\ & 3x_2 & -x_3 & +7x_4 & = 2 \end{array}$$

Solution:
(a) $A = \begin{bmatrix} 2 & -3 & 5 \\ 9 & -1 & 1 \\ 1 & 5 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 0 & -3 & 1 \\ 5 & 1 & 0 & -8 \\ 2 & -5 & 9 & -1 \\ 0 & 3 & -1 & 7 \end{bmatrix}, b = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

17. In each part, determine whether block multiplication can be used to compute AB from the given partitions. If so, compute the product by block multiplication. [***Note*** See Exercise 15 of book.]
(a)

$$A = \left[\begin{array}{ccc|c} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right], B = \left[\begin{array}{cc|c} 2 & 1 & 4 \\ -3 & 5 & 2 \\ \hline 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

(b)

$$A = \left[\begin{array}{cccc} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right], B = \left[\begin{array}{c|c|c} 2 & 1 & 4 \\ -3 & 5 & 2 \\ 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

Solution:

(a) The matrix AB can not be computed using block multiplication as the partitioned matrix A_{11} is an 2x3 matrix whereas the partitioned B_{11} is an 2x2 matrix. So matrix multiplication is not possible as the number of columns of A_{11} are not equal to the number of rows of B_{11}.

(b) The matrix A_{11} is an 2x4 matrix, A_{21} is an 1x4 matrix, and B_{11}, B_{12}, and B_{13} are all 4x1 matrices. So matrix multiplication is possible between A_{11} and all partitions of B, and A_{21} and all partitions of B.
The resultant matrix should be a 3x3 matrix with 6 partitions as follows:

$$A = \left[\begin{array}{c|c|c} A_{11}B_{11} & A_{11}B_{12} & A_{11}B_{13} \\ \hline A_{21}B_{11} & A_{21}B_{12} & A_{21}B_{13} \end{array}\right]$$

$$A_{11}B_{11} = \begin{bmatrix} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 37 \end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 23 \\ -13 \end{bmatrix}$$

$$A_{11}B_{13} = \begin{bmatrix} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 1 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 29 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 1 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 23 \end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix} 1 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 41 \end{bmatrix}$$

Then combining our results,

$$AB = \left[\begin{array}{c|c|c} \begin{bmatrix} -1 \\ 37 \end{bmatrix} & \begin{bmatrix} 23 \\ -13 \end{bmatrix} & \begin{bmatrix} -10 \\ 8 \end{bmatrix} \\ \hline [29] & [23] & [41] \end{array}\right] = \begin{bmatrix} -1 & 23 & -10 \\ 37 & -13 & 8 \\ 29 & 23 & 41 \end{bmatrix}$$

18 (a) Show that if A has a row of zeroes and B is any matrix for which AB is defined, then AB also has a row of zeroes.

Solution:

Let $A = \begin{bmatrix} 0 & 0 & ... & 0 \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1k} \\ b_{21} & b_{22} & ... & b_{2k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & ... & b_{nk} \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 0b_{11} + 0b_{21} + ... + 0b_{n1} & 0b_{12} + 0b_{22} + ... + 0b_{n2} & ... & 0b_{1k} + 0b_{2k} + ... + 0b_{nk} \\ a_{21}b_{11} + a_22b_{21} ... + a_{2n}b_{n1} & a_{21}b_{12} + a_22b_{22} ... + a_{2n}b_{n2} & ... & a_{21}b_{1k} + a_22b_{2k} ... + a_{2n}b_{nk} \\ \vdots & \vdots & & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + ... + a_{mn}b_{n1} & a_{m1}b_{12} + a_{m2}b_{22} + ... + a_{mn}b_{n2} & & a_{m1}b_{1k} + a_{m2}b_{2k} + ... + a_{mn}b_{nk} \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & ... & 0 \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Hence shown that if A has a row of zeroes and B is any matrix for which AB is defined, then AB also has a row of zeroes. The position of the row of zeroes would not matter in A, since the row of zeroes would occur at the same position in AB.