Linear Algebra Spring 23
Exercise 9 and 10

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Chapter 2: Determinants and Matrix Properties

Ex Set 2.1: Determinants by Cofactor Expansion

Question 1 Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

(a) Find all the minors of A (b) Find all the Cofactors of A

Solution:

(a) $M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 7(4) - (-1) = 29, M_{12} = 6(4) - (3) = 21, M_{13} = 27, M_{21} = -11, M_{22} = 13, M_{23} = -5, M_{31} = -19, M_{32} = -19, M_{33} = 19$

(b) $C_{11} = i^{1+1}M_{11} = 29, C_{12} = -21, C_{13} = 27, C_{21} = 11, C_{22} = 13, C_{23} = 5, C_{31} = -19, C_{32} = 19, C_{33} = 19$

Question 8 Evaluate $det(A)$ by Cofactor expansion along a row or column of your choice.

$A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

Solution: Expanding on the first row, $|A| = (k+1)\begin{vmatrix} k-3 & 4 \\ k+1 & k \end{vmatrix} - (k-1)\begin{vmatrix} 2 & 4 \\ 5 & k \end{vmatrix} + 7\begin{vmatrix} 2 & k-3 \\ 5 & k+1 \end{vmatrix} \implies |A| = (k+1)(k^2 - 3k - 4k - 4) - (k-1)(2k - 20) + 7(2k + 1 - 5k + 15) = (k+1)(k^2 - 7k - 4) - (2k^2 - 22k + 20) + (-21k + 112) = k^3 - 6k^2 - 11k - 4 - 2k^2 + 22k - 20 - 21k + 112 = k^3 - 8k^2 - 10k + 88$

Question 22 Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible for all values of θ;then find A^{-1} using Theorem 2.1.2.

Solution:

Theorem 2.1.2: *If A is an invertible matrix, then* $A^{-1} = \dfrac{1}{det(A)} adj(A)$

We can take the determinant on the last row for our ease, since 2 coefficients are zero so will reduce to 0.

$det(A) = 0 + 0 + 1 \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1(\cos^2\theta + \sin^2\theta) = 1(1) = 1$

Since $det(A) \neq 0$, the matrix is invertible.

Then $\text{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question 24 Let $Ax = B$ be the system:

$$4x + y + z + w = 6$$

$$3x + 7y - z + w = 1$$

$$7x + 3y - 5z + 8w = -3$$

$$x + y + z + 2w = 3$$

(a) Solve by Cramer's Rule
(b) Solve by Gauss-Jordan Elimination
(c) Which method involves faster computations?

Solution:

(a) $A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 6 \\ 1 \\ -3 \\ 3 \end{pmatrix}$ and $det(A) = -424$

Then by Crammer's Rule;

$$A_1 = \begin{pmatrix} 6 & 1 & 1 & 1 \\ 1 & 7 & -1 & 1 \\ -3 & 3 & -5 & 8 \\ 3 & 1 & 1 & 2 \end{pmatrix} \implies |A_1| = -424 \implies x = \frac{|A_1|}{A} = \frac{-424}{-424} = 1$$

$$A_2 = \begin{pmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{pmatrix} \implies |A_2| = 0 \implies y = 0$$

$$A_3 = \begin{pmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{pmatrix} \implies |A_3| = -848 \implies z = \frac{-848}{-424} = 2$$

$$A_4 = \begin{pmatrix} 4 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & -3 \\ 1 & 1 & 1 & 3 \end{pmatrix} \implies |A_4| = 0 \implies w = 0$$

Then the solution becomes $x = 1, y = 0, z = 2, w = 0 \implies x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

(b) Gauss-Jordan Elimination blah blah blah lamba chora kaam for $4_\times 4$

(c) Crammer's Rule invoves fewer computations.

Question 25 Prove that if $det(A) = 1$ and all the entries in A are integers, then all the entries in A^{-1} are integers.

Solution:

If a matrix has all integer entries, then each minor is an integer and it follows that each CoFactor is also an integer. Then the determinant of the matrix A is also an integer if the matrix A has all integer entries.

Then from the theorem, it follows that $A^{-1} = \frac{1}{det(A)}\text{adj}(A)$. Since $det(A) = 1$, it follows that $A^{-1} = adj(A)$. We know that A has all integer entries, therefore the adjoint of A, that is $adj(A)$ would also have integer entries as integers are closed under multiplication, addition, and subtraction. Therefore A^{-1} also has only integer entries.

Hence proved.

Question 26 Let $Ax = B$ be a system of n linear equations in n unknowns with integer coefficients and integer constants. Prove that if $det(A) = 1$, the solution x has integer entries.

Solution: By Crammer's Rule, the solution of $Ax = b$ is:

$$x_1 = \frac{det(A_1)}{det(A)}, x_2 = \frac{det(A_2)}{det(A)}, x_3 = \frac{det(A_3)}{det(A)}$$

where $A_i(i = 1, 2, 3)$ is the matrix obtained by replacing the i^{th} column by b.

As $det(A) = 1$, therefore, $x_1 = det(A_1), x_2 = det(A_2), x_3 = det(A_3)$.

Since A and b both have only integer entries, therefore A_1, A_2, and A_3 also only have integer entries. Since A has integer entries, then $det(A)$ is an integer, therefore $det(A_1), det(A_2), det(A_3)$ are also integers. Hence the solution to x that is x_1, x_2, x_3 are also all integers.

Therefore the solution x only has integer entries.

Ex Set 2.2: Evaluating Determinants by Row Reduction

Question 12 Given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$, find:

(a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$ (b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$ (c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix}$

(d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$

Solution:

(a) There have been two row operations, that is two rows have been swapped twice. Therefore $det = -6 * -1 * -1 = -6$

(b) There have been a few row operations, R_1 has been multiplied by 3, R_2 multiplied with -1, and R_3 multiplied with 4. Therefore $det = -6 * 3 * -1 * 4 = 72$

(c) Since we have $R_1 \to R_1 + R_3$, therefore determinant remains unchanged. $det = -6$

(d) Here a few row operations have been performed. $R_1 * -3$, $R3 \to R_3 - 4R_2$ So $det = -6 * -3 = 18$

Question 13 Use row reduction to show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

Solution: We can make the following matrix an upper triangular matrix.

$R_2 = R_2 - aR_1, R_3 = R_3 - a^2R_1 \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$ $[b^2 - a^2 = (b-a)(b+a)$, then $-(b-a)(b+a) = -(b^2-a^2)]$

$R_3 = R_3 - (b+a)R_2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c^2-a^2)-(c-a)(b+a) \end{vmatrix}$

$\implies det = 1 * (b-a) * [(c-a)(c+a) - (c-a)(b+a)] \implies (b-a)(c-a)(c+a-b-a)$
$\implies (b-a)(c-a)(c-b)$

Hence shown

Question 14 Use an argument like that in the proof of Theorem 2.1.3 to show that:

(a) $det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -a_{13}a_{22}a_{31}$ (b) $det \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{14}a_{23}a_{32}a_{41}$

Solution: Theorem 2.1.3: *If A is an $n_\times n$ traingular matrix(upper, lower or diagonal), then $det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $det(A) = a_{11}a_{22}...a_{nn}$*

(a) If we expand on column 1, we get:

$det = 0 - 0 + a_{31} \begin{bmatrix} 0 & a_{13} \\ a_{22} & a_{23} \end{bmatrix} = a_{31}(-a_{13}a_{22}) = -a_{31}a_{13}a_{22}$

Hence shown.

(b) On expansion along the first column, we get:

$det = 0 - 0 + 0 - a_{41} \begin{bmatrix} 0 & 0 & a_{14} \\ 0 & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix} = -a_{41} \left(0 - 0 + a_{32} \begin{bmatrix} 0 & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \right)$

$= -a_{41}(a_{32}(-a14a_{23})) = a_{41}a_{32}a_{14}a_{23}$

Hence shown.

Question 15 Prove the following special cases of Theorem 2.2.3

(a) $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(b) $\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Solution:

Theorem 2.2.3: *Let A be an $n_\times n$ matrix.*

(a) *If B is the matrix that results when a single row or a single column of A is multiplied by a scalar k, then $det(B) = kdet(A)$.*

(b) *If B is the matrix that results when two rows or two columns of A are interchanged, then $det(B) = -det(A)$*

(c) *If B is the matrix that resulst when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $det(B) = det(A)$*

Solution:

$det(A) = \sum \pm a_{1j_1}a_{2j_2}a_{3j_3}$

(a) Then $det(A) = \sum \pm a_{2j_1}a_{1j_2}a_{3j_3} = \sum \pm a_{1j_1}a_{2j_1}a_{3j_3} = -det(A)$

Hence shown

(b) Then $det(A) = \sum \pm (a_{1j_1} + ka_{2j_2})a_{2j_1}a_{3j_3} = \sum \pm a_{1j_1}a_{2j_2}a_{3j_3} = det(A)$

Hence shown