Linear Algebra Spring 23
Assignment 1 — Lecture 3

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Chapter 1 : Linear Equations and Matrices

Q1. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$

(a) Find A^2 and A^3

(b) Show that $A^3 = A^2 + A - 5I$

(c) Using (b) without doing anymore multiplication, prove that:

(i) $A^4 = 2A^2 - 4A - 5I$

(ii) $A^{-1} = \frac{1}{5}(I + A - A^2)$

Solution:

(a) $A^2 = AA = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+1 & 0+0+-2 & 0+0+0 \\ 0+2+0 & 0+1+0 & 2+0+0 \\ 0-4+0 & 0-2+0 & 1+0+0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix}$

$A^3 = AAA = A^2A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0-4+0 & 0-2+0 & 1+0+0 \\ 0+2+2 & 0+1-4 & 2+0+0 \\ 0-4+1 & 0-2-2 & -4+0+0 \end{bmatrix}$

$A^3 = \begin{bmatrix} -4 & -2 & 1 \\ 4 & -3 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

(b) $A^3 = A^2 + A - 5I$

Then $A^3 = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 2 \\ -4 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 2 & 2 \\ -3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\implies A^3 = \begin{bmatrix} -4 & -2 & 0 \\ 4 & -3 & 2 \\ -3 & -4 & -4 \end{bmatrix}$ which is the same as the result we found in part (a). Hence shown.

(c) (i) $A^4 = 2A^2 - 4A - 5I$.
We know that $A^3 = A^2 + A - 5I$. Then $A^4 = A^3A = A^2A + AA - 5IA$
$\implies A^4 = A^3 + A^2 - 5A$
$\implies A^4 = A^2 + A - 5I + A^2 + -5A$
$\implies A^4 = 2A^2 - 4A - 5I$ which is the same as the question. Hence shown.

(ii) $A^{-1} = \frac{1}{5}(I + A - A^2)$
We know that $A^3 = A^2 + A - 5I$. Then $A^3A^{-1} = A^2A^{-1} + AA^{-1} - 5IA^{-2}$
$\implies A^2 = A + I - 5A^{-1}$
$\implies 5A^{-1} = A + I - A^2$
$\implies A^{-1} = \frac{1}{5}(I + A - A^2)$ Hence shown!

Q2. Show that the most general matrix that commutes with $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is of the form $\begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$

Solution: Let the matrix be A that commutes with P. Then $PA = AP$.
Let $A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix}$.
Then $PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} e & f & g \\ h & i & j \\ 0 & 0 & 0 \end{bmatrix}$
and $AP = \begin{bmatrix} a & b & c \\ e & f & g \\ h & i & j \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$
For P to commute with A, $PA = AP$
$\implies \begin{bmatrix} e & f & g \\ h & i & j \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$
By comparison, $e = 0$, $f = a$, $g = b$, $h = 0$, $i = e = 0$, $j = f = a$ where a has some value, b has some value, c has some value [we consider c since c can have any value, even though c

does not appear in either PA or AP].
Then by plugging in the values in the matrix A,

$$A = \begin{bmatrix} a & b & c \\ e=0 & f=a & g=b \\ h=0 & i=0 & j=a \end{bmatrix} \implies A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} \text{ hence shown!}$$

Q3.

(a) If A is be a square matrix then $A + A^T$ is symmetric and $A - A^T$ is skew symmetric.

(b) If A is $m \times n$ matrix, then prove that AA^T and A^TA are both symmetric [See Q7.]

(c) If $A^2 = A$, A^{-1} exists, then $A = I$.

(d) If A is invertible, then prove that $(A^{-1})^T = (A^T)^{-1}$

Solution:

(a) By the definition, a matrix A is symmetric if $A = A^T$.
Following the definition, if A is a square matrix, then $A + A^T$ is symmetric.
$A + A^T = (A + A^T)$ [By the definition]
$A + A^T = A^T + (A^T)^T$ [$\because (A^T)^T = A$]
$A + A^T = A^T + A \implies A + A^T = A + A^T$. Hence proved is symmetric.
By the definition, a matrix A is skew symmetric if $A = -A^T$
Following the definition, if A is a square matrix, then $A - A^T$ is skew symmetric.
$A - A^T = -(A - A^T)^T$ [By the definition]
$A - A^T = -A^T - (-A^T)^T$ [$\because (A^T)^T = A$]
$A - A^T = -A^T + A \implies A - A^T = A - A^T$. Hence proved is skew-symmetric.

Q4. Question 2 from Exercise 1.3

Solution: Already done in previous homework

Q5. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Show that if the matrix $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ satisfies the equation $AX = XB$, then X is a scalar multiple of $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution: $AX = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ x+2z & y+2t \end{bmatrix}$
$XB = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2x-y & -x+2y \\ 2z-t & -z+2t \end{bmatrix}$
$AX = XB \implies \begin{bmatrix} x & y \\ x+2z & y+2t \end{bmatrix} = \begin{bmatrix} 2x-y & -x+2y \\ 2z-t & -z+2t \end{bmatrix}$
By comparison:
1 $x = 2x - y \implies x = y$
2 $y = -x + 2y \implies x = y$
3 $x + 2z = 2z - t \implies x = -t = y$
4 $y + 2t = -z + 2t \implies y = -z = x$
Then $X = \begin{bmatrix} x & x \\ -x & -x \end{bmatrix} = x \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \forall x \mid x \in \mathbb{R}$ Hence shown.

Q6. If A is a square matrix of order 3 such that $A^T = -A$, then prove that the diagonal entries of $A = 0$.

Solution: Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.
Then $-A^T = \begin{bmatrix} -a & -d & -g \\ -b & -e & -h \\ -c & -f & -i \end{bmatrix}$.
Then $A^T = -A \implies \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$.
By comparison, $a = -a$, $d = -b$, $g = -c$, $b = -d$, $e = -e$, $h = -f$, $c = -g$, $f = -h$, and $i = -i$. Then $a = 0$, $e = 0$, and $i = 0$ as that can only be the case when both matrices are equal, therefore proved.
Then $A = \begin{bmatrix} 0 & b & c \\ d & 0f & \\ g & h & 0 \end{bmatrix}$

Q7. If A, B are matrices such that AB is defined then $(AB)^T = B^T A^T$.
Check this result for 2×2 matrices by taking general entries. [Note: The transpose of a product of any number of matrices is equal to the product of their transposes in the reverse order. i.e $(A_1 A_2 ... A_n)^T = A_n^T ... A_2^T A_1^T$]

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.
Then $AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.
Then $(AB)^T = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.
$B^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$ and $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$.
Then $B^T A^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} & b_{11}a_{21} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{12} & b_{12}a_{21} + b_{22}a_{22} \end{bmatrix}$
$\implies B^T A^T = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.
Hence shown $(AB)^T = B^T A^T$ for general 2×2 matrices.