Linear Algebra
Homework 2 part ii

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Chapter 1 : Linear Equations and Matrices

Ex Set 1.5 Elementary Matrices and a Method for finding A^{-1}

2. Find a row operation that will restore the given matrix to an identity matrix

(a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:
(a) $R_3 \longrightarrow R_3 + 3R_1$
(b) $R_3 \longrightarrow \frac{1}{3}R_3$
(c) $R_3 \longleftrightarrow R_1$
(d) $R_1 \longrightarrow R_1 + \frac{1}{7}R_3$

3. Consider the matrices:

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find the elementary matrices $E_1, E_2, E_3,$ and E_4 such that
(a) $E_1A = B$ (b) $E_2B = A$ (c) $E_3A = C$ (d) $E_4C = A$

Solution:
(a) A can be converted to B with the following row operation: $R_1 \longleftrightarrow R_3$.
$\implies E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) B can be converted to A with the following row operation: $R_1 \longleftrightarrow R_3$.

$$\implies E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) A can be converted to C with the following row operation: $R_3 \longrightarrow R_3 - 2R_1$.

$$\implies E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(d) C can be converted to A with the following row operation: $R_3 \longrightarrow R_3 + 2R_1$.

$$\implies E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

9. Find the inverse of the following 4×4 matrices, where k_1, k_2, k_3, k_4 and k are all nonzero

(a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

Solution:

(a) Using row operations, we can obtain the inverse by:

$$\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$$

The following operations give us the inverse:

$R_1 \to \frac{1}{k_1} R_1 \qquad R_2 \to \frac{1}{k_2} R_2 \qquad R_3 \to \frac{1}{k_3} \qquad R_4 \to \frac{1}{k_4} R_4$

$$\implies \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$

Then our inverse is $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$

(b) Using row operations, we can obtain the inverse by:

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

First we swap the rows: $R_1 \longleftrightarrow R_4$ and $R_2 \longleftrightarrow R_3$.

$$\Longrightarrow \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right]$$

Next we can perform the following row operations, $R_1 \to \frac{1}{k_4}R_1,\ R_2 \to \frac{1}{k_3}R_2,\ R_3 \to \frac{1}{k_2}R_3,\ R_4 \to \frac{1}{k_1}R_4$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 0 & 0 & 0 \end{array}\right]$$

Then our inverse is $\begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}$

10. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

(a) Find the elementary matrices E_1 and E_2 such that $E_2E_1A = I$

(b) Write A^{-1} as a product of two elementary matrices.

(c) Write A as a product of two elementary matrices.

Solution:

(a) To reduce A to identity, we can observe the following row operations: $R_2 \to R_2 + 5R_1$ followed by $R_2 \to \frac{1}{2}R_2$. Then our elementary matrices in order become $E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ as first A will get premultiplied by E_1 and then followed by premultiplication with E_2.

(b) Since $A^{-1}A = I$ holds true, and A is reduced to an identity matrix following the two row operations, the matrix A^{-1} can be defined as $A^{-1} = E_2E_1$.

(c) Similarly, $E_2E_1A = I \implies E_1^{-1}E_2^{-1}E_2E_1A = E_1^{-1}E_2^{-1}I \implies A = E_1^{-1}E_2^{-1}$

16. Show that $A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$ is not invertible for any values of the entries.

[*Hint: The row containing b must become the first row after ERO have been carried out and the one with g must be the last row. Why? this caused problems when trying to reduce the middle column to only one nonzero entry.*]

Solution: It is sufficient to show that A can have a row of zeroes through elementary row operations to show that A is not invertible.
We can perform the following row operations: $R_1 \to \frac{d}{a}R_1$ and $R_5 \to \frac{e}{h}R_5$.

$$\implies A = \begin{bmatrix} 0 & d & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & e & 0 \end{bmatrix}$$

Then by the following row operations, $R_3 \to R_3 - R_1$ and $R_3 \to R_3 - R_5$ we get

$$A = \begin{bmatrix} 0 & d & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & e & 0 \end{bmatrix}$$

Therefore, we are left with a row of zeroes which shows that A is not invertible for any values of the entries as we get a row of zeroes through elementary row operations.