Linear Algebra Spring 23
Homework 5

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Chapter 2: Determinants and Matrix Properties

Ex Set 2.3: Properties of the Determinant Function

Question 3 By inspection, explain why $\det(A) = 0$

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

Solution: By inspection, the following row operation can be done

$$R_2 \to R_2 + R_1 \implies A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 1 & 10 & 6 & 5 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

We can clearly see that there are two identical rows:

$$R_2 =\to R_2 - R_3 \implies A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

By using EROS, we get a row of zeroes, therefore, $\det(A) = 0$.

Question 4 Use Theorem 2.3.3 to determine which of the following matrices are invertible.

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} \sqrt{2} & -\sqrt{7} & 0 \\ 3\sqrt{2} & -3\sqrt{7} & 0 \\ 5 & -9 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 0 & 1 \\ 5 & 0 & 6 \\ 8 & 0 & 3 \end{bmatrix}$

Solution:

Theorem 2.3.3: *A square matrix is invertible if and only if* $det(A) \neq 0$

Further, suppose each matrix to be as A

(a) $\det(A) = 1\begin{bmatrix} -1 & 4 \\ 9 & -1 \end{bmatrix} - 0 - 1\begin{bmatrix} 9 & -1 \\ 8 & 9 \end{bmatrix} = 1(1-36) + 0 - 1(81+8) \neq 0$

Therefore is invertible.

(b) $\det(A) = 4\begin{bmatrix} 1 & -4 \\ 1 & 6 \end{bmatrix} + -2\begin{bmatrix} -2 & -4 \\ 3 & 6 \end{bmatrix} + 8\begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix} = 4(6+4) - 2(-12+12) + 8(-2-3) = 40 - 40 = 0 \therefore det(A) = 0$

Also, it can be clearly seen that R_1 is a scalar multiple of R_2. Therefore this matrix is not invertible.

(c) The last column is a column of zeroes, therefore the determinant is zero. Hence this matrix is not invertible.

(d) Column of zeroes, determinant is zero, therefore not invertible.

Question 5 Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Assuming that $\det(A) = -7$, find:

(a) $\det(3A)$ (b) $\det(A^{-1})$ (c) $\det(2A^{-1})$ (d) $\det((2A)^{-1})$ (e) $\det\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}$

Solution: By a basic property, $\det(kA) = k^n \det(A)$

(a) $det(3A) = 3^3 det(A) = 27 * -7 = -189$

(b) $det(A^{-1}) = \frac{1}{det(A)} = \frac{1}{-7}$

(c) $det(2A^{-1}) = 2^3 det(A^{-1}) = \frac{8}{det(A)} = -\frac{8}{7}$

(d) $det((2A)^{-1}) = \dfrac{1}{det(2A)} = \dfrac{1}{2^3 det(A)} = -\dfrac{1}{8*7} = -\dfrac{1}{56}$

(e) From inspection, if we take the tranpose of the matrix, we get $\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$. Then by swapping R_2 with R_3, we get $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ which is the same as the matrix given in the question. Therefore $det = -(-7)$ as we took the tranpose. So $det = 7$.

Question 6 Without directly evaluating, show that $x = 0$ and $x = 2$ satisfy $\begin{vmatrix} x^2 & x & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 0$

Solution:
(1) $x = 0$: $\begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} \implies R_3 \to R_3 + \frac{5}{2}R_1 \begin{vmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$
Since we have a row of zeroes, the determinant is also 0. Hence $x = 0$ satisfies the matrix.
(2) $x = 2$ $\begin{vmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix} \implies R_1 \to \frac{1}{2}R_1 \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{vmatrix}$
Since we have two identical rows, the determinant of the matrix is 0. Hence shown that $x = 2$ satisfies the matrix.

Question 7 Without directly evaluating show that $det \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$

Solution:
By inspection, $R_1 \to R_1 + R_2 \implies \begin{bmatrix} a+b+c & a+b+c & a+b+c \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$
Then $R_1 \to \frac{1}{a+b+c} \implies \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$
Hence we are left with two identical matrices. Therefore the determinant is zero.

Question 8 Prove the identity without evaluating the determinants

$$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solution: Column operations have no effect on the determinant:

LHS: $C_3 \to C_3 - C_2 - C_1 \implies \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$ RHS. Hence proved.

Question 9 Prove the identity without evaluating determinants.

$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Solution: LHS: $C_1 \to C_1 + C_2 \implies \begin{vmatrix} 2a_1 & a_1-b_1 & c_1 \\ 2a_2 & a_2-b_2 & c_2 \\ 2a_3 & a_3-b_3 & c_3 \end{vmatrix}$

We can take 2 common from the first column, and then apply the column operations:

$C_2 \to C_2 - C_1 \implies 2\begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} =$ RHS

Further taking -1 common from the second column, we get $-2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Hence proved.

Question 11 Prove the identity without evaluating determinants.

$$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Solution: LHS: $C_2 \to C_2 - tC_1, c_3 \to C_3 - sC_1 \implies \begin{vmatrix} a_1 & b_1 & c_1+rb_1 \\ a_2 & b_2 & c_2+rb_2 \\ a_3 & b_3 & c_3+rb_3 \end{vmatrix}$

$C_3 \to C_3 - rC_2 \implies \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ by tranpose $\implies \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ $[det(A) = det(A^T)]$

Hence proved.

Question 13 Use Theorem 2.3.3 to show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of $\alpha, \beta,$ and γ.

Solution: By $R_1 \to R_1 + R_2$ we will get $\sin^2\alpha + \cos^2\alpha, \sin^2\beta + \cos^2\beta, \sin^2\gamma + \cos^2\gamma$ in the first row. And by basic trigonometric identity, it will reduce to 1. So the first row becomes a row of 1s, and so does the third row. Since we have two identical rows, the determinant is 0. Therefore, the matrix is not invertible for any values of α, β, γ.

Question 16 Let A and B be $n_{\times}n$ matrices. Show that if A is invertible, then $det(B) = det(A^{-1}BA)$.

Solution: If A is invertible, then $det(A) \neq 0 \therefore det(A^{-1}) = \frac{1}{det(A)}$ is defined.
$det(B) = det(A^{-1}) * det(B) * det(A)$ by the theorem.
$\implies det(B) = \frac{1}{det(A)} * det(B) * det(A)$
$\implies det(B) = det(B)$ Hence proved.

Question 21 Let A and B be $n_{\times}n$ matrices. You know from earlier work that AB is invertible if A and B are invertible. What can you say about the invertibility of AB if one or both of the factors are singular? Explain your reasoning.

Solution: If A and B are $n_{\times}n$ matrices, then $det(AB) = det(A) * det(B)$.
If either one of the matrices are not invertible, then either $det(A) = 0$ or $det(B) = 0$ or both equal to 0.
Then $det(AB) = det(A) * det(B) \implies det(AB) = 0$.
Then AB is also not invertible. So if one or both of the factors are singular, then AB is not invertible.