Linear Algebra Spring 23
Assignment 2 — Lecture 5

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Chapter 1 : Linear Equations and Matrices

Q1. (a) Under what conditions $AB = BA$ where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

(b) If A is a matrix then $A^r A^s = A^{r+s}$ $[\forall r, s \in \mathbb{Z}^+]$.

Is this result true for negative integers also? Justify your answer

(c) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$ and $c = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$. Then $AB = BC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ but $B \neq C$.

Why?

Solution:

(a) For $AB = BA$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

Therefore, by comparison:

$a_{11}b_{11} + a_{12}b_{21} = b_{11}a_{11} + b_{12}a_{21}$
$a_{11}b_{12} + a_{12}b_{22} = b_{11}a_{12} + b_{12}a_{22}$
$a_{21}b_{11} + a_{22}b_{21} = b_{21}a_{11} + b_{22}a_{21}$
$a_{21}b_{12} + a_{22}b_{22} = b_{21}a_{12} + b_{22}a_{22}$

So:

$a_{12}b_{21} = b_{12}a_{21}$
$a_{11}b_{12} + a_{12}b_{22} = b_{11}a_{12} + b_{12}a_{22}$
$a_{21}b_{11} + a_{22}b_{21} = b_{21}a_{11} + b_{22}a_{21}$
$a_{21}b_{12} = b_{21}a_{12}$

(b) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $r = 1, s = -1$

Then $A^{1-1} = A^1 A^{-1}$
$\implies A^0 = A^1 A^{-1}$
$\implies I = A^1 A^{-1}$

Then, $A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^{-1}$, however, A^{-1} does not exist as A has a row of zeroes, so determinant of A is zero. Hence we can conclude that it is not valid for negative numbers.

(c) Since A has a column of zeroes, then A^{-1} does not exist. Hence, $AB = AC \implies A^{-1}AB = A^{-1}AC \implies IB = IC \implies B = C$ does not hold as A^{-1} does not exist. Therefore, $B \neq C$.

Q2. Using the technique of forming a block matrix $[A \mid I]$ and performing EROS such that $[A \mid I] \xrightarrow{EROS} [I \mid A^{-1}]$. Find the inverse of the following where A is given by

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$

Solution: $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$

$R_2 - 2R_1, R_3 - 4R_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$ $R_2 \times -1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$

$R_2 - R_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$ $R_3 \rightleftarrows R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$

$R_1 - 2R_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$

(b) follows just like the above part

Q3. Solve the following system of equations by reducing them to Echelon form (Guassian Elimination method)
$x + y + 2z = 9$
$2x + 4y - 3z = 1$
$3x + 6y - 5z = 0$

Solution: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$

we follow the same steps as the above question to get to the Identity matrix at the left hand side of the line, then at the right hand side we get the solution which is:
$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$

Q4. Solve the following system by Gauss-Jordan elimination (reduced row Echelon form)
$2x_1 + 2x_2 + 2x_3 = 0$
$-2x_1 + 5x_2 + 2x_3 = 1$
$8x_1 + x_2 + 4x_3 = -1$

Solution: Same method as the above part

Q5. Reduce $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 7 \\ 3 & 4 & 5 \end{bmatrix}$ to reduced row echelon form without introducing any fractions.

Solution: Same as above part

Q6. Find two different row Echelon forms of $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

Solution:

One possible solution is:

$R_2 - 2R_1 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Another solution could be:

$\frac{R_2}{2} \begin{bmatrix} 1 & 3 \\ 1 & \frac{7}{2} \end{bmatrix} \qquad R_2{-}R_1 \begin{bmatrix} 1 & 3 \\ 0 & \frac{1}{2} \end{bmatrix}$

Q7. Exercise set 1.3 Q25

Solution: Done in Homework 3 i

Q8. Exercise set 1.3 Q18 and 19

Solution: Done in Homework 3 i