# Kruskal’s Algorithm

Minimum Spanning Tree

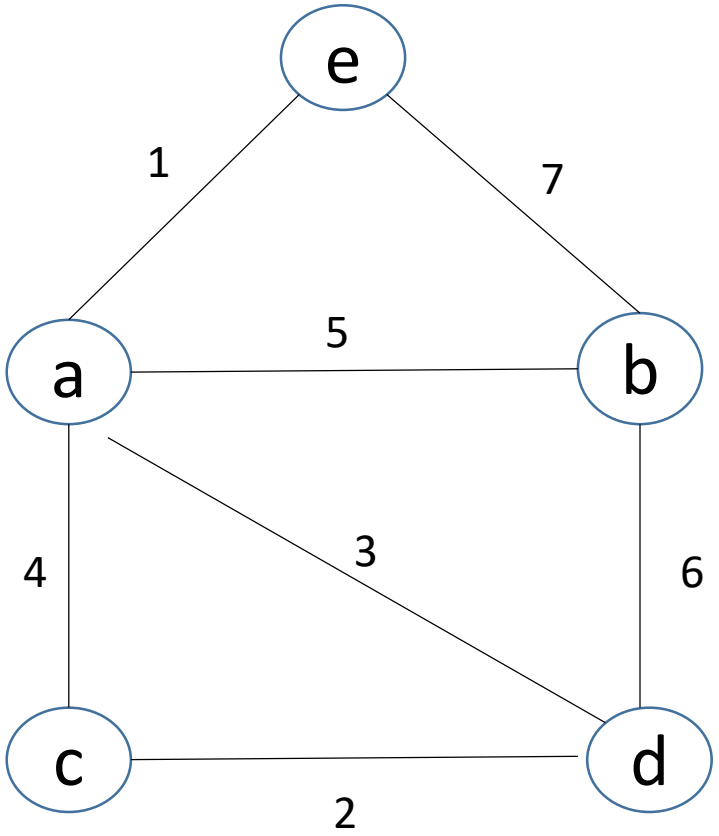
# MST Review

- Input: Undirected graph G = (V; E), edge costs $c_e$
- Output: Min-cost spanning tree (no cycles, connected)
- Assumptions: G is connected, distinct edge costs.
- Cut Property: If e is the cheapest edge crossing some cut (A; B), then e belongs to the MST.

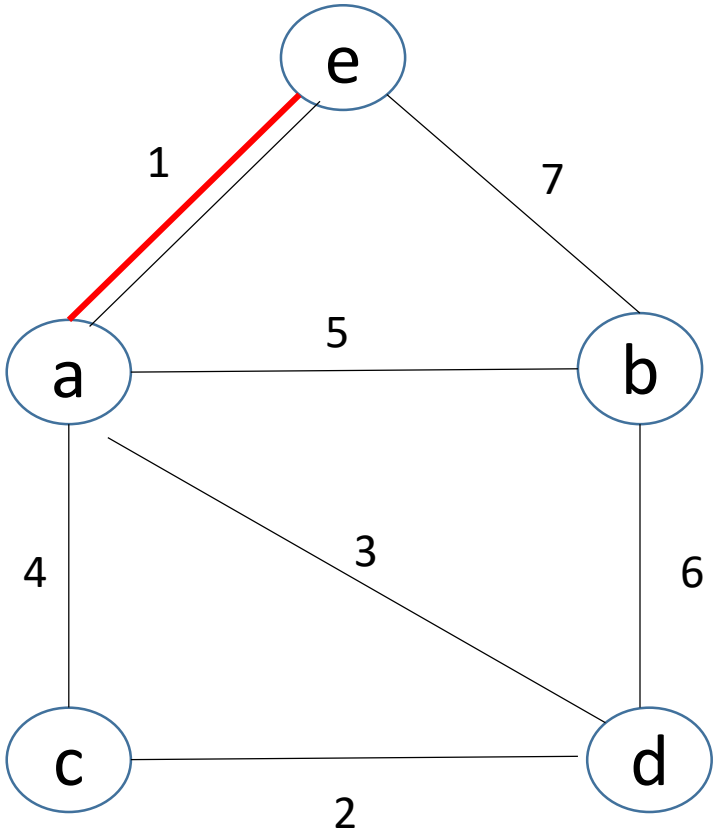
# Kruskal's MST Algorithm

- Sort edges in order of increasing cost
  [Rename edges 1; 2; : : : ;m so that c1 < c2 < : : : < cm]
- T = Ø
- For i = 1 to m
  - If T Ụ {i} has no cycles
  - Add i to T
- Return T

# Kruskal's Algorithm Dry Run



# Kruskal's Algorithm Dry Run



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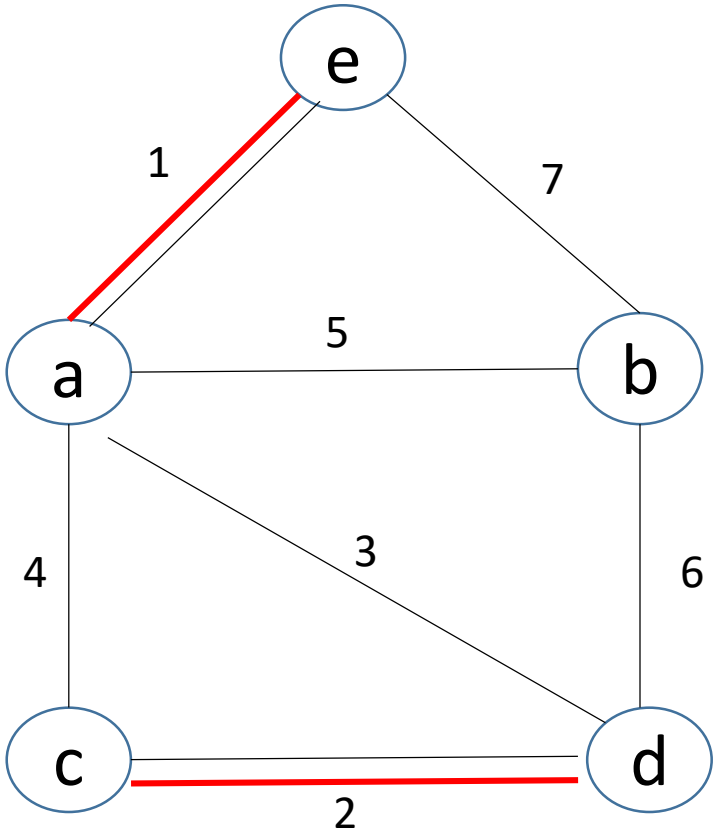
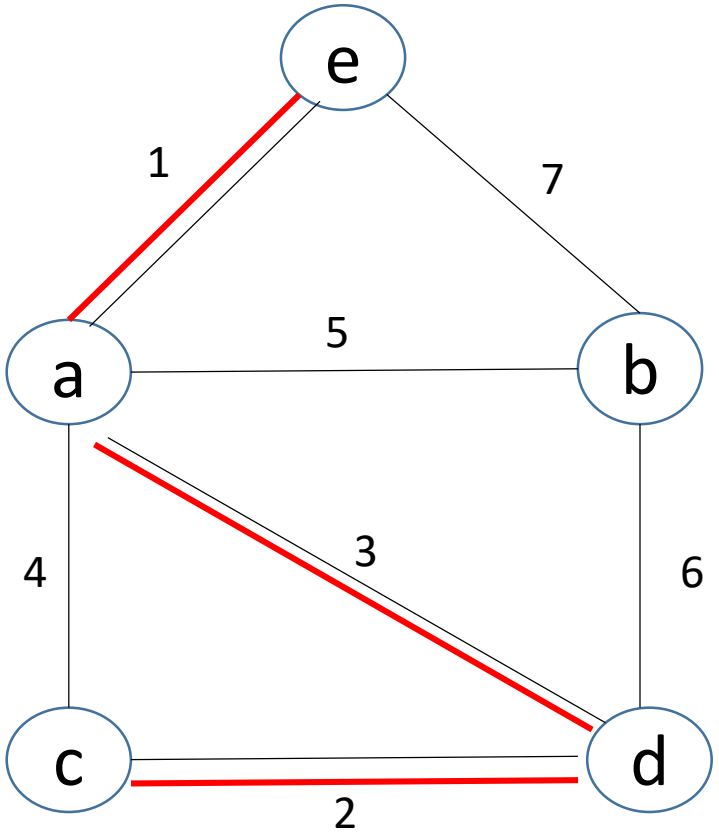


# Kruskal's Algorithm Dry Run



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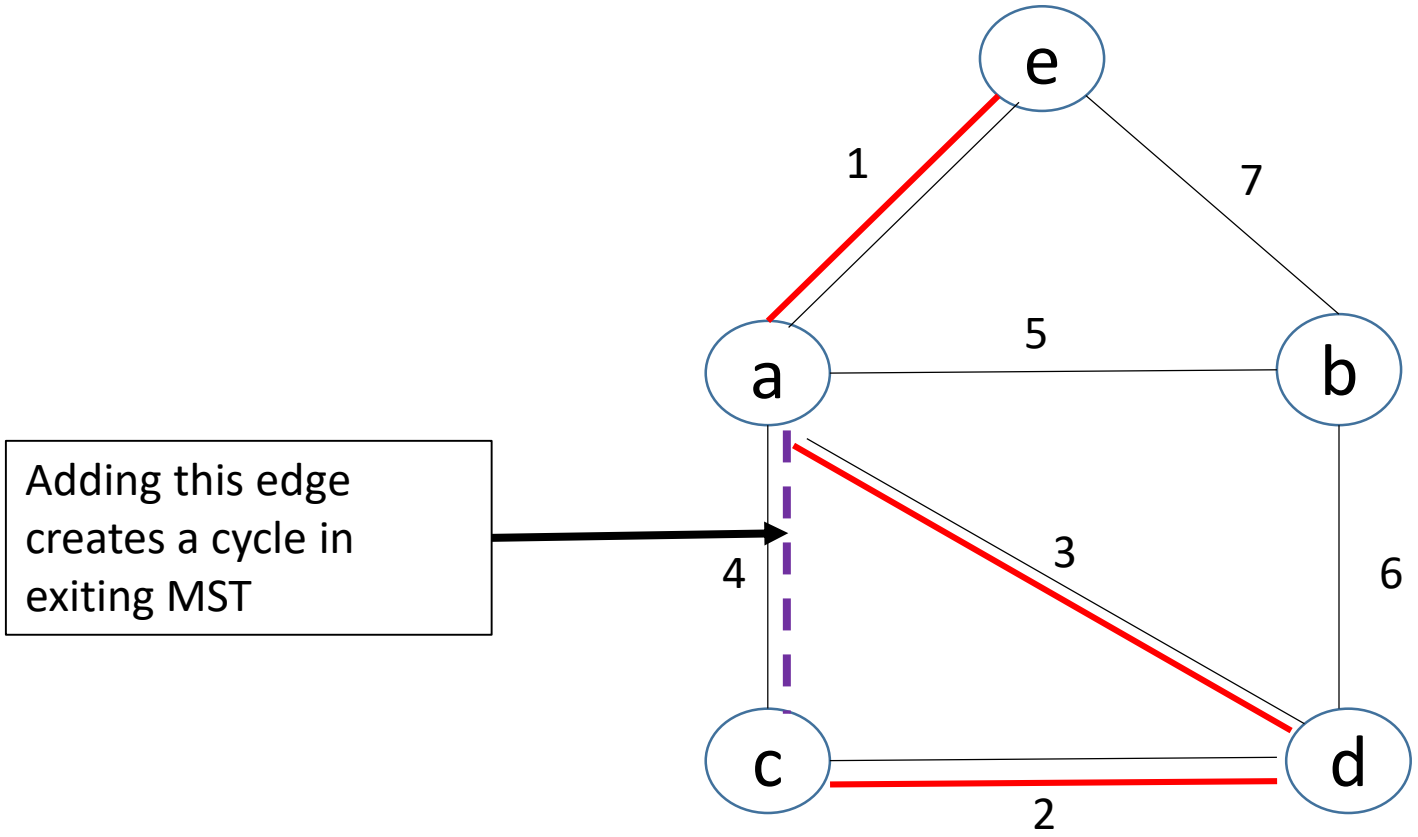
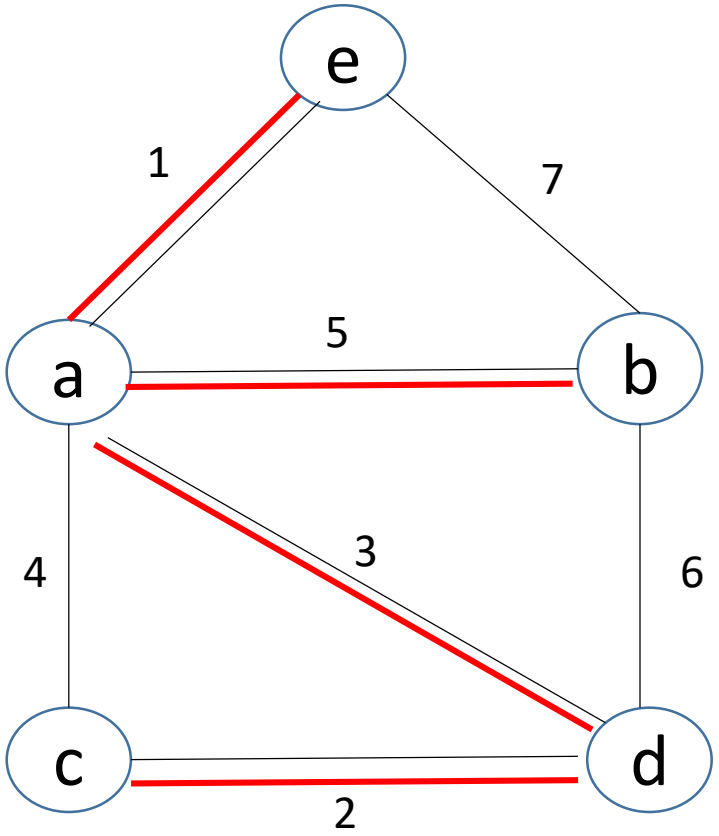


# Kruskal's Algorithm Dry Run



# How to check Cycles in MST?

# Kruskal's MST Algorithm

- Sort edges in order of increasing cost. ($O(m \log n)$, recall $m = O(n^2)$ assuming nonparallel edges)
- $T = \emptyset$
  - For $i = 1$ to $m$ ($O(m)$ iterations) → O (m )
    - If $T \cup \{i\}$ has no cycles ($O(n)$ time to check for cycle [Use BFS or DFS in the graph $(V, T)$ which contains $\leq n - 1$ edges]) → O (n )
      - Add $i$ to $T$
- Return $T$

Running time of straightforward implementation: ($m = \#$ of edges, $n = \#$ of vertices) $O(m \log n) + O(mn) = O(mn)$

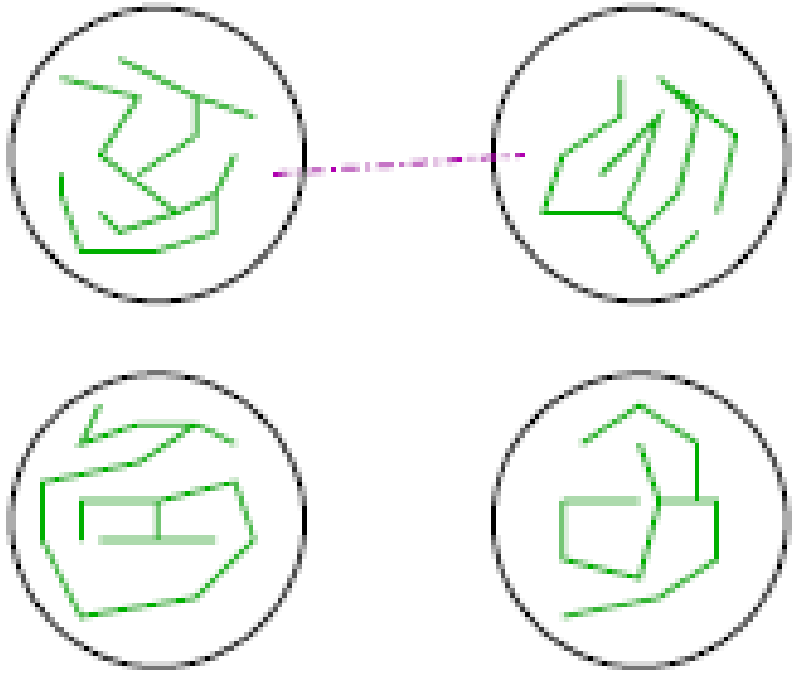
Plan: Data structure for $O(1)$-time cycle checks $\Rightarrow$ $O(m \log n)$ time.

# Kruskal's Algorithm

- Can we improve the time to check cycles?
- Can we use some data structure to speed up Kruskal's algorithm?
- What kind of data structure can check efficiently if adding an edge creates a cycle

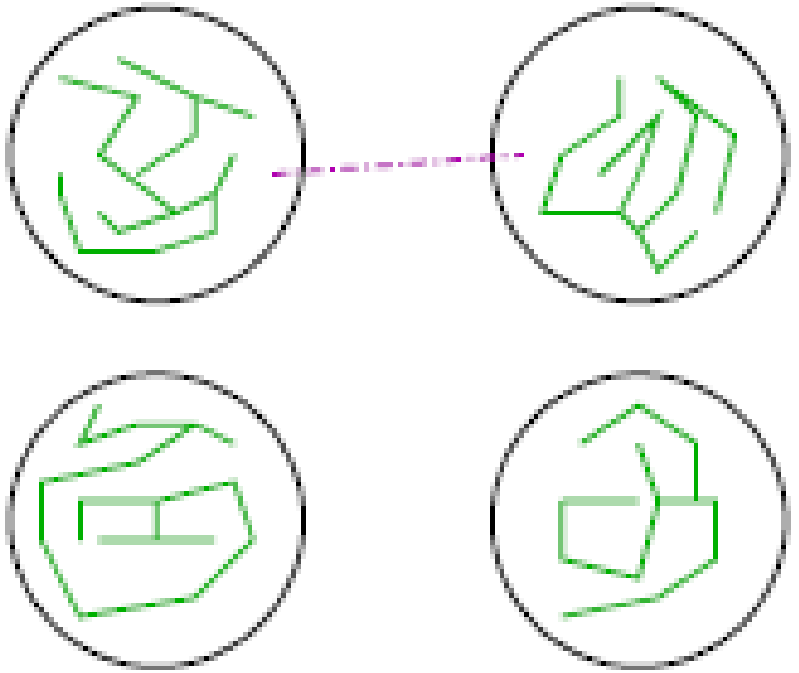
# How to Check Cycle ?

- In Kruskal's algorithm, the partial MST is in form of different connected components



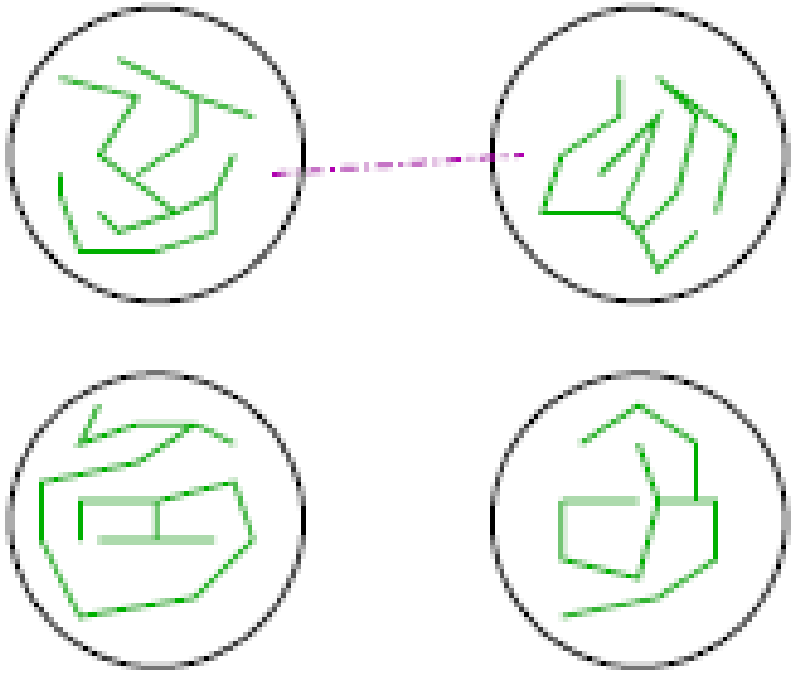
# How to Check Cycle ?

- If the edge is added between vertices of different components then it will not create a cycle.
- If the edge is added between vertices of same component then it will create a cycle.



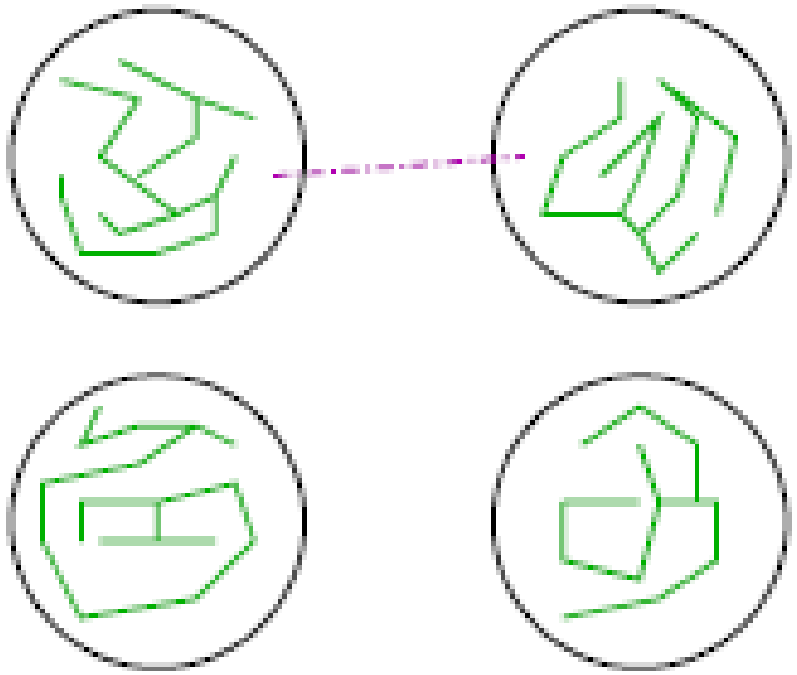
# How to Check Cycle ?

- So we need a data structure that can keep track of different components created in the MST (created so far)



# How to Check Cycle ?

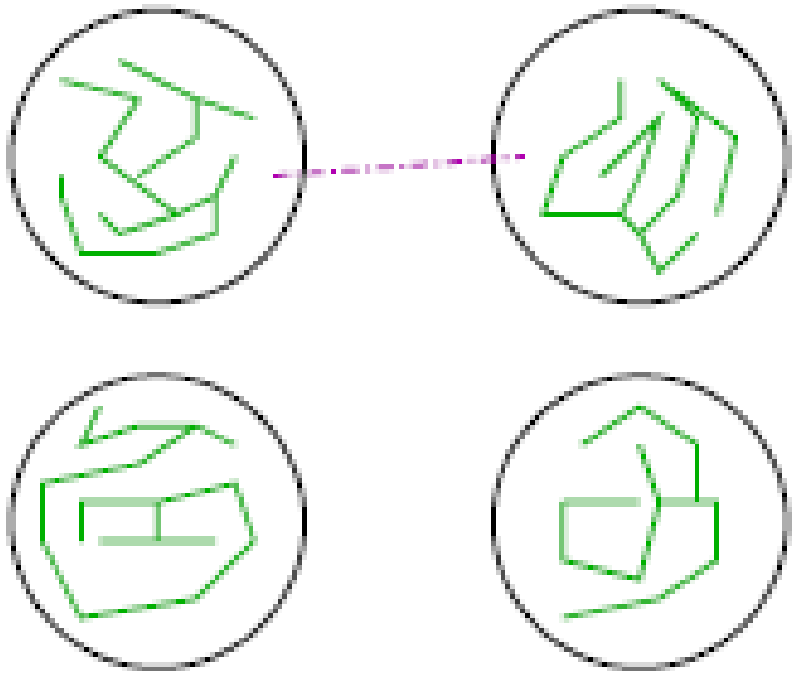
- Suppose you are given a data structure that can keep track of different components of MST in Kruskal's algorithm
- What operations must be supported by that data structure ?



# Operation of Data Structure

- Two Operations
  1. Given a vertex check its component // needed for cycle check
  2. What is second operation?

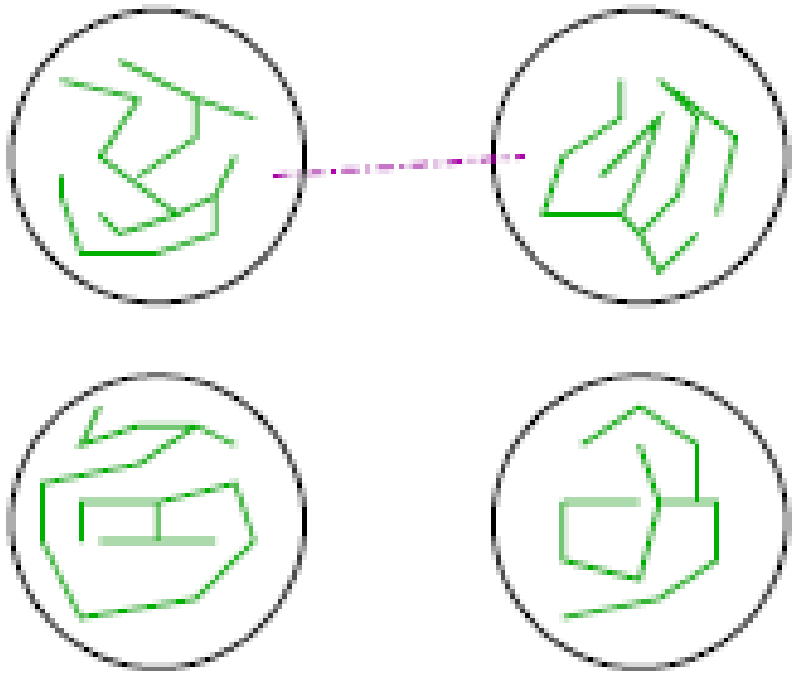
After cycle check, if an edge is selected to be added in MST, then should we make any update?



# Operation of Data Structure

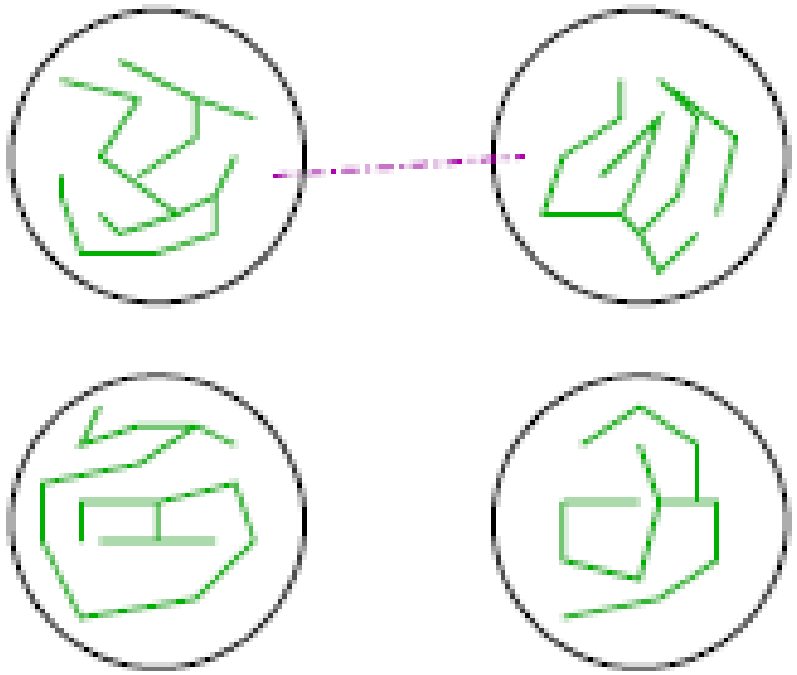
- Two Operations
  1. Given a vertex check its component
  2. What is second operation?

When an edge is added, we should merge the two components since now they become one connected component due to the new edge



# Operation of Data Structure

- Two Operations
  1. Given a vertex check its component
  2. Merge two components into one

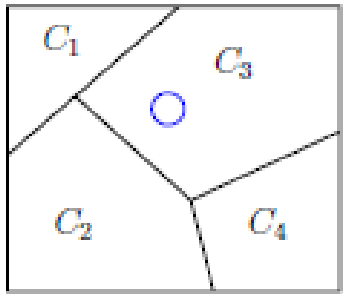


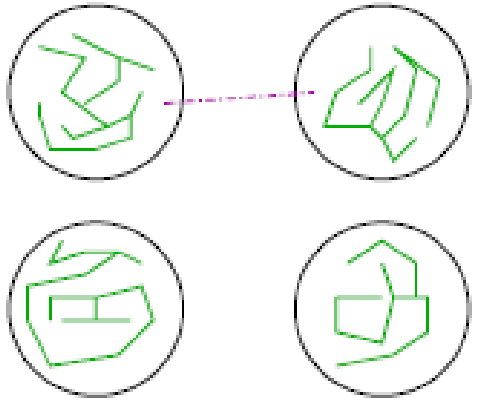
# The Union-Find Data Structure

union-find data structure: Maintain partition of a set of objects.

FIND($X$): Return name of group that $X$ belongs to.

UNION($C_i, C_j$): Fuse groups $C_i, C_j$ into a single one.



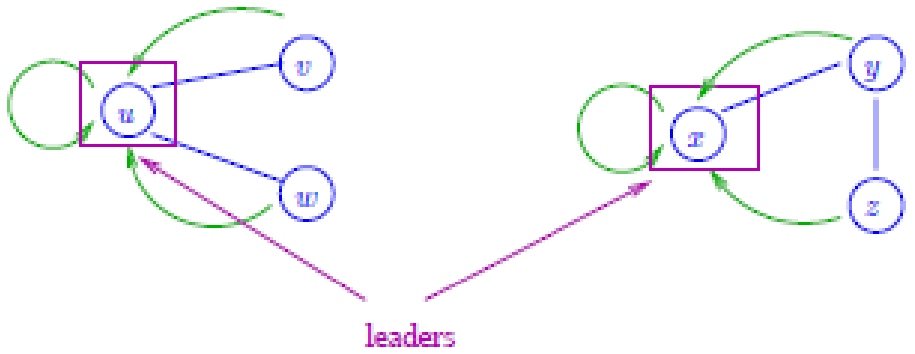
Why useful for Kruskal's algorithm: Objects = vertices

- Groups = Connected components w.r.t. chosen edges $T$.
- Adding new edge $(u, v)$ to $T \iff$ Fusing connected components of $u, v$.

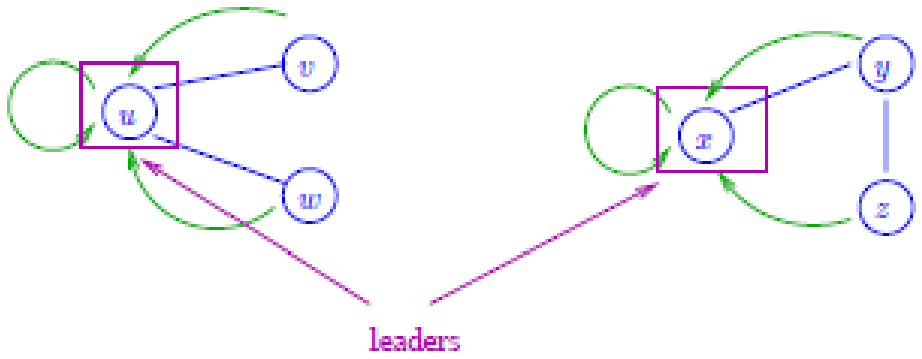
# Union Find Basics

Motivation: $O(1)$-time cycle checks in Kruskal's algorithm.

Idea #1: - Maintain one linked structure per connected component of $(V, T)$.
- Each component has an arbitrary leader vertex.



# Union Find Basics




Invariant: Each vertex points to the leader of its component ["name" of a component inherited from leader vertex]

Key point: Given edge $(u, v)$, can check if $u$ & $v$ already in same component in $O(1)$ time. [if and only if leader pointers of $u, v$ match, i.e., FIND($u$)=FIND($v$)] $\Rightarrow$ $O(1)$-time cycle checks!

# Maintaining the Invariant

Note: When new edge (u; v) added to T, connected components of u & v merge.

Question: How many leader pointer updates are needed to restore the invariant in the worst case?

a) (1)
b) (log n)
c) (n) (e.g., when merging two components with n/2 vertices each)
d) (m)

# Maintaining the Invariant

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d) (m)

# Maintaining the Invariant

Idea #2: When two components merge, have smaller one inherit the leader of the larger one. [Easy to maintain a size field in each component to facilitate this]

Question: How many leader pointer updates are now required to restore the invariant in the worst case?

a) O(1)

b) O(log n)

c) O(n) (for same reason as before, i.e., when merging two components with n/2 vertices each)

d) (m)

# Maintaining the Invariant

Idea #2: When two components merge, have smaller one inherit the leader of the larger one. [Easy to maintain a size field in each component to facilitate this]

Question: How many leader pointer updates are now required to restore the invariant in the worst case?

a) (1)

b) (log n)

c) (n) (for same reason as before, i.e., when merging two components with n/2 vertices each)

d) (m)

# Updating Leader Pointers

But: How many times does a single vertex v have its leader pointer updated over the course of Kruskal's algorithm?

a) (1)

b) (log n)

c) (n)

d) (m)

# Updating Leader Pointers

But: How many times does a single vertex v have its leader pointer updated over the course of Kruskal's algorithm?

a) (1)

b) (log n)

c) (n)

d) (m)

Reason: Every time v's leader pointer gets updated, population of its component at least doubles ) ⇒ Can only happen ≤ $\log_2$ n times.

# Kruskal's MST Algorithm

- Sort edges in order of increasing cost

  [Rename edges 1; 2; : : : ;m so that c1 < c2 < : : : < cm]
- T = Ø

- For i = 1 to m

  - If Find (u) ≠Find (v)  // edge i is between vertices u and v

    - Add i to T
    - Union (Find (u) , Find (v) )

- Return T

# Running Time of Fast Implementation

Scorecard:

O(mlog n) time for sorting

O(m) times for cycle checks [O(1) per iteration]

O(n log n) time overall for leader pointer updates

O(m log n) total (Matching Prim's algorithm)

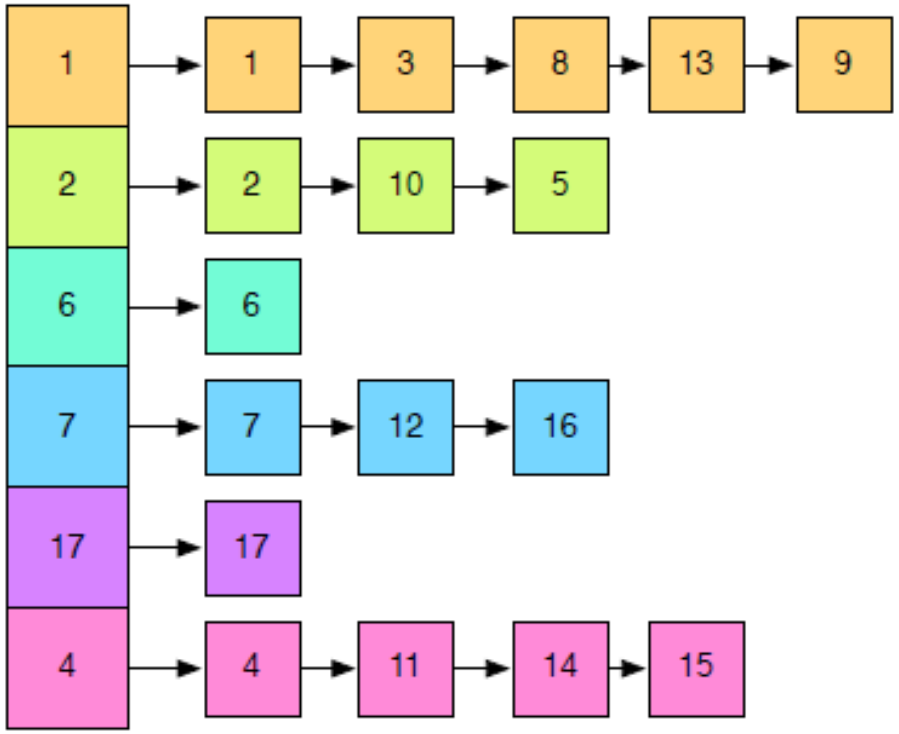
# Union-Find Abstract Data Type

The Union-Find abstract data type supports the following operations:

- UF.create(S): create the data structure containing |S| sets, each containing one item from S.
- UF.Find(i): return the “name” of the set containing item i .
- UF.union(a,b): merge the sets with names a and b into a single set.

# A Union Find Data Structure

**UF Items:**




**UF Sizes:**

| | |
|---|---|
| 1 | 5 |
| 2 | 3 |
| 6 | 1 |
| 7 | 3 |
| 17 | 1 |
| 4 | 4 |

**UF Sets Array:**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 4 | 2 | 6 | 7 | 1 | 1 | 2 | 4 | 7 | 1 | 4 | 4 | 7 | 17 |

Slide Credit: Carl Kingsford

# Implementing the union & find operations

- make union find(S) Create data structures on previous slide.

  Takes time proportional to the size of S (number of vertices).

- find(i) Return UF.sets[i].

  Takes a constant amount of time.

- union(x,y) Use the “size” array to decide which set is smaller.

  Assume x is smaller.

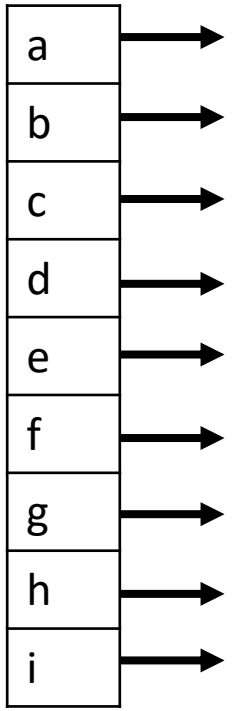
  Walk down elements i in set x (UF linked Items), setting sets[i ] = y.

  Set size[y] = size[y] + size[x].

  Takes time proportional to size[x].

# Kruskal Algorithm Dry Run Using Union Find

**UF Items**



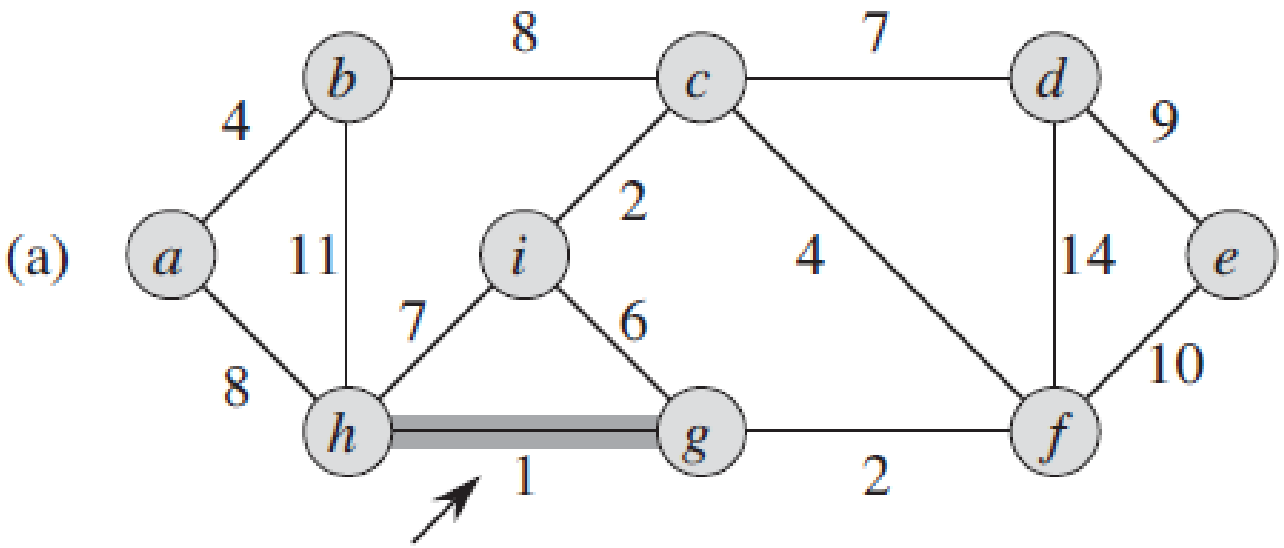

**UF Sizes**

| | |
|---|---|
| a | 1 |
| b | 1 |
| c | 1 |
| d | 1 |
| e | 1 |
| f | 1 |
| g | 1 |
| h | 1 |
| i | 1 |

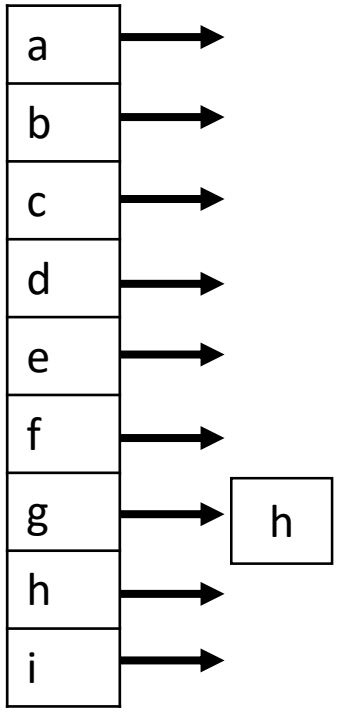
**UF Sets Array**

| a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




**UF Items**



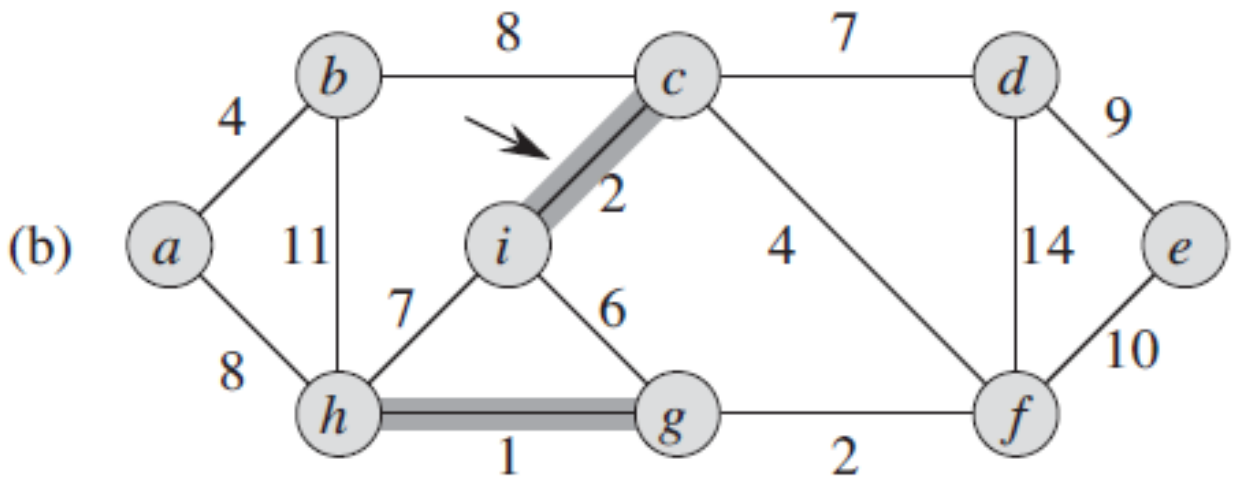

**UF Sizes**

| | |
|---|---|
| a | 1 |
| b | 1 |
| c | 1 |
| d | 1 |
| e | 1 |
| f | 1 |
| g | 2 |
| h | 0 |
| i | 1 |

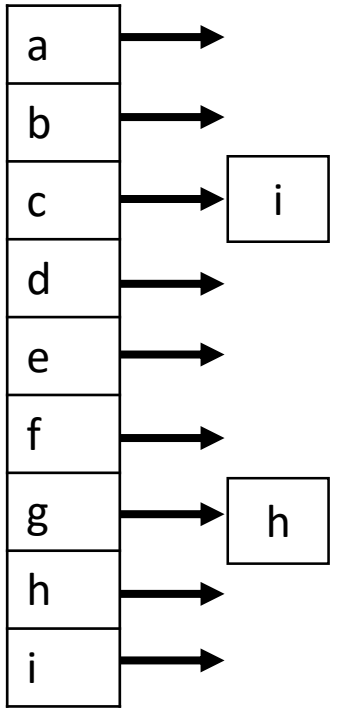
**UF Sets Array**

| a | b | c | d | e | f | g | g | i |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




**UF Items**



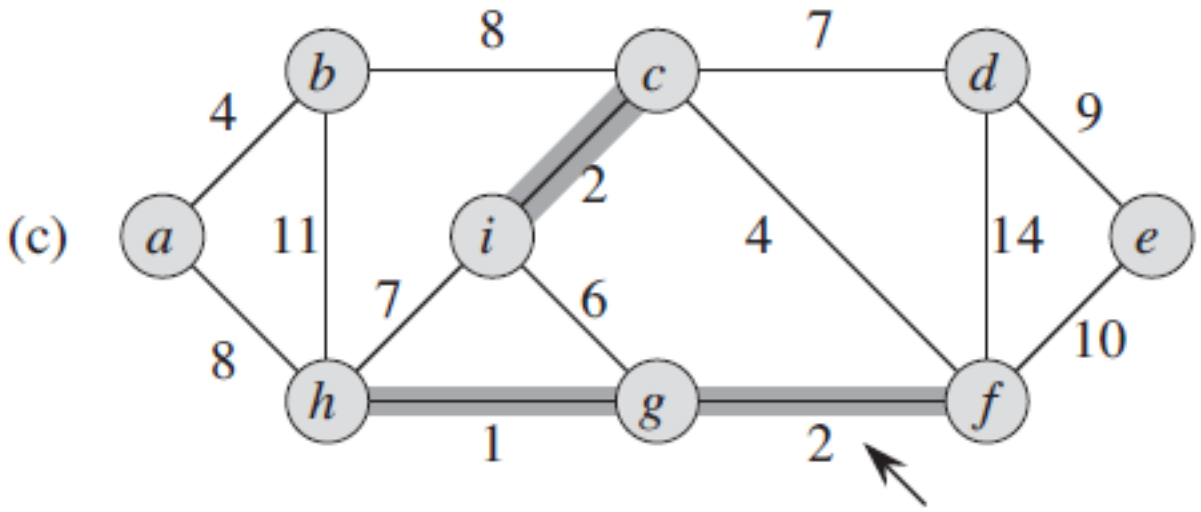

**UF Sizes**

| | |
|---|---|
| a | 1 |
| b | 1 |
| c | 2 |
| d | 1 |
| e | 1 |
| f | 1 |
| g | 2 |
| h | 0 |
| i | 0 |

**UF Sets Array**

| a | b | c | d | e | f | g | g | c |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find

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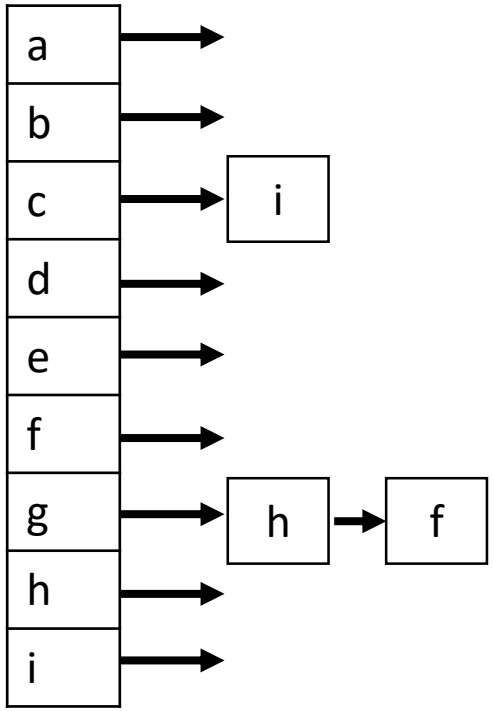


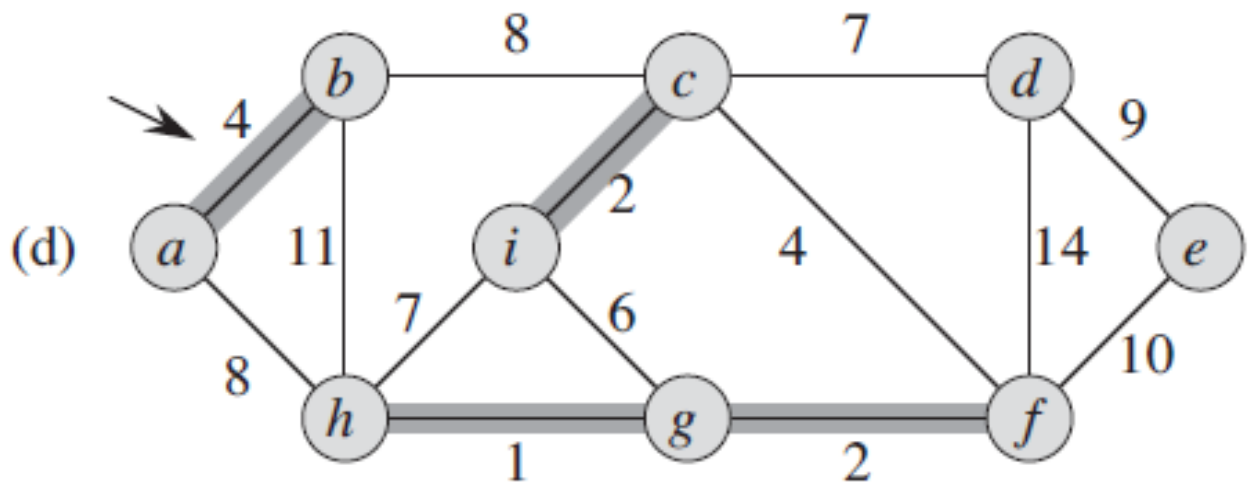

**UF Sizes**

| | |
|---|---|
| a | 1 |
| b | 1 |
| c | 2 |
| d | 1 |
| e | 1 |
| f | 0 |
| g | 3 |
| h | 0 |
| i | 0 |

**UF Sets Array**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | g | g | g | c |
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find

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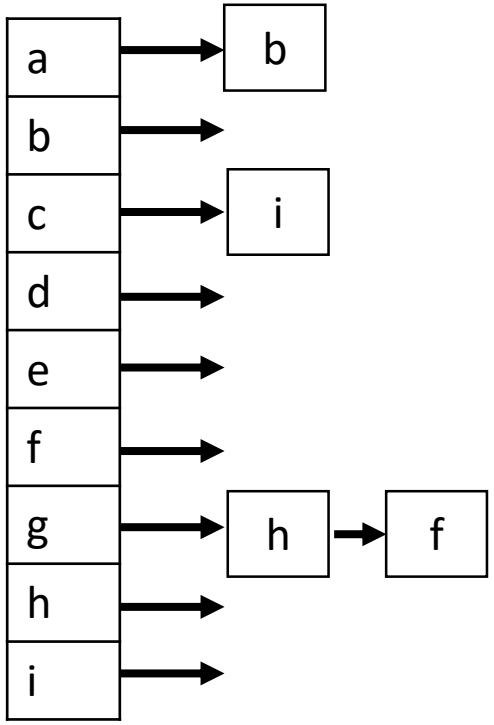


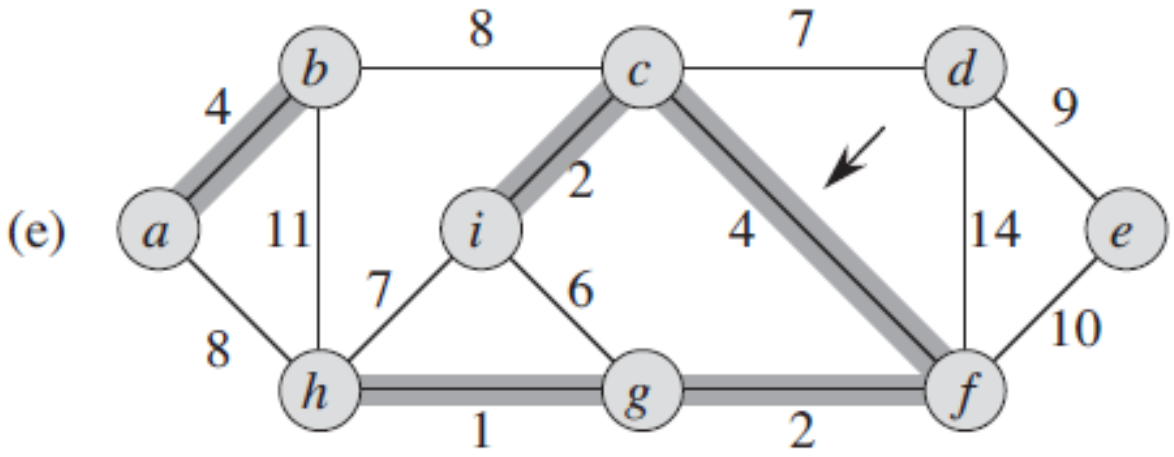

**UF Sizes**

| | |
|---|---|
| a | 2 |
| b | 0 |
| c | 2 |
| d | 1 |
| e | 1 |
| f | 0 |
| g | 3 |
| h | 0 |
| i | 0 |

**UF Sets Array**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| a | a | c | d | e | g | g | g | c |
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find

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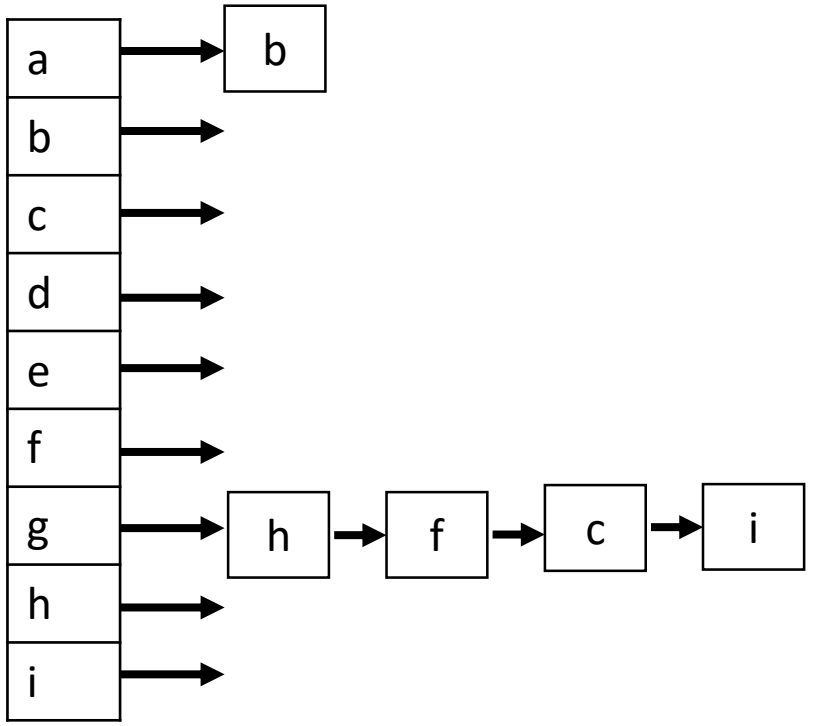


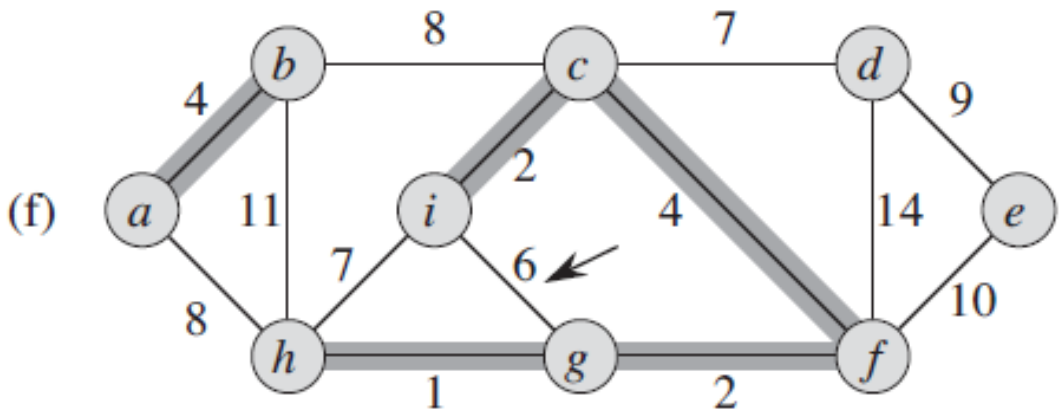

**UF Sizes**

| | |
|---|---|
| a | 2 |
| b | 0 |
| c | 0 |
| d | 1 |
| e | 1 |
| f | 0 |
| g | 5 |
| h | 0 |
| i | 0 |

**UF Sets Array**

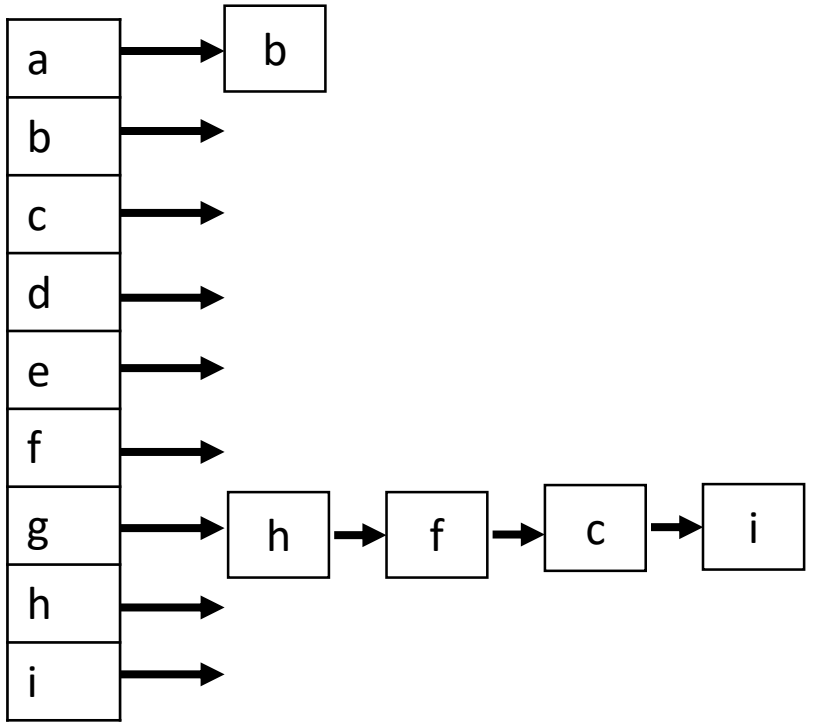
| a | a | g | d | e | g | g | g | g |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




Edge (i, g) creates cycle so it will not be added in MST

**UF Items**



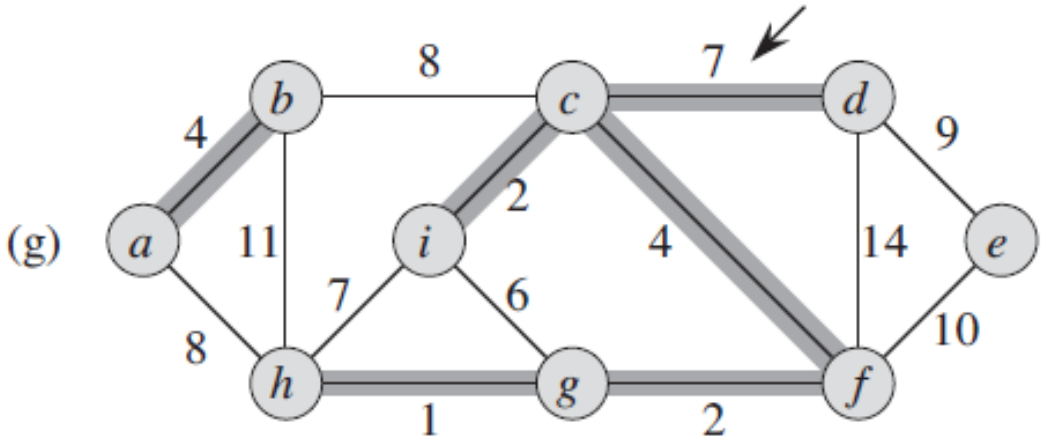

**UF Sizes**

| | |
|---|---|
| a | 2 |
| b | 0 |
| c | 0 |
| d | 1 |
| e | 1 |
| f | 0 |
| g | 5 |
| h | 0 |
| i | 0 |

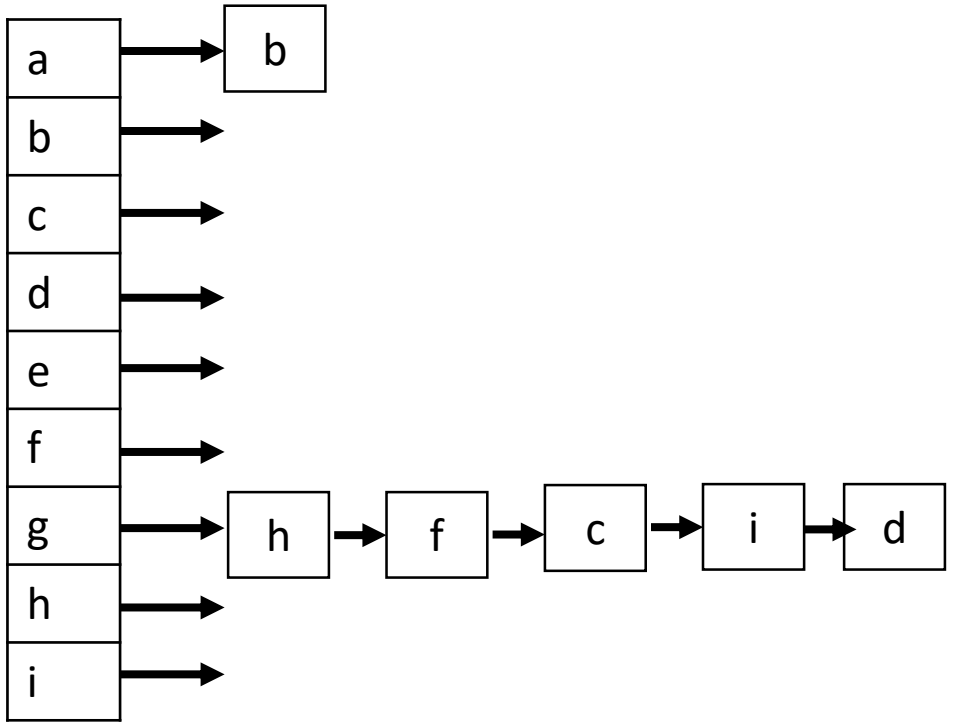
**UF Sets Array**

| a | a | g | d | e | g | g | g | g |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




**UF Items**




**UF Sizes**

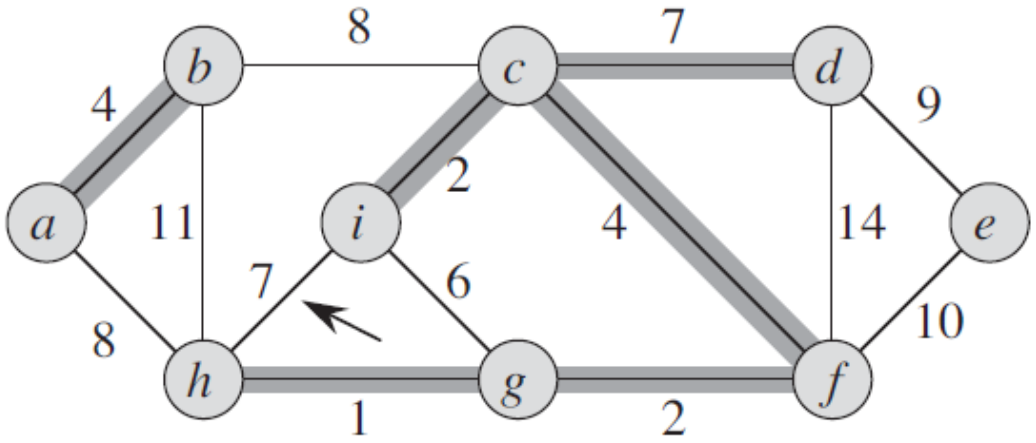
| | |
|---|---|
| a | 2 |
| b | 0 |
| c | 0 |
| d | 0 |
| e | 1 |
| f | 0 |
| g | 6 |
| h | 0 |
| i | 0 |

**UF Sets Array**

| a | a | g | g | e | g | g | g | g |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

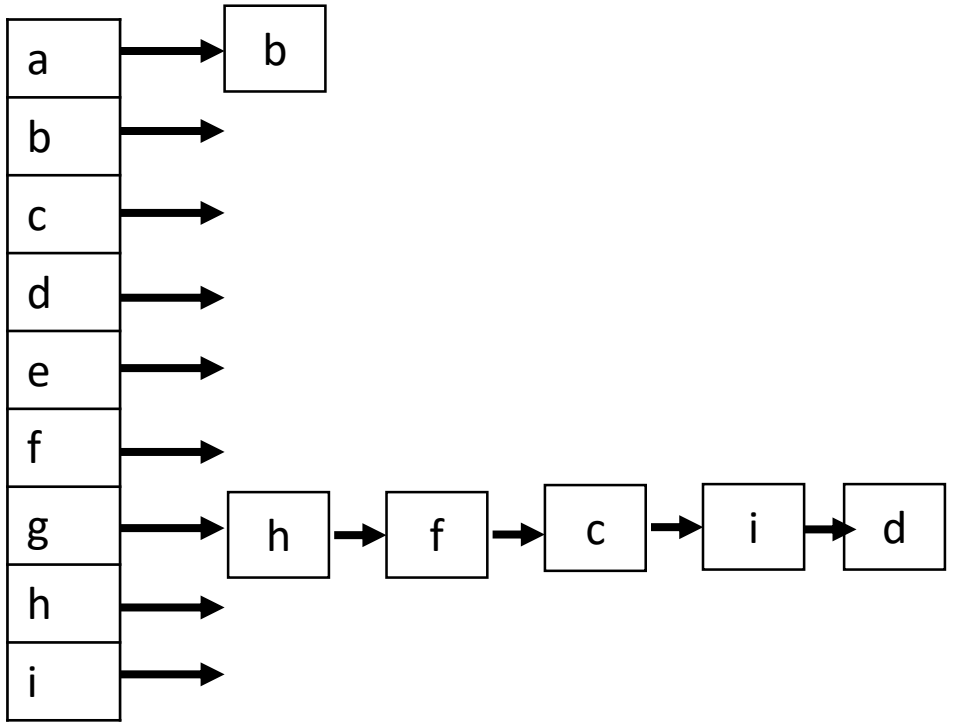
# Kruskal Algorithm Dry Run Using Union Find

(h)




Edge (i, h) creates cycle so it will not be added in MST

**UF Items**



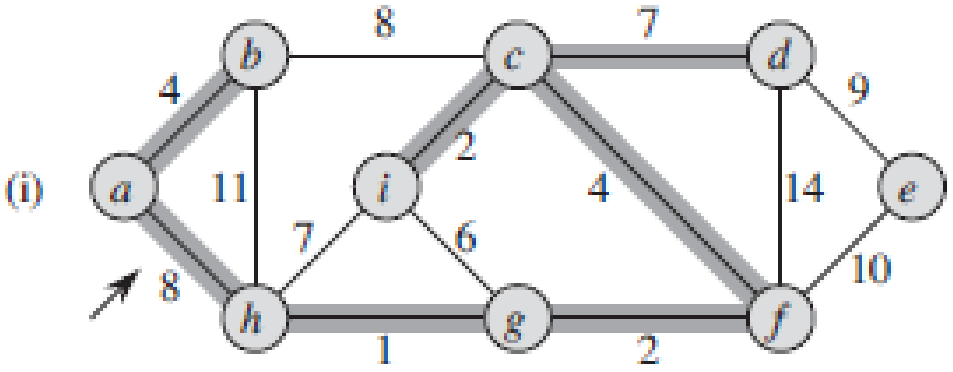

**UF Sizes**

| | |
|---|---|
| a | 2 |
| b | 0 |
| c | 0 |
| d | 0 |
| e | 1 |
| f | 0 |
| g | 6 |
| h | 0 |
| i | 0 |

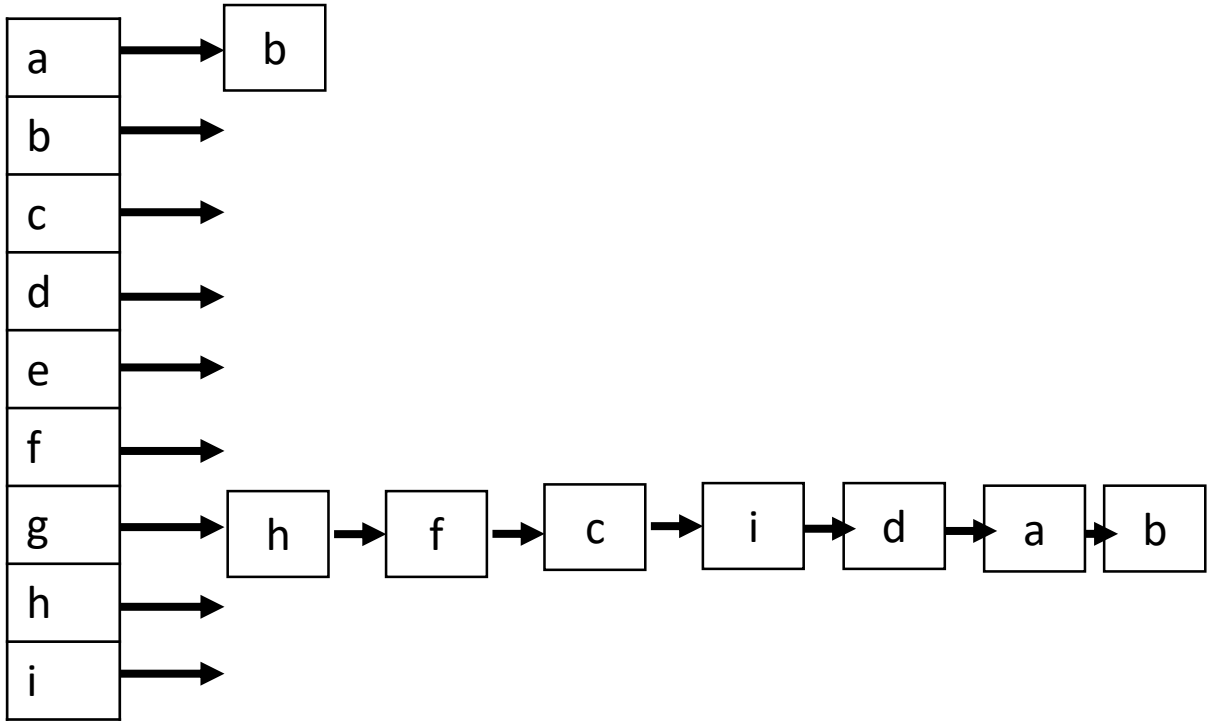
**UF Sets Array**

| a | a | g | g | e | g | g | g | g |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




**UF Items**



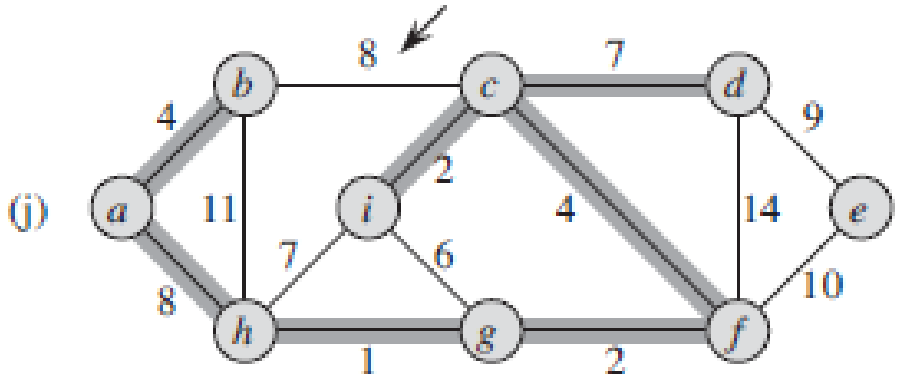

**UF Sizes**

| | |
|---|---|
| a | 0 |
| b | 0 |
| c | 0 |
| d | 0 |
| e | 1 |
| f | 0 |
| g | 8 |
| h | 0 |
| i | 0 |

**UF Sets Array**

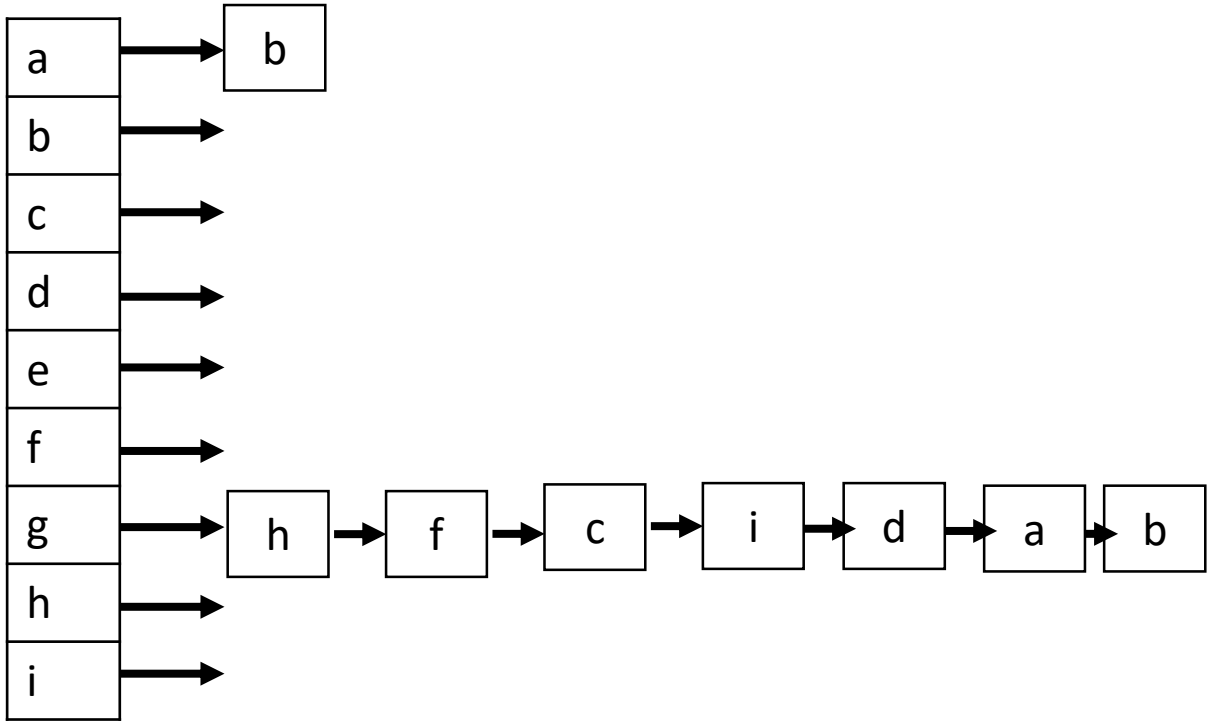
| a | b | c | d | e | f | g | h | i |
|---|---|---|---|---|---|---|---|---|
| g | g | g | g | e | g | g | g | g |

# Kruskal Algorithm Dry Run Using Union Find




Edge (b, c) creates cycle so it will not be added in MST

**UF Items**



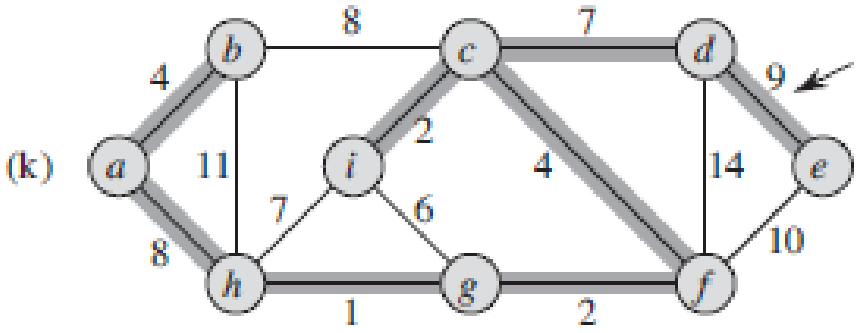

**UF Sizes**

| | |
|---|---|
| a | 0 |
| b | 0 |
| c | 0 |
| d | 0 |
| e | 1 |
| f | 0 |
| g | 8 |
| h | 0 |
| i | 0 |

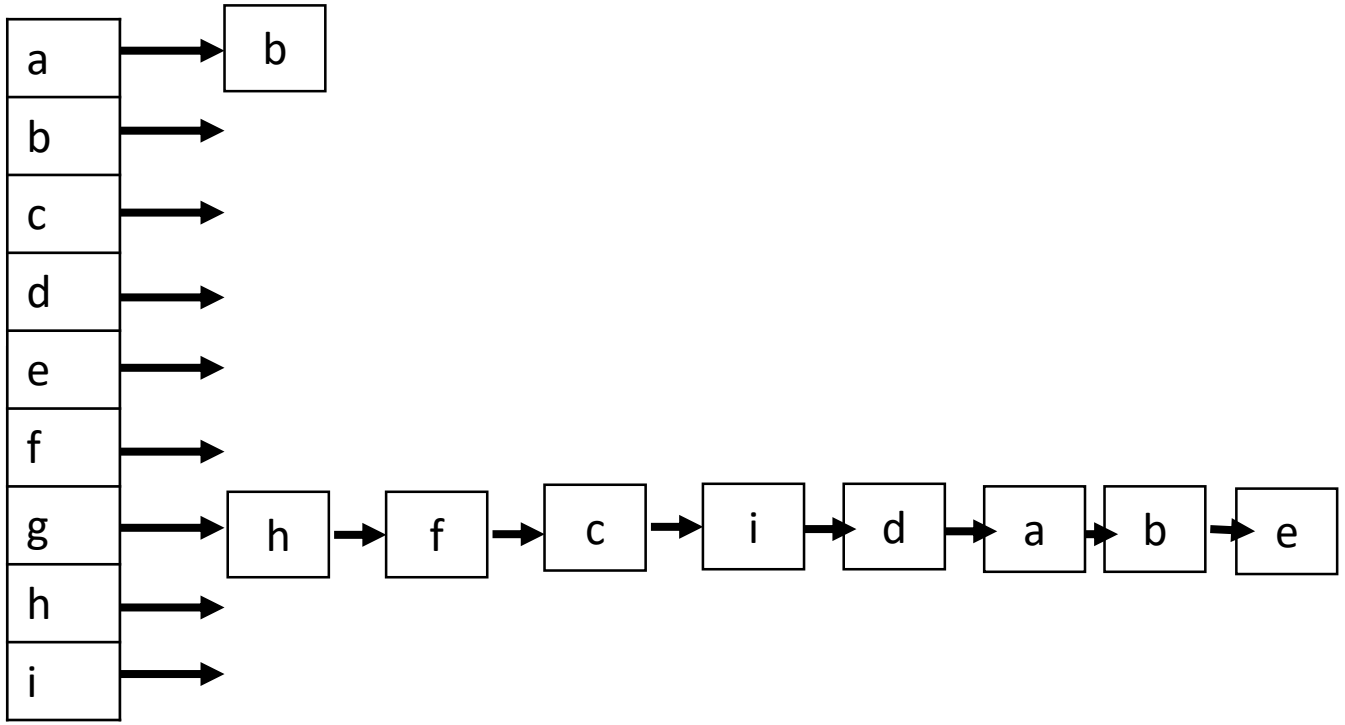
**UF Sets Array**

| g | g | g | g | e | g | g | g | g |
|---|---|---|---|---|---|---|---|---|
| a | b | c | d | e | f | g | h | i |

# Kruskal Algorithm Dry Run Using Union Find




**UF Items**




**UF Sizes**

| | |
|---|---|
| a | 0 |
| b | 0 |
| c | 0 |
| d | 0 |
| e | 0 |
| f | 0 |
| g | 9 |
| h | 0 |
| i | 0 |

**UF Sets Array**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| g | g | g | g | g | g | g | g | g |
| a | b | c | d | e | f | g | h | i |