Comparison Based Sorting

(Lower Bound)

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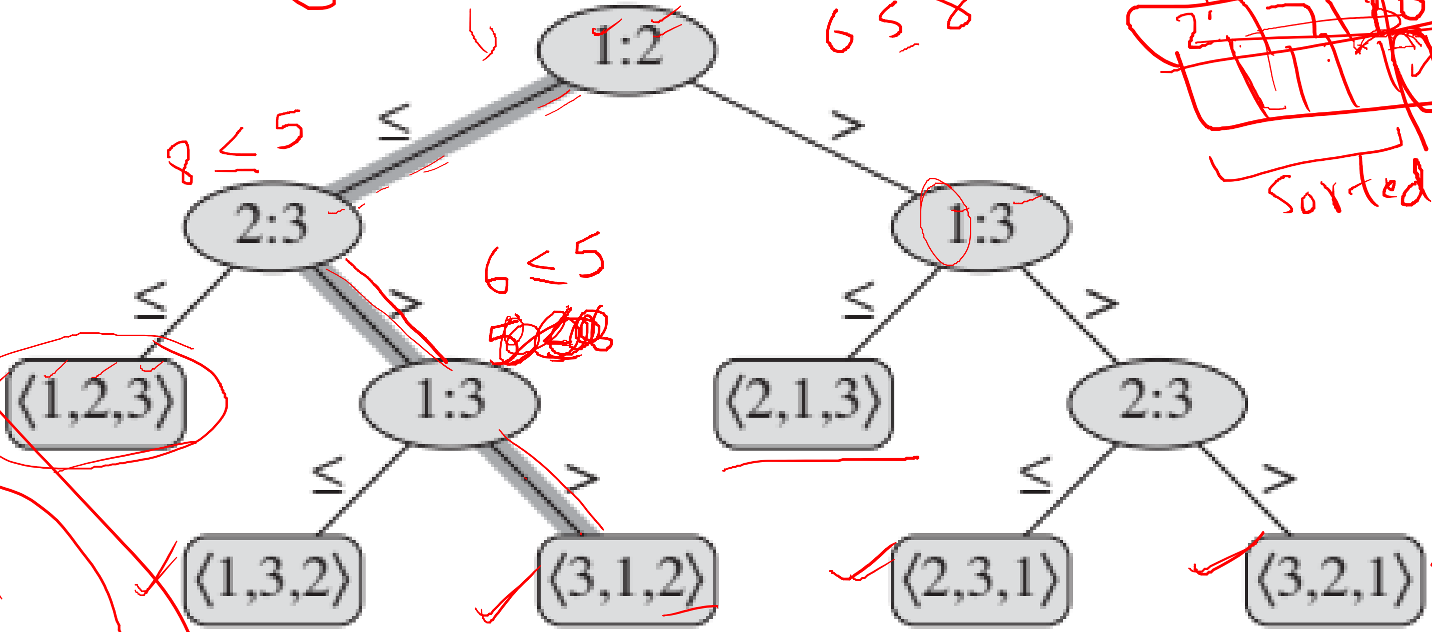


The decision tree for insertion sort operating on three elements.

original array

1 2 3
6 8 5

$6 \le 8$

$6 \le 8$

$8 \le 5$

$6 \le 5$

height of Tree

worst case running time = height of longest branch

3 1 2
5 6 8

5 6 8

1 2 3
10 7 2

7 10 2

7 2 10

2 7 10

sorted

n values permutations

n!

$n=3$

$n! = 3!$
$= 3 \times 2 \times 1$
$= 6$

Comparison Based Sorting
(Lower Bound)

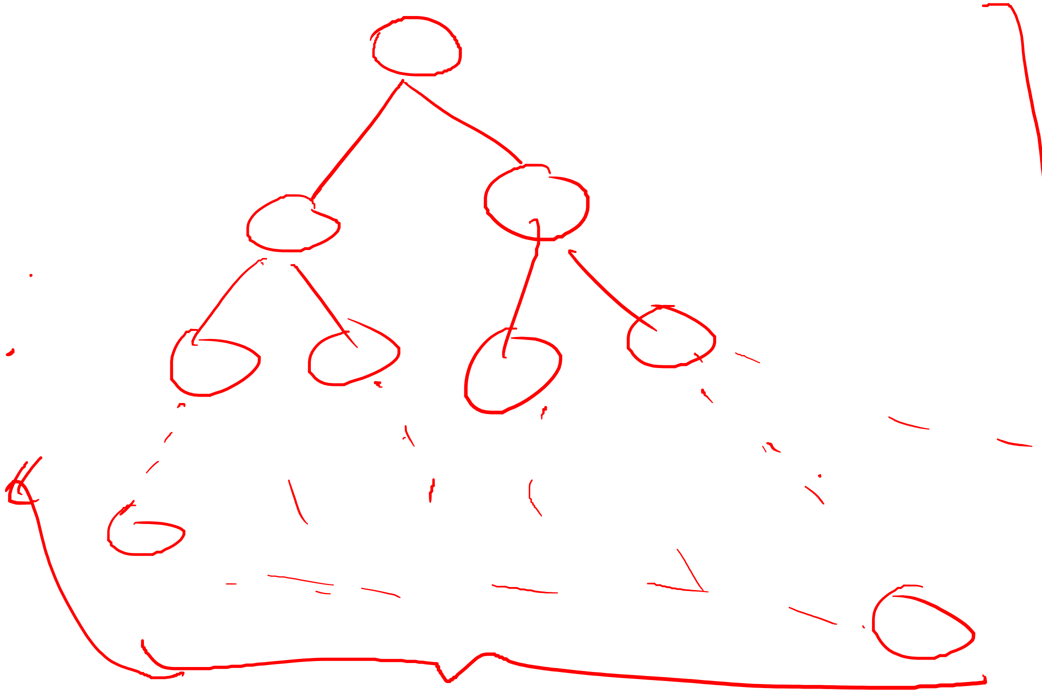
$l \le 2^h$

$$\boxed{n! \le l \le 2^h}$$

- The decision tree has l leaves and h height.
- Each permutation of array appears as a leaf of the decision tree.
- There are $n!$ possible permutations of an input array of size n.

$\boxed{n! \le l}$

Number of leaves = l = ?



height = h = ?

How many leaves does a binary tree of height h has?

$$h \geq \frac{n}{2}\log_2\frac{n}{2} = \Omega(n\lg n)$$

$$\boxed{h = \Omega(n\lg n)}$$

10 terms

$10!$

$n/2 = 5$

$$\boxed{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}$$

$5 = n/2$ terms

$$5\times 5\times 5\times 5\times 5$$

$$(5)^5$$

$n = 10$

$n/2 = 5$

$$\left(\frac{n}{2}\right)^{n/2}$$

$$n! \leq l \leq 2^h$$

$$\boxed{n! \leq 2^h}$$

$$2^h \geq n!$$

$$\log_2 2^h \geq \log_2(n!)$$

$$h \geq \log_2(n!)$$

$$h \geq \log_2(n!) \geq \log_2\left(\frac{n}{2}\right)^{n/2}$$

$$h \geq \log_2\left(\frac{n}{2}\right)^{n/2}$$

$$n! = \underbrace{(n)\times(n-1)\times\cdots\left(\frac{n}{2}\right)}_{n/2\text{ terms}}\underbrace{\left(\frac{n}{2}-1\right)\times\cdots 2\times 1}_{n/2\text{ terms}}$$

$$\geq \left(\frac{n}{2}\right)^{n/2}$$

$n/2$ terms

$$(n)\times(n-1)\times\cdots\left(\frac{n}{2}\right)\times\left(\frac{n}{2}-1\right)\times\cdots 2\times 1 \geq \left(\frac{n}{2}\right)^{n/2}$$

$$n/2 \times n/2 \times \cdots \times n/2 = \left(n/2\right)^{n/2}$$

$$\boxed{n! \geq \left(n/2\right)^{n/2}}$$

$$\log_2 n! \geq \log_2 \left(n/2\right)^{n/2}$$

$$h \geq \frac{n}{2}\log_2\frac{n}{2}$$

$$h \geq \frac{n}{2}\left(\log_2 n - \log_2 2\right)$$