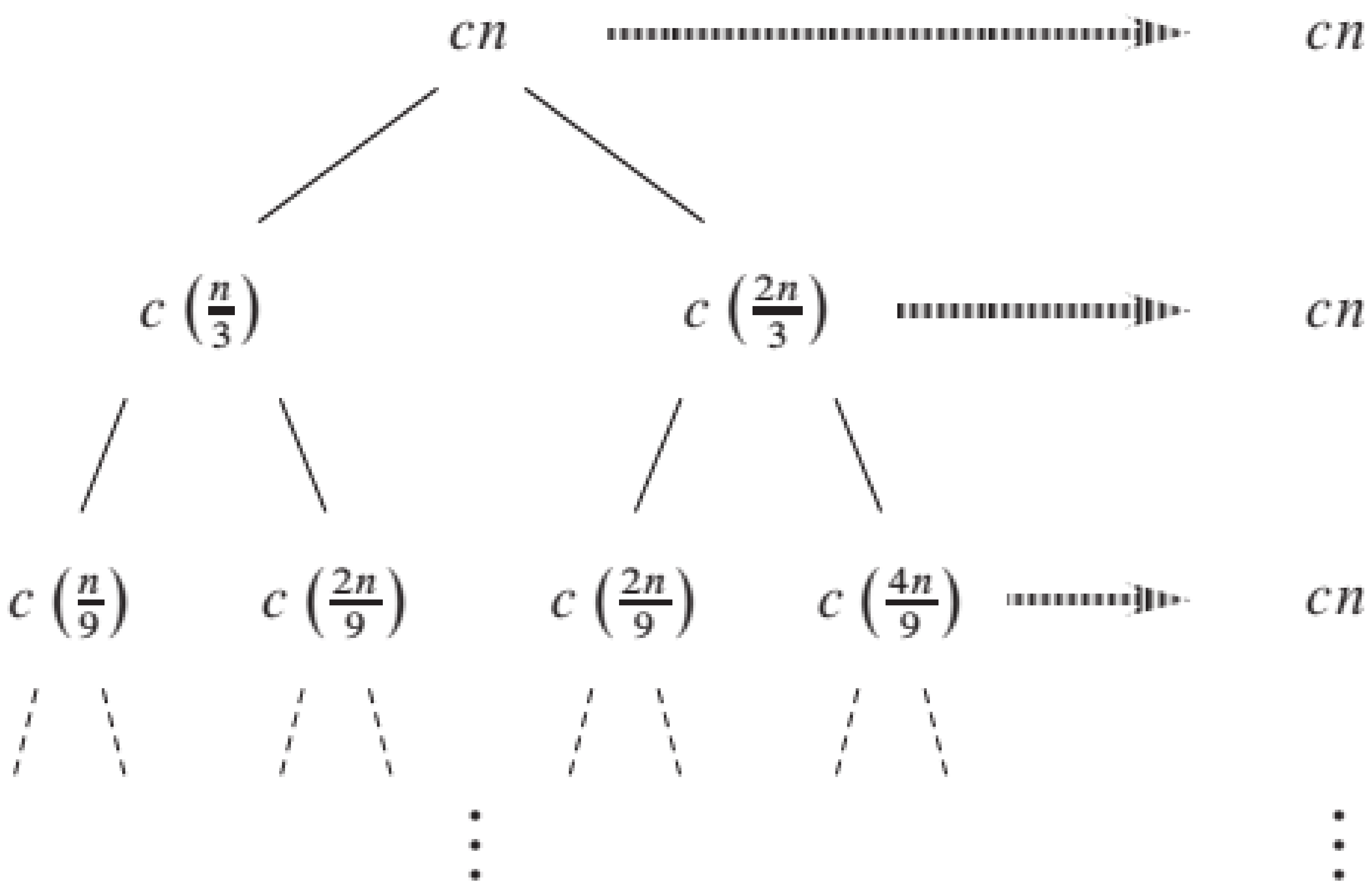
Solving Recurrences

Lecture 4

$$T(n) = T(n/3) + T(2n/3) + cn.$$

$$\frac{\frac{n}{3}}{2} = \frac{2n}{3}$$

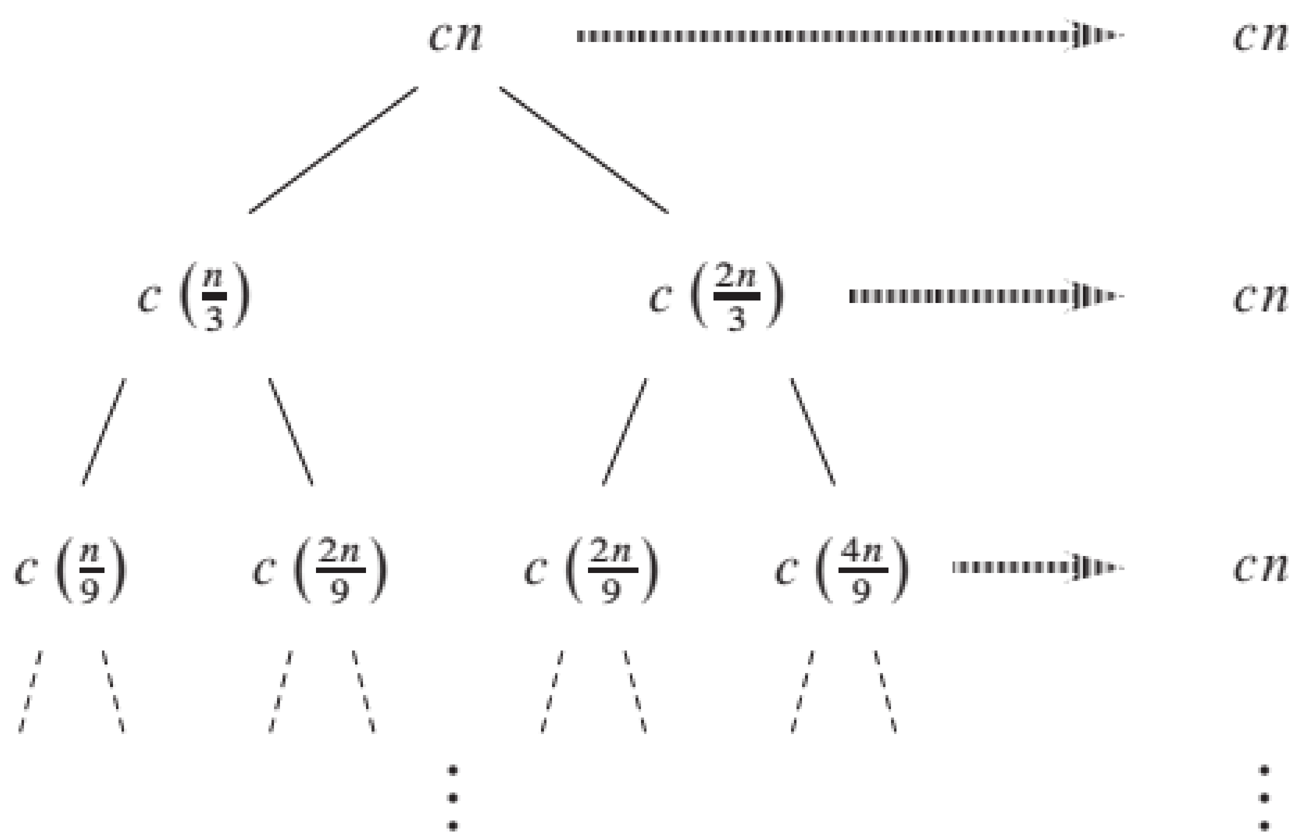
$$T(n) = T(n/3) + T(2n/3) + cn.$$



$$T(n) = T(n/3) + T(2n/3) + cn.$$

- What is height of tree?

$$(2/3)^k n = 1$$



$$T(n) = T(n/3) + T(2n/3) + cn.$$

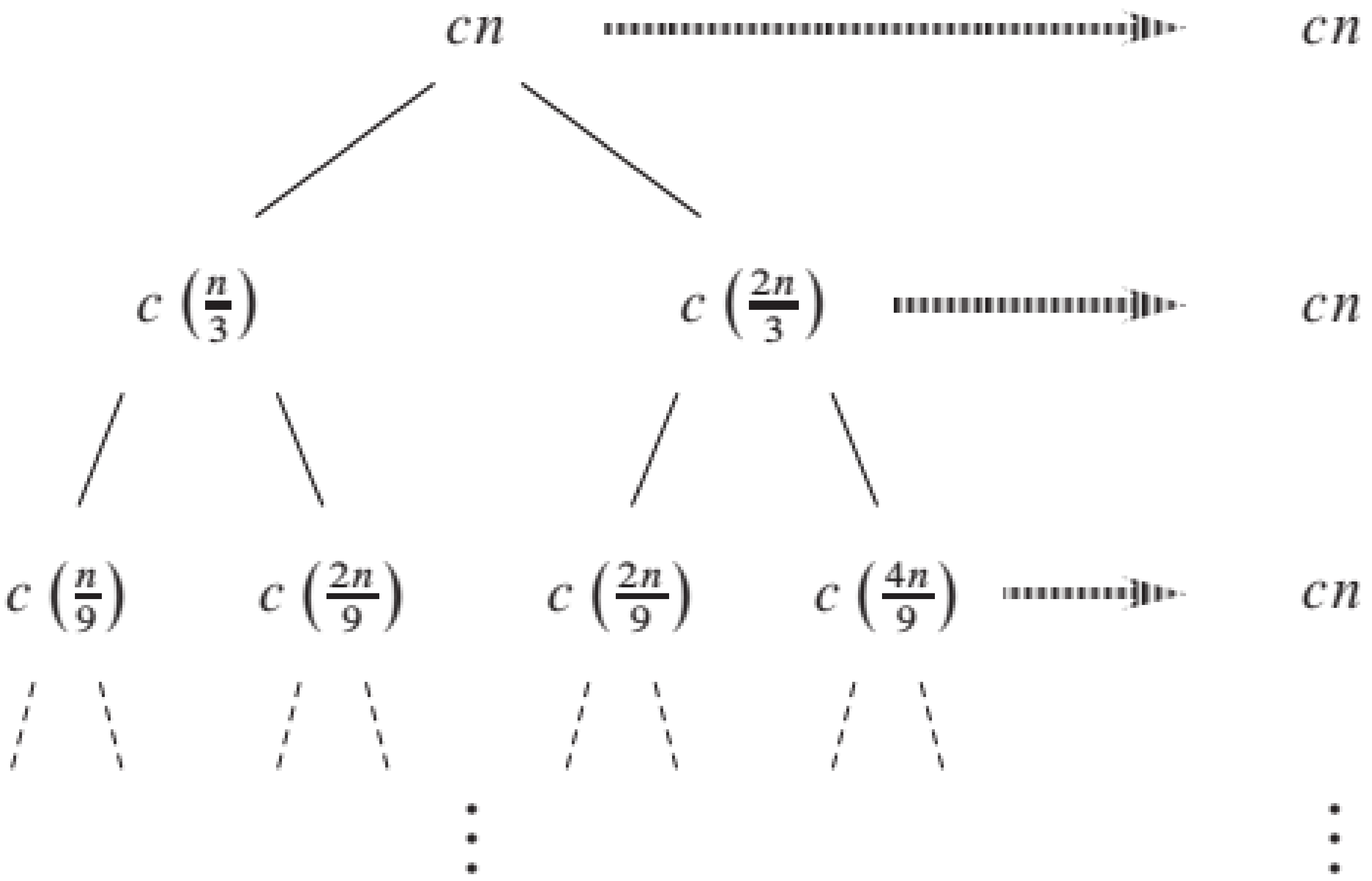
- What is height of tree?

$$(2/3)^k n = 1$$

$$n = 3/2^k$$

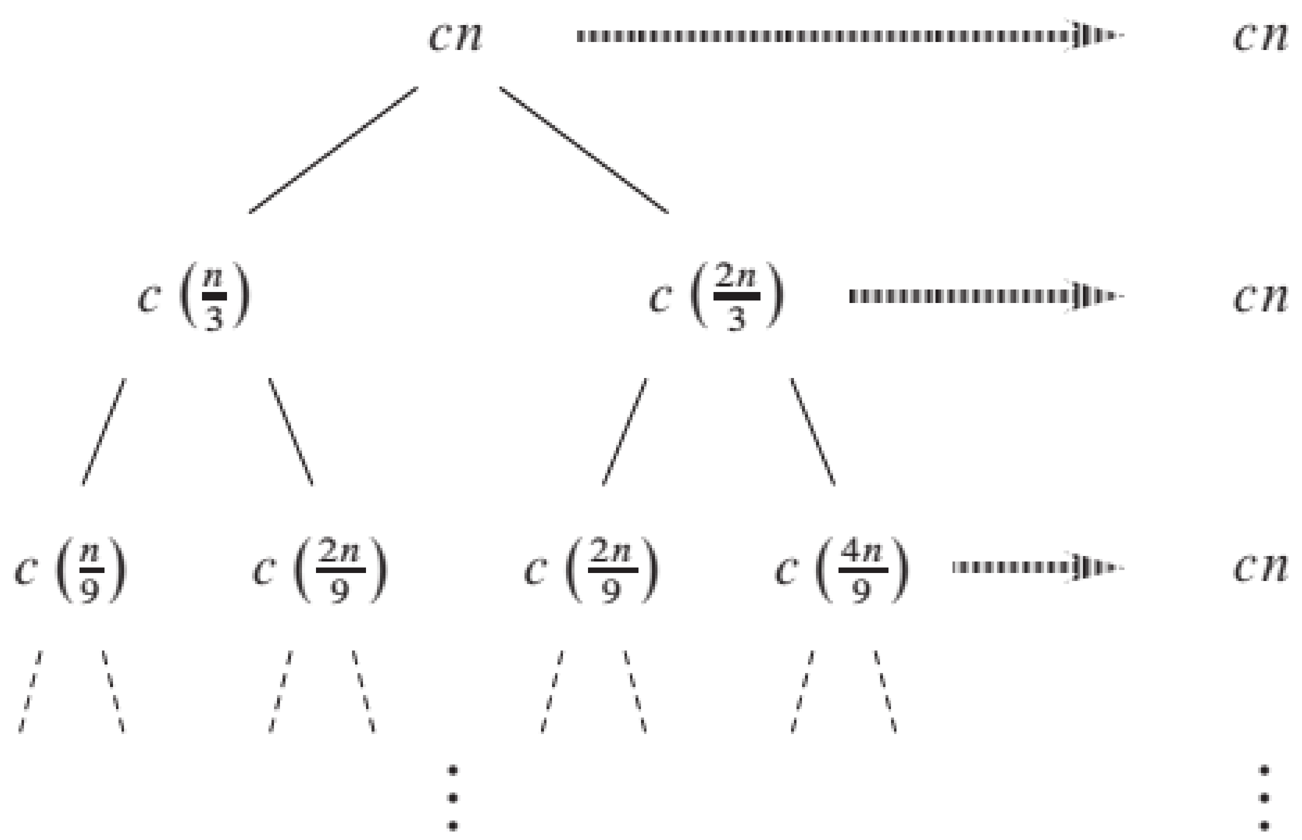
$$\log_{3/2} n = \log_{3/2} 3/2^k$$

$$k = \log_{3/2} n$$

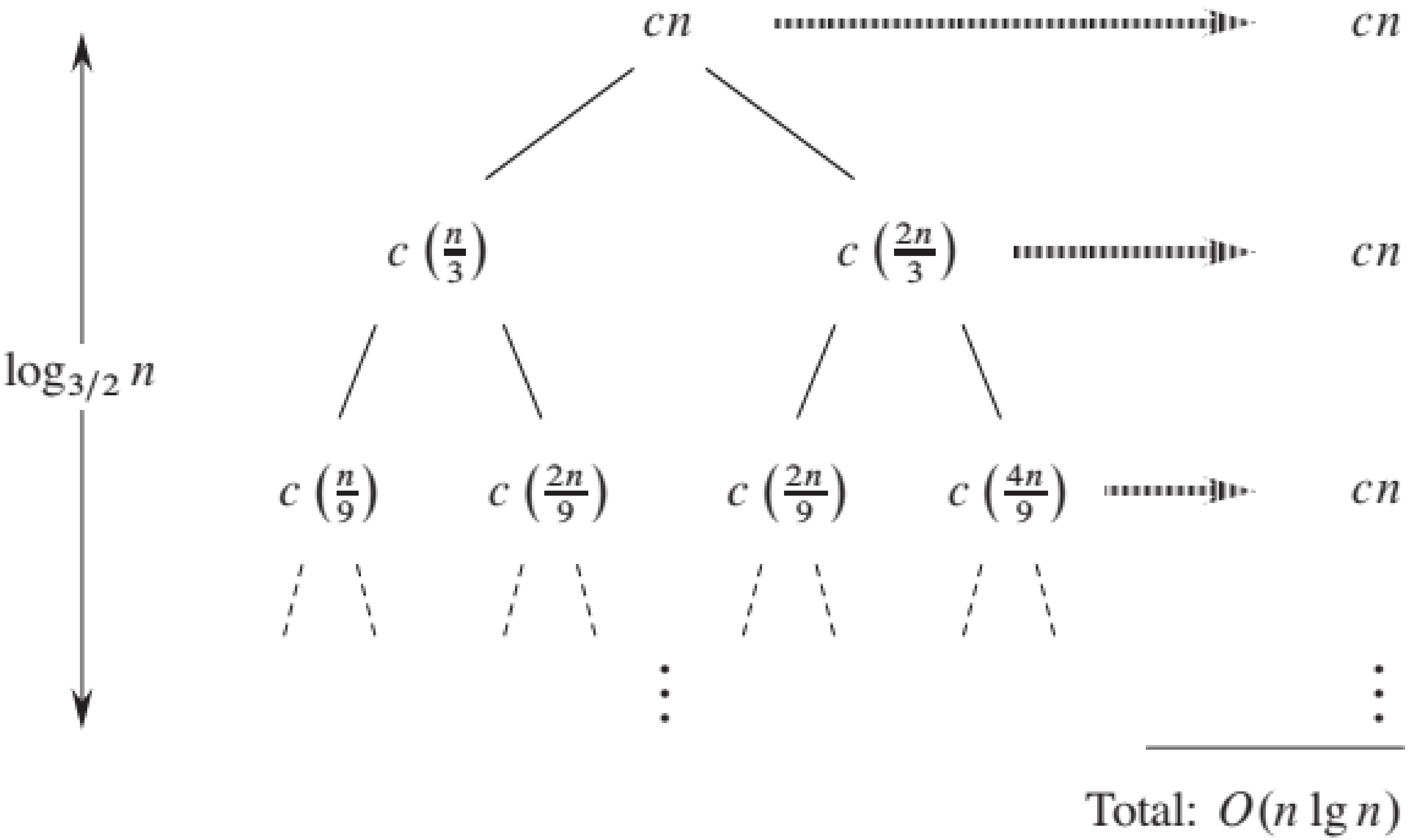


$$T(n) = T(n/3) + T(2n/3) + cn.$$

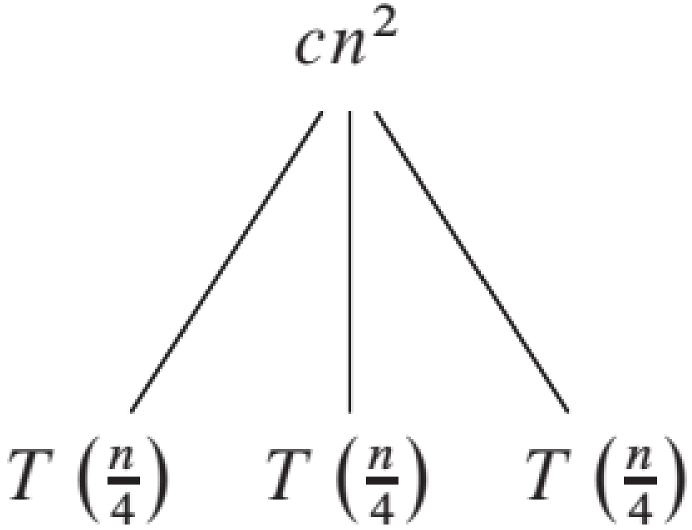
- What is height of tree?
- N is being divided by 3/2 at every level so height is $\log_{3/2} n$



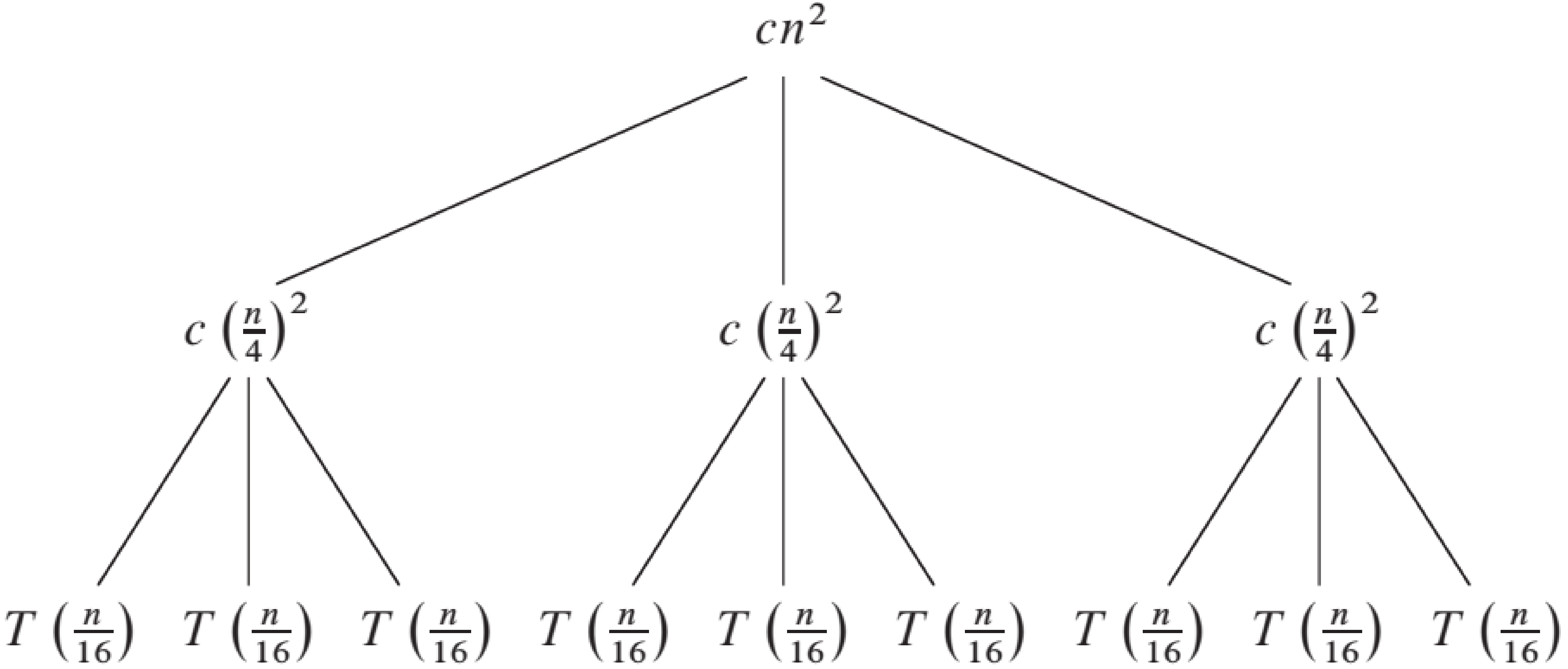
$$T(n) = T(n/3) + T(2n/3) + cn.$$



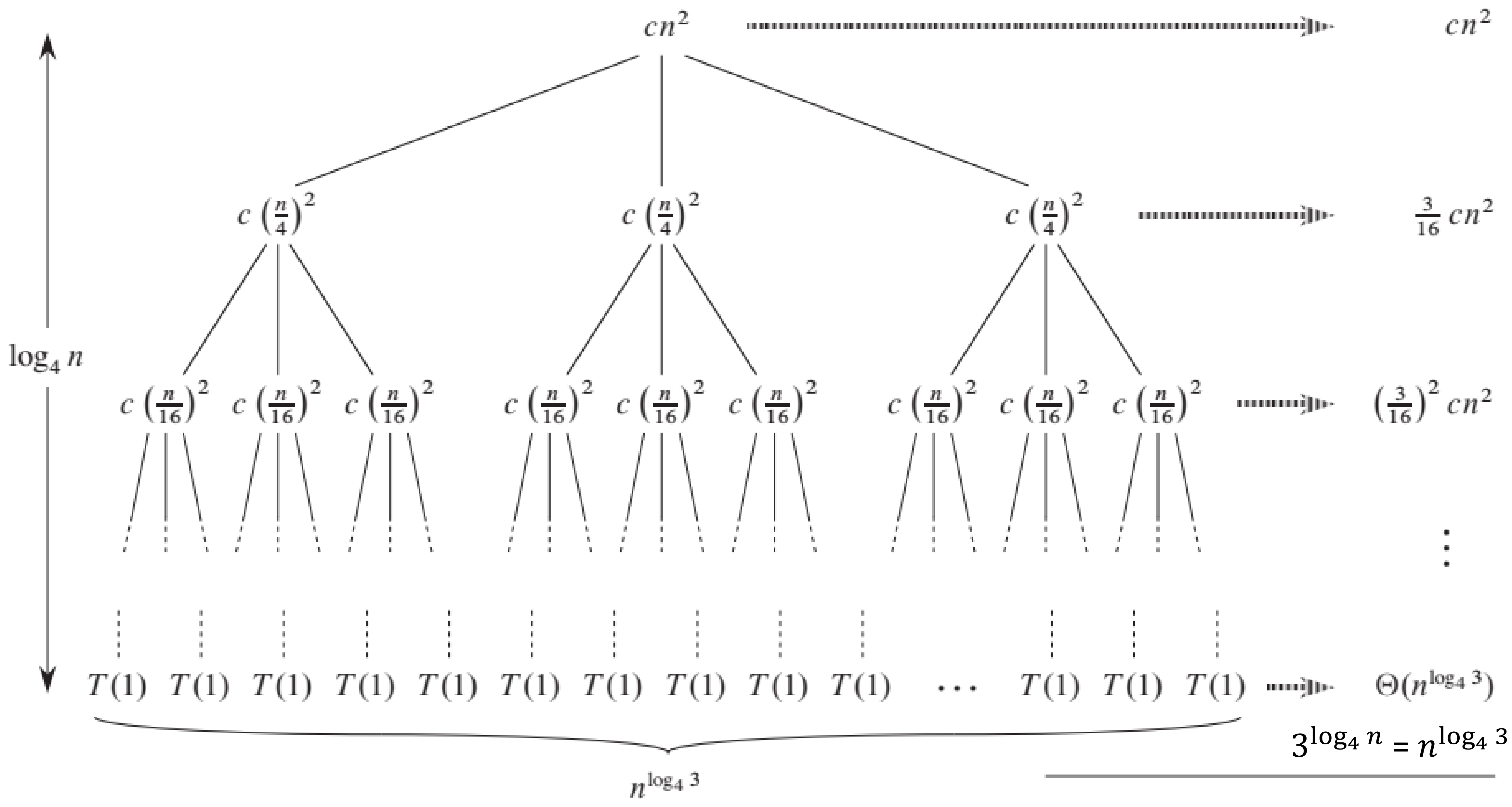
$T(n) = 3T(n/4) + cn^2$

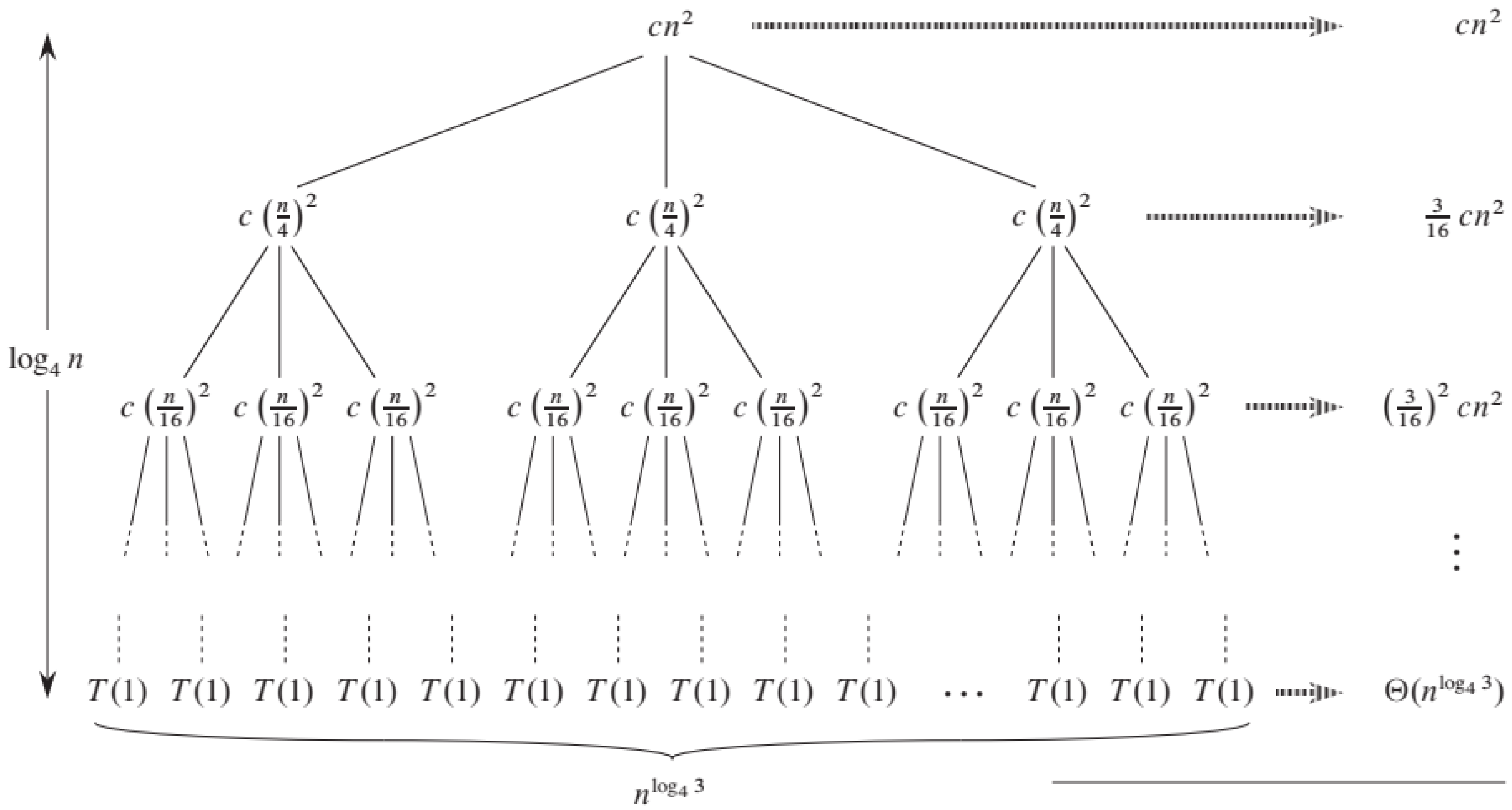

cn^2
$T\left(\frac{n}{4}\right)$
$T\left(\frac{n}{4}\right)$
$T\left(\frac{n}{4}\right)$

$T(n) = 3T(n/4) + cn^2$


cn^2
$c\left(\frac{n}{4}\right)^2$ $c\left(\frac{n}{4}\right)^2$ $c\left(\frac{n}{4}\right)^2$
$T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$ $T\left(\frac{n}{16}\right)$

$T(n) = 3T(n/4) + cn^2$


cn^2
$c\left(\frac{n}{4}\right)^2$
$\frac{3}{16}cn^2$
$c\left(\frac{n}{16}\right)^2$
$\left(\frac{3}{16}\right)^2 cn^2$
$\log_4 n$
$T(1)$
$\Theta(n^{\log_4 3})$
$n^{\log_4 3}$
$3^{\log_4 n} = n^{\log_4 3}$



$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3})$$

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n-1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})
\end{aligned}
$$

Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n \ ,$$

is an ***arithmetic series*** and has the value

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{1}{2}n(n+1) \\ &= \Theta(n^2) \ .\end{aligned}$$

Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} . \tag{A.5}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} . \tag{A.6}$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1-(3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2)\ .
\end{aligned}
$$

The substitution method for solving recurrences

1. Guess the form of the solution.
2. Use mathematical induction to find the constants and show that the solution works.

The substitution method for solving recurrences

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$

The substitution method requires us to prove that $T(n) \leq c\ \mathrm{n lg}\, n$ for an appropriate choice of the constant c > 0.

We start by assuming that this bound holds for all positive m < n, in particular for m = n/2, yielding

$$T(n) \leq c\ \mathrm{nlg}\, n$$

$$T(\lfloor n/2 \rfloor) \leq c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$

The substitution method for solving recurrences

$$\log a - \log b = \frac{\log a}{\log b}$$

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$ .... Equation 1

$$T(\lfloor n/2 \rfloor) \le c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$$ .... Equation 2

$$T(n) \le 2(c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n$$ .... Plug in Equation 2 in Equation 1

$$\le cn \lg(n/2) + n$$ .... Floor function is removed so result greater or equal

$$= cn \lg n - cn \lg 2 + n$$

$$= cn \lg n - cn + n$$ .... since *lg* 2 is 1

$$\le cn \lg n ,$$ .... $For\ all\ c \ge 1$

Proof using substitution method

Now we can use the substitution method to verify that our guess was correct, that is, $T(n) = O(n^2)$ is an upper bound for the recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$. We want to show that $T(n) \le dn^2$ for some constant $d > 0$. Using the same constant $c > 0$ as before, we have

$$T(n) \le 3\,d\left(\frac{n}{4}\right)^2 + cn^2$$

$$(3/16)d + c \le d$$

$$c \le d\left(\frac{13}{16}\right)$$

$$T(n) \le ((3/16)d + c)\, n^2$$

$$T(n) \le dn^2$$

Lets suppose $T\left(\frac{n}{4}\right) \le d(n/4)^2$

Proof using substitution method

$$
\begin{aligned}
T(n) &\le 3T(\lfloor n/4 \rfloor) + cn^2 \\
&\le 3d \lfloor n/4 \rfloor^2 + cn^2 \\
&\le 3d(n/4)^2 + cn^2 \\
&= \frac{3}{16} dn^2 + cn^2 \\
&\le dn^2 ,
\end{aligned}
$$

Proof using substitution method

$$\begin{aligned} T(n) &\le 3T(\lfloor n/4 \rfloor) + cn^2 \\ &\le 3d \lfloor n/4 \rfloor^2 + cn^2 \\ &\le 3d(n/4)^2 + cn^2 \\ &= \frac{3}{16} dn^2 + cn^2 \\ &\le dn^2 , \end{aligned}$$

$$[\left(\frac{3}{16}\right) d + c]n^2$$

$$\left(\frac{3}{16}\right) d + c \le d$$

$$c \le d - \left(\frac{3}{16}\right) d$$

$$c \le \left(\frac{13}{16}\right) d$$

Where the last step holds as long as $c \le \left(\frac{13}{16}\right) d$

Proof using substitution method

$$\log a - \log b = \frac{\log a}{\log b}$$

$$\begin{aligned} T(n) &\le T(n/3) + T(2n/3) + cn \\ &\le d(n/3)\lg(n/3) + d(2n/3)\lg(2n/3) + cn \\ &= (d(n/3)\lg n - d(n/3)\lg 3) \\ &\quad + (d(2n/3)\lg n - d(2n/3)\lg(3/2)) + cn \end{aligned}$$

$$T(n) \le d\,\mathrm{nlg}\,n$$

Proof using substitution method

$$dnlg\ n \ \geq dnlgn \ - dn\ (lg3 \ - 2/3) + cn$$

$$0 \geq \ -dn\ (lg3 \ - 2/3) + cn$$

$$\begin{aligned}
T(n) &\leq T(n/3) + T(2n/3) + cn \\
&\leq d(n/3)\lg(n/3) + d(2n/3)\lg(2n/3) + cn \\
&= (d(n/3)\lg n - d(n/3)\lg 3) \\
&\qquad + (d(2n/3)\lg n - d(2n/3)\lg(3/2)) + cn \\
&= dn\lg n - d((n/3)\lg 3 + (2n/3)\lg(3/2)) + cn \\
&= dn\lg n - d((n/3)\lg 3 + (2n/3)\lg 3 - (2n/3)\lg 2) + cn \\
&= dn\lg n - dn(\lg 3 - 2/3) + cn \\
&\leq dn\lg n\ ,
\end{aligned}$$

as long as $d \geq c/(\lg 3 - (2/3))$.

$$c \ \leq d(\lg 3 \ - 2/3)$$