CSCI 406: Algorithms
Solution to Assignment 2

2-07 **[10 pts]**

1. **[5 pts]** Yes! Using the definition of big-Oh, we pick the constants $c = 2$ and $n_0 = 1$. Clearly, $2^{n+1} = 2.2^n \leq 2.2^n$ for all $n \geq 1$.
2. **[5 pts]**
   No! $2^{2n} = 2^n.2^n$. For the statement to be true, we would need to find a constant c such that $2^n.2^n \leq c.2^n$ (i.e., $2^n \leq c$) for all $n \geq n_0$. This is not possible.

2-23 **[20 pts, 5 pts for each part]**

Recall that big-Oh notation only provides an upper bound (and possibly a very loose bound at that). So if an algorithm is said to take $O(n^2)$ time, it could actually take much less time. Recall that Θ notation provides a lower and an upper bound. Also notice that the problem focuses on the worst-case complexity; i.e., the run time elicited by the worst possible input. It doesn't specify how the algorithm performs on all inputs.

(a) Yes.

(b) Yes, there is no requirement for the function in the big-Oh to be tight. So we might say $O(n^2)$ but it's possible that all inputs take $O(n)$ time.

(c) Yes, some inputs can certainly take less time than the worst case $\Theta(n^2)$ time specified in the problem.

(d) No, it is not possible for the algorithm to take $O(n)$ on all inputs because n^2 is not in $O(n)$ and there must be at least one input that elicits the worst case n^2 behavior.

2-35 **[10 pts: 4pts for getting the correct summation, 6 pts for solving correctly]**

(a)

$$T(n) = \sum_{i=1}^{n} \sum_{j=i}^{2i} 1$$

(b)

$$\begin{aligned}
T(n) &= \sum_{i=1}^{n} \sum_{j=i}^{2i} 1 \\
&= \sum_{i=1}^{n} (i+1) \\
&= \left(\sum_{i=1}^{n} i \right) + n \\
&= \frac{n(n+1)}{2} + n \\
&= \frac{n(n+3)}{2}
\end{aligned}$$

2-39c **[5 pts]**

We are asked to prove that $\log_a x = \frac{\log_b x}{\log_b a}$.

Let $\xi = \log_b x$ and $\alpha = \log_b a$. From the definition of log, we have

$$b^{\xi} = x \tag{1}$$

and

$$b^\alpha = a \tag{2}$$

From Equation 2, we get $a^{1/\alpha} = b$. Combining this with Equation 1, we get

$$\begin{aligned} \left(a^{1/\alpha}\right)^\xi &= x \\ a^{\xi/\beta} &= x \end{aligned}$$

Taking $\log_a$ on both sides gives us the result.

2-47 **[5 pts]** Choose 1 coin from Bag 1, 2 coins from Bag 2, 3 coins from Bag 3, ..., 10 coins from Bag 10. This is 55 coins in all. If all coins were legit, then the total weight would be 550 grams. Let Bag i contain the fake coins. Since i coins are chosen from Bag i (each weighing 1 gram less than the expected 10 grams) the actual total weight is $550 - i$ grams. For instance, if the weight was 546 grams,this would mean that Bag 4 contains the fake coins.