پاسخ تمرینات فصل ۱۰

گسسته گریمالدی

علی نظری
۹۶۳۱۰۷۵

۲- الف) $a_n = (1/5)^n a_0 , n \ge 0$ ب) $a_n = (1/25)^n a_1 , n \ge 1$

ت) $a_n = (\frac{4}{3})^n 5 , n \ge 1$ ث) $a_n = (\frac{3}{2})^n 6 , n \ge 1$

۵- n را به عنوان هر ۳ ماه انتخاب کردیم پس تعداد n در ۳ ماه ها را بدست می دهد.

$$a_n = (1.06)a_{n-1} \;\; ; \;\; n \ge 0 \, , a_0 = 100$$

$$a_n = 100(1.06)^n \;\; ; \;\; n \ge 0$$

$$a_n = 2a_0 = 200 = (1.06)^n 100 \;\; \rightarrow \;\; (1.06)^n = 2 \;\; \rightarrow n = 12((1.06)^{11} = 1.89 \, , (1.06)^{12} = 2.012)$$

$$Months = 3 * 12 = 36$$

۶-

$P_n = (1.02)^n P_0$, $n = 15*4 = 60$ ➔ $P_{60} = (1.02)^{60} P_0 = 7218.27$ ➔ P_0 =2200تومان

۷-

الف)

$$19 + 18 + 17 + \cdots + 10 = 145$$

ب)

$$9 + 8 + 7 + \cdots + 1 = 45$$

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الف)

$a_n - 5a_{n-1} - 6a_{n-1} = 0 \;\rightarrow\; r^2 - 5r - 6 = 0 \;\rightarrow\; (r-6)(r+1) = 0 \;\rightarrow\; r = 6\,, r = -1$

$a_n = c_1(-1)^n + c_2 6^n \;\rightarrow\; \begin{cases} a_0 = 1 = c_1 + c_2 \\ a_1 = 3 = 6c_2 - c_1 \end{cases} \;\rightarrow\; c_1 = \frac{3}{7}\,, c_2 = \frac{4}{7} \;\rightarrow\; a_n = \frac{3}{7}(-1)^n + \frac{4}{7}6^n$

ب)

$2r^2 - 11r + 5 = 0 \;\rightarrow\; (2r-1)(r-5) = 0 \;\rightarrow\; r = \frac{1}{2}\,, r = 5$

$a_n = c_1(\frac{1}{2})^n + c_2 5^n \;\rightarrow\; \begin{cases} a_0 = 2 = c_1 + c_2 \\ a_1 = -8 = 5c_2 + \frac{1}{2}c_1 \end{cases} \;\rightarrow\; c_1 = 4\,, c_2 = -2 \;\rightarrow\; a_n = 4(-1)^n - 2(6^n)$

پ)

$3a_{n+1} - 2a_n - a_{n-1} = 0 \;\rightarrow\; 3r^2 - 2r - 1 = 0 \;\rightarrow\; (3r+1)(r-1) = 0 \;\rightarrow\; r = -\frac{1}{3}\,, r = 1$

$a_n = c_1(-\frac{1}{3})^n + c_2 1^n \;\rightarrow\; \begin{cases} a_0 = 7 = c_1 + c_2 \\ a_1 = 3 = c_2 - \frac{1}{3}c_1 \end{cases} \;\rightarrow\; c_1 = 3\,, c_2 = 4 \;\rightarrow\; a_n = 3\left(-\frac{1}{3}\right)^n + 4$

ت)

$r^2 + 1 = 0 \;\rightarrow\; r^2 = -1 \;\rightarrow\; r = \pm i \;\rightarrow\; a_n = c_1(i)^n + c_2(-i)^n = c_1\left(\cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right)^n +$

$c_2\left(\cos\left(\frac{\pi}{2}\right) + isin\left(-\frac{\pi}{2}\right)\right)^n = Acos\left(\frac{n\pi}{2}\right) + Bsin\left(\frac{n\pi}{2}\right) \;\rightarrow$

$\begin{cases} a_0 = 0 = A \\ a_1 = 3 = Bsin\left(\frac{\pi}{2}\right) = B \end{cases}$

$a_n = 3\sin\left(\frac{n\pi}{2}\right)\;;\; n \geq 0$

ث)

$r^2+4=0 \ \rightarrow \ r^2=-4 \ \rightarrow \ r=\pm 2i \ \rightarrow \ a_n=c_1(2i)^n+c_2(-2i)^n=$
$2^n[c_1\left(\cos\left(\frac{\pi}{2}\right)+isin\left(\frac{\pi}{2}\right)\right)^n$

$+c_2(\cos\left(\frac{\pi}{2}\right)-isin(\frac{\pi}{2}))^n]=2^n[Acos\left(\frac{n\pi}{2}\right)+Bsin\left(\frac{n\pi}{2}\right)]$

$\rightarrow \begin{cases} a_0=1=A \\ a_1=1=2Bsin\left(\frac{\pi}{2}\right) \ \rightarrow \ B=\frac{1}{2} \end{cases} \rightarrow \ a_n=2^n[\cos\left(\frac{n\pi}{2}\right)+\frac{1}{2}\sin\left(\frac{n\pi}{2}\right)]$

ج)

$r^2-6r+9=0 \ \rightarrow \ (r-3)^2=0 \ \rightarrow \ r=3\,,r=3 \ \rightarrow \ a_n=c_1 3^n+$
$c_2 n(3^n)$

$\begin{cases} a_0=5=c_1 \\ a_1=12=3c_1+3c_2 \end{cases} \rightarrow \ c_1=5\,,c_2=-1 \ \rightarrow \ a_n=(5-n)3^n$

چ)

$r^2+2r+2=0 \ \rightarrow \ r=-1\pm i \ \rightarrow \ -1+i=\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+isin\left(\frac{3\pi}{4}\right)\right)$

$,-1-i=\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+isin\left(\frac{3\pi}{4}\right)\right) \ \rightarrow \ a_n=\left(\sqrt{2}\right)^n[A\cos\left(\frac{3n\pi}{4}\right)+$
$Bsin\left(\frac{3n\pi}{4}\right)]$

$\rightarrow \begin{cases} a_0=1=A \\ a_1=3=\sqrt{2}\left[Acos\left(\frac{3\pi}{4}\right)+Bsin\left(\frac{3\pi}{4}\right)\right] \ \rightarrow \ B=4 \end{cases} \rightarrow a_n=a_n=$
$\left(\sqrt{2}\right)^n[\cos\left(\frac{3n\pi}{4}\right)+4sin\left(\frac{3n\pi}{4}\right)]$

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$4+b=0\rightarrow b=-4$

$37-16+c=0\rightarrow c=-21$

$r^2-4r-21=0\rightarrow(r-7)(r+3)=0\rightarrow r=7\,,r=-3\rightarrow a_n$
$=c_1 7^n+c_2(-3)^n$

$\begin{cases} a_0=0=c_1+c_2 \\ a_1=1=7c_1-3c_2 \end{cases} \rightarrow c_1=\frac{1}{10}\,,c_2=-\frac{1}{10}\rightarrow a_n=\frac{1}{10}(7^n-(-3)^n) \ ; \ n\geq 0$

-۴

$a_n = a_{n-1} + a_{n-2}\ ;\ n \geq 2\ ,\ a_0 = a_1 = 1$

$r^2 - r - 1 = 0\ ;\ r = \frac{1\pm\sqrt{5}}{2} \ \rightarrow\ a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$\begin{cases} a_0 = 1 \ \rightarrow\ A + B = 1 \\ a_1 = 1 \ \rightarrow\ \frac{1}{2} + \frac{(A-B)\sqrt{5}}{2} = 1 \end{cases} \rightarrow\ A = \frac{\sqrt{5}+1}{2\sqrt{5}}\ ,\ B = \frac{\sqrt{5}-1}{2\sqrt{5}}\ \rightarrow\ a_n = \frac{1}{\sqrt{5}}[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}]$

-۵

الف)

$F_1 = F_2 - F_0 \ \rightarrow\ F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

$n = 1 \rightarrow F_1 = F_2 = 1\ true$

$S(k):\ F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}\ ;\ k \geq 1$

$S(k+1):\ F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2k+1} = F_{2k} + F_{2k+1} = F_{2k+2}\ true$

ب)

$F_0 = F_1 + F_2 \ \rightarrow\ F_1 + F_2 + F_3 + \cdots + F_{2n-1} = F_{2n+1} - 1$

$n = 0 \rightarrow F_0 = F_1 - 1 = 0$

$S(k):\ F_0 + F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1\ \ true$

$S(k+1):\ F_0 + F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} = F_{2k+1} + F_{2k+2} - 1 = F_{2k+3} - 1\ \ true$

-۶

الف)

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}]}{\frac{1}{\sqrt{5}}[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}]} = \frac{1+\sqrt{5}}{2}$$

برای حل بالا بر عبارت دوم به ۰ میل می کند پس به جواب می رسیم.

ب)

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$$\frac{AC}{AX} = \frac{\sin(AXC)}{\sin(ACX)} = \frac{\sin(108)}{\sin(36)} = 2\cos(36)$$

۲-

$$\cos(18) = \sin(72) = 2\sin(36)\cos(36) = 4\sin(18)\cos(18)(1 - 2sin^2(18))$$

$$-8sin^3(18) + 4\sin(18) = 1 \;\rightarrow\; 8sin^3(18) - 4\sin(18) + 1 = 0 \;\rightarrow 8x^3 - 4x + 1 = 0$$

$$(2x-1)(4x^2+2x-1) = 0 \;\rightarrow\; 4x^2 + 2x - 1 = 0 \;\rightarrow\; x = \frac{-2\pm\sqrt{20}}{8} = \frac{-1\pm\sqrt{5}}{4}$$

$$\sin(18) > 0 \;\rightarrow\; \sin(18) = \frac{-1+\sqrt{5}}{4}$$

پ)

$$\frac{AC}{AX} = 2\cos(36) = 2\left(1 - 2sin^2(18)\right) = 2\left(1 - 2 * \frac{6-2\sqrt{5}}{16}\right) = 2 - \frac{3-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

۱۰-

a'_n : *n number with blue one on top*

a''_n :: *n number without blue one on top*

$$a_n = ka'_n + (k-1)a''_n = (k-1)\left(a'_n + a''_n\right) + a'_n = (k-1)a_n + (k-1)a_{n-1}$$

۱۳-

$$a_0 = 1 \;,\; a_1 = 2 \;,\; a_2 = 2 \;,\; a_3 = 2^2 \;,\; a_4 = 2^3 \;,\; a_5 = 2^5 \;\rightarrow\; a_n = 2^{F_n}$$

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با تقسیم بندی به ۲ حالت داریم :

$a_1 = 0\,, a_2 = 1\,, a_3 = 1\,, \ldots$

$a_n :\ n = x_1 + x_2 + \cdots x_t\ ;\ \ 1 \le t \le \left[\frac{n}{2}\right]\,, x_i \ge 2$

$\begin{cases} x_1 = 2 \to a_{n-2} :\ n-2 = x_2 + x_3 + \cdots x_t \\ x_1 > 2 \to a_{n-1} : n-1 = (x_1 - 1) + x_2 + \cdots x_t \end{cases} \to a_n = a_{n-1} + a_{n-2} = F_{n-1}\ ;\ \ n \ge 4$

-۱۵

$x_{n+2} - x_{n+1} = 2(x_{n+1} - x_n) \ \to\ x_{n+2} - 3x_{n+1} + 2x_n = 0 \ \to\ r^2 - 3r + 2 = 0 \ \to\ r = 1\,, r = 2$

$x_n = A2^n + B \ \to\ \begin{cases} x_0 = 1 \to A + B = 1 \\ x_1 = 5 \to 2A + B = 5 \end{cases} \ \to\ A = 4\,, B = -3 \ \to\ x_n = 4(2^n) - 3$

-۱۶

$D_n = 2D_{n-1} - D_{n-2} \ \to\ D_n - 2D_{n-1} + D_{n-2} = 0 \ \to\ r^2 - 2r + 1 = 0 \ \to\ r = 1\,, r = 1$

$D_n = A + nB \ \to\ \begin{cases} D_1 = 2 \ \to\ A + B = 2 \\ D_2 = 3 \ \to\ A + 2B = 3 \end{cases} \ \to\ A = 1\,, B = 1 \ \to\ D_n = 1 + n\ ;\ \ n \ge 1$

-۱۷

$b_n = a_n^2 \ \to\ (r-4)(r-1) = 0 \ \to\ r = 4\,, r = 1$

$b_n = A4^n + B \ \to\ \begin{cases} b_0 = 16 = A + B \\ b_1 = 169 = 4A + B \end{cases} \ \to\ A = 51\,, B = -35 \ \to\ b_n = 51(4^n) - 35$

$a_n = \sqrt{b_n} = \sqrt{51(4^n) - 35}\ ;\ \ n \ge 0$

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-۱

الف)

$a_n = a_0 + 2[1+2+\cdots+(n-1)] + 3n = 1 + n^2 - n + 3n = (n-1)^2 \;;\; n \geq 0$

یا

$f(n+1) = 2n+3 \rightarrow f(n) = 2n+1 \rightarrow a_n = a_0 + \sum_{i=1}^{n}(2i+1) = 1 + n + 2 * \frac{n(n+1)}{2} =$

$n^2 + 2n + 1 = (n+1)^2$

ب)

$f(n+1) = 3n^2 - n \rightarrow f(n) = 3n^2 - 7n + 4 \rightarrow a_n = a_0 + \sum_{i=1}^{n} 3i^2 - 7i + 4$

$= 3 + 3 * \frac{(n)(n+1)(2n+1)}{6} - 7 * \frac{(n)(n+1)}{2} + 4n = 3 + n(n-1)^2$

پ)

$a_1 = 2+5 \;,\; a_2 = 2^2 + (2+1)5 \;,\; a_3 = 2^3 + (2^2+2+1)5$

$a_n = 2^n + (2^{n-1} + 2^{n-2} + \cdots + 1)5 = 2^n + \frac{(1-2^n)}{1-2}5 = 6(2^n) - 5 \;;\; n \geq 0$

ت)

$a_1 = 2+1 \,,\; a_2 = 2^2 + 2 * 2^1 \;,\; a_3 = 2^3 + 3 * 2^2$

$a_n = 2^n + n * 2^{n-1}$

-۳

الف)

$a_n = a_{n-1} + n \;;\; a_0 = 1 \quad a_n^{(h)} = A \;,\; a_n^{(p)} = Bn + Cn^2 \;\rightarrow\; Bn + Cn^2 = B(n-1) + C(n-1)^2 + n$

$B = C = \frac{1}{2} \;,\; A = a_0 = 1 \;\rightarrow\; a_n = 1 + \frac{1}{2}(n)(n+1) \;;\; n \geq 0$

ب)

$b_n = b_{n-1} + 2 \ ; \ \ n \geq 2\,, b_1 = 2$

$b_n = 2n \ ; \ \ n \geq 1\,, b_0 = 1$

-۵

الف)

$a_n = cr^n \to (r+2)(r+1) = 0 \to r = -1, r = -2$

$a_n^{(h)} = A(-1)^n + B(-2)^n, a_n^{(p)} = C(3^n) \ ; \ \ a_n^{(p)} \ in\ equation \ \to C = \frac{1}{20}$

$\begin{cases} a_0 = 0 = A + B + \frac{1}{20} \\ a_1 = 1 = -A - 2B + \frac{3}{20} \end{cases} \to B = -\frac{4}{5}\,, A = -B - \frac{1}{20} = \frac{3}{4} \ \to \ \ a_n = \frac{3}{4}(-1)^n - \frac{4}{5}(-2)^n + \frac{1}{20}(3^n)$

ب)

$a_n = \frac{2}{9}(-2)^n - \frac{5}{6}n(-2)^n + \frac{7}{9} \ \ ; \ \ n \geq 0$

پ)

$(r-1)(r+1) = 0 \to r = 1, r = -1 \ \ \to \ \ a_n^{(h)} = A + B(-1)^n\,, a_n^{(p)} = Csin\left(\frac{n\pi}{2}\right) + Dcos\left(\frac{n\pi}{2}\right)$

$a_n^{(p)} \ in\ equation \ \to C = -\frac{1}{2}\,, D = 0$

$\begin{cases} a_0 = 1 = A + B \\ a_1 = 1 = A - B - \frac{1}{2} \end{cases} \to A = \frac{5}{4}\,, B = -\frac{1}{4} \ \ \to \ \ a_n = \frac{5}{4} - \frac{1}{4}(-1)^n - \frac{1}{2}\sin\left(\frac{n\pi}{2}\right) \ ; \ \ n \geq 0$

-۶

$a_n^{(h)} = A3^n + Bn3^n \ , a_n^{(p)} = C2^n + Dn^2 3^n$

$a_n^{(p)}$ *in equation* $\rightarrow C = 3 \, , D = \frac{7}{18}$

$\begin{cases} a_0 = 1 \\ a_1 = 4 \end{cases} \rightarrow A = -1 \, , B = \frac{17}{18} \ \rightarrow \ a_n = -2 * 3^n + \frac{17}{18} n3^n + \frac{7}{18} n^2 3^n + 3 * 2^n$

-۷

$(r-1)^3 = 0 \rightarrow r = 1 \, , r = 1 \, , r = 1 \ \rightarrow \ a_n^{(h)} = A + Bn + Cn^2 \, , a_n^{(p)} =$
$Dn^3 + En^4$

$a_n^{(p)}$ *in equation* $\rightarrow D = -\frac{3}{4} \, , D = \frac{5}{24} \ \rightarrow \ a_n = A + Bn + Cn^2 - \frac{3}{4} n^3 +$
$\frac{5}{24} n^4 \ ; \ n \geq 0$

-۸

$a_n^{(h)} = A3^n \, , a_n^{(p)} = Bn3^n \ \rightarrow \ a_n = A3^n + Bn3^n$

$a_n^{(p)}$ *in equation* $\rightarrow B = \frac{1}{3} \ , A = a_0 = 1 \rightarrow a_n = 3^n + n3^{n-1}$

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الف)

$b_n = a_n^2 \rightarrow b_n^{(h)} = A3^n + B2^n \, , b_n^{(p)} = Cn + D$

$b_n^{(p)}$ *in equation* $\rightarrow C = \frac{7}{2} \ , D = \frac{21}{4}$

$b_0 = b_1 = 1 \rightarrow A = \frac{3}{4} \, , B = -5 \ \rightarrow \ a_n = \sqrt{\left(\frac{3}{4}\right) 3^n - 5(2^n) + \frac{7}{2} n + \frac{21}{4}}$

ب)

$b_n = \frac{a_n}{n!} \to b_n + b_{n-1} = 1 \to b_n = \frac{1}{2}[(-1)^n + 1] \to a_n = \frac{n!}{2}[(-1)^n + 1]$

پ)

$a_n^2 = 2a_{n-1} \to 2\log_2 a_n = 1 + \log_2 a_{n-1} \to b_n = \log_2 a_n \to 2b_n - b_{n-1} - 1 = 0$

$b_n = A(\frac{1}{2})^n + 1\,, b_0 = A + 1 = 1 \to A = 0 \to b_n = 1 \to a_n = 2$

-۱۲

الف)

$s_{n+1} - s_n = t_{n+1} = \frac{1}{2}(n^2 + 3n + 2)$

$s_{n+1}^{(h)} - s_n^{(h)} = 0 \to s_n^{(h)} = A\,, s_n^{(p)} = n(Bn^2 + Cn + D) = Bn^3 + Cn^2 + Dn$

$s_n^{(p)}\ in\ equation \to B = \frac{1}{6}\,, C = \frac{1}{2}\,, D = \frac{1}{3}\,, s_1 = 1 \to A = 0$

$s_n = \frac{1}{6}(n)(n+1)(n+2)$

ب)

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$s_{10000000}$ اتم

-۲

$s_{99999} - s_{10000} = 1.665 * 10^{14}$ اتم