Subject:
Year. Month. Date.

باسمه تعالی

تمرین سری دو آمار و احتمال مهندسی

پرهام چمی - ۹۵۳۱۰۳۱

(۱) الف)

$E[x] = \sum_{x=0}^{\infty} x p(x) \geq \sum_{x\geq a}^{\infty} x p(x) \geq \sum_{x\geq a}^{\infty} a p(x)$

$E[x] \geq a P(x\geq a) \Rightarrow \frac{E[x]}{a} \geq P(x\geq a)$

ب)

$\frac{E[x]}{a} \geq P(x\geq a)$ $\quad x \sim (x-\mu)^2$, $\quad a \sim a^2$

$\frac{E[(x-\mu)^2]}{a^2} \geq P((x-\mu)^2 \geq a^2) \to \frac{var(x)}{a^2} \geq P(|x-\mu| \geq a)$

ج)

$\sqrt{var(x)} = \sigma \Rightarrow var(x) = \sigma^2$

$4 \to P(5-0 > a) < \frac{4}{a} \to a = 5 \Rightarrow P(5) < \frac{4}{5} \Rightarrow P(5) < 0.8$

٪۸۰

(۲) الف)

$E(ax+b) = \sum_{x=0}^{\infty} (aX+b) f(x) = a\sum_{x=0}^{\infty} x f(x) + b\sum_{x=0}^{\infty} f(x)$

$= aE[x] + b = aE[x] + b$

ب)

$E[xy] = E[x]E[y]$

$E[xy] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xy f(x,y) = \sum_{x,y=0}^{\infty} xy\, g(x) h(y) = \sum_{x=0}^{\infty} x g(x) \sum_{y=0}^{\infty} y h(y) \overset{*}{=} E[x]E[y]$

* x, y مستقل اند

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$\sigma^2[(aX+bY+c)^2] = var[(aX+bY+c)^2] =$ (2.

$E[(aX+bY+c)^2] - (E[aX+bY+c])^2 =$

$= E[a^2X^2+b^2Y^2+c^2+2abXY+2acX+2bcY] - (E[aX+bY+c])^2$

$= a^2E[X^2]+b^2E[Y^2]+c^2+2abE[XY]+2acE[X]+2bcE[Y]$

$-(aE[X]+bE[Y]+c)^2 =$

$a^2(E[X^2]-E[X]^2)+b^2(E[Y^2]-E[Y]^2)+2ab(E[XY-E[X]E[Y]]$

$+2ac\,E[X]+2bc\,E[Y]) = a^2\sigma^2(X)+b^2\sigma^2(Y)+2ab\sigma(XY)$

$\sigma^2_{aX-bY} = a^2\sigma_X^2+b^2\sigma_Y^2$ (3

$var[aX-bY] = E[(aX-bY)^2] - E[aX-bY]^2 =$

$E[a^2X^2+b^2Y^2-2abXY]-(aE[X]-bE[Y])^2 =$

$= a^2\sigma^2[X]+b^2\sigma^2[Y]-2ab(E[X]E[Y]-E[X]E[Y]) = a^2\sigma^2(x)b^2\sigma^2(y)$

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(۳ الف)

$$var\left(\frac{X}{\sigma(x)}+\frac{Y}{\sigma(Y)}\right)=۲+۲\rho(X,Y)$$

$$\left.\begin{array}{l}\sigma^2\left[\frac{X}{\sigma(x)}+\frac{Y}{\sigma(Y)}\right]\\ \frac{1}{\sigma(x)}=a \quad \frac{1}{\sigma_Y}=b\end{array}\right\} \to \sigma^2[aX+bY]=a^2\sigma^2_{(X)}+b^2\sigma^2_{(Y)}+2ab\sigma_{XY}=$$

$$=\frac{1}{\sigma_X^2}\times\sigma_X^2+\frac{1}{\sigma_Y^2}\sigma^2_{(Y)}+\frac{2}{\sigma_X\sigma_Y}$$

$$=۲+۲\rho(X,Y)$$

$$var\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right)=۲-۲\rho(x,y)$$

$$\left.\begin{array}{l}\sigma^2\left[\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right]\\ \frac{1}{\sigma_X}=a \quad \frac{1}{\sigma_Y}=b\end{array}\right\} \Rightarrow \sigma^2(aX-bY)=a^2\sigma_X^2+b^2\sigma_Y^2-2ab\,\sigma_{XY}=$$

$$۲-۲\rho(x,y)$$

(ب

$$\left.\begin{array}{l}var\left[\frac{X}{\sigma_X}+\frac{1}{\sigma_Y}\right]=۲+۲\rho(x,y)\\ var\left[\frac{X}{\sigma_X}+\frac{Y}{\sigma_Y}\right]\geq ۰\end{array}\right\} \Rightarrow ۲+۲\rho(x,y)\geq ۰ \Rightarrow \rho(x,y)\geq -1$$

$$\left.\begin{array}{l}var\left[\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right]=۲-۲\rho(x,y)\\ var\left[\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right]\geq ۰\end{array}\right\} \Rightarrow ۲-۲\rho(x,y)\geq ۰$$

$$-1\leq\rho(x,y)\leq 1$$

(ج

$$Y=aX+b$$

$$Cov(x,y)=E[xy]-E[x]E[Y]=E[X(aX+b)]-E[X]E[aX+b]$$

$$=E[aX^2+bX]-E[X](aE[X]+b)=aE[X^2]+bE[X]-aE[X]^2-bE[X]$$

$$=a\sigma_X^2$$

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$$X = \frac{Y}{a} - \frac{b}{a} \quad cov(x,y) = E\left[Y\left(\frac{Y}{a} - \frac{b}{a}\right)\right] - E[Y]E\left[\frac{Y}{a} - \frac{b}{a}\right]$$

$$= E\left[\frac{Y^2}{a} - \frac{Yb}{a}\right] - E[Y]\left(\frac{E[Y]}{a} - \frac{b}{a}\right) = \frac{E[Y^2]}{a} - \frac{b}{a}E[Y] - \frac{E[Y]^2}{a} + \frac{b}{a}E[Y]$$

$$= \frac{\sigma_Y^2}{a} \Rightarrow a\sigma_X^2 = \frac{\sigma_Y^2}{a} \Rightarrow \sigma_Y^2 = a^2\sigma_X^2 \Rightarrow a\sigma_X = \sigma_Y$$

$$var\left[\frac{-b}{a\sigma_X}\right] = var[const] = 0 \Rightarrow ۲ - ۲\rho(X,Y) = 0 \Rightarrow \rho(X,Y) = 1$$

$Y = aX + b$ ~~...~~ (۲

$$var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = ۲ - ۲\rho(X,Y) = 0 \Rightarrow \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = cte \Rightarrow Y = \frac{\sigma_Y}{\sigma_X}X + C\sigma_Y \Rightarrow Y = aX + b$$

$$a = \frac{\sigma_Y}{\sigma_X} = \frac{\sqrt{Var(Y)}}{\sqrt{Var(X)}} > 0$$

۴

$$P(\underbrace{x > y}_{A}) = ۰/۷ \quad P(\underbrace{y > z}_{B}) = ۰/۷ \quad P(\underbrace{z > x}_{C}) = ۰/۷$$

$$P(A \cap B) = P(x > y \cap y > z) = P(x > z) = ۰/۷ \times ۰/۷ = ۰/۴۹$$

$$\underbrace{P(x > z)}_{۰/۴۹} + \underbrace{P(z > x)}_{۰/۷} = P(U) = 1$$

$۱/۱۹ \neq 1$ ✗

الف) ۵

$$~~...~~\ f(k:P) = \begin{cases} P & if\ k=1 \\ 1-P & if\ k=0 \end{cases}$$

$$E[X] = 1 \times P + 0 \times (1-P) = P$$

$$var[x] = E[X^2] - E[X]^2 = 1^2 P + 0 \times (1-P) - (1 \times P + 0 \times (1-P))^2 = P(1-P)$$

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$P(k;n,p)=\binom{n}{k}p^{k}(1-p)^{n-k}$ $\quad E[X]=\sum_{k=0}^{n}\binom{n}{x}p^{x}(1-p)^{n-x}=np$

$Var[X]=E[X^2]-E[X]^2=\sum_{x=0}^{n}x^2\binom{n}{x}p^{x}(1-p)^{n-x}-\left(\sum_{x=0}^{n}x\,p^{x}(1-p)^{n-x}\right)^2=npq$

$f(k;\lambda)=\frac{\lambda^{k}e^{-\lambda}}{k!}\quad k=0,1,2,\dots$ ۱۲

$E[X]=\sum_{k=0}^{\infty}ke^{-\lambda}\frac{\lambda^k}{k!}=e^{-\lambda}\sum_{k=1}k\frac{\lambda^k}{k!}=e^{-\lambda}\lambda\sum_{k=0}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=e^{-\lambda}\lambda e^{\lambda}=\lambda$

$E[X^2]=\sum_{k=0}^{\infty}k^2e^{-\lambda}\frac{\lambda^k}{k!}=\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\lambda^{k-1}}{(k-1)!}=\lambda e^{-\lambda}\frac{d}{d\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}=\lambda e^{-\lambda}\frac{d}{d\lambda}\lambda e^{\lambda}$

$=\lambda e^{-\lambda}(e^{\lambda}+\lambda e^{\lambda})=\lambda+\lambda^2$

$Var[X]=E[X^2]-E[X]^2=\lambda^2+\lambda-\lambda^2=\lambda$

$f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$ ۱۳

$E[X]=\int_0^{\infty}x\lambda e^{-\lambda x}dx\longrightarrow -xe^{-\lambda x}\Big|_0^{\infty}+\int_0^{\infty}e^{-\lambda x}dx=\frac{1}{\lambda}$

$E[X^2]=\int_0^{\infty}x^2\lambda e^{-\lambda x}dx=-x^2e^{-\lambda x}\Big|_0^{\infty}+\int_0^{\infty}2xe^{-\lambda x}dx=\frac{2}{\lambda^2}$

$\Rightarrow Var[X]=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$

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۴) الف)

$P(x,y) = P(y)\,P(x|y)$

$E[x|Y=y] = g(y) \Rightarrow E[X|Y] = g(Y) = g \circ Y$

$E[E[X|Y]] = \int_{y=-\infty}^{\infty} g(y)\, f_Y(y)\, dy$

$g(y) = \int_{x=-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx = \int_{x=-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)}\, dx$

$E[E[X|Y]] = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)}\, dx\, f_Y(y)\, dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f(x,y)\, dx\, dy$
$= E[X]$

ب)

احتمال: $\frac{1}{2}$: A : میانگین تعداد غلط‌های چاپ کل کتاب ۲ ؛ توزیع احتمال پواسون
۶ : $\frac{1}{2}$: B " " " " " ۵

$E[E[A|B']] + E[E[B|A']] = E[A] + E[B] = 2 + 5 = 7$

۷)

$\frac{1}{\lambda} = 5 = E[X]$

$E[T] = E[t_1] + E[t_2] = 10$

اتوبوس: $-20 + \frac{20}{4} = -\frac{100}{4}$ ؛ $-\frac{100}{4} > -5$

تاکسی: -5

با تاکسی برود هزینه کمتری می‌پردازد.

۸) الف)

$E(T_F \mid start = A) = 1 + E(T_F|B)$ (I)

$E[T_F|B] = \frac{1}{3}E[T_F|C] + \frac{1}{3}E[T_F|A] + \frac{1}{3}E[T_F|D] + 1$ (II)

$E[T_F|C] = \frac{1}{2}E[T_F|B] + \frac{1}{2}E[T_F|E] + 1 = E[T_F|D]$ (III)

$E[T_F|E] = \frac{1}{3}E[T_F|C] + \frac{1}{3}E[T_F|D] + \frac{1}{3}E[T_F|E] + 1$ (IV)

(I), (II) $\Rightarrow E[T_F|A] - 1 = \frac{1}{3}E[T_F|C] + \frac{1}{3}[T_F|A] + \frac{1}{3}E[T_F|D] + 1$

(II) $\Rightarrow \frac{2}{3}E[T_F|A] = 2 + \frac{2}{3}E[T_F|C] \Rightarrow E[T_F|C] = E[T_F|A] - 3$ (V)

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$\Rightarrow E[T_F|C] = \frac{1}{2}\left[E[T_F|A]-1\right] + \frac{1}{2}E[T_F|E] + 1$

$\text{I}, \text{III}, \text{IV}, \text{V} \Rightarrow E[T_F|E] = \frac{2}{3}\left[E[T_F|A]-2\right] + \frac{1}{3}E[T_F|F] + 1$ (VI)

$= \frac{2}{3}E[T_F|A] - 1 + \frac{1}{3}E[T_F|F]$ (VII)

$\text{V}, \text{IV} \Rightarrow E(T_F|A) = \frac{1}{2}E[T_F|A] - \frac{1}{2} + \frac{1}{2}E[T_F|E] + 1$

$\frac{1}{2}E[T_F|A] = \frac{7}{2} + \frac{1}{2}E[T_F|E] \quad E[T_F|E] = E[T_F|A] - 7$ (VIII)

$\text{VIII}, \text{VII} \Rightarrow \frac{2}{3}E[T_F|A] - 1 = E[T_F|A] - 7$

$\frac{1}{3}E[T_F|A] = 6 \Rightarrow E[T_F|A] = 18 \Rightarrow E[T_F|B] = 17$

$E[T_F|E] = 11$

$E[T_F|C] = 15$

$E[T_F|D] = 15$

ب) $E[T_F|C] + 2 = 15 + 2 = 17$

ج) $\left\{\begin{array}{l} ABCEF: 1\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3} = \frac{1}{18} \\ ABDEF: 1\times\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3} = \frac{1}{18} \end{array}\right. \rightarrow \frac{1}{18} + \frac{1}{18} = \frac{1}{9}$

د) چنین مسیری وجود ندارد

هر مسیر چهار قدمی را که بشماریم هر کدام B یا C را ندارد سپس باید از B به C یا C به B برود

و اگر B به F برسد با ۳ قدم (تا A) و ۳ قدم از A تا F ← ۶ قدم نشان هست نه ۵ قدم

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۹) الف) اگر x را تعداد دقایقی در نظر بگیریم که طول بکشد تا آزاد شود داریم:

$$E[X]=\frac{5}{10}(E[X]+2)+\frac{3}{10}(E[X]+3)=\frac{8}{10}E[X]+1+\frac{9}{10}=E[X]$$
$$\Rightarrow E[X]=9.5$$

ب)

$$E[X]=\frac{1}{3}(2+E(x|y))+\frac{1}{3}(3+E(x|z))$$
$$\frac{1}{3}(2+\frac{1}{2}\times 3+\frac{1}{2}\times 0)+\frac{1}{3}(3+\frac{1}{2}\times 2+\frac{1}{2}\times 0)=\frac{10}{4}=\frac{5}{2}$$

ج) الف)

$$E[x^2]=\frac{5}{10}(\underbrace{E[x]+2}_{11.5})^2+\frac{3}{10}(E[x]+3)^2=\frac{5}{10}\times 132.25+\frac{3}{10}\times 156.25=112$$

$$var[X]=E[X^2]-E[X]^2=112-90.25=220.75$$

ب)

$$E[X^2]=\frac{1}{3}(2+\frac{1}{2}\times 3^2)^2+\frac{1}{12}(3+\frac{1}{2}\times 2^2)^2=\frac{249}{12}$$

$$var[X]=E[X^2]-E[X]^2=\frac{249}{12}-\frac{25}{4}=\frac{194}{12}=\frac{97}{6}$$

۱۰) الف) تابع توزیع حاشیه‌ای $f_{X_1}(x_1)$:

$$f_{X_1}(x_1)=\int_0^{x_1}6x_2\,dx_2=3x_2^2\Big|_0^{x_1}=3x_1^2\quad 0<x_1<1$$

$$\left.\begin{array}{l}0<f_{X_1}(x_1)<1\\ \int_{-\infty}^{+\infty}f(x_1)\,dx_1=\int_0^1 3x_1^2\,dx_1=x_1^3\Big|_0^1=1\end{array}\right\}\Rightarrow$$ تابع توزیع حاشیه‌ای باشد

ب)

$$P(x_2<0.5|x_1=0.7)=\int_0^{0.5}6x_2\,dx_2=3x_2^2\Big|_0^{0.5}=3\times 0.25$$
$$3x_1^2\xrightarrow{x_1=0.7}3\times(0.7)^2=0.49\times 3\Rightarrow P(x_2<0.5|x_1=0.7)=\frac{3\times 0.25}{3\times 0.49}=\frac{25}{49}=0.51$$

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