1 - الف) $E[x] = \sum_{x=0}^{\infty} x\,p(x) \geq \sum_{x \geq a}^{\infty} x\,p(x) \geq \sum_{x \geq a}^{\infty} a\,p(x)$

$a\,p(x \geq a) \leq \cancel{E[x]} \rightarrow E[x] \geq a\,p(x \geq a) \rightarrow \frac{E[x]}{a} \geq p(x \geq a)$

ب) $\frac{E[x]}{a} \geq P(x \geq a) \qquad x \sim (x-\mu)^2 \qquad a \sim a^2$

$\frac{E[(x-\mu)^2]}{a^2} \geq P((x-\mu)^2 \geq a^2) \rightarrow \frac{var(x)}{a^2} \geq P(x-\mu \geq a)$

ج) ~~$var(x) = \sqrt{\sigma} = \sqrt{2} \rightarrow \frac{\sqrt{2}}{a} \geq P(5-0 \geq a) \rightarrow a = 5 \rightarrow$~~

~~$\mu = 0$~~

~~$P(5) \leq \frac{\sqrt{2}}{5}$~~ $\sqrt{var(x)} = \cancel{\sqrt{\sigma}} \rightarrow var(x) = \sigma^2$

$= 4 \rightarrow P(5-0 \geq a) \leq \frac{4}{a} \rightarrow a = 5 \rightarrow P(5) \leq \frac{4}{5}$

$P(5) \leq 80\%$

2 - الف) $E(ax+b) = \sum_{x=0}^{\infty} (ax+b) f(x) = a\sum_{x=0}^{\infty} x f(x) + b\sum_{x=0}^{\infty} f(x)$

$= aE[x] + b \times 1 = aE[x] + b$

ب) $E[xy] = E[x]E[y]$: حکم   فرض : x, y مستقل

$E[xy] = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xy\,f(x,y) = \sum_{x,y=0}^{\infty} xy\,g(x)\,h(y) = \sum_{x=0}^{\infty} x\,g(x) \sum_{y=0}^{\infty} y\,h(y) = E[x]\,E[y]$

چون x و y مستقل اند پس
$f(x,y) = g(x)\,h(y)$

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2) $\sigma^2[(aX+bY+c)^2] = var[(aX+bY+c)^2] =$

$E[(aX+bY+c)^2] - E[aX+bY+c]^2 =$

$= E[a^2X^2+b^2Y^2+c^2+2abXY+2acX+2bcY] - E[aX+bY+c]^2 =$

$= a^2E[X^2]+b^2E[Y^2]+c^2+2abE[XY]+2acE[X]+2bcE[Y]$

$-(aE[X]+bE[Y]+c)^2 = a^2E[X^2]+b^2E[Y^2]+\cancel{c^2}+2abE[XY]$

$+\cancel{2acE[X]}+\cancel{2bcE[Y]} - (a^2E[X]^2+b^2E[Y]^2+\cancel{c^2}+2abE[X]E[Y]$

$+\cancel{2acE[X]}+\cancel{2bcE[Y]}) = a^2(E[X^2]-E[X]^2)+b^2(E[Y^2]-E[Y]^2)$

$+2ab(E[XY]-E[X]E[Y]) = a^2\sigma^2(X)+b^2\sigma^2(Y)$

$+2ab\sigma(XY)$ ✓

3) $\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$

$var[aX-bY] = E[(aX-bY)^2] - E[aX-bY]^2$

$E[a^2X^2+b^2Y^2-2abXY] - (aE[X]-bE[Y])^2 =$

$= a^2 E[X^2] + b^2 E[Y^2] - 2ab\, E[XY] - (a^2 E[X]^2 + b^2 E[Y]^2$

$- 2ab\, E[X]E[Y]) = a^2\sigma^2(x) + b^2\sigma^2(y) - 2ab(E[XY]$

$- E[X]E[Y]) = a^2\sigma^2[X] + b^2\sigma^2[Y] - 2ab(E[X]E[Y]$

$- E[X]E[Y]) = a^2\sigma^2[X] + b^2\sigma^2[Y]$

3- الف) $\mathrm{var}\left(\dfrac{X}{\sigma_X} + \dfrac{Y}{\sigma_Y}\right) = 2 + 2\rho(X,Y)$

$\sigma^2\left[\dfrac{X}{\sigma_X} + \dfrac{Y}{\sigma_Y}\right] \to \sigma^2[aX + bY] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY} =$

$\dfrac{1}{\sigma_X} = a \quad \dfrac{1}{\sigma_Y} = b$

$= \dfrac{1}{\cancel{\sigma_X^2}} \times \cancel{\sigma_X^2} + \dfrac{1}{\cancel{\sigma_Y^2}} \times \cancel{\sigma_Y^2} + 2\dfrac{1}{\sigma_X} \times \dfrac{1}{\sigma_Y} \times \sigma_{XY} =$

$= 2 + 2\,\dfrac{Cov(X,Y)}{\sigma_X \sigma_Y} = 2 + 2\rho(x,Y)$

$\mathrm{Var}\left(\dfrac{X}{\sigma_X} - \dfrac{Y}{\sigma_Y}\right) = 2 - 2\rho(x,Y)$

$\sigma^2\left[\dfrac{X}{\sigma_X} - \dfrac{Y}{\sigma_Y}\right] \to \sigma^2(aX - bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 - 2ab\,\sigma_{XY} =$

$\dfrac{1}{\sigma_X} = a \quad \dfrac{1}{\sigma_Y} = b$

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$= \frac{1}{\cancel{\sigma_X^2}} \cancel{\sigma_X^2} + \frac{1}{\cancel{\sigma_Y^2}} \cancel{\sigma_Y^2} - 2 \frac{1}{\sigma_X} \times \frac{1}{\sigma_Y} \times \cancel{\sigma_{XY}} Cov(X, Y) =$

$= 2 - 2\rho(X, Y)$

$\Rightarrow$ 1) $Var\left[\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right] = 2 + 2\rho(X, Y)$
$Var\left[\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right] \geq 0$ $\Big\} \Rightarrow 2 + 2\rho(X, Y) \geq 0$

$\rightarrow \rho(X, Y) \geq -1$ ①

$Var\left[\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right] = 2 - 2\rho(X, Y)$
$Var\left[\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right] \geq 0$ $\Big\} \Rightarrow 2 - 2\rho(X, Y) \geq 0$

$\rightarrow 1 \geq \rho(X, Y)$ ② ①, ②: $1 \geq \rho(X, Y) \geq -1$

ج) $Y = aX + b$: اگر $\rho(X, Y) = 1$, ب
$a > 0$

$Cov(X, Y) = E[XY] - E[X]E[Y] = E[X(aX + b)] - E[X]E[aX + b]$

$= E[aX^2 + bX] - E[X](aE[X] + b) = aE[X^2] + b\cancel{E[X]} - aE[X]^2 - b\cancel{E[X]}$

$= a\sigma_X^2$

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$X = \frac{Y}{a} - \frac{b}{a}$ $\quad Cov(x,y) = E[Y(\frac{Y}{a} - \frac{b}{a})] - E[Y]E[\frac{Y}{a} - \frac{b}{a}]$

$= E[\frac{Y^2}{a} - \frac{Yb}{a}] - E[Y](\frac{E[Y]}{a} - \frac{b}{a}) =$

$= \frac{E[Y^2]}{a} - \cancel{\frac{b}{a}E[Y]} - \frac{E[Y]^2}{a} + \cancel{\frac{b}{a}E[Y]} =$

$= \frac{\sigma_y^2}{a} \rightarrow a\sigma_X^2 = \frac{\sigma_y^2}{a} \rightarrow a^2\sigma_X^2 = \sigma_Y^2$

$\rightarrow \boxed{a\sigma_X = \sigma_Y}$

$Var[\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}] = Var[\frac{X}{\sigma_X} - \frac{aX+b}{a\sigma_X}] = Var[\frac{X}{\sigma_X} - \frac{X}{\sigma_X} - \frac{b}{a\sigma_X}]$

$= Var[\frac{-b}{a\sigma_X}] = Var[$ ثابت عدد $] = 0 \rightarrow 2 - 2\rho(X,Y) = 0$

$\rightarrow \boxed{\rho(X,Y) = 1}$

2-) $Y = aX + b$ ⇐ $\rho(X,Y) = 1$ ، فرض

$var(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}) = 2 - 2\rho(X,Y) = 0 \rightarrow \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = C$ عدد ثابت

$\frac{Y}{\sigma_Y} = \frac{X}{\sigma_X} - C \rightarrow Y = \underbrace{\frac{\sigma_Y}{\sigma_X}}_{a} X \underbrace{- C\sigma_Y}_{b} \rightarrow \boxed{Y = aX + b}$

$a = \frac{\sigma_Y}{\sigma_X}$

$\sigma_Y = \sqrt{var(Y)}$

$\sigma_X = \sqrt{var(X)}$

$\rightarrow \sigma_Y > 0, \ \sigma_X > 0 \rightarrow \boxed{a > 0}$

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4 _ $P(\underbrace{x>y}_{A}) = 0.7 \quad P(\underbrace{y>z}_{B}) = 0.7 \quad P(\underbrace{z>x}_{C}) = 0.7$: فرض

$P(A \cap B) = P(x>y \cap y>z) = P(x>z) = 0.7 \times 0.7 = 0.49$

$\rightarrow P(x>z) = 0.49 \Rightarrow P(C') + P(C) > 1 \rightarrow$ تناقض

$P(z>x) = 0.7$

5 _ الف) $f(k;p) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$

$E[X] = 1 \times p + 0 \times (1-p) = p$

$var[X] = E[X^2] - E[X]^2 = 1^2 \times p + 0^2 \times (1-p) - (1 \times p + 0(1-p))^2 =$

$= p - p^2 = p(1-p)$

ب) $Pr(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np$

$var[X] = E[X^2] - E[X]^2 = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} -$

$- \left(\sum_{x=0}^{n} x\, p^x (1-p)^{n-x}\right)^2 = npq$