تمرین ۳ ۹۶۳۱۰۷۵ علی نظری

(1)

a) $x(t) = \sum_{k=-\infty}^{\infty} 3\delta(2t - kT_0)$

نوعی قطار ضربه است.

$$T = \frac{T_0}{2} \Rightarrow \omega_0 = \frac{2\pi}{\frac{T_0}{2}} = \frac{4\pi}{T_0}$$

$$a_i = \frac{1}{\frac{T_0}{2}} \int_{-\frac{T_0}{4}}^{+\frac{T_0}{4}} \underbrace{\left(\sum_{k=-\infty}^{\infty} 3\delta(2t - kT_0) \right)}_{-\frac{T_0}{4} < t < \frac{T_0}{4} \Rightarrow t = 0} e^{-ji\omega_0 t} \, dt$$

$$= \frac{2}{T_0} \times 3 = \frac{6}{T_0} \Rightarrow a_i = \frac{6}{T_0}$$

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b) $x(t) = \cos\left(\frac{3}{4}t\right)\left(3 + 2\cos t\right) = 3\cos\left(\frac{3t}{4}\right) + \cos\left(\frac{7t}{4}\right) + \cos\left(\frac{t}{4}\right)$

$$= \frac{3}{2}\left(e^{\frac{3}{4}tj} + e^{-\frac{3}{4}tj}\right) + \frac{1}{2}\left(e^{\frac{7}{4}tj} + e^{-\frac{7}{4}tj} + e^{\frac{t}{4}j} + e^{-\frac{t}{4}j}\right)$$

$$= \frac{1}{2}e^{j(-7)\frac{t}{4}} + \frac{3}{2}e^{j(-3)\frac{t}{4}} + \frac{1}{2}e^{j(-1)\frac{t}{4}} + \frac{1}{2}e^{j\frac{t}{4}} + \frac{3}{2}e^{j(3)\frac{t}{4}}$$

$$+ \frac{1}{2}e^{j(7)\frac{t}{4}} \Rightarrow a_k = \begin{cases} \frac{1}{2} & k = \pm 1 , \pm 7 \\ \frac{3}{2} & k = \pm 3 \\ 0 & \text{o.w} \end{cases}$$

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c) $x(t) = e^{j\frac{\pi}{4}t}\left(\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{6}t + \frac{\pi}{4}\right)\right)$

$$= e^{j\frac{\pi}{4}t}\left(\frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{2} + \frac{e^{j(\frac{\pi}{6}t+\frac{\pi}{4})} - e^{-j(\frac{\pi}{6}t+\frac{\pi}{4})}}{2j}\right)$$

$$= \frac{1}{2}e^{-j\frac{3\pi}{12}t} - \frac{e^{-\frac{\pi}{4}j}}{2j}e^{j\frac{\pi}{12}t} + \frac{e^{\frac{\pi}{4}j}}{2j}e^{j\frac{5\pi}{12}t} + \frac{1}{2}e^{j\frac{9\pi}{12}t}$$

$$\Rightarrow \begin{cases} a_{-3} = \frac{1}{2} \\ a_1 = -\frac{e^{-\frac{\pi}{4}j}}{2j} \end{cases} \quad \begin{cases} a_5 = \frac{e^{\frac{\pi}{4}j}}{2j} \\ a_9 = \frac{1}{2} \end{cases} \quad a_{\text{بقیه}} = 0$$

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d) $x(t) = \cos(2\pi t) + 7\sin(t)$

$\frac{2\pi}{2\pi} = 1$, $\to 2\pi$

نسبت به هم گویا نیستند $\Rightarrow$ غیر متناوب $\Rightarrow$ سری فوریه ندارد

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e) $x[n] = e^{j\frac{\pi}{3}n}\cos\left(\frac{\pi}{8}n\right) = \frac{1}{2}e^{j\frac{5\pi}{24}n} + \frac{1}{2}e^{j\frac{11\pi}{24}n}$

$T = 48 \to 0 \leq k \leq 47$

$\Rightarrow a_5 = a_{11} = \frac{1}{2}$, $a_{\text{بقیه}} = 0$ → در یک دوره تناوب

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f) $x[n] = e^{j\frac{4\pi}{5}n}$ $\qquad T = 5 \to 0 \leq k \leq 4$

$= e^{j(2)(\frac{2\pi}{5})n} \Rightarrow a_2 = 1$, $a_{\text{بقیه}} = 0$

در یک دوره تناوب

g) $x[n] = \sin(\frac{\pi}{2}n) + \cos(\frac{3\pi}{5}n)$ $\qquad$ ک.م.م $= (4, 10) = 20 = T$ $\rightarrow$ $0 \leq k \leq 19$

$\frac{2\pi}{\frac{\pi}{2}} = 4 = T_1$

$\frac{2\pi}{\frac{3\pi}{5}} = \frac{10}{3} \rightarrow T_2 = 10$

$$x[n] = \frac{e^{j\frac{5\pi}{10}n} - e^{-j\frac{5\pi}{10}n}}{2j} + \frac{e^{j\frac{6\pi}{10}n} + e^{-j\frac{6\pi}{10}n}}{2}$$

$$= \frac{1}{2}e^{-j\frac{6\pi}{10}n} - \frac{1}{2j}e^{-j\frac{5\pi}{10}n} + \frac{1}{2j}e^{j\frac{5\pi}{10}n} + \frac{1}{2}e^{j\frac{6\pi}{10}n}$$

$$\Rightarrow \begin{cases} a_{-6} = a_{-6+20} = a_{14} = \frac{1}{2} & a_5 = \frac{1}{2j} \\ a_{-5} = a_{-5+20} = a_{15} = \frac{-1}{2j} & a_6 = \frac{1}{2} \end{cases} \quad , a_{\text{بقیه}} = 0$$

در دوره تناوب

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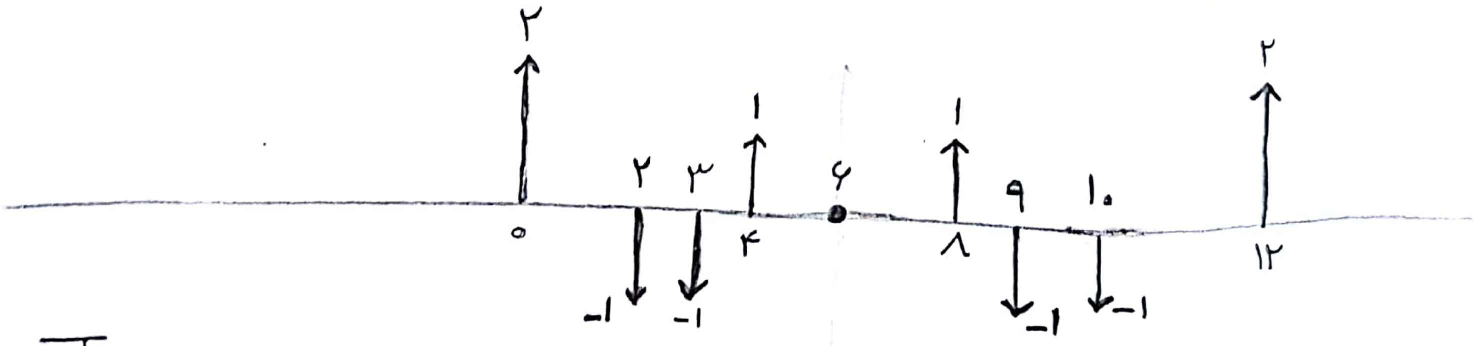
h) $x[n] = \sum_{m=-\infty}^{\infty} (-1)^m \left[\delta[n-2m] + \delta[n+3m]\right]$



$T = 12$

(۲)

$$\left.\begin{aligned} P(t) &= (x(t))^2 = x(t) \times x^*(t) \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \end{aligned}\right\} \Rightarrow P(t) = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\Rightarrow P(t) = \sum_{k=-\infty}^{\infty} 2^{-2|k|} \underset{\text{زوج}}{=} \left(2\sum_{k=1}^{\infty} 2^{-2k}\right) + 1$$

$$= 2\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right) + 1 = \frac{2}{3} + 1 = \frac{5}{3}$$

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(۳)

$$P[n] = \text{طبق پارسوال} = \frac{1}{N}\sum_{k=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

$$x[n] = \cos^2\left(\frac{n\pi}{10}\right) = \frac{1+\cos\left(\frac{n\pi}{5}\right)}{2} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{n\pi}{5}\right)$$

$$\Rightarrow \frac{2\pi}{\frac{\pi}{5}} = 10 = N$$

$$x[n] = \frac{1}{2} + \frac{1}{4}\left(e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}\right) = \frac{1}{4}e^{-j\frac{\pi}{5}n} + \frac{1}{2} + \frac{1}{4}e^{j\frac{\pi n}{5}}$$

$$a_{-1} = a_9 = \frac{1}{4}, \quad a_0 = \frac{1}{2}, \quad a_1 = \frac{1}{4}$$

$$\Rightarrow P[n] = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{3}{8}$$

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(۴)

$$b_k = \begin{cases} 1+j^k & 1 \leqslant k \leqslant 3 \\ j^0 = 1 & k = 0 \end{cases}$$

$$\Rightarrow P[n] = \sum_{k=\langle 4\rangle} |b_k|^2 = 1 + (\sqrt{2})^2 + (1-1)^2 + (\sqrt{2})^2$$

$$= 1 + 2 + 0 + 2 = 5 \Rightarrow P[n] = 5$$

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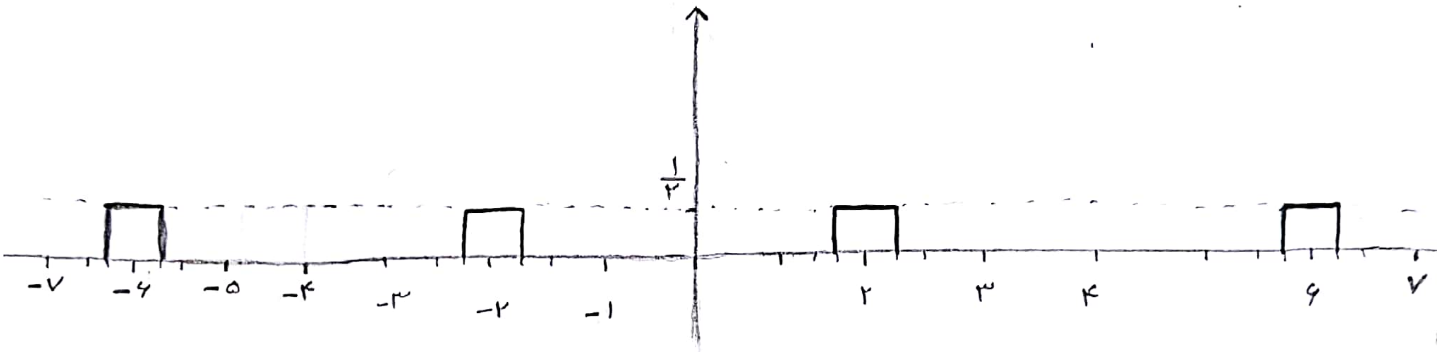
$$a_k = \frac{(-1)^k}{۱۶} \frac{\sin \frac{k\pi}{۸}}{\frac{k\pi}{۸}} = \frac{(-1)^k}{۱۶} \text{Sinc}\left(\frac{k}{۸}\right)$$

$$\text{duty cycle} = \frac{1}{۸} = d \Rightarrow a_k = \frac{(-1)^k}{۲} d \,\text{Sinc}(kd)$$

$$\omega_0 = \frac{\pi}{۲} = \frac{۲\pi}{T} \Rightarrow T = ۴ \Rightarrow d = \frac{۲T_1}{T} \Rightarrow T_1 = \frac{1}{۴}$$

$$a_k = \frac{1}{۲} \underbrace{e^{-jk\pi}}_{e^{-j\frac{۲\pi}{۴}k(۲)}} \text{Sinc}(kd)$$

$\longrightarrow$ ۲ واحد شیفت زمانی



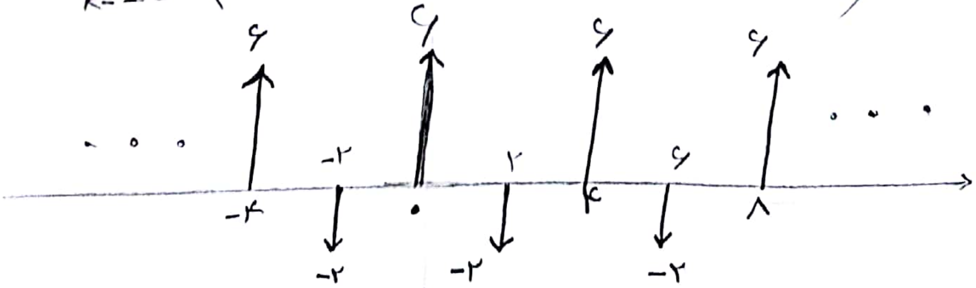
$\{a_k$ ثابت $\Rightarrow$ قطار ضربه $\displaystyle\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{FS} \frac{1}{T}$ ۶

$$a_k = b_k - c_k \longleftrightarrow x(t) = y(t) - z(t)$$

$$b_k = ۲ \Rightarrow y(t) = ۸\sum_{k=-\infty}^{\infty} \delta(t-۴k)$$

$$c_k = \begin{cases} ۱ & \text{زوج } k \\ ۰ & \text{فرد } k \end{cases} \Rightarrow z(t) = ۲\sum_{k=-\infty}^{\infty} \delta(t-۲k)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} \left(۸\delta(t-۴k) - ۲\delta(t-۲k) \right)$$



$$y(t)=\sum_{k=-\infty}^{\infty} a_k H(k\omega_0)e^{jk\omega_0 t} \qquad T=4 \Rightarrow \omega_0=\frac{\pi}{2} \qquad (7)$$

$$a_k=\frac{1}{T}\int_{-1}^{3} x(t)e^{-jk\omega_0 t}=\frac{1}{4}\left(\int_{-1}^{-\frac{1}{2}} -e^{-j\frac{k\pi}{2}t}+\int_{-\frac{1}{2}}^{\frac{1}{2}} 2te^{-j\frac{k\pi}{2}t}\right.$$

$$\left.+\int_{\frac{1}{2}}^{1} e^{-j\frac{k\pi}{2}t}\right)=\frac{1}{2k\pi j}e^{-j\frac{k\pi}{2}t}\Big|_{-1}^{-\frac{1}{2}}$$

$$-\frac{1}{2}\left(\frac{te^{-j\frac{k\pi}{2}t}}{j\frac{k\pi}{2}}+\frac{e^{-j\frac{k\pi}{2}t}}{(j\frac{k\pi}{2})^2}\right)\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}}-\frac{1}{2k\pi j}e^{-j\frac{k\pi}{2}t}\Big|_{\frac{1}{2}}^{1}$$

$$=\frac{e^{j\frac{k\pi}{4}}-e^{j\frac{k\pi}{2}}}{2k\pi j}-\frac{\cancel{e^{-j\frac{k\pi}{4}}}-e^{j\frac{k\pi}{4}}}{2k\pi j}+2\,\frac{e^{-j\frac{k\pi}{4}}-e^{j\frac{k\pi}{4}}}{k^2\pi^2}$$

$$-\frac{e^{-j\frac{k\pi}{2}}-\cancel{e^{-j\frac{k\pi}{4}}}}{2k\pi j}=\frac{-2\cos(\frac{k\pi}{2})+2\sin(\frac{k\pi}{4})}{2k\pi j}$$

$$\frac{-2\sin(\frac{k\pi}{4})}{k^2\pi^2}=\frac{-1}{k\pi j}\left(\cos(\frac{k\pi}{2})-\sin(\frac{k\pi}{4})+\frac{2\sin(\frac{k\pi}{4})}{k\pi j}\right)$$

$$b_k=H(k\omega_0)a_k=H(\frac{k\pi}{2})a_k \longrightarrow -1\leq\frac{k\pi}{2}\leq 1 \Rightarrow k=0$$

$$y(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Leftarrow b_k=a_k \Leftarrow H(\frac{k\pi}{2})=1 \hookleftarrow$$

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(۸) خیلی از فرکانس ها توسط این فیلتر به صفر نگاشت می‌شوند

$\Leftarrow$ سیستم وارون پذیر نیست.

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$$b_k = \underbrace{j^k}_{c_k} \; \underbrace{\frac{\sin\frac{k\pi}{۴}}{k\pi}}_{d_k}$$

$$\begin{cases} a_k \longleftrightarrow x(t) \\ b_k \longleftrightarrow y(t) \\ d_k \longleftrightarrow z(t) \end{cases}$$

$$\begin{cases} d_k = \dfrac{\sin\frac{k\pi}{۴}}{k\pi} = \dfrac{1}{۴} \operatorname{Sin}\left(\dfrac{k}{۴}\right) = d \operatorname{Sinc}(kd) \,,\; d = d_0 = \dfrac{1}{۴} \\ d = \dfrac{۲T_1}{T} = \dfrac{۲T_1}{۴} = \dfrac{1}{۴} \Rightarrow T_1 = \dfrac{1}{۲} \end{cases}$$

$$\left\{ c_k = j^k = \left(j \sin\left(\frac{\pi}{۲}\right)\right)^k = e^{j\frac{k\pi}{۲}} \,,\quad \omega_0 = \frac{\pi}{۲} \right.$$

$$y(t) = z(t-1) \longleftrightarrow b_k \,,\quad b_0 = d_0 = \frac{1}{۴}$$

$$\Rightarrow \begin{cases} x(t) = y(t) - \dfrac{1}{۴} = z(t-1) - \dfrac{1}{۴} \\ a_0 = 0 \end{cases}$$

$z(t)$ ← موج مربعی با دوره تناوب $T = ۴$ ، duty cycle $= \frac{1}{۴}$

⇐ $x(t)$ هم انتقال یافته $z(t)$ است ، در نتیجه: