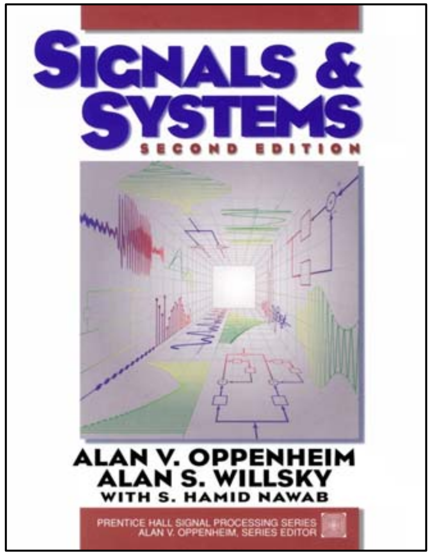
Spring 2011

信號與系統
Signals and Systems

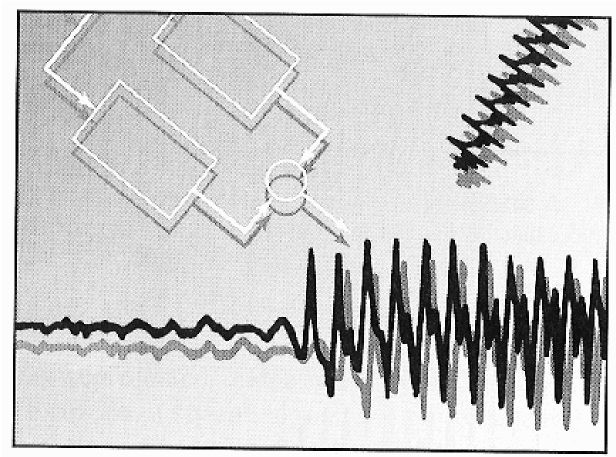
Chapter SS-1
Signals and Systems



Feng-Li Lian

NTU-EE

Feb11 – Jun11

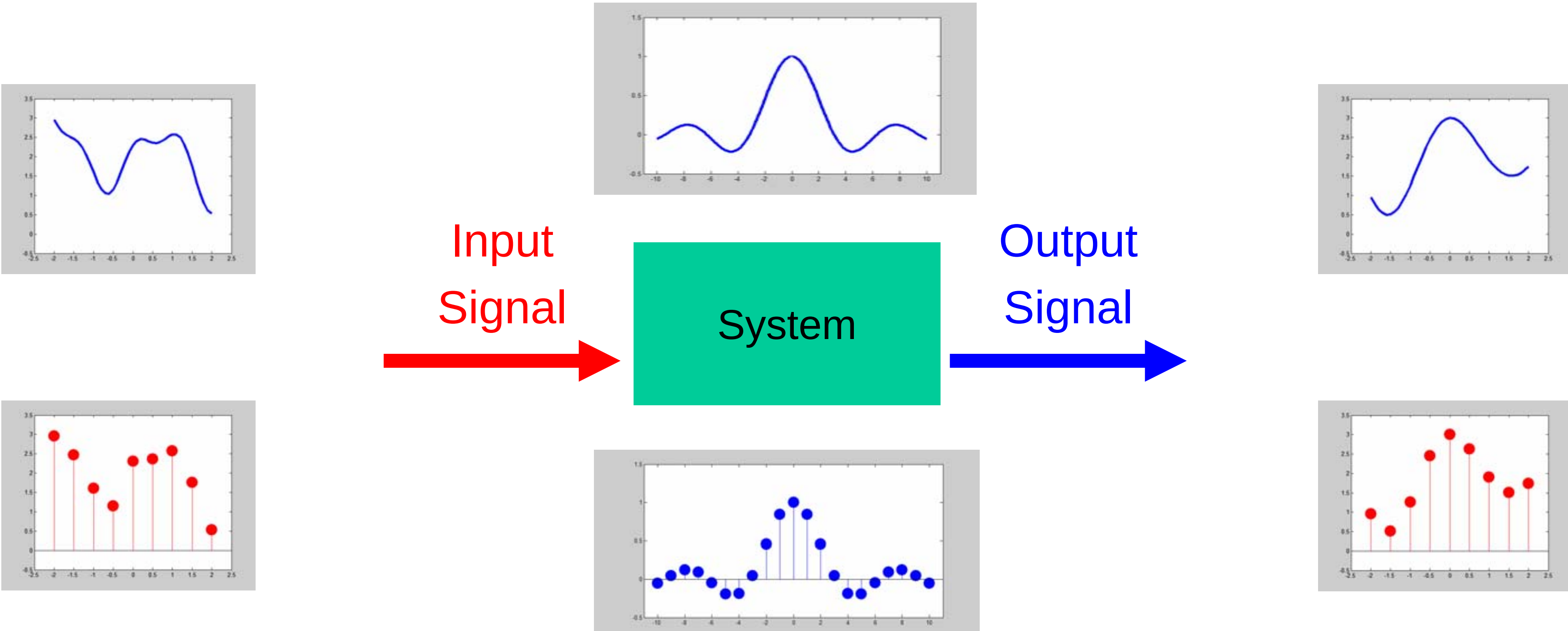


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

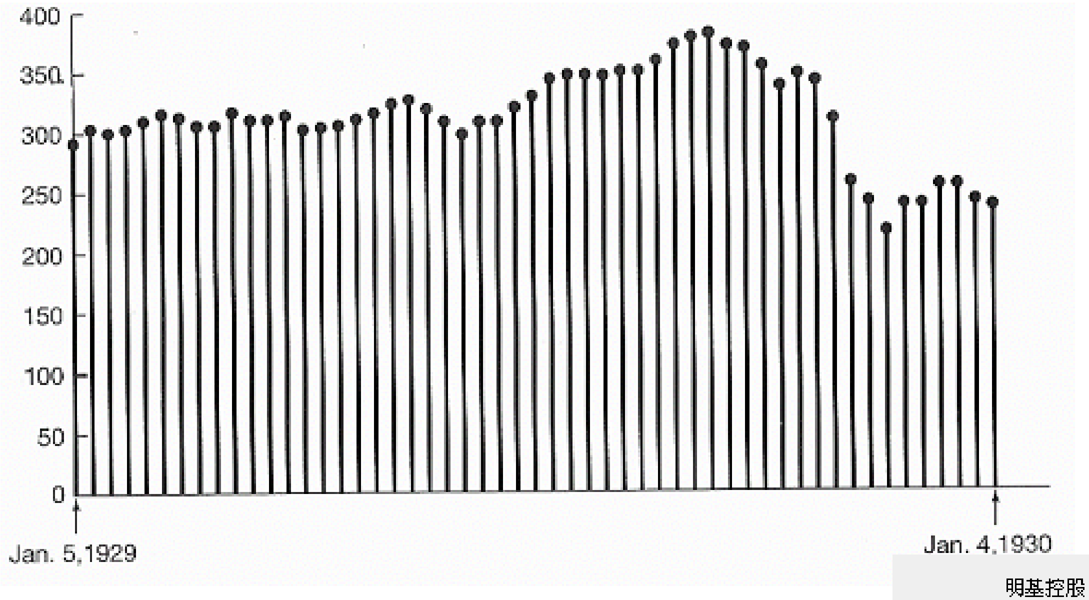
- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

Signals & Systems:

- Is about using mathematical techniques to help describe and analyze systems which process signals
  - Signals are variables that carry information
  - Systems process input signals to produce output signals

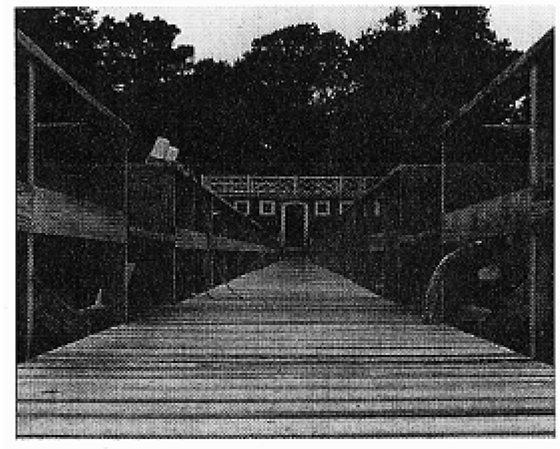


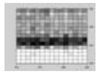
- **Discrete-Time Signals:**
  - The weekly Dow-Jones stock market index

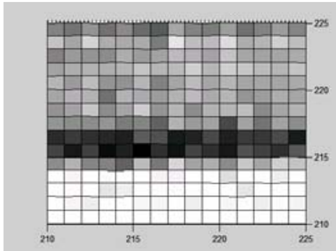


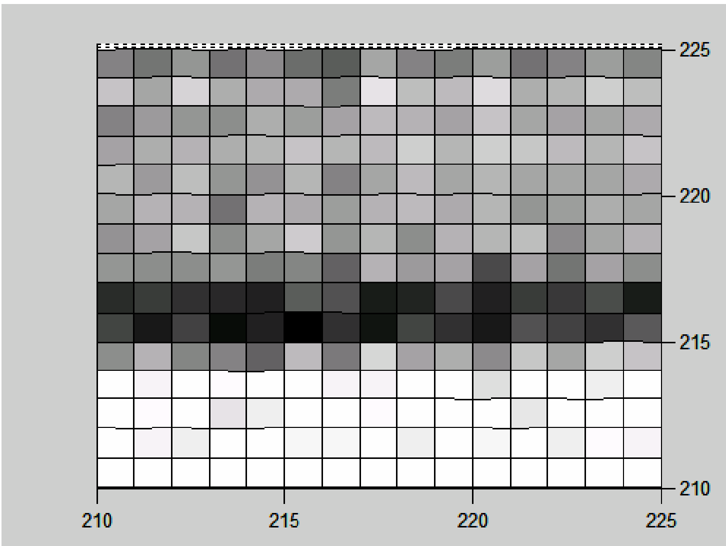
http://big5.jrj.com.cn/

- Discrete-Time Signals:
  - A monochromatic picture



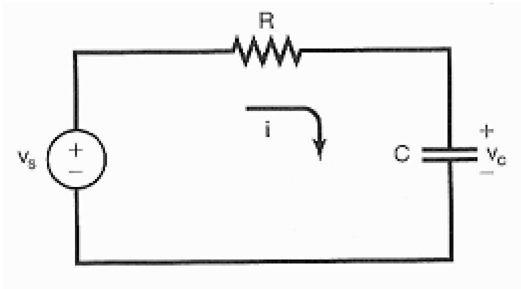




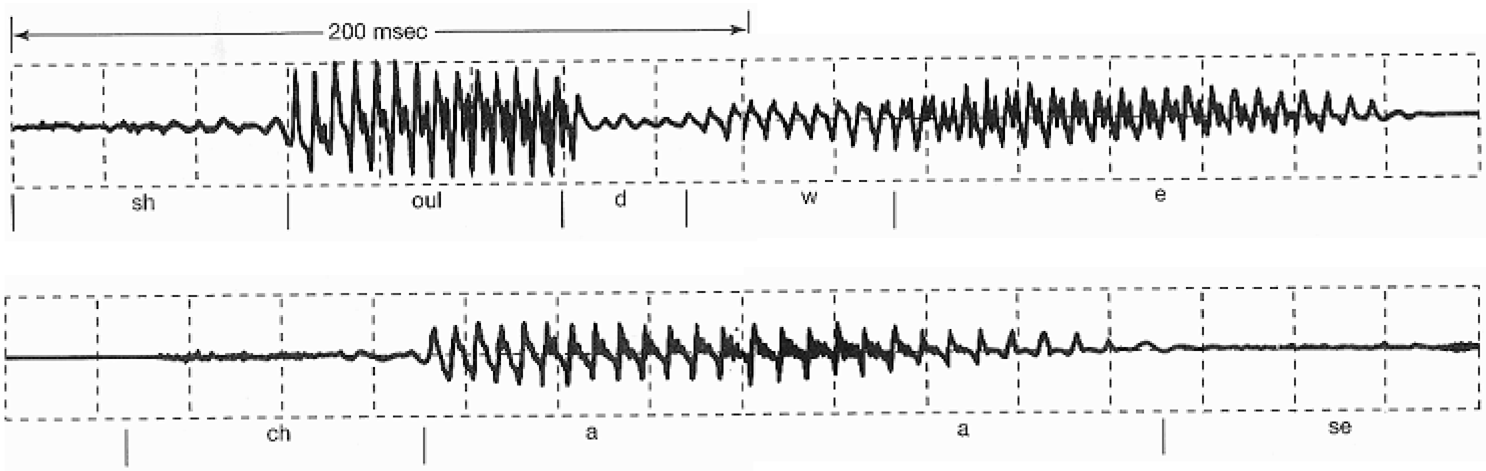


117	151	112	136	110	95	166	131	125	159	114	130	159	133
164	209	173	171	168	126	225	189	184	219	172	180	206	188
154	151	143	174	156	160	186	179	163	193	167	161	167	168
173	179	172	182	195	181	187	205	183	205	196	186	180	193
155	188	149	145	181	131	166	185	166	181	164	164	166	171
176	176	113	178	169	159	177	184	171	182	148	159	174	164
162	198	141	164	202	149	181	142	176	182	189	136	165	176
143	143	148	127	132	97	177	152	160	74	163	119	162	143
62	51	40	32	95	82	28	39	75	35	60	58	77	28
26	67	14	34	3	49	22	69	48	26	81	67	49	91
176	134	129	97	185	120	212	160	173	139	197	166	207	192
243	255	250	255	254	242	242	255	254	222	255	253	239	252
248	255	226	238	255	255	249	255	255	255	230	252	255	255
243	238	255	253	245	247	255	238	255	244	255	237	249	241

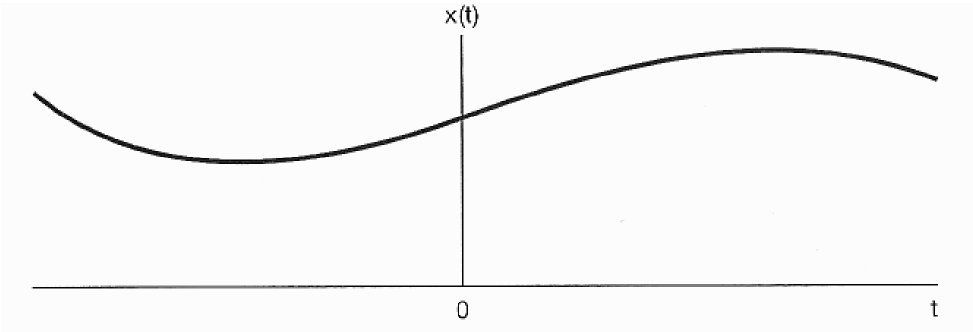
- Continuous-Time Signals:
  - Source voltage & capacity voltage in a simple RC circuit



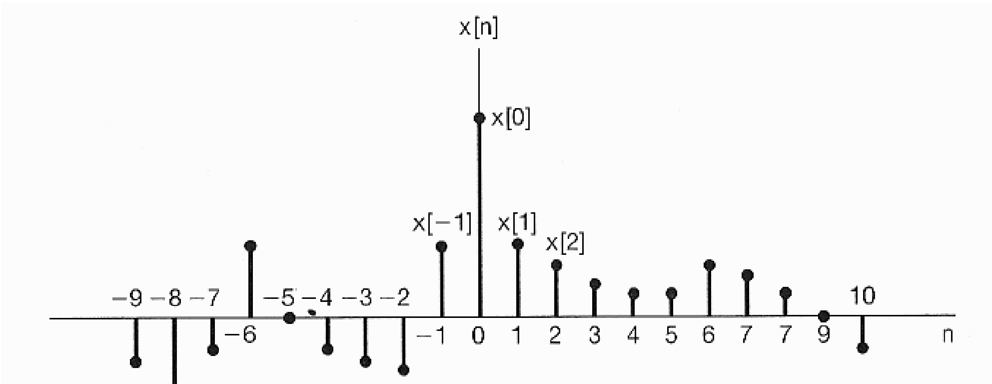
  - Recording of a speech signal



- Graphical Representations of Signals:
  - Continuous-time signals $x(t)$ or $x_c(t)$

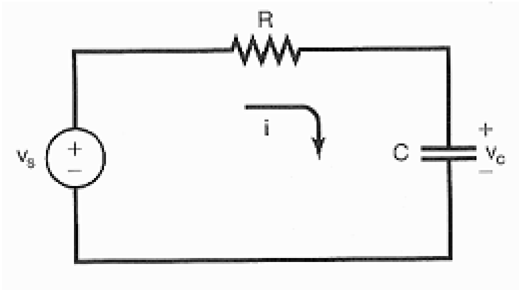


  - Discrete-time signals $x[n]$ or $x_d[n]$



(b)

- Energy & Power of a resistor:



  - Instantaneous power

$$p(t) = v(t)i(t) = \frac{1}{R}v^2(t)$$

  - Total energy over a finite time interval

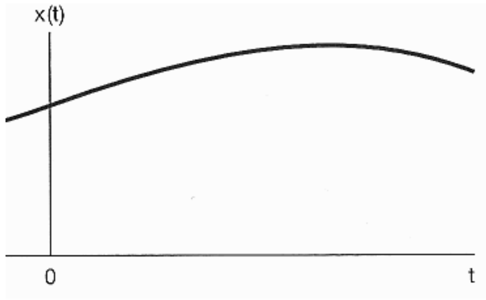
$$\int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

  - Average power over a finite time interval

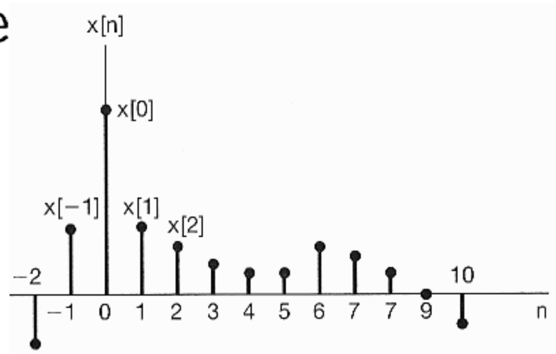
$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p(t)dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{1}{R}v^2(t)dt$$

■ Signal Energy & Power:

- Total energy over a finite time interval



$$E \triangleq \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$



$$E \triangleq \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

- Time-averaged power over a finite time interval

$$P \triangleq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt \qquad \text{continuous-time}$$

$$P \triangleq \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2 \qquad \text{discrete-time}$$

Signal Energy & Power:

- Total energy over an infinite time interval

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Time-averaged power over an infinite time interval

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[t]|^2$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Three Classes of Signals:

- Finite total energy & zero average power

$$0 \le E_\infty < \infty \quad \Rightarrow \quad P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$

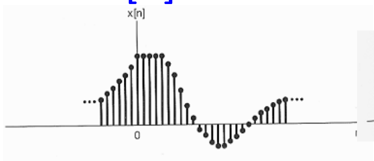
- Finite average power & infinite total energy

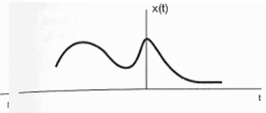
$$0 \le P_\infty < \infty \quad \Rightarrow \quad E_\infty = \infty \ \ (\text{if } P_\infty > 0)$$

- Infinite average power & infinite total energy

$$P_\infty = \infty \quad \& \quad E_\infty = \infty$$

$x[n]$ $x(t)$





- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift
  - Time Reversal
  - Time Scaling
  - Periodic Signals
  - Even & Odd Signals
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
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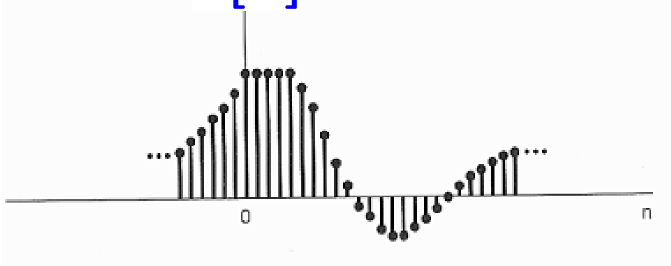
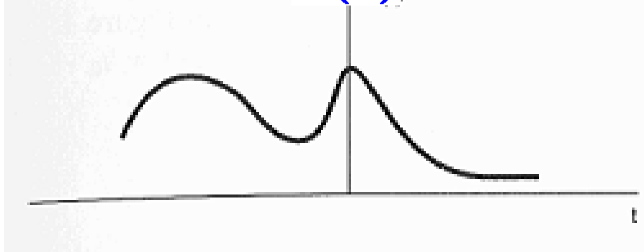
- Time Shift:

$$\begin{cases} n_0,\ t_0 > 0: & \text{delay} \\ n_0,\ t_0 < 0: & \text{advance} \end{cases}$$

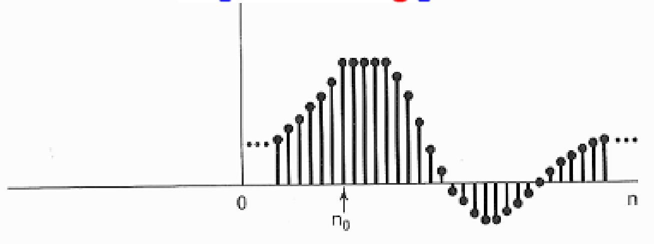
$x[n]$



$x(t)$



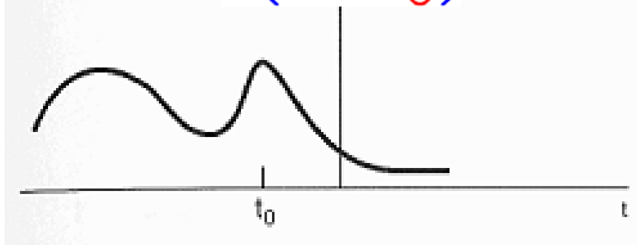
$x[n-n_0]$



$n_0 > 0$

$x[n-8]$

$x(t-t_0)$

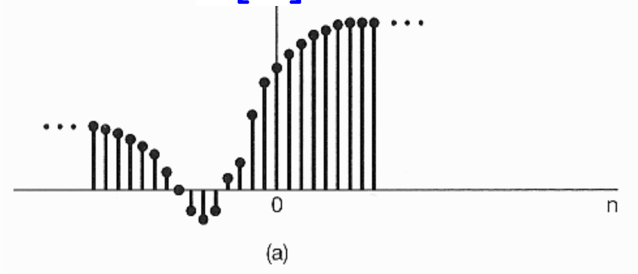


$t_0 < 0$

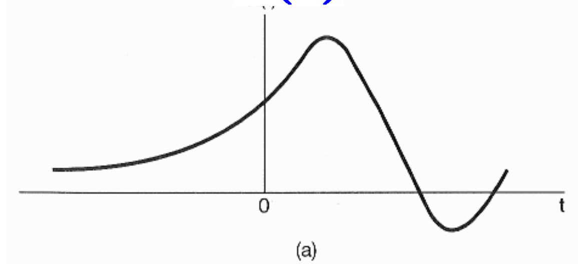
$x(t+5)$

▪ Time Reversal:

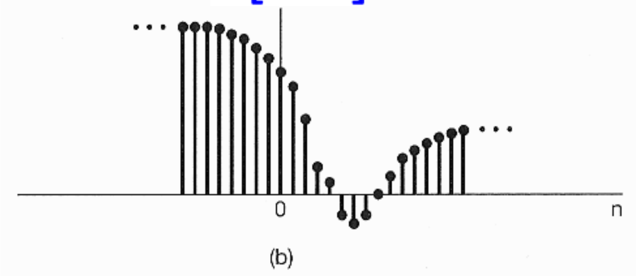
$x[n]$



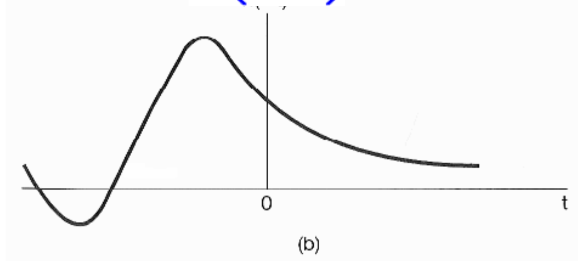
$x(t)$



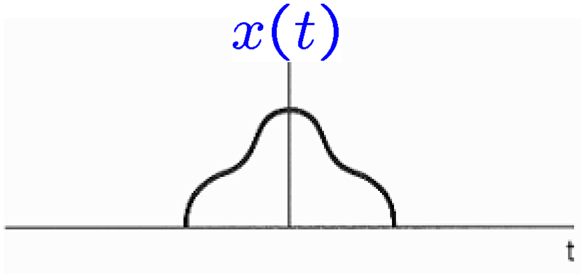
$x[-n]$



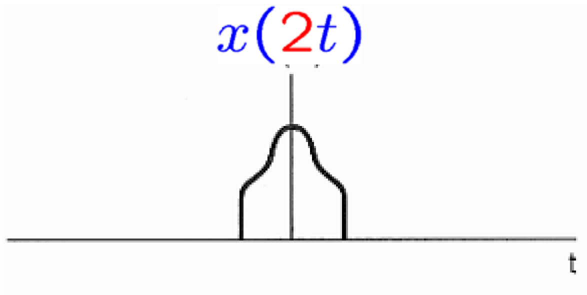
$x(-t)$



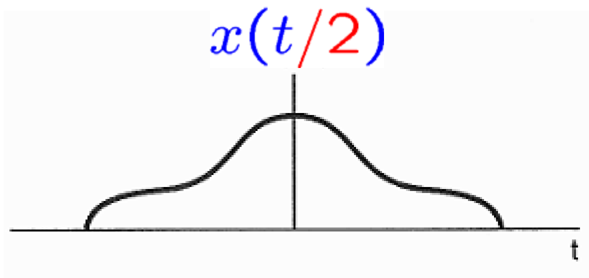
- Time Scaling:



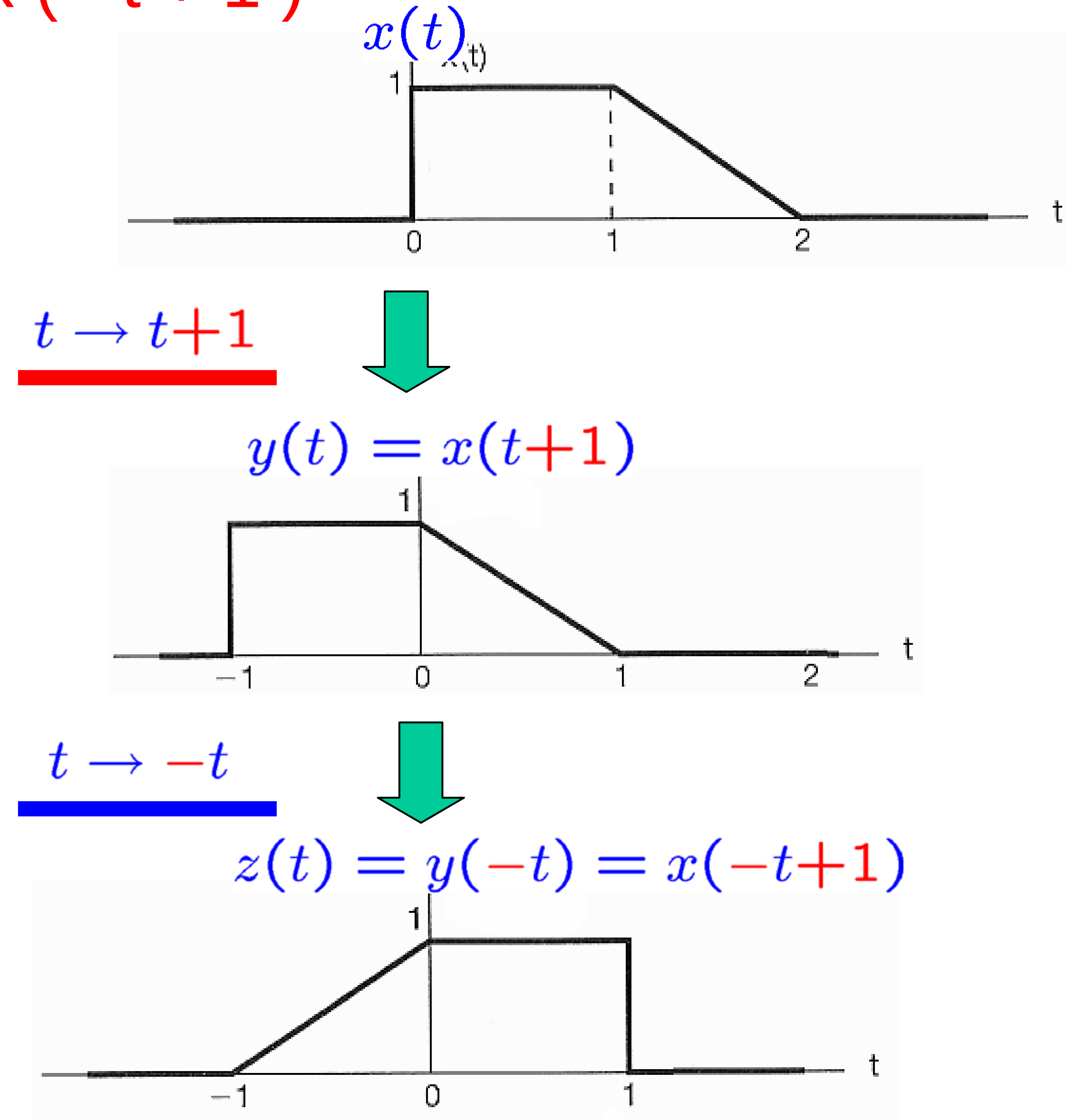
$t \longrightarrow 2t$



$t \longrightarrow t/2$

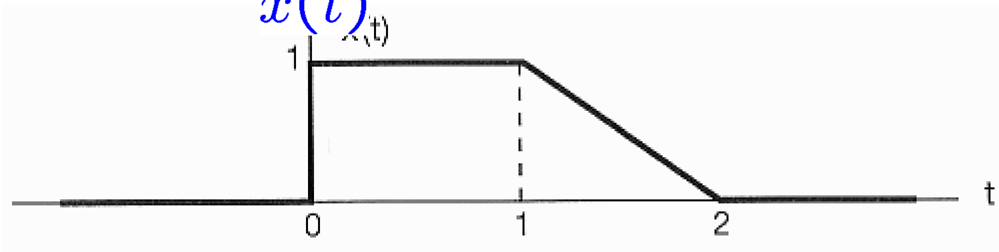


- $x(t) \rightarrow x(-t+1)$



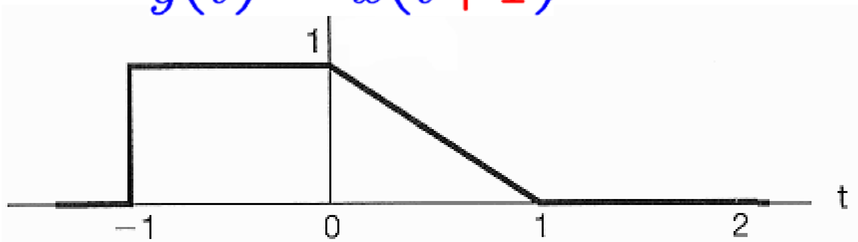
- $x(t) \rightarrow x(-t+1)$

$x(t)$



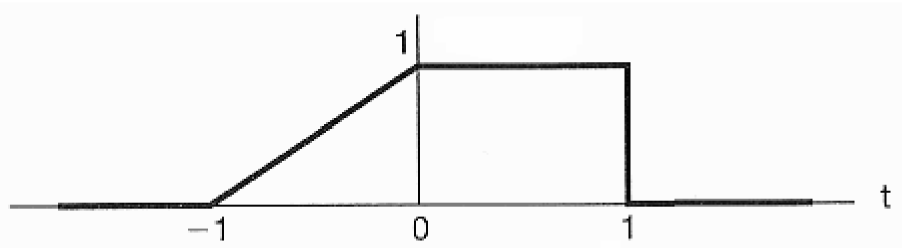
$t \rightarrow t+1$

$y(t) = x(t+1)$



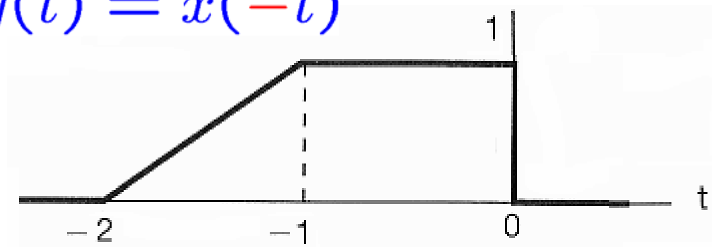
$t \rightarrow -t$

$z(t) = y(-t) = x(-t+1)$



$t \rightarrow -t$

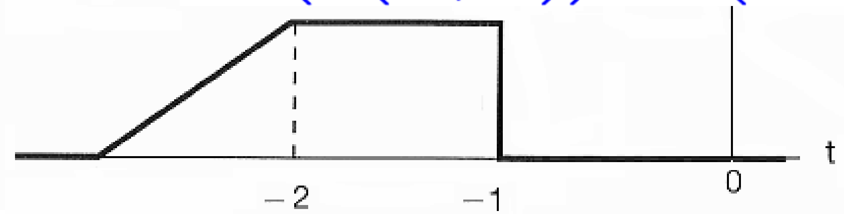
$y(t) = x(-t)$



$t \rightarrow t+1$

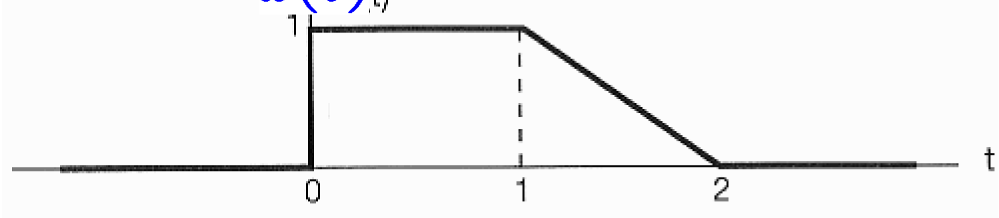
$z(t) = y(t+1)$

$= x(-(t+1)) = x(-t-1)$



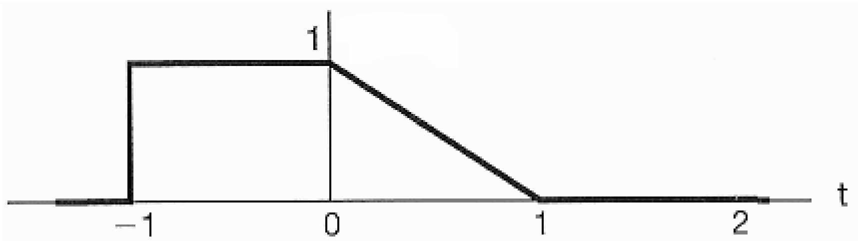
- $x(t) \rightarrow x(3/2\,t + 1)$

$x(t)$



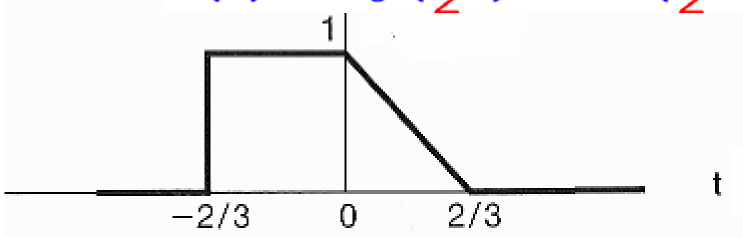
$t \rightarrow t+1$

$y(t) = x(t+1)$



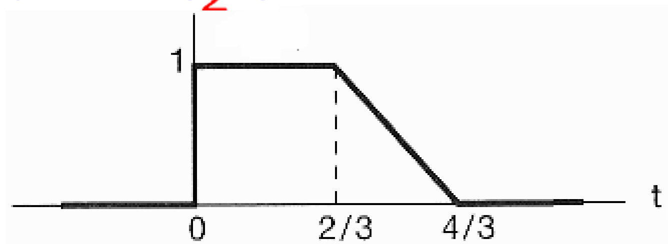
$t \rightarrow \frac{3}{2}t$

$z(t) = y(\frac{3}{2}t) = x(\frac{3}{2}t+1)$



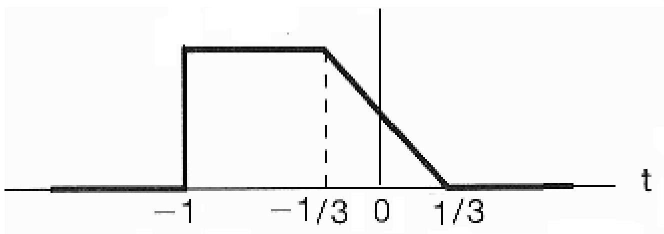
$t \rightarrow \frac{3}{2}t$

$y(t) = x(\frac{3}{2}t)$



$t \rightarrow t+1$

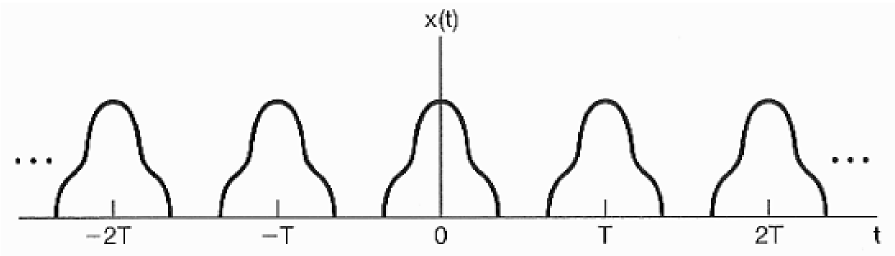
$z(t) = y(t+1) = x(\frac{3}{2}t+\frac{3}{2})$

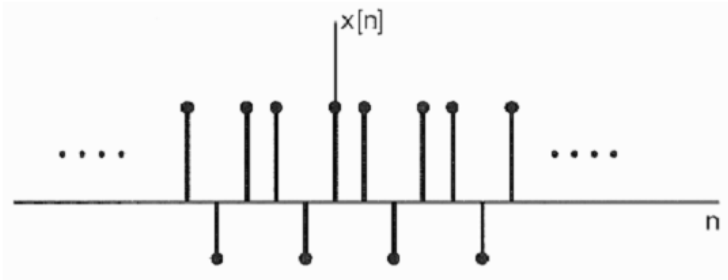


- $x(t) \rightarrow x(at - b)$
  - $|a| < 1$: linearly stretched
  - $|a| > 1$: linearly compressed
  - $a < 0$: time reversal
  - $b > 0$: delayed time shift
  - $b < 0$: advanced time shift

- Problems:
  - P1.21 for CT
  - P1.22 for DT

▪ CT & DT Periodic Signals:





$N = 3$

$$x(t) = x(t + T) \quad \text{for } T > 0 \text{ and all values of } t$$

$$x[n] = x[n + N] \quad \text{for } N > 0 \text{ and all values of } n$$

▪ Periodic Signals:

$$x(t) = x(t+T) \quad \text{for } T > 0 \text{ and all values of } t$$

$$x[n] = x[n+N] \quad \text{for } N > 0 \text{ and all values of } n$$

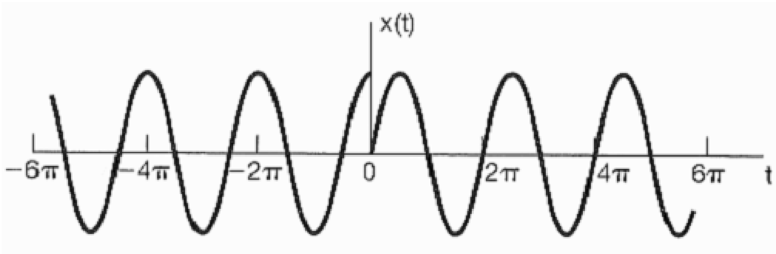
- A periodic signal is unchanged by a time shift of T or N
- They are also periodic with period
  - 2T, 3T, 4T, …
  - 2N, 3N, 4N, …
- T or N is called the fundamental period

  denoted as T_0 or N_0

- Periodic signal ?

$$x(t) = x(t+T) \quad \forall t, \quad T > 0$$

$$x(t) = \begin{cases} \cos(t), & \text{if } t < 0 \\ \sin(t), & \text{if } t \geq 0 \end{cases}$$

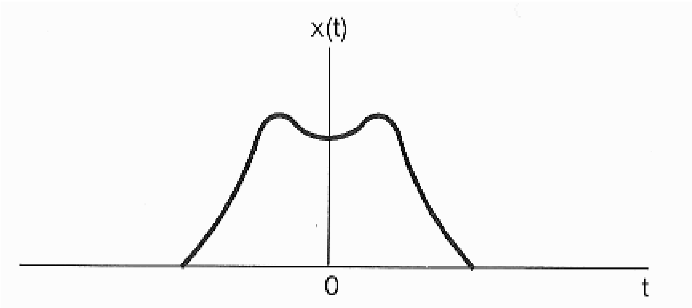


- Problems:
    - P1.25 for CT
    - P1.26 for DT

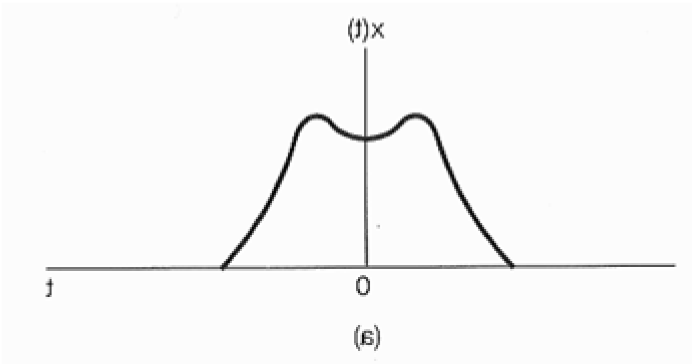
▪ Even & odd signals:

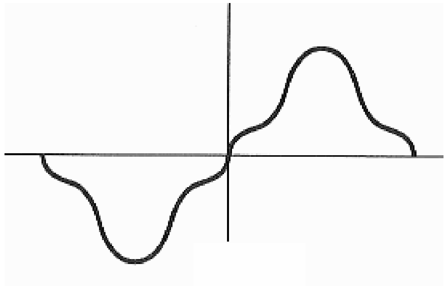
A signal is even if $x(-t) = x(t)$ or $x[-n] = x[n]$

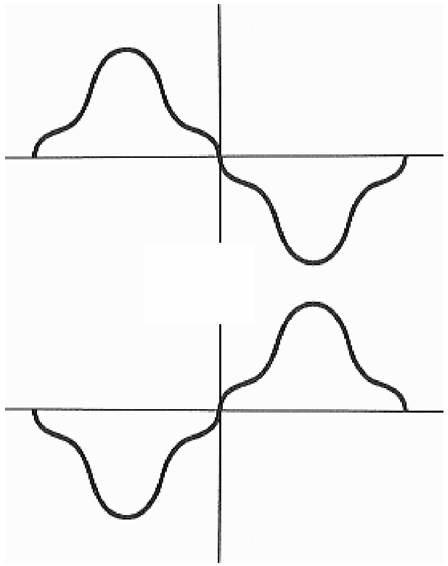
A signal is odd if $x(-t) = -x(t)$ or $x[-n] = -x[n]$



(a)







Even-odd decomposition of a signal:

- Any signal can be broken into a sum of one even signal and one odd signal

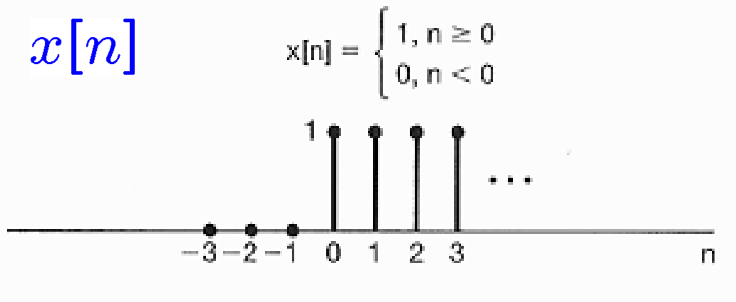
$$\mathcal{E}v\left\{x(t)\right\} = \frac{1}{2}\left[\, x(t) + x(-t) \,\right] = \frac{1}{2}\left[\, x(-t) + x(t) \,\right]$$

$$\mathcal{O}d\left\{x(t)\right\} = \frac{1}{2}\left[\, x(t) - x(-t) \,\right] = -\frac{1}{2}\left[\, x(-t) - x(t) \,\right]$$

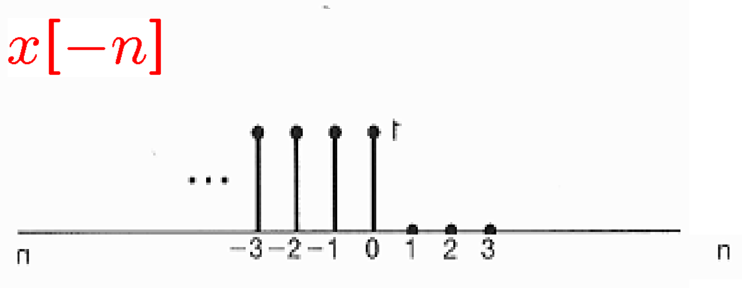
$$\Rightarrow x(t) = \mathcal{E}v\left\{x(t)\right\} + \mathcal{O}d\left\{x(t)\right\}$$

Even-odd decomposition of a DT signal:

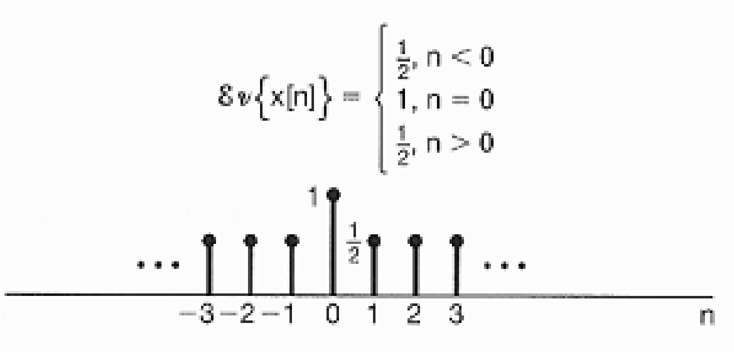
$x[n]$



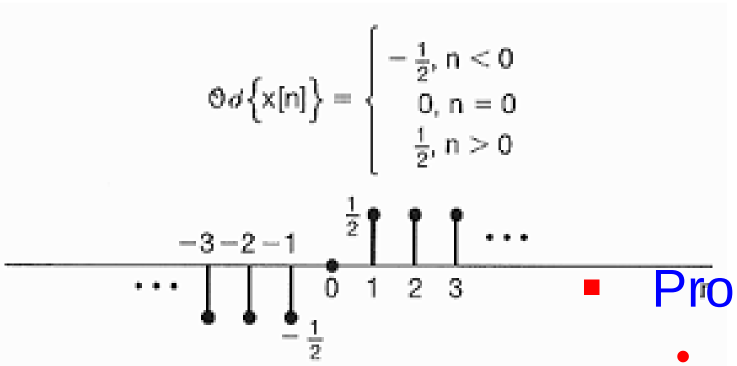
$x[-n]$



$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$



$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$



- Problems:
  - P1.23 for CT
  - P1.24 for DT

- Uniqueness of even-odd decomposition:

$$\text{Assume that } x(t) = \mathcal{E}v_1(t) + \mathcal{O}d_1(t)$$

$$\text{and } x(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$$

$$\text{So, } \mathcal{E}v_1(t) + \mathcal{O}d_1(t) = \mathcal{E}v_2(t) + \mathcal{O}d_2(t)$$

$$\text{and } \mathcal{E}v_1(-t) + \mathcal{O}d_1(-t) = \mathcal{E}v_2(-t) + \mathcal{O}d_2(-t)$$

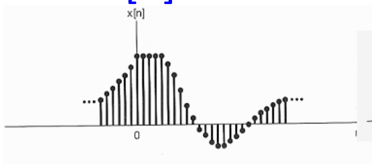
$$\text{Because } \begin{cases} \mathcal{E}v_1(-t) = \mathcal{E}v_1(t) \\ \mathcal{E}v_2(-t) = \mathcal{E}v_2(t) \end{cases} \text{ and } \begin{cases} \mathcal{O}d_1(-t) = -\mathcal{O}d_1(t) \\ \mathcal{O}d_2(-t) = -\mathcal{O}d_2(t) \end{cases}$$

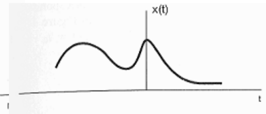
$$\text{Then, } \mathcal{E}v_1(t) - \mathcal{O}d_1(t) = \mathcal{E}v_2(t) - \mathcal{O}d_2(t)$$

$$\Rightarrow 2\mathcal{E}v_1(t) = 2\mathcal{E}v_2(t) \quad \text{or, } \mathcal{E}v_1(t) = \mathcal{E}v_2(t)$$

$$\Rightarrow 2\mathcal{O}d_1(t) = 2\mathcal{O}d_2(t) \quad \text{or, } \mathcal{O}d_1(t) = \mathcal{O}d_2(t)$$

$x[n]$ $x(t)$





- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift $x[n-n_0]$ $x(t-t_0)$
  - Time Reversal $x[-n]$ $x(-t)$
  - Time Scaling $x[an]$ $x(at)$
  - Periodic Signals $x(t)=x(t+T)$ $x[n]=x[n+N]$
  - Even & Odd Signals

$x(-t) = x(t), x[-n] = x[n]$

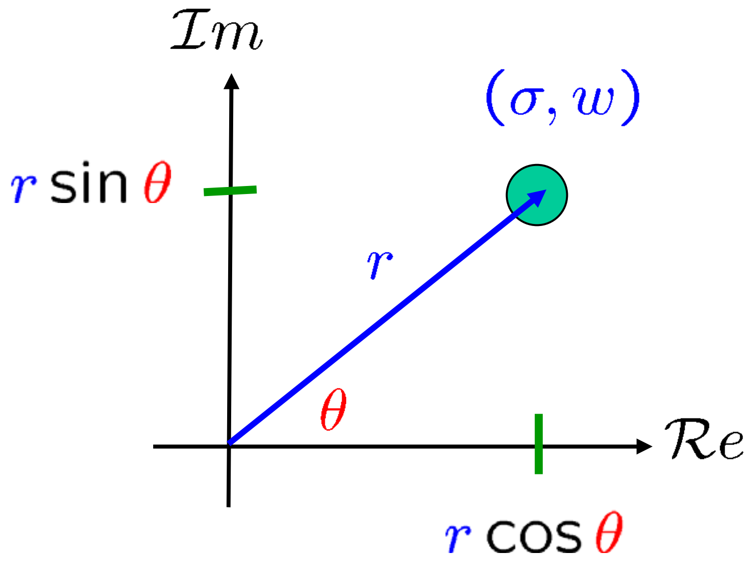
$x(-t) = -x(t), x[-n] = -x[n]$

$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$

$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$

- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

- ## Magnitude & Phase Representation:



$$j = \sqrt{-1}$$

$$\sigma + jw \quad \Rightarrow \quad \begin{cases} r = \sqrt{\sigma^2 + w^2} \\ \tan(\theta) = \frac{w}{\sigma} \end{cases}$$

$$\Rightarrow \quad \sigma + jw \quad = \quad r\, e^{j\theta}$$

- ## Euler's relation:

$$e^{j\theta} = \cos\theta + j \sin\theta$$

$$\Rightarrow \quad \sigma + jw = r \left(\cos\theta + j \sin\theta \right)$$

$$= (r \cos\theta) + j(r \sin\theta)$$

CT Complex Exponential Signals:

$$x(t) = Ce^{at}$$

- where C & a are, in general, complex numbers

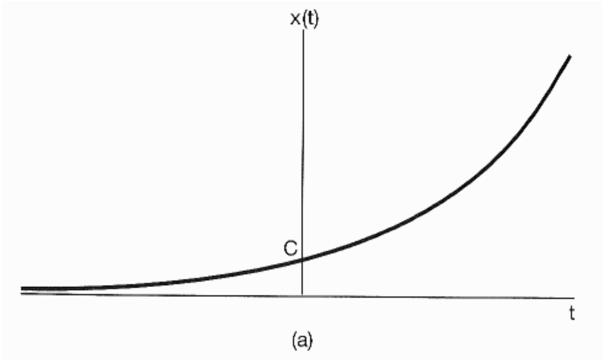
$$a = \sigma + jw$$

$$C = |C|\, e^{j\theta}$$

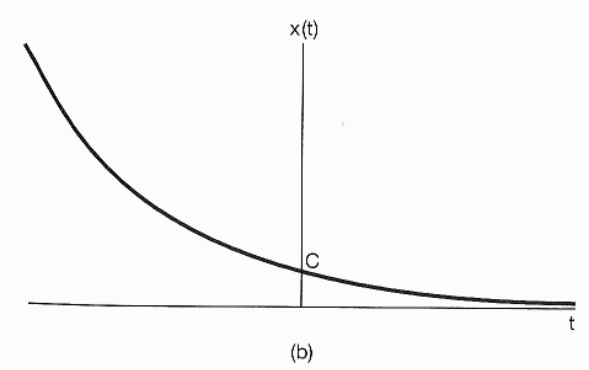
- Real exponential signals:

  - If C & a are real

$$x(t) = Ce^{at}$$



$a > 0$



$a < 0$

- **Periodic complex exponential signals:** $e^{j\theta} = \cos\theta + j\sin\theta$

  - If a is purely imaginary

$$a = \sigma + jw$$

$$x(t) = e^{jw_0 t}$$

  - It is periodic
    - Because let $T_0 = \frac{2\pi}{|w_0|}$
    - Then

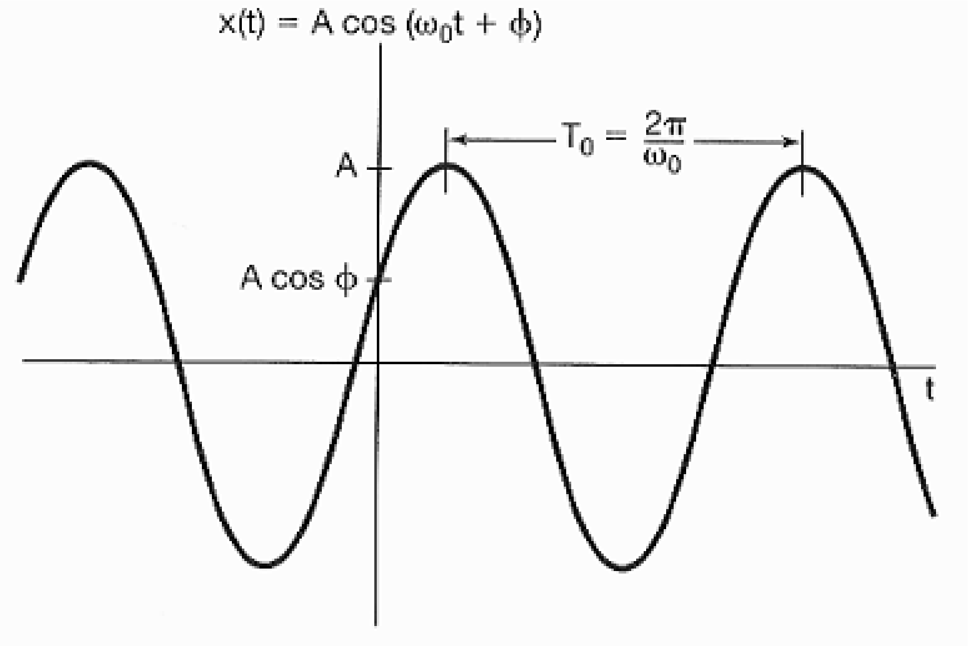
$$e^{jw_0 T_0} = e^{jw_0 \frac{2\pi}{w_0}} = \cos(2\pi) + j\sin(2\pi) = 1$$

    - Hence

$$x(t + T_0) = x(t) \qquad e^{jw_0(t+T_0)} = e^{jw_0 t} e^{jw_0 T_0} = e^{jw_0 t}$$

- Periodic sinusoidal signals:

$$x(t) = A\cos(w_0 t + \phi)$$



$$w_0 = 2\pi f_0$$

$$T_0 = \frac{2\pi}{w_0}$$

$$T_0 = \frac{1}{f_0}$$

$T_0 : (sec)$

$w_0 : (rad/sec)$

$f_0 : (1/sec = Hz)$

Period & Frequency:

$$T_0 = \frac{2\pi}{w_0}$$

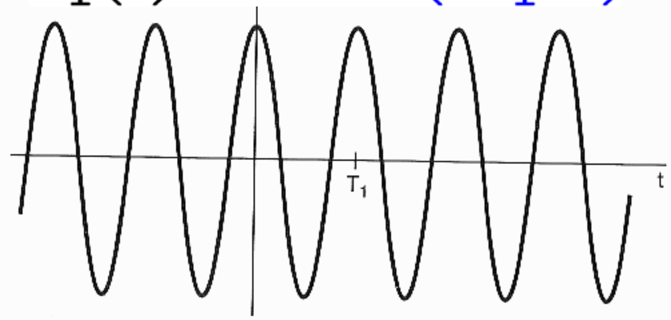
$$w_0 = 2\pi f_0$$

$$T_0 = \frac{1}{f_0}$$

$w_1 \quad w_2 \quad w_3$

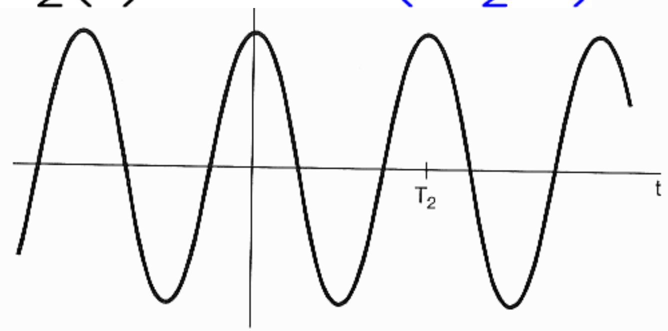
$T_1 \quad T_2 \quad T_3$

$x_1(t) = \cos(w_1 t)$



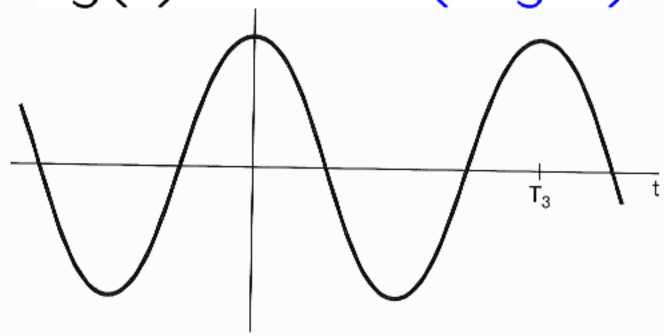
(a)

$x_2(t) = \cos(w_2 t)$



(b)

$x_3(t) = \cos(w_3 t)$



(c)

Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(\theta) = \mathcal{R}e\left\{e^{(j\theta)}\right\}$$

$$\sin(\theta) = \mathcal{I}m\left\{e^{(j\theta)}\right\}$$

$$e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta)$$

$$= \cos(\theta) - j\sin(\theta)$$

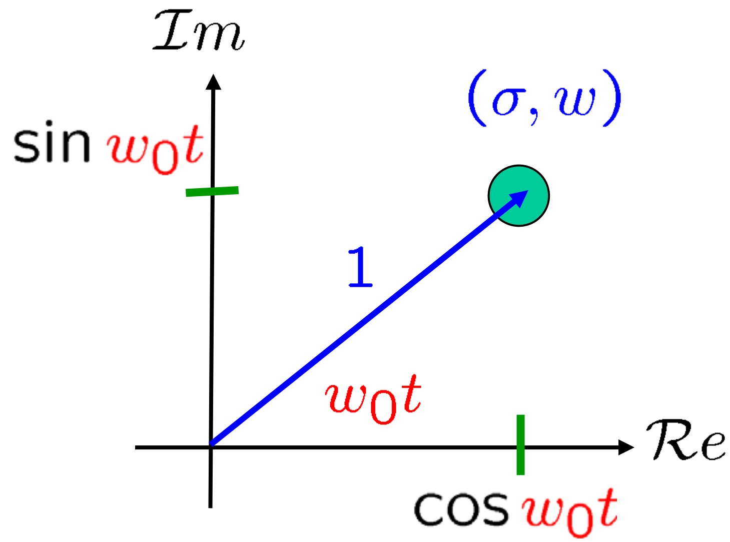
$$\Rightarrow \cos(\theta) = \frac{e^{(j\theta)} + e^{-(j\theta)}}{2}$$

$$\Rightarrow \sin(\theta) = \frac{e^{(j\theta)} - e^{-(j\theta)}}{2j}$$

$$\Rightarrow A\cos(w_0 t + \phi) = \frac{A}{2}\, e^{j(\phi + w_0 t)} + \frac{A}{2}\, e^{-j(\phi + w_0 t)}$$

$$= \frac{A}{2}\, e^{j\phi}\, e^{j w_0 t} + \frac{A}{2}\, e^{-j\phi}\, e^{-j w_0 t}$$

- Total energy & average power:



$$E_{\text{period}} = \int_0^{T_0} \left| e^{jw_0 t} \right|^2 dt$$

$$= \int_0^{T_0} 1 \cdot dt = T_0$$

$$P_{\text{period}} = \frac{1}{T_0} E_{\text{period}} = 1$$

$$E_\infty = \infty$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e^{jw_0 t} \right|^2 dt = 1$$

- Problem:
  - P1.3

▪ Harmonically related periodic exponentials

$$e^{j0w_0t}, \quad e^{j1w_0t}, \quad e^{j2w_0t}, \quad e^{j3w_0t}, \quad \cdots,$$

$$e^{j(-1)w_0t}, \quad e^{j(-2)w_0t}, \quad e^{j(-3)w_0t}, \quad \cdots$$

$$\phi_k(t) = e^{\,j\,k\,w_0\,t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- For $k = 0$, $\phi_k(t)$ is constant
- For $k \neq 0$, $\phi_k(t)$ is periodic with

  fundamental frequency $|k|w_0$ and

  fundamental period $\frac{T_0}{|k|}$

General complex exponential signals:

$$C e^{at} = (|C| e^{j\theta})(e^{(r+jw_0)t})$$

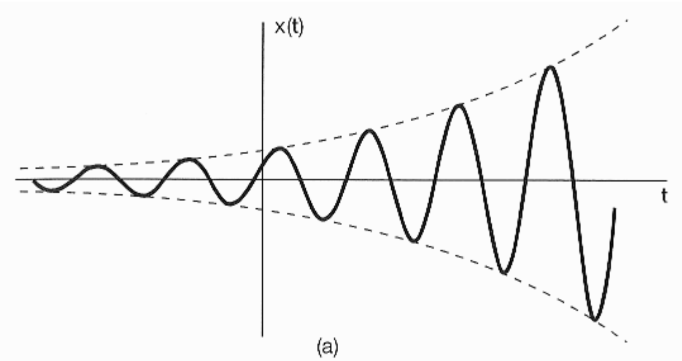
$$\sigma + jw \quad = \quad r \ e^{j\theta}$$

$$= (\ |C| \ e^{j\theta} \ ) \ (\ e^{rt} \ e^{jw_0 t} \ )$$

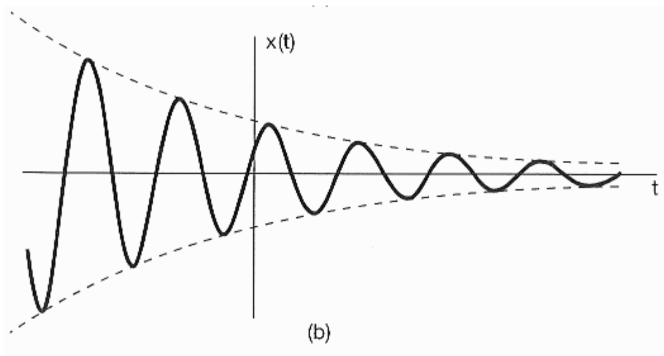
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$= |C| \ e^{rt} \ e^{j(w_0 t + \theta)}$$

$$= \quad |C| e^{rt} \ \cos(w_0 t + \theta) \quad + \quad j |C| e^{rt} \ \sin(w_0 t + \theta)$$



(a)



(b)

▪ DT complex exponential signal or sequence:

$$x(t) = Ce^{at} \qquad x[n] = Ce^{bn}$$

$$= C(e^{b})^{n} \quad \text{with } a = e^{b}$$

$$x[n] = Ca^{n}$$

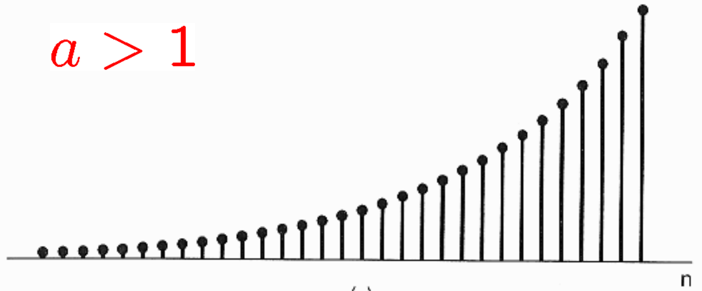
- where C & a are, in general, complex numbers

Real exponential signals:

- If C & a are real

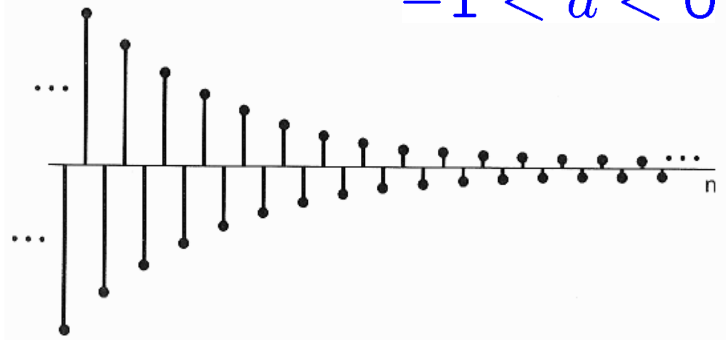
$$x[n] = Ca^n$$

$a > 1$



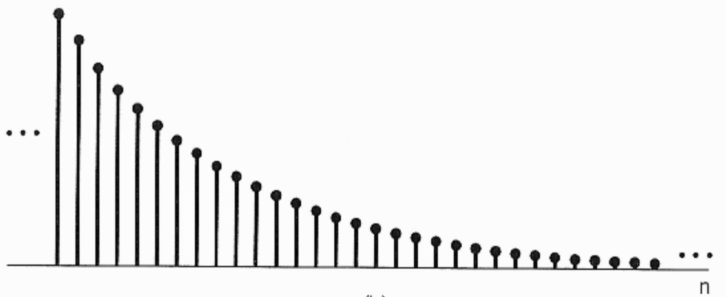
(a)

$-1 < a < 0$



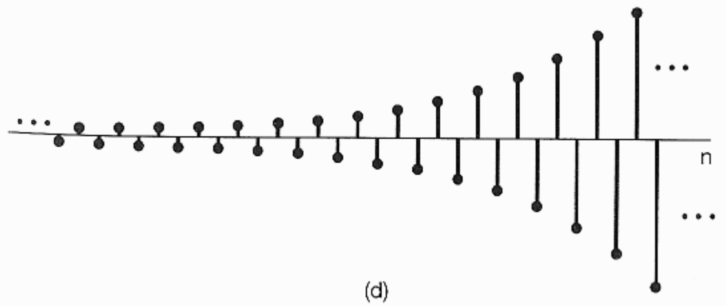
(c)

$0 < a < 1$



(b)

$a < -1$



(d)

DT Complex Exponential & Sinusoidal Signals

- If b is purely imaginary (or |a| = 1)

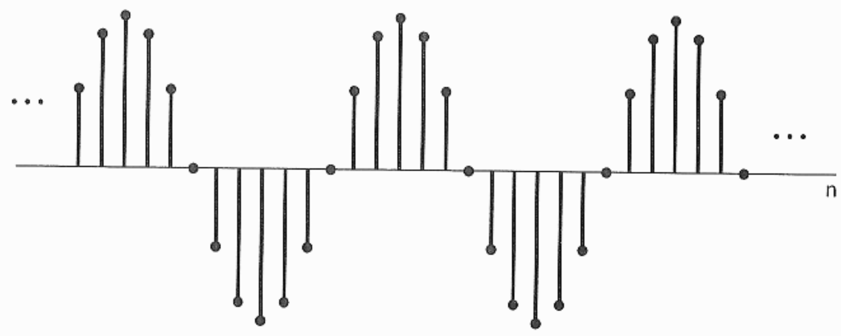
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$x[n] = e^{jw_0 n}$$

$$= \cos(w_0 n) + j\sin(w_0 n)$$

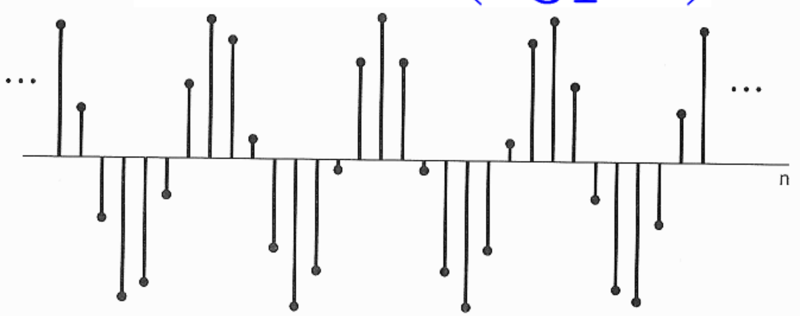
$$x[n] = A\cos(w_0 n + \phi)$$

$$x[n] = \cos\left(\frac{2\pi}{12}n\right)$$



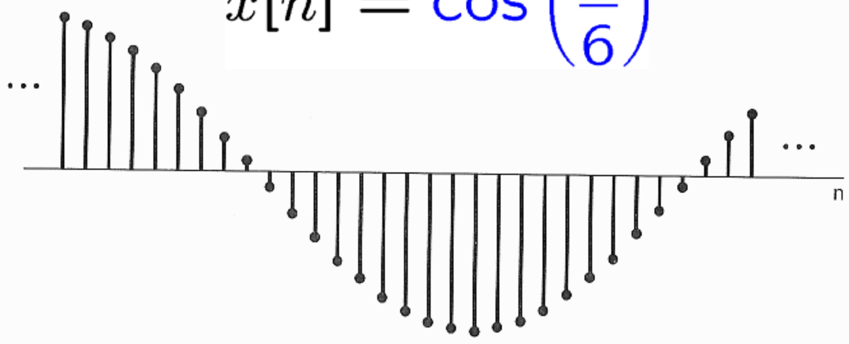
(a)

$$x[n] = \cos\left(\frac{4 \cdot 2\pi}{31}n\right)$$



(b)

$$x[n] = \cos\left(\frac{n}{6}\right)$$



(c)

- Euler's relation:

$$e^{jw_0 n} = \cos w_0 n + j \sin w_0 n$$

  - And,

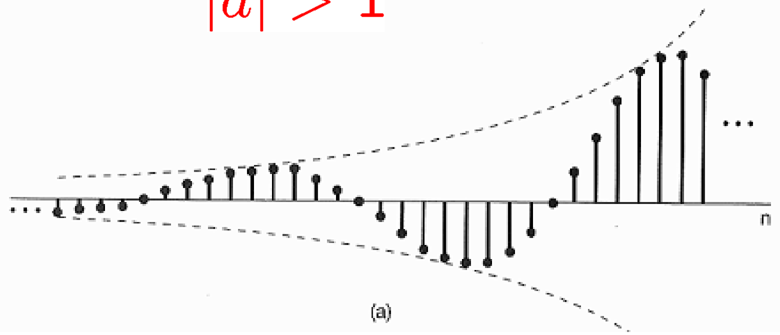
$$A \cos(w_0 n + \phi) = \frac{A}{2}\, e^{j\phi}\, e^{jw_0 n} + \frac{A}{2}\, e^{-j\phi}\, e^{-jw_0 n}$$

- General complex exponential signals:

$$Ca^n = (|C|e^{j\theta})((|a|e^{jw_0})^n)$$

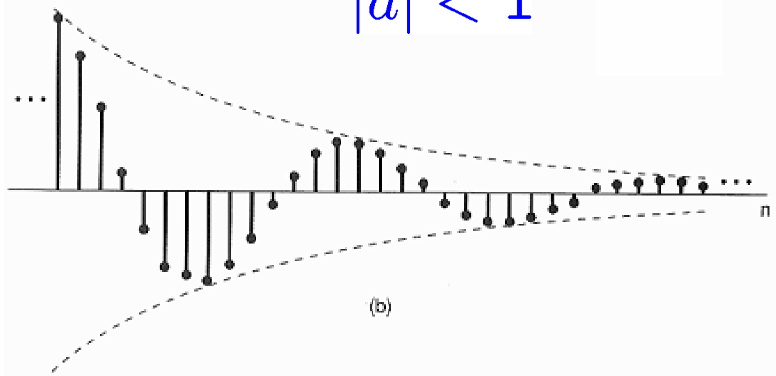
$$= |C||a|^n \ \cos(w_0 n+\theta) \ + \ j|C||a|^n \ \sin(w_0 n+\theta)$$

$|a| > 1$



(a)

$|a| < 1$



(b)

▪ Periodicity properties of DT complex exponentials:

$$e^{jw_0 n}$$

$$e^{j2\pi n} = \cos 2\pi n + j \sin 2\pi n$$

$$e^{j(w_0+2\pi)n} = e^{j2\pi n}\, e^{jw_0 n} = e^{jw_0 n}$$

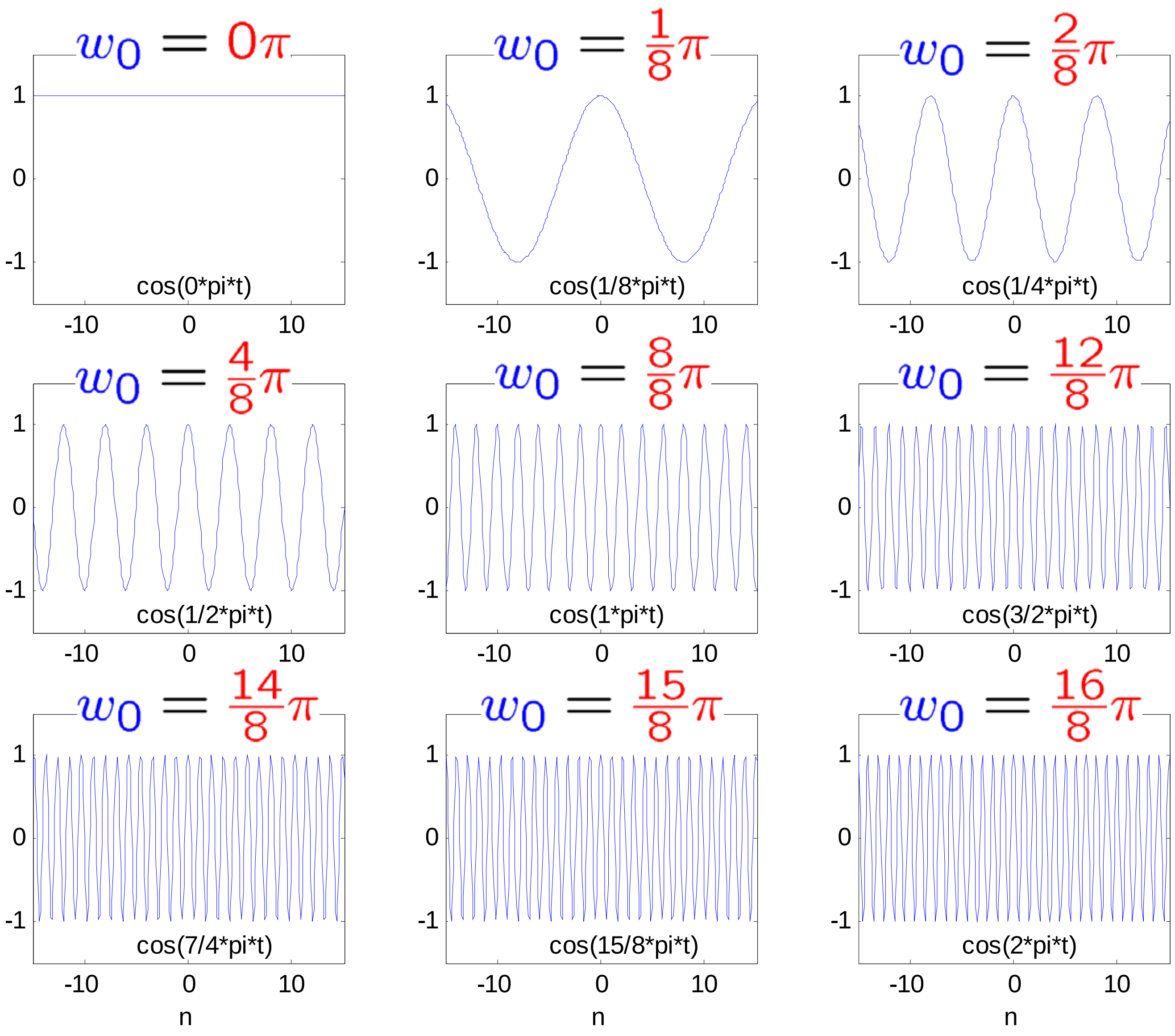
- The signal with frequency ω_0 is identical to the signals with frequencies $w_0 \pm 2\pi,\ w_0 \pm 4\pi,\ w_0 \pm 6\pi,\ \cdots$

- Only need to consider a frequency interval of length 2π
  - Usually use $0 \le w_0 < 2\pi$ or $-\pi \le w_0 < \pi,$

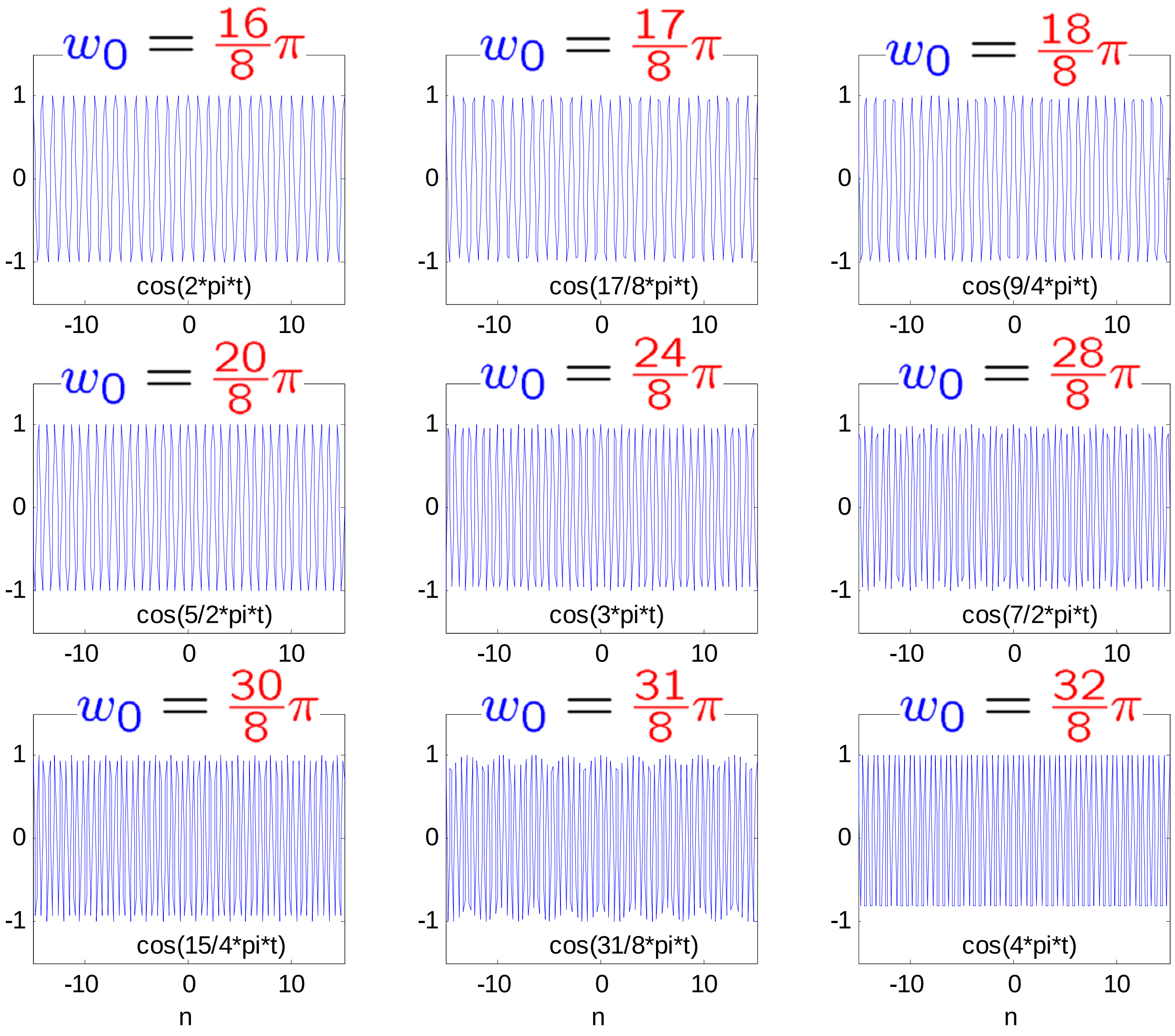
- The low frequencies are located at $w_0 = 0,\ \pm 2\pi,\ \cdots$
  The high frequencies are located at $w_0 = \pm\pi, \pm 3\pi,\ \cdots$

$$e^{j(0)n} = 1 \quad \text{and} \quad e^{j(\pi)n} = (e^{j(\pi)})^n = (-1)^n$$

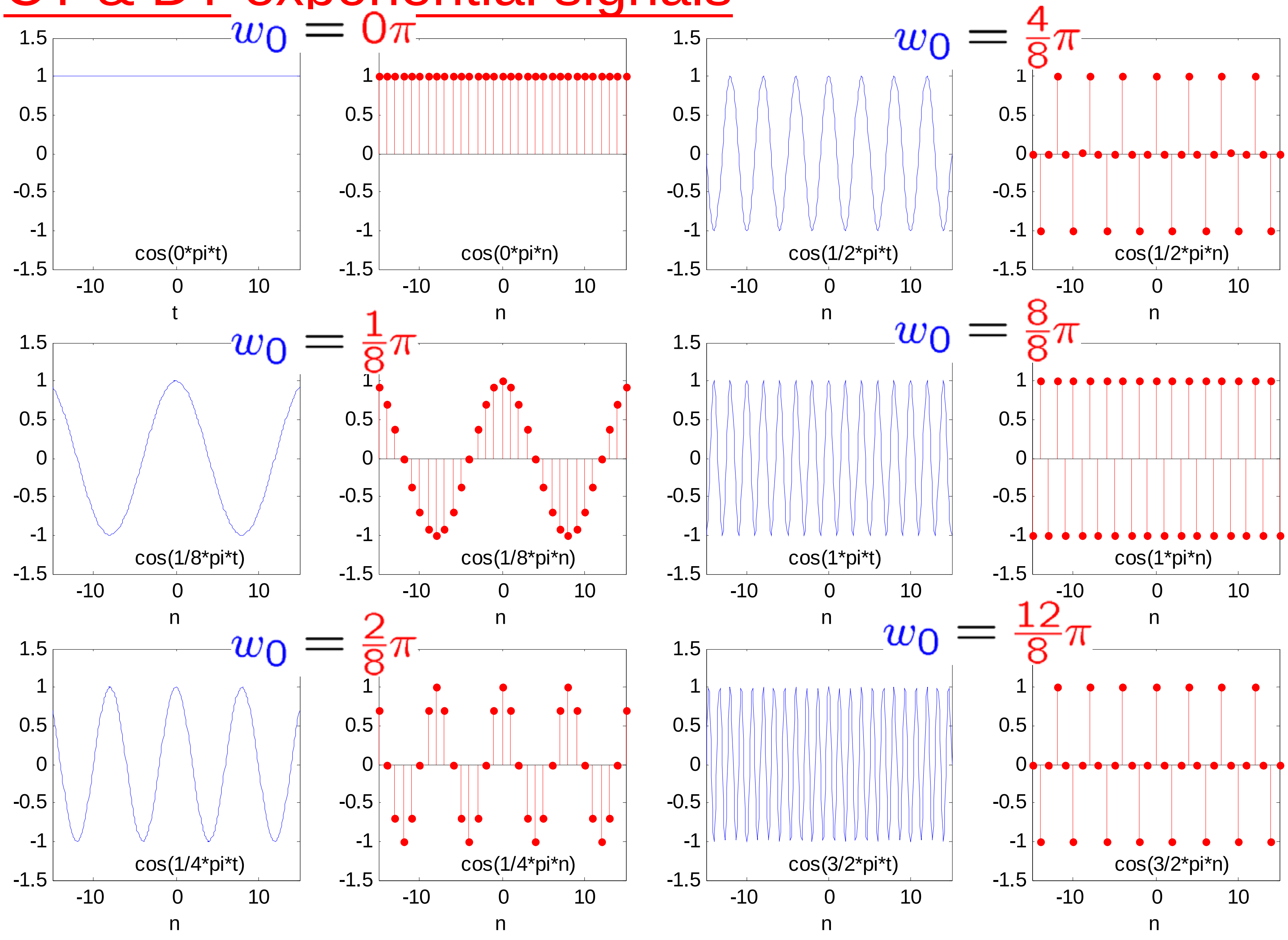
▪ CT exponential signals



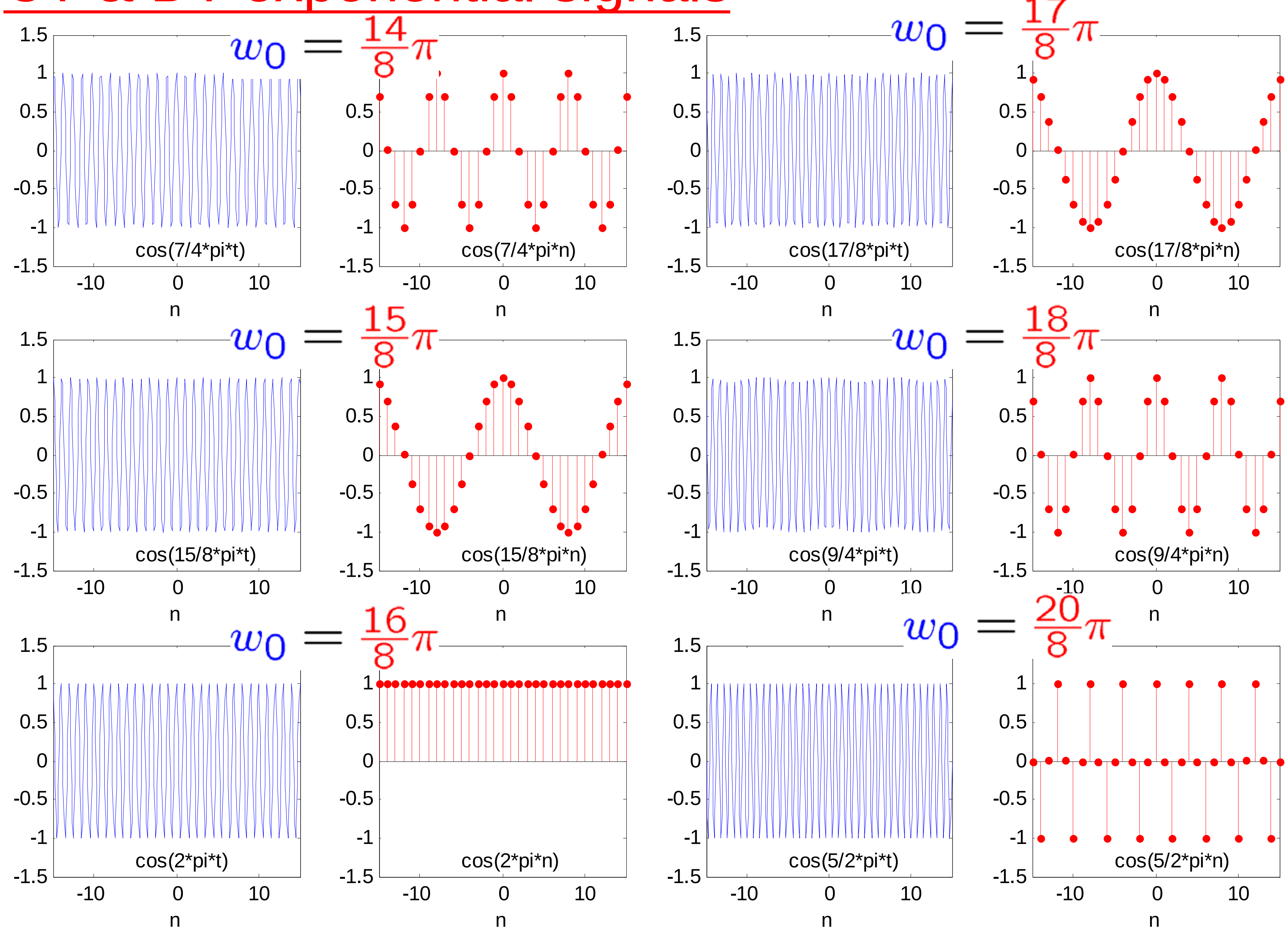
- CT exponential signals



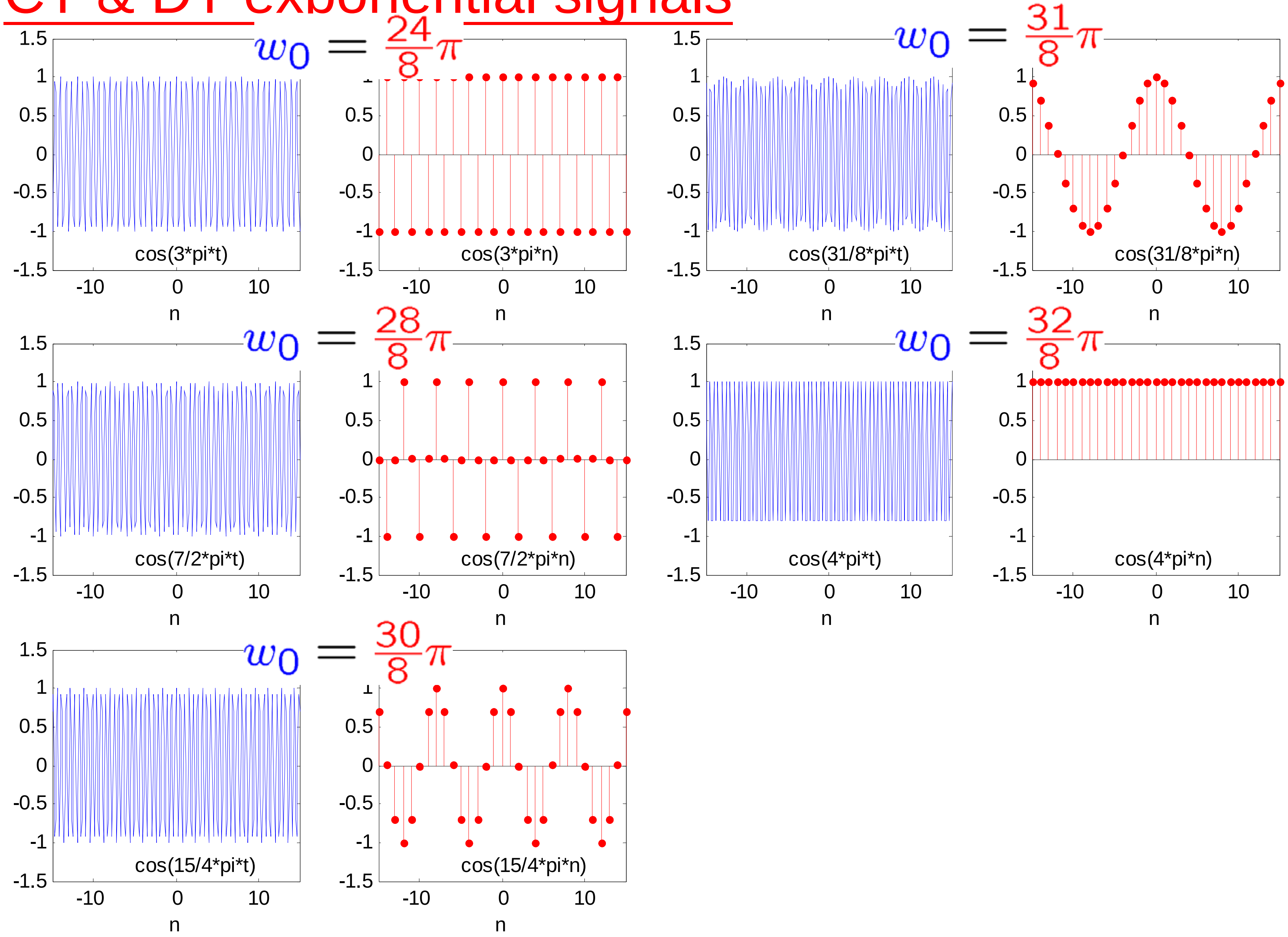
- CT & DT exponential signals



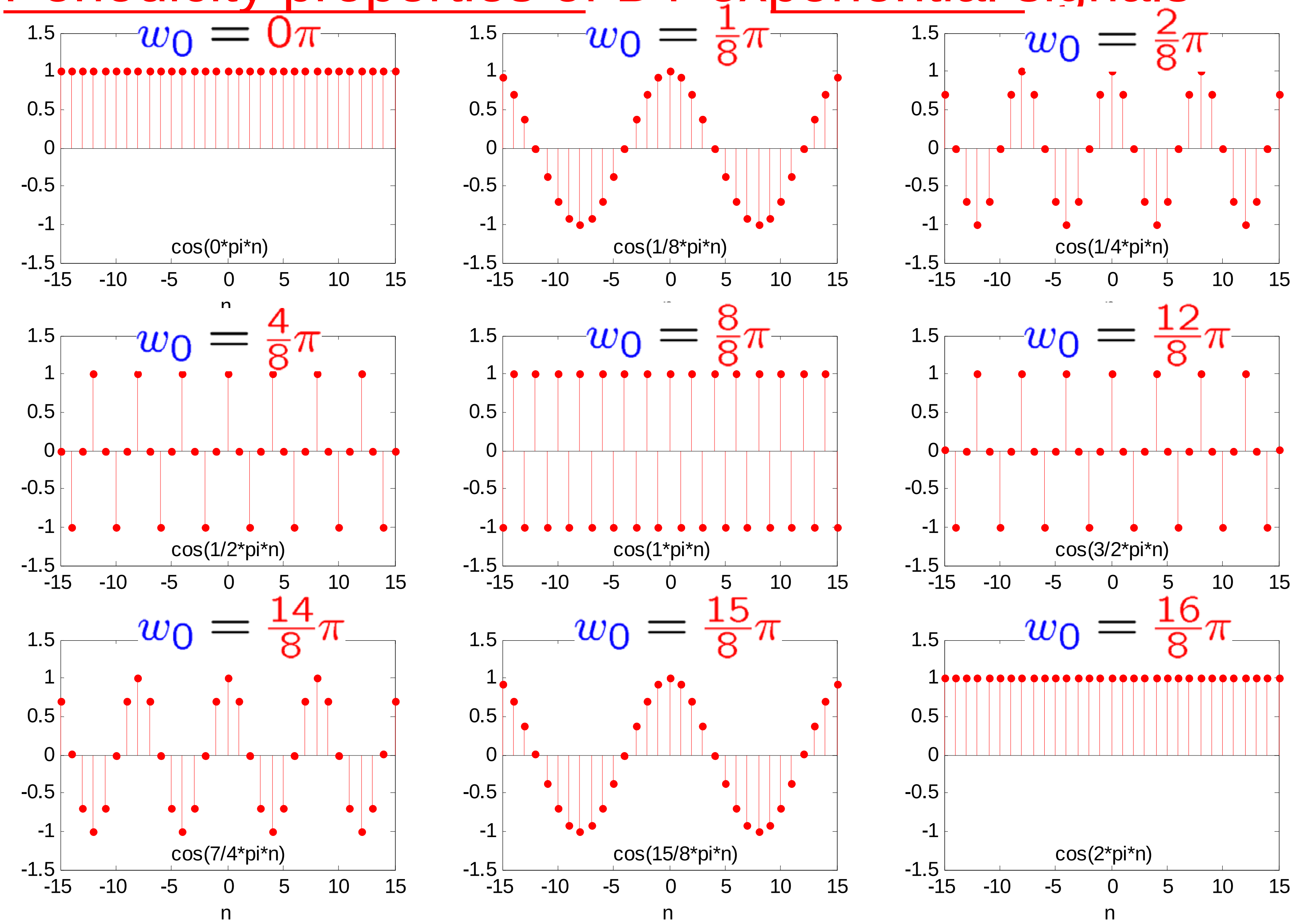
- CT & DT exponential signals



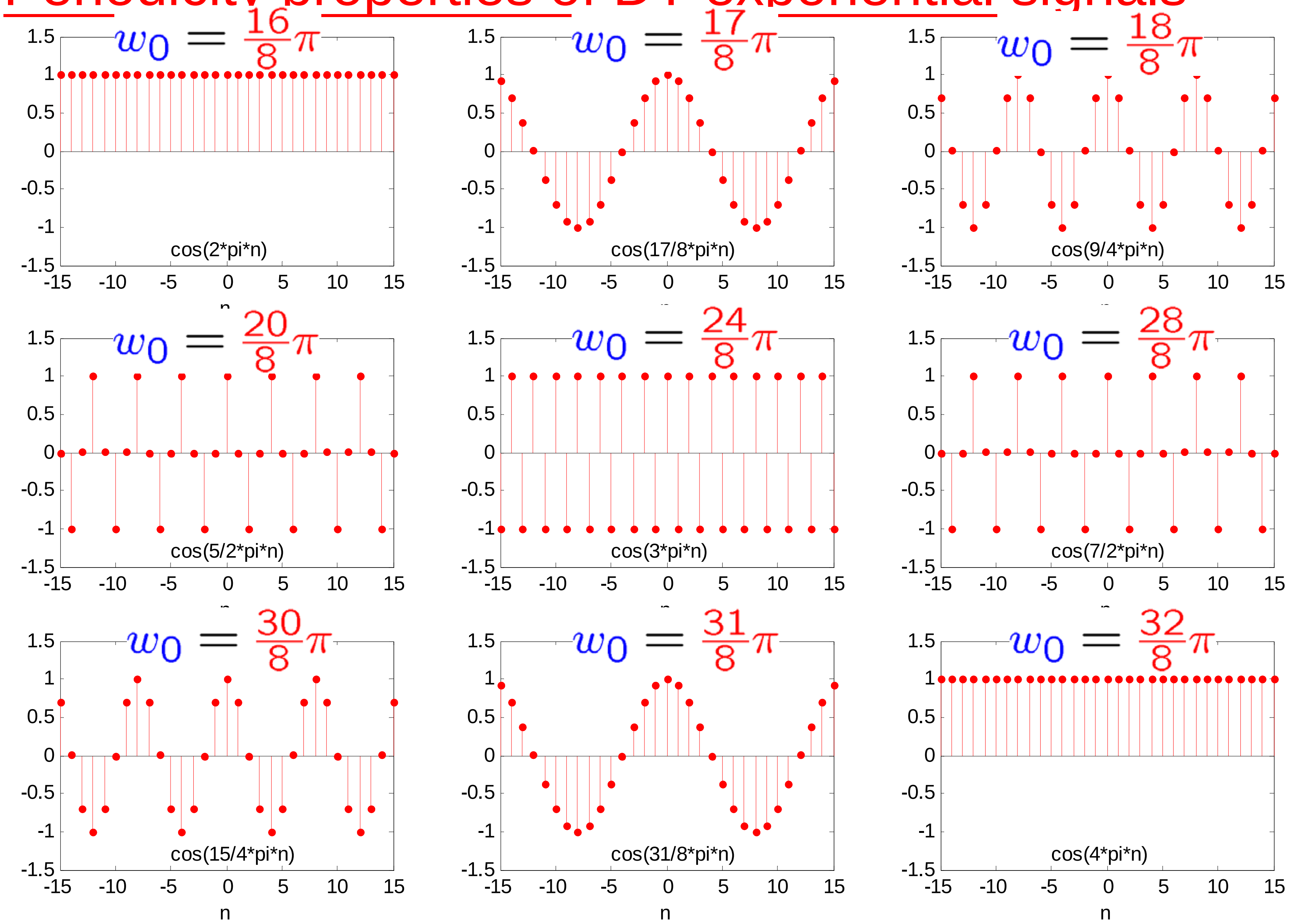
- CT & DT exponential signals



- Periodicity properties of DT exponential signals



▪ Periodicity properties of DT exponential signals



Periodicity properties of DT exponential signals

- Periodicity of N > 0

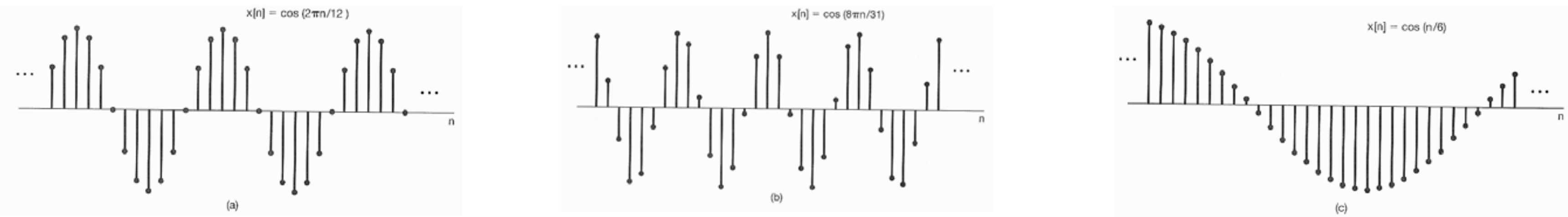
- Problem:
  - P1.35

$$e^{jw_0 n} \qquad e^{jw_0(n+N)} = e^{jw_0 n} \; e^{jw_0 N} = e^{jw_0 n} \quad \text{or} \quad e^{jw_0 N} = 1$$

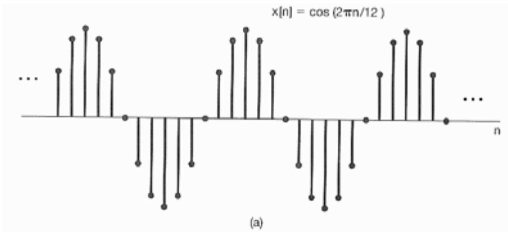
- That is, $\quad w_0 N = 2\pi m \quad$ or $\quad \dfrac{w_0}{2\pi} = \dfrac{m}{N}$

- Hence, $\quad e^{jw_0 n}$ is periodic $\quad$ if $\frac{w_0}{2\pi}$ is a rational number

$$x[n] = \cos\left(\frac{2\pi}{12} n\right) \qquad x[n] = \cos\left(\frac{4 \cdot 2\pi}{31} n\right) \qquad x[n] = \cos\left(\frac{n}{6}\right)$$

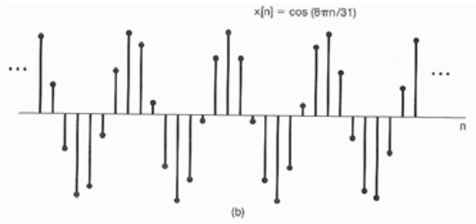


▪ Periodicity properties of DT exponential signals



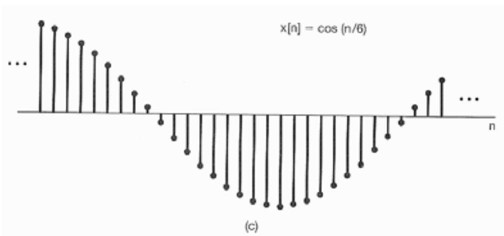
$$x(t) = \cos\left(\frac{2\pi}{12}t\right) \qquad T = 12?$$

$$x[n] = \cos\left(\frac{2\pi}{12}n\right) \qquad N = 12?$$



$$x(t) = \cos\left(\frac{4\cdot 2\pi}{31}t\right) \qquad T = \frac{31}{4}?$$

$$x[n] = \cos\left(\frac{4\cdot 2\pi}{31}n\right) \qquad N = \frac{31}{4}?$$



$$x(t) = \cos\left(\frac{1}{6}t\right) \qquad T = 12\pi?$$

$$x[n] = \cos\left(\frac{1}{6}n\right) \qquad N = 12\pi?$$

- Harmonically related periodic exponentials

$$\phi_k[n] = e^{jk(w_0)n}, \quad = e^{jk(\frac{2\pi}{N})n}, \quad k = 0, \pm 1, \pm 2, \dots$$

$$\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n}$$

$$= e^{jk(\frac{2\pi}{N})n}\, e^{jN(\frac{2\pi}{N})n} = \phi_k[n]$$

- Only N distinct periodic exponentials in the set

$$\phi_0[n] = 1, \quad \phi_1[n] = e^{j(\frac{2\pi}{N}n)}, \quad \phi_2[n] = e^{j(2\frac{2\pi}{N}n)},$$

$$\dots, \quad \phi_{N-1}[n] = e^{j(N-1)\frac{2\pi}{N}n}$$

$$\phi_N[n] = e^{j(N)\frac{2\pi}{N}n} = e^{j2\pi n} = 1 = \phi_0[n], \ ; \quad \phi_{N+1}[n] = \phi_1[n], \dots$$

Comparison of CT & DT signals:

TABLE 1.1 Comparison of the signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$.

CT $e^{jw_0 t}$	DT $e^{jw_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency* ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$	Fundamental period* $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

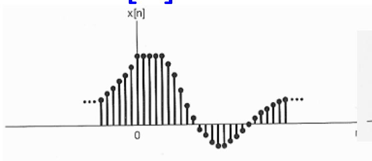
*Assumes that m and N do not have any factors in common.

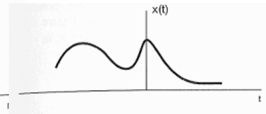
DT $\quad 1, \quad e^{j(\frac{2\pi}{N}n)}, \quad e^{j(2\frac{2\pi}{N}n)}, \quad \ldots, \quad e^{j(N-1)\frac{2\pi}{N}n}$

CT $\quad 1, \quad e^{j1w_0 t}, \quad e^{j2w_0 t}, \quad e^{j3w_0 t}, \quad \cdots,$

$e^{j(-1)w_0 t}, \quad e^{j(-2)w_0 t}, \quad e^{j(-3)w_0 t}, \quad \cdots$

- Problem:
  - P1.36

$x[n]$ 

$x(t)$ 

- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift $\quad x[n-n_0] \quad x(t-t_0)$
  - Time Reversal $\quad x[-n] \quad x(-t)$
  - Time Scaling $\quad x[an] \quad x(at)$
  - Periodic Signals $\quad x(t) = x(t+T) \quad x[n] = x[n+N]$
  - Even & Odd Signals

$x(-t) = \quad x(t), x[-n] = \quad x[n]$

$x(-t) = -x(t), x[-n] = -x[n]$

$\mathcal{E}v\{x[n]\} = \frac{1}{2}\left[x[n] + x[-n]\right]$

$\mathcal{O}d\{x[n]\} = \frac{1}{2}\left[x[n] - x[-n]\right]$

$\phi_k(t) = e^{jkw_0t}, k = 0, \pm 1, \ldots$

$\phi_k[n] = e^{jkw_0n}, k = 0, \ldots, N-1$

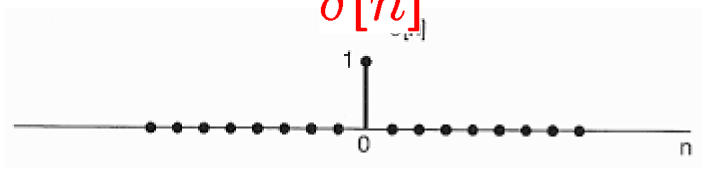
- Exponential & Sinusoidal Signals
- The Unit Impulse & Unit Step Functions
- Continuous-Time & Discrete-Time Systems
- Basic System Properties

DT Unit Impulse & Unit Step Sequences

Unit impulse (or unit sample)

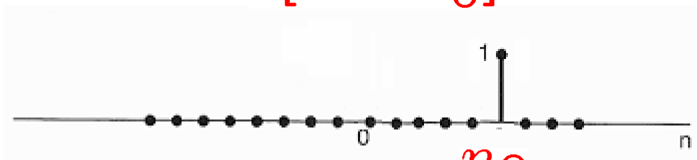
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$\delta[n]$



$$\delta[n- \quad] = \begin{cases} 1, & n = \\ 0, & n \neq \end{cases}$$

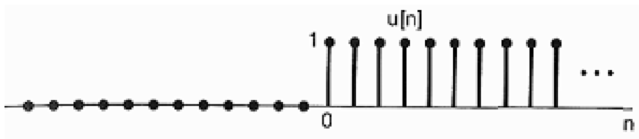
$\delta[n - n_0]$



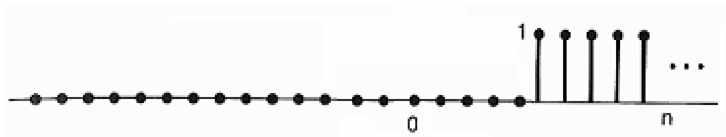
n_0

Unit step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

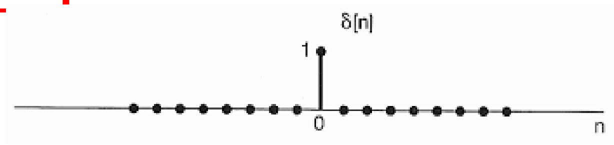


$$u[n- \quad] = \begin{cases} 0, & n < \\ 1, & n \geq \end{cases}$$

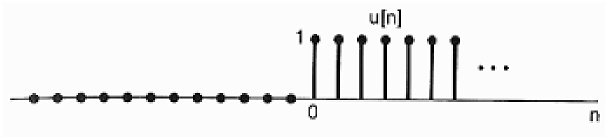


Relationship Between Impulse & Step

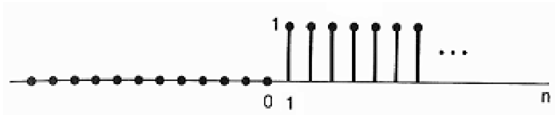
First difference



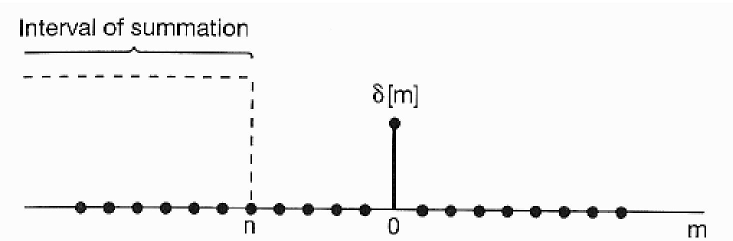
$$\delta[n] = u[n] - u[n-1]$$



Running sum

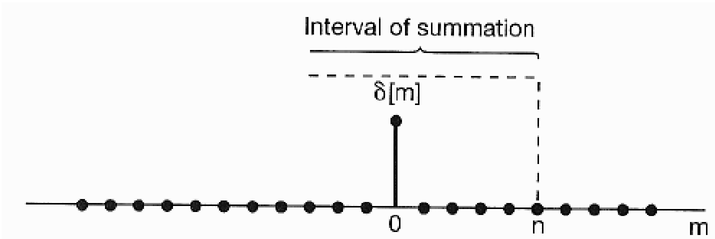


$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \qquad = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



(a)

$n < 0$



(b)

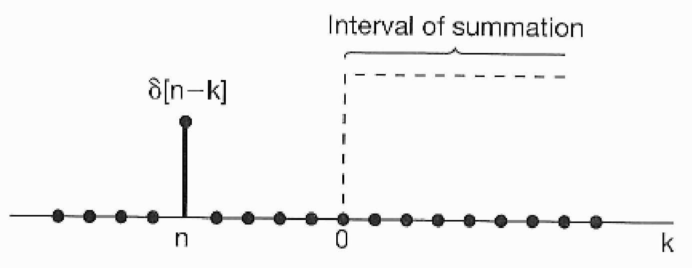
$n > 0$

Relationship Between Impulse & Step

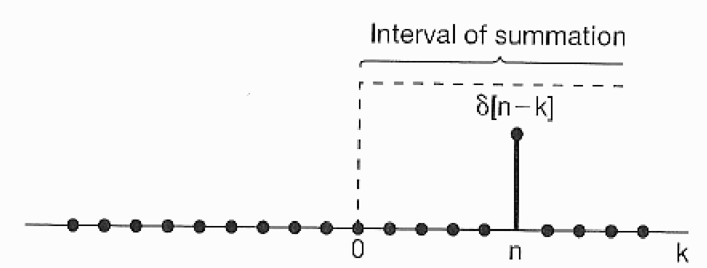
- Alternatively,

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k], \quad \text{with } m = n - k$$

or, $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



(a)
$n < 0$

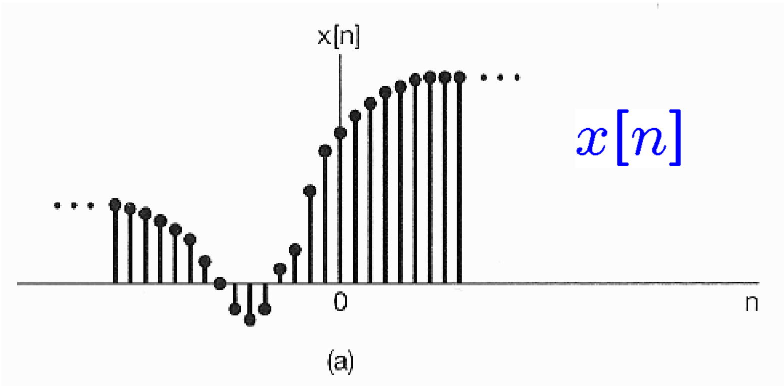


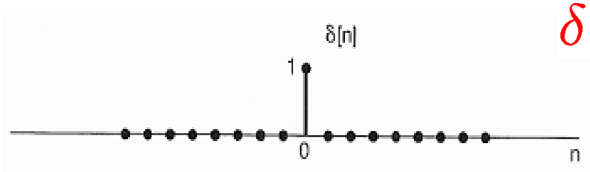
(b)
$n > 0$

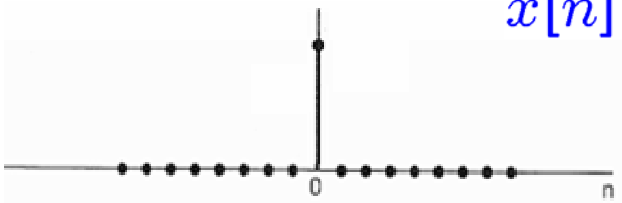
Sample by Unit Impulse

- For x[n]

$$x[n]\delta[n] = x[0]\delta[n]$$

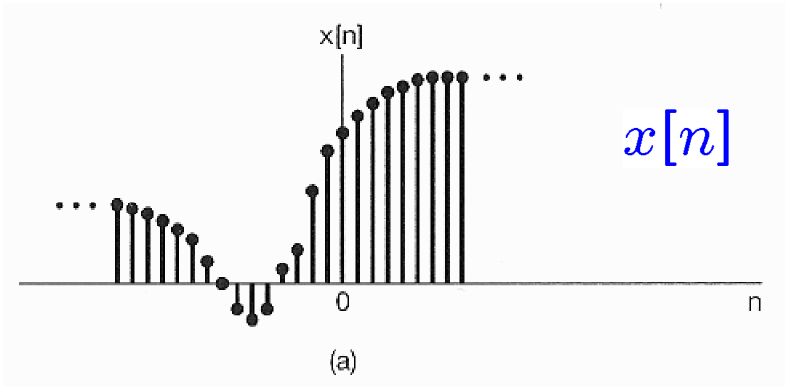


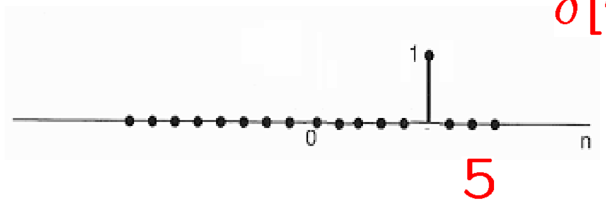


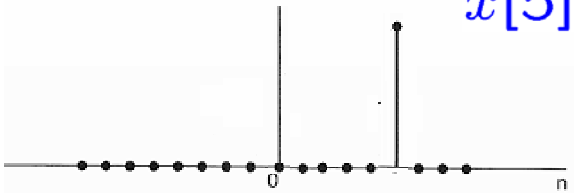


- More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



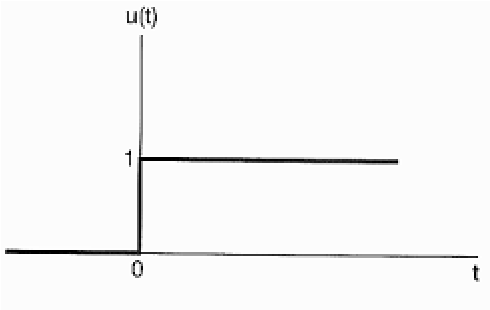




CT Unit Impulse & Unit Step Functions

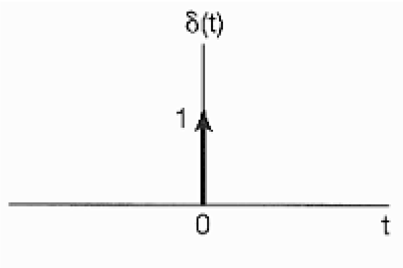
- Unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



- Unit impulse function

$$\delta(t)$$



Relationship Between Impulse & Step

- Running integral

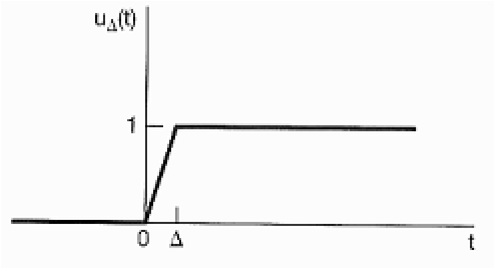
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \qquad = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

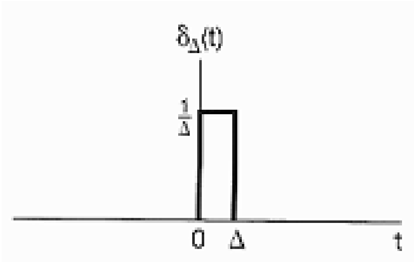
- First derivative

$$\delta(t) = \frac{du(t)}{dt}$$

  - But, u(t) is discontinuous at t = 0, hence, not differentiable
  - Use approximation

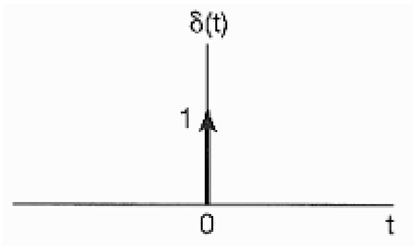
- Relationship Between Impulse & Step
  - Use approximation

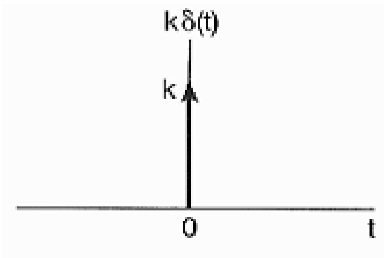




$$\delta_{\Delta}(t) = \frac{du_{\Delta}(t)}{dt}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)$$





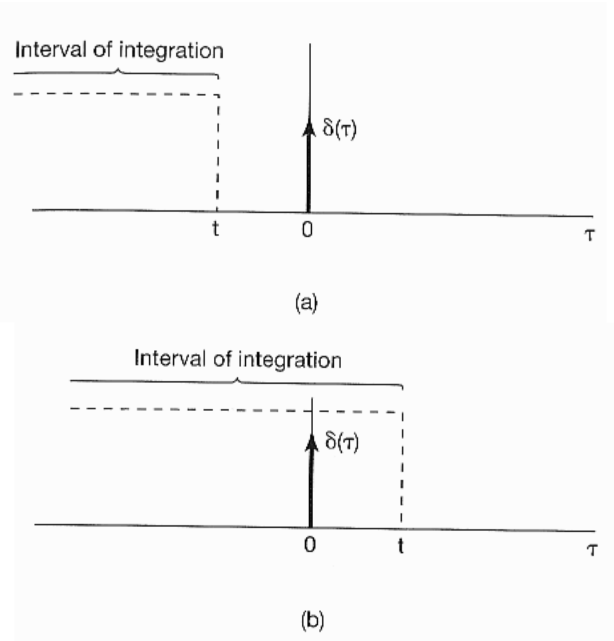
Relationship Between Impulse & Step

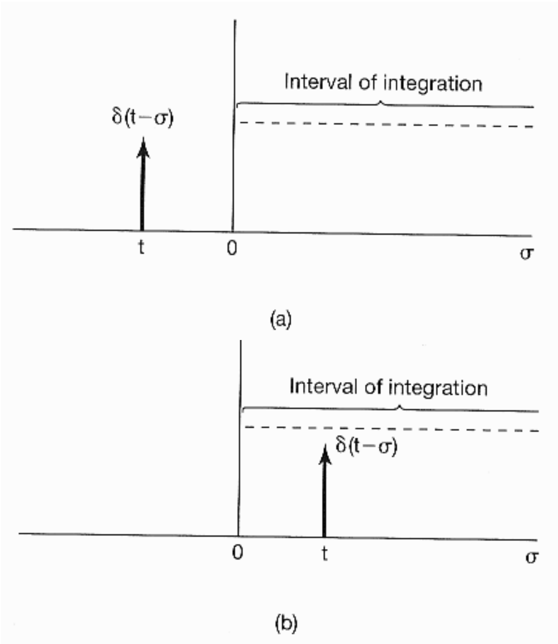
$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \quad = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma) \quad = \int_{0}^{\infty} \delta(t-\sigma)(d\sigma)$$

$$\tau = t - \sigma$$

$$d\tau = -d\sigma$$

$$= \int_{0}^{\infty} \delta(t-\tau)(d\tau)$$





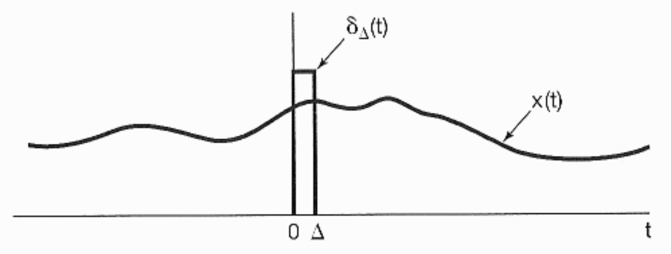
Sample by Unit Impulse Function

- For x(t)

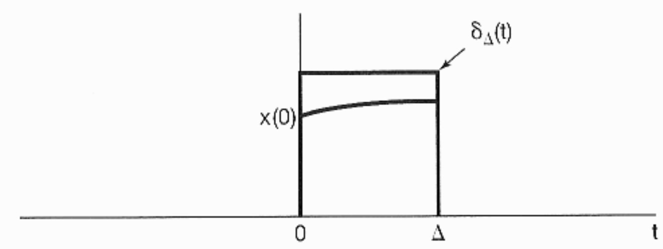
$$x(t)\delta(t) = x(0)\delta(t)$$

- More generally,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$



(a)

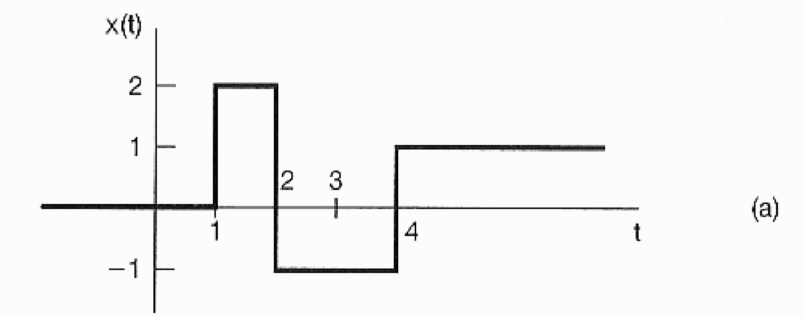


(b)

Example 1.7:

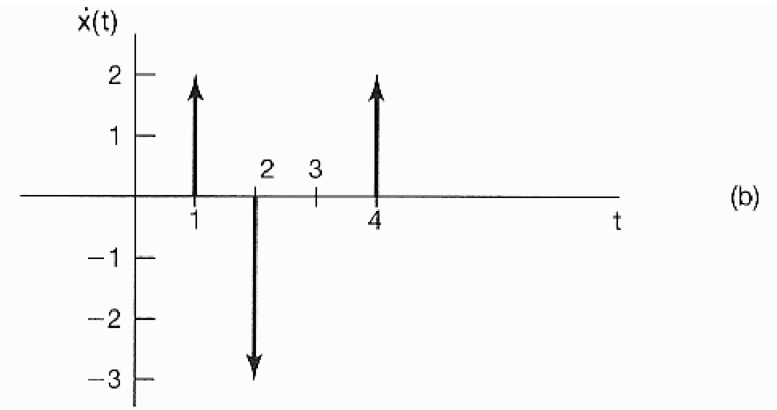
$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

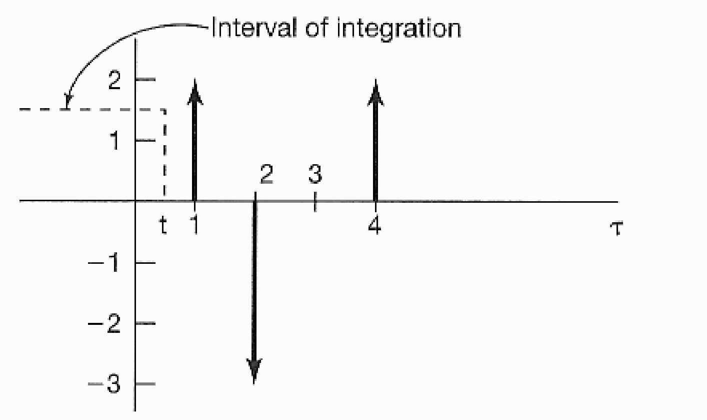


(a)

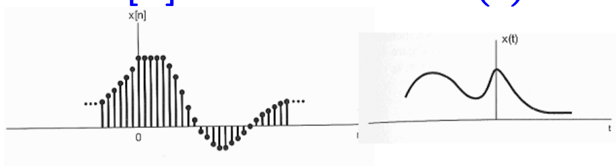
$$x(t) = \int_{0}^{t} \dot{x}(\tau) d\tau$$



(b)



(c)



- Introduction
- Continuous-Time & Discrete-Time Signals
- Transformations of the Independent Variable
  - Time Shift $x[n-n_0]$ $x(t-t_0)$
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  - Even & Odd Signals

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$x(-t) = -x(t), x[-n] = -x[n]$

$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$

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- Exponential & Sinusoidal Signals

$\phi_k(t) = e^{jkw_0t}, k = 0, \pm 1, ...$

$\phi_k[n] = e^{jkw_0n}, k = 0, ..., N-1$

- The Unit Impulse & Unit Step Functions

$\delta[n], u[n]$

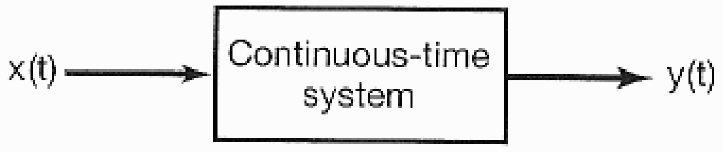
$\delta(t), u(t)$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties

▪ Physical Systems & Mathematical Descriptions

- Examples of physical systems are
  signal processing, communications,
  electromechanical motors, automotive vehicles,
  chemical-processing plants

- A system can be viewed as a process
  in which input signals are transformed by the system
  or cause the system to respond in some way,
  resulting in other signals or outputs



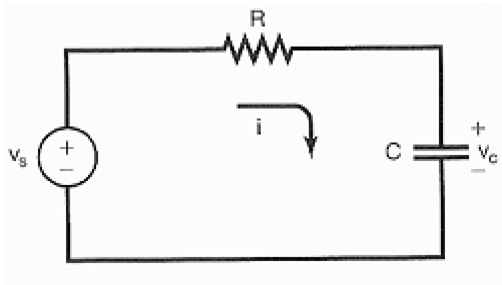
(a)

(b)

Simple examples of CT systems

- RC circuit

Input signal: $v_s(t)$

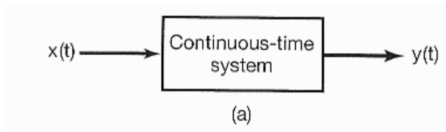


Output signal: $v_c(t)$

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \qquad i(t) = C\frac{dv_c(t)}{dt}$$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

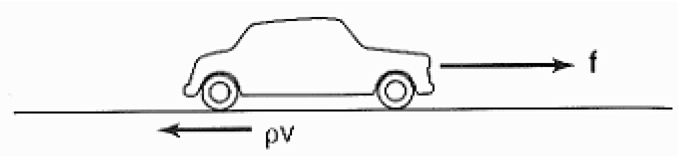
$$\Rightarrow \frac{dy(t)}{dt} + a y(t) = b x(t)$$



$$x(t) \rightarrow y(t)$$

Simple examples of CT systems

- Automobile



Input signal: $f(t)$

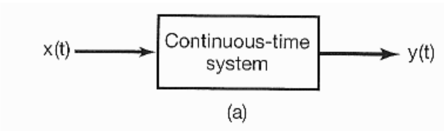
Output signal: $v(t)$

$$f(t) - \rho v(t) = m\frac{dv(t)}{dt}$$

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)]$$

$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$



$$x(t) \rightarrow y(t)$$

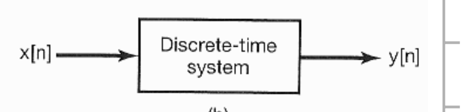
Simple examples of DT systems

- Balance in a bank account

$$y[n] = 1.01y[n-1] + x[n]$$

or, $y[n] - 1.01y[n-1] = x[n]$

$$\Rightarrow \quad y[n] + ay[n-1] = bx[n]$$



(b)

$$x[n] \rightarrow y[n]$$

Simple examples of DT systems

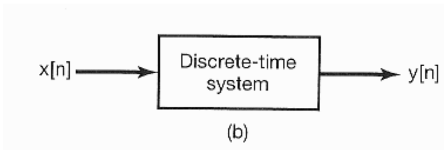
Digital simulation of differential equation

$$\frac{dv(t)}{dt} \approx \frac{v(n\Delta) - v((n-1)\Delta)}{\Delta} = \frac{v[n] - v[n-1]}{\Delta},$$

$$t = n\Delta$$

$$\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

$$\Rightarrow \quad v[n] - \frac{m}{m + \rho\Delta} v[n-1] = \frac{\Delta}{m + \rho\Delta} f[n]$$

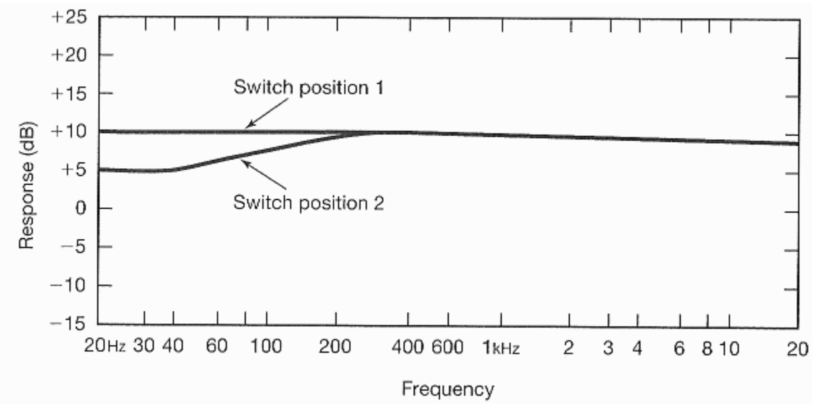


$$\Rightarrow \quad y[n] + a y[n-1] = b x[n]$$

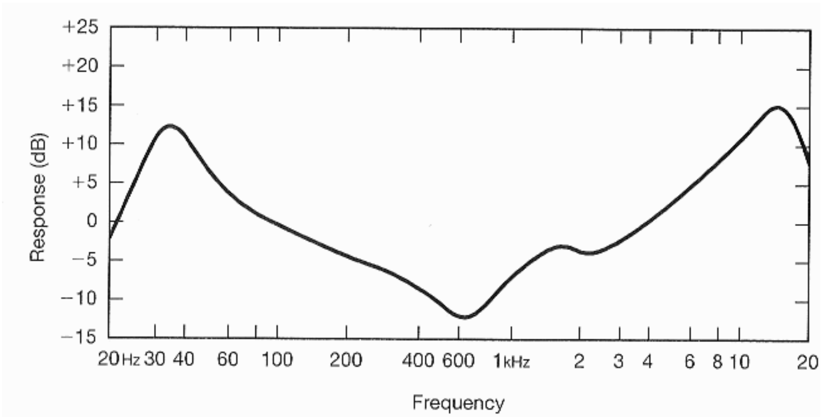
$$x[n] \rightarrow y[n]$$

Interconnections of Systems:

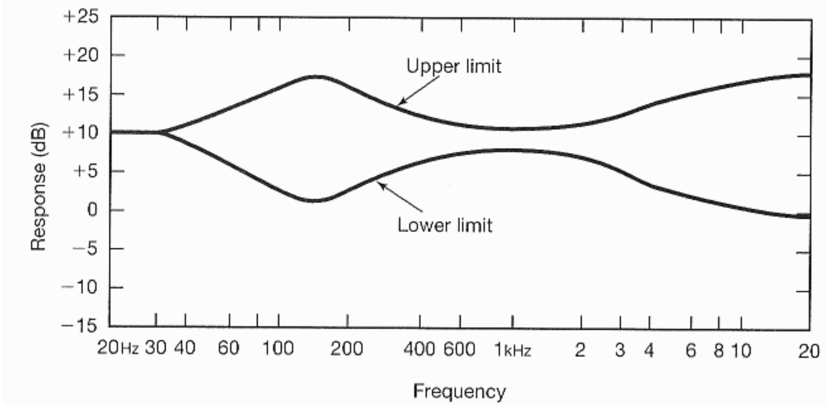
- Audio System:



(a)



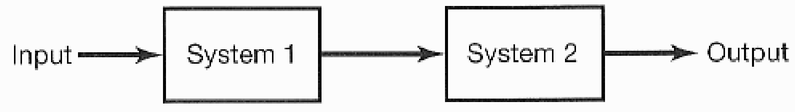
(c)



(b)

Interconnections of Systems

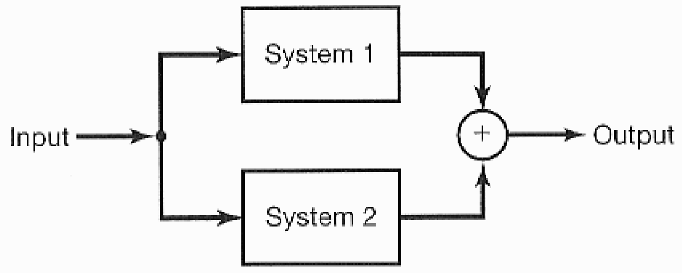
- Series or cascade interconnection of 2 systems



(a)

> e.x. radio receiver + amplifier

- Parallel interconnection of 2 systems

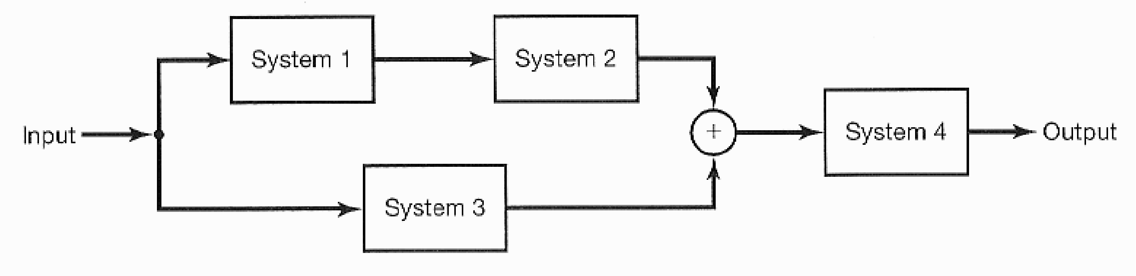


(b)

> e.x. audio system with several microphones or speakers

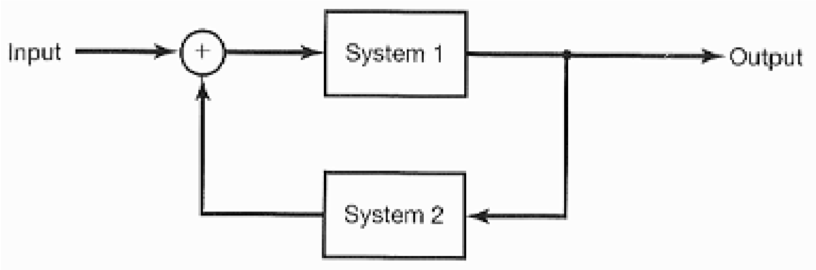
Interconnections of Systems

- Series-parallel interconnection



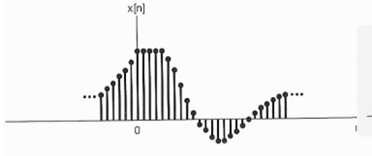
(c)

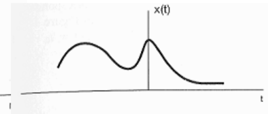
- Feedback interconnection



> e.x. cruise control, electrical circuit

- Introduction
- Continuous-Time & Discrete-Time Signals

$x[n]$ 

$x(t)$ 

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  - Time Reversal $x[-n]$ $x(-t)$
  - Time Scaling $x[an]$ $x(at)$
  - Periodic Signals $x(t) = x(t+T)$
  - Even & Odd Signals $x[n] = x[n+N]$

$x(-t) = x(t), x[-n] = x[n]$

$x(-t) = -x(t), x[-n] = -x[n]$

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$\phi_k(t) = e^{jkw_0 t}, k = 0, \pm 1, ...$

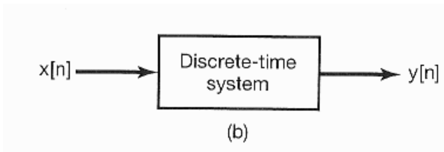
$\phi_k[n] = e^{jkw_0 n}, k = 0, ..., N-1$

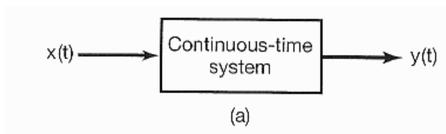
- The Unit Impulse & Unit Step Functions

$\delta[n], u[n]$

$\delta(t), u(t)$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties
  - Systems with or without memory
  - Invertibility & Inverse Systems
  - Causality
  - Stability
  - Time Invariance
  - Linearity





$x[n] \rightarrow y[n]$

$x(t) \rightarrow y(t)$

Systems with or without memory

Memoryless systems

- Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t)$$ (resistor)

Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ (accumulator)

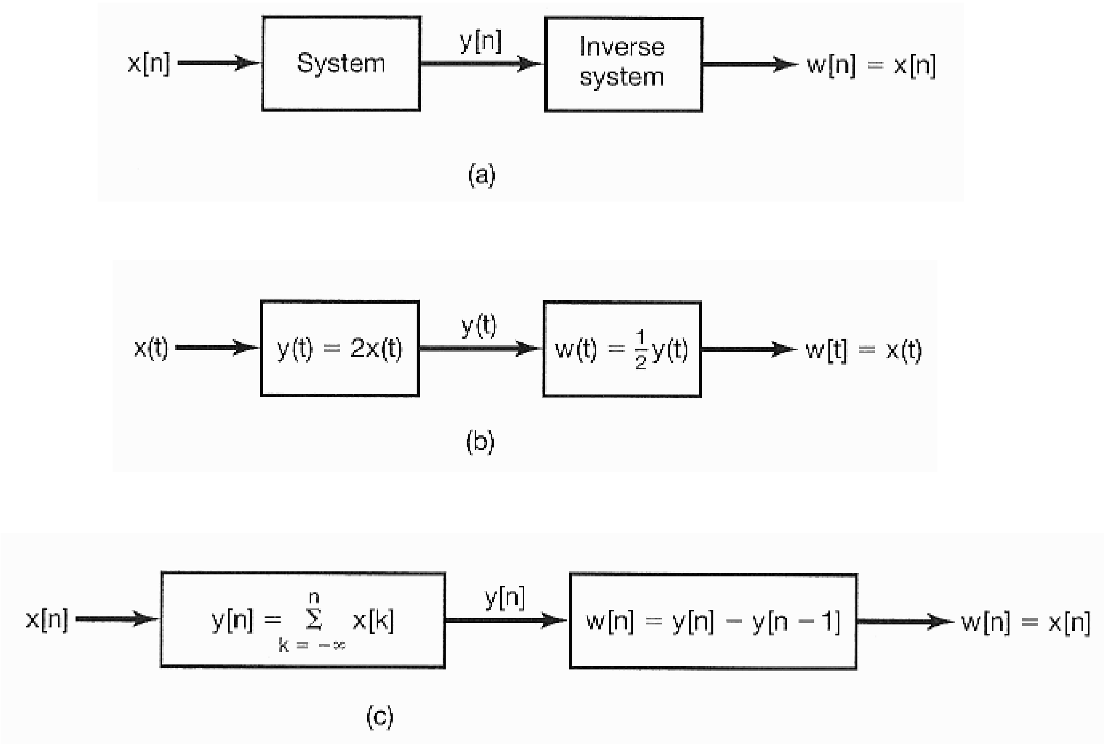
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = x[n-1]$$ (delay)

Invertibility & Inverse Systems

Invertible systems

- Distinct inputs lead to distinct outputs

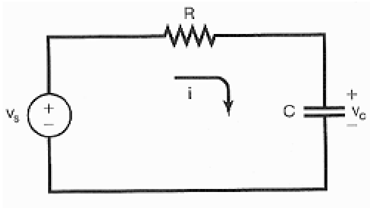


$y(t) = x(t)^2$ is not invertible

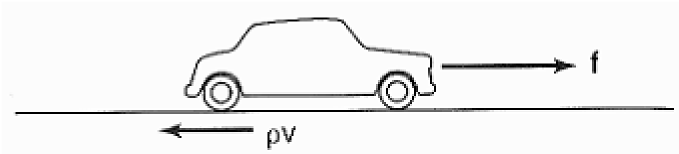
Causality

Causal systems

- Output depends only on input at present time & in the past



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

- Non-causal systems

$$y[n] = x[n] - x[n+1]$$

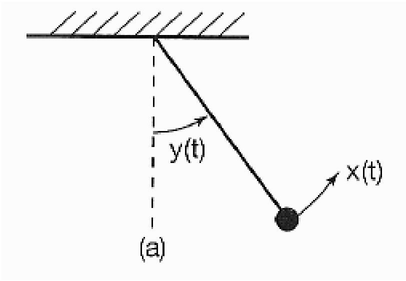
$$y(t) = x(t+1)$$

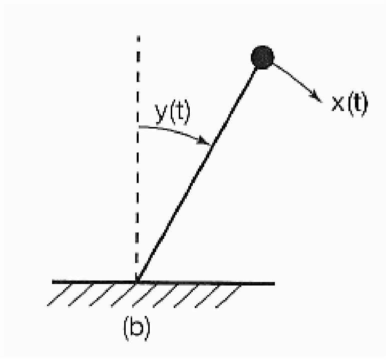
$$y(t) = x(t)\cos(t+1) \quad ???$$

Stability

Stable systems

- Small inputs lead to responses that do not diverge
- Every bounded input excites a bounded output
  - Bounded-input bounded-output stable (BIBO stable)
  - For all $|x(t)| < a$, then $|y(t)| < b$, for all t





- Balance in a bank account? $y[n] = 1.01y[n-1] + x[n]$

- Example 1.13: Stability

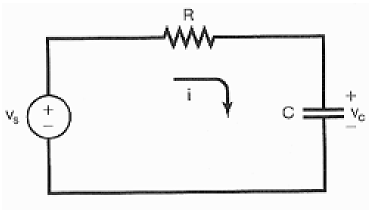
$$S_1: \quad y(t) = t\, x(t)$$

$$S_2: \quad y(t) = e^{x(t)}$$

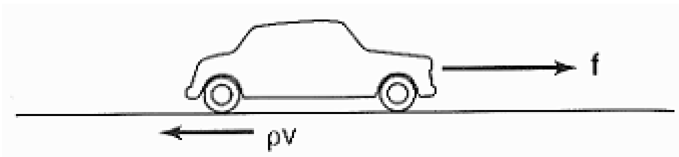
Time Invariance

Time-invariant systems

- Behavior & characteristics of system are fixed over time



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$



$$\Rightarrow \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

- A time shift in the input signal results in an identical time shift in the output signal

$$x[n] \to y[n] \quad \Longleftrightarrow \quad x[n-n_0] \to y[n-n_0]$$

- Time Invariance
  - Example of time-invariant system (Example 1.14)

$$y(t) = \sin[x(t)]$$

$x_1(t)$ $\qquad$ $y_1(t) = \sin[x_1(t)]$

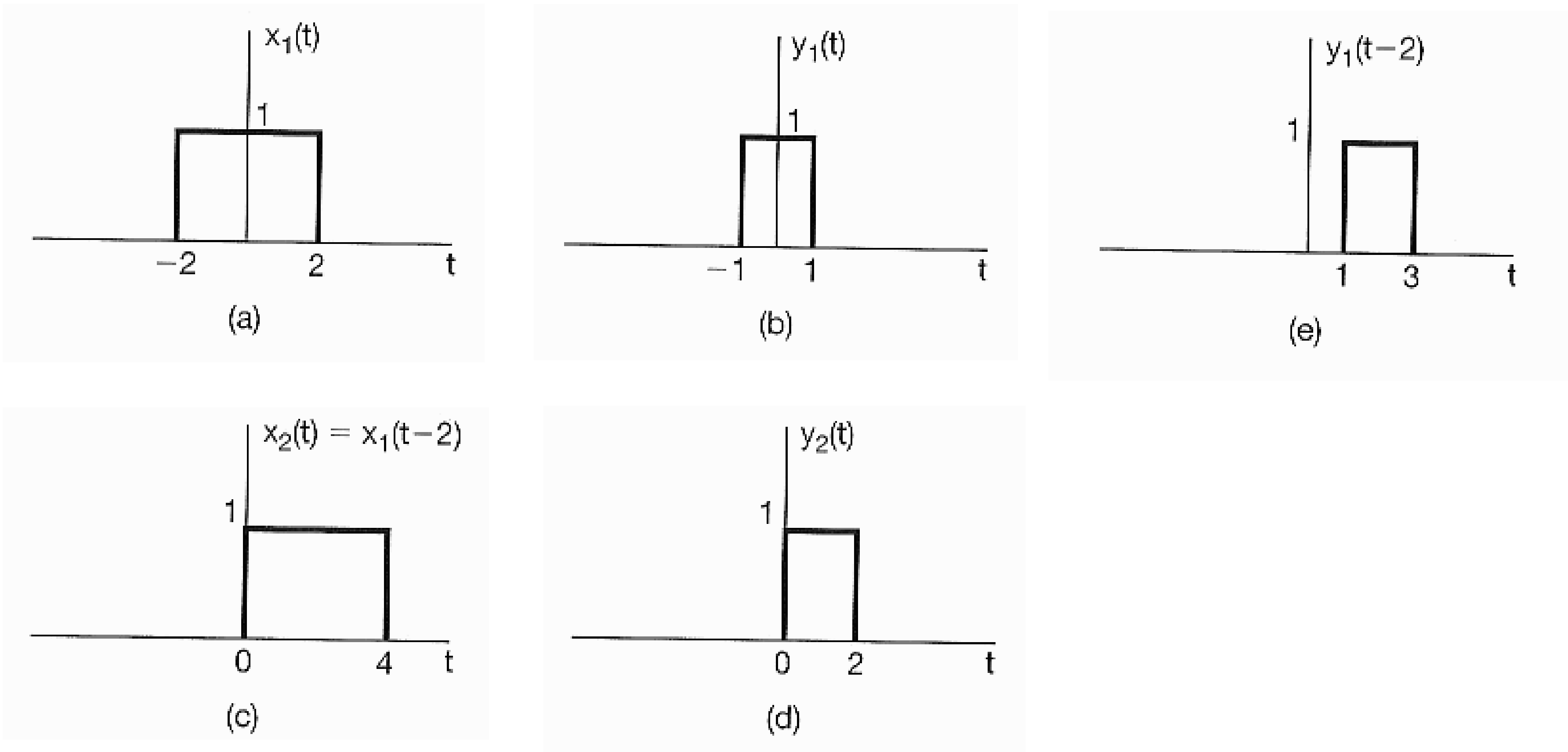
$x_2(t) = x_1(t - t_0)$ $\qquad$ $y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]$

$$y_1(t - t_0) = \sin[x_1(t - t_0)]$$

$$y_2(t) = y_1(t - t_0)$$

- Time Invariance
  - Example of time-varying system (Example 1.16)

$$y(t) = x(2t)$$



Linearity

Linear systems

- If an input consists of the weighted sum of several signals, then the output is the superposition of the responses of the system to each of those signals

$$x_1[n] \rightarrow y_1[n]$$

$$x_2[n] \rightarrow y_2[n]$$

IF (1) $x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$ (additivity)

(2) $a\, x_1[n] \longrightarrow a\, y_1[n]$ (scaling or homogeneity)

a: any complex constant

THEN, the system is linear

- Linearity
  - Linear systems
    - In general,

a, b: any complex constants

$$a x_1[n] + b x_2[n] \rightarrow a y_1[n] + b y_2[n] \qquad \text{for DT}$$

$$a x_1(t) + b x_2(t) \rightarrow a y_1(t) + b y_2(t) \qquad \text{for CT}$$

    - OR,

$$x[n] = \sum_k a_k x_k[n] = a_1 x_1[n] + a_2 x_2[n] + \ldots$$

$$\longrightarrow \ y[n] = \sum_k a_k y_k[n] = a_1 y_1[n] + a_2 y_2[n] + \ldots$$

This is known as the superposition property

- Linearity
  - Example 1.17: $S: \; y(t) = t x(t)$

$$x_1(t) \rightarrow y_1(t) = t x_1(t)$$

$$x_2(t) \rightarrow y_2(t) = t x_2(t)$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$\rightarrow \quad y_3(t) = t x_3(t)$$

$$= t(a x_1(t) + b x_2(t)) = a t x_1(t) + b t x_2(t)$$

$$= a y_1(t) + b y_2(t)$$

- Linearity
  - Example 1.18: $S: \; y(t) = (x(t))^2$

$$x_1(t) \quad \rightarrow \quad y_1(t) = (x_1(t))^2$$

$$x_2(t) \quad \rightarrow \quad y_2(t) = (x_2(t))^2$$

$$x_3(t) = a x_1(t) + b x_2(t)$$

$$\rightarrow \quad y_3(t) = (x_3(t))^2 \qquad = (a x_1(t) + b x_2(t))^2$$

$$= a^2 (x_1(t))^2 + b^2 (x_2(t))^2 + 2ab x_1(t) x_2(t)$$

$$= a^2 y_1(t) + b^2 y_2(t) + 2ab x_1(t) x_2(t)$$

- Linearity
  - Example 1.20: $S: \; y[n] = 2x[n] + 3$

$$x_1[n] \quad \rightarrow \quad y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \quad \rightarrow \quad y_2[n] = 2x_2[n] + 3$$

$$x_3[n] = ax_1[n] + bx_2[n]$$

$$\begin{aligned}
\rightarrow \quad y_3[n] &= 2x_3[n] + 3 \\
&= 2(ax_1[n] + bx_2[n]) + 3 \\
&= a(2x_1[n] + 3) + b(2x_2[n] + 3) + 3 - 3a - 3b \\
&= ay_1[n] + by_2[n] + 3(1 - a - b)
\end{aligned}$$

- Linearity
  - Example 1.20: $S: \; y[n] = 2x[n] + 3$

$$x_1[n] \quad \rightarrow \quad y_1[n] = 2x_1[n] + 3$$

$$x_2[n] \quad \rightarrow \quad y_2[n] = 2x_2[n] + 3$$

  - However,

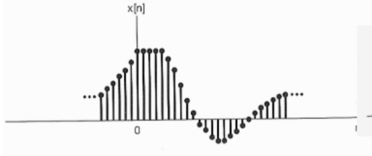
$$y_1[n] - y_2[n] = \left(2x_1[n] + 3\right) - \left(2x_2[n] + 3\right)$$

$$= 2\left[x_1[n] - x_2[n]\right]$$

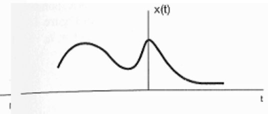
It is a incrementally linear system

- Introduction
- Continuous-Time & Discrete-Time Signals

$x[n]$



$x(t)$



- Transformations of the Independent Variable
  - Time Shift $\quad x[n-n_0] \quad x(t-t_0)$
  - Time Reversal $\quad x[-n] \quad x(-t)$
  - Time Scaling $\quad x[an] \quad x(at)$
  - Periodic Signals $\quad x(t) = x(t+T)$
  - Even & Odd Signals $\quad x[n] = x[n+N]$

$$x(-t) = x(t), x[-n] = x[n]$$

$$x(-t) = -x(t), x[-n] = -x[n]$$

$$\mathcal{E}v\left\{x[n]\right\} = \frac{1}{2}\left[x[n] + x[-n]\right]$$

$$\mathcal{O}d\left\{x[n]\right\} = \frac{1}{2}\left[x[n] - x[-n]\right]$$

- Exponential & Sinusoidal Signals

$$\phi_k(t) = e^{jkw_0 t}, k = 0, \pm 1, \ldots$$

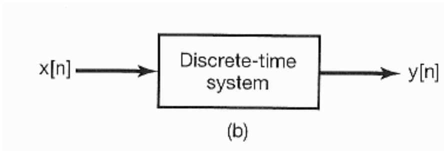
$$\phi_k[n] = e^{jkw_0 n}, k = 0, \ldots, N-1$$

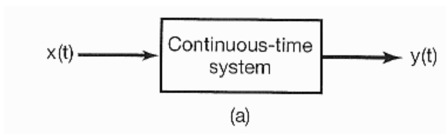
- The Unit Impulse & Unit Step Functions

$$\delta[n], u[n]$$

$$\delta(t), u(t)$$

- Continuous-Time & Discrete-Time Systems
- Basic System Properties
  - Systems with or without memory
  - Invertibility & Inverse Systems
  - Causality
  - Stability
  - Time Invariance
  - Linearity





$$x[n] \rightarrow y[n] \qquad x(t) \rightarrow y(t)$$

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3)
- CT
- DT

Aperiodic

FT
- CT (Chap 4)
- DT (Chap 5)

Unbounded/Non-convergent

LT
- CT (Chap 9)

zT
- DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)

- Problem 1.26 (Page 61)

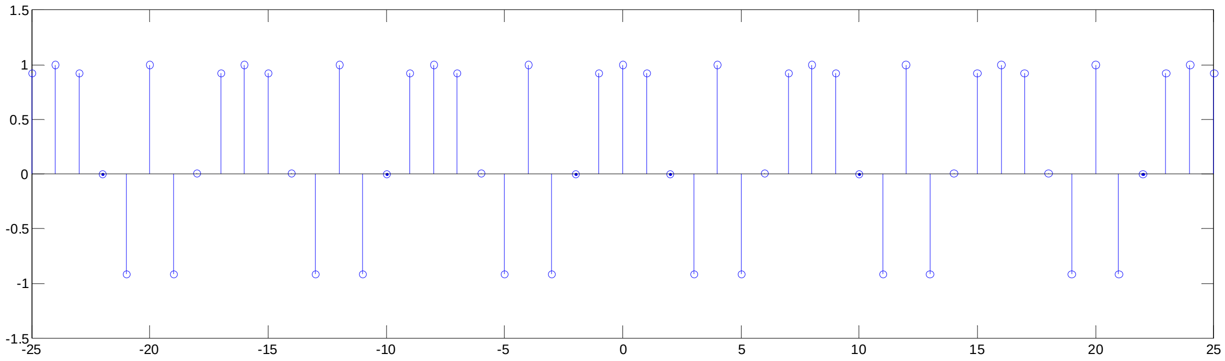
$$x[n] = \cos\left(\frac{\pi}{8}\, n^2\right)$$

```
L = 25;
n = -L:L;

x = cos( pi/8 * (n.^2) );

figure(1)
stem( n, x, 'o' ); hold on;

axis([-L L -1.5 1.5])
```



- Problem 1.27 (Page 62)

$(a)\ y(t) = x(t-2) + x(2-t)$ Time-Invariant?

$x_1(t) \rightarrow y_1(t)$ $\quad y_1(t) = x_1(t-2) + x_1(2-t)$

$x_2(t) \rightarrow y_2(t)$ $\quad y_2(t) = x_2(t-2) + x_2(2-t)$

$$x_2(t) = x_1(t-t_0)$$

$$\Rightarrow y_2(t) = x_2(t-2) + x_2(2-t)$$

$$= x_1((t-t_0)-2) + x_1(2-(t-t_0))$$

$$= x_1(t-t_0-2) + x_1(2-t+t_0))$$

$$\Rightarrow y_2(t) = x_1(t-2-t_0) + x_1(2-t-t_0)$$

$$= x_1(t-t_0-2) + x_1(2-t-t_0)$$

- Problem 1.27 (Page 62)

$(a)\ y(t) = x(t-2) + x(2-t)$ Time-Invariant?

$x_1(t) = \delta(t)$

$y_1(t) = \delta(t-2) + \delta(2-t)$

$x_2(t) = \delta(t-3)$

$$y_2(t) = \delta(t-2-3) + \delta(2-t-3)$$
$$= \delta(t-5) + \delta(-1-t)$$

$$\Rightarrow y_1(t-3) = \delta(t-3-2) + \delta(2-(t-3))$$
$$= \delta(t-5) + \delta(5-t)$$