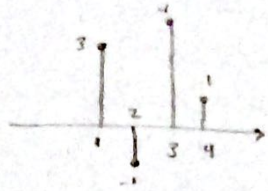
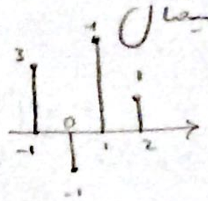
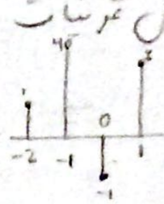
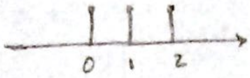
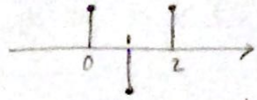
حل تمرینات سری 2 سیگنال

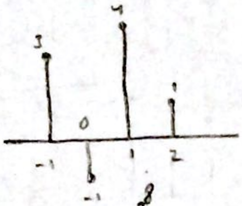
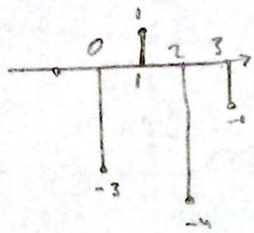
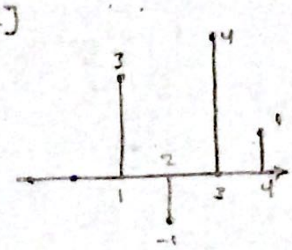
سؤال 1

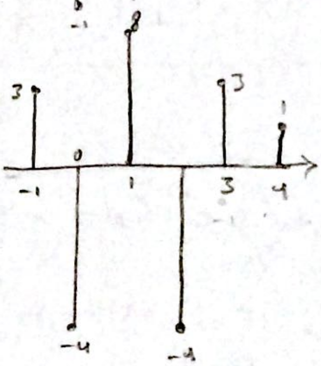
$h[n-2] =$  $\Rightarrow h[n] =$  $, \; h[-n] =$ 

a) $x[n] = [\cos(\pi n) + j\sin(\pi n)]\Big($  $\Big) =$ 

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{2} x[k]\,h[n-k] = \overset{1}{\cancel{x[0]}}h[n-0] + \underset{-1}{\cancel{x[1]}}h[n-1]$$

$$+ \overset{1}{\cancel{x[2]}}h[n-2] \Rightarrow y[n] = h[n] - h[n-1] + h[n-2]$$

$\Rightarrow y[n] =$  $+$  $+$ 

$=$ 

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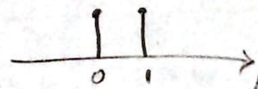
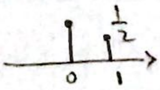
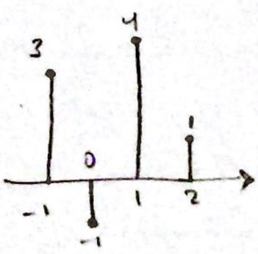
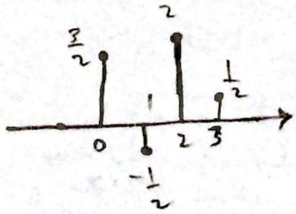
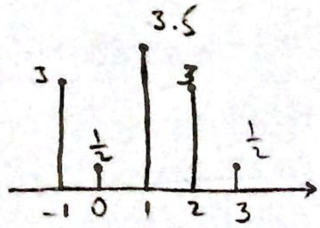
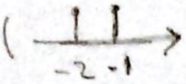
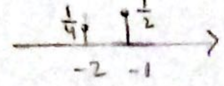
b) $x[n] = \left(\frac{1}{2}\right)^n \Big($  $\Big) =$ 

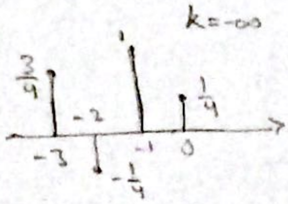
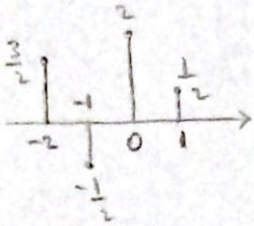
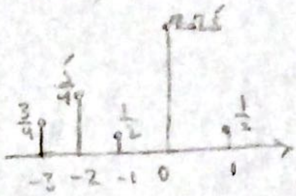
$$y[n] = x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k] = \overset{1}{\cancel{x[0]}}h[n] + \overset{\frac{1}{2}}{\cancel{x[1]}}h[n-1]$$

$$\Rightarrow y[n] = h[n] + \frac{1}{2}h[n-1]$$

$\Rightarrow y[n] =$  $+$  $=$ 

c) $x[n] = 2^n\Big($  $\Big) =$ 

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[-2]\,h[n+2] + x[-1]\,h[n+1]$$

$\Rightarrow y[n] =$  $+$  $=$ 

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d) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \Rightarrow y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k]$

$$= \cdots + x[3]\,\underbrace{h[-2]}_{0} + x[2]\,h[-1] + x[1]\,h[0] + x[0]\,h[1] + \underbrace{x[-1]\,h[2]}_{0} + \cdots$$

$$\Rightarrow y[1] = 3 + 1 + 4 = 8$$

سؤال 2

a) $y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} u(\tau)\,e^{t-\tau}\,d\tau = \int_{0}^{\infty} e^{t-\tau}\,d\tau = -e^{t-\tau}\Big|_0^{\infty}$

$$= 0 - (-e^{t}) = e^{t}$$

---

b) $y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} u(\tau)\,e^{t-\tau}\,u(t-\tau)\,d\tau = \int_{0}^{\infty} e^{t-\tau}\,u(t-\tau)\,d\tau$

$$u(t-\tau) = \begin{cases} 0, & \tau > t \\ 1, & \tau \leq t \end{cases} \Rightarrow y(t) = \int_0^t e^{t-\tau}\,d\tau = -e^{t-\tau}\Big|_0^t = -1 - (-e^{t}) = e^{t} - 1$$

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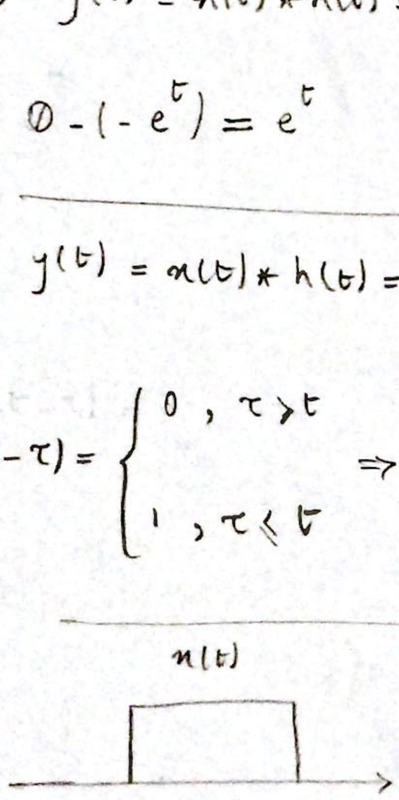
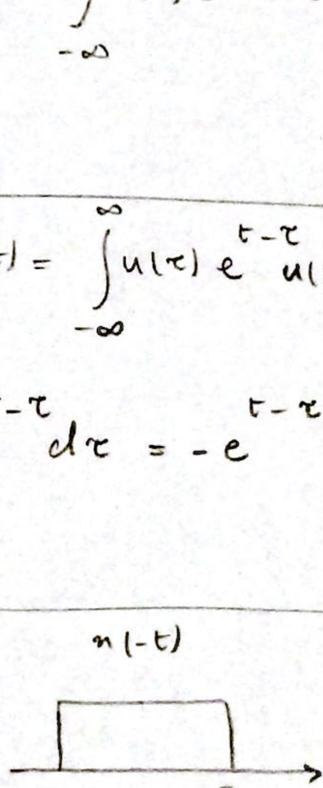
c)

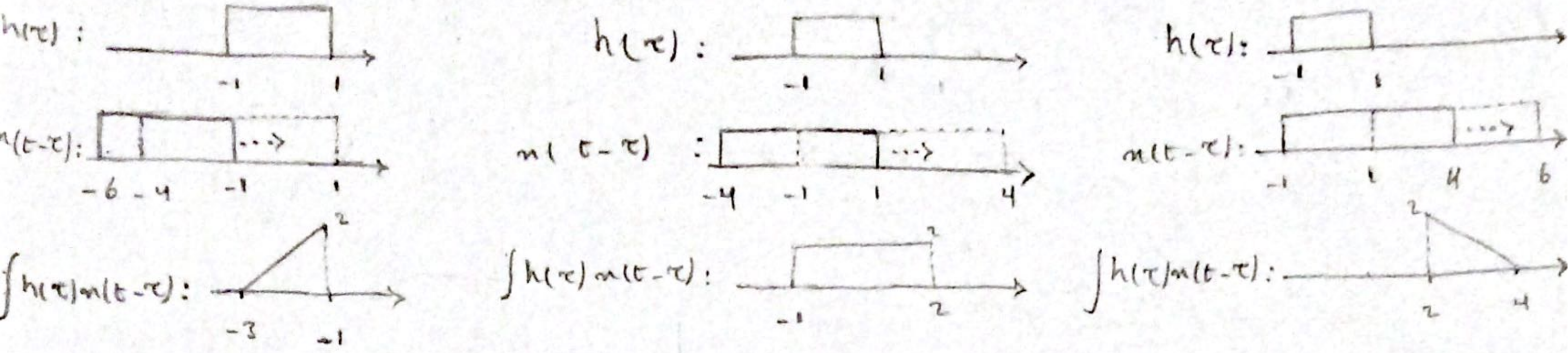
$x(t)$

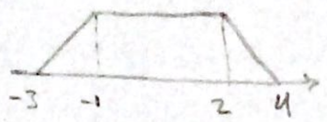


$\rightarrow$

$x(-t)$





$\Rightarrow x(t) * h(t) =$ 

d) $x(t) * \delta(t+t_0) = x(t+t_0) \Rightarrow x(t) * h(t) = x(t+\frac{3}{2}) + 2x(t+\frac{5}{2})$

$$= \begin{cases} 2(t+\frac{3}{2})+1 & , -1 \leq t+\frac{3}{2} < 1 \\ \ln(t+\frac{3}{2}) & , 1 \leq t+\frac{3}{2} < 2 \\ e^{t+\frac{3}{2}} & , 2 \leq t+\frac{3}{2} < 3 \\ 0 & , \text{O.W} \end{cases} + \begin{cases} 4(t+\frac{5}{2})+2 & , -1 \leq t+\frac{5}{2} < 1 \\ 2\ln(t+\frac{5}{2}) & , 1 \leq t+\frac{5}{2} < 2 \\ 2e^{t+\frac{5}{2}} & , 2 \leq t+\frac{5}{2} < 3 \\ 0 & , \text{O.W} \end{cases}$$

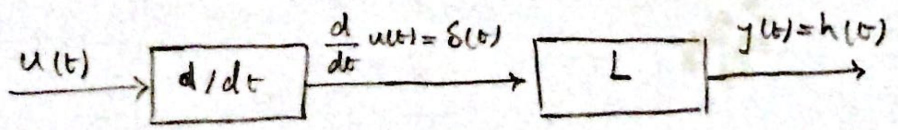
$$= \begin{cases} 4(t+\frac{5}{2})+2 & , -3.5 \leq t < -2.5 \\ 2(t+\frac{3}{2})+1+4(t+\frac{5}{2})+2 & , -2.5 \leq t < -1.5 \\ 2(t+\frac{3}{2})+1+2\ln(t+\frac{5}{2}) & , -1.5 \leq t < -0.5 \\ \ln(t+\frac{3}{2})+2e^{t+\frac{5}{2}} & , -0.5 \leq t < 0.5 \\ e^{t+\frac{3}{2}} & , 0.5 \leq t < 1.5 \end{cases}$$

سؤال 3

a) $y(t) = \frac{d}{dt} x(t)$ خطی: $\frac{d}{dt}\left(x(t)+u(t)\right) = \frac{d}{dt}x(t) + \frac{d}{dt}u(t)$

تغییرناپذیر با زمان: $\frac{d}{dt}x(t) = y(t) \Rightarrow \frac{d}{dt}x(t+t_0) = y(t+t_0)$

می‌دانیم که سیستم‌های LTI سری، به دلیل خاصیت جابه‌جایی کانولوشن می‌توان جای سیستم‌ها را عوض کرد و خروجی نهایی سیستم همان خواهد بود:



$\Rightarrow h(t) = \frac{d}{dt}\delta(t)$

سؤال 4

* یک سیستم LTI اگر بخواهد بی‌حافظه باشد، باید پاسخ ضربه‌ی آن به فرم $k\delta(t)$ باشد
$k\delta[n]$

* یک سیستم LTI اگر بخواهد علّی باشد، باید شرط $\forall t<0 , h(t)=0$ برقرار باشد.

* یک سیستم LTI اگر بخواهد پایدار باشد، باید پاسخ ضربه‌ی آن مطلقاً جمع‌پذیر (گسسته) یا مطلقاً انتگرال‌پذیر (پیوسته) باشد:

$$\sum_{-\infty}^{\infty} |h[k]| < \infty \quad , \quad \int_{-\infty}^{\infty} |h(t)| < \infty$$

---

a) $h(t) \neq k\,\delta(t) \Rightarrow$ حافظه‌دار است

$h(-1) = e^{-4} \neq 0 \Rightarrow$ علّی نیست

$$\int_{-\infty}^{\infty} |e^{-4|t|}| dt = \int_{-\infty}^{0} e^{4t} dt + \int_{0}^{\infty} e^{-4t} dt = \frac{e^{4t}}{4}\Big|_{-\infty}^{0} + \frac{-e^{-4t}}{4}\Big|_{0}^{\infty} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \Rightarrow$$ پایدار است

---

b) $te^{-t}u(t) \neq k\,\delta(t) \Rightarrow$ حافظه‌دار است

$u(t)=0 , t<0 \Rightarrow te^{-t}u(t)=0 , t<0 \Rightarrow$ علّی است

$$\int_{-\infty}^{\infty} |te^{-t}u(t)|\, dt = \int_{0}^{\infty} \underbrace{t}_{u}\, \underbrace{e^{-t} dt}_{dv} = \left[-te^{-t} - \int -e^{-t} dt\right]\Big|_{0}^{\infty} = -te^{-t} - e^{-t}\Big|_{0}^{\infty}$$

$$= -\underbrace{\lim_{t\to\infty} \frac{t}{e^{t}}}_{0} + \cancel{e^{-t}} + 1 < \infty \Rightarrow$$ پایدار است

---

c) $h(t) \neq k\,\delta(t) \Rightarrow$ حافظه‌دار است

$h(-\frac{1}{2}) = \cos(-\pi)\, u(0.5) = -1 \neq 0 \Rightarrow$ علّی نیست

$$\int_{-\infty}^{\infty} |\cos(2\pi t)\, u(t+1)|\, dt = \int_{-1}^{\infty} |\cos(2\pi t)|\, dt \;,\; \lim_{t\to\infty} |\cos(2\pi t)| \neq 0 \Rightarrow$$ انتگرال آن به بی‌نهایت همگرا نیست

$\Rightarrow$ سیستم پایدار نیست

d) $|\sin(t)| \leq 1 \Rightarrow \forall t>1 , \left|\frac{\sin(t)}{t}\right| \leq \left|\frac{1}{t}\right|$

این عبارت مطلقاً انتگرال پذیر است پس این عبارت هم انتگرالش همگرا و محدود است $\Rightarrow \int_1^{\infty} \left|\frac{\sin(t)}{t}\right| dt < \infty$ I

$\lim_{t\to 0} \frac{\sin(t)}{t} = 1$

$\frac{\sin(t)}{t}$ در بازه $[0,1]$

مخرجش ریشه ندارد و به

دو نهایت میل نمی کند

$\Rightarrow \int_0^1 \left|\frac{\sin(t)}{t}\right| dt < \infty$ II

$\xrightarrow{\text{I و II}} \int_0^{\infty} \left|\frac{\sin(t)}{t}\right| dt < \infty$

$\Rightarrow$ پایدار است

$h(t) \neq k\delta(t) \Rightarrow$ حافظه دارد

$h(t) = 0,\ t<0 \Rightarrow$ علی است

---

e) $h[n] \neq k\delta[n] \Rightarrow$ حافظه دار است

$h[-1] = \left(\frac{1}{2}\right)^{-1} u[1] = 2 \neq 0 \Rightarrow$ علی نیست

$\sum_{-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u[-n]\right| = \sum_{-\infty}^{0} \left(\frac{1}{2}\right)^n$

$\lim_{n\to -\infty} 2^{-n} = \infty$

$\Rightarrow$ پایدار نیست

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f) $\delta[2n] = \delta[n] \Rightarrow h[n] = \delta[n] \Rightarrow$ بدون حافظه است

$\delta[n] = 0,\ n<0 \Rightarrow$ علی است

$\sum_{-\infty}^{\infty} |\delta[n]| = 1 \Rightarrow$ پایدار است

---

g) $h[n] \neq k\delta[n] \Rightarrow$ حافظه دار است

$n<-1 : u[n+1] = 0$

$n=-1 : \cos\left(\frac{\pi}{2}n\right) = 0$

$\Rightarrow h[n] = 0,\ n<0 \Rightarrow$ علی است

$\sum_{-\infty}^{\infty} \left|\cos\left(\frac{\pi}{2}n\right) u[n+1]\right| = \sum_{-1}^{\infty} \left|\cos\left(\frac{\pi}{2}n\right)\right| = \infty \Rightarrow$ پایدار نیست

h) $h[n] \neq K\delta[n] \Rightarrow$ حافظه دار است

$u[n]=0,\ n<0 \Rightarrow h[n]=0,\ n<0 \Rightarrow$ علی است

$$\sum_{-\infty}^{\infty} |e^{2n}u[n]| = \sum_{0}^{\infty} e^{2n} = \infty \Rightarrow$$ پایدار نیست

$\boxed{\text{سؤال ۴}}$

$$x(t)*h(t) = h(t)*x(t) = \int_{-\infty}^{\infty} [2u(\tau)+4u(\tau-2)+8u(\tau-4)]\, e^{2t-2\tau} u(1-t+\tau)\, d\tau$$

$$= 2\int_0^2 e^{2t-2\tau} u(1-t+\tau)\, d\tau + 6\int_2^4 e^{2t-2\tau} u(1-t+\tau)\, d\tau + 14\int_4^{\infty} e^{2t-2\tau} u(1-t+\tau)\, d\tau$$

$$u(1-t+\tau) = \begin{cases} 0, & \tau < t-1 \\ 1, & \tau \geq t-1 \end{cases} \quad \text{I}$$

$$\xrightarrow{\text{I}} 2\int_0^2 e^{2t-2\tau} u(1-t+\tau) = \begin{cases} 2\int_0^2 e^{2t-2\tau}\, d\tau, & t<1 \\ 2\int_{t-1}^2 e^{2t-2\tau}\, d\tau, & 1 \leq t < 3 \\ 0, & t \geq 3 \end{cases} = \begin{cases} -e^{2t-4}+e^{2t}, & t<1 \\ -e^{2t-4}+e^{2}, & 1\leq t<3 \\ 0, & t\geq 3 \end{cases}$$

$$\xrightarrow{\text{I}} 6\int_2^4 e^{2t-2\tau} u(1-t+\tau)\, d\tau = \begin{cases} 6\int_2^4 e^{2t-2\tau}\, d\tau, & t<3 \\ 6\int_{t-1}^4 e^{2t-2\tau}\, d\tau, & 3\leq t<5 \\ 0, & t\geq 5 \end{cases} = \begin{cases} -3e^{2t-8}+3e^{2t-4}, & t<3 \\ -3e^{2t-8}+3e^{2}, & 3\leq t<5 \\ 0, & t\geq 5 \end{cases}$$

$$\xrightarrow{\text{I}} 14\int_4^{\infty} e^{2t-2\tau} u(1-t+\tau)\, d\tau = \begin{cases} 14\int_4^{\infty} e^{2t-2\tau}\, d\tau, & t<5 \\ 14\int_{t-1}^{\infty} e^{2t-2\tau}\, d\tau, & t\geq 5 \end{cases} = \begin{cases} 7e^{2t-8}, & t<5 \\ 7e^{2}, & t\geq 5 \end{cases}$$

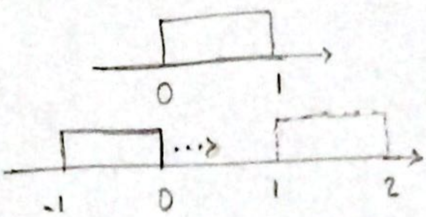
$$\Rightarrow x(t) * h(t) = \begin{cases} -e^{2t-4} + e^{2t} + 3e^{2t-4} - 3e^{2t-8} + 7e^{2t-8} & , t<1 \\ -e^{2t-4} + e^{2} - 3e^{2t-8} + 3e^{2t-4} + 7e^{2t-8} & , 1 \leqslant t < 3 \\ -3e^{2t-8} + 3e^{2} + 7e^{2t-8} & , 3 \leqslant t < 5 \\ 7e^{2} & , t \geqslant 5 \end{cases}$$

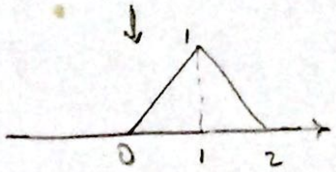
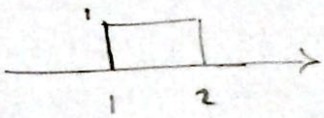
سؤال 6

$$h_{eq} = h_1(t) + h_2(t) * [h_3(t) + h_4(t)]$$

$$= e^{-t}u(t) + [u(t)-u(t-1)] * [u(t)-u(t-1)] + [u(t)-u(t-1)] * \delta(t-1)$$

نکته ← این نوع کانولوشن ها را سعی کنید اکثر امتحان با شکل حل کنید. وگرنه دقت کم می آورید.



$$= e^{-t}u(t) +$$  $$+$$ 

$$= \begin{cases} 0 & , t<0 \\ e^{-t} + t & , 0 \leqslant t < 1 \\ e^{-t} - t + 3 & , 1 \leqslant t < 2 \\ e^{-t} & , t \geqslant 2 \end{cases}$$

سؤال 7

$$x[n] * h[n] = \sum_{-\infty}^{\infty} x[k]h[n-k] = y[n]$$

$$x[n] * h[n-1] , \sum_{-\infty}^{\infty} x[k]h[n-k-1] = \sum_{-\infty}^{\infty} x[k]h[n-1-k] = y[n-1]$$

$$x[n] * \alpha^{n} u[n] = y[n] \quad (\mathrm{I})$$

$$x[n] * \alpha^{n-1} u[n-1] = y[n-1] \Rightarrow x[n] * \alpha^{n} u[n-1] = \alpha y[n-1] \quad (\mathrm{II})$$

$$\xrightarrow{(\mathrm{I}) - (\mathrm{II})} x[n] * (\alpha^{n} u[n] - \alpha^{n} u[n-1]) = y[n] - \alpha y[n-1]$$

$$x[n] * (\alpha^{n} \underbrace{(u[n]-u[n-1])}_{\delta[n]}) = x[n] * \underbrace{(\alpha^{n}\delta[n])}_{\alpha^{0}\delta[n]} = x[n] * \alpha^{0}\delta[n] = x[n] * \delta[n] = x[n]$$

$$\Rightarrow \boxed{x[n] = y[n] - \alpha y[n-1]}$$

سؤال 8

* دو سیستم LTI، معکوس یکدیگرند هرگاه: $h_1(t) * h_2(t) = \delta(t)$

a) $e^{-t}u(t) * (\delta(t) + \delta'(t)) = e^{-t}u(t) * \delta(t) + e^{-t}u(t) * \delta'(t) = e^{-t}u(t) + (e^{-t}u(t))' * \delta(t)$

$= e^{-t}u(t) + [-e^{-t}u(t) + e^{-t}\delta(t)] * \delta(t) = e^{-t}\delta(t) = e^{0}\delta(t) = \delta(t)$

b) $(\delta[n] - \delta[n-1]) * u[n] = \delta[n] * u[n] - \delta[n-1] * u[n] = u[n] - u[n-1] = \delta[n]$