دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Signals and Systems

Assignment 3

Fall 2019 - Group 1

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Question 1

Determine the Fourier Series coefficients a_k for the following periodic signals:

(a) $x(t) = 2cos(\frac{2\pi t}{3} + \frac{\pi}{6})$

$$\omega_0 = \frac{2\pi}{3}$$

$$x(t) = 2cos(\omega_0 t + \frac{\pi}{6})$$

$$x(t) = e^{j(\omega_0 t + \frac{\pi}{6})} + e^{-j(\omega_0 t + \frac{\pi}{6})}$$

$$\Rightarrow x(t) = e^{j\omega_0 t}e^{j\frac{\pi}{6}} + e^{-j\omega_0 t}e^{-j\frac{\pi}{6}}$$

$$\Rightarrow a_1 = e^{j\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + j\frac{1}{2}$$

$$\Rightarrow a_{-1} = e^{-j\frac{\pi}{6}} = \frac{\sqrt{3}}{2} - j\frac{1}{2}$$

$$k \neq -1, 1 \Rightarrow a_k = 0$$

(b) $x(t) = 2cos(\frac{2\pi t}{3} + \frac{\pi}{6}) + 5sin(\frac{2\pi t}{6})$

$$T_1 = \frac{2\pi}{\frac{2\pi}{3}} = 3, T_2 = \frac{2\pi}{\frac{2\pi}{6}} = 6 \Rightarrow T = 6$$

$$\Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$x(t) = e^{j(\frac{2\pi}{3}t + \frac{\pi}{6})} + e^{-j(\frac{2\pi}{3}t + \frac{\pi}{6})} + \frac{5}{2j}\left(e^{j\frac{2\pi}{6}t} - e^{-j\frac{2\pi}{6}t}\right)$$

$$\Rightarrow x(t) = e^{j\frac{2\pi}{3}t}e^{j\frac{\pi}{6}} + e^{-j\frac{2\pi}{3}t}e^{-j\frac{\pi}{6}} + \frac{5}{2j}e^{j\frac{2\pi}{6}t} - \frac{5}{2j}e^{-j\frac{2\pi}{6}t}$$

$$\Rightarrow x(t) = e^{j2\omega_0 t}e^{j\frac{\pi}{6}} + e^{-j2\omega_0 t}e^{-j\frac{\pi}{6}} + \frac{5}{2j}e^{j\omega_0 t} - \frac{5}{2j}e^{-j\omega_0 t}$$

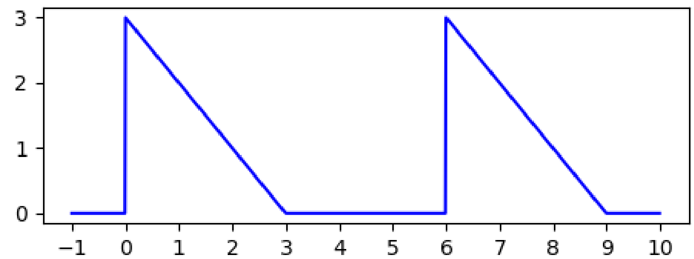
$$a_2 = e^{j\frac{\pi}{6}}$$

$$a_{-2} = e^{-j\frac{\pi}{6}}$$

$$a_1 = \frac{5}{2j}$$

$$a_{-1} = -\frac{5}{2j}$$

(c) .



$$T = 6 \Rightarrow \omega_0 = \frac{\pi}{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$\Rightarrow a_k = \frac{1}{6}\int_0^3 (-t+3)e^{-jk\frac{\pi}{3}t}dt$$

$$\Rightarrow a_k = \frac{1}{6}(-\int_0^3 te^{-jk\frac{\pi}{3}t}dt + 3\int_0^3 e^{-jk\frac{\pi}{3}t}dt)$$

$$\Rightarrow a_k = \frac{1}{6}(-I_1 + 3I_2)$$

- I_1

$$I_1 = \int_0^3 udv = uv\Big|_0^3 - \int_0^3 vdu$$
$$u = t \Rightarrow du = dt$$
$$dv = e^{-jk\frac{\pi}{3}t}dt \Rightarrow v = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t}$$
$$uv = \frac{-3t}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_0^3 = \frac{-9}{jk\pi}(cos(k\pi))$$
$$\int_0^3 vdu = \frac{-3}{jk\pi}(\frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t})\Big|_0^3 = \frac{-9}{k^2\pi^2}(cos(k\pi)-1)$$
$$\Rightarrow I_1 = \frac{9}{k\pi}\Big(\frac{1}{k\pi}(cos(k\pi)-1) - \frac{1}{j}(cos(k\pi))\Big)$$

- I_2

$$I_2 = \frac{-3}{jk\pi}e^{-jk\frac{\pi}{3}t}\Big|_0^3 = \frac{-3}{jk\pi}(e^{-jk\pi}-1) = \frac{-3}{jk\pi}(cos(k\pi)-1)$$

$$\Rightarrow a_k = \frac{1}{6}(-I_1 + 3I_2) = \frac{1}{6}\Big[\frac{-9}{k\pi}\Big(\frac{1}{k\pi}(cos(k\pi)-1) - \frac{1}{j}(cos(k\pi))\Big) + \frac{-9}{jk\pi}(cos(k\pi)-1)\Big]$$

k zero:

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{6}\frac{9}{2} = \frac{3}{4}$$

k even:

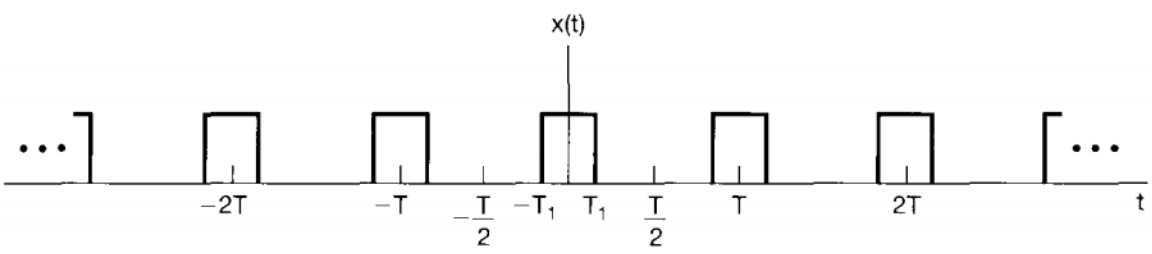
$$a_k = \frac{3}{2jk\pi}$$

k odd:

$$a_k = \frac{3}{2jk\pi} + \frac{3}{k^2\pi^2}$$

Question 2

Determine the Fourier Series coefficients a_k for $x(t)$:



substituting from eq. (3.41), we have first, for $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}. \tag{3.42}$$

As mentioned previously, a_0 is interpreted to be the average value of $x(t)$, which in this case equals the fraction of each period during which $x(t) = 1$. For $k \neq 0$, we obtain

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1},$$

which we may rewrite as

$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]. \tag{3.43}$$

Noting that the term in brackets is $\sin k\omega_0 T_1$, we can express the coefficients a_k as

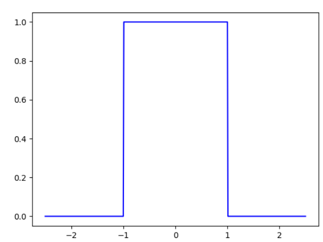
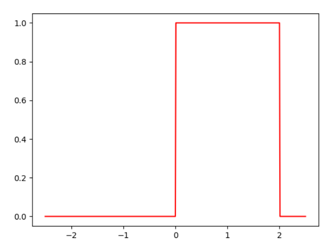
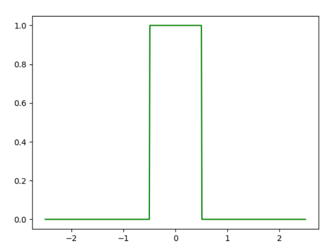
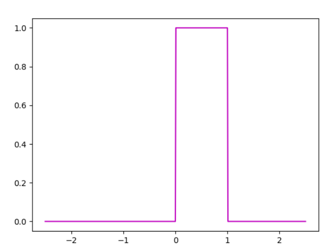
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0, \tag{3.44}$$

where we have used the fact that $\omega_0 T = 2\pi$.

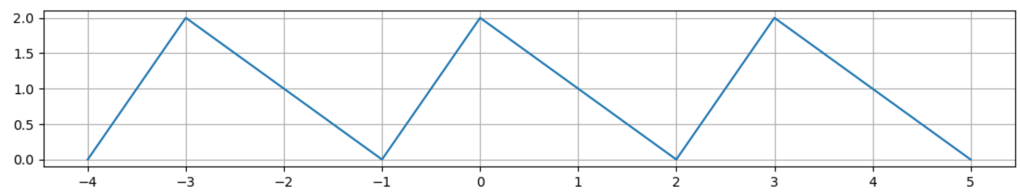
(a) .

$$T = 3 \Rightarrow \omega_0 = \frac{2\pi}{3}$$

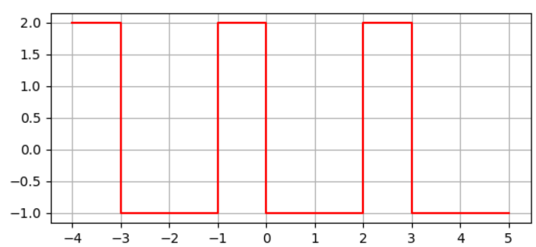
$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{3}3 = 1$$

Signal	Plot	Params	FS Coeffs
$x_1(t)$		$T = 3$, $\omega_0 = \frac{2\pi}{3}$, $T_1 = 1$	$b_k = \frac{sin(k\frac{2\pi}{3}(1))}{k\pi}$
$x_2(t)$		$x_2(t) = x_1(t-1)$	$c_k = e^{-jk\frac{2\pi}{3}}b_k$
$x_3(t)$		$T = 3$, $\omega_0 = \frac{2\pi}{3}$, $T_1 = 0.5$	$d_k = \frac{sin(k\frac{2\pi}{3}(0.5))}{k\pi}$
$x_4(t)$		$x_4(t) = x_3(t-0.5)$	$e_k = e^{-jk\frac{2\pi}{3}(0.5)}d_k$
$x(t)$		$x(t) = x_2(t) + x_4(t)$	$a_k = c_k + e_k$

(b) .



If we take the derivative of $x(t)$, $x^{'}(t)$ would be like this:



Since $x^{'}(t)$ is periodic, it has a Fourier Series representation with coefficients c_k. Since $c_0 = 0$ ($\int_T x^{'}(t)dt = 0$), we can use a property from Fourier Series which implies a_k will be equal to:

$$a_k = \frac{1}{jk\omega_0} c_k = \frac{1}{jk\frac{2\pi}{3}} c_k$$

Keep in mind that:

$$x(t) = \int_{-\infty}^{t} x^{'}(\tau)d\tau$$

Now we should compute c_k: Choosing $T = 3$ and $T_1 = \frac{1}{2}$ leaves us with $y(t)$ with Fourier coefficients d_k:

$$d_0 = \frac{1}{T}\int_T y(t)dt = \frac{1}{3}$$

$$d_k = \frac{sin(k\frac{\pi}{3})}{k\pi}$$

Choosing $T = 3$ and $T_1 = 1$ leaves us with $w(t)$ with Fourier coefficients e_k:

$$e_0 = \frac{1}{T}\int_T w(t)dt = \frac{2}{3}$$

$$e_k = \frac{sin(k\frac{2\pi}{3})}{k\pi}$$

Combining these two signals in a specific way, gives us $x^{'}(t)$ with Fourier coefficients c_k (As mentioned before).

$$x^{'}(t) = 2y(t + \frac{1}{2}) - w(t - 1)$$

$$y(t) \overset{\text{FS}}{\longleftrightarrow} d_k$$
$$y(t + \frac{1}{2}) \overset{\text{FS}}{\longleftrightarrow} e^{-jk\frac{2\pi}{3}\frac{-1}{2}} d_k = e^{jk\frac{\pi}{3}} d_k$$
$$w(t) \overset{\text{FS}}{\longleftrightarrow} e_k$$
$$w(t - 1) \overset{\text{FS}}{\longleftrightarrow} e^{-jk\frac{2\pi}{3}(1)} e_k = e^{-jk\frac{2\pi}{3}} e_k$$
$$x^{'}(t) = 2y(t + \frac{1}{2}) - w(t - 1) \overset{\text{FS}}{\longleftrightarrow} 2e^{jk\frac{\pi}{3}} d_k - e^{-jk\frac{2\pi}{3}} e_k = c_k$$
$$\int_{-\infty}^{t} x^{'}(\tau) d\tau \overset{\text{FS}}{\longleftrightarrow} \frac{1}{jk\frac{2\pi}{3}} c_k$$

Question 3

(Textbook Section 3.8 - Fourier Series and LTI Systems)

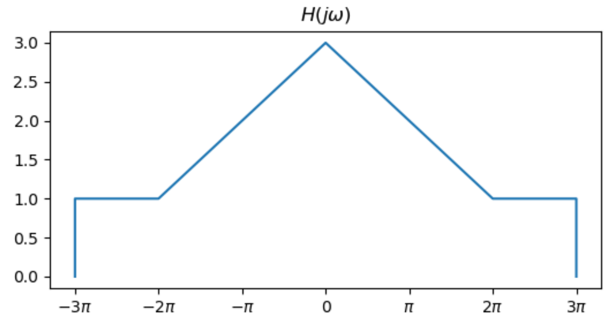
Imagine we have a signal $x(t)$ with Fourier Series representation like this:

$$a_{-2} = a_2 = \frac{1}{4}$$

$$a_{-1} = a_1 = \frac{1}{2}$$

$$a_0 = 1$$

And otherwise $a_k = 0$. Keep in mind that $T = 2$. Consider a LTI System with frequency response $H(j\omega)$ as plotted below.



(a) Determine the output $y(t)$, and its Fourier Series coefficients b_k, if we apply $x(t)$ as input.

$$T = 2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi$$

We know that if $x(t)$ has a Fourier Series representation, if we apply it to a system with frequency response $H(j\omega)$, the output would look like this:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

We know that:

$$x(t) = \frac{1}{4}e^{-j2\pi t} + \frac{1}{2}e^{-j1\pi t} + 1 + \frac{1}{2}e^{j1\pi t} + \frac{1}{4}e^{j2\pi t}$$

$$\Rightarrow y(t) = \frac{1}{4}H(j(-2\pi))e^{-j2\pi t}+\frac{1}{2}H(j(-\pi))e^{-j1\pi t}+1H(j(0))+\frac{1}{2}H(j(1\pi))e^{j1\pi t}+\frac{1}{4}H(j(2\pi))e^{j2\pi t}$$

Now we should determine $H(j\omega)$ values:

$$H(j\omega) = \begin{cases} 1 & -3\pi \leq \omega < -2\pi \\ \frac{1}{\pi}\omega + 3 & -2\pi \leq \omega < 0 \\ \frac{-1}{\pi}\omega + 3 & 0 \leq \omega < 2\pi \\ 1 & 2\pi \leq \omega < 3\pi \\ 0 & otherwise \end{cases}$$

- $H(j(-2\pi)) = 1$
- $H(j(-\pi)) = 2$
- $H(j(0)) = 3$
- $H(j(1\pi)) = 2$
- $H(j(2\pi)) = 1$

$$\Rightarrow y(t) = \frac{1}{4}e^{-j2\pi t} + e^{-j1\pi t} + 3 + e^{j1\pi t} + \frac{1}{4}e^{j2\pi t}$$

$$\Rightarrow b_{-2} = \frac{1}{4}, b_{-1} = 1, b_0 = 3, b_1 = 1, b_2 = \frac{1}{4}$$

(b) Using Parseval's relation, determine the average power of $y(t)$.

$$AvgPower = \frac{1}{T}\int_T |y(t)|^2 = \sum_{k=-\infty}^{\infty} |b_k|^2$$

$$\Rightarrow AvgPower = \frac{1}{16} + 1 + 9 + 1 + \frac{1}{16}$$