حل تمرین مبحث تبدیل فوریه گسسته.

1) a) $x[2-n] \overset{F}{\longleftrightarrow} e^{-2j\omega} X(e^{-j\omega})$

$x[1-n] \overset{F}{\longleftrightarrow} e^{-j\omega} X(e^{-j\omega})$

$\Rightarrow X_1(e^{j\omega}) = (e^{-2j\omega} - e^{-j\omega}) X(e^{j\omega})$

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b) $n^2 x[n] \overset{F}{\longleftrightarrow} -\dfrac{d^2 X(e^{j\omega})}{d\omega^2}$

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c) $\sum_{k=-\infty}^{n} x[k] \overset{F}{\longleftrightarrow} \dfrac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l)$

$\Rightarrow e^{j\omega_0 n} \sum_{k=-\infty}^{n} x[n] \overset{F}{\longleftrightarrow} \dfrac{1}{1-e^{-j(\omega-\omega_0)}} X(e^{j(\omega-\omega_0)}) + \pi X(e^{j0}) \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$

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2) a) $x[n] = \frac{1}{4} \times \left(\frac{1}{2}\right)^{n-2} u[n-2] \Rightarrow X(e^{j\omega}) = \frac{1}{4} \times \dfrac{e^{-2j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

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b) $x[n] = \dfrac{(n+2)!}{n! \times 2!} \left(\frac{1}{4}\right)^n u[n] \Rightarrow X(e^{j\omega}) = \dfrac{1}{(1-\frac{1}{4}e^{-j\omega})^3}$

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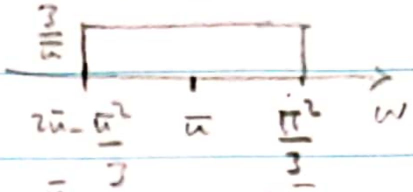
c) $\dfrac{\sin(\frac{\pi^2}{3} n)}{\frac{\pi^2}{3} n} \times \cos\left(\frac{\pi}{6} n\right) = \frac{3}{\pi} \dfrac{\sin\left(\frac{\pi^2}{3} n - \pi n + \pi n\right)}{\pi n} \times \cos\left(\frac{\pi}{6} n\right)$

$= \left(\frac{3}{\pi} \dfrac{\sin[(\frac{\pi^2}{3} - \pi) n]}{\pi n} \times \cos \pi n + \frac{3}{\pi} \dfrac{\cos[(\frac{\pi^2}{3} - \pi) n]}{\pi n} \cancel{\sin \pi n}^{\,0}\right) \cos\left(\frac{\pi}{6} n\right)$

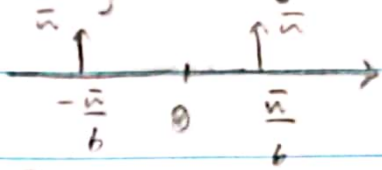
$= \frac{3}{\pi} \dfrac{\sin[(\frac{\pi^2}{3} - \pi) n]}{\pi n} e^{j\pi n} \times \cos\left(\frac{\pi}{6} n\right)$

$\frac{3}{\pi}\,\frac{\sin[(\frac{\pi^2}{3}-\pi)n]}{\pi n}\, e^{j\pi n} \xleftrightarrow{F}$



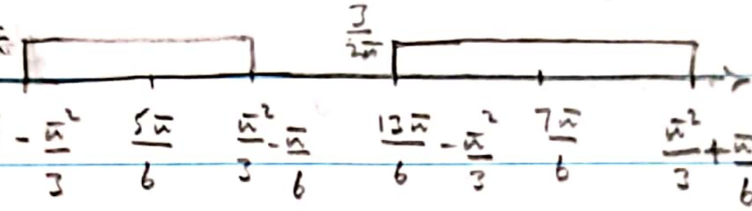
periodic, $T=2\pi$

$\cos(\frac{\pi}{6}n) \xleftrightarrow{F}$



periodic, $T=2\pi$

$\Rightarrow \hat{X}(e^{j\omega}) =$



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d) $u[n+3] \xleftrightarrow{F} \frac{e^{3j\omega}}{1-e^{-j\omega}} + \sum_{l=-\infty}^{\infty} e^{3j\omega}\pi\delta(\omega-2\pi l)$

$u[n-4] \xleftrightarrow{F} \frac{e^{-4j\omega}}{1-e^{-j\omega}} + \sum_{l=-\infty}^{\infty} e^{-4j\omega}\pi\delta(\omega-2\pi l)$

* $e^{t_0 j\omega}\delta(\omega-2\pi l) = \delta(\omega-2\pi l)$

$\Rightarrow X(e^{j\omega}) = \frac{e^{3j\omega}-e^{-4j\omega}}{1-e^{-j\omega}} = \frac{\sin\frac{7\omega}{2}}{\sin\frac{\omega}{2}}$

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e) $x[n] = 1+\cos[\frac{\pi}{8}n]$

$\Rightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty}\left\{2\pi\delta(\omega-2\pi l) + \pi\delta(\omega-\frac{\pi}{8}-2\pi l) + \pi\delta(\omega+\frac{\pi}{8}-2\pi l)\right\}$

3) a) $-\pi < \omega < \pi$: $u[n] \xleftrightarrow{F} \dfrac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{F} \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{2}{2-e^{-j\omega}}$$

$$\Rightarrow u[n] * \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{F} \frac{1}{1-e^{-j\omega}} \cdot \frac{2}{2-e^{-j\omega}} + \frac{2}{2-1} \times \pi\delta(\omega)$$

$$u[n] * \left(\frac{1}{2}\right)^n u[n] = \sum_{m=-\infty}^{\infty} u[n-m] \times \left(\frac{1}{2}\right)^m u[m] = \sum_{m=0}^{\infty} u[n-m] \times \left(\frac{1}{2}\right)^m$$

$$= \begin{cases} \sum_{0}^{n} \left(\frac{1}{2}\right)^m & , n \geq 0 \\ 0 & , n < 0 \end{cases} = \begin{cases} \dfrac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} & , n \geq 0 \\ 0 & , n < 0 \end{cases} = \left[2-\left(\frac{1}{2}\right)^n\right] u[n]$$

b) $\delta[n-k] \xleftrightarrow{F} e^{-jk\omega} \Rightarrow \sum_{k=0}^{\infty} \delta[n-k] \longleftrightarrow \sum_{k=0}^{\infty} e^{-j\omega k}$

c) $e^{-j\omega} = A \quad X(e^{j\omega}) = \dfrac{2-12A}{1-12A+35A^2} = \dfrac{2-12A}{(7A-1)(5A-1)}$

$$= \frac{-1}{7A-1} + \frac{-1}{5A-1} = \frac{1}{1-7e^{-j\omega}} + \frac{1}{1-5e^{-j\omega}}$$

تبدیل شناخته

d) $\cos\omega = \dfrac{e^{j\omega}+e^{-j\omega}}{2} \Rightarrow \cos^2\omega = \dfrac{1}{4}\left(e^{2j\omega}+e^{-2j\omega}+2\right)$

$\sin 3\omega = \dfrac{e^{3j\omega}-e^{-3j\omega}}{2j} \Rightarrow \sin^2 3\omega = \dfrac{1}{4}\left(-e^{6j\omega}-e^{-6j\omega}+2\right)$

$$\Rightarrow x[n] = \delta[n] + \frac{1}{4}\delta[n+2] + \frac{1}{4}\delta[n-2] - \frac{1}{4}\delta[n-6] - \frac{1}{4}\delta[n+6]$$

4) a) $y[n] + \frac{1}{2}y[n-1] = x[n] \Rightarrow Y(e^{j\omega}) + \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

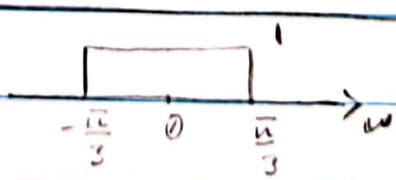
b) $h[n] = \left(-\frac{1}{2}\right)^n u[n]$

c) $Y(e^{j\omega}) = X(e^{j\omega}) \times H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \times \frac{1}{1+\frac{1}{2}e^{-j\omega}}$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1+\frac{1}{2}e^{-j\omega}} \Rightarrow y[n] = \left(\frac{1}{2}\right)^{n+1}u[n] + \frac{1}{2}\times\left(-\frac{1}{2}\right)^n u[n]$$

d) $Y(e^{j\omega}) = X(e^{j\omega}) \times H(e^{j\omega}) = \left(1+\frac{1}{2}e^{-j\omega}\right) \times \left(\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right) = 1$

$\Rightarrow y[n] = \delta[n]$

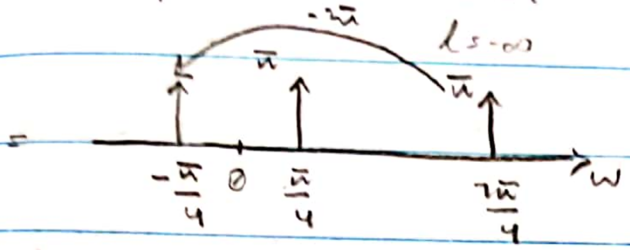
5) a) $H(e^{j\omega}) =$ 

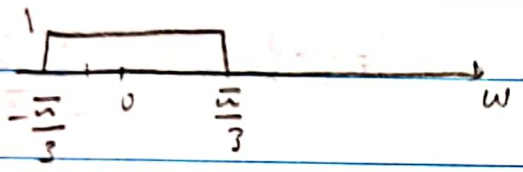
periodic, $T = 2\pi$

b) $H(e^{j\omega}) \times \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\left\{\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) - \delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)\right\} = 0$

5) c) $(-1)^n \cos(\frac{3\pi}{4}n) = e^{j\pi n}\cos(\frac{3\pi}{4}n)$

$$e^{j\pi n}\cos(\tfrac{3\pi}{4}n) \xleftrightarrow{F} \pi\sum_{l=-\infty}^{\infty}\left\{\delta(\omega-\tfrac{7\pi}{4}-2\pi l)+\delta(\omega-\tfrac{\pi}{4}-2\pi l)\right\}$$

$=$ 

$H(e^{j\omega}) =$ 

$\Rightarrow H(e^{j\omega}) \times X(e^{j\omega}) = X(e^{j\omega})$

$\Rightarrow y[n] = x[n]$

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6) a) $\cos(\frac{\pi}{3}n) + j\sin(\frac{2\pi}{3}n) = \frac{e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n}}{2} + \frac{e^{j\frac{2\pi}{3}n}-e^{-j\frac{2\pi}{3}n}}{2}$

$$\Rightarrow X(e^{j\omega}) = \pi\sum_{l=-\infty}^{\infty}\left\{\delta(\omega-\tfrac{\pi}{3}-2\pi l)+\delta(\omega+\tfrac{\pi}{3}-2\pi l)+\delta(\omega-\tfrac{2\pi}{3}-2\pi l)-\delta(\omega+\tfrac{2\pi}{3}-2\pi l)\right\}$$

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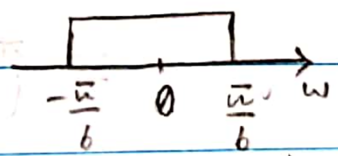
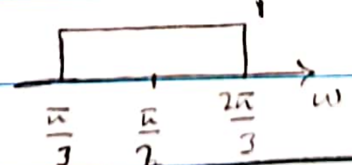
b) $\frac{\sin(\frac{\pi}{6}n)}{\pi n} \xleftrightarrow{F}$  , periodic, $T = 2\pi$

$\Rightarrow e^{j\frac{\pi}{2}n}\frac{\sin(\frac{\pi}{6}n)}{\pi n} \xleftrightarrow{F}$  , periodic, $T = 2\pi$

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c) $W(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \pi\sum_{l=-\infty}^{\infty} \delta(\omega-\frac{\pi}{3}-2\pi l)+\delta(\omega-\frac{2\pi}{3}-2\pi l)$

6) d) $w[n] = \frac{e^{j\frac{\pi}{3}n}}{2} + \frac{e^{j\frac{2\pi}{3}n}}{2}$

e) $h_2[n] = \delta[n-1] - 2\delta[n-2]$

f) $y[n] = w[n] * h_2[n] = \frac{e^{j\frac{\pi}{3}(n-1)}}{2} + \frac{e^{j\frac{2\pi}{3}(n-1)}}{2} - e^{j\frac{\pi}{3}(n-2)} - e^{j\frac{2\pi}{3}(n-2)}$

g) $Y(e^{jw}) = 2\pi \sum_{l=-\infty}^{\infty} \left\{ \frac{\delta(w-\frac{\pi}{3}-2\pi l)}{2e^{j\frac{\pi}{3}}} + \frac{\delta(w-\frac{2\pi}{3}-2\pi l)}{2e^{j\frac{2\pi}{3}}} - \frac{\delta(w-\frac{\pi}{3}-2\pi l)}{e^{j\frac{2\pi}{3}}} - \frac{\delta(w-\frac{2\pi}{3}-2\pi l)}{e^{j\frac{4\pi}{3}}} \right\}$

h) $H_{eq}(e^{jw}) = H_1(e^{jw}) \times H_2(e^{jw}) = \begin{cases} e^{-jw} - 2e^{-2jw}, & |w-\frac{\pi}{2}| < \frac{\pi}{6} \\ 0, & \text{o.w} \end{cases}$

periodic, $T = 2\pi$

i) $h_{eq}[n] = h_1[n] * h_2[n] = e^{j\frac{\pi}{2}n} \frac{\sin(\frac{\pi}{6}n)}{\pi n} * (\delta[n-1] - 2\delta[n-2])$

$= e^{j\frac{\pi}{2}(n-1)} \frac{\sin(\frac{\pi}{6}n - \frac{\pi}{6})}{\pi n - \pi} - 2e^{j\frac{\pi}{2}(n-2)} \frac{\sin(\frac{\pi}{6}n - \frac{\pi}{3})}{\pi n - 2\pi}$