① a) $x[n] = 3 + \sin\left(\frac{\pi}{6}n + \frac{\pi}{8}\right)$

$\left(\omega_0 = \frac{\pi}{6}\right)$ $(N=12)$: x[n] متناوب است پس نمایش سری فوریه دارد

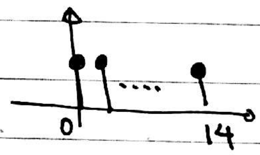
$$x[n] = 3 + \frac{1}{2j}\left(e^{j\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)} - e^{-j\left(\frac{\pi}{6}n+\frac{\pi}{8}\right)}\right) \Rightarrow \begin{cases} a_0 = 3 \\ a_1 = \frac{1}{2j}e^{j\frac{\pi}{8}} \\ a_{-1} = -\frac{1}{2j}e^{-j\frac{\pi}{8}} = a_{11} \end{cases}$$

چون x گسسته است ، $X(e^{j\omega})$ متناوب و پیوسته با پریود 2π است .

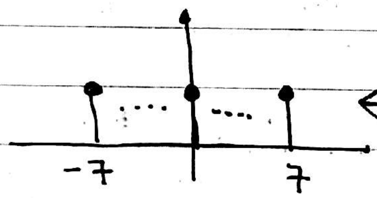
$X(e^{j\omega})$ در یک پریود را با $\hat{X}(e^{j\omega})$ نمایش می دهیم

$$\hat{X}(e^{j\omega}) = 2\pi \sum_{k=0}^{11} a_k \delta\left(\omega - k\frac{\pi}{6}\right) = 6\pi\delta(\omega) + \frac{\pi}{j}e^{j\frac{\pi}{8}}\delta\left(\omega - \frac{\pi}{6}\right) - \frac{\pi}{j}e^{-j\frac{\pi}{8}}\delta\left(\omega + \frac{\pi}{6}\right)$$

b) $x[n] = u[n] - u[n-15]$



می دانیم

$y[n] =$ 

$$\longleftrightarrow \frac{\sin\left(\frac{15}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} = Y(e^{j\omega})$$

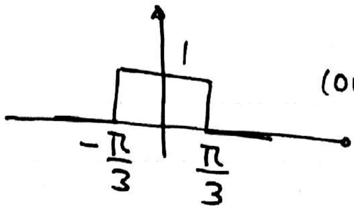
$$\xrightarrow{\text{time shift}} y[n-7] = x[n] \longleftrightarrow X(e^{j\omega}) = e^{-j\omega 7}\,Y(e^{j\omega})$$

c) $x[n] = \dfrac{\sin(\frac{\pi}{3}n)}{\pi n}$

می‌دانیم

$$\frac{\sin(Wn)}{\pi n} \longleftrightarrow \hat{X}(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le W \\ 0 & W \le |\omega| \le \pi \end{cases}$$

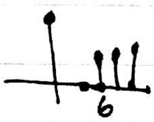
$$\Rightarrow \quad \frac{\sin(\frac{\pi}{3}n)}{\pi n} \longleftrightarrow$$



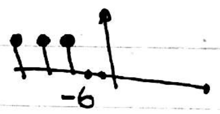
(one period)

⟸ اگر سیگنال به فرم $\dfrac{\sin(\alpha n)}{\pi n}$ داشتیم باید α را با استفاده از قواعد مثلثاتی از π کوچکتر کنیم.

d) $x[n] = (\frac{1}{3})^{|n|} u[-n-6]$ $\quad u[n-6]$:



$\rightarrow u[-n-6]$:



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} (x[n]) e^{-jn\omega} = \sum_{n=-\infty}^{+\infty} (\tfrac{1}{3})^{|n|} u[-n-6] e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{-6} (\tfrac{1}{3})^{|n|} e^{-jn\omega} = \sum_{n=-\infty}^{-6} (\tfrac{1}{3})^{-n} e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{-6} (\tfrac{1}{3} e^{j\omega})^{-n} = \sum_{n=6}^{+\infty} (\tfrac{1}{3} e^{j\omega})^{n}$$

$$= \sum_{m=0}^{+\infty} (\tfrac{1}{3} e^{j\omega})^{m+5} = \frac{1}{243} e^{j5\omega} \frac{1}{1 - \frac{1}{3} e^{j\omega}}$$

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e) $x[n] = 3^n \sin(\frac{\pi}{6} n)\, u[-n]$

می دانیم : $\alpha^n u[n] \quad |\alpha|<1 \longleftrightarrow \dfrac{1}{1-\alpha e^{-j\omega}}$

در این مثال $\alpha = 3$ است که شرط $|\alpha|<1$ را برآورده نمی کند. راه حل :
تبدیل فوریه $y[n] = x[-n]$ را به دست می آوریم سپس از قاعده $Y(e^{j\omega}) = X(e^{-j\omega})$
استفاده می کنیم.

$$y[n] = x[-n] = 3^{-n} \times (-\sin(\tfrac{\pi}{6} n))\, u[n]$$

$$= \underbrace{-(\tfrac{1}{3})^n u[n]}_{r[n]}\ \underbrace{\sin(\tfrac{\pi}{6} n)}_{s[n]}$$

$$r[n] = -(\tfrac{1}{3})^n u[n] \longleftrightarrow R(e^{j\omega}) = \frac{-1}{1-\frac{1}{3}e^{-j\omega}}$$

$$s[n] = \sin(\tfrac{\pi}{6} n) \longleftrightarrow \hat{S}(e^{j\omega}) = \frac{\pi}{j}\left(\delta(\omega-\tfrac{\pi}{6}) - \delta(\omega+\tfrac{\pi}{6})\right)$$

$$y[n] = r[n]\, s[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\left(R(e^{j\omega}) * \hat{S}(e^{j\omega})\right)$$

$$= \underbrace{\frac{1}{2j}\left(R(e^{j(\omega-\frac{\pi}{6})}) - R(e^{j(\omega+\frac{\pi}{6})})\right)}_{X(e^{-j\omega})}$$

به جای ω بگذارید $-\omega$ تا $X(e^{j\omega})$ بدست آید

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② a) $x[2-n] + x[-3-n]$

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$\xrightarrow{n \leftarrow n+2} x[n+2] \leftrightarrow e^{j2\omega} X(e^{j\omega})$$

$$\xrightarrow{n \leftarrow -n} x[-n+2] \leftrightarrow e^{-j2\omega} X(e^{-j\omega}) \longrightarrow$$ پاسخ بخش اول

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$\xrightarrow{n \leftarrow n-3} x[n-3] \leftrightarrow e^{-j3\omega} X(e^{j\omega})$$

$$\xrightarrow{n \leftarrow -n} x[-n-3] \leftrightarrow e^{j3\omega} X(e^{-j\omega}) \longrightarrow$$ پاسخ بخش دوم

$$\Rightarrow x[2-n] + x[-3-n] \leftrightarrow \left(e^{-j2\omega} + e^{j3\omega}\right) X(e^{-j\omega})$$

b) $(n-1)^3 x[n]$

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$n x[n] \leftrightarrow j \frac{d}{d\omega} X(e^{j\omega})$$

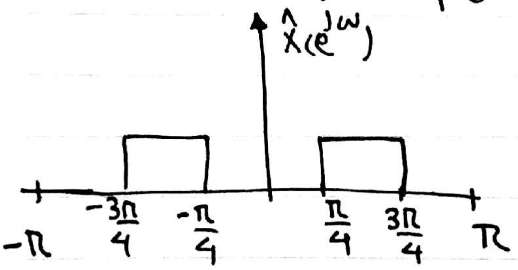
$$n^2 x[n] \leftrightarrow j^2 \frac{d^2}{d\omega^2} X(e^{j\omega}) = -\frac{d^2}{d\omega^2} X(e^{j\omega})$$

$$n^3 x[n] \leftrightarrow j^3 \frac{d^3}{d\omega^3} X(e^{j\omega}) = -j \frac{d^3}{d\omega^3} X(e^{j\omega})$$

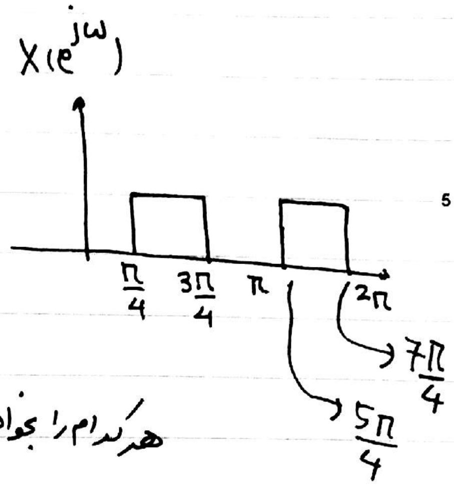
$$(n-1)^3 x[n] = n^3 x[n] - 3n^2 x[n] + 3n\, x[n] + \cancel{\ \ } x[n]$$

تبدیل فوریه $\Rightarrow$ $-jX'''(e^{j\omega}) + 3X''(e^{j\omega}) + 3jX'(e^{j\omega}) + X(e^{j\omega})$

③ a) $\hat{X}(e^{j\omega}) = \begin{cases} 1 & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & \text{otherwise} \end{cases}$



بازه بین صفر تا
$\xrightarrow{\quad 2\pi \quad}$



هر کدام را بخواهیم می‌توانیم برای انتگرال استفاده کنیم

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{jn\omega} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{jn\omega} d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{jn\omega} d\omega \right] = \frac{1}{2\pi jn} \left(e^{jn\omega} \Big|_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} + e^{jn\omega} \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \right)$$

$$= \frac{1}{2\pi jn} \left(e^{jn(-\frac{\pi}{4})} - e^{jn(\frac{-3\pi}{4})} + e^{jn(\frac{3\pi}{4})} - e^{jn(\frac{\pi}{4})} \right)$$

$$= \frac{1}{\pi n} \left(\sin(\frac{3\pi}{4} n) - \sin(\frac{\pi}{4} n) \right)$$

b) $1 + 3e^{-j\omega} + 8e^{j6\omega}$

$$x[n] = \delta[n] + 3\delta[n-1] + 8\delta[n+6]$$

c) $X(e^{j\omega}) = \dfrac{1 - \frac{1}{3}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}}$

$e^{-j\omega} = k \Rightarrow X(e^{j\omega}) = \dfrac{\frac{1}{3}k - 1}{\frac{1}{8}k^2 + \frac{1}{4}k - 1} \underset{\times \frac{8}{8}}{=} \dfrac{\frac{8}{3}k - 8}{k^2 + 2k - 8}$

$= \dfrac{\frac{8}{3}k - 8}{(k+4)(k-2)} = \dfrac{A}{k+4} + \dfrac{B}{k-2} \Rightarrow \begin{cases} A + B = \frac{8}{3} \\ -2A + 4B = -8 \end{cases} \Rightarrow \begin{cases} A = \frac{28}{9} \\ B = \frac{-4}{9} \end{cases}$

$\Rightarrow X(e^{j\omega}) = \dfrac{28/9}{e^{-j\omega} + 4} - \dfrac{4/9}{e^{-j\omega} - 2} = \dfrac{7/9}{\frac{1}{4}e^{-j\omega} + 1} - \dfrac{2/9}{\frac{1}{2}e^{-j\omega} - 1}$

$= \dfrac{7/9}{1 + \frac{1}{4}e^{-j\omega}} + \dfrac{2/9}{1 - \frac{1}{2}e^{-j\omega}}$

$\Rightarrow x[n] = \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{2}{9}\left(\frac{1}{2}\right)^n u[n]$

$$X \longrightarrow \boxed{H_1} \longrightarrow \boxed{H_2} \longrightarrow Y \equiv X \longrightarrow \boxed{H=H_1H_2} \longrightarrow Y$$

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④ a) $H(e^{j\omega}) = H_1(e^{j\omega})\,H_2(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

$$H(e^{j\omega}) = \frac{2-e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j\omega 2})} = \frac{2-e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{j2\omega}+\frac{1}{2}e^{-j\omega}-\frac{1}{4}e^{-j2\omega}+\frac{1}{8}e^{-j3\omega}}$$

$$= \frac{2(1-\frac{1}{2}e^{-j\omega})}{1+\frac{1}{8}e^{-j3\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\Rightarrow Y(e^{j\omega}) + \frac{1}{8}e^{-j3\omega}\,Y(e^{j\omega}) = 2X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})$$

$$\xrightarrow[\text{Fourier}]{\text{Inverse}} y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1]$$

b) $e^{-j\omega} = k \Rightarrow H(e^{j\omega}) = \dfrac{2(1-\frac{1}{2}k)}{1+\frac{1}{8}k^3} = \dfrac{16(1-\frac{1}{2}k)}{k^3+8}$

$$= \frac{16(1-\frac{1}{2}k)}{(k+2)(k^2-2k+4)} = \frac{16(1-\frac{1}{2}k)}{(k+2)(k-r_1)(k-r_2)} \qquad \begin{cases} r_1 = 1+j\sqrt{3} \\ r_2 = 1-j\sqrt{3} \end{cases}$$

$\Rightarrow \dfrac{A}{k+2} + \dfrac{B}{k-r_1} + \dfrac{C}{k-r_2} = \dfrac{16-8k}{(k+2)(k-r_1)(k-r_2)} \Rightarrow$ حل سه معادله سه مجهول $\Rightarrow \dots$

$$H(e^{j\omega}) = \frac{4/3}{1+\frac{1}{2}e^{-j\omega}} + \frac{\frac{1+j\sqrt{3}}{3}}{1-\frac{1}{2}e^{j120}e^{-j\omega}} + \frac{\frac{1-j\sqrt{3}}{3}}{1-\frac{1}{2}e^{-j120}e^{-j\omega}}$$

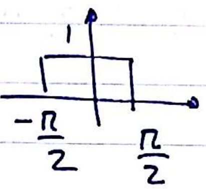
$$\Rightarrow h[n] = \frac{4}{3}\left(\frac{-1}{2}\right)^n u[n] + \frac{1+j\sqrt{3}}{3}\left(\frac{1}{2}e^{j120}\right)^n u[n] + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j120}\right)^n u[n]$$

(5) $y[n] = x[n] * h[n] + (-1)^n \left(x[n] * h[n]\right)$

$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) + X(e^{j(\omega-\pi)}) H(e^{j(\omega-\pi)})$$

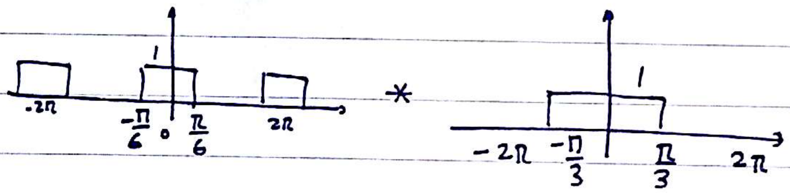
$$\xrightarrow[X(e^{j\omega})=1]{x[n]=\delta[n]} Y(e^{j\omega}) = H(e^{j\omega}) + H(e^{j(\omega-\pi)})$$

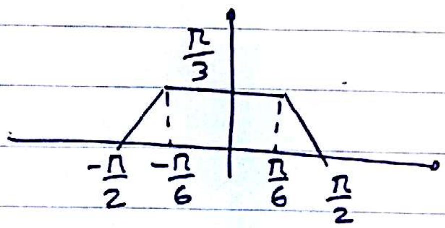
$$\Rightarrow y[n] = h[n] + (-1)^n h[n] = \begin{cases} 2h[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

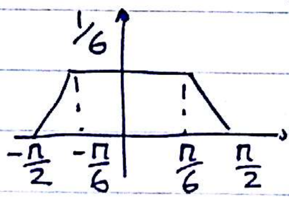
$\hat{H}(e^{j\omega}) =$  $\Rightarrow h[n] = \dfrac{\sin(\frac{\pi}{2} n)}{\pi n}$

(6)
$$h[n] = \frac{\sin(\frac{\pi}{6} n)}{\pi n} \; \frac{\sin(\frac{\pi}{3} n)}{\pi n} = h_1[n]\, h_2[n]$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{2\pi}\left(H_1(e^{j\omega}) * \hat{H}_2(e^{j\omega})\right)$$

$= \dfrac{1}{2\pi}\Big($  $\Big)$

$= \dfrac{1}{2\pi}\Big($  $\Big)$

$=$  $= \hat{H}(e^{j\omega})$

$$\hat{X}(e^{j\omega}) = \frac{\pi}{j}\left(\delta(\omega-\tfrac{\pi}{8}) - \delta(\omega+\tfrac{\pi}{8})\right) - 2\pi\left(\delta(\omega-\tfrac{\pi}{4}) + \delta(\omega+\tfrac{\pi}{4})\right)$$

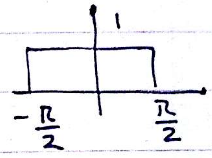
$$\Rightarrow \hat{Y}(e^{j\omega}) = \underbrace{\hat{H}(e^{j\frac{\pi}{8}})}_{1/6} \sin(\tfrac{\pi}{8} n) - 2\underbrace{\hat{H}(e^{j\frac{\pi}{4}})}_{1/8} \cos(\tfrac{\pi}{4} n)$$

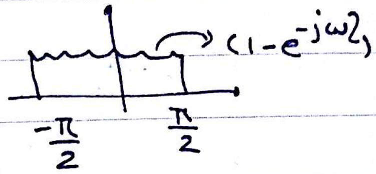
به معادله خط :

7 a) $w[n] = x[n] - x[n-2] \xrightarrow{F} W(e^{j\omega}) = X(e^{j\omega}) - e^{-j2\omega} X(e^{j\omega})$

b) $W(e^{j\omega}) = H_1(e^{j\omega})\, X(e^{j\omega}) = (1 - e^{-j\omega 2})\, X(e^{j\omega})$

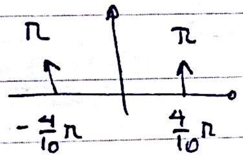
$\Rightarrow H_1(e^{j\omega}) = (1 - e^{-j\omega 2})$

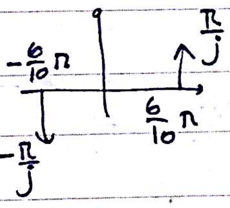
c) $H_{eq}(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega}) = (1 - e^{-j\omega 2}) \times$ 

$=$ 

d) $x[n] = \cos(\frac{4}{10}\pi n) + \sin(\frac{6}{10}\pi n) + 2\delta[n-2] = x_1[n] + x_2[n] + x_3[n]$

$\rightarrow X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega}) + X_3(e^{j\omega})$

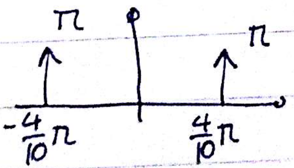
$=$ 

$+$ 

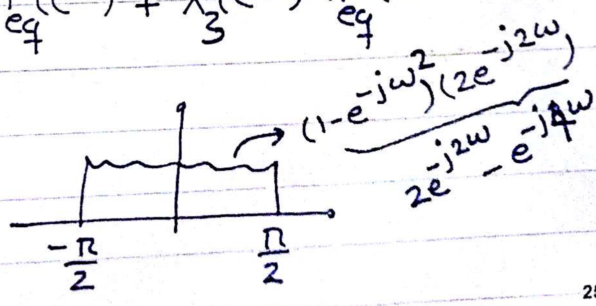
$+ 2e^{-j2\omega}$

e) $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H_{eq}(e^{j\omega}) \longrightarrow$ حذف می‌شوند $X_2(e^{j\omega})$ و $x_2[n]$

$= (X_1(e^{j\omega}) + X_3(e^{j\omega}))\, H_{eq}(e^{j\omega}) = X_1(e^{j\omega})\, H_{eq}(e^{j\omega}) + X_3(e^{j\omega})\, H_{eq}(e^{j\omega})$

$= \underbrace{(1 - e^{-j\omega 2}) \times}_{}$ 

$Y_1(e^{j\omega})$

$+$ 

$Y_2(e^{j\omega})$

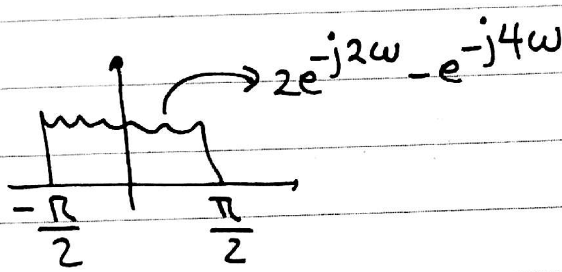
f) $Y(e^{j\omega}) = Y_1(e^{j\omega}) + Y_2(e^{j\omega})$

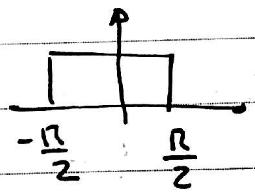
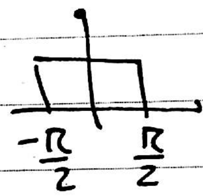
اگر بگیریم $c[n] = \cos[\frac{4}{10}\pi n]$ باشد داریم :

$$\star\ Y_1(e^{j\omega}) = (1 - e^{-j\omega 2})C(e^{j\omega}) = C(e^{j\omega}) - e^{-j\omega}C(e^{j\omega})$$

$$\Rightarrow y_1[n] = c[n] - c[n-2]$$

$$= \cos(\frac{4}{10}\pi n) - \cos(\frac{4}{10}\pi(n-2))$$

$\star\ Y_2(e^{j\omega}) =$ 

$=$  $\times 2e^{-j2\omega}$ $-$  $\times e^{-j4\omega}$

$$\Rightarrow y_2[n] = 2\frac{\sin(\frac{\pi}{2}(n-2))}{\pi(n-2)} - \frac{\sin(\frac{\pi}{2}(n-4))}{\pi(n-4)}$$

$$\longrightarrow y[n] = y_1[n] + y_2[n]$$