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1.(a) $x(t)=1+\cos(6\pi t+\frac{\pi}{8})$

Periodic: $\omega_0=6\pi$

$$x(t)=1+\frac{1}{2}\left(e^{j(\omega_0 t+\frac{\pi}{8})}+e^{-j(\omega_0 t+\frac{\pi}{8})}\right)$$

$$=\underbrace{1}_{a_0}+\underbrace{\frac{1}{2}e^{j\frac{\pi}{8}}}_{a_1}e^{j\omega_0 t}+\underbrace{\frac{1}{2}e^{-j\frac{\pi}{8}}}_{a_{-1}}e^{-j\omega_0 t}$$

$$x(t)=\sum a_k e^{jk\omega_0 t}\xleftrightarrow{FT}X(j\omega)=\sum 2\pi a_k\delta(\omega-k\omega_0)$$

$$\Rightarrow X(j\omega)=2\pi\delta(\omega)+2\pi\frac{1}{2}e^{j\frac{\pi}{8}}\delta(\omega-6\pi)$$

$$+2\pi\frac{1}{2}e^{-j\frac{\pi}{8}}\delta(\omega+6\pi$$

1.(b) $x(t)=(te^{-2t}\sin(4t))u(t)$

$$\longrightarrow x(t)=(te^{-2t}u(t))(\sin(4t))=p(t)\,S(t)$$

$$x(t)=p(t)s(t)\xleftrightarrow{FT}X(j\omega)=\frac{1}{2\pi}\left(P(j\omega)*S(j\omega)\right)$$

$*\ p(t)=te^{-2t}u(t)$

we know: $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\xleftrightarrow{FT}\frac{1}{(a+j\omega)^n}$

$$\xrightarrow{a=2=n} p(t)\longleftrightarrow\frac{1}{(2+j\omega)^2}=P(j\omega)$$

$$*\ S(t)=\sin(4t)\Longrightarrow S(j\omega)=\frac{\pi}{j}\left(\delta(\omega-4)-\delta(\omega+4)\right)$$

$$\Rightarrow X(j\omega)=\frac{1}{2j}\left(P(j(\omega-4))-P(j(\omega+4))\right)$$

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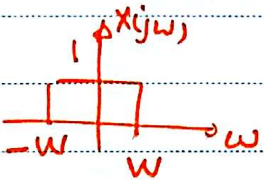
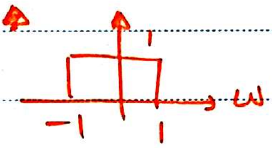
1.(C) $x(t) = t\left(\frac{\sin(t)}{\pi t}\right)^2 \longrightarrow x(t) = t\,\frac{\sin(t)}{\pi t}\,\frac{\sin(t)}{\pi t}$

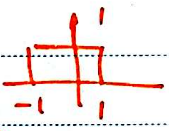
* می‌دانیم $x(t) = \frac{\sin(Wt)}{\pi t} \overset{FT}{\longleftrightarrow}$



$\Rightarrow \frac{\sin(t)}{\pi t} \longleftrightarrow$



$\Rightarrow \left(\frac{\sin(t)}{\pi t}\right)^2 \longleftrightarrow \frac{1}{2\pi}\Bigg($



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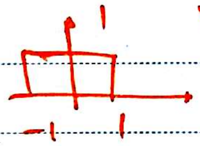
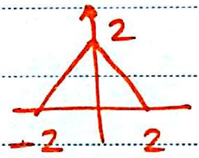


$\Bigg)$

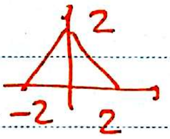
$= \frac{1}{2\pi}$



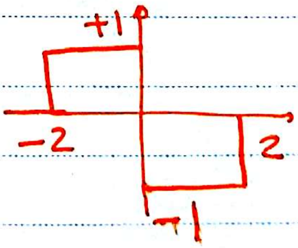
* $x(t) \longleftrightarrow X(j\omega)$

$t\,x(t) \longleftrightarrow j\frac{d}{d\omega}X(j\omega)$

$\Rightarrow \left(\frac{\sin(t)}{\pi t}\right)^2 \longleftrightarrow \frac{1}{2\pi}$



$t\left(\frac{\sin(t)}{\pi t}\right)^2 \longleftrightarrow \frac{j}{2\pi}$



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1.(d) $x(t) = \dfrac{4t}{(1+t^2)^2}$

Duality: $f(t) \longleftrightarrow F(j\omega)$

$F(t) \longleftrightarrow 2\pi f(-\omega)$

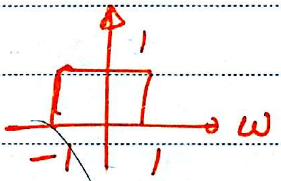
we know: $e^{-|t|} \longleftrightarrow \dfrac{2}{1+\omega^2}$

$\xrightarrow{\text{Duality}} \quad \dfrac{2}{1+t^2} \longleftrightarrow 2\pi e^{-|\omega|}$

$\xrightarrow{\text{مشتق}} \quad \dfrac{4t}{(1+t^2)^2} \longleftrightarrow j\omega\left(2\pi e^{-|\omega|}\right)$

1.(e) $x(t) = \dfrac{\sin(t-2\pi)}{\pi(t-2\pi)}$

$x(t) = \dfrac{\sin(t)}{\pi t} \longleftrightarrow$



$= X(j\omega)$

$\xrightarrow{\text{Shift}} \quad x(t-2\pi) \longleftrightarrow e^{-j\omega(2\pi)} \, X(j\omega)$

$= X(j\omega)$

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2.(a) $X(j\omega) = 2\delta(\omega+6)$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

$$\xrightarrow[\text{Shift}]{\text{Frequency}} e^{j(-6)t} \longleftrightarrow 2\pi\delta(\omega-(-6))$$

$$\xrightarrow{\div\pi} \frac{1}{\pi}e^{-j6t} \longleftrightarrow 2\delta(\omega+6)$$

2.(b) $\dfrac{7j\omega+46}{-\omega^2+13j\omega+42} = \dfrac{7(j\omega)+46}{(j\omega)^2+13j\omega+42}$

$$\Rightarrow X(j\omega) = \frac{7j\omega+46}{(j\omega+7)(j\omega+6)} = \frac{A}{j\omega+7} + \frac{B}{j\omega+6}$$

$$\Rightarrow \begin{cases} A+B=7 \\ 6A+7B=46 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=4 \end{cases}$$

$$\frac{1}{j\omega+7} \longleftrightarrow e^{-7t}u(t)$$

$$\frac{1}{j\omega+6} \longleftrightarrow e^{-6t}u(t)$$

$$\Rightarrow X(j\omega) = \frac{3}{j\omega+7} + \frac{4}{j\omega+6} \longleftrightarrow \left(3e^{-7t}+4e^{-6t}\right)u(t)$$

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2.(c) $X(j\omega) = \pi e^{-3|\omega|}$

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}$$

$$\xrightarrow{\text{Duality}} \frac{2a}{a^2+t^2} \longleftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

$$\xrightarrow{a=3} \frac{6}{9+t^2} \longleftrightarrow 2\pi e^{-3|\omega|}$$

$$\overset{\div 2}{\Longrightarrow} \frac{3}{9+t^2} \longleftrightarrow \pi e^{-3|\omega|}$$

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3. $$\frac{d^2}{dt^2}y(t)+6\frac{d}{dt}y(t)+8y(t)=2x(t)$$

(a) $$\xrightarrow[\text{Transform}]{\text{Fourier}} (j\omega)^2Y(j\omega)+{}^{6}(j\omega)Y(j\omega)+8Y(j\omega)=2X(j\omega)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{2}{(j\omega)^2+6j\omega+8}=\frac{2}{(j\omega+4)(j\omega+2)}$$

$$=\frac{A}{j\omega+4}+\frac{B}{j\omega+2}\Rightarrow\begin{cases}B=1\\A=-1\end{cases}$$

$$\Rightarrow H(j\omega)=\frac{1}{j\omega+2}-\frac{1}{j\omega+4}$$

$$\Rightarrow h(t)=\left(e^{-2t}-e^{-4t}\right)u(t)$$

(b) $$x(t)=te^{-2t}u(t)\Longrightarrow X(j\omega)=\frac{1}{(2+j\omega)^2}$$

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$$\boxed{\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)\longleftrightarrow\frac{1}{(a+j\omega)^n}}$$

$$\Rightarrow Y(j\omega)=X(j\omega)H(j\omega)$$

$$=\frac{1}{(j\omega+2)^2}\,\frac{2}{(j\omega+2)(j\omega+4)}$$

$$=\frac{2}{(j\omega+2)^3(j\omega+4)}$$

تجزیه کسر : $\frac{2}{(jw+2)^3(jw+4)} = \frac{A}{(jw+2)^3} + \frac{B}{(jw+2)^2} + \frac{C}{(jw+2)} + \frac{D}{(jw+4)}$

$\Rightarrow A(jw+4) + B(jw+2)(jw+4) + C(jw+2)^2(jw+4) + D(jw+2)^3 = 2$

① $A(jw+4) = Ajw + 4A$

② $B(jw+2)(jw+4) = B(jw)^2 + 6B(jw) + 8B$

③ $C(jw+2)^2(jw+4) = C((jw)^2 + 4jw + 4)(jw+4)$

$= C\left((jw)^3 + 8(jw)^2 + 20jw + 16\right)$

④ $D(jw+2)^3 = D\left((jw)^3 + 6(jw)^2 + 12jw + 8\right)$

$(jw)^3$ ضرایب • $C + D = 0$

$(jw)^2$ ضرایب • $B + 8C + 6D = 0$

jw ضرایب • $A + 6B + 20C + 12D = 0$

ثابت ضرایب • $4A + 8B + 16C + 8D = 2$

$\Rightarrow$ حل $\begin{cases} A = 1 \\ B = -0.5 \\ C = 0.25 \\ D = -0.25 \end{cases} \Rightarrow Y(jw) = \frac{\overset{=1}{A}}{(jw+2)^3} + \frac{\overset{=-0.5}{B}}{(jw+2)^2} + \frac{\overset{=0.25}{C}}{(jw+2)}$

$+ \frac{\overset{=-0.25}{D}}{(jw+4)}$

$\Rightarrow y(t) = \frac{t^2}{2} e^{-2t} u(t) - \frac{1}{2} t e^{-2t} u(t) + \frac{1}{4} e^{-2t} u(t) + \frac{-1}{4} e^{-4t} u(t)$

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4\. $H(j\omega) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \dfrac{j\omega + 4}{(j\omega)^2 + 5j\omega + 6}$

(a) $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$

$$\Longrightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+4}{(j\omega)^2+5j\omega+6}$$

$$\Longrightarrow (j\omega)^2 Y(j\omega) + 5j\omega Y(j\omega) + 6Y(j\omega) = j\omega X(j\omega) + 4X(j\omega)$$

$$\xRightarrow{F^{-1}} \frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)$$

(b) $H(j\omega) = \dfrac{j\omega+4}{(j\omega+3)(j\omega+2)} = \dfrac{A}{(j\omega+3)} + \dfrac{B}{(j\omega+2)} \Rightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$

$$\Longrightarrow h(t) = -e^{-3t}u(t) + e^{-2t}u(t)\cdot(2)$$

(c) $x(t) = e^{-4t}u(t) - te^{-4t}u(t) \Rightarrow X(j\omega) = \dfrac{1}{j\omega+4} - \dfrac{1}{(j\omega+4)^2}$

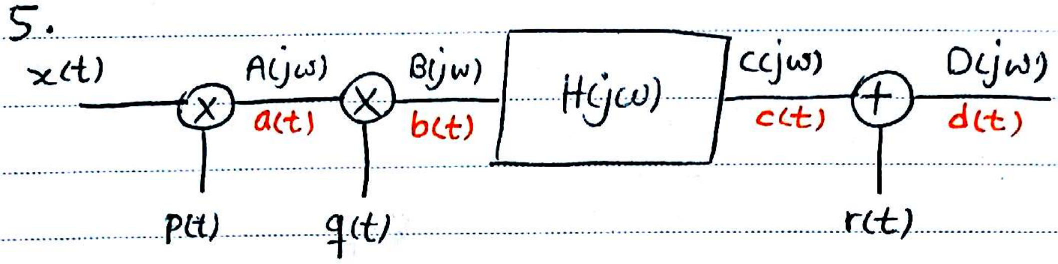
$$\Rightarrow X(j\omega) = \frac{1}{j\omega+4}\left(1 - \frac{1}{j\omega+4}\right) = \frac{1}{j\omega+4}\left(\frac{j\omega+3}{j\omega+4}\right)$$

- $Y(j\omega) = X(j\omega)H(j\omega) = \dfrac{1}{j\omega+4}\left(\dfrac{j\omega+3}{j\omega+4}\right)\left(\dfrac{j\omega+4}{(j\omega+3)(j\omega+2)}\right)$

$$= \frac{1}{(j\omega+4)(j\omega+2)} = \frac{-1}{2}\frac{1}{j\omega+4} + \frac{1}{2}\frac{1}{j\omega+2}$$

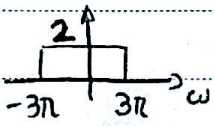
- $y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$

5.



$x(t) = \frac{\sin(\pi t)}{\pi t}$ $p(t) = \frac{\sin(2\pi t)}{\pi t}$ $q(t) = \cos(3\pi t)$

$H(j\omega) =$

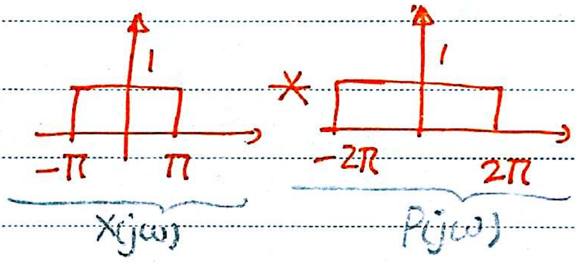


$r(t) = \frac{\sin(\pi t)}{\pi t}$

* محاسبه A(jω)

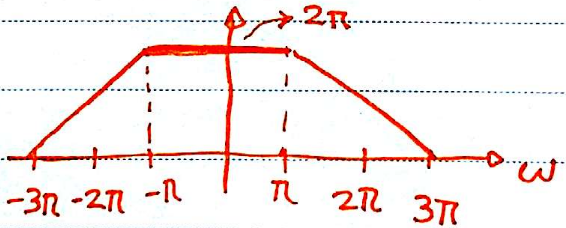
$a(t) = x(t)\,p(t) \Rightarrow A(j\omega) = \frac{1}{2\pi}\left(X(j\omega) * P(j\omega)\right)$

$\Rightarrow A(j\omega) = \frac{1}{2\pi}$



* برای انجام این convolution، از کادر موجود در سولوشن سوال 2c تمرین 2 استفاده کنید

$\Rightarrow A(j\omega) = \frac{1}{2\pi}$



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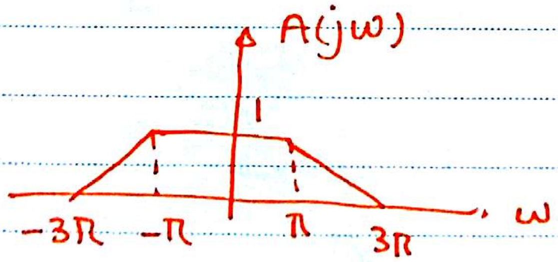

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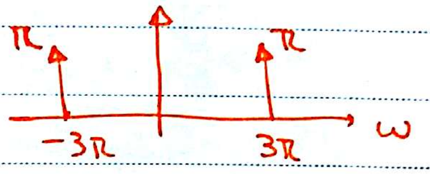
B(jω) محاسبه * $b(t) = a(t)\, q(t) \Rightarrow B(j\omega) = \frac{1}{2\pi}\left(A(j\omega) * Q(j\omega)\right)$

$q(t) = \cos(3\pi t) \Rightarrow Q(j\omega) = \pi\left(\delta(\omega - 3\pi) + \delta(\omega + 3\pi)\right)$

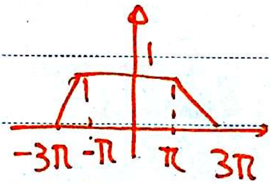


$\text{convolution}\left(\delta(\omega - \omega_0), X(j\omega)\right) = X(j(\omega - \omega_0))$: یادآوری *

$\Rightarrow B(j\omega) = \frac{1}{2\pi}\left(A(j\omega) * \pi\left(\delta(\omega - 3\pi) + \delta(\omega + 3\pi)\right)\right)$

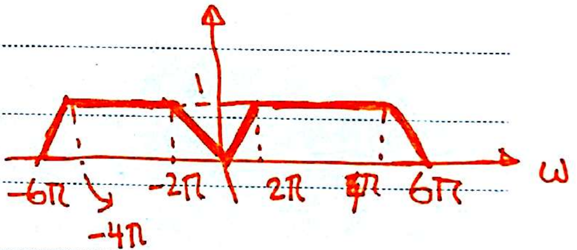
$= \frac{1}{2}\left(A(j(\omega - 3\pi)) + A(j(\omega + 3\pi))\right)$

A(jω) :

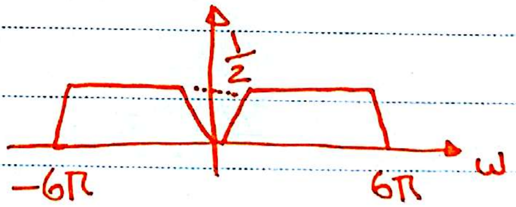


B(jω) =

$\Rightarrow \frac{1}{2}\left(A(j(\omega - 3\pi)) + A(j(\omega + 3\pi))\right) = \frac{1}{2}$



$\Rightarrow B(j\omega) =$



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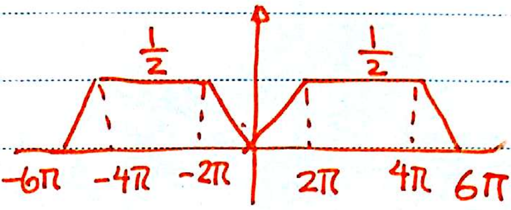
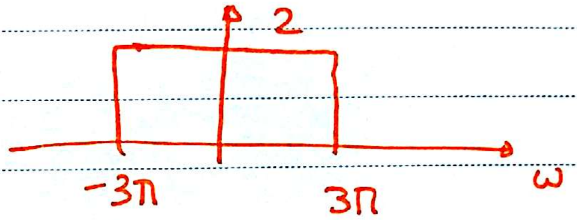
$C(j\omega)$ *محاسبه $\qquad C(j\omega) = B(j\omega)\,H(j\omega)$

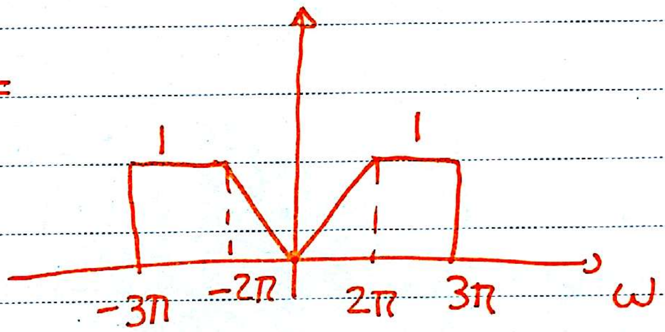
$B(j\omega)$:



$H(j\omega)$:



$\Rightarrow C(j\omega) =$



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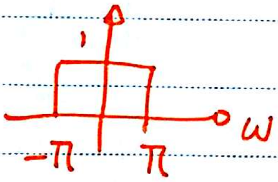
D(jω) محاسبه * $d(t) = c(t) + r(t) \Rightarrow D(j\omega) = C(j\omega) + R(j\omega)$

$r(t) = \sin(\pi t) \times \frac{1}{\pi t} \Rightarrow R(j\omega) =$



$\Rightarrow D(j\omega) = C(j\omega) + R(j\omega) =$

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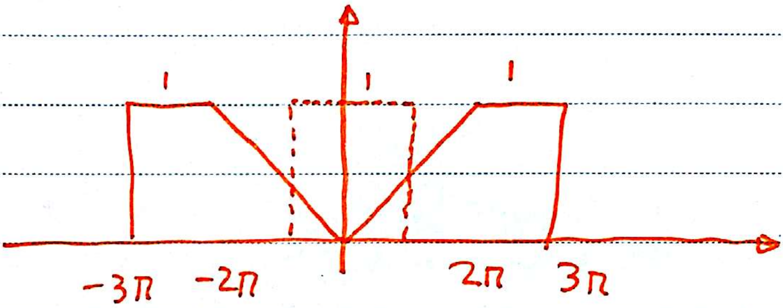
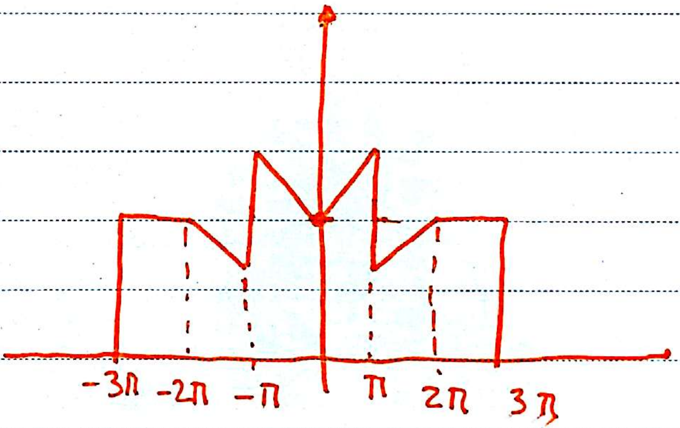


$\Rightarrow D(j\omega) =$



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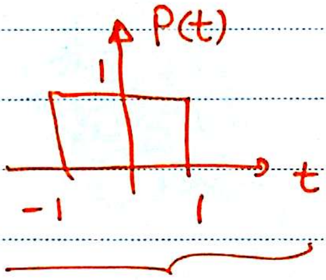
6. $\int_{-\infty}^{+\infty} \left(\frac{\sin(\omega)}{\omega}\right)^2 d\omega = \pi$

Parseval : $\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$

می دانیم :



$\longleftrightarrow \quad P(j\omega) = \frac{2\sin(\omega)}{\omega}$

$\hookrightarrow \int_{-\infty}^{+\infty} |p(t)|^2 dt = \int_{-1}^{1} dt = 2$

$\Rightarrow 2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{2\sin(\omega)}{\omega}\right)^2 d\omega$

دقت : چون $P(j\omega)$ حقیقی است، $|P(j\omega)|^2 = (P(j\omega))^2$ برقرار است.

$\Rightarrow 4\pi = \int_{-\infty}^{+\infty} \left(\frac{2\sin(\omega)}{\omega}\right)^2 d\omega$

$\Rightarrow 4\pi = 4 \int_{-\infty}^{+\infty} \left(\frac{\sin(\omega)}{\omega}\right)^2 d\omega$

$\Rightarrow \int_{-\infty}^{+\infty} \left(\frac{\sin(\omega)}{\omega}\right)^2 d\omega = \pi$