دانشگاه صنعتی امیرکبیر
( پلی تکنیک تهران )

# Signals and Systems

Assignment 1 Solutions

Fall 2019 - Group 1

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## Question 2

For each of the signals listed below, find the even and odd components $Ev\{x(t)\}$ and $Od\{x(t)\}$.

(a) $x(t) = e^{-5t}sin(t)u(t)$

$$x(t) = e^{-5t}sin(t)u(t)$$
$$x(-t) = e^{5t}sin(-t)u(-t) = -e^{5t}sin(t)u(-t)$$
$$Ev\{x(t)\} = \frac{x(t) + x(-t)}{2} \Rightarrow Ev\{x(t)\} = \frac{e^{-5t}sin(t)u(t) - e^{5t}sin(t)u(-t)}{2}$$
$$Od\{x(t)\} = \frac{x(t) - x(-t)}{2} \Rightarrow Od\{x(t)\} = \frac{e^{-5t}sin(t)u(t) + e^{5t}sin(t)u(-t)}{2}$$
$$\Rightarrow Ev\{x(t)\} = \frac{sin(t)}{2}(e^{-5t}u(t) - e^{5t}u(-t))$$
$$\Rightarrow Od\{x(t)\} = \frac{sin(t)}{2}e^{-5|t|}$$

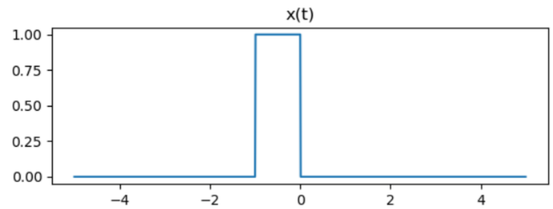
(b) $x(t) = e^{-3|t|}cos(t)$

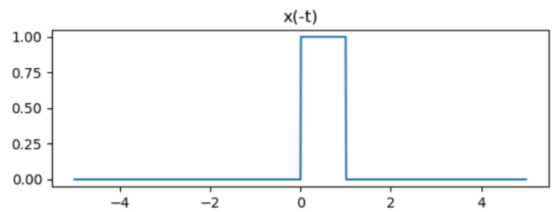
$$x(t) = e^{-3|t|}cos(t)$$
$$x(-t) = e^{-3|-t|}cos(-t) = e^{-3|t|}cos(t) = x(t)$$
$$\Rightarrow Ev\{x(t)\} = x(t)$$
$$\Rightarrow Od\{x(t)\} = 0$$

(c) $x(t) = \Pi(t + \frac{1}{2})$, (Solve by sketching) Hint:

$$\Pi(t) = rect(t) = \text{unit pulse} = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$$

$$Ev\{x(t)\} = \begin{cases} 0.5, |x| < 1 \\ 0, otherwise \end{cases}$$




$$Od\{x(t)\} \begin{cases} 0.5, -1 < x < 0 \\ -0.5, 0 < x < 1 \\ 0, otherwise \end{cases}$$

## Question 3

Determine if each signal below is periodic or not. If the signal is periodic, determine the fundamental period and the fundamental frequency,

(a) $x(t) = cos^2(3t - \frac{\pi}{3})$

$$cos(2\alpha) = 2cos^2(\alpha) - 1 \Rightarrow cos^2(\alpha) = \frac{1 + cos(2\alpha)}{2}$$

$$\Rightarrow x(t) = \frac{1 + cos(6t + \frac{2\pi}{3})}{2}$$

Since $cos(6t + \frac{2\pi}{3})$ is periodic with fundamental period $\frac{2\pi}{6} = \frac{\pi}{3}$, $x(t)$ is also periodic with the same fundamental period.

(b) $x(t) = e^{-|t|}cos(5t)$

Not Periodic

(c) $x(t) = e^{j(2t+\frac{\pi}{10})}$

$$\omega_0 = 2 \Rightarrow T_0 = \frac{2\pi}{\omega_0} = \pi$$

(d) $x(t) = sin^3(2t)$

$$x(t) = sin^3(2t)$$

$$sin(3\alpha) = 3sin(\alpha) - 4sin^3(\alpha)$$

$$sin^3(2t) = \frac{1}{4}[3sin(2t) - sin(6t)]$$

$$Period_1 = \frac{2\pi}{2} = \pi, Period_2 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$\Rightarrow T_0 = \pi$$

(e) $x[n] = 5cos(3\pi n)$

$$\frac{\omega_0}{2\pi} = \frac{3\pi}{2\pi} = \frac{3}{2} \Rightarrow \text{Rational} \Rightarrow N_0 = 2$$

(f) $x[n] = 5cos(3n)$

$$\frac{\omega_0}{2\pi} = \frac{3}{2\pi} \Rightarrow \text{Irrational} \Rightarrow NotPeriodic$$

(g) $x[n] = e^{j\frac{n}{2}} + e^{j\frac{n}{3}}$

Neither $e^{j\frac{n}{2}}$ nor $e^{j\frac{n}{3}}$ is Periodic $\Rightarrow$ Not Periodic

(h) $x[n] = e^{j\frac{n\pi}{2}} + e^{j\frac{n\pi}{3}}$

$$Period_1 : \frac{\omega_0}{2\pi} = \frac{1}{4} \Rightarrow Period_1 = 4$$

$$Period_2 : \frac{\omega_0}{2\pi} = \frac{1}{6} \Rightarrow Period_2 = 6$$

$$\Rightarrow T_0 = 12$$

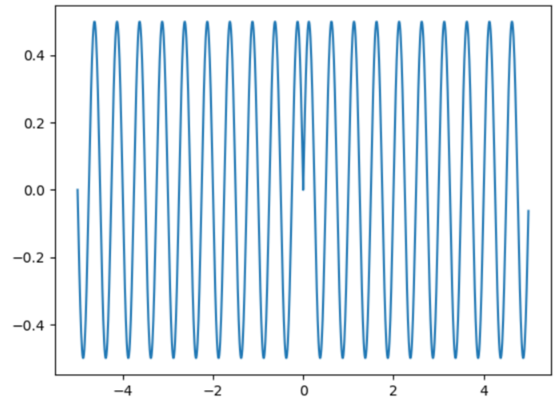
(i) $x(t) = Ev\{sin(4\pi t)u(t)\}$

Not Periodic (Look at the figure)

(j) $x(t) = Ev\{cos(4\pi t)u(t)\}$

$$Ev\{x(t)\} = \frac{cos(4\pi t)(u(t) + u(-t))}{2} = \frac{cos(4\pi t)}{2} \Rightarrow T_0 = \frac{2\pi}{4\pi} = \frac{1}{2}$$

Figure 1: $3(i)$




## Question 4

Determine whether these systems have memory or are memoryless, are causal or noncausal, are time-invariant or time-varying, are stable or unstable, are linear or nonlinear.

(a) $y(t) = e^{x(t)}$

- Memoryless: Yes.
- Causal: Yes.
- Time-Invariant: Yes.

$$input = x_1(t) \Rightarrow y_1(t) = e^{x_1(t)} \xrightarrow{Shift} y_1(t-t_0) = e^{x_1(t-t_0)}$$

$$input = x_2(t) = x_1(t-t_0) \Rightarrow y_2(t) = e^{x_2(t)} = e^{x_1(t-t_0)}$$

- Stable: Yes, BIBO applies.
- Linear: No.

$$input = x_1(t) \Rightarrow y_1(t) = e^{x_1(t)}$$

$$input = x_2(t) \Rightarrow y_2(t) = e^{x_2(t)}$$

$$input = x_3(t) = ax_1(t) + bx_2(t) \Rightarrow y_3(t) = e^{ax_1(t)+bx_2(t)} = e^{ax_1(t)}e^{bx_2(t)} \neq ay_1(t) + by_2(t)$$

(b) $y(t) = x(sin(t))$

- Memoryless: No. $y(\frac{3\pi}{2}) = x(-1)$.
- Causal: No. $y(-\pi) = x(0)$ while $-\pi < 0$
- Time-Invariant: No.

$$input = x_1(t) \Rightarrow y_1(t) = x_1(sin(t)) \xrightarrow{Shift} y_1(t-t_0) = x_1(sin(t-t_0))$$

$$input = x_2(t) = x_1(t-t_0) \Rightarrow y_2(t) = x_2(sin(t)) = x_1(sin(t) - t_0)$$

- Stable: Yes, BIBO applies.
- Linear: Yes.

$$input = x_1(t) \Rightarrow y_1(t) = x_1(sin(t))$$

$$input = x_2(t) \Rightarrow y_2(t) = x_2(sin(t))$$

$$input = x_3(t) = ax_1(t)+bx_2(t) \Rightarrow y_3(t) = x_3(sin(t)) = ax_1(sin(t))+bx_2(sin(t)) = ay_1(t)+by_2(t)$$

(c) $y(t) = \frac{dx(t)}{dt}$

- Memoryless: No, $\frac{dx(t)}{dt} = \lim_{h\to 0} \frac{x(t)-x(t-h)}{h}$
- Causal: Yes, $\frac{dx(t)}{dt} = \lim_{h\to 0} \frac{x(t)-x(t-h)}{h}$
- Time-Invariant: Yes.

$$a$$

- Stable: No. A finite signal can have an infinite derivative.
- Linear: Yes.

$$input = x_1(t) \Rightarrow y_1(t) = \frac{dx_1(t)}{dt}$$

$$input = x_2(t) \Rightarrow y_2(t) = \frac{dx_2(t)}{dt}$$

$$input = x_3(t) = ax_1(t) + bx_2(t) \Rightarrow y_3(t) = \frac{dx_3(t)}{dt} = \frac{d(ax_1(t) + bx_2(t))}{dt} = ay_1(t) + by_2(t)$$

(d) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$

- Memoryless: No, $y(1) = \int_{-\infty}^{2} \dots$
- Causal: No, $y(1) = \int_{-\infty}^{2} \dots$
- Time-Invariant: No.

$$input = x_1(t) \Rightarrow y_1(t) = \int_{-\infty}^{2t} x_1(\tau)d\tau \xrightarrow{Shift} y_1(t-t_0) = \int_{-\infty}^{2t-2t_0} x_1(\tau)d\tau$$

$$input = x_2(t) = x_1(t-t_0) \Rightarrow y_2(t) = \int_{-\infty}^{2t} x_1(\tau - t_0)d\tau$$

- Stable: No, if $x(t) = 1$, $y(0) = \int_{-\infty}^{0} d\tau = \infty$
- Linear: Yes.

$$input = x_1(t) \Rightarrow y_1(t) = \int_{-\infty}^{2t} x_1(\tau)d\tau$$

$$input = x_2(t) \Rightarrow y_2(t) = \int_{-\infty}^{2t} x_2(\tau)d\tau$$

$$input = x_3(t) = ax_1(t) + bx_2(t) \Rightarrow y_3(t) = \int_{-\infty}^{2t} (ax_1(\tau) + bx_2(\tau))d\tau = ay_1(t) + by_2(t)$$

(e) $y[n] = (n-1)x[n]$

- Memoryless: Yes.
- Causal: Yes.
- Time-Invariant:

$$input = x_1[n] \Rightarrow y_1[n] = (n-1)x_1[n] \xrightarrow{Shift} y_1[n-n_0] = (n-n_0-1)x_1[n-n_0]$$

$$input = x_2[n] = x_1[n-n_0] \Rightarrow y_2[n] = (n-1)x_2[n] = (n-1)x_1[n-n_0]$$

- Stable: No. If $x[n] = 1$, $y[\infty] = \infty$ so it's not bounded.
- Linear: Yes.

$$input = x_1[n] \Rightarrow y_1[n] = (n-1)x_1[n]$$

$$input = x_2[n] \Rightarrow y_2[n] = (n-1)x_2[n]$$

$$input = x_3[n] = ax_1[n] + bx_2[n] \Rightarrow y_3[n] = (n-1)\Big(ax_1[n] + bx_2[n]\Big) = ay_1[n] + by_2[n]$$

(f) $y[n] = x[3n+2]$

- Memoryless: No, $y[1] = x[5]$
- Causal: No, $y[1] = x[5]$
- Time-Invariant: No.

$$input = x_1[n] \Rightarrow y_1[n] = x_1[3n+2] \xrightarrow{Shift} y_1[n-n_0] = x_1[3n-3n_0+2]$$

$$input = x_2[n] = x_1[n-n_0] \Rightarrow y_2[n] = x_2[3n+2] = x_1[3n+2-n_0]$$

- Stable: Yes, BIBO applies.
- Linear: Yes.

$$input = x_1[n] \Rightarrow y_1[n] = x_1[3n+2]$$

$$input = x_2[n] \Rightarrow y_2[n] = x_2[3n+2]$$

$$input = x_3[n] = ax_1[n] + bx_2[n] \Rightarrow y_3[n] = x_3[3n+2] = ax_1[3n+2] + bx_2[3n+2] = ay_1[n] + by_2[n]$$

(g) $y[n] = Od\{x[n-1]\}$

Hint: first determine whether $Od\{x[n-n_0]\}$ is equal to $\frac{x[n-n_0]-x[-n-n_0]}{2}$ or $\frac{x[n-n_0]-x[-(n-n_0)]}{2}$

$$Od\{x[n-n_0]\} = \frac{x[n-n_0]-x[-n-n_0]}{2}$$

- Memoryless: No, $y[0]$ depends on $x[-1]$
- Causal: No, $y[-1]$ depends on $x[0]$
- Time-Invariant: No.

$$input = x_1[n] \Rightarrow y_1[n] = \frac{x_1[n-1]-x_1[-n-1]}{2} \xrightarrow{Shift} y_1[n-n_0] = \frac{x_1[n-n_0-1]-x_1[-n+n_0-1]}{2}$$

$$input = x_2[n] = x_1[n-n_0] \Rightarrow y_2[n] = \frac{x_2[n-1]-x_2[-n-1]}{2} = \frac{x_1[n-1-n_0]-x_1[-n-1-n_0]}{2}$$

- Stable: Yes, BIBO applies.
- Linear: Yes.

$$input = x_1[n] \Rightarrow y_1[n] = \frac{x_1[n-1]-x_1[-n-1]}{2}$$

$$input = x_2[n] \Rightarrow y_2[n] = \frac{x_2[n-1]-x_2[-n-1]}{2}$$

$$input = x_3[n] = ax_1[n]+bx_2[n] \Rightarrow y_3[n] = \frac{(ax_1[n-1]+bx_2[n-1])-(ax_1[-n-1]+bx_2[-n-1])}{2}$$

$$= a\frac{x_1[n-1]-x_1[-n-1]}{2} + b\frac{x_2[n-1]-x_2[-n-1]}{2} = ay_1[n] + by_2[n]$$

(h) $y[n] = x[n]\sum_{k=-\infty}^{\infty} \delta[n-2k]$, (Under what circumstances might this system be wrongly considered time-invariant?)

$$y[n] = \begin{cases} \text{x[n]} & \text{n is even} \\ 0 & \text{n is odd} \end{cases}$$

- Memoryless: Yes.
- Causal: Yes.

- Time-Invariant: No.

$$input = x_1[n] \Rightarrow y_1[n] = x_1[n] \sum_{k=-\infty}^{\infty} \delta[n-2k] \xrightarrow[Shift]{} y_1[n-n_0] = x_1[n-n_0] \sum_{k=-\infty}^{\infty} \delta[n-n_0-2k]$$

$$input = x_2[n] = x_1[n-n_0] \Rightarrow y_2[n] = x_2[n] \sum_{k=-\infty}^{\infty} \delta[n-2k] = x_1[n-n_0] \sum_{k=-\infty}^{\infty} \delta[n-2k]$$

Under the wrong assumption of $n_0 = 2k$, this system is time-invariant

- Stable: Yes, BIBO applies.
- Linear: Yes.

$$input = x_1[n] \Rightarrow y_1[n] = x_1[n] \sum_{k=-\infty}^{\infty} \delta[n-2k]$$

$$input = x_2[n] \Rightarrow y_2[n] = x_2[n] \sum_{k=-\infty}^{\infty} \delta[n-2k]$$

$$input = x_3[n] = ax_1[n] + bx_2[n] \Rightarrow y_3[n] = (ax_1[n] + bx_2[n]) \sum_{k=-\infty}^{\infty} \delta[n-2k] = ay_1[n] + by_2[n]$$

## Question 5

Determine if each of the following systems is invertible. If it is, construct the inverse system. if it is not, find two input signals to the system that have the same output.

(a) $y[n] = x[n]$

Not Invertible

$$x_1[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k] \Rightarrow y_1[n] = 1$$

$$x_2[n] = 1 \Rightarrow y_2[n] = 1$$

(b) $y[n] = \begin{cases} x[\frac{n}{2}], \text{n even} \\ 0, \text{otherwise} \end{cases}$

Invertible

$$g[n] = y[2n]$$

(c) $y(t) = \frac{dx(t)}{dt}$

Not Invertible

$$x_1(t) = 5t \Rightarrow y_1(t) = 5$$

$$x_2(t) = 5t + 3 \Rightarrow y_2(t) = 5$$

(d) $y(t) = Ev\{x(t)\}$

Not invertible

$$x_1(t) = cos(t) \Rightarrow y_1(t) = cos(t)$$

$$x_2(t) = cos(t) + sin(t) \Rightarrow y_2(t) = cos(t)$$

# Question 6

Determine the values of $P_\infty$ and $E_\infty$ for each of the following signals.

(a) $x(t) = e^{-4t}u(t)$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\Rightarrow E_\infty = \int_{0}^{+\infty} e^{-8t} dt = \frac{-1}{8}\Big(0 - 1\Big) = \frac{1}{8}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt$$

$$\Rightarrow P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{+T} e^{-8t} dt = \frac{-1}{8} \lim_{T\to\infty} \frac{1}{2T}(e^{-8T} - 1) = \frac{0}{\infty} = 0$$

(b) $x(t) = e^{j(2t + \frac{\pi}{4})}$

$$x(t) = cos(\dots) + jsin(\dots) \Rightarrow |x(t)| = \sqrt{cos^2(\dots) + sin^2(\dots)} = 1$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\Rightarrow E_\infty = \int_{-\infty}^{\infty} dt = \infty$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$\Rightarrow P_\infty = \lim_{T\to\infty} \frac{1}{2T} 2T = 1$$

(c) $x[n] = (\frac{1}{2})^n u[n]$

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

$$\Rightarrow E_\infty = \sum_{n=0}^{\infty} ((\frac{1}{2})^n)^2 = \sum_{n=0}^{\infty} (\frac{1}{4})^n = 1 + \frac{1}{4} + \frac{1}{16} + \dots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

$$\Rightarrow P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (\frac{1}{4})^n = \frac{\frac{4}{3}}{\infty} = 0$$

(d) $x[n] = e^{j(\frac{\pi}{2}n + \frac{\pi}{8})}$ Just like 6.$b$ with discrete formulas :)