دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

Signals and Systems

Assignment 2 Solutions

Fall 2019 - Group 1

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Question 1

(a) The impulse response of a CTLTI system is

$$h(t) = \delta(t) - \delta(t-1)$$

Determine and sketch the response of this system to the triangular waveform shown in Figure 1.

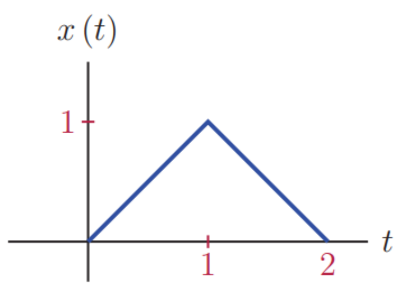
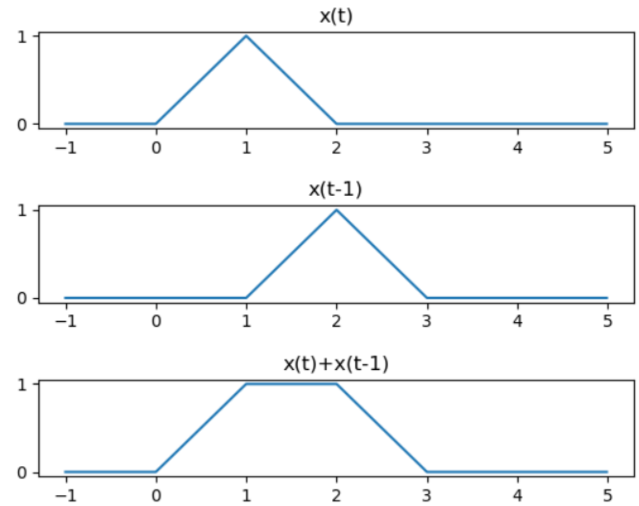


Figure 1: $x(t)$ for question $1(a)$

$$y(t) = x(t) * h(t) = x(t) * \Big(\delta(t) - \delta(t-1)\Big)$$

$$\Rightarrow y(t) = x(t) * \delta(t) + x(t) * \delta(t-1) = x(t) + x(t-1)$$



(b) A CTLTI system has the impulse response

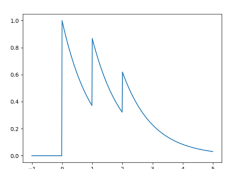
$$h(t) = \delta(t) + 0.5\delta(t-1) + 0.3\delta(t-2)$$

Determine and sketch the response of this system to the exponential input signal

$$x(t) = e^{-t}u(t)$$

$$y(t) = e^{-t}u(t) * \Big(\delta(t) + 0.5\delta(t-1) + 0.3\delta(t-2)\Big)$$

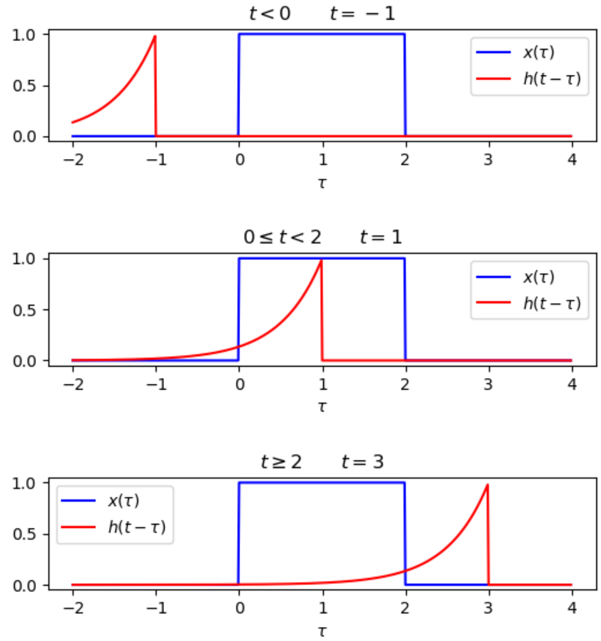
$$\Rightarrow y(t) = e^{-t}u(t) + 0.5e^{-(t-1)}u(t-1) + 0.3e^{-(t-2)}u(t-2)$$



Question 2

For each pair of signals $x(t)$ and $h(t)$ given below, find the convolution $y(t) = x(t) * h(t)$

(a) $x(t) = u(t) - u(t-2)$, $h(t) = e^{-2t}u(t)$



- $t < 0$

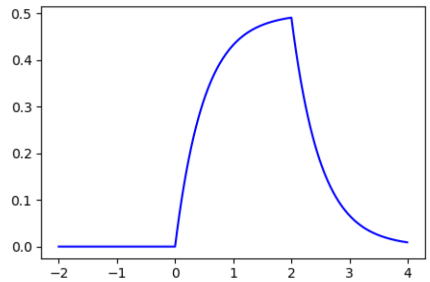
$$y(t) = 0$$

- $0 \le t < 2$

$$y(t) = \int_0^t e^{-2(t-\tau)} d\tau = \frac{1}{2}\left(e^{2(\tau-t)}\right)\Big|_0^t = \frac{1-e^{-2t}}{2}$$

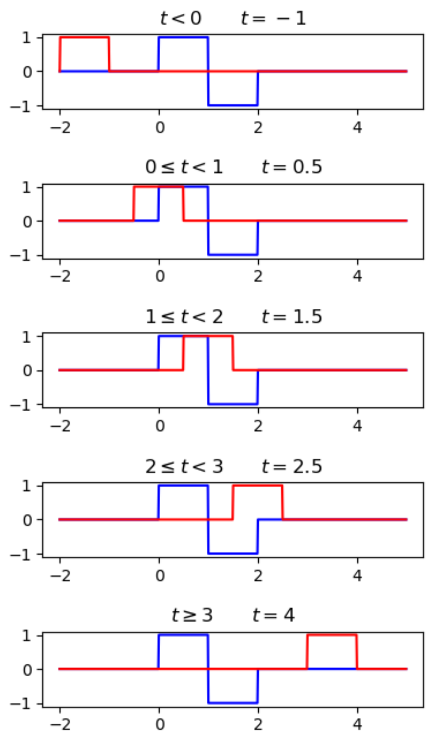
- $t \ge 2$

$$y(t) = \int_0^2 e^{-2(t-\tau)} d\tau = \frac{1}{2}\left(e^{2(\tau-t)}\right)\Big|_0^2 = \frac{e^{2(2-t)} - e^{-2t}}{2}$$



0.5
0.4
0.3
0.2
0.1
0.0
−2
−1
0
1
2
3
4

(b) $x(t) = \Pi(t - \frac{1}{2}) - \Pi(t - \frac{3}{2})$, $h(t) = u(t) - u(t-1)$



- $t < 0$ or $t \geq 3$

$$y(t) = 0$$

- $0 \leq t < 1$

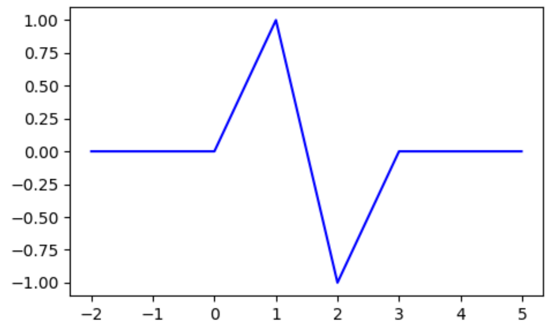
$$y(t) = \int_0^t d\tau = t$$

- $1 \leq t < 2$

$$y(t) = \int_{t-1}^1 d\tau + \int_1^t -d\tau = 2 - t + 1 - t = 3 - 2t$$

- $2 \leq t < 3$

$$y(t) = \int_{t-1}^2 -d\tau = -(2 - (t-1)) = t - 3$$


1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−2
−1
0
1
2
3
4
5

(c) Figure 2

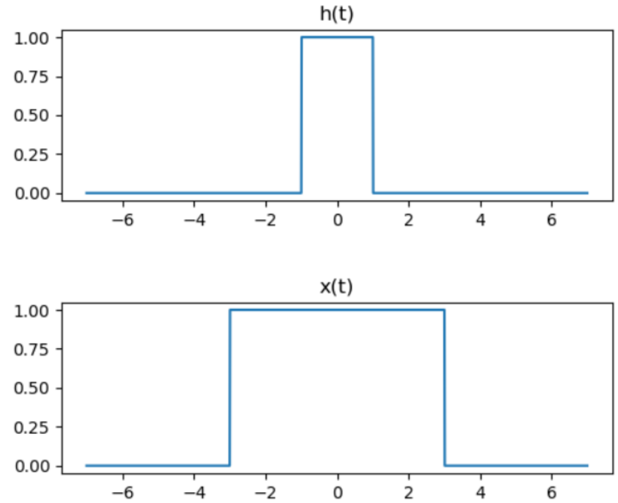
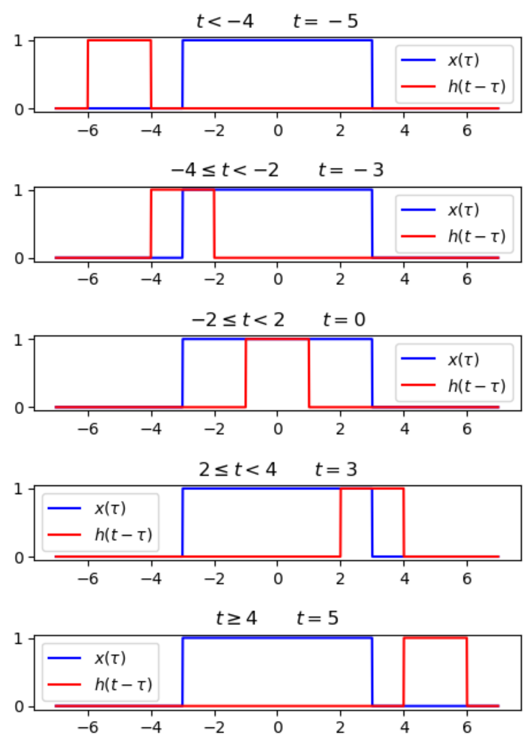


Figure 2: This type of convolution is widely used in the upcoming chapters



- $t < -4$ or $t \geq 4$

$$y(t) = 0$$

- $-4 \leq t < -2$

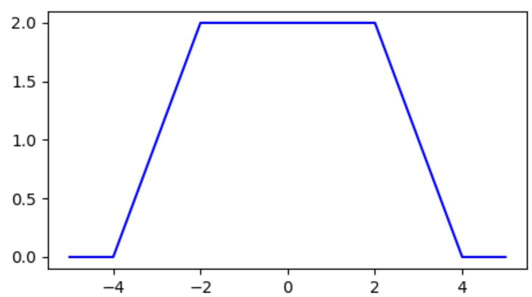
$$y(t) = \int_{-3}^{t+1} d\tau = t + 4$$

- $-2 \leq t < 2$

$$y(t) = \int_{t-1}^{t+1} d\tau = 2$$

- $2 \leq t < 4$

$$y(t) = \int_{t-1}^{3} d\tau = 4 - t$$



DO NOT MEMORIZE! DERIVE WHILE KEEPING THIS IN MIND!

General Case: if $\alpha > \beta > 0$

$$x_1(t) = u(t + \beta) - u(t - \beta)$$

$$x_2(t) = u(t + \alpha) - u(t - \alpha)$$

$$\Rightarrow x_1(t) * x_2(t) = \begin{cases} 0 & t < -(\alpha + \beta) \\ t + (\alpha + \beta) & -(\alpha + \beta) \leq t < -(\alpha - \beta) \\ 2\beta & -(\alpha - \beta) \leq t < (\alpha - \beta) \\ (\alpha + \beta) - t & (\alpha - \beta) \leq t < (\alpha + \beta) \\ 0 & t \geq (\alpha + \beta) \end{cases}$$

Question 3

(a) Consider a system with the impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

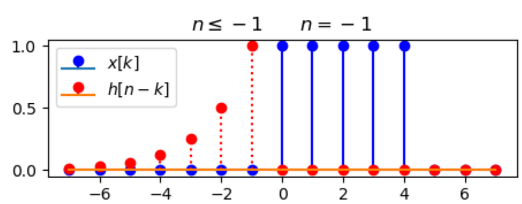
Determine the output if the input is defined as follows:

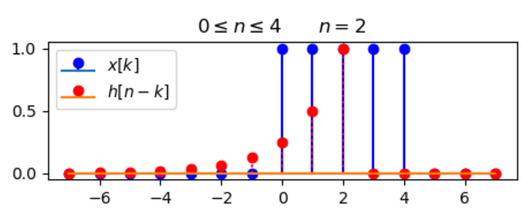
$$x[n] = u[n] - u[n-5]$$

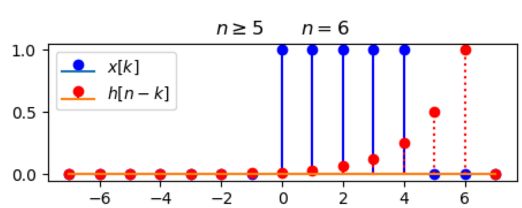
$$\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$$

$$|\alpha| < 1 \Rightarrow \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

$$\sum_{k=m}^{N} \alpha^k \xrightarrow{k'=k-m} \quad = \sum_{k'=0}^{N-m} \alpha^{k'+m} = \alpha^m \sum_{k'=0}^{N-m} \alpha^{k'} = a^m \frac{1-\alpha^{(N-m)+1}}{1-\alpha}$$







- $n \leq -1$

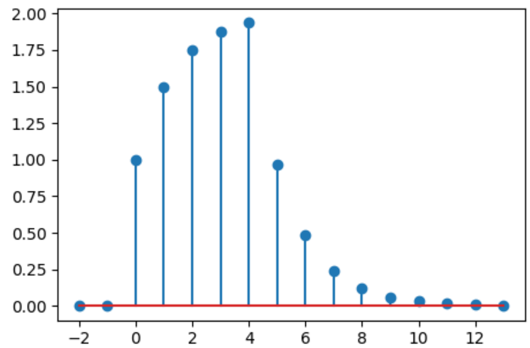
$$y[n] = 0$$

- $0 \leq n \leq 4$

$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}}$$

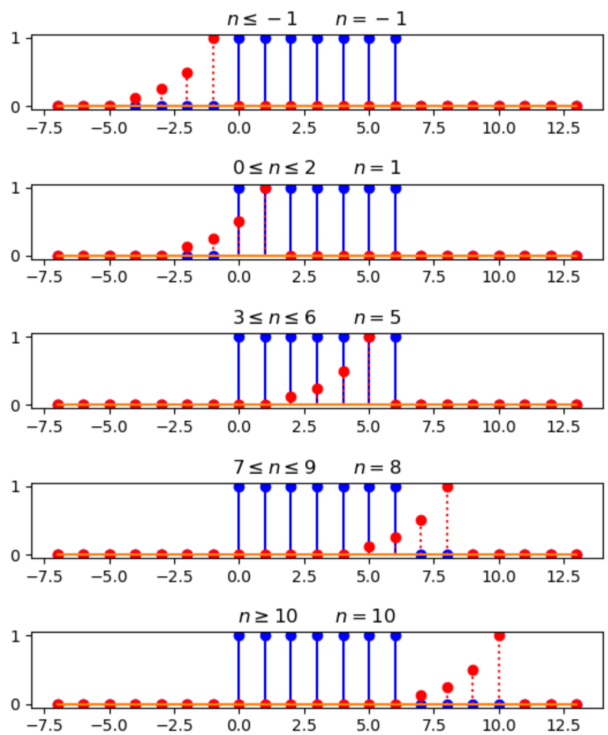
- $n \geq 5$

$$y[n] = \sum_{k=n-4}^{n} \left(\frac{1}{2}\right)^k = \sum_{k'=0}^{4} \left(\frac{1}{2}\right)^{k'+n-4} = \left(\frac{1}{2}\right)^{n-4} \sum_{k'=0}^{4} \left(\frac{1}{2}\right)^{k'} = \left(\frac{1}{2}\right)^{n-4} \frac{1-(\frac{1}{2})^5}{1-\frac{1}{2}}$$

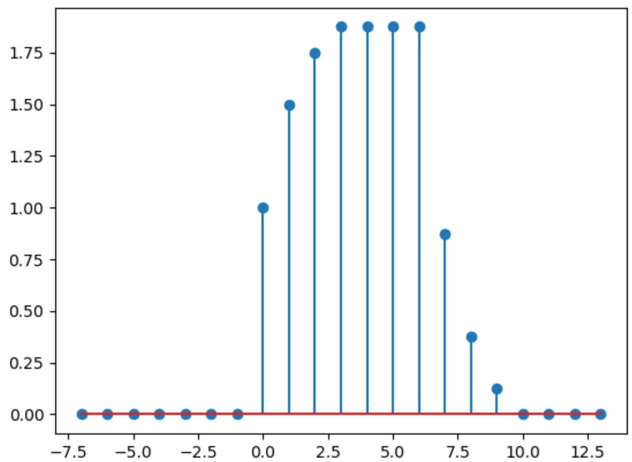


(b) Convolve:

$$h[n] = \left(\frac{1}{2}\right)^n (u[n] - u[n-4])$$
$$x[n] = u[n] - u[n-7]$$



- $n \leq -1$
$$y[n] = 0$$
- $0 \leq n \leq 2$
$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}}$$
- $3 \leq n \leq 6$
$$y[n] = \sum_{k=0}^{3} \left(\frac{1}{2}\right)^k = \frac{1 - (\frac{1}{2})^{4}}{1 - \frac{1}{2}}$$
- $7 \leq n \leq 9$
$$y[n] = \sum_{k=n-6}^{3} \left(\frac{1}{2}\right)^k = \sum_{k'=0}^{3-n+6} \left(\frac{1}{2}\right)^{n+n-6} = \left(\frac{1}{2}\right)^{n-6} \frac{1 - (\frac{1}{2})^{10-n}}{1 - \frac{1}{2}}$$
- $n \geq 10$
$$y[n] = 0$$



1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5

Question 4

For each of the following impulse responses, determine whether the corresponding system is memoryless, causal and stable. Justify your answers.

(a) $h(t) = u(t) - u(t-3)$

- Memoryless: No, $h(t)$ has nonzero values for nonzero t.
- Causal: Yes, $h(t)$ is zero for $t < 0$.
- Stable: Yes, $\int_{-\infty}^{\infty} |h(\tau)| d\tau$ is finite.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = 3$$

(b) $h(t) = \Pi(t)$

- Memoryless: No, $h(t)$ has nonzero values for nonzero t.
- Causal: No, $h(t)$ is not always zero for $t < 0$.
- Stable: Yes, $\int_{-\infty}^{\infty} |h(\tau)| d\tau$ is finite.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = 1$$

(c) $h(t) = e^{-3|t|}$

- Memoryless: No, $h(t)$ has nonzero values for nonzero t.
- Causal: No, $h(t)$ is not zero for $t < 0$.
- Stable: Yes, $\int_{-\infty}^{\infty} |h(\tau)| d\tau$ is finite.

$$\int_{-\infty}^{\infty} h(\tau) d\tau = \int_{-\infty}^{0} e^{3\tau} d\tau + \int_{0}^{\infty} e^{-3\tau} d\tau = \frac{1}{3}(1-0) + \frac{-1}{3}(0-1) = \frac{2}{3}$$

(d) $h(t) = sin(2\pi t)u(t)$

- Memoryless: No, $h(t)$ has nonzero values for nonzero t.
- Causal: Yes, $h(t)$ is zero for $t < 0$.
- Stable: No, $\int_{-\infty}^{\infty} |h(\tau)| d\tau$ is infinite.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{0}^{\infty} |sin(2\pi\tau)| d\tau = \infty$$

(e) $h[n] = (\frac{1}{2})^n u[n]$

- Memoryless: No, $h[n]$ has nonzero values for nonzero n.
- Causal: Yes, $h[n]$ is zero for $n < 0$.
- Stable: Yes, $\sum_{k=-\infty}^{\infty} |h[k]|$ is finite.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1-\frac{1}{2}} = 2$$

(f) $h[n] = 5^n u[3-n]$

$$u[3-n] = u[-n+3] = \begin{cases} 1 & n \leq 3 \\ 0 & otherwise \end{cases}$$

- Memoryless: No, $h[n]$ has nonzero values for nonzero n.

- Causal: No, $h[n]$ is nonzero for $n < 0$.
- Stable: Yes, $\sum_{k=-\infty}^{\infty} |h[k]|$ is finite.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{3} 5^k = 5^1 + 5^2 + 5^3 + \sum_{k=0}^{\infty} (\frac{1}{5})^k = 5^1 + 5^2 + 5^3 + \frac{1}{1 - \frac{1}{5}}$$

(g) $h[n] = cos(n\pi)u[n + 5]$

- Memoryless: No, $h[n]$ has nonzero values for nonzero n.
- Causal: No, $h[n]$ is not always zero for $n < 0$.
- Stable: No, $\sum_{k=-\infty}^{\infty} |h[k]|$ is infinite.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-5}^{\infty} |cos(k\pi)| = 1 + 1 + 1 + \cdots = \infty$$

Question 5

Find the step response for systems with following impulse responses:

Step Response for CTLTI Systems:

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Step Response for DTLTI Systems:

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{n} h[k]$$

(a) $h(t) = \delta(t) - \delta(t-5)$

- $t < 0$

$$s(t) = 0$$

- $0 \leq t < 5$

$$s(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = 1$$

- $t \geq 5$

$$s(t) = \int_{-\infty}^{t} (\delta(\tau) - \delta(\tau-5))d\tau = 1 - 1 = 0$$

(b) $h(t) = \delta(t) + \delta(t-5)$

- $t < 0$

$$s(t) = 0$$

- $0 \leq t < 5$

$$s(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = 1$$

- $t \geq 5$

$$s(t) = \int_{-\infty}^{t} (\delta(\tau) + \delta(\tau-5))d\tau = 1 + 1 = 2$$

(c) $h(t) = e^{-|t|}$

- $t < 0$

$$s(t) = \int_{-\infty}^{t} e^{\tau}d\tau = e^{t}$$

- $t \geq 0$

$$s(t) = \int_{-\infty}^{0} e^{\tau}d\tau + \int_{0}^{t} e^{-\tau}d\tau = 1 - (e^{-t} - 1) = 2 - e^{-t}$$

(d) $h[n] = (\frac{1}{5})^n u[n]$

- $n < 0$

$$s[n] = 0$$

- $n \geq 0$

$$s[n] = \sum_{k=0}^{n} \left(\frac{1}{5}\right)^k = \frac{1 - (\frac{1}{5})^{n+1}}{1 - \frac{1}{5}}$$

Question 6

Consider the CTLTI system shown in Figure 3

$$h_1(t) = e^{-t}u(t)$$

$$h_2(t) = h_3(t) = u(t) - u(t-1)$$

$$h_4(t) = \delta(t-1)$$

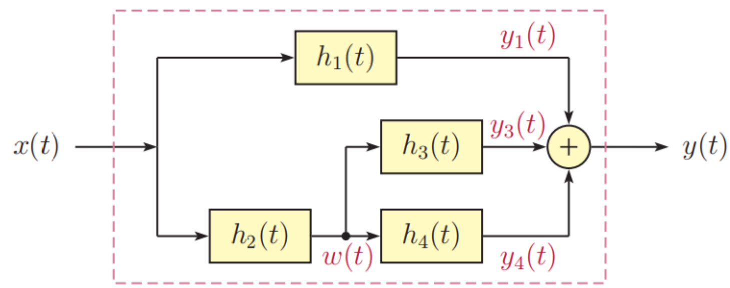


Figure 3: An integrated system

(a) Determine the impulse response $h_{eq}(t)$ of the equivalent system.

$$y(t) = y_1(t) + y_3(t) + y_4(t)$$

$$y_1(t) = x(t) * h_1(t)$$

$$w(t) = x(t) * h_2(t)$$

$$y_3(t) = w(t) * h_3(t) = x(t) * (h_2(t) * h_3(t))$$

$$y_4(t) = w(t) * h_4(t) = x(t) * (h_2(t) * h_4(t)) = x(t) * h_2(t-1)$$

$$\Rightarrow y(t) = x(t) * h_1(t) + x(t) * (h_2(t) * h_3(t)) + x(t) * h_2(t-1)$$

$$\Rightarrow y(t) = x(t) * [h_1(t) + h_2(t) * h_3(t) + h_2(t-1)]$$

$$\Rightarrow h_{eq}(t) = h_1(t) + h_2(t) * h_2(t) + h_2(t-1)$$

(b) Let the input signal be a unit-step, that is, $x(t) = u(t)$. Determine and sketch the signals $w(t), y_1(t), y_3(t)$ and $y_4(t)$.

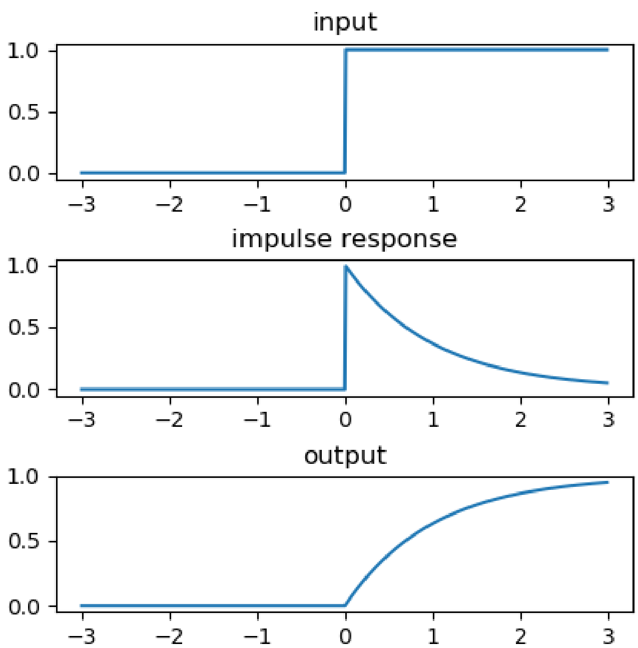
$$x(t) = u(t)$$

- $y_1(t) = u(t) * e^{-t}u(t)$
  * $t < 0$

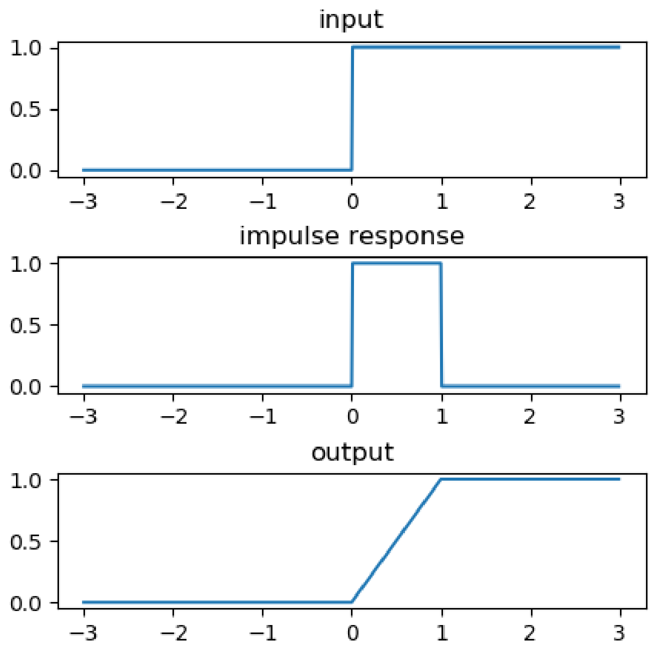
$$y_1(t) = 0$$

  * $t \geq 0$

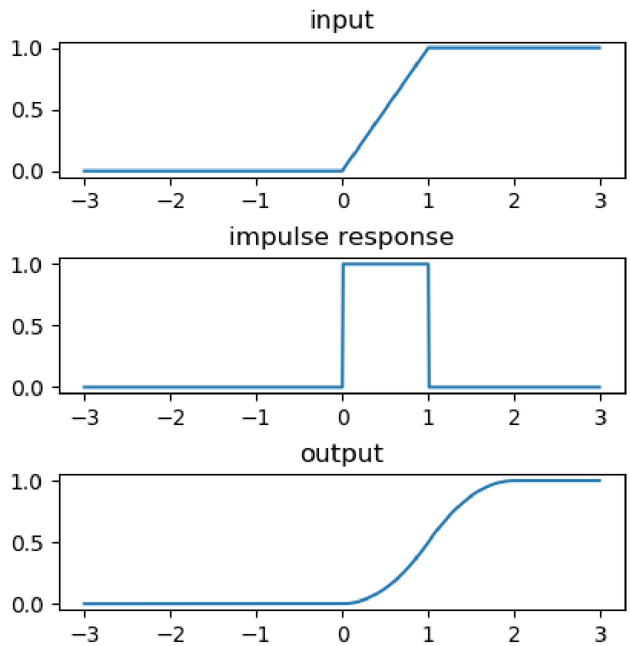
$$y_1(t) = \int_0^t e^{\tau - t} d\tau = 1 - \frac{1}{e^t}$$



- $w(t) = u(t) * (u(t) - u(t-1))$
  * $t < 0$
  $$w(t) = 0$$
  * $0 \leq t < 1$
  $$w(t) = t$$
  * $t \geq 1$
  $$w(t) = 1$$



- $y_3(t) = w(t) * (u(t) - u(t-1))$
  - * $t < 0$
$$y_3(t) = 0$$
  - * $0 \le t < 1$
$$y_3(t) = \int_0^t \tau d\tau = \frac{t^2}{2}$$
  - * $1 \le t < 2$
$$y_3(t) = \int_{t-1}^1 \tau d\tau + \int_1^t d\tau = \frac{1}{2} - \frac{(t-1)^2}{2} + t - 1 = t - \frac{(t-1)^2}{2} - \frac{1}{2}$$
  - * $t \ge 2$
$$y_3(t) = 1$$



- $y_4(t) = w(t) * \delta(t-1) = w(t-1)$
  No need to sketch. Just shift $w(t)$ one unit to the right.