حل سوالات تمرین سری چهارم

1) a) $x(t) \xleftrightarrow{F.T.} X(j\omega) \Rightarrow x(t+1) \xleftrightarrow{F.T.} e^{j\omega} X(j\omega) \Rightarrow x(1-t) \xleftrightarrow{F.T.} e^{-j\omega} X(-j\omega)$

$\Rightarrow \frac{d}{dt} x(1-t) \xleftrightarrow{F.T.} j\omega e^{-j\omega} X(-j\omega)$

b) $x(t) \xleftrightarrow{F.T.} X(j\omega) \Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F.T.} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$

c) $2x(t) \xleftrightarrow{F.T.} 2X(j\omega)$, $x(3-t) \xleftrightarrow{F.T.} e^{-3j\omega} X(-j\omega)$ $\Rightarrow 2x(t) + x(3-t) \xleftrightarrow{F.T.} 2X(j\omega) + e^{-3j\omega} X(-j\omega)$

d) $\frac{d^2}{dt^2} x(t) \xleftrightarrow{F.T.} (j\omega)^2 X(j\omega) \Rightarrow \frac{d^2}{dt^2} x(t+2) \xleftrightarrow{F.T.} -\omega^2 e^{2j\omega} X(j\omega)$

e) $x(2t) \xleftrightarrow{F.T.} \frac{1}{2} X\left(\frac{j\omega}{2}\right) \Rightarrow t\,x(2t) \xleftrightarrow{F.T.} \frac{j}{2}\frac{d}{d\omega} X\left(\frac{j\omega}{2}\right) \Rightarrow t^2 x(2t) \xleftrightarrow{F.T.} -\frac{1}{2}\frac{d^2}{d\omega^2} X\left(\frac{j\omega}{2}\right)$

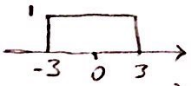
f) $x(-2t) \xleftrightarrow{F.T.} \frac{1}{2} X\left(\frac{j\omega}{-2}\right)$, $x(3t) \xleftrightarrow{F.T.} \frac{1}{3} X\left(\frac{j\omega}{3}\right)$ $\Rightarrow x(-2t) * x(3t) \xleftrightarrow{F.T.} \frac{1}{6} X\left(-\frac{j\omega}{2}\right) X\left(\frac{j\omega}{3}\right)$

2) a) $\cos\left(\frac{\pi}{2}t\right) \xleftrightarrow{F.T.} \pi\left[\delta\left(\omega-\frac{\pi}{2}\right)+\delta\left(\omega+\frac{\pi}{2}\right)\right] \Rightarrow \cos\left(\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right) \xleftrightarrow{F.T.} e^{j\omega/2}\pi\left[\delta\left(\omega-\frac{\pi}{2}\right)+\delta\left(\omega+\frac{\pi}{2}\right)\right]$

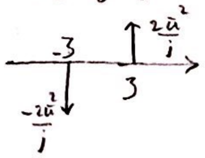
$\sin(t) \xleftrightarrow{F.T.} \frac{\pi}{j}\left[\delta(\omega-1)-\delta(\omega+1)\right] \Rightarrow x(t) \xleftrightarrow{F.T.} \frac{\pi}{j}\left[\delta(\omega-1)-\delta(\omega+1)\right] + e^{j\omega/2}\pi\left[\delta\left(\omega-\frac{\pi}{2}\right)+\delta\left(\omega+\frac{\pi}{2}\right)\right]$

b) $x(t) = \frac{\sin(3t)}{\pi t} \times 2\pi \sin(3t)$

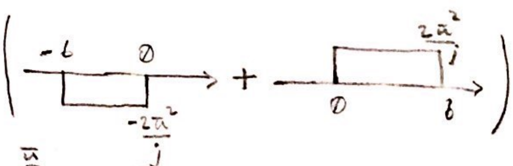
$\frac{\sin(3t)}{\pi t} \xleftrightarrow{F.T.}$



$2\pi\sin(3t) \xleftrightarrow{F.T.}$

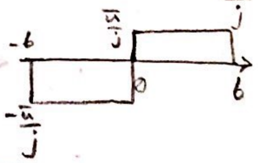


$\Rightarrow x(t) \xleftrightarrow{F.T.} \frac{1}{2\pi}\Big($



$\Big)$

$\Rightarrow x(t) \xleftrightarrow{F.T.}$



c) $x(t) = e^{-t}u(t) \times t^2\cos(t)$, $e^{-t}u(t) \xleftrightarrow{F.T.} \frac{1}{1+j\omega} \Rightarrow t^2 e^{-t}u(t) \xleftrightarrow{F.T.} -\frac{d^2}{d\omega^2}\frac{1}{1+j\omega} = \frac{2}{(1+j\omega)^3}$

$\Rightarrow t^2 e^{-t}u(t)\cos(t) \xleftrightarrow{F.T.} \frac{1}{(1+j(\omega-1))^3} + \frac{1}{(1+j(\omega+1))^3}$

2) d) $x(t) = t \times \frac{2}{1+t^2}$

$$e^{-|t|} \xleftrightarrow{F.T.} \frac{2}{1+\omega^2}$$

$$\frac{2}{1+t^2} \xleftrightarrow{F.T.} 2\pi e^{-|\omega|}$$

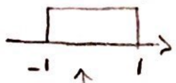
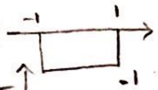
$$\Rightarrow t \times \frac{2}{1+t^2} \xleftrightarrow{F.T.} j\frac{d}{d\omega} 2\pi e^{-|\omega|}$$

e) $$\frac{1}{1+t^2} \xleftrightarrow{F.T.} \pi e^{-|\omega|} \Rightarrow \int_{-\infty}^{t} \frac{d\tau}{1+\tau^2} \xleftrightarrow{F.T.} \frac{\pi e^{-|\omega|}}{j\omega} + \pi^2 \delta(\omega)$$

f) $$e^{-3|t|+j\frac{\pi}{6}t} = e^{j\frac{\pi}{6}t} e^{-3|t|}$$

$$e^{-3|t|} \longleftrightarrow \frac{6}{3+\omega^2}$$

$$\Rightarrow x(t) \xleftrightarrow{F.T.} \frac{6}{3+(\omega-\frac{\pi}{6})^2}$$

g) $x(t) =$  $+ t \times$ 

$$\frac{2\sin\omega}{\omega} \qquad + j\frac{d}{d\omega}\, -\frac{2\sin(\omega)}{\omega}$$

$$\Rightarrow x(t) \xleftrightarrow{F.T.} \frac{2\sin\omega}{\omega} - 2j\frac{\omega\cos\omega - \sin(\omega)}{\omega^2}$$

3) a) $$e^{j\omega_0 t} \xleftrightarrow{F.T.} 2\pi\delta(\omega-\omega_0) \Rightarrow \frac{3}{2\pi} e^{3jt} \xleftrightarrow{F.T.} 3\delta(\omega-3)$$

b) $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \Rightarrow x(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\omega|} e^{j\omega t} d\omega = \frac{1}{2}\int_{-\infty}^{0} e^{(jt+1)\omega} d\omega + \frac{1}{2}\int_{0}^{\infty} e^{(jt-1)\omega} d\omega$$

$$= \frac{1}{2}\frac{1}{jt+1} e^{(jt+1)\omega}\Big|_{-\infty}^{0} + \frac{1}{2}\frac{1}{jt-1} e^{(jt-1)\omega}\Big|_{0}^{\infty} = \frac{1}{2}\frac{1}{jt+1} - \frac{1}{2}\frac{1}{jt-1} = \frac{1}{2}\frac{-2}{-t^2-1} = \frac{1}{t^2+1}$$

c) $$\frac{7j\omega+32}{-\omega^2+9j\omega+20} = \frac{7j\omega+32}{(\omega j+4)(\omega j+5)} = \frac{4j\omega+20+3j\omega+12}{(\omega j+4)(\omega j+5)} = \frac{3}{\omega j+5} + \frac{4}{\omega j+4}$$

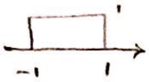
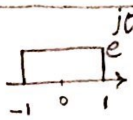
$$\frac{3}{\omega j+5} \xleftrightarrow{F.T.} 3e^{-5t}u(t) \qquad \frac{4}{\omega j+4} \xleftrightarrow{F.T.} 4e^{-4t}u(t)$$

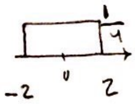
$$\Rightarrow x(t) \xleftrightarrow{F.T.} 3e^{-5t}u(t) + 4e^{-4t}u(t)$$

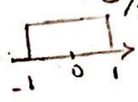
d) $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\omega\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right) e^{j\omega t} d\omega$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{j\omega(t+1)} d\omega + \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{j\omega(t-1)} d\omega = \frac{1}{4\pi}\frac{e^{j\omega(t+1)}}{j(t+1)}\Big|_{-\pi}^{\pi} + \frac{1}{4\pi}\frac{e^{j\omega(t-1)}}{j(t-1)}\Big|_{-\pi}^{\pi}$$

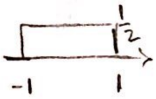
3) d) $= \frac{1}{2\pi}\frac{\cos(\pi t+\pi)}{jt+j} + \frac{1}{2\pi}\frac{\cos(\pi t-\pi)}{jt-j} = -\frac{1}{2\pi}\left[\frac{\cos(\pi t)}{jt+j} + \frac{\cos(\pi t)}{jt-j}\right] = \frac{jt\cos(\pi t)}{\pi(t^2-1)}$

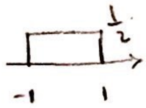
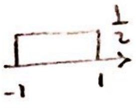
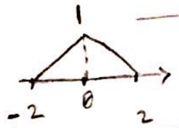
e)  $\xleftrightarrow{F.T.} \frac{2\sin w}{w} \Rightarrow$  $\xleftrightarrow{F.T.} \frac{2\sin(w-1)}{w-1}$

 $\xleftrightarrow{F.T.} \frac{\sin(2w)}{w}$

$\Rightarrow$  $\xleftrightarrow{F.T.} X(jw)$

f) $\frac{\sin(-w)}{w} \times \frac{\sin(-w)}{w} = X(jw)$

 $\xleftrightarrow{F.T.} \frac{\sin(-w)}{w}$

$\Rightarrow x(t) =$  $*$  $=$ 

g) $X(jw) = \frac{1}{j}\frac{d}{dw}\left[\frac{\cos(\pi w)+j\sin(\pi w)}{1+2jw}\right] = \frac{1}{j}\frac{d}{dw}\left[\frac{e^{j\pi w}}{1+2jw}\right]$

$\frac{1}{2}e^{-\frac{t}{2}}u(t) \xleftrightarrow{F.T.} \frac{1}{1+2jw} \Rightarrow \frac{1}{2}e^{-\frac{t}{2}-\frac{\pi}{2}}u(t+\pi) \xleftrightarrow{F.T.} \frac{e^{j\pi w}}{1+2jw} \Rightarrow \frac{t}{2}e^{-\frac{t}{2}-\frac{\pi}{2}}u(t+\pi) \xleftrightarrow{F.T.} j\frac{d}{dw}\left[\frac{e^{j\pi w}}{1+2jw}\right]$

h) $\frac{t^5}{(5-1)!}e^{-t}u(t) \xleftrightarrow{F.T.} \frac{1}{(1+jw)^6}$

4) a) $-(jw)^2Y(jw) - 7jwY(jw) - 10Y(jw) = 2jwX(jw) + 13X(jw)$

$H(jw) = \frac{Y(jw)}{X(jw)} \Rightarrow H(jw) = \frac{2jw+13}{w^2-7jw-10} = -\frac{2wj+13}{(wj+2)(wj+5)} = -\frac{3wj+15-wj-2}{(wj+2)(wj+5)} = \frac{1}{wj+5} - \frac{3}{wj+2}$

b) $h(t) \xleftrightarrow{F.T.} \frac{1}{wj+5} - \frac{3}{wj+2} \Rightarrow h(t) = e^{-5t}u(t) - 3e^{-2t}u(t)$

c) $x_1(t) = te^{-t}u(t) \Rightarrow X_1(jw) = \frac{1}{(1+jw)^2}$

$Y_1(jw) = X_1(jw)H(jw)$

$\Rightarrow Y_1(jw) = \frac{1}{(1+jw)^2}\times\left(\frac{1}{jw+5} - \frac{3}{jw+2}\right)$

$= \frac{1}{1+jw}\left(\frac{1}{1+jw}\frac{1}{5+jw} - \frac{1}{1+jw}\frac{3}{2+jw}\right)$

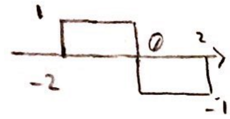
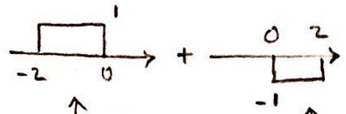
$= \frac{1}{1+jw}\left(\frac{\frac{1}{4}}{1+jw} - \frac{\frac{1}{4}}{5+jw} - \frac{3}{1+jw} + \frac{3}{2+jw}\right) = \frac{1}{1+jw}\left(-\frac{11}{4}\frac{1}{1+jw} + \frac{3}{1+2jw} - \frac{1}{4}\frac{1}{5+jw}\right)$

$= -\frac{11}{4}\frac{1}{(1+jw)^2} + \frac{3}{1+jw} - \frac{3}{2+jw} - \frac{1}{16}\frac{1}{1+jw} + \frac{1}{16}\frac{1}{5+jw} \Rightarrow y_1(t) = -\frac{11}{4}te^{-t}u(t) + \frac{47}{16}e^{-t}u(t) - 3e^{-2t}u(t) + \frac{e^{-5t}u(t)}{16}$

4) d) $G(j\omega) = \frac{1}{H(j\omega)} \Rightarrow G(j\omega) = \frac{\omega^2 - 7j\omega - 10}{2\omega j + 13} = \frac{\omega^2 - 6.5\omega j - 0.5\omega j - 10}{2\omega j + 13} = -\frac{\omega j}{2} - \frac{\omega j + 20}{4\omega j + 26}$

$= -\frac{\omega j}{2} - \frac{\omega j + 6.5 + 13.5}{4\omega j + 26} = -\frac{\omega j}{2} - \frac{1}{4} - \frac{13.5}{4\omega j + 26} \Rightarrow \boxed{g(t) = -\frac{\delta'(t)}{2} - \frac{\delta(t)}{4} - 3.375\, e^{-6.5t} u(t)}$

e) $G(j\omega) = \frac{(j\omega)^2 - 7j\omega - 10}{2\omega j + 13} \Rightarrow 2\frac{d}{dt}y(t) + 13y(t) = -\frac{d^2}{dt^2}x(t) - 7\frac{d}{dt}x(t) - 10x(t)$

5) a) $\frac{d}{dt}x(t) =$  $=$ 

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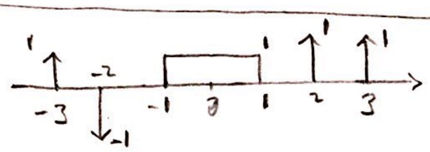
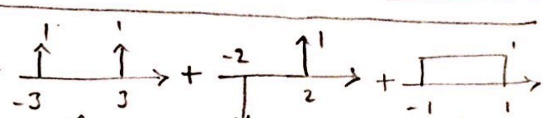
$e^{j\omega}\frac{2\sin\omega}{\omega} + -e^{-j\omega}\frac{2\sin\omega}{\omega}$

$\Rightarrow \frac{d}{dt}x(t) \xleftrightarrow{F.T.} \frac{2\sin\omega}{\omega} \times 2j\sin(\omega)$

$\Rightarrow \boxed{x(t) \xleftrightarrow{F.T.} 4\frac{\sin^2\omega}{\omega^2}}$

b) $\frac{d}{dt}x(t) =$  $=$ 

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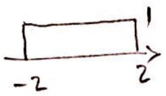
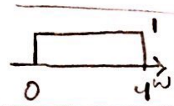
$j\omega X(j\omega) = 2\cos(-3\omega) + 2j\sin(-2\omega) + \frac{2\sin\omega}{\omega}$

$\Rightarrow X(j\omega) = \frac{2}{j\omega}\cos(-3\omega) + \frac{2}{\omega}\sin(-2\omega) + \frac{2\sin\omega}{j\omega^2}$

6) $x(t) \times p(t) = a(t) \Rightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega) = A(j\omega)$

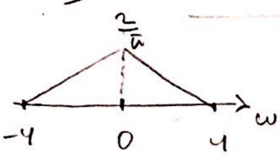
$x(t) = \frac{\sin 2t}{\pi t} \Rightarrow X(j\omega) =$ 

$P(t) = \cos(2t) + j\sin(2t) \Rightarrow P(j\omega) = 2\pi\delta(\omega - 2)$

$\Rightarrow A(j\omega) =$ 

$B(j\omega) = \frac{1}{2\pi} Q(j\omega) * A(j\omega)$

$Q(j\omega) =$ 

$\Rightarrow B(j\omega) =$ 

$c(t) = b(t) * h(t) \Rightarrow C(j\omega) = B(j\omega) H(j\omega)$

$h(t) = \begin{cases} 1, & |t| < 1 \\ 0, & \text{o.w.} \end{cases} \Rightarrow H(j\omega) = \frac{\sin 2\omega}{\omega}$

$\Rightarrow C(j\omega) = \begin{cases} \left[\frac{\omega}{\pi} + \frac{4}{\pi}\right]\frac{\sin\omega}{\omega}, & -4 \leq \omega < 0 \\ \left[-\frac{\omega}{\pi} + \frac{4}{\pi}\right]\frac{\sin\omega}{\omega}, & 0 \leq \omega < 4 \\ 0, & \text{o.w.} \end{cases}$

$c(t) + r(t) = D(t) \Rightarrow D(j\omega) = C(j\omega) + R(j\omega)$

$r(t) = \frac{\delta(2t)}{2} \Rightarrow R(j\omega) = \frac{1}{4}$

$\Rightarrow D(j\omega) = C(j\omega) + \frac{1}{4}$

7) $\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$

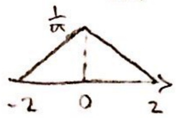
a) $|X(j\omega)|^2 = \frac{1}{(\omega^2+a^2)^2} \Rightarrow X(j\omega) = \frac{1}{\omega^2+a^2} \Rightarrow x(t) = \frac{e^{-a|t|}}{2a} \Rightarrow |x(t)|^2 = \frac{e^{-2a|t|}}{4a^2}$

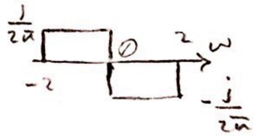
$$\Rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2+a^2)^2} = 2\pi \int_{-\infty}^{\infty} \frac{e^{-2a|t|}}{4a^2} \, dt = \pi \int_{-\infty}^{0} \frac{e^{2at}}{2a^2} \, dt + \pi \int_{0}^{\infty} \frac{e^{-2at}}{2a^2} \, dt = \pi \frac{e^{2at}}{4a^3} \Big|_{-\infty}^{0} - \pi \frac{e^{-2at}}{4a^3} \Big|_{0}^{\infty} = \boxed{\frac{\pi}{2a^3}}$$

b) $|x(t)|^2 = t^2 \left(\frac{\sin(t)}{\pi t}\right)^4 \Rightarrow x(t) = t\left(\frac{\sin(t)}{\pi t}\right)^2 = t \times \frac{\sin(t)}{\pi t} \times \frac{\sin(t)}{\pi t}$

$\frac{\sin^2(t)}{\pi^2 t^2} \xleftrightarrow{F.T.} \frac{1}{2\pi} \times$ 

$*$ 

$=$ 

$t \frac{\sin^2(t)}{\pi^2 t^2} \xleftrightarrow{F.T.}$ 

$= X(j\omega) \Rightarrow |X(j\omega)| = \begin{cases} \frac{1}{2\pi} & |\omega| < 2 \\ 0 & o.w. \end{cases}$

$\Rightarrow |X(j\omega)|^2 = \begin{cases} \frac{1}{4\pi^2} & |\omega| < 2 \\ 0 & o.w. \end{cases} \Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \boxed{\frac{1}{2\pi^3}}$