Date: #HW1_ Sub: ali nikkhah 99102445 ①

(25) Matrix Operations

$$v=\begin{bmatrix}3\\-2\\1\end{bmatrix}\quad u=\begin{bmatrix}2\\1\\1\end{bmatrix}\quad A=\begin{bmatrix}2&1&1\\0&3&0\\0&0&1\end{bmatrix}\quad B=\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$$

1. calculate first, second, infinity norms of vector $\vec{v}$ and explain

$$\|x\|_p \triangleq \left[\sum_{i=1}^{N}(x_i)^p\right]^{1/p}$$

$$\|v\|_1 = 3+2+1=\boxed{6}$$

$$\|v\|_2=\sqrt{3^2+2^2+1^2}=\boxed{\sqrt{14}}$$

$$\|v\|_\infty = \lim_{p\to\infty}\underbrace{\left(\sum_{i=1}^{N}(x_i)^p\right)^{1/p}}_{\{\max\{x_i\}^p\}^{1/p}} = \max\{x_i\}=\boxed{3}$$

$$\boxed{\|v\|_1 > \|v\|_2 > \|v\|_\infty}$$

2. inner product of vectors, cautchy shwartz inequality

$$\langle v,u\rangle = v^T u = \begin{bmatrix}3 & -2 & 1\end{bmatrix}\begin{bmatrix}2\\1\\1\end{bmatrix} = 5$$

$$|\langle v,u\rangle|^2 \leq \langle u,u\rangle\cdot\langle v,v\rangle$$

$$25 \overset{\checkmark}{\leq} \underset{\underline{84}}{14\times 6}$$

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3. determinant, trace, inverse of (A)

$$tr(A) = \sum_{i=1}^{N} a_{ii} = 6$$

$$|A| = (-1)^{6} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}{6} = \begin{bmatrix} 1/6 & -1/6 & 1/6 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

4. multiplication product (Bu) in two approaches:

Approach1:

$$A\underline{x} = \underset{3\times3}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} \sum_j x_j a_{1j} = 6 \\ \sum_j x_j a_{2j} = 3 \\ \sum_j x_j a_{3j} = 1 \end{bmatrix}}$$

Approach2:

$$A\underline{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = x_1 \underline{a_1} + x_2 \underline{a_2} + x_3 a_3 =$$

$$= 2\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

5. calculate matrix multiplication AB in 4 approaches:

Approach 3:

$$AB = \underset{3\times3}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}} = \text{make each element:} \begin{bmatrix} 7 & 14 & 21 \\ 6 & 12 & 18 \\ 3 & 6 & 9 \end{bmatrix}$$

$a_{11} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 7$ $a_{12} = 4 + 4 + 6 = 14$ $a_{13} = 21$

$a_{21} = 6$ $a_{22} = 12$ $a_{23} = 18$

$a_{31} = 3$ $a_{32} = 6$ $a_{33} = 9$

Approach 2:

$$AB = A\left[b_1 \ldots b_n\right] = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\underline{b_1} \quad \underline{b_2} \quad \underline{b_3}\right] = \left[A\underline{b_1}, A\underline{b_2}, A\underline{b_3}\right]$$

$$A\underline{b_1} = \underset{3\times3}{A} \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}} \qquad = \begin{bmatrix} 7 & 14 & 21 \\ 6 & 12 & 18 \\ 3 & 6 & 9 \end{bmatrix}$$

$$A\underline{b_2} = A \underset{3\times1}{\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}} = \begin{bmatrix} 14 \\ 12 \\ 6 \end{bmatrix}$$

$$A\underline{b_3} = A \underset{3\times1}{\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}} = \begin{bmatrix} 21 \\ 18 \\ 9 \end{bmatrix}$$

Approach 3:

$$AB = \begin{bmatrix} \underline{a_1}^T \\ \underline{a_2}^T \\ \underline{a_3}^T \end{bmatrix} B = \begin{bmatrix} a_1^T B \\ a_2^T B \\ a_3^T B \end{bmatrix} = \begin{bmatrix} 7 & 14 & 21 \\ 6 & 12 & 18 \\ 3 & 6 & 9 \end{bmatrix}$$

$$a_1^T B = \underset{1\times3}{\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}} = \begin{bmatrix} 7 & 14 & 21 \end{bmatrix}$$

$$a_2^T B = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix} \quad " \quad = \begin{bmatrix} 6 & 12 & 18 \end{bmatrix}$$

$$a_3^T B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad " \quad = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$$

Approach 4:

$$AB = \sum_K \underset{3\times3}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \underset{3\times3}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \sum_k \underline{a_K}\,\underline{b_K}^T$$

$$= \underline{a_1}\ \underline{b_1}^T + \underline{a_2}\ \underline{b_2}^T + \underline{a_3}\ \underline{b_3}^T = \begin{bmatrix} 7 & 14 & 21 \\ 6 & 12 & 18 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 12 & 18 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Calculate tr(BA)

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 4 \\ 4 & 14 & 8 \\ 6 & 21 & 12 \end{bmatrix}$$

$$tr(BA) = \sum_{i}^{3} c_{i,i} = \underline{28}$$

7. Calculate det of block matrix $C = \begin{bmatrix} (A+B)^2 & B^3 \\ 0_{3\times3} & A^2B \end{bmatrix}$

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$$\det(C) = \left[\det\left((A+B)^2\right)\right]\left[\det(A^2B)\right]$$

$$= |A+B|^2 \, |A|^2 \, |\underset{0}{\cancel{B}}| = 0$$

$$|A| = (-1)^6 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$$

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ \overset{0}{\cancel{2-2}} & \overset{0}{\cancel{4-4}} & \overset{0}{\cancel{6-6}} \\ 3 & 6 & 9 \end{vmatrix} = 0$$

(20) Extraordinary matrices :

$Q_{n\times n} \longrightarrow$ singular

$B_{n\times n} \longrightarrow$ symmetric

$C_{n\times n}$

1. $\underline{q_i}^H q_j = \delta_{ij}\ ,\ \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$Q^H = Q^{-1}$

$QQ^{-1} = Q^{-1}Q = I$

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \qquad Q^H = \begin{bmatrix} \overline{q_{11}} & \overline{q_{12}} & \overline{q_{13}} \\ \overline{q_{21}} & \overline{q_{22}} & \overline{q_{23}} \\ \overline{q_{31}} & \overline{q_{32}} & \overline{q_{33}} \end{bmatrix}^T = \begin{bmatrix} \overline{q_{11}} & \overline{q_{21}} & \overline{q_{31}} \\ \overline{q_{12}} & \overline{q_{22}} & \overline{q_{32}} \\ \overline{q_{13}} & \overline{q_{23}} & \overline{q_{33}} \end{bmatrix}$$

$Q^H Q = I \Longrightarrow [q_{11}\ q_{12}\ q_{13}]$

$Q Q^H = I$

$\longrightarrow QQ^H = I \longrightarrow \underline{q_i}\ \overline{\underline{q_j}}^T = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = \delta_{ij}$

$\longrightarrow \boxed{\overline{\underline{q}_j}^T\ \underline{q}_i = \delta_{ij}}$

2. $\|Qx\|_2 = \|x\|_2$

Prove that this matrix doesn't change length

$y \triangleq Qx$

$\|Qx\|_2 = y^H y = (Qx)^H (Qx) = x^H Q^{-1} Q x = x^H x = \|x\|_2$

3. $P = q_i q_i^H$

$P^2 = (q_i q_i^H)^2 = (\underline{q_i} \overset{I}{\boxed{\underline{q_i}^H \underline{q_i}}} \underline{q_i}^H) = \underline{q_i}\underline{q_i}^H$

$P^n_{n>2}$: $P^n = P^{n-2} P^2 = P^{n-2} P \longrightarrow P^n = P^{n-1}$ & $P^2 = P^1 \longrightarrow P^n = P^1$

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4. $|A|$ is real

Theorem1: for $A_{n\times n}$ Product of n eignvalues is determinant of A

Theorem2: eignvalues of Hermitian matrix are real

A Hermitian $\Longleftrightarrow A = A^H \Rightarrow A = \overline{A^T}$

$A = \overline{A^T} \rightarrow a_{i,i} = \overline{a_{i,i}}$ ⊛ $a_{i,i} = 6 + j\omega = \overline{6 + j\omega} = 6 - j\omega \Rightarrow \omega = 0$

$A \begin{cases} a_{i,i} \\ a_{i,j} \end{cases} \rightarrow A^T \begin{cases} a_{i,i} \\ a_{j,i} \end{cases} \rightarrow \overline{A^T} \begin{cases} \overline{a_{i,i}} \longrightarrow a_{i,i} \overset{\in}{\text{is}} \mathbb{R} \\ \overline{a_{j,i}} \longrightarrow \text{⊛ } a_{i,j} = \overline{a_{j,i}} \end{cases}$

Solution 1:

eign values of Hermitian matrix are allways real;

$\det(H) = \underbrace{a_{11}}_{\mathbb{R}} a_{22} \cdots \underbrace{a_{nn}}_{\mathbb{R}} = \underbrace{\prod_{i=1}^{N} a_{i,i}}_{\mathbb{R}}$

then product of some real valued eignvalues, will be real.

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5. $A^2 = I$

$$\begin{bmatrix} \sqrt{2} & +j \\ -j & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -j \\ j & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^2 = I \ \& \ A \neq \pm I$$

6. check if these matrices are symmetric or asymmetric

(a) $B^2 - C^2$

for B & C we have $b_{ij} = b_{ji}$ & $C_{ij} = C_{ji}$ $\quad B^T = B$ & $C^T = C$

$(B^2)^T = (B \times B)^T = (B^T)^2$

$(C^2)^T = (C \times C)^T = (C^T)^2$

$(B^2 - C^2)^T = (B^2)^T - (C^2)^T = (B^T)^2 - (C^T)^2 \xrightarrow[C^T=C]{B^T=B} = B^2 - C^2 \rightarrow$ symmetric

(b) $(B+C)(B-C)$

$((B+C)(B-C))^T = (B-C)^T(B+C)^T = (B^T - C^T)(B^T + C^T) \xrightarrow[C^T=C]{B^T=B}$

$= (B-C)(B+C) \longrightarrow$ symmetric

(c) CBC

$(CBC)^T = C^T(CB)^T = C^T B^T C^T \xrightarrow[C^T=C]{B^T=B} CBC \longrightarrow$ symmetric

(d) $CB \cdot CB$

$(CBCB)^T = (CB)^T(CB)^T = B^T C^T B^T C^T \xrightarrow[C^T=C]{B^T=B} BCBC \longrightarrow$ not always symmetric

(15) Block matrix

$$X = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

1. calculate (X^{-1}) matrix

A & B are not singular

$E_1 = A - CB^{-1}0 = A$ $\quad$ $E_2 = B - 0A^{-1}C = B$

$$\rightarrow X^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ -B^{-1}\cancel{0A^{-1}} & B^{-1} \end{bmatrix}$$

2. $$X = \begin{array}{c} \begin{array}{ccc} A & & C \end{array} \\ \begin{bmatrix} \boxed{2} & \boxed{3 \quad -1} \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 & -2 \\ -3 & 5 \end{matrix}} \end{bmatrix} \\ B \end{array}$$

$$X^{-1} = \begin{bmatrix} {A^{-1}}_{1\times1} & -A^{-1}CB^{-1}{}_{1\times2} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times1} & \begin{bmatrix} B^{-1} \end{bmatrix}_{2\times2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 5/11 & 2/11 \\ 3/11 & 1/11 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{12}{22} & -\frac{5}{22} \\ 0 & \frac{5}{11} & \frac{2}{11} \\ 0 & \frac{3}{11} & \frac{1}{11} \end{bmatrix}$$

$$-A^{-1}CB^{-1} = -\begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 5/11 & 2/11 \\ 3/11 & 1/11 \end{bmatrix}_{2\times2} = \begin{bmatrix} -\frac{12}{22} & -\frac{5}{22} \end{bmatrix}_{1\times2}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \end{bmatrix}_{1\times2} \times$$

(20) Matrix norm

1. $$A = \begin{bmatrix} 1 & 5 & 7 \\ -3 & 0 & 2 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| = \max(\overset{4}{\cancel{1+3}}, \overset{9}{\cancel{5+4}}, \overset{10}{\cancel{7+2+1}}) = 10$$

$$\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| = \max(\overset{13}{\cancel{1+5+7}}, \overset{5}{\cancel{3+2}}, \overset{5}{\cancel{4+1}}) = 13$$

$$\|A\|_2 = \max \frac{\sqrt{(x_1+5x_2+7x_3)^2 + (-3x_1+2x_3)^2 + (4x_2-x_3)^2}}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A\underline{x} = \underset{3\times 3}{\begin{bmatrix} 1 & 5 & 7 \\ -3 & 0 & 2 \\ 0 & 4 & -1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} x_1+5x_2+7x_3 \\ -3x_1+2x_3 \\ 4x_2-x_3 \end{bmatrix}$$

$$\|A\|_2 = \sqrt{\max_\lambda (A^T A)}$$

$$|A-\lambda I| = 0 \rightarrow (1-\lambda)(\lambda^2+\lambda-8)+5(-3\lambda)+7\times 12 = 0$$

$$\lambda = \{79.39, 8.36, 17.23\} \rightarrow \max_\lambda = \boxed{79.39} \rightarrow \sqrt{79.39} = 8.91$$

$$\|A\|_2 = \boxed{8.91}$$

2. Prove $\|A\|_F^2 = tr(AA^T)$

$$\begin{array}{cc} A & A^T \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ & \ddots \end{bmatrix} & \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ \vdots & \end{bmatrix} \end{array}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

$$\|A\|_F^2 = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \cdot a_{ij}}$$

$$(AA^T)_{i,i} = \sum_{k=1}^{N} a_{i,k}^2$$

$$= \sum_{j=1}^{N} a_{i,j} * a_{i,i}$$

$$= \sqrt{\sum_{i=1}^{m} (AA^T)_{i,i}}$$

$$= \sqrt{tr(AA^T)} \longrightarrow \|A\|_F^2 = tr(AA^T) \quad ✓$$

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(20) Matrix algebra $B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

1. find A,B matrices that satisfy:

① $AB \neq BA$ ✓

② $b_{ij} \neq 0 \quad B^2 = 0$ ✓

③ $A^2 = -I$ ✓

$$AB = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} j & -j \\ j & -j \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} = \begin{bmatrix} -j & j \\ -j & j \end{bmatrix}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \rightarrow A^2 = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \; A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \longrightarrow B^2 = \begin{bmatrix} b_{1,1}^2 + b_{1,2}b_{2,1} & b_{1,1}b_{1,2} + b_{1,2}b_{22} \\ b_{2,1}b_{22} + b_{2,2}b_{1,2} & b_{2,1}b_{2,2} + b_{2,2}b_{1,2} \end{bmatrix}$$

$$= \begin{bmatrix} b_{1,1}^2 + b_{1,2}b_{2,1} & b_{1,2}(b_{1,1} + b_{2,2}) \\ b_{2,2}\boxed{(b_{1,2} + b_{2,1})} & b_{2,2}\boxed{(b_{2,1} + b_{1,2})} \end{bmatrix}$$

2. Prove that for no A,B matrices $AB - BA = I$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$AB - BA = I_2$

$tr(I_2) = 2$

$tr(AB - BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0$ ✗

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