Linear Algebra

EE25872

Spring Semester 1402-03
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c

1 (15pt)

Vector $\mathbf{v}$ is defined in $\mathbb{R}^3$ space as $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

1.1 elaborate this vector by basis of **B**

Since b is the basis vectors of the 3D space, (or the transformation matrix is the identity) which results:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{v}$$

$\mathbf{v} = a_1 b_1 + a_2 b_2 + a_3 b_3 \rightsquigarrow \mathbf{A} = \mathbf{v}^T = [3 \quad 2 \quad -1]$

$a_1 = 3, a_2 = 2, a_3 = -1$

1.2 elaborate this vector by basis of **C**

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{v}$$

Which Is a linear system of equations

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Which we solve using gaussian elimination method:

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Whee we conclude

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

1.3 find matrix P in $\mathbf{v} = P\mathbf{w}$

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = P \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

As we wrote down, the matrix $\mathbf{P} = [c_1 \quad c_2 \quad c_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the transformation matrix

1.4 then we try to find inverse:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$\det(A) = 1 \cdot (0 \cdot 1 - 1 \cdot 1) - 1 \cdot (1 \cdot 1 - 0 \cdot 1) + 0 \cdot (1 \cdot 1 - 0 \cdot 1) = -2$$

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{v} = P^{-1}\mathbf{w} \rightsquigarrow \mathbf{P}^{-1}\mathbf{w} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Which is pretty unreasonable and I think the question meant that :

$$P^{-1}\mathbf{v} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \mathbf{w}$$

2 (10pt)

2.1 subset of $\mathbb{R}^2$ close to sum/subtraction but not close to scaler multiplication:

$\mathbf{V} = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}$

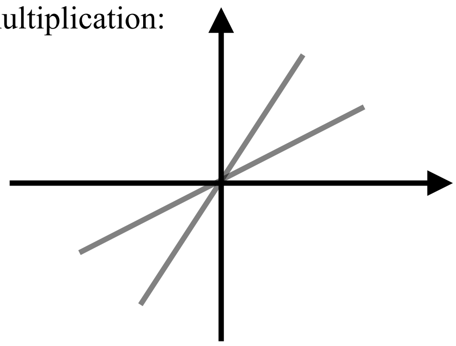
$(x_1, y_1) \in \mathbf{V}$

$(x_2, y_2) \in \mathbf{V}$

$(x_1 + x_2), (y_1 + y_2) \in \mathbb{Z}$

But for example (1,-1) and scalar value 0.5 we have (0.5,-0.5) which is not inside the set

2.2 subset of $\mathbb{R}^2$ not-close to sum/subtraction but close to scaler multiplication:



Imagine the span of two intersecting lines

$\mathbf{V} = \{(x, y) \in \mathbb{R} : y = 0.5x \cup y = 2x\}$

$(x_1, 0.5x_1) \in \mathbf{V} \rightsquigarrow (kx_1, k/2x_1) \in \mathbf{V}$

$(x_1, 2x_1) \in \mathbf{V} \rightsquigarrow (kx_1, 2kx_1) \in \mathbf{V}$

$(2x_1, 2.5x_1) \notin \mathbf{V}$

3 (15 pt)

$C = \{(x_1, x_2) | x_1, x_2 \in \mathbb{C}\}$

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$

$\alpha(x_1, x_2) = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1)$

3.1 what is the zero vector?

$-1(x_1, x_2) = (-x_1 - 2, -x_2 - 2)$

$-(x_1, x_2) + (x_1, x_2) = (-x_1 - 2 + x_1 + 1, x_2 - 2 + x_2 + 1) = (-1, -1)$

3.2 prove the identity $\alpha(u + v) = \alpha u + \alpha v$

$\alpha(u + v) = \alpha(u_1 + v_1 + 1, u_2 + v_2 + 1) = (\alpha u_1 + \alpha v_1 + \alpha + \alpha - 1, \alpha u_2 + \alpha v_2 + \alpha + \alpha - 1)$

$= (\alpha u_1 + \alpha v_1 + 2\alpha - 1, \alpha u_2 + \alpha v_2 + 2\alpha - 1) = (\alpha u_1 + \alpha v_1 + 2\alpha - 2 + 1, \alpha u_2 + \alpha v_2 + 2\alpha - 2 + 1) =$

$= (x_1, x_2) + (y_1, y_2)$ where $x_1 = \alpha u_1 + \alpha - 1, x_2 = \alpha u_2 + \alpha - 1$ and $y_1 = \alpha v_1 + v_1 - 1, y_2 = \alpha v_2 + v_2 - 1$

$(x_1, x_2) = \alpha(u_1, u_2) = \alpha u$

$(y_1, y_2) = \alpha(v_1, v_2) = \alpha v$

$\rightsquigarrow (\alpha u_1 + \alpha v_1 + 2\alpha - 2 + 1, \alpha u_2 + \alpha v_2 + 2\alpha - 2 + 1) = (x_1, x_2) + (y_1, y_2) = \alpha u + \alpha v = \alpha(u + v)$

4 (15 pt)

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

4.1 Four main sub spaces:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 0 & 0 & 13/3 & 26/3 & -26/3 \end{bmatrix} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}(A) = Span(\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix})$$

$$\mathbf{C}(A^T) = Span(\begin{bmatrix} -3 \\ 6 \\ -1 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \\ -1 \end{bmatrix})$$

$$\mathbf{N}(A) = Span(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix})$$

$$\mathbf{N}(A^T) = Span(\begin{bmatrix} -1/5 \\ -13/5 \\ 1 \end{bmatrix}$$

4.2

$rank(A) = 2$

$dim(C(A)) = r = 2$

$dim(C(A^T)) = r = 2$

$dim(N(A)) = n - r = 3$

$dim(N(A^T)) = m - r = 1$

4.3

The basis vectors of Part one are the ones written down earlier:

$$\mathbf{C}(A) \rightsquigarrow \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \dots$$

4.4

$$[-1 \quad 2 \quad 5]\begin{bmatrix} -1/5 \\ -13/5 \\ 1 \end{bmatrix} = 0,\ldots$$ and the result is zero for every other inner product of this type

$\rightarrow C(A) \perp N(A^T)$

$$[-3 \quad 6 \quad -1 \quad 1 \quad -7]\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = 0,\ldots$$ and the result is zero for every other inner product of this type

$\rightarrow C(A^T) \perp N(A)$

5 (15 pt)

By using the definition of Null Space we have:

$A\underline{x} = 0$

Which yields these immediate results:

$\underline{x}$ is perpendicular to matrix A rows

$\underline{x}$ is perpendicular to matrix A combination of rows

$\underline{x}$ is perpendicular to matrix A row space

$C(A^T) \perp N(A)$

$dim(C(A^T)) + dim(N(A)) = n$

$dim(N(A)) = n - r$

$dim(N(A)) + rank(A) = n$

6 (20 pt)

6.1 Null space basis:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow x_n = x_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

6.2 the number of free variables will increase this way:

$$\hat{A} = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 1 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \hat{A} = \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2 can take infinite different values thus we have infinite distinct basis values. In other words, we need another vector to span the space (a unit vector in the direction of added column).

6.3 Null space will have no changes, the only change will be on left null space and column space

$$\overline{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6.4 $C(A) = span(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix})$

$dim(C(A)) = 2$

$C(A) \in \mathbb{R}^3$

6.5

$b = \begin{bmatrix} \alpha \\ 6 \\ 1 \end{bmatrix}$ and $Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \alpha \\ 1 & 2 & 4 & 6 & 6 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & \alpha \\ 0 & 0 & 1 & 1 & 6-\alpha \\ 0 & 0 & 0 & 0 & \alpha - 5 \end{bmatrix}$$

$\rightsquigarrow \alpha = 5$

6.6

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$x_p = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

6.7

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N(A^T) = span(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}) \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} = 0$$

$\underline{b} \perp N(A^T)$

7 (10 pt)

For r<m one column will be zero and for some values of b we won’t have answers

For m=r and n>r we will have infinity answers according to Null Space special answers

For m>r and n=r we will have zero / or / one answer

For n=m=r we will have exactly one answer for b