Closure is a property which describes when we combine any two elements of the set; the result is also included in the set.

If we multiply two integer numbers, we will get another integer number. Since this process is always true, it is said that the integer numbers are 'closed under the operation of multiplication'. There is simply no way to escape the set of integer numbers when multiplying.

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\}$ is a set of integer numbers.

$12 = 2$

$23 = 6$

$52 = 10$

All are included in the set of integer numbers.

We can conclude that integer numbers are closed under the operation of multiplication.

Theorem 5.3: Two REs L_1 and L_2 over Σ are closed under union operation.
Proof: We have to prove that if L_1 and L_2 are regular over Σ,then their union, i.e.,$L_1 \cup L_2$ will be also regular.
As L_1 and L_2 are regular over Σ, there must exist FA $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ such that $L_1 \varepsilon M_1$ and $L_2 \varepsilon M_2$
Assume that there is no common state between Q_1 and Q_2, i.e., $Q_1 \cap Q_2 = \emptyset$.
Defi ne another FA, $M_3 = (Q, \Sigma, \delta, q_0, F)$ where
1. $Q = Q_1 \cup Q_2 \cup \{q_0\}$, where q_0 is a new state $\notin Q_1 \cup Q_2$
2. $F = F_1 \cup F_2$
3. Transitional function δ is defi ned as $\delta(q_0, \varepsilon) \rightarrow \{q_{01}, q_{02}\}$
and $$\delta(q, \Sigma) \rightarrow \delta_1(q, \Sigma) \text{ if } q \varepsilon Q_1$$

$$\delta(q, \Sigma) \rightarrow \delta_2(q, \Sigma) \text{ if } q \varepsilon Q_2$$

It is clear from the previous discussion that from q_0 we can reach either the initial state q_1 of M_1 or the initial state q_2 of M_2.

Transitions for the new FA, M, are similar to the transitions of M_1 and M_2.

As $F = F_1 \cup F_2$, any string accepted byM_1orM_2 will also be accepted by M.

Therefore, $L_1 \cup L_2$ is also regular.

Example:

Let $$L_1 = a^*(a + b)b^*$$

$$L_2 = ab(a + b)b^*$$

The FA $M1$ accepting $L1$ is as shown in Fig. 5.38(a).

The FA $M2$ accepting $L2$ is as shown in Fig. 5.38(b).

The machine M produced by combining$M1$and$M2$is as shown in Fig. 5.38(c).

It accepts $L_1 \cup L_2$.

Theorem 5.4: The complement of an RE is also regular.

If L is regular, we have to prove that L^T is also regular.

As L is regular, there must be an FA, $M = (Q, \Sigma, \delta, q_0, F)$, accepting L.

M is an FA, and so M has a transitional system.

Let us construct another transitional system $M\prime$ with the state diagram of M but reversing the direction of the directed edges. $M\prime$ can be designed as follows:

1. The set of states of $M\prime$ is the same as M
2. The set of input symbols of $M\prime$ is the same as M