TECHNISCHE UNIVERSITEIT DELFT
Process and energy department

Assignments week 4 CHAOS, 2023
Hamiltonian Chaos

Write a small report about the assignments listed below. Add (your extensions to) the computer scripts as an appendix.

In this assignment we study two Hamiltonian systems: the periodically kicked rotator, and fluid flow due to vortices which are switched periodically. In both systems, chaos leads to intense mixing in phase space. For the kicked rotator, we can even have an analogy to diffusion, while the vortex flow can mix added tracer particles. In both cases transport is possible due to nonlinearity and time dependence.

1. **Blinking vortex**

   In this exercise we are going to look at fluid mixing through chaotic advection. We will restrict ourselves to the situation of two-dimensional incompressible flow, where the velocity field satisfies $\nabla \cdot \boldsymbol{v} = 0$. This is the same as saying that there is a stream function Ψ, with

$$\begin{aligned} v_x &= -\frac{\partial \Psi}{\partial y} \\ v_y &= \frac{\partial \Psi}{\partial x}. \end{aligned} \tag{1}$$

   These two equations are Hamilton's canonical equations for one degree of freedom. We can identify the stream function Ψ with the Hamiltonian H, x with the momentum p and y with the 'position' q.

   An example of such a system is the 'blinking vortex' flow, described by Aref in [1]. The flow consists of two vortices of equal strength, Ω, one located at $(x, y) = (b, 0)$ and the other located at $(x, y) = (-b, 0)$. Now we will switch the vortices on and off with time period T. This is the 'blinking' of the vortices. In this time-periodic flow field, passive tracer particles follow very complicated trajectories. For these particles the equations of motion are:

$$\begin{aligned} \dot{x} &= -\frac{\Omega\, y}{x_s^2 + y^2} \\ \dot{y} &= \frac{\Omega\, x_s}{x_s^2 + y^2}, \end{aligned} \tag{2}$$

   with the position $x_s = x + b$ for $0 < t < T/2$ and $x_s = x - b$ when $T/2 < t < T$.

   We represent the dynamical state of this system by a stroboscopic map where we register the position of a particle at $t = kT$, with k an integer number. Such a registration still completely determines the dynamical state of the system. The attractor of this map depends on the dimensionless parameter $\mu = \Omega T/b^2$ (check it !). We expect that for small μ the map is very close to the integrable case, where we only have fixed points and limit cycles. For larger μ chaotic behavior is expected, and the area occupied by the attractor will grow. In this case there is mixing by chaotic advection.

There is a program showing the tractories of fluid parcels `blink_w.m`, and a template program of the stroboscopic map `blink_map.m`. You can read more about the blinking vortex model in [1] and [2] [1]

(1.a) To get started: what is the stream function Ψ leading to Eq.(2) ? Then: explore with `blink_w.m` the trajectories of a tracer for $\Omega = 10, T = 10$ and $b = 5$ (the program asks for T, Ω). Collect some pretty pictures.

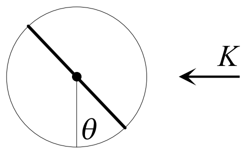
(1.b) We just explained how a map can be created by sampling the position of the particles at $t = kT$. That is done in the template program `blink_map.m`. Create a map for $\Omega = 10$, $b = 5$ and $T = 0.5$, starting from a *range of initial conditions* (which still has to be programmed).

(1.c) Now we want to vary the strength parameter μ. You can do this by varying T and keep $\Omega = 10$ and $b = 5$ fixed. Show maps for different values of μ, starting from $T = 0.125$. Collect some pretty pictures.

(1.d) In the chaotic region the distance between a pair of close points grows exponentially fast, on average. Pick many pairs of close points, distance δ_0, and see how this distance grows as $\delta(t)$, with $\delta(t = 0) = \delta_0$. Then take many averages $\langle \ldots \rangle$ and plot the average: $\langle \ln(\delta(t)/\delta_0) \rangle$. For not too large t this will be a straight line, meaning that $\delta(t) \sim \mathrm{e}^{\Lambda t}$, with Λ the Lyapunov exponent. The value of Λ will depend on T, that is, how chaotic the dynamics is. Check ! Get inspiration from Chapter VI of the lecture notes.

(1.e) Show the fate of a small square blob of initial conditions: how does that blob evolve with time ? Get inspiration from the article [1].

2. The periodically kicked rotor.



(From the lecture notes) Consider a a frictionless rotor, which at times $t = n\tau$ receives a kick with strength K in the $x-$direction. We can derive a mapping for this system. The Hamiltonian does not explicitly depend on time; it is

$$H(p_\theta, \theta, t) = \frac{p_\theta^2}{2I} + K\cos\theta \sum_n \delta(t - n\tau),$$

[1]The program integrates the ordinary differential equations Eq. 3, but for this system the trajectories are pieces of circles, so that we could have done the integration analytically. It then amounts to inverting sines and cosines, which is not very insightful.

with I the moment of inertia of the rotor. The equations of motion then become

$$\begin{aligned}\frac{\mathrm{d}p_\theta}{\mathrm{d}t} &= -\frac{\partial H}{\partial \theta} = K\sin\theta\sum_n \delta(t-n\tau)\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}.\end{aligned} \tag{3}$$

In between two kicks p_θ is constant, while the angle θ increases with a constant velocity. We can integrate the equations of motion from kick to kick

$$\begin{aligned}p_{n+1}-p_n &= \int_{n\tau+0}^{(n+1)\tau+0} K\sin\theta\sum_n \delta(t-n\tau)\,\mathrm{d}\tau = K\sin\theta_{n+1}\\ \theta_{n+1}-\theta_n &= p_n\,\frac{\tau}{I}\end{aligned}$$

When we set $\tau/I = 1$, we arrive at the *standard map*,

$$\begin{aligned}p_{n+1} &= p_n + K\sin\theta_{n+1}\\ \theta_{n+1} &= (\theta_n + p_n) \bmod 2\pi.\end{aligned} \tag{4}$$

A computer program `standard.m` is provided that lets you play. It provides a beautiful illustration of chaos in Hamiltonian systems.

(2.a) Make a picture for a small value of K. Then prove (mathematically) that for $K > 0$ the origin $(p_n, \theta_n) = (0, 0)$ is a hyperbolic point, and the edge points $(p_n, \theta_n) = (0, \pm\pi)$ are elliptic points. This is exactly what you see.

(2.b) Explore further ! Larger K reveals Poincaré Birkhoff chains and a KAM surface. Make pretty pictures of both.

(2.c) Verify that your KAM surface has a winding number close to the golden mean $W_g = (5^{1/2} - 1)/2$. For this you still need to program the winding number. While the KAM surface winding number is *irrational*, other elliptic points have a rational winding number. Find a few.

Chaotic transport

The famous Tokamak fusion plasma reactor, now being built in the south of France, is our hope to solve the energy crisis. I will generate unlimited energy from nuclear fusion using water as a fuel. The tokamak (a torus) is an instance of a Hamiltonian system. The hot plasma is trapped in stable islands. A great concern, however, is the breakup of islands in a chaos transition. The leakage of energy can be understood using the standard map, see the article by Rechester *et al.* [4].

When K is large there are no visible KAM surfaces present in the standard map, and the entire region of p modulo 2π versus θ is covered by a single chaotic orbit. Also we see that the the change in momentum, according to equation 4, is large. As a result we expect θ to vary wildly and we can treat θ_n as random and uniformly distributed [3].

In other words, p_n increases with random jumps ξ_n (with average 0), $p_{n+1} = p_n + \xi_n$. After m steps,

$$p_{n+m} = p_n + \sum_{i=1}^{m} \xi_{n+i},$$

with root mean square (rms) distance

$$\Delta p_m^2 = \left\langle (p_{n+m} - p_n)^2 \right\rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} \langle \xi_{i+n} \xi_{j+n} \rangle = m \left\langle \xi^2 \right\rangle = m\, D, \tag{5}$$

because ξ_i and ξ_j are uncorrelated. Therefore the squared momentum (half the energy) increases linearly with time: diffusion [3]. When using $\langle \sin^2 \theta_{n+1} \rangle = \frac{1}{2}$ we find for the diffusion coefficient

$$D = \frac{K^2}{2},$$

see Edward Ott's nice book on Chaos [3]. When we take initial conditions uniformly spread in θ and p, the momentum distribution function will follow a Gaussian distribution (which must be checked !).

(2.d) Plot p modulo 2π versus θ for orbits with $K = 1$ and the following five initial conditions:
$(\theta_0, p_0) = (\pi, \pi/5); (\pi, 4\pi/5); (\pi, 6\pi/5); (\pi, 8\pi/5); (\pi, 2\pi)$.

(2.e) For $K = 21$ plot the average value of p^2 versus iterate number. Average over 100 different initial conditions:
$(\theta_0, p_0) = (2n\pi/11, 2m\pi/11)$ for $n = 1, 2, ..., 10$ and $m = 1, 2, ..., 10$ and estimate the diffusion coefficient D from equation 5.
How well does your numerical result agree with the quasi-linear value $D = K^2/4$? (Possibly, you should not take p modulo 2π when looking at its spread)

References

[1] Hassan Aref. Stirring by chaotic advection. *Journal of Fluid Mechanics*, 143(1):1–21, 1984.

[2] G Károlyi and T Tél. Chaotic tracer scattering and fractal basin boundaries in a blinking vortex-sink system. *Physics Reports*, 290:125–147, 1997.

[3] Edward Ott. Chaos in Hamiltonian systems. In *Chaos in dynamical systems*, pages 208–264. 1993.

[4] A. B. Rechester, M. N. Rosenbluth, and R. B. White. Electron heat transport in a Tokamak with destroyed magnetic surfaces. *Phys. Rev. Lett.*, 40:38, 1978.