TECHNISCHE UNIVERSITEIT DELFT
Process and energy department

Assignment 4: Turbulence
Chaos in Dynamical Systems, ME45190 (2023-2024)

Write a small report about the assignments listed below. Please add the computer scripts as an appendix. There is a lot of reading to do for this final assignment (7 pages). The context is explained in the last chapter of the lecture notes, so you must read that too.

Hints for using GOY.m

This is a real turbulence simulation, so you will have to run it a long time, say 2×10^7 time steps. The time step dt is updated regularly to the smallest of $dt_n \approx \left|u\frac{\partial u}{\partial x}\right|_n / \left|\frac{\partial u}{\partial t}\right|_n \approx 1/k_n|u_n|, n = 1, \ldots, N$. This is put in the file tau.dat, and it is fun to look at it. The time step has a minimum at the "integral scale", then increases again at large n, that is dissipative length scales where the velocity shells are damped stronger and stronger.

Because the program runs long, you dump all results in files, and look at them afterwards, or as they are updated (say every 10^3 time steps). Program first all the computations necessary for the assignments. Then relax and sit back and see the statistics improve as time progresses.

First look at the energy and play with switching off forcing (after a while), or both forcing and viscosity (although the integration gets unstable in the latter case).

Then draw a spectrum and order-p structure functions. *The ultimate goal of this assignment is to show them, and say something intelligent about them, such as scaling exponents, and the range of scaling.*

I. The GOY shell model

In this exercise we will investigate the energy cascade mechanism in turbulence using a shell model. This shell model is the GOY shell model (after Gledzer, Ohkitani and Yamada) and it is already programmed in the matlab script **goy.m** [2]. The Lecture Notes tell us that turbulence is characterized by an energy flow to smaller and smaller scales. We will investigate this *energy-cascade* by analyzing scaling of the energy spectrum and structure functions. In this exercise we will see that the shell model shows many similarities with Navier-Stokes dynamics. It is a true turbulence model, that amazingly shows the same statistics as real-life turbulence.

Let us first tell you how this model follows from the Navier Stokes equation in one dimension, without pressure (so we will not write vectors, and the velocity is simply $u(x,t)$)

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u = \nu\frac{\partial^2}{\partial x^2}u + f, \tag{1}$$

where ν is the kinematic viscosity and f is the forcing of the fluid. Without forcing, but with damping (viscosity), all motion stops.

A very common approach to solving the Navier Stokes equation (at least numerically) is expanding the velocity field in spatial Fourier modes

$$u(x,t) = \sum_n u_n(t)\,\mathrm{e}^{-ik_n x}. \tag{2}$$

Each Fourier amplitude $u_n(t)$ lives in a "shell", that is why we will be talking about the "shell model". Now

$$\begin{aligned} \frac{\partial}{\partial x} &\rightarrow -i\sum_n k_n\,u_n(t)\,\mathrm{e}^{-ik_n x} \\ \frac{\partial^2}{\partial x^2} &\rightarrow -\sum_n k_n^2\,u_n(t)\,\mathrm{e}^{-ik_n x} \end{aligned}$$

so that the Navier Stokes equation Eq. 1 for the shell with wavenumber k_n becomes

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}\mathrm{e}^{-ik_n x} + \underbrace{\sum_k u_k\mathrm{e}^{-ik_k x}\sum_l -ik_l\,u_l\,\mathrm{e}^{-ik_l x}}_{S_n} = \underbrace{-\nu\,k_n^2\,u_n}_{D_n}\,\mathrm{e}^{-ik_n x} + \underbrace{f_n}_{F_n}\,\mathrm{e}^{-ik_n x}. \tag{3}$$

The nonlinear advection term involves two different wavenumbers k_k and k_l. Since *all* terms must have $\mathrm{e}^{-ik_n x}$, they must add up to k_n, so

$$k_k + k_l = k_n \tag{4}$$

It is the nonlinear advection term, the second term on the left of Eq. 1, that couples all shells together. That is exactly which makes the Navier Stokes equation so special. This term takes care of the energy transport to smaller and smaller length scales (larger and larger wave numbers).

In the GOY shell model the velocities of the Navier-Stokes dynamics are placed on a one-dimensional array of wave vectors. The set of shells are defined such that the nth shell has wave vector

$$k_n = k_0\lambda^n, \tag{5}$$

where $n = 0, 1, 2, ..., N$. We choose $\lambda = 2$, then you could say that all wave numbers are "harmonics" of the lowest one k_0. Each shell is described by a complex variable, the complex velocity $u_n(t)$ (remember, this is a Fourier transform). Thus, we have reduced the Navier-Stokes equation to a set of coupled ordinary differential equations. This is great, because we can now view turbulence as chaos, and use all our acquired skills in doing and analyzing nonlinear ODE's to understand turbulence.

We can now write the Navier Stokes equation Eq. 1 as

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n = D_n + F_n + S_n, \tag{6}$$

where $D_n = -\nu k_n^2 u_n$ represents the dissipation term. The forcing term is F_n; we will force the largest scale only, $F_n = f\ \delta_{n,0}$. That makes a lot of sense since turbulence is normally stirrred at large scales.

The nonlinear coupling between shells is S_n. Equation Eq. 4 says that every shell k_n can interact with all other shell (under the restriction that $k_k + k_l = k_n$. This is really too much ! Therefore, we make a *model* in wich we restrict interaction to be between neighboring shells only, so

$$S_n = i\left(a\, k_n\, u^*_{n+1}\, u^*_{n+2} + b\, k_{n-1} u^*_{n-1}\, u^*_{n+1} + c\, k_{n-2} u^*_{n-1}\, u^*_{n-2}\right), \tag{7}$$

where the superscript $*$ denotes complex conjugation. In the program, **goy.m**, the terms D_n are computed at lines 31 and 32, and used in the special numerical scheme. The terms S_n are computed in the function **goy_rhs()**. The numerical scheme [3] is explained at the end of this assignment. The viscosity ν and forcing f are set at the beginning of the program, together with k_0 and λ, as used in equation 5, and N, the number of shells.

We do not have complete liberty in modelling the nonlinear advection term. We make a special choice for the constants a and b in Eq. 7,

$$\begin{aligned} a &= 1, \\ b &= -\delta, \\ c &= -1 + \delta. \end{aligned} \tag{8}$$

With this choice Eq. 6 now conserves two important quantities, the same quantities that are also conserved in reality by the original Navier Stokes equation. First, if viscosity is set to 0 ($\nu = 0$) and in the absence offorcing ($f = 0$), the resulting equation should conserve energy. In the Navier Stokes equation energy is

$$E = \tfrac{1}{2} \int |\boldsymbol{u}(\boldsymbol{x})|^2\, \mathrm{d}\boldsymbol{x}. \tag{9}$$

In addition, we can also show that the Navier Stokes equation conserves helicity,

$$H = \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}. \tag{10}$$

For the shell model, we can define the spectral analogues of these quantities [2],

$$E_N = \frac{1}{2}\sum_{n=1}^{N} |u_n|^2 , \quad \text{and} \quad H_N = \sum_{n}^{N} (\delta - 1)^{-n+1} |u_n|^2 . \tag{11}$$

Actually, the expression for E_N follows from Parseval's equality, which you may remember from spectral analysis.

> **Running your GOY simulation I**
>
> This is the exploration part, see your program produce sensible numbers. Key is 2: the dissipation rate ϵ. From it compute the Kolmogorov scale, and compare it to the smallest length scale in this simulation $2\pi/k_N$. Save the proof of (3) for later, it involves a trick that can be told easily.

1. Run the simulation and make a picture of a few snapshots of the velocity field.

$$U_N(x,t) = \Re \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}k_n x} u_n(t),$$

with $\Re$ the real part. Select the times t of the snapshots far enough apart. (A template is provided in **goy.m**) You will see that the velocity fluctuates (wildly) as a function of the spatial coordinate x. The question is about the correlation length of these fluctuations. Compute the correlation length from the correlation function $C(r)$:

$$C(r) = \langle U_N(x+r,t) U_N(x,t) \rangle_{x,t} , \tag{12}$$

with the average $\langle \cdots \rangle$ over x and t. The equation Eq. 12 is the hard way; but there is a much easier way using the famous Wiener-Khintchine theorem. What is it ?

2. Compute the instantaneous energy E_N (Eq. 11), and the instantaneous energy dissipation rate

$$\epsilon_N(t) = \nu \int (\mathrm{d}u/\mathrm{d}x)^2 \mathrm{d}x = \nu \sum_{n=1}^{N} k_n^2 u_n^*(t) u_n(t).$$

On average, the dissipation rate should be equal to the energy input in the forced shell 1, $\langle \epsilon_N \rangle = |\langle f u_1^* \rangle|$, where $\langle \ldots \rangle$ are averages done over time. (Templates are provided in **goy.m**.)

Make pictures of $E_N(t)$ and $\epsilon_N(t)$, and verify that the average ϵ_N indeed equals the average energy input.

3. Show analytically that for inviscid ($\nu = 0$) and unforced flow, the GOY model satisfies energy conservation, $\frac{\mathrm{d}E_N}{\mathrm{d}t} = 0$. *Hint:* Write Eq. 6 for u_n, multiply with u_n^* (to get $|u_n|^2$), then again write Eq. 6 for u_n^*, and multiply with u_n, and add them so that $u_n^* \frac{\mathrm{d}u_n}{\mathrm{d}t} + u_n \frac{\mathrm{d}u_n^*}{\mathrm{d}t} = \frac{\mathrm{d}|u_n|^2}{\mathrm{d}t}$, and grind on. Also use that u_n is nonzero only for $n = 1, \ldots, N$.

II. Structure functions and the energy spectrum

Turbulence must be characterized by statistical quantities. Important statistical properties of the energy cascade are given by the structure functions $S_p(R)$, which are moments of the velocity differences in the direction $\boldsymbol{R}$ [1]. For the Navier-Stokes equations the longitudinal structure function is defined as:

$$S_p(R) = \left\langle \left[\left(\boldsymbol{u}\left(\boldsymbol{x} + \boldsymbol{R}\right) - \boldsymbol{u}(\boldsymbol{x}) \right) \cdot \widehat{\boldsymbol{R}} \right]^p \right\rangle , \tag{13}$$

with $\widehat{\boldsymbol{R}}$ the unit vector in the direction of $\boldsymbol{R}$. Do not worry about the vector character of $\boldsymbol{u}$, $\boldsymbol{r}$ and $\boldsymbol{R}$, but this is just to show that the structure function is actually a *tensor*. The shell model lives in a one-dimensional world, and the shell amplitudes u_n may be interpreted as velocity differences over a length $2\pi/k$. Therefore, the analogous definition of the structure function of the shell amplitudes would be the simple:

$$S_p(k_n) = \langle |u_n|^p \rangle. \tag{14}$$

Using dimension arguments Kolmogorov suggested that the structure functions of all orders p scale as follows in the inertial range of turbulence:

$$S_p(R) = \langle (u(x+R) - u(x))^p \rangle \sim \epsilon^{p/3} \, R^{p/3} \tag{15}$$

In analogy, this would translate in the shell model as

$$S_p(k_n) \sim k_n^{-p/3}. \tag{16}$$

For large values of p this Kolmogorov scaling is however not satisfied. A model that captures the scaling for larger values of p is the She-Lévêque model [4], where

$$\zeta_p = \frac{p}{9} + 2 \left[1 - \left(\frac{2}{3} \right)^{p/3} \right]. \tag{17}$$

Of course, the statistical analysis of turbulence starts with the energy spectrum,

$$E(k) = \left\langle \left| \int \mathrm{e}^{-i\,kx} u(x,t) \, \mathrm{d}x \right|^2 \right\rangle_t , \tag{18}$$

where $\langle \cdots \rangle_t$ is an average over time and realizations. From spectral analysis you may remember that this is the same as the Fourier transform of the correlation function $\langle u(x+R,t) u(x,t) \rangle_t$ (or the structure function S_2). The nice thing about the shell model is that the (spatial) Fourier transform to get the spectrum is already done. Then, the spectrum is determined by just the shell amplitudes:

$$E(k_n) = \frac{1}{k_n} \langle |u_n|^2 \rangle = \frac{1}{k_n} S_2(k_n), \tag{19}$$

where the factor $1/k_n$ takes care of the fact that the *integral* over wavenumbers must be equal the total energy. It is just a dimension argument.

Running your GOY simulation II

This part of the assignment is central: it connects to chapter X of the lecture notes, and to the notion of fractals and scaling in chapter V. It is all embodied by the dependence on the order p of the scaling exponent ζ_p. You should see that it is *nonlinear*.

4. Compute $S_p(k_n)$ for different values of p. You can vary p as $p = 0, 1, ..., 9$. A suggestion is to organize your program such that all $S_p(k_n)$, with $p = 0, ..., 9$ are updated at once, otherwise, this would take many CPU minutes.

5. Compute all scaling exponents $\zeta_p, p = 1, \ldots, 9$ by plotting $S_p(k_n) \propto k_n^{-\zeta_p}$. in log log plots and fitting straight lines (a ruler and pencil suffice). Then plot ζ_p as a function of p.

6. Compare your ζ_p to Kolmogorov's prediction. The She-Lévêque model is interesting. If you want, compare it to your ζ_p, but then you must say something about the background of the model. This means that you have to dig out the original paper, and explain it in a few lines of text.

7. Structure functions and energy spectrum are in real (wavenumber) space. Turbulence also has a *frequency* spectrum. This follows from the Fourier transform of $u(x = 0, t)$, so we have:

$$E_N(f) = \left\langle \left| \int U^N(x = 0, t) \mathrm{e}^{-2\pi i\, f\, t} \, \mathrm{d}t \right|^2 \right\rangle$$

Compute this frequency spectrum in your simulation and show how it scales with frequency f. More specifically, what is the exponent ζ_f in $E_N(f) \sim f^{-\zeta_f}$. Provide a prediction for ζ_f and compare it to the result of your simulation.

In Eqs. 9 and 10 two conserved quantities for the shell model were shown. The analogy between the first conserved quantity and energy conservation in the Navier-Stokes dynamics, was already discussed in assignment 1. For the second conserved quantity, H_N, we need an analogy with helicity. In the GOY shell model the closest analog to helicity is the quantity $\sum_n (-1)^n k_n |U_n|^2$ [2]. To match this quantity to the conserved quantity from equation 11, the following condition should be satisfied

$$\lambda = \frac{1}{1 - \delta} \tag{20}$$

8. (*Bonus.*) For the simulations performed so far, the parameters $\lambda = 2.0$ and $\delta = 0.5$, do indeed satisfy Eq. 20. Now vary the free parameter δ (and also λ if you like) and show what happens if equation 20 is not statisfied anymore. You can do this, for example, by showing plots of $S_p(k_n)$ for different combinations of parameters λ and δ.

III. Numerics

The numerical integration of Eq. 6 uses the fast damping of high wavenumbers [3]. The following scheme can be recognized in the template program. We first write equation 6 as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_n^2\right) u_n = g_n(t).$$

By introducing $\widetilde{u}_n(t) = u_n(t)\, \mathrm{e}^{\nu k_n^2 t}$, this can be formally integrated to

$$\widetilde{u}_n(t+\delta t) - \widetilde{u}_n(t) = \int_t^{t+\delta t} \mathrm{d}t'\, \mathrm{e}^{\nu k_n^2 t'}\, g_n(t').$$

we next take out $g_n(t')$ from the integral using the Adams-Bashfort scheme, so that

$$u_n(t+\delta t) = \mathrm{e}^{-\nu k_n^2 \delta t} u_n(t) + \frac{1}{\nu k_n^2}\left(1 - \mathrm{e}^{-\nu k_n^2 \delta t}\right)\left(\tfrac{3}{2} g_n(t) - \tfrac{1}{2} g_n(t-\delta t)\right).$$

In this scheme we must remember one previous timestep. This you will recognize in the program (with the u_1, u_2 and $du_1 (= g_n(t-\delta t))$ and $du_2 (= g_n(t))$.

References

[1] Luca Biferale. Shell models of energy cascade in turbulence. *Ann. Review Fluid Mech.*, 35:441–68, 2003.

[2] L. Kadanoff, D. Lohse, and J. Wang. Scaling and dissipation in the GOY shell model. *Phys. Fluids*, 7:617–629, 1995.

[3] D. Pisarenko, L. Biferale, D. Courvoiser, U. Frisch, and M. Vergassola. Further results on multifractality in shell models. *Phys. Fluids*, 5:2533–2538, 1993.

[4] Z-S She and E. Leveque. Universal scaling laws in fully developed turbulence. *Phys. Rev. Lett.*, 72:336–339, 1994.