TECHNISCHE UNIVERSITEIT DELFT
Process and energy department

Assignment synchronization, CHAOS, 2023-2024
Synchronization

Synchronization is the tendency of two chaotic systems to become the same, although chaos implies the strong wish to diverge. Synchronization is now a very popular subject of pursuit in chaos. We will do practical experiments on the synchronization of coupled maps, and on the synchronization of coupled pendula.

Guide to the lecture notes

This assignment crosses several chapters of the lecture notes. It starts with the notion of invariant measure (**read V.9**), followed by the idea of "averages", for example the *averaged sensitivity to variation of initial conditions* (**read VI.2**). This information allows you to answer the first few questions. Next, you are asked to gather the statistics of outbreaks. This is described in Chapter VIII (**read the entire chapter**).

You will need a small matlab program to compute *statistics*. With the following few Matlab lines you can compute a histogram of values in array z, then plot the normalized histogram. This may come handy.

```
[yn xn] = hist(z,50);
yn = yn/trapz(xn,yn);
figure
plot(xn,yn,'-ob')
```

After that, we will work on applying "sensitivity", or "unpredictability" notion to real coupled pendula, described by ordinary differential equations (ODE's). You will need your now refreshed matlab skills to solve coupled ODE's. A template is provided in `example_ode45.m`.

1. Consider the system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{L} \begin{pmatrix} f(x_n) \\ f(y_n) \end{pmatrix}, \text{ with } \mathbf{L} = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}. \tag{1}$$

and

$$f(x) = \begin{cases} x/a, & 0 \le x \le a \\ (1-x)/(1-a), & a < x \le 1. \end{cases} \tag{2}$$

Prove that the invariant measure of the chaos generated by the map $f(x)$ is $\mu(x) \equiv 1$, and convincingly prove that the Lyapunov exponent is

$$\lambda = -a\ln(a) - (1-a)\ln(1-a).$$

The arguments are already in the lecture notes. *Just provide a brief review.*

2. Now iterate this system on the computer for $a = 0.7$ and $\epsilon = 0.2$, which is *below* the critical coupling strength ϵ_c. *Show a picture of the outbreaks.* Sometimes, for some of you, the system can accidentally get stuck into the synchronized state, which it never leaves. Why ? Think about numerical precision. [1] [2]

3. Avoid this being trapped inside the computer by taking the parameters a slightly different for the two coupled maps. Compute the histogram of outbreaks $P(z)$, $z = \ln|(x_n - y_n)/2|$, and verify its exponential behavior, $P(z) \propto \exp(\kappa z)$.

   This histogram, on semilog axies, and its dramatic change with ϵ as it moves through the critical point, is key !

   As explained in the lecture notes, κ should be proportional to the *transverse* Lyapunov exponent $\lambda_\perp$ (for which an analytical expression exists: Eq.8.4 of the lecture notes). *You will report on this proportionality by running the systems for various coupling strengths, both below and above criticality.* The proportionality factor involves the second (curvature) moment of the distribution of the finite-time Lyapunov exponents λ of the *individual* system. You can either compute the analytic form (Eq.8.14 of the lecture notes), or find it from simulating the (single) map and looking at the distribution of λ.

   You must try very hard to verify this relation. This is a challenge ! Get inspired by the lecture notes. Please notice: the statistics of outbreaks is the central theme of this assignment

Maps are quick, but it is interesting to see this theory work for differential equations. It also provides confidence in handling continuous-time systems through integration of coupled ordinary differential equations. You will find that the phenomenology is exactly the same, so, if you understood the previous questions, you will know exactly what to do in the ODE system.

$$\frac{\mathrm{d}^2\theta_m}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\theta_m}{\mathrm{d}t} + \sin\theta_m = \Gamma_0\cos(\Omega t) + c(\sin\theta_m - \sin\theta_s)/2 \quad (3)$$

$$\frac{\mathrm{d}^2\theta_s}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\theta_s}{\mathrm{d}t} + \sin\theta_s = \Gamma_0\cos(\Omega t) + c(\sin\theta_s - \sin\theta_m)/2 \quad (4)$$

with damping $\gamma = 0.2$, $\Gamma_0 = 1.2$ and $\Omega = 0.5$. It is the synchronization of symmetrically coupled driven pendula. It so happens (which you may verify) that the critical coupling strength $c_{\mathrm{cr}} = 0.7948\ldots$.

4. Compute the histogram of the difference between slave and master, $z(t) = \ln(\{(\theta_s - \theta_m)^2 + (\dot\theta_s - \dot\theta_m)^2\}^{1/2})$ *slightly* below criticality ($c_{\mathrm{cr}} = 0.7948$). Avoid the "getting

[1] This has resulted in great confusion in the literature, see the remarkable paper *Reconsideration of intermittent synchronization in coupled chaotic pendula*, by S. Rim *et al.*, Phys. Rev. E **64**, 060101(R) (2001). See also the references at the end of Chapter VIII of the lecture notes. The articles are on Brightspace.

[2] MATLAB has an option to compute with more than 16 digits, but this has little to do with physics.

stuck" pitfall. Also realize that θ and $\theta + 2\pi$ are exactly the same state, so you want to take θ modulo 2π.

Show a picture of the histogram for couplings below, at and above criticality.

5. Now we must verify from the value of the transverse Lyapunov exponent $\lambda_\perp$ that the number above is indeed the critical coupling strength. *Compute $\lambda_\perp$ and show that it changes sign if we move through $c_{\rm cr}$.* See the section "how to compute Lyapunov exponents".

First prove that the transverse (difference) dynamics $\delta_\perp = \theta_m - \theta_s$ satisfies the linear equation

$$\ddot{\delta}_\perp + \gamma\dot{\delta}_\perp + (1-c)\delta_\perp \cos\theta(t) = 0, \tag{5}$$

with driving force $\cos\theta(t)$ from the synchronized state. Add this ODE to your system of ODE's (see the last page of this assignment) and use it to compute the Lyapunov exponent $\lambda_\perp$.

In order to evade the locking pitfall, you should add noise. For example, uniformly distributed random numbers on the interval $[-5 \times 10^{-11}, 5 \times 10^{-11}]$ to θ_m and θ_s after each 10 integration steps (0.1 s). Of course this implies that z can never become smaller than $\approx 10^{-10}$. In this sense, synchronization is the ability of the coupled systems to overcome tiny noise that tries to push them apart while it is amplified by their positive Lyapunov exponent.

Prove the value of the critical strength $c_{\rm cr}$ from the computation of the transverse Lyapunov exponents.

6. Finally, establish the relation between the "shape" κ of the histogram of outbreaks and $\lambda_\perp$.
Do the same as in (3), i.e. relate the proportionality factor between κ and $\lambda_\perp$ to the width of to the Gaussian distribution of long-time *Lyapunov exponents Λ_T.*

This is really ultimate and you must do several things: find κ from the outbreak histograms, then find the shape of the PDF of the finite-time Lyapunov exponent of the single map, and finally relate them.

Lyapunov exponents.

In this problem we must compute the chaotic system $\theta(t)$, together with its linearized version, which we need in order to compute the fluctuating finite-time Lyapunov exponent. We also need the Lyapunov exponent of the linearized difference $\delta_\perp(t)$, and both the master and slave systems $\theta_m(t)$ and $\theta_s(t)$.

Let us illustrate the computation of the Lyapunov exponent of the *individual* chaotic oscillator. The best is to introduce a time dependent vector $y_i(t)$, with $y_1(t) = \theta(t)$, for the angle θ, $y_2(t)$ for its derivative $\dot{\theta}$, and $y_3(t) = \delta(t)$ for the perturbation and $y_4(t)$ for its derivative $\dot{\delta}$:

$$\begin{aligned}
\dot{y_1}(t) &= y_2(t) \\
\dot{y_2}(t) &= \Gamma_0 \cos(\Omega t) - \gamma y_2(t) - \sin(y_1(t)) \\
\dot{y_3}(t) &= y_4(t) \\
\dot{y_4}(t) &= -\gamma y_4(t) - y_3(t)\ \cos(y_1(t))
\end{aligned}$$

This can be fed to any standard first-order numerical integrator. We start the linearized system with a vector of unit length, for example $(y_3, y_4) = (1, 0)$. The length of this vector will grow in time, while it rotates into the direction of fastest growth. After a time Δt, its length will have increased to, say, η. The finite-time Lyapunov exponent $\Lambda_{\Delta T}$ then is $\Lambda_{\Delta T} = (\log \eta)/\Delta T$. When we continue the integration over the next time interval, we reset $(y_3, y_4) \Rightarrow (y_3, y_4)/(y_3^2 + y_4^2)^{1/2}$, which sets its length to 1, but keeps its direction, etc.

The true Lyapunov exponent is the long-time average of the fluctuating instantaneous ones $\Lambda_{\Delta T}$. Since we do the linear system, we do not care about the size of η, as long as it fits in the computer. We could then also set ΔT very large to find this average. By the way, to compute the second (next-largest) Lyapunov exponent we have to co-evolve *two* perpendicular vectors (y_3, y_4) and (y_5, y_6) and the normalization now involves a re-orthogonalization procedure.

The result for the PDF of the Lyapunov exponent $\Lambda_{\Delta T}$ with $\Delta T = 10$, and for the PDF of the separation $z(t)$ below, at, and above criticality, $c = c_{\text{cr}}$ are shown in the (stylished) figure below.

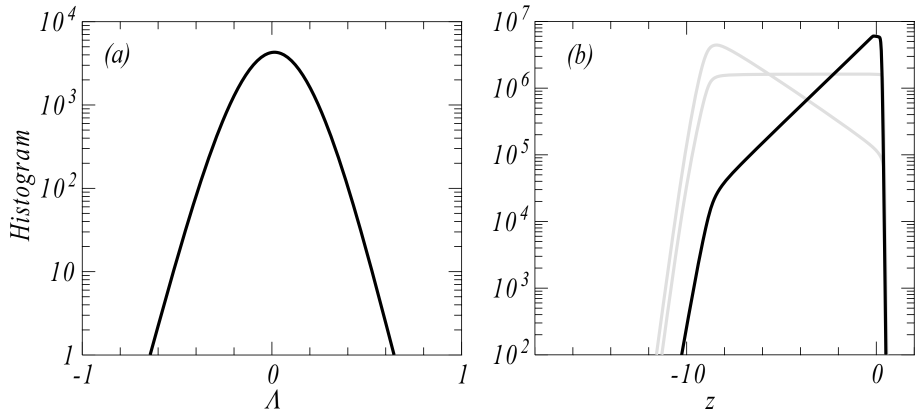

Figure 1: (a) PDF of the fluctuating finite-time Lyapunov exponents of the isolated driven swing $\Lambda_{T=10}$ is a Gaussian. *Try hard to see it, running programs for a long time.* (b) PDF of $z = \frac{1}{2}\log\{(\theta_m - \theta_s)^2 + (\dot{\theta}_m - \dot{\theta}_s)^2\}$ for coupling strengths $c = c_{\text{cr}} - 0.02$, $c = c_{\text{cr}}$, and $c = c_{\text{cr}} + 0.02$, with $c_{\text{cr}} = 0.7948$. *Try to see the dramatic change in shape: rising, level, and falling.* This is accompanied by a change of sign of the transverse Lyapunov exponent, just as in the case of the map.

These results require some numerical effort, but they beautifully illustrate that the analytical theory for map synchronisation carries over to that of ordinary differential equations.

A system of 10 ordinary differential equations

As engineers you are perfectly equipped to model dynamical systems using ODE's. In our case it is efficient to collect all ODE's in one large system, although not everyone depends on everyone else.

So far we have seen $y_1(t)$ and $y_2(t)$ for the uncoupled oscillator alone, and $y_3(t)$ and $y_4(t)$ for its linearized version. Together they allow the computation of the (largest) Lyapunov exponent of the isolated oscillator. To do this, the linearized version has to be integrated along the trajectory $y_1(t)$ of the uncoupled oscillator, as we have already seen:

$$
\begin{aligned}
\dot{y_1}(t) &= y_2(t) \\
\dot{y_2}(t) &= \Gamma_0 \cos(\Omega t) - \gamma\, y_2(t) - \sin(y_1(t)) \\
\dot{y_3}(t) &= y_4(t) \\
\dot{y_4}(t) &= -\gamma\, y_4(t) - y_3(t)\ \cos(y_1(t))
\end{aligned} \tag{6}
$$

Our coupled oscillator problem needs more: we need the Lyapunov exponent $\lambda_\perp$ of the linearized *difference* problem, driven by the isolated one, as was derived by you in Eq. 5:

$$\ddot{\delta}_\perp + \gamma\dot{\delta}_\perp + (1-c)\delta_\perp \cos\theta(t) = 0,$$

and finaly we need the full coupled problem.

We do $y_5(t)$ and $y_6(t)$ for the linearized difference, driven by the (isolated) y_1, $y_7(t)$ and $y_8(t)$ for oscillator 1 coupled to 2, and $y_9(t)$ and $y_{10}(t)$ for oscillator 2 coupled to oscillator 1.

$$
\begin{aligned}
\dot{y_5}(t) &= y_6(t) \\
\dot{y_6}(t) &= -\gamma\, y_6(t) - y_5(t)\,(1-c)\ \cos(y_1(t)) \\
\dot{y_7}(t) &= y_8(t) \\
\dot{y_8}(t) &= \Gamma_0 \cos(\Omega t) - \gamma\, y_8(t) - \sin(y_7(t)) + c\,(\sin(y_7(t)) - \sin(y_9(t)))/2 \\
\dot{y_9}(t) &= y_{10}(t) \\
\dot{y_{10}}(t) &= \Gamma_0 \cos(\Omega t) - \gamma\, y_{10}(t) - \sin(y_9(t)) + c\,(\sin(y_9(t)) - \sin(y_7(t)))/2
\end{aligned} \tag{7}
$$

We could make separate programs for Eq. 6 and Eq. 7, but it is more efficient to let everybody ride along in this 10-dimensional code.