Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

6.041 Quiz 2 Solutions:
November 2, 2011

Problem 1. (17 points)

(a) **(9 points)** Since X is a sum of independent normal random variables, X is also a normal random variable. Its mean, μ_X, variance, σ_X^2, are

$$\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}[3U + 4V] \\
&= 3\mathbf{E}[U] + 4\mathbf{E}[V] \\
&= 0. \\
\mathrm{var}(X) &= \mathrm{var}(3U + 4V) \\
&= 9\mathrm{var}(U) + 16\mathrm{var}(V) \\
&= 25.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{P}(X \geq 8) &= \mathbf{P}\left(\frac{X-0}{5} \geq \frac{8}{5}\right) \\
&= 1 - \Phi(1.6) \\
&= 1 - 0.9452 \\
&= 0.0548.
\end{aligned}$$

(b) **(8 points)**

$$\begin{aligned}
\mathbf{E}[XY] &= \mathbf{E}[(3U + 4V)(U + W)] \\
&= \mathbf{E}[3U^2 + 3UW + 4UV + 4VW] \\
&= 3\mathbf{E}[U^2] + 3\mathbf{E}[U]\mathbf{E}[W] + 4\mathbf{E}[U]\mathbf{E}[V] + 4\mathbf{E}[V]\mathbf{E}[W] \\
&= 3\mathbf{E}[U^2] \\
&= 3\left(\mathrm{var}(U) + (\mathbf{E}[U])^2\right) \\
&= 3.
\end{aligned}$$

Problem 2. (68 points)

(a) **(5 points)** The distribution must integrate to 1.

$$\begin{aligned}
\int_1^\infty \frac{c}{x^3}\,dx &= -\frac{c}{2x^2}\Big|_1^\infty \\
&= \frac{c}{2}.
\end{aligned}$$

Therefore, $c = 2$.

(b) **(5 points)**

$$\begin{aligned}
\mathbf{P}(2 \leq X \leq 3) &= \int_2^3 \frac{2}{x^3}\,dx \\
&= \frac{5}{36}.
\end{aligned}$$

(c) **(8 points)**

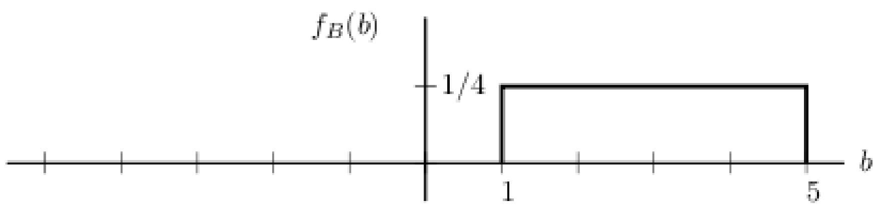
$$\begin{aligned}\mathbf{E}[X^3e^{-X}] &= \int_1^\infty x^3 e^{-x}\frac{2}{x^3}\,dx \\ &= \int_1^\infty 2e^{-x}\,dx \\ &= \frac{2}{e}.\end{aligned}$$

(d) **(8 points)** The distribution of U is $f_U(u) = 1/2$ for $0 \leq u \leq 2$. Let $B = 2U + 1$. Since B is a linear transformation of U, the distribution of B can be written as

$$f_B(b) = \frac{1}{|2|} f_U\left(\frac{b-1}{2}\right).$$

Therefore, B is uniform for $1 \leq b \leq 5$ and

$$f_B(b) = \begin{cases} \frac{1}{4}, & 1 \leq b \leq 5 \\ 0, & \text{otherwise.}\end{cases}$$



(e) **(10 points)** By independence, $f_{X,U}(x,u) = f_X(x)\cdot f_U(u)$. To compute $\mathbf{P}(X \leq U)$, integrate the joint PDF over the proper set.

$$\begin{aligned}\mathbf{P}(X \leq U) &= \int_1^2\int_1^u f_X(x)f_U(u)\,dx\,du \\ &= \int_1^2\int_1^u \frac{1}{x^3}\,dx\,du \\ &= \frac{1}{4}.\end{aligned}$$

(f) **(10 points)** Let $D = 1/X$. Since X takes on values $[1,\infty)$, D takes on values $[0,1]$. Using derived distribution to the find the CDF of D on $0 \leq d \leq 1$,

$$\begin{aligned}F_D(d) &= \mathbf{P}(D \leq d) \\ &= \mathbf{P}(X \geq 1/d) \\ &= \int_{1/d}^\infty \frac{2}{x^3}\,dx \\ &= d^2.\end{aligned}$$

The complete CDF of D is

$$F_D(d) = \begin{cases} 0, & d < 0 \\ d^2, & 0 \leq d \leq 1 \\ 1, & d > 1. \end{cases}$$

Differentiating the CDF gives the PDF of D

$$f_D(d) = \begin{cases} 2d, & 0 \leq d \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, the same result can be achieved without explicitly computing the CDF of X:

$$\begin{aligned} F_D(d) &= \mathbf{P}(D \leq d) \\ &= \mathbf{P}(X \geq 1/d) \\ &= 1 - F_X(1/d) \\ f_D(d) &= -f_X(1/d) \cdot \frac{(-1)}{d^2} \text{ (chain rule for differentiation)} \end{aligned}$$

$$f_D(d) = \begin{cases} 2d, & 0 \leq d \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

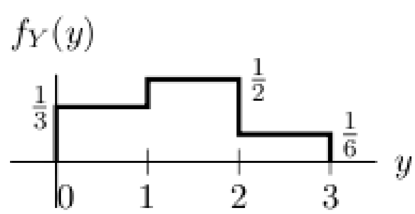
(g) **(11 points)** The PDF of Y can be computed using the total probability theorem:

$$f_Y(y) = \mathbf{P}(K = 0) f_{Y|K}(y \mid 0) + \mathbf{P}(K = 1) f_{Y|K}(y \mid 1).$$

If $K = 0$, $Y = U$ and $f_{Y|K}(y \mid 0)$ is uniform for $y \in [0, 2]$. If $K = 1$, $Y = U + 1$ and $f_{Y|K}(y \mid 1)$ is uniform for $y \in [1, 3]$.

The terms can be added carefully to compute $f_Y(y)$ as

$$f_Y(y) = \begin{cases} 1/3, & 0 \leq y \leq 1 \\ 1/2, & 1 < y \leq 2 \\ 1/6, & 2 < y \leq 3. \end{cases}$$



Alternatively, one can use derived distribution to find the CDF of Y:

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \leq y) \\ &= \mathbf{P}(U + K \leq y) \\ &= \mathbf{P}(U + K \leq y \mid K = 0)\mathbf{P}(K = 0) + \mathbf{P}(U + K \leq y \mid K = 1)\mathbf{P}(K = 1) \\ &= \mathbf{P}(U \leq y \mid K = 0) \cdot \frac{2}{3} + \mathbf{P}(U \leq y - 1 \mid K = 1) \cdot \frac{1}{3} \\ &= \mathbf{P}(U \leq y) \cdot \frac{2}{3} + \mathbf{P}(U \leq y - 1) \cdot \frac{1}{3}. \end{aligned}$$

The CDF of Y has three distinct regions listed below.

$$F_Y(y) = \begin{cases} \frac{y}{2} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{y}{3}, & 0 \le y \le 1 \\ \frac{y}{2} \cdot \frac{2}{3} + \frac{y-1}{2} \cdot \frac{1}{3} = \frac{y}{2} - \frac{1}{6}, & 1 < y \le 2 \\ 1 \cdot \frac{2}{3} + \frac{y-1}{2} \cdot \frac{1}{3} = \frac{y}{6} + \frac{1}{2}, & 2 < y \le 3. \end{cases}$$

By differentiating the CDF, the PDF is the same as above.

(h) **(11 points)** For $y \in [0,1]$, K must be 0. Similarly, for $y \in [2,3]$, K must be 1. For $y \in [1,2]$,

$$p_{K|Y}(k \mid y) = \frac{p_K(k) f_{Y|K}(y \mid k)}{f_Y(y)}.$$

For $k = 0$ or $k = 1$ and $y \in [1,2]$, $f_{Y|K} = f_Y(y) = 1/2$. Therefore, $p_{K|Y}(k \mid y) = p_K(k)$.

Putting it altogether,

$$p_{K|Y}(k \mid y) = \begin{cases} \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise.} \end{cases} & 0 \le y \le 1 \\ \begin{cases} 2/3, & k = 0 \\ 1/3, & k = 1 \\ 0, & \text{otherwise.} \end{cases} & 1 < y \le 2 \\ \begin{cases} 1, & k = 1 \\ 0, & \text{otherwise.} \end{cases} & 2 < y \le 3. \end{cases}$$

Problem 3. (15 points)

The covariance of A and B is defined as $\text{cov}(A,B) = \mathbf{E}[AB] - \mathbf{E}[A]\mathbf{E}[B]$. Using the law of iterated expectations,

$$\begin{aligned} \mathbf{E}[AB] &= \mathbf{E}[\mathbf{E}[AB \mid N]] \\ &= \mathbf{E}\left[\mathbf{E}\left[(X_1 + X_2 + \cdots + X_N)(Y_1 + Y_2 + \cdots Y_N) \mid N\right]\right] \\ &= \mathbf{E}[N^2 \mathbf{E}[X_1 Y_1]] \\ &= \mathbf{E}[N^2 \mathbf{E}[X_1]\mathbf{E}[Y_1]] \\ &= \mu_X \mu_Y \mathbf{E}[N^2] \\ &= \mu_X \mu_Y (\sigma_N^2 + \mu_N^2), \end{aligned}$$

where the third equality holds since there are N^2 cross-terms, the $X_i's$ are identically distributed and the $Y_j's$ are identically distributed. The fourth equality holds by independence.

$$\begin{aligned} \mathbf{E}[A] &= \mathbf{E}[\mathbf{E}[X_1 + X_2 + \cdots + X_N \mid N]] \\ &= \mathbf{E}[N\mu_X] \\ &= \mu_X \mu_N. \end{aligned}$$

$$\begin{aligned}
\mathbf{E}[B] &= \mathbf{E}[\mathbf{E}[Y_1 + Y_2 + \cdots + Y_N \mid N]] \\
&= \mathbf{E}[N\mu_Y] \\
&= \mu_Y \mu_N.
\end{aligned}$$

The covariance of A and B is

$$\begin{aligned}
\mathrm{cov}(A, B) &= \mu_X \mu_Y (\sigma_N^2 + \mu_N^2) - \mu_X \mu_Y \mu_N^2 \\
&= \mu_X \mu_Y \sigma_N^2.
\end{aligned}$$