Problem Set 1: Solutions
Due: September 16, 2009

1. We are given that $\mathbf{P}(A^c) = 0.6$, $\mathbf{P}(B) = 0.3$, and $\mathbf{P}(A \cap B) = 0.2$. Since $\mathbf{P}(A) + \mathbf{P}(A^c) = 1$, we find

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1 - 0.6 = 0.4$$

From a property of probability laws proved in lecture, we have

$$\mathbf{P}(A \cup B) \quad = \quad \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 0.4 + 0.3 - 0.2 = \boxed{0.5}$$

Drawing a Venn diagram can be helpful in these kinds of problems.

2. (a) By the additivity axiom,

$$\mathbf{P}(X_1) = \mathbf{P}((X_1 \cap X_2) \cup (X_1 \cap X_2^c)) = \mathbf{P}(X_1 \cap X_2) + \mathbf{P}(X_1 \cap X_2^c)$$

Using the non-negativity axiom, we obtain

$$\mathbf{P}(X_1 \cap X_2) \leq \mathbf{P}(X_1)$$

(b) Similar to the argument in the previous part,

$$\mathbf{P}(X_1 \cup X_2) = \mathbf{P}(X_1) + \mathbf{P}(X_1^c \cap X_2) \geq \mathbf{P}(X_1)$$

(c) Let $A = X_1 \cap X_2^c, B = X_2 \cap X_1^c, C = X_1 \cap X_2$. Since A, B, C are disjoint by construction,

$$\mathbf{P}(X_1 \cup X_2) = \mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$$

and

$$\mathbf{P}(X_1) + \mathbf{P}(X_2) = \mathbf{P}(A \cup C) + \mathbf{P}(B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + 2 \cdot \mathbf{P}(C)$$

The result follows. This is also property c) on page 14 of the textbook. The proof is on page 15, in the the caption to Figure 1.6.

3. Since all outcomes are equally likely, we apply the discrete uniform probability law to solve the problem. To solve for any event we simply count the number of elements in the event and divide by the total number of elements in the sample space.

There are 2 possible outcomes for each flip, and 3 flips. Thus there are $2^3 = 8$ elements (or sequences) in the sample space.

(a) Any sequence has probability of 1/8. Therefore $\mathbf{P}(HHH) = \boxed{1/8}$

(b) This is still a single sequence, thus $\mathbf{P}(THT) = \boxed{1/8}$

(c) The event of interest has 4 unique sequences, thus $\mathbf{P}(\{HHH, HTH, THT, TTT\}) = 4/8 = \boxed{1/2}$

(d) The sequences where there are more heads than tails are $A = \{HHH, HHT, HTH, THH\}$. 4 unique sequences gives us $\mathbf{P}(A) = \boxed{1/2}$

4. The easiest way to solve this problem is to make a table of some sort, similar to the one below.

Die 1	Die 2	Sum	**P**(Sum)
1	1	2	2p
1	2	3	3p
1	3	4	4p
1	4	5	5p
2	1	3	3p
2	2	4	4p
2	3	5	5p
2	4	6	6p
3	1	4	4p
3	2	5	5p
3	3	6	6p
3	4	7	7p
4	1	5	5p
4	2	6	6p
4	3	7	7p
4	4	8	8p
		Total	80p

$$\mathbf{P}(\text{All outcomes}) = 80p \text{ (Total from the table)}$$

and therefore

$$p = \frac{1}{80}$$

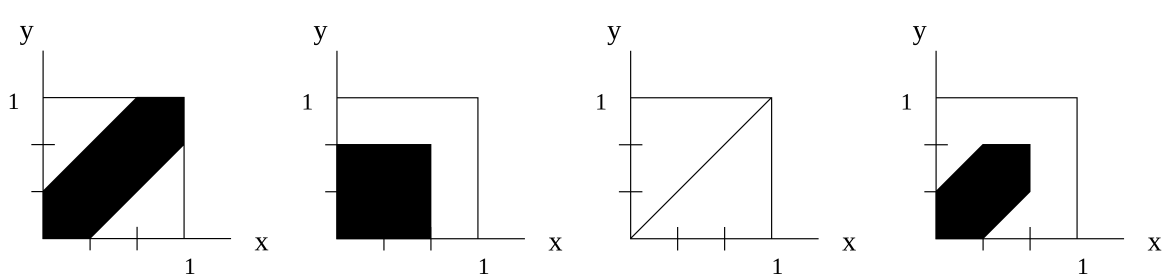
(a)

$$\mathbf{P}(\text{Even sum}) \quad = \quad 2p + 4p + 4p + 6p + 4p + 6p + 6p + 8p = 40p = \boxed{1/2}$$

(b)

$$\mathbf{P}(\text{Rolling a 4 and a 1}) \quad = \quad \mathbf{P}(1,4) + \mathbf{P}(4,1) = 5p + 5p = 10p = \boxed{1/8}$$

5. The events A, B, C, and $A \cap B$ are shown in the shaded areas of the sample spaces below. The physical description of the events given in the problem can be related to mathematical inequalities to define certain regions of the sample space.. Be sure that you are able to match regions of the sample space with the descriptions of the events. Note that the region for the event $A \cap B$ is the intersection of the regions for A and B.



A: {| x - y | <= 1/3} **B : {x <= 2/3, y <= 2/3}** **C : {x = y}** **A ∩ B**

Since we are dealing with a "uniform probability law", we know that probability is proportional to the area of the shaded regions. Since the whole space is the unit square, and the unit square has area equal to one, the probability of any event is exactly the area of its respective region. (If the whole space were a square of different size, we would need a normalizing factor.)

Thus, finding $\mathbf{P}(A)$, $\mathbf{P}(B)$, $\mathbf{P}(A \cap B)$, and $\mathbf{P}(C)$ amounts to finding the area of the regions. These turn out to be

$$\mathbf{P}(A) = \frac{5}{9} \qquad \mathbf{P}(B) = \frac{4}{9} \qquad \mathbf{P}(A \cap B) = \frac{1}{3} \qquad \mathbf{P}(C) = 0.$$

Note that $\mathbf{P}(C)$ is zero. Continuous random variables tend to have this peculiar problem when dealing with exact values. This will be explored later.

6. (a) The probability of Mike scoring 50 points is proportional to the area of the inner disk. Hence, it is equal to $\alpha\pi R^2 = \alpha\pi$, where α is a constant to be determined.

   Since the probability of landing the dart on the board is equal to one, $\alpha\pi 10^2 = 1$, which implies that $\alpha = 1/(100\pi)$.

   Therefore, the probability that Mike scores 50 points is equal to $\pi/(100\pi) = \boxed{0.01}$

   (b) In order to score exactly 30 points, Mike needs to place the dart between 1 and 3 inches from the origin. An easy way to compute this probability is to look first at that of scoring *more* than 30 points, which is equal to $\alpha\pi 3^2 = 0.09$.

   Next, since the 30 points ring is disjoint from the 50 points disc, probability of scoring more than 30 points is equal to the probability of scoring 50 points plus that of scoring exactly 30 points. Hence, the probability of Mike scoring exactly 30 points is equal to $0.09 - 0.01 = \boxed{0.08}$

   (c) For the part (a) question. The probability of John scoring 50 points is equal to the probability of throwing in the right half of the board and scoring 50 points plus that of throwing in the left half and scoring 50 points.

   The first term in the sum is proportional to the area of the right half of the inner disk and is equal to $\alpha\pi R^2/2 = \alpha\pi/2$, where α is a constant to be determined.

   Similarly, the probability of him throwing in the left half of the board and scoring 50 points is equal to $\beta\pi/2$, where β is a constant (not necessarily equal to α).

   In order to determine α and β, let us compute the probability of throwing the dart in the right half of the board. This probability is equal to

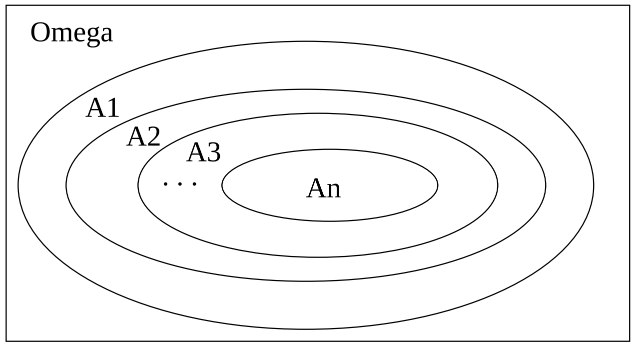
   $$\alpha\pi R^2/2 = \alpha\pi 10^2/2 = \alpha 50\pi.$$

   Since that probability is equal to $2/3$, $\alpha = 1/(75\pi)$. In a similar fashion, β can be determined to be $1/(150\pi)$. Consequently, the total probability is equal to $1/150 + 1/300 = \boxed{0.01}$

   For the part (b), The probability of scoring exactly 30 points is equal to that of scoring more than 30 points minus that of scoring exactly 50. By applying the same type of analysis as in (b) above, the probability is found to be equal to $\boxed{0.08}$

   These numbers suggest that John and Mike have similar skills, and are equally likely to win the game. The fact that Mike's better control (or worst, depending on how you look at it) of the direction of his throw does not increase his chances of winning can be explained by the observation that both players' control over the distance from the origin is identical.

G1†. (a) If we define $A_n = [a_n, b_n]$ for all n, it is easy to see that the sequence $A_1, A_2, \ldots$ is "monotonically decreasing," i.e., $A_{n+1} \subset A_n$ for all n:



Furthermore, $\cap_n^\infty A_n = [a, b]$.

By the continuity property of probabilities (see Problem 1.13, page 56 of the text),

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

(b) No. Consider the following example. Let $a_n = a + \frac{1}{n}$, $b_n = b - \frac{1}{n}$ for all n. Then $\{a_n\}$ is a decreasing sequence that converges to a, and $\{b_n\}$ is an increasing sequence that converges to b. If we define a probability law that places non-zero probability only on points a and b, then $\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = 0$, but $\mathbf{P}([a, b]) = 1$.

This example is closely related to the continuity property of probabilities. In this case, if we define $A_n = [a_n, b_n]$, then $A_1, A_2, \ldots$ is "monotonically increasing," i.e., $A_n \subset A_{n+1}$, but $A = (\cup_n^\infty A_n) = (a, b)$, which is an open interval whose probability is 0 under our probability law.