LECTURE 6

- **Readings:** Sections 2.4–2.6

Lecture outline

- Review
- Poisson random variable
- Variance
- Conditioning
- Joint PMF of two random variables

Review

- Random variable: assignment of number to each outcome
- **Discrete** random variable has probability mass function:

$$p_X(x) \;=\; \mathbf{P}(X = x)$$

- Expectation: (r.v. X, function g, constants a, b)

$$\mathbf{E}[X] \;=\; \sum_x x p_X(x)$$

$$\mathbf{E}[g(X)] \;=\; \sum_x g(x) p_X(x)$$

$$\mathbf{E}[aX + b] \;=\; a\mathbf{E}[X] + b$$

Poisson random variable

- With parameter $\lambda > 0$:

$$p_X(k) \;=\; e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \ldots$$

- Validity?
- $\mathbf{E}[X] \;=$

Variance and standard deviation

- $\mathbf{E}\big[X - \mathbf{E}[X]\big] \;=$
- Definitions:

$$\text{var}(X) \;=\; \mathbf{E}\left[(X - \mathbf{E}[X])^2\right], \qquad \sigma_X \;=\; \sqrt{\text{var}(X)}$$

- Alternative expressions:

$$\text{var}(X) \;=\; \sum_x (x - \mathbf{E}[X])^2 p_X(x)$$

$$\text{var}(X) \;=$$

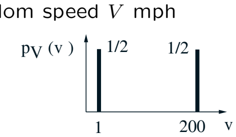
Properties of variance

$$\text{var}(X) \;=\; \sum_x (x - \mathbf{E}[X])^2 p_X(x)$$

- What does $\text{var}(X) = 0$ imply?
- $\text{var}(aX + b) \;=$

Example: Random speed

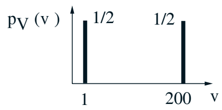
- Consider a random speed V mph



- $\mathbf{E}[V] \;=$
- $\text{var}(V) \;=$
- $\sigma_V \;=$

Average speed vs. average time

- Traverse 200 miles at single random speed V mph



- Let T be the time in hours, $\quad T = t(V) =$
- $\mathbf{E}[T] \;=\; \mathbf{E}[t(V)] \;=\; \sum_v t(v) p_V(v) \;=$
- $\mathbf{E}[TV] \;=$

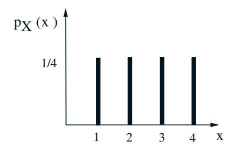
Conditioning

- Conditioning on A with $\mathbf{P}(A) > 0$ gives a probability law
  - Conditional PMF:

$$p_{X|A}(x) \;=\; \mathbf{P}(X = x \mid A)$$

  - Conditional expectation:

$$\mathbf{E}[X \mid A] \;=\; \sum_x x p_{X|A}(x)$$



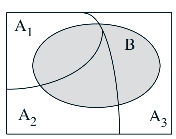
Let $A = \{X \geq 2\}$

$p_{X|A}(x) \;=$

$\mathbf{E}[X \mid A] \;=$

Total probability and total expectation

- Consider conditioning on each element of a **partition** $A_1, A_2, \ldots, A_n$ of sample space



$$\mathbf{P}(B) \;=\; \mathbf{P}(A_1)\mathbf{P}(B \mid A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B \mid A_n)$$

$$p_X(x) \;=\; \sum_{i=1}^{n} \mathbf{P}(A_i) p_{X|A_i}(x)$$

$$\mathbf{E}[X] \;=\; \mathbf{P}(A_1)\mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X \mid A_n]$$

Geometric PMF mean

- X: number of independent coin tosses until first head

$$p_X(k) \;=\; (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

$$\mathbf{E}[X] \;=\; \sum_{k=1}^{\infty} k p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

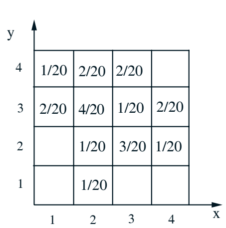
- What does conditioning on $A = \{X > n\}$ tell us?
- Use $n = 1$ and total expectation:

$$\begin{aligned}\mathbf{E}[X] \;&=\; \mathbf{P}(A)\mathbf{E}[X \mid A] + \mathbf{P}(A^c)\mathbf{E}[X \mid A_c] \\ &=\; \mathbf{P}(X > 1)\mathbf{E}[X \mid X > 1] + \mathbf{P}(X = 1)\mathbf{E}[X \mid X = 1]\end{aligned}$$

Joint PMFs

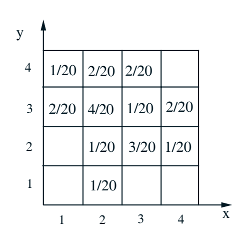
- Pair of discrete random variables have a joint PMF:

$$p_{X,Y}(x, y) \;=\; \mathbf{P}(\{X = x\} \cap \{Y = y\})$$



- $\sum_x \sum_y p_{X,Y}(x, y) \;=$

Joint, marginal, and conditional PMFs



- $p_X(x) \;=\; \sum_y p_{X,Y}(x, y)$
- $p_{X|Y}(x \mid y) \;=\; \mathbf{P}(X = x \mid Y = y) \;=\; \dfrac{p_{X,Y}(x, y)}{p_Y(y)}$