LECTURE 7

• Readings: Finish Chapter 2

Lecture outline

- Multiple random variables
- Joint PMF
- Conditioning
- Independence
- More on expectations
- Binomial distribution revisited
- A hat problem

Review

$$p_X(x) = P(X = x)$$

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y)$$

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

$$p_{X,Y}(x,y) = p_X(x)p_{Y\mid X}(y\mid x)$$

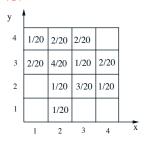
Independent random variables

$$p_{X,Y,Z}(x, y, z) = p_X(x)p_{Y|X}(y \mid x)p_{Z|X,Y}(z \mid x, y)$$

 Random variables X, Y, Z are independent if:

$$p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)$$

for all x, y, z



- Independent?
- What if we condition on $X \le 2$ and $Y \ge 3$?

Expectations

$$\begin{aligned} \mathbf{E}[X] &= \sum_{x} x p_X(x) \\ \mathbf{E}[g(X,Y)] &= \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y) \end{aligned}$$

- In general: $\mathbf{E}[g(X,Y)] \neq g(\mathbf{E}[X],\mathbf{E}[Y])$
- $E[\alpha X + \beta] = \alpha E[X] + \beta$
- E[X + Y + Z] = E[X] + E[Y] + E[Z]
- If X, Y are independent:
- $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
- $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

Variances

- $Var(aX) = a^2Var(X)$
- Var(X + a) = Var(X)
- Let Z = X + Y. If X, Y are independent:

$$Var(X + Y) = Var(X) + Var(Y)$$

• Examples:

- If
$$X = Y$$
, $Var(X + Y) =$

- If
$$X = -Y$$
, $Var(X + Y) =$

- If
$$X$$
, Y indep., and $Z = X - 3Y$,
 $Var(Z) =$

Binomial mean and variance

- X = # of successes in n independent trials
- probability of success p

$$E[X] = \sum_{k=0}^{n} k {n \choose k} p^k (1-p)^{n-k}$$

- $\bullet \ \ \, X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases}$
- $E[X_i] =$
- \bullet E[X] =
- $Var(X_i) =$
- Var(X) =

The hat problem

- *n* people throw their hats in a box and then pick one at random.
- X: number of people who get their own hat
- Find $\mathbf{E}[X]$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

- $X = X_1 + X_2 + \dots + X_n$
- $P(X_i = 1) =$
- $E[X_i] =$
- Are the X_i independent?
- \bullet E[X] =

Variance in the hat problem

•
$$Var(X) = E[X^2] - (E[X])^2 = E[X^2] - 1$$

$$X^{2} = \sum_{i} X_{i}^{2} + \sum_{i,j:i \neq j} X_{i}X_{j}$$

• $E[X_i^2] =$

$$P(X_1X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 1 \mid X_1 = 1)$$

- $E[X^2] =$
- Var(X) =