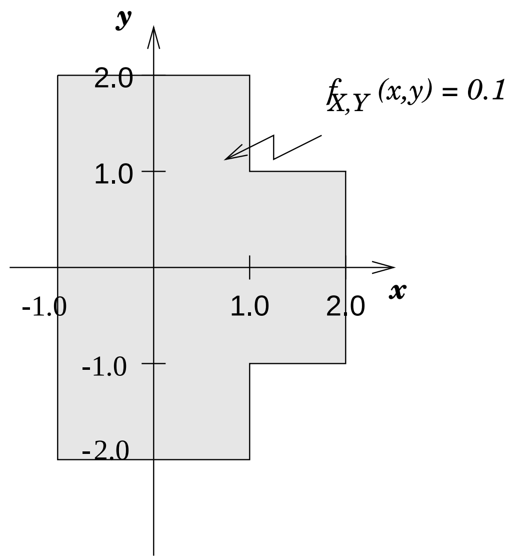
Problem Set 5
Due October 19, 2009

1. Random variables X and Y have the joint PDF shown below:



   (a) Prepare neat, fully labeled sketches of $f_X(x)$, $f_Y(y)$, $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.
   (b) Are X and Y independent?
   (c) Find $f_{X,Y|A}(x,y)$, where the event A corresponds to points (x,y) within the unit circle centered at the origin.
   (d) Find $\mathbf{E}[X \mid Y = y]$ and $\text{var}(X \mid Y = y)$.

2. Let X be a normal random variable with mean 1 and variance 4. Find the PDF of the random variable $Y = 3X - 1$. Also find the mean of the random variable $W = Y^2$.

3. Random variables X and Y are independent and are described by the probability density functions $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \begin{cases} 1, & 0 < x \leq 1; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_Y(y) = \begin{cases} 1, & 0 < y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

   Stations A and B are connected by two *parallel* message channels. One message from A to B is sent over each of the channels at the same time. Random variables X and Y represent the message delays in hours over parallel channels 1 and 2, respectively.

   A message is considered "received" as soon as it arrives on any one channel and it is considered "verified" as soon as it has arrived over both channels.

   (a) Determine the probability that a message is received within 15 minutes after it is sent.
   (b) Determine the probability that the message is received but not verified within 15 minutes after it is sent.

(c) Let T represent the time in hours between transmission at A and verification at B. Determine the CDF $F_T(t)$, and then differentiate it to obtain the PDF $f_T(t)$.

(d) If the attendant at B leaves for a 15-minute coffee break right after the message is received, what is the probability that he is present at the proper time for verification?

(e) The management wishes to have the maximum probability of having the attendant present for *both* reception and verification. Would they do better to let him take his coffee break as described above or simply allow him to go home 45 minutes after transmission?

4. Consider the following problem and a purported solution. Either declare the solution to be correct or explain the flaw.

   Question: Let X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} 1, & x \in [0,1] \quad \text{and} \quad y \in [x, x+1]; \\ 0, & \text{otherwise.} \end{cases}$$

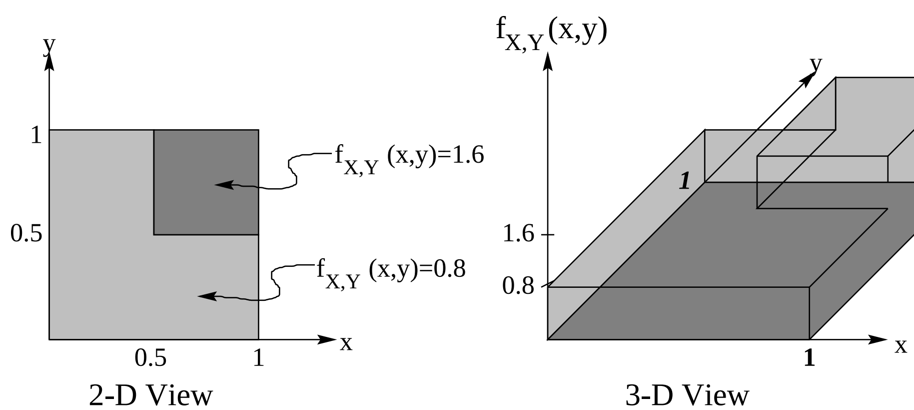
   Find $f_X(x)$, $f_Y(y)$, and $f_{Y|X}(y|x)$. Are X and Y independent?

   Solution:

$$f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_x^{x+1} 1 \cdot dy = 1.$$
$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_0^1 1 \cdot dx = 1.$$
$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{1} = 1.$$

   Since $f_{Y|X}(y|x)$ does not depend on x, we have that X and Y are independent. Alternatively, X and Y are independent because $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

5. Continuous random variables X and Y each take on experimental values between zero and one, with the joint pdf indicated below (the cutoff between probability density 0.8 and 1.6 occurs at $x = 0.5$ and $y = 0.5$):



(a) Are X and Y independent? Present a convincing argument for your answer.

(b) Prepare neat, fully labelled plots for $f_X(x)$ and $f_{Y|X}(y \mid 0.75)$.

(c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.

(d) Let $W = \min\{X, Y\}$ and determine the cumulative distribution function (CDF) of W. You should be able to reason out this part without doing any formal integrals.

6. Beginning at time $t = 0$ we begin using bulbs, one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a Type-A bulb and a Type-B bulb.

   The lifetime, X, of any particular bulb of a particular type is an independent random variable with the following PDF:

$$\text{For Type-A Bulbs:} \quad f_{X|A}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{For Type-B Bulbs:} \quad f_{X|B}(x) = \begin{cases} 3e^{-3x} & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the expected time until the first failure.

   (b) Find $\mathbf{P}(D)$, the probability that there are no bulb failures during the first τ hours of this process.

   (c) Given that there are no failures during the first τ hours of this process, determine $\mathbf{P}(A|D)$, the conditional probability that the first bulb used is a Type-A bulb.

   (d) Given that there are no failures during the first τ hours of this process, determine the total expected time until the first failure (i.e., the expected time elapsed from $t = 0$ until the first bulb fails).

7. Let Q be a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability Q. Furthermore, given the value of Q, the status of the machine on different days is independent.

   (a) Find the probability that the machine is functional on a particular day.

   (b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of Q. You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}$$

   (c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

   (Hint: See Problem 3.34 in course text.)

G1†. Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference of C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

G2†. (a) Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{2n}$ be independent and identically distributed random variables.

Find

$$\mathbf{E}[X_1 \mid X_1 + X_2 + \ldots + X_n = x_0],$$

where x_0 is a constant.

(b) Define

$$S_k = X_1 + X_2 + \ldots + X_k, 1 \leq k \leq 2n.$$

Find

$$\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}],$$

where $s_n, s_{n+1}, \ldots, s_{2n}$ are constants.

†Required for 6.431; optional challenge problem for 6.041