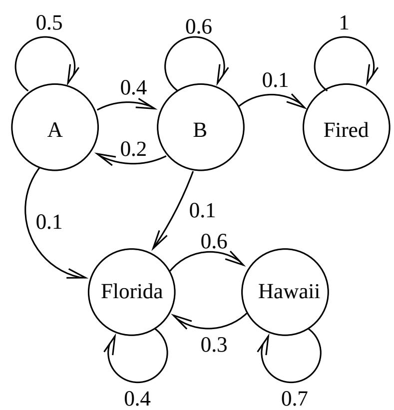
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 10
Due: May 6, 2009

1. Consider a man who works at company X. Every day, he works on one of two projects, A and B. On any given day while at the company, there is a probability of 0.1 that he may receive an inheritance, at which point he moves to Florida and spends the rest of his days between Florida and Hawaii. On any given day while at the company, if he is working on project B, there is a 0.1 probability of getting Fired.



Assume that the person starts working at the company on project A on day 1. Provide numerical values for the following quantities:

(a) The expected time until a recurrent state is reached.

(b) The approximate probability of being in Florida when the number of transitions is large enough.

2. A telephone company establishes a direct connection between two cities expecting Poisson traffic with rate c calls/minute. The durations of calls are independent and exponentially distributed with mean $\frac{1}{d}$ minutes. Interarrival times are independent of call durations. The system can handle up to n calls, meaning that any calls that are attempted when the system has n calls on line will be blocked. Each time a call comes in (regardless of whether or not it is blocked), a red light flashes. Each time a call (that was not blocked) ends, a green light flashes. Assume we start observing the system long after the process has started.

(a) Given that the number of calls on the system just changed, find the PMF for the number of calls on the system immediately after the change.

(b) Given that the number of calls on the system just changed, find the probability the number of calls decreased.

(c) Given that a light just flashed, find the PMF for the number of calls on the system immediately after the light flashed.

(d) Given that a light just flashed, find the probability the light was green.

(e) Given that the red light just flashed, fine the PMF for the number of calls on the system immediately after the light flashed.

3. (a) **Chernoff Bound:** Let X be a random variable and b be a scalar. Use the Markov inequality on the random variable e^{sX} to show that

$$\mathbf{P}(X \geq b) \leq e^{-sb} M_X(s),$$

for every $s > 0$, where $M_X(s)$ is the transform of X. Since the inequality is true for every $s > 0$, the tightest upper bound is

$$\mathbf{P}(X \geq b) \leq \min_{s>0} [\, e^{-sb} M_X(s) \,].$$

(b) Let Y be the sum of n independent, identically distributed random variables,

$$Y = X_1 + X_2 + \cdots + X_n,$$

where each X_i is normal with mean m and unit variance. Obtain an upper bound on $P(Y \geq \alpha)$ for our sum-of-normal random variable Y using the Chernoff bound. Choose a value for s which yields the best (i.e., smallest) upper bound. What is this smallest upper bound? For what values of α is your best bound valid? (Note that the exact value of $P(Y \geq \alpha)$ can be expressed as an integral, which cannot be evaluated in a closed form.)

4. (a) Find a non-negative random variable X for which the Markov bound is achieved as an equality, *i.e.*, given a value for the mean μ_X, and a value of $a > \mu_X$, find a random variable X such that $P[X \geq a] = \frac{\mu_X}{a}$.

(b) Find a random variable Y for which the Chebyshev bound is achieved as an equality, *i.e.*, given values for the mean μ_Y and standard deviation σ_Y and a constant $b > \sigma_Y$, find a random variable Y such that $P[|Y - \mu_Y| \geq b] = \frac{\sigma_Y^2}{b^2}$.

(c) Let Z be a non-negative random variable with mean 10 and standard deviation 2. Sketch the Markov and Chebyshev upper bounds on $P[Z \geq z]$ as a function of z for $z > 10$. Over what region does the Markov bound give the tighter upper bound? Over what region does the Chebyshev bound give the lower upper bound?

5. Suppose we know that a sequence with terms X_n is convergent in probability to a constant c. Does that mean that the limit of $\text{var}(X_n)$ as n goes to infinity must be zero? To help us think about this, consider the PMF for the members of the sequence:

$$p_{X_n}(x) = \begin{cases} 1 - \frac{1}{n}, & x = 0, \\ \frac{1}{n}, & x = n. \end{cases}$$

(a) Check whether this sequence is convergent in probability.

(b) Determine the behavior of the variance of X_n as n grows.

G1†. A sequence X_n of random variables is said to converge to a number c **in the mean square**, if

$$\lim_{n \to \infty} E[(X_n - c)^2] = 0$$

(a) Show that convergence in the mean square implies convergence in probability.

(b) Give an example that shows that convergence in probability does not imply convergence in the mean square.

†Required for 6.431; optional for 6.041