Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 8 Solutions
Due November 18, 2009

1. The described process for cars passing the checkpoint is a Poisson process with an arrival rate of $\lambda = 2$ cars per minute.

   (a) The first and third moments are, respectively,

   $$\mathbf{E}[T] = \frac{1}{\lambda} = \frac{1}{2} \qquad \mathbf{E}[T^3] = \int_0^\infty t^3 2e^{-2t} dt = \frac{3!}{2^3} \underbrace{\int_0^\infty \frac{2^4 t^3 e^{-2t}}{3!} dt}_{=1} = \frac{3}{4}$$

   where we recognized the integrand to be a 4th-order Erlang PDF and therefore its integral over the entire range of the random variable must be unity.

   (b) The Poisson process is memoryless, and thus the history of events in the previous 4 minutes does not affect the future. So, the conditional PMF for K is equivalent to the unconditional PMF that describes the number of Poisson arrivals in an interval of time, which in this case is $\tau = 6$ minutes and thus $(\lambda\tau) = 12$:

   $$p_K(k) = \frac{12^k e^{12}}{k!} \ , \quad k = 0, 1, 2, \ldots \qquad .$$

   (c) The first dozen computer cards are used up upon the 36th car arrival. Letting D denote this total time, $D = T_1 + T_2 + \ldots + T_{36}$, where each independent T_i is exponentially distributed with parameter $\lambda = 2$, the distribution for D is therefore a 36th-order Erlang distribution with PDF

   $$f_D(d) = \frac{2^{36} d^{35} e^{-2d}}{35!} \ , \quad d \geq 0 \ .$$

   and expected value

   $$\mathbf{E}[D] = 36\mathbf{E}[T] = 18 \ .$$

   (d) In both experiments, because a card completes after registering three cars, we are considering the amount of time it takes for three cars to pass the checkpoint. In the second experiment, however, note that the manner with which the particular card is selected is biased towards cards that are in service longer. That is, the time instant at which we come to the corner is more likely to fall within a longer interarrival period – one of the three interarrival times that adds up to the total time the card is in service is selected by *random incidence* (see the end of Section 6.2 in text).

   i. The service time of any particular completed card is given by $Y = T_1 + T_2 + T_3$, and thus Y is described by a 3rd-order Erlang distribution with parameter $\lambda = 2$:

   $$\mathbf{E}[Y] = \frac{3}{\lambda} = \frac{3}{2} \qquad \mathrm{var}(Y) = \frac{3}{\lambda^2} = \frac{3}{4} \quad .$$

   ii. The service time of a particular completed card with one of the three interarrival times selected by random incidence is $W = T_1 + T_2 + L$, where L is the interarrival period that contains the time instant we arrived at the corner. Following the arguments in the text, L is Erlang of order two and thus W is described by a 4th-order Erlang distribution with parameter $\lambda = 2$:

   $$\mathbf{E}[W] = \frac{4}{\lambda} = 2 \qquad \mathrm{var}(W) = \frac{4}{\lambda^2} = 1 \quad .$$

2. The event $\{X < Y < Z\}$ can be expressed as $\{X < \min\{Y,Z\}\} \cap \{Y < Z\}$. Let Y and Z be the 1st arrival times of two independent Poisson processes with rates μ and ν. By merging the two processes, it should be clear that $Y < Z$ if and only if the first arrival of the merged process comes from the original process with rate μ, and thus

$$\mathbf{P}(Y < Z) = \frac{\mu}{\mu+\nu}\ .$$

Let X be the 1st arrival time of a third independent Poisson process with rate λ. Now $\{X < \min\{Y,Z\}\}$ if and only if the first arrival of the Poisson process obtained by merging the two processes with rates λ and $\mu+\nu$ comes from the original process with rate λ, and thus

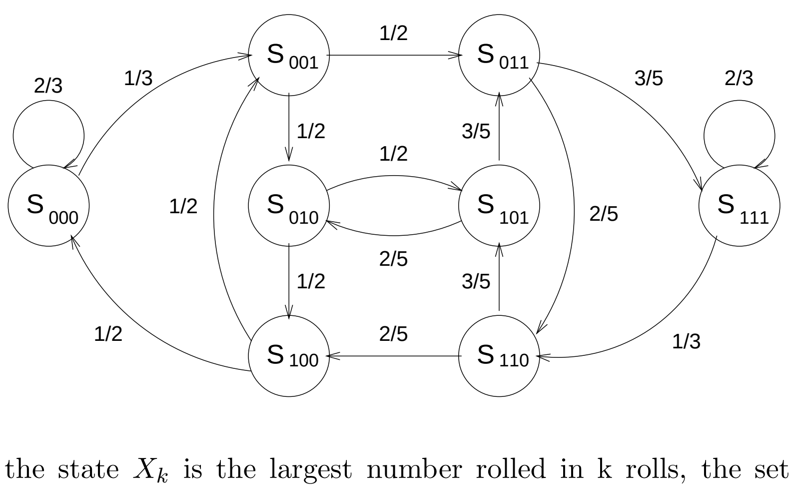
$$\mathbf{P}(X < \min\{Y,Z\}) = \frac{\lambda}{\lambda+\mu+\nu}\ .$$

Note that the event $\{X < \min\{Y,Z\}\}$ is independent of the event $\{Y < Z\}$, as the time of the first arrival of the merged process with rate $\mu+\nu$ is independent of whether that first arrival comes from the process with rate μ or the process with rate ν. Hence,

$$\begin{aligned}\mathbf{P}(X < Y < Z) &= \mathbf{P}(X < \min\{Y,Z\}) \cdot \mathbf{P}(Y < Z) \\ &= \frac{\lambda\mu}{(\lambda+\mu+\nu)(\mu+\nu)}\ .\end{aligned}$$

3. Since the probability of success depends on the results of the previous three trials, we need a separate state for every possible result in the previous three trials. Therefore we need $2^3 = 8$ states. We label each state S_{ijl}, where the triplet (i,j,l) represents the result of the last three trials, with l being the most recent trial. Each component of the triplet has a value of 1 if the trial was successful, and a value of 0 if the trial was unsuccessful.

Now we can easily draw the state transition diagram. Note that the transition between any two states is $\frac{k+1}{k+3}$ where k is the number of successes in the last three trials, if the transition is leading to a success, and $\frac{2}{k+3}$ if the transition is leading to a failure.



4. (a) i. Since the state X_k is the largest number rolled in k rolls, the set of states $S = \{1,2,3,4,5,6\}$. The probability of the largest number rolled in the first $(k+1)$

trials is only dependent to the what the largest number that was rolled in the first$k$ trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} 0 & , \quad j < i \\ \frac{i}{6} & , \quad j = i \\ \frac{1}{6} & , \quad j > i \end{cases}$$

ii. Since the state X_k is the number of sixes in the first k rolls, the set of states $S = \{0, 1, 2, ...\}$. The probability of getting a six in a given trial is 1/6. The number of sixes rolled in the first $(k + 1)$ trials is only dependent to the number of sixes rolled in the first k trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = i + 1 \\ \frac{5}{6} & , \quad j = i \\ 0 & , \quad \text{otherwise} \end{cases}$$

iii. Since the state X_k is the number of rolls since the most recent six, the set of states $S = \{0, 1, 2, ...\}$. If the roll of the die is 6 on the next trial the chain goes to state 0. If not, the state goes to the next higher state. Therefore, the probability of the next state depends on the past only through the present state. Clearly, this satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = 0 \\ \frac{5}{6} & , \quad j = i + 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(b) i. For $X_k = Y_{r+k}$, and by the Markov property of Y

$$\begin{aligned} \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i, ...Y_r = i_r) \\ &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i) \\ \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. Also we can see that, X_k is a delayed process by r of Y_k. Therefore, they should have the same transition probability p_{ij}, So, we have:

$$p_{ij} = q_{ij} \; .$$

ii. For $X_k = Y_{2k}$, and by the Markov property of Y

$$\begin{aligned} &\mathbf{P}(X_{k+1} = j | X_k = i, X_{k-1} = i_{k-1}.....X_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i, Y_{2k-2} = i_{2k-2}.....Y_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. The transition probabilities p_{ij} are given by:

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{k+1} = j | X_k = i) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= r^y_{ij}(2) \end{aligned}$$

where $r_{ij}^{y}(n)$ is the n step transition probability of Y.

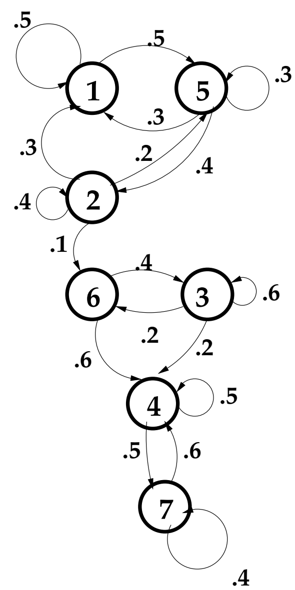
iii.

$$\begin{aligned}
&\mathbf{P}(X_{k+1} = (n,l) | X_0 = (i_0, i_1), X_1 = (i_1, i_2), ..., X_k = (i_k, n)) \\
&= \mathbf{P}(X_{k+1} = (n,l) | Y_0 = i_0, Y_1 = i_1, Y_2 = i_2 ...., Y_k = i_k, Y_{k+1} = n) \\
&= \mathbf{P}(X_{k+1} = (n,l) | Y_{k+1} = n) \\
&= \mathbf{P}(X_{k+1} = (n,l) | X_k = (i_k, n))
\end{aligned}$$

Letting $i = (i_k, i_{k+1})$ and $j = (n, l)$, the transition probabilities p_{ij} are given by:

$$p_{ij} = \mathbf{P}(X_{k+1} = (n,l) | X_k = (i_k, i_{k+1})) = \begin{cases} q_{nl} & , \quad i_{k+1} = n \\ 0 & , \quad i_{k+1} \neq n \end{cases}$$

5. Let i ($i = 1, \ldots, 7$) be the states of the Markov chain. From the graphical representation of the transition matrix it is easy to see the following:



   (a) $\{4, 7\}$ are recurrent and the rest are transient. All states are aperiodic.
   (b) There is only one class formed by the recurrent states.

6. (a) Let W_{t+1} denote the number of newly infected individuals at time t. Let I_t and S_t denote the number of infected and susceptible individuals, respectively, at time t. We are asked to compute $p_{W_{t+1}|I_t,S_t}(w|k,s)$.
   Each susceptible individual becomes infected if he/she has contacts with more than two infected individuals (which occur independently with probability p). Thus for a given individual,

$$\begin{aligned}
\mathbf{P}(\text{infected at } t+1 | I_t = k, \text{ susceptible at } t) &= \sum_{l=2}^{k} \binom{k}{l} p^l (1-p)^{k-l} \\
&= \begin{cases} 1 - (1-p)^k - kp(1-p)^{k-1} & , \ k \geq 2 \\ 0 & , \ k < 2 \end{cases}
\end{aligned}$$

Letting $\rho(k) = 1 - (1-p)^k - kp(1-p)^{k-1}$, we then can compute our conditional PMF,

$$p_{W_{t+1}|I_t,S_t}(w|k,s) = \begin{cases} \binom{s}{w}\rho(k)^w(1-\rho(k))^{s-w} & , \quad k \geq 2 \\ 0 & , \quad k < 2 \end{cases} .$$

(b) Each state can be characterized by the number of infected individuals, I, and the number of susceptible individuals, S. Note that the sum of the number of infected and susceptible individuals can be any integer between 0 and n. There are a total of $l+1$ distinct states whereby $I + S = l$, and hence, the minimum total number of states must be

$$1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2} .$$

(c) Any state with $I = 0$ is recurrent, and thus there are $n+1$ recurrent states. Each of these recurrent states is its own recurrent class.

(d) Each state is identified with a particular assignment of I and S, the total number of infected and susceptible individuals, respectively.