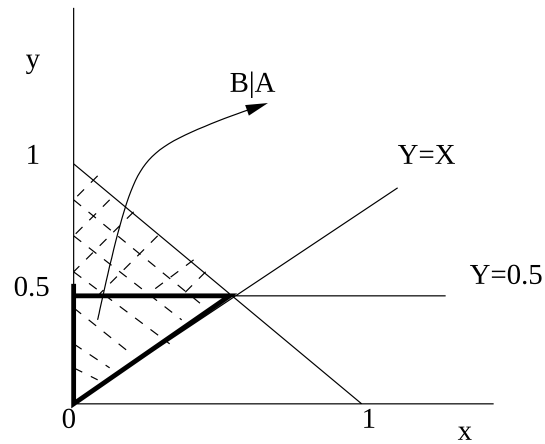
Problem Set 6: Solutions
Due October 28, 2009

1. (a)

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \mid B)\mathbf{P}(B)}{\mathbf{P}(A)} = \frac{(.5)(.5)}{.75} = \frac{1}{3}$$

Another way of seeing this is to realize that in the universe of A, $f_{X,Y}(x,y)$ is still uniform and from figure shown $(B \mid A)$ is $\frac{1}{3}$ of (A).



(b)

$$f_{X|Y}(x \mid 0.5) = \frac{f_{X,Y}(x, 0.5)}{f_Y(0.5)}$$

$$\begin{aligned} f_Y(y) &= \int_0^{1-y} f_{X,Y}(x,y)\,dx \\ &= \int_0^{1-y} 2dx \\ &= 2 - 2y \end{aligned}$$

$$f_Y(0.5) = 2 - 2(0.5) = 1$$

$$f_{X|Y}(x \mid 0.5) = \begin{cases} 2 & 0 \le x \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Another way to think about this is to first take a slice of the joint PDF at $y = 0.5$ and normalize the cross section such that the area under the curve is 1.

By symmetry, we see that

$$\mathbf{E}[X \mid Y = 0.5] = 0.25$$

$$\begin{aligned} \text{var}(X \mid Y = 0.5) &= \mathbf{E}\left[(X - \mathbf{E}[X \mid Y = 0.5])^2\right] \\ &= \int_0^{0.5} f_{X|Y}(x \mid 0.5)(x - 0.25)^2\,dx \\ &= \int_0^{0.5} 2(x - .25)^2\,dx \\ &= \frac{1}{48} \end{aligned}$$

Alternatively, realize that for a uniform random variable X with value $\frac{1}{b-a}$ in the region $a \le x \le b$, the variance is $\frac{(b-a)^2}{12}$. So

$$\text{var}(X \mid Y = 0.5) = \frac{(0.5-0)^2}{12} = \frac{1}{48}$$

(c)

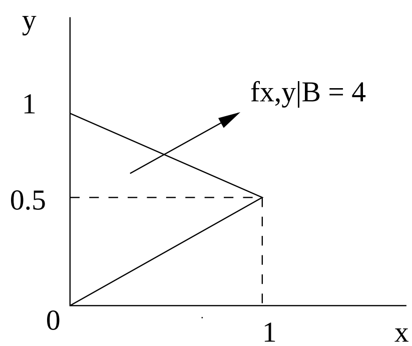
$$f_{X|B}(x \mid B) = \int_{Y \in B} f_{X,Y|B}(x, y \mid B)\, dy, \quad x \in B,$$

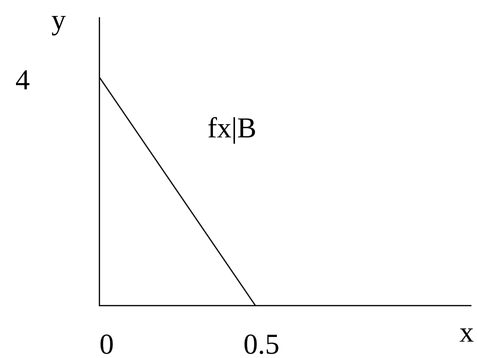
$$f_{X,Y|B}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{\mathbf{P}(B)}, & x, y \in B, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{X,Y|B}(x,y) = \begin{cases} \frac{2}{0.5} = 4, & x > 0, x + y \le 1, y > x, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{X|B}(x \mid B) = \int_x^{1-x} 4dy, \quad 0 \le x \le 0.5,$$

$$f_{X|B}(x \mid B) = \int_x^{1-x} 4\, dy = 4(1-x-x) = \begin{cases} 4 - 8x, & 0 \le x \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

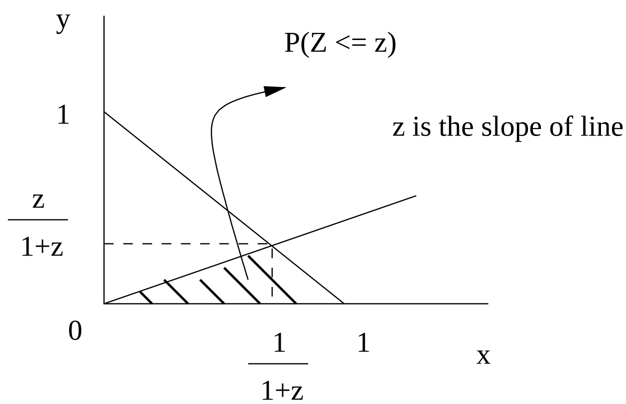




(d)

$$\begin{aligned} \mathbf{E}[XY] &= \int_0^1 \int_0^{1-y} xy f_{X,Y}(x,y)\, dx\, dy \\ &= \int_0^1 \int_0^{1-y} 2xy\, dx\, dy \\ &= \int_0^1 (y - 2y^2 + y^3)\, dy \\ &= \frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4}\Big|_0^1 \\ &= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

(e) Let $Z = \frac{Y}{X}$



$$
\begin{aligned}
F_Z(z) &= \mathbf{P}(Z \le z) \\
&= \frac{z}{1+z} \cdot \frac{1}{2} \cdot 1 \cdot 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dz} F_Z(z) &= \frac{-z}{(1+z)^2} + \frac{1}{z+1} \\
&= \frac{-z+1+z}{(1+z)^2} \\
&= \frac{1}{(1+z)^2}
\end{aligned}
$$

$$
f_Z(z) = \begin{cases} \frac{1}{(1+z)^2} & z \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

2. To find the joint PDF of S and L, we first find their joint CDF and then take the second-order cross derivative, as shown on page 162 of the text.
   Let's consider the event $\{S \ge s, L \le \ell\}$, which corresponds to the smallest number being at least s and the largest number being at most ℓ. In other words, all n random variables fall between s and ℓ. Since the X_i's are independent, this is equal to the product of the probabilities of each X_i falling between s and ℓ.

$$
\mathbf{P}(S \ge s, L \le \ell) = \prod_{i=1}^{n} \mathbf{P}(s \le X_i \le \ell) = \prod_{i=1}^{n} (\ell - s) = (\ell - s)^n, \quad 0 \le s \le \ell \le 1.
$$

   Since the event $\{L \le \ell\}$ is equal to the event $\{S \le s, L \le \ell \cup S \ge s, L \le \ell\}$, we have

$$
F_{S,L}(s,\ell) = \mathbf{P}(S \le s, L \le \ell) = \mathbf{P}(L \le \ell) - \mathbf{P}(s \ge s, L \le \ell) = \ell^n - (\ell - s)^n,
$$

   which we can then differentiate to find the joint PDF:

$$
\begin{aligned}
f_{S,L}(s,\ell) &= \frac{\partial^2 F_{S,L}}{\partial s \partial \ell}(s,\ell) \\
&= \frac{\partial^2 (\ell^n - (\ell - s)^n)}{\partial s \partial \ell} \\
&= \frac{\partial(-n(\ell - s)^{n-1}(-1))}{\partial \ell} \\
&= \begin{cases} n(n-1)(\ell - s)^{n-2}, & 0 \le s \le \ell \le 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

3. (a) Y can take on values in the interval $[0, 2)$, but values in the interval $[0, 1]$ are twice as likely as values in the interval $(1, 2)$ because in the former case, both positive and negative values of X are mapped to the same value of Y. Thus, we have

$$f_Y(y) = \begin{cases} \frac{2}{3}, & 0 \le y \le 1, \\ \frac{1}{3}, & 1 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since X can take on only positive values, $Y = X$ in this case and so the PDF of Y is the same as the PDF of X:

$$f_Y(y) = \begin{cases} 2e^{-2y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) In the general case, we can use our usual method of calculating derived distributions via differentiating the CDF. For $y \ge 0$, we have

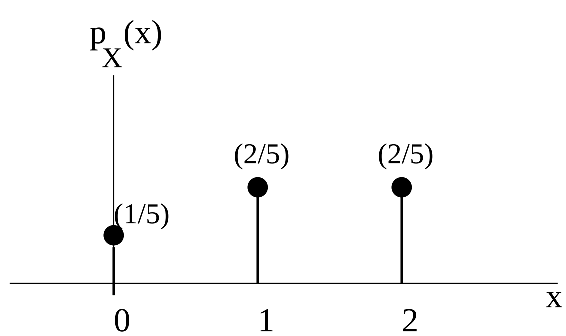
$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(|X| \le y) = \mathbf{P}(-y \le X \le y) = F_X(y) - F_X(-y),$$

which we can then differentiate to obtain

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & y \ge 0, \\ 0, & y < 0. \end{cases}$$

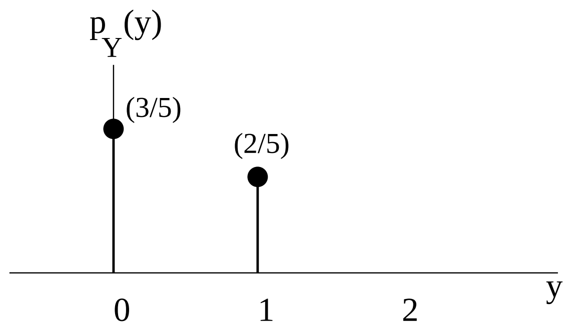
4. (a)

$$\begin{aligned} p_X(x) &= \sum_{y=0}^{1} p_{X,Y}(x, y) \\ &= \begin{cases} 1/5, & x = 0, \\ 2/5, & x = 1, \\ 2/5, & x = 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$



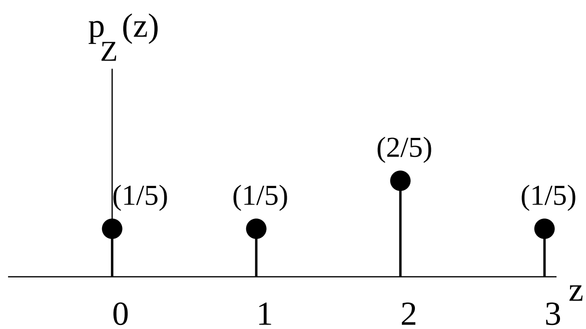
(b)

$$
\begin{aligned}
p_Y(y) &= \sum_{x=0}^{2} p_{X,Y}(x,y) \\
&= \begin{cases} 3/5, & y = 0, \\ 2/5, & y = 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$



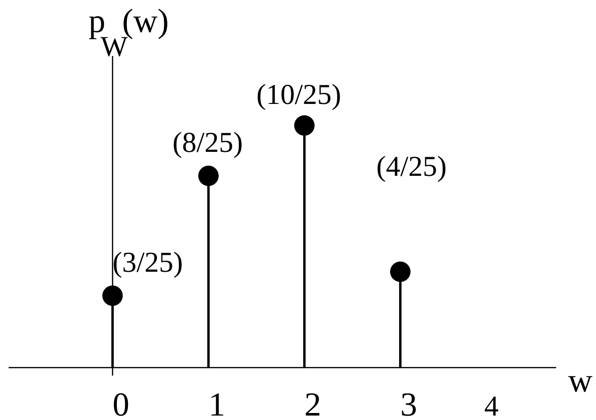
(c)

$$
\begin{aligned}
p_Z(z) &= \sum_{x,y|z=x+y} p_{X,Y}(x,y) \\
&= \begin{cases} 1/5, & z = 0, \\ 1/5, & z = 1, \\ 2/5, & z = 2, \\ 1/5, & z = 3, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$



(d)

$$\sum_x p_X(x)p_Y(w-x) = \begin{cases} 3/25, & w=0, \\ 8/25, & w=1, \\ 10/25, & w=2, \\ 4/25, & w=3, \\ 0, & \text{otherwise.} \end{cases}$$



The convolution differs from the PMF of Z found in part (c) because X and Y are dependent random variables, hence the PMF for their sum is not the convolution of their individual PMFs.

5. Let $W = Y - Z$ and $X = |Y - Z|$. We find the PDF of W by convolution of the exponential with parameter 1 with the uniform in the interval $[-1, 0]$. We obtain

$$f_W(w) = \begin{cases} \displaystyle\int_w^{w+1} e^{-x}dx = e^{-w} - e^{-(w+1)}, & \text{if } w \geq 0, \\ \displaystyle\int_0^{w+1} e^{-x}dx = 1 - e^{-(w+1)}, & \text{if } -1 \leq w \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We then note that

$$f_X(x) = \begin{cases} f_W(x) + f_W(-x), & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and obtain the PDF of X. The validity of the above formula is established by using the relation

$$F_X(x) = \mathbf{P}(|W| \leq x) = \mathbf{P}(-x \leq W \leq x) = F_X(x) - F_X(-x),$$

and differentiating with respect to x.

6. Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = \frac{1}{k} \cdot \frac{1}{k} \qquad \text{for} \quad s \neq t.$$

Thus,

$$\begin{aligned}
\mathbf{E}[X_1X_2] &= \mathbf{E}\left[(A_1+\cdots+A_n)(B_1+\cdots+B_n)\right] \\
&= n\mathbf{E}\left[A_1(B_1+\cdots+B_n)\right] \;=\; n(n-1)\cdot\frac{1}{k}\cdot\frac{1}{k}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{cov}(X_1,X_2) &= \mathbf{E}[X_1X_2]-\mathbf{E}[X_1]E[X_2] \\
&= \frac{n(n-1)}{k^2}-\frac{n^2}{k^2} \;=\; -\frac{n}{k^2}.
\end{aligned}$$

7. By linearity of expectations, we calculate that

$$\mathbf{E}[Y] = a + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] = a + b\cdot 1 + c\cdot 0 = a+b,$$

and that

$$\mathbf{E}[XY] = a\mathbf{E}[X] + b\mathbf{E}[X^3] + c\mathbf{E}[X^4] = a\cdot 0 + b\cdot 0 + c\cdot 3 = 3c,$$

and so

$$\mathrm{cov}(X,Y) = \mathbf{E}[XY]-\mathbf{E}[X]\mathbf{E}[Y] = 3c - 0\cdot(a+b) = 3c.$$

We can also calculate that

$$\mathrm{var}(X) = \mathbf{E}[X^2]-(\mathbf{E}[X])^2 = 1-0 = 1.$$

and that

$$\begin{aligned}
\mathrm{var}(Y) &= \mathbf{E}\left[(Y-\mathbf{E}[Y])^2\right] \\
&= \mathbf{E}\left[(b(X^2-1)+cX^3)^2\right] \\
&= \mathbf{E}\left[b^2(X^2-1)^2+2bc(X^5-X^3)+c^2X^6\right] \\
&= b^2\mathbf{E}[X^4-2X^2+1]+2bc(\mathbf{E}[X^5]-\mathbf{E}[X^3])+c^2\mathbf{E}[X^6] \\
&= b^2(3-2\cdot 1+1)+2bc(0-0)+c^2\cdot 15 \\
&= 2b^2+15c^2,
\end{aligned}$$

and so

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{3c}{\sqrt{1\cdot(2b^2+15c^2)}} = \frac{3c}{\sqrt{2b^2+15c^2}}.$$

Note that $0 < |\rho(X,Y)| < 1$ because b and c are nonzero constants.

8. **A financial parable.**

   (a) The bank becomes insolvent if the asset's gain $R \le -5$ (i.e., it loses more than 5%). This probability is the CDF of R evaluated at -5. Since R is normally distributed, we can convert this CDF to be in terms of a standard normal random variable by subtracting away the mean and dividing by the standard deviation, and then look up the value in a standard normal CDF table.

$$\begin{aligned}
\mathbf{E}[R] &= 7, \\
\text{var}(R) &= 10^2 = 100, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{10} \leq \frac{-5-7}{10}\right) = \Phi(-1.2) \approx 0.115.
\end{aligned}$$

Thus, by investing in just this one asset, the bank has a 11.5% chance of becoming insolvent.

(b) If we model the R_i's as **independent** normal random variables, then their sum $R = (R_1 + \cdots + R_{20})/20$ is also a normal random variable (see Example 4.11 on page 214 of the text). Thus, we can calculate the mean and variance of this new R and proceed as in part (a). Note that since the random variables are assumed to be independent, the variance of their sum is just the sum of their individual variances.

$$\begin{aligned}
\mathbf{E}[R] &= (\mathbf{E}[R_1] + \cdots + \mathbf{E}[R_{20}])/20 = 7, \\
\text{var}(R) &= \frac{1}{20^2}(\text{var}(R_1) + \cdots + \text{var}(R_{20})) = \frac{20 \cdot 100}{400} = 5, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{\sqrt{5}} \leq \frac{-5-7}{\sqrt{5}}\right) = \Phi(-5.367) \approx 0.0000000439 = 4.39 \cdot 10^{-8}.
\end{aligned}$$

Thus, by diversifying and assuming that the 20 assets have **independent** gains, the bank has seemingly decreased its probability of becoming insolvent to a palatable value.

(c) Now, if the gains R_i are positively correlated, then we can no longer sum up the individual variances; we need to account for the covariance between pairs of random variables. The covariance is given by

$$\text{cov}(R_i, R_j) = \rho(R_i, R_j)\sqrt{\text{var}(R_i)\text{var}(R_j)} = \frac{1}{2}\sqrt{10^2 \cdot 10^2} = 50.$$

From page 220 in the text, we know that the variance in this case is

$$\begin{aligned}
\text{var}(R) &= \text{var}\left(\frac{1}{20}\sum_{i=1}^{20} R_i\right) = \frac{1}{400}\left(\sum_{i=1}^{20}\text{var}(R_i) + \sum_{\{(i,j)|i\neq j\}}\text{cov}(R_i, R_j)\right) \\
&= \frac{1}{400}(20 \cdot 100 + 380 \cdot 50) = 52.5.
\end{aligned}$$

Since we assume that $R = (R_1 + \cdots + R_{20})/20$ is still normal, we can again apply the same steps as in parts (a) and (b):

$$\begin{aligned}
\mathbf{E}[R] &= (\mathbf{E}[R_1] + \cdots + \mathbf{E}[R_{20}])/20 = 7, \\
\text{var}(R) &= 52.5, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{\sqrt{52.5}} \leq \frac{-5-7}{\sqrt{52.5}}\right) = \Phi(-1.656) \approx 0.0488.
\end{aligned}$$

Thus, by taking into account the positive correlation between the assets' gains, we are no longer as comfortable with the probability of insolvency as we thought we were in part (b).