Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2011)

Problem Set 7
Due: April 4, 2011

1. Consider a sequence of mutually independent, identically distributed, probabilistic trials. Any particular trial results in either a success (with probability p) or a failure.

   (a) Obtain a simple expression for the probability that the ith success occurs before the jth failure. You may leave your answer in the form of a summation.

   (b) Determine the expected value and variance of the number of successes which occur before the jth failure.

   (c) Let L_{17} be described by a Pascal PMF of order 17. Find the numerical values of a and b in the following equation. Explain your work.

$$\sum_{l=42}^{\infty} p_{L_{17}}(l) = \sum_{x=0}^{a} \binom{b}{x} p^x (1-p)^{(b-x)}$$

2. The probability that Iwana Passe fails any quiz is $\frac{1}{4}$. Iwana's performance on each quiz is independent of her performance on all other quizzes.

   (a) Determine the probability that Iwana fails exactly two of the next six quizzes.

   (b) Find the expected number of quizzes that Iwana will pass before she has failed three times.

   (c) Find the probability that the second and third time Iwana fails a quiz will occur when she takes her 8^{th} and 9^{th} quizzes, respectively.

   (d) Determine the probability that Iwana fails two quizzes in a row before she passes two quizzes in a row. *Hint: Write the event of interest as the union of two events, depending on whether or not she passes or fails the first quiz.*

3. Iwana Passe is taking a multiple-choice exam. You may assume that the number of questions is infinite. *Simultaneously, but independently,* her conscious and subconscious faculties are generating answers for her, each in a Poisson manner. (Her conscious and subconscious are always working on different questions.) Conscious responses are generated at the rate λ_c responses per minute. Subconscious responses are generated at the rate λ_s responses per minute. Assume $\lambda_c \neq \lambda_s$. Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct. Iwana responds only once to each question, and you can assume that her time for recording these conscious and subconscious responses is negligible.

   (a) Determine $p_K(k)$, the probability mass function for the number of *conscious responses* Iwana makes in an interval of T minutes.

   (b) If we pick any question to which Iwana has responded, what is the probability that her answer to that question:

      i. Represents a conscious response

      ii. Represents a conscious correct response

   (c) If we pick an interval of T minutes, what is the probability that in that interval Iwana will make exactly r conscious responses *and* s subconscious responses?

(d) The papers are to be collected as soon as Iwana has completed exactly N responses. Determine:

i. The expected number of questions she will answer correctly

ii. The probability mass function for L, the number of questions she answers correctly.

(e) Repeat part (f) for the case in which the exam papers are to be collected at the end of a fixed interval of T minutes.

4. Let T_1 and T_2 be exponential random variables with parameter λ, and let S be an exponential random variable with parameter μ. We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1+T_2, S\}$. *Hint:* See Problem 6.19 in the text.

5. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geq 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

$$\mathbf{P}\Big(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \,|\, N(G) = k\Big) .$$

G1†. Consider a poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t)$ and M be the number of arrivals in $(0, t + s)$.

(a) Find the joint PMF of N and M, $p_{N,M}(n, m)$.

(b) Find $E[NM]$.

†Required for 6.431 students and optional (not graded) for 6.041.