Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2011)

Problem Set 5

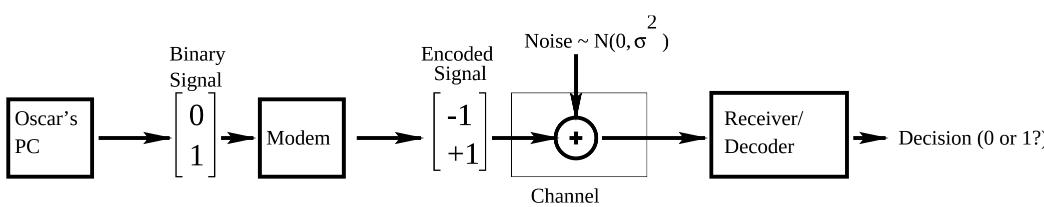
Due: March 14, 2011 (Note that this is a MONDAY.)

1. Often times, a particular process may produce a continuous random variable, yet the more interesting, or more pertinent information, may be some discretized version of the ouput. As a simple example, imagine a plane that flies one hour journeys. We may know that the engine's lifetime has a particular distribution, but what we really care about is the distribution of the trips-to-go until the engine dies. The "Greatest Integer" function maps from the real numbers to the integers. Given any number, its output is the greatest integer less than that number, so for example, Int(e)=2. Now suppose that X is exponentially distributed with rate $\lambda = 1$. Find $\mathbf{E}[\text{Int}(X)]$.

   Hint 1: Start with the definition of expectation. Rewrite the integral as an infinite sum and an integral with finite boundaries.

   Hint 2: Use properties and manipulations of infinite geometric series to find the value of the infinite sum.

2. Oscar's modem transmits bits in such a fashion that -1 is sent if a given bit is zero and $+1$ is sent if a given bit is one. The channel has additive zero-mean Gaussian (normal) noise with variance σ^2 (so, the receiver on the other end gets a signal which is the sum of the transmitted signal and the channel noise). The value of the noise is assumed to be independent of the encoded signal value.



   We assume that the probability of the modem sending -1 is p and the probability of sending 1 is $1 - p$.

   (a) Suppose we conclude that an encoded signal of -1 was sent when the value received on the other end of the line is less than a (where $-1 < a < +1$), and conclude $+1$ was sent when the value is more than a. What is the probability of making an error?

   (b) Answer part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

3. To help find a pile of gold buried in a swamp, a 6.041 graduate uses a satellite scan to compute the following a priori probability density function for the X and Y coordinates of the treasure:

$$f_{X,Y}(x,y) = \begin{cases} Qx^2y & \text{if } 0 \le x \le 10,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   where Q is a constant. Just then, a librarian rushes in with a fragment of a treasure map. The map says that the X coordinate of the gold is either 6 or 8, but the number is so smudged that the two treasure hunters decides that both possibilities are equally likely.

   Using this information, calculate and sketch the probability density function for the Y coordinate of the gold's location.

4. Random variables X and Y are independent and are described by the probability density functions $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \begin{cases} 1, & 0 < x \leq 1; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad f_Y(y) = \begin{cases} 1, & 0 < y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Stations A and B are connected by two *parallel* message channels. One message from A to B is sent over each of the channels at the same time. Random variables X and Y represent the message delays in hours over parallel channels 1 and 2, respectively.

A message is considered "received" as soon as it arrives on any one channel and it is considered "verified" as soon as it has arrived over both channels.

(a) Determine the probability that a message is received within 15 minutes after it is sent.

(b) Determine the probability that the message is received but not verified within 15 minutes after it is sent.

(c) Let T represent the time in hours between transmission at A and verification at B. Determine the CDF $F_T(t)$, and then differentiate it to obtain the PDF $f_T(t)$.

(d) If the attendant at B leaves for a 15-minute coffee break right after the message is received, what is the probability that he is present at the proper time for verification?

(e) The management wishes to have the maximum probability of having the attendant present for *both* reception and verification. Would they do better to let him take his coffee break as described above or simply allow him to go home 45 minutes after transmission?

G1†. Suppose n runners run a race. At one point, all the runners are uniformly distributed on a stretch of one mile that starts at point A and ends at point B. Number the runners from 1 to n, according to the order in which they are at this particular moment. Let X_i denote the distance between point A and the ith runner and let $X_0 = 0$ and $X_{n+1} = 1$.

(a) For $n = 2$, find $\mathbf{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, 3$ and $0 < t < 1$.

(b) For $n \geq 1$, find $\mathbf{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, \ldots, n+1$ and $0 < t < 1$. (Hint: The answer does not depend on i.)

(c) Find the expected distance between the runners. In other words, find $\mathbf{E}(|X_i - X_{i-1}|)$ where $i = 1, 2, \ldots, n+1$.

†Required for 6.431; optional for 6.041