6.041/6.431 Fall 2010 Quiz 2 Solutions

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.

$$\begin{aligned}\mathbf{E}[X-3Y] &= \mathbf{E}[X] - 3\mathbf{E}[Y] \\ &= 2 - 3\cdot\frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\text{var}(X-3Y) &= \text{var}(X) + 9\text{var}(Y) \\ &= \frac{(4-0)^2}{12} + 9\cdot\frac{1}{2^2} \\ &= \frac{43}{12}.\end{aligned}$$

2. **(10 points)** Find the probability that $Y \geq X$.
(Let c be the answer to this question.)

The PDFs for X and Y are:

$$f_X(x) = \begin{cases} 1/4, & \text{if } 0 \leq x \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} 2e^{-2y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Using the total probability theorem,

$$\begin{aligned}\mathbf{P}(Y \geq X) &= \int_x f_X(x)\mathbf{P}(Y \geq X \mid X = x)\, dx \\ &= \int_0^4 \frac{1}{4}(1 - F_Y(x))\, dx \\ &= \int_0^4 \frac{1}{4}e^{-2x}\, dx \\ &= \frac{1}{8}\int_0^4 2e^{-2x}\, dx \\ &= \frac{1}{8}(1 - e^{-8}).\end{aligned}$$

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
(You may express your answer in terms of the constant c from the previous part.)

Let A be the event that $Y \geq X$. Since X and Y are independent,

$$\begin{aligned} f_{X,Y|A}(x,y) &= \frac{f_{X,Y}(x,y)}{\mathbf{P}(A)} = \frac{f_X(x)f_Y(y)}{\mathbf{P}(A)} \text{ for } (x,y) \in A \\ &= \begin{cases} \frac{4e^{-2y}}{1-e^{-8}}, & \text{if } 0 \leq x \leq 4,\ y \geq x \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

4. **(10 points)** Find the PDF of $Z = X + Y$.

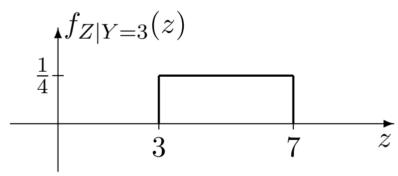
Since X and Y are independent, the convolution integral can be used to find $f_Z(z)$.

$$\begin{aligned} f_Z(z) &= \int_{\max(0,z-4)}^{z} \frac{1}{4} 2e^{-2t}\, dt \\ &= \begin{cases} 1/4 \cdot \left(1 - e^{-2z}\right), & \text{if } 0 \leq z \leq 4, \\ 1/4 \cdot \left(e^{8} - 1\right) e^{-2z}, & \text{if } z > 4, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.

Given that $Y = 3$, $Z = X + 3$ and the conditional PDF of Z is a shifted version of the PDF of X. The conditional PDF of Z and its sketch are:

$$f_{Z|\{Y=3\}}(z) = \begin{cases} 1/4, & \text{if } 3 \leq z \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$



6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.

The conditional PDF $f_{Z|Y=y}(z)$ is a uniform distribution between y and $y + 4$. Therefore,

$$\mathbf{E}[Z \mid Y = y] = y + 2.$$

The above expression holds true for all possible values of y, so

$$\mathbf{E}[Z \mid Y] = Y + 2.$$

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

The joint PDF of Z and Y can be expressed as:

$$\begin{aligned} f_{Z,Y}(z,y) &= f_Y(y) f_{Z|Y}(z \mid y) \\ &= \begin{cases} 1/2 \cdot e^{-2y}, & \text{if } y \geq 0,\ y \leq z \leq y + 4, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

   Let X be a Bernoulli random variable for the result of the fair coin where $X = 1$ if the coin lands "heads". Because the coin is fair, $\mathbf{P}(X = 1) = \mathbf{P}(X = 0) = 1/2$. Furthermore, the conditional PDFs of W given the value of X are:

$$\begin{aligned} f_{W|X=1}(w) &= f_Y(w) \\ f_{W|X=0}(w) &= f_Y(w-2). \end{aligned}$$

   Using the appropriate variation of Bayes' Rule:

$$\begin{aligned} \mathbf{P}(X = 1 \mid W = 3) &= \frac{\mathbf{P}(X = 1) f_{W|X=1}(3)}{\mathbf{P}(X = 1) f_{W|X=1}(3) + \mathbf{P}(X = 0) f_{W|X=0}(3)} \\ &= \frac{\mathbf{P}(X = 1) f_Y(3)}{\mathbf{P}(X = 1) f_Y(3) + \mathbf{P}(X = 0) f_Y(1)} \\ &= \frac{\mathbf{P}(X = 1) f_Y(3)}{\mathbf{P}(X = 1) f_Y(3) + \mathbf{P}(X = 0) f_Y(1)} \\ &= \frac{e^{-6}}{e^{-6} + e^{-2}}. \end{aligned}$$

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \le |X| \le 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.

$$\begin{aligned} \mathbf{P}(1 \le |X| \le 1 + \delta) &= 2\mathbf{P}(1 \le X \le 1 + \delta) \\ &\approx 2 f_X(1)\delta. \end{aligned}$$

   Therefore,

$$\begin{aligned} \alpha &= 2 f_X(1) \\ &= 2 \cdot \frac{1}{\sqrt{9 \cdot 2\pi}} e^{-\frac{1}{2} \cdot \frac{(1-0)^2}{9}} \\ &= \frac{2}{3\sqrt{2\pi}} e^{-\frac{1}{18}}. \end{aligned}$$

2. **(10 points)** Find the variance of S.

   Using the Law of Total Variance,

$$\begin{aligned} \text{var}(S) &= \mathbf{E}[\text{var}(S \mid N)] + \text{var}(\mathbf{E}[S \mid N]) \\ &= \mathbf{E}[9 \cdot N] + \text{var}(0 \cdot N) \\ &= 9\mathbf{E}[N] \;=\; 18. \end{aligned}$$

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

   The covariance of S and N is

$$\begin{aligned}
\text{cov}(S,N) &= \mathbf{E}[SN] - \mathbf{E}[S]\mathbf{E}[N] \\
&= \mathbf{E}[\mathbf{E}[SN \mid N]] - \mathbf{E}[\mathbf{E}[S \mid N]]\mathbf{E}[N] \\
&= \mathbf{E}[\mathbf{E}[\sum_{i=1}^{N} X_i N \mid N]] - \mathbf{E}[\mathbf{E}[\sum_{i=1}^{N} X_i \mid N]]\mathbf{E}[N] \\
&= \mathbf{E}[X_1]\mathbf{E}[N^2] - \mathbf{E}[X_1]\mathbf{E}[N] \\
&= 0
\end{aligned}$$

   since the $\mathbf{E}[X_1]$ is 0. Therefore, S and N are uncorrelated.

4. **(5 points)** Are N and S independent? Justify your answer.

   S and N are not independent.

   Proof: We have $\text{var}(S \mid N) = 9N$ and $\text{var}(S) = 18$, or, more generally, $f_{S|N}(s \mid n) = N(0, 9n)$ and $f_S(s) = N(0, 18)$ since a sum of an independent normal random variables is also a normal random variable. Furthermore, since $\mathbf{E}[N^2] = 5 \neq (\mathbf{E}[N])^2 = 4$, N must take more than one value and is not simply a degenerate random variable equal to the number 2. In this case, N can take at least one value (with non-zero probability) that satisfies $\text{var}(S \mid N) = 9N \neq \text{var}(S) = 18$ and hence $f_{S|N}(s \mid n) \neq f_S(s)$. Therefore, S and N are not independent.