6.041/6.431 Fall 2010 Quiz 2

Tuesday, November 2, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Question	Score	Out of
1.1		10
1.2		10
1.3		10
1.4		10
1.5		10
1.6		10
1.7		10
1.8		10
2.1		10
2.2		10
2.3		5
2.4		5
Your Grade		110

Name: ______________________________

Recitation Instructor: ______________________________

TA: ______________________________

- **For full credit, answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.**
- This quiz has 2 problems, worth a total of 110 points.
- You may tear apart page 3, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed two two-sided, handwritten, 8.5 by 11 formula sheets. Calculators are not allowed.
- You have 120 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/4.

Problem 0: (0 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Uzoma Orji	10 & 11 AM
Peter Hagelstein	Ahmad Zamanian	12 & 1 PM
Ali Shoeb	Shashank Dwivedi	2 PM
Dimitri Bertsekas (6.431)	Aliaa Atwi	2 & 3 PM

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.
2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)
3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
   (You may express your answer in terms of the constant c from the previous part.)
4. **(10 points)** Find the PDF of $Z = X + Y$.
5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.
6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.
7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.
8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.
2. **(10 points)** Find the variance of S.
3. **(5 points)** Are N and S uncorrelated? Justify your answer.
4. **(5 points)** Are N and S independent? Justify your answer.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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