LECTURE 12

- **Readings:** Section 4.3, parts of Section 4.5 (mean and variance only; no transforms)

Lecture outline

- Review and small correction
- Conditional expectation and variance revisited
  - Law of iterated expectations
  - Law of total variance
- Sum of a random number of independent RVs
  - mean, variance

Small correction: Correlation coefficient

- Dimensionless version of covariance:

$$\rho = \mathbf{E}\left[\frac{(X-\mathbf{E}[X])}{\sigma_X}\cdot\frac{(Y-\mathbf{E}[Y])}{\sigma_Y}\right] = \frac{\mathrm{cov}(X,Y)}{\sigma_X\sigma_Y}$$

- $|\rho| = 1 \Leftrightarrow (X-\mathbf{E}[X]) = c(Y-\mathbf{E}[Y])$ (linearly related)

Review: Linear least mean squares estimation

- Let X and Y be jointly distributed with $\mathbf{E}[X] = \mathbf{E}[Y] = 0$
- What linear function $g(X) = aX$ minimizes $\mathbf{E}[(Y-g(X))^2]$?

$$a = \rho\frac{\sigma_y}{\sigma_x}$$

Expectation as least mean squares estimation

- What number c minimizes $\mathbf{E}[(X-c)^2]$?

- How shall we interpret $\mathbf{E}[X \mid Y = y]$?

$\mathbf{E}[X \mid Y]$ and its expectation

- $\mathbf{E}[X \mid Y]$ is a **random variable** that takes the value $\mathbf{E}[X \mid Y = y]$ when $Y = y$
- Apply expected value rule
  - discrete case:

$$\mathbf{E}[\,\mathbf{E}[X \mid Y]\,] = \sum_y \mathbf{E}[X \mid Y = y]\, p_Y(y) = \mathbf{E}[X]$$

  - continuous case:

$$\mathbf{E}[\,\mathbf{E}[X \mid Y]\,] = \int_{-\infty}^{\infty} \mathbf{E}[X \mid Y = y]\, f_Y(y)\, dy = \mathbf{E}[X]$$

  (in either case, total expectation theorem)

- **Law of iterated expectations**: $\mathbf{E}[\,\mathbf{E}[X \mid Y]\,] = \mathbf{E}[X]$

Stick breaking revisited (L09)

- Break a stick of length ℓ (an interval $[0, \ell]$) twice:
  - break at a uniformly chosen random point Y
  - break remaining stick $[0, Y]$ at a uniformly chosen point X
- Find $\mathbf{E}[X]$

- $\mathbf{E}[X \mid Y = y] = \dfrac{y}{2}$ (number)
- $\mathbf{E}[X \mid Y] = \dfrac{Y}{2}$ (random variable, with Y uniform on $[0, \ell]$)
- $\mathbf{E}[X] = \mathbf{E}[\,\mathbf{E}[X \mid Y]\,] = \mathbf{E}[Y/2] = \ell/4$

$\mathrm{var}(X \mid Y)$ and its expectation

- $\mathrm{var}(X \mid Y)$ is a **random variable** that takes the value $\mathrm{var}(X \mid Y = y)$ when $Y = y$
- $\mathrm{var}(X \mid Y = y) = \mathbf{E}\left[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y\right]$
- **Law of total variance**:

$$\mathrm{var}(X) = \mathbf{E}[\mathrm{var}(X \mid Y)] + \mathrm{var}(\mathbf{E}[X \mid Y])$$

$$\mathrm{var}(X \mid Y) = \mathbf{E}[X^2 \mid Y] - (\mathbf{E}[X \mid Y])^2$$

$$\mathbf{E}[\mathrm{var}(X \mid Y)] = \mathbf{E}[X^2] - \mathbf{E}[\,(\mathbf{E}[X \mid Y])^2\,]$$

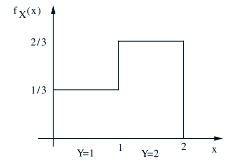
$$\mathrm{var}(\mathbf{E}[X \mid Y]) = \mathbf{E}[\,(\mathbf{E}[X \mid Y])^2\,] - (\mathbf{E}[X])^2$$

Stick breaking ad nauseam

- Find var(X)
- $\mathbf{E}[X \mid Y = y] = y/2$ (number)
- $\mathbf{E}[X \mid Y] = Y/2$ (random variable, with Y uniform on $[0, \ell]$)
- $\text{var}(\mathbf{E}[X \mid Y]) = \text{var}(Y/2) = \frac{1}{4}\text{var}(Y) = \frac{1}{4} \cdot \frac{1}{12}\ell^2$
- $\text{var}(X \mid Y = y) = y^2/12$ (number)
- $\text{var}(X \mid Y) = Y^2/12$ (random variable)
- $\mathbf{E}[\text{var}(X \mid Y)] = \mathbf{E}[Y^2/12] = \frac{1}{12}\mathbf{E}[Y^2] = \frac{1}{12} \cdot \frac{1}{3}\ell^2$

$$\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]) = \frac{1}{36}\ell^2 + \frac{1}{48}\ell^2 = \frac{7}{144}\ell^2$$

Example



$\mathbf{E}[X \mid Y = 1] =$

$\mathbf{E}[X \mid Y = 2] =$

$\text{var}(X \mid Y = 1) =$

$\text{var}(X \mid Y = 2) =$

$\mathbf{E}[\text{var}(X|Y)] =$

$\mathbf{E}[X] =$

$\text{var}(\mathbf{E}[X \mid Y]) =$

$\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]) =$

Section means and variances

30 students in sections $y = 1$, $y = 2$

$$y = 1: \ \frac{1}{10}\sum_{i=1}^{10} x_i = 90 \qquad y = 2: \ \frac{1}{20}\sum_{i=11}^{30} x_i = 60$$

$$\mathbf{E}[X] = \frac{1}{30}\sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$$

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \tfrac{1}{3} \cdot 90 + \tfrac{2}{3} \cdot 60 = 70 = \mathbf{E}[X]$$

$$\text{var}(\mathbf{E}[X \mid Y]) = \tfrac{1}{3}(90 - 70)^2 + \tfrac{2}{3}(60 - 70)^2 = \tfrac{600}{3} = 200$$

Section means and variances (continued)

$$\frac{1}{10}\sum_{i=1}^{10}(x_i - 90)^2 = 10 \qquad \frac{1}{20}\sum_{i=11}^{30}(x_i - 60)^2 = 20$$

$$\text{var}(X \mid Y) = \begin{cases} 10, & \text{w.p. } 1/3 \\ 20, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbf{E}[\text{var}(X \mid Y)] = \frac{1}{3} \cdot 10 + \frac{2}{3} \cdot 20 = \frac{50}{3}$$

$$\begin{aligned} \text{var}(X) &= \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y]) \\ &= \frac{50}{3} + 200 \\ &= \text{(average variability \textbf{within} sections)} \\ &\quad + \text{(variability \textbf{between} sections)} \end{aligned}$$

Sum of a random number of independent RVs

- N: nonnegative integer random variable
- $X_1, X_2, \ldots$: Random variables with $\mathbf{E}[X_i]$ all equal
- $N, X_1, X_2, \ldots$ independent
- Let $Y = X_1 + \cdots + X_N$
- Compute mean and variance of Y

Sum of a random number of independent RVs

$$\begin{aligned} \mathbf{E}[Y \mid N = n] &= \mathbf{E}[X_1 + X_2 + \cdots + X_n \mid N = n] \\ &= \mathbf{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbf{E}[X_1] + \mathbf{E}[X_2] + \cdots + \mathbf{E}[X_n] \\ &= n\,\mathbf{E}[X] \end{aligned}$$

$$\mathbf{E}[Y] = \mathbf{E}[\,\mathbf{E}[Y \mid N]\,] = \mathbf{E}[\,N\,\mathbf{E}[X]\,] = \mathbf{E}[N]\,\mathbf{E}[X]$$

$$\text{var}(\mathbf{E}[Y \mid N]) = (\mathbf{E}[X])^2\,\text{var}(N)$$

$$\text{var}(Y \mid N = n) = n\,\text{var}(X)$$

$$\mathbf{E}[\text{var}(Y \mid N)] = \mathbf{E}[N]\,\text{var}(X)$$

$$\begin{aligned} \text{var}(Y) &= \mathbf{E}[\text{var}(Y \mid N)] + \text{var}(\mathbf{E}[Y \mid N]) \\ &= \mathbf{E}[N]\,\text{var}(X) + (\mathbf{E}[X])^2\,\text{var}(N) \end{aligned}$$