Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2009)

Problem Set 9 Solutions
Due: April 29, 2009

1. (a) All states are recurrent.

   (b) Since the chain is a single recurrent class and since the class is aperiodic, steady state probabilities exist and are given by solving the balance and normalization equations:

$$\begin{aligned}
\pi_1 &= 1/2 \cdot \pi_1 + 1/2 \cdot \pi_2 + 1/3 \cdot \pi_3 \\
\pi_2 &= 1/2 \cdot \pi_1 + 1/4 \cdot \pi_2 + 1/3 \cdot \pi_3 \\
\pi_3 &= 0 \cdot \pi_1 + 1/4 \cdot \pi_2 + 1/3 \cdot \pi_3 \\
1 &= \pi_1 + \pi_2 + \pi_3
\end{aligned}$$

   giving $\pi_1 = 0.48, \pi_2 = 0.38$ and $\pi_3 = 0.14$.

   (c) The number of transitions X until it first leaves state 2 is a geometric random variable with parameter $p = 1/2$, and so the number of transitions until it first enters state 2 is $X + 1$; and the number of transitions Y until it first leaves state 2 is geometric with parameter $p = 1/4$. The total number of transitions until the process makes a transition out of state 2 is thus $N = X + Y + 1$, where random variables X and Y are independent. The mean of N is given by:

$$\mathbf{E}[N] = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{3}{4}} = \frac{10}{3}$$

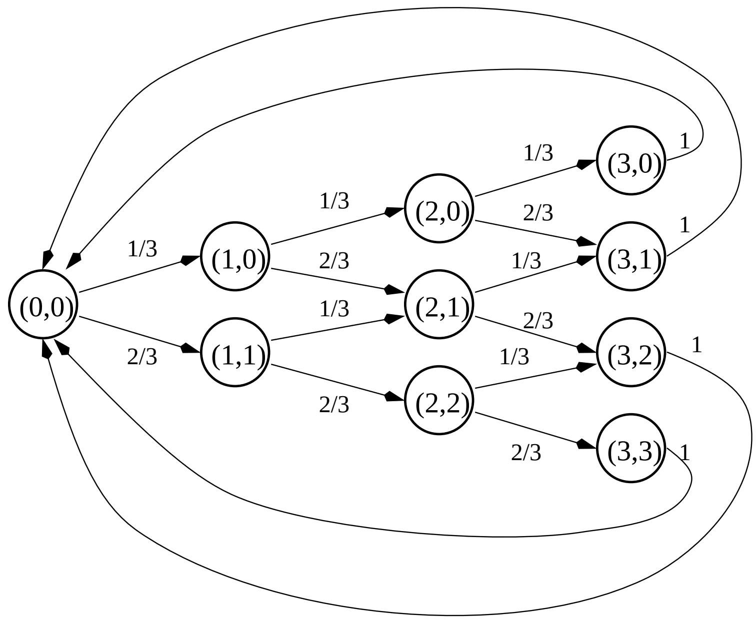
   And the variance of N is given by:

$$\begin{aligned}
\mathrm{Var}(N) &= \mathrm{Var}(X) + \mathrm{Var}(Y) \\
&= \frac{1-\frac{1}{2}}{(\frac{1}{2})^2} + \frac{1-\frac{3}{4}}{(\frac{3}{4})^2} \\
&= 22/9
\end{aligned}$$

2. (a) The state of the system has to capture all the relevant information about the system. In this case, this is the number of drops that are in thimble A and in thimble B. Since the state must capture *all* the relevant information, we cannot say that it is just the number of drops in A or just the number of drops in B. Instead, we must think of the state of the system as the combined description of the number of drops in A *and* the number of drops in B.

   Since at every point in time we get a drop in A, we know that the number of drops in B must be less than or equal to the number of drops in A since B sometimes gets a drop and sometimes does not. So, let us denote the state of the system as a pair (x, y) where x represents the number of drops in A, and y represents the number of drops in B.

   The automatic mechanism basically implies that the maximum number of drops in A is 3. And since B receives a drop at any time with probability $\frac{2}{3}$, we get the following Markov chain.



(b) Notice that this Markov chain is periodic with period 4. A periodic Markov chain has no steady-state probabilities. We are asked to find the approximate probability that there is exactly one drop in both thimbles after exactly $10,001$ seconds when we started observing when both of them were empty.

Relating the above word explanation to states in our Markov chain, we are asked to find the approximate probability that we end up in state $(1,1)$ after $10,001$ transitions if we started at state $(0,0)$. Notice that after $10,001$ transitions, we will be in either state $(1,0)$ or in state $(1,1)$. So the approximate probability we end up in state $(1,1)$ is $\boxed{\frac{2}{3}}$.

3. (a) The recurrent classes are $\{1\}$ and $\{5,6\}$. They are both aperiodic since the self transition probabilities are greater than 0, i.e., $p_{ii} > 0$ for $i = 1,5,6$.

(b) Let a_i denote the probability of absorption into State 1 starting from state i. Then, it's clear that $a_1 = 1$ and $a_3 = 0$. Direct application of the Total Probability Theorem yields

$$a_2 = \frac{1}{3}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 = \frac{1}{3} + \frac{1}{6}a_2.$$

Therefore $a_2 = \frac{2}{5}$.

Let b_i denote the absorption probability into $\{5,6\}$. Note that $a_i + b_i = 1$. Therefore $b_2 = \frac{3}{5}$.

(c) Note that we need to only compute the a_i's. We already have the values for $i = 1,2,3$ from Part (b). It's easy that verify that $a_4 = a_5 = a_6 = 0$, and hence $b_4 = b_5 = b_6 = 1$. The results are summarized in the following table.

i	1	2	3	4	5	6
a_i	1	2/5	0	0	0	0
b_i	0	3/5	1	1	1	1

(d) Let π_j^i denote the steady state probability for state j starting from the recurrent class $\{i\}$. The steady state probabilities for the recurrent class $\{5,6\}$ are obtained by solving the following balance equations:

$$\begin{aligned}
\pi_5^{\{5,6\}} &= \frac{1}{2}\pi_6^{\{5,6\}} + \frac{1}{4}\pi_5^{\{5,6\}} \\
\pi_6^{\{5,6\}} &= \frac{3}{4}\pi_5^{\{5,6\}} + \frac{1}{2}\pi_6^{\{5,6\}} \\
1 &= \pi_5^{\{5,6\}} + \pi_6^{\{5,6\}}
\end{aligned}$$

Solving for the steady-state probabilities we get:

$$\pi_5^{\{5,6\}} = \frac{2}{5}$$

$$\pi_6^{\{5,6\}} = \frac{3}{5}$$

If the Markov chain starts in state 1, then $\pi_1^{\{1\}} = 1$.
Therefore,

$$\begin{aligned}
\lim_{n\to\infty} P(X_n = j|X_0 = i) &= \lim_{n\to\infty}\{P(X_n = j|X_0 = i, \text{absorption class is } \{1\}) \\
&\quad P(\text{absorption class is } \{1\}|X_0 = i) \\
&+ P(X_n = j|X_0 = i, \text{absorption class is } \{5,6\}) \\
&\quad P(\text{absorption class is } \{5,6\}|X_0 = i)\}
\end{aligned}$$

and

$$r_{ij}(\infty) = a_i\pi_j^{\{1\}} + b_i\pi_j^{\{5,6\}}.$$

$$r_{ij}(\infty) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2/5 & 0 & 0 & 0 & 6/25 & 9/25 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \\ 0 & 0 & 0 & 0 & 2/5 & 3/5 \end{bmatrix}$$

(e) N is a geometric random variable with $\frac{5}{6}$ probability of success at each time step. Therefore, $E[N] = \frac{6}{5}$ and $var(N) = \frac{6}{25}$.

(f) To solve this probem we can calculate the expected number of transitions needed to get to state 3 and add the expected number of transitions from state 3 until absorption by $\{5,6\}$. Conditioning on eventually entering the recurrent class $\{5,6\}$ changes the transition probabilities p_{22}, p_{21}, and p_{23}.
Define A as the event that the recurrent class 5,6 is eventually entered.

$$P(X_{n+1} = 3|X_n = 2, A) = \frac{P(X_{n+1} = 3, A|X_n = 2)}{P(A|X_n = 2)}$$

However, since the event that $X_{n+1} = 3$ implies that A is true, the probability can be simplified:

$$
\begin{aligned}
P(X_{n+1} = 3 | X_n = 2, A) &= \frac{P(X_{n+1} = 3, A | X_n = 2)}{P(A | X_n = 2)} \\
&= \frac{P(X_{n+1} = 3 | X_n = 2)}{P(A | X_n = 2)} \\
&= \frac{p_{23}}{\frac{p_{23}}{p_{23}+p_{21}}} = p_{23} + p_{21} = \frac{5}{6}.
\end{aligned}
$$

The self loop probability of state 2 remains the same as can be seen by:

$$
\begin{aligned}
P(X_{n+1} = 2 | X_n = 2, A) &= \frac{P(X_{n+1} = 2, A | X_n = 2)}{P(A | X_n = 2)} \\
&= \frac{P(A | X_{n+1} = 2, X_n = 2) P(X_{n+1} = 2 | X_n = 2)}{P(A | X_n = 2)} \\
&= P(X_{n+1} = 2 | X_n = 2) = p_{22} = \frac{1}{6}.
\end{aligned}
$$

Note that the rest of the transition probabilities won't be affected because the recurrent class does not depend on the rest of the chain. Also, the transition probabilities out of the transient states within the chain remain the same.

Therefore, the expected number of transitions to get to state 3 from state 2 is $\frac{6}{5}$.

To calculate the expected number of transitions until absorption by the recurrent class $\{5, 6\}$, we combine $\{5, 6\}$ into one absorbing state and use the expected time to absorption equations.

$$
\begin{aligned}
\mu_{\{5,6\}} &= 0 \\
\mu_3 &= 1 + \frac{1}{2}\mu_4 + \frac{1}{2}\mu_{\{5,6\}} \\
\mu_4 &= 1 + \frac{3}{4}\mu_3 + \frac{1}{4}\mu_{\{5,6\}}
\end{aligned}
$$

Therefore, $\mu_3 = \frac{12}{5}$ and $E[M] = \frac{6}{5} + \mu_3 = \frac{18}{5}$.

4. (a) The process is in state S_3 immediately before the first transition. After leaving state S_3 for the first time, the process cannot go back to state S_3 again. Hence J, which represents the number of transitions up to and including the transition on which the process leaves state S_3 for the last time is a geometric random variable with success probability equal to 0.6. The variance for J is given by:

$$\sigma_J^2 = \frac{1-p}{p^2} = \frac{10}{9}$$

(b) There is a positive probability that we never enter state S_4; i.e., $P(K < \infty) < 1$. Hence the expected value of K is ∞.

(c) The Markov chain has 3 different recurrent classes. The first recurrent class consists of states $\{S_1, S_2\}$, the second recurrent class consists of state $\{S_7\}$ and the third recurrent class consists of states $\{S_4, S_5, S_6\}$. The probability of getting absorbed into the first recurrent class starting from the transient state S_3 is,

$$\frac{1/10}{1/10 + 2/10 + 3/10} = \frac{1}{6}$$

which is the probability of transition to the first recurrent class given there is a change of state. Similarly, probability of absorption into second and third recurrent classes are $\frac{3}{6}$ and $\frac{2}{6}$ respectively.

Now, we solve the balance equations within each recurrent class, which give us the probabilities conditioned on getting absorbed from state S_3 to that recurrent class. The long term probabilities of being in a given state are found by weighing the steady-state probabilities within a recurrent class by the probability of absorption to the recurrent classes.

The first recurrent class is a birth-death process. We write the following equations and solve for the steady state probabilities within the recurrent class, where p_i is the steady state probability within the recurrent class of being in state S_i.

$$p_1 = \frac{p_2}{2}$$

$$p_1 + p_2 = 1$$

Solving these equations, we get $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$. For the second recurrent class, $p_7 = 1$. The third recurrent class is also a birth-death process, we can find the steady-state probabilities within a recurrent class as follows,

$$p_4 = 2p_5$$

$$p_5 = 2p_6$$

$$p_4 + p_5 + p_6 = 1$$

and thus, $p_4 = \frac{4}{7},\ p_5 = \frac{2}{7},\ p_6 = \frac{1}{7}$.

Using these data, the long term probability π_i of being in state S_i for $i \in \{1, 2, \ldots, 7\}$ are found as follows:

$$\pi_1 = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

$$\pi_2 = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$$

$$\pi_3 = 0 \text{ (transient state)}$$

$$\pi_7 = 1 \cdot \frac{3}{6} = \frac{1}{2}$$

$$\pi_4 = \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21}$$

$$\pi_5 = \frac{2}{7} \cdot \frac{2}{6} = \frac{2}{21}$$

$$\pi_6 = \frac{1}{7} \cdot \frac{2}{6} = \frac{1}{21}$$

G1[†]. a) First let the p_{ij}'s be the transition probabilities of the Markov chain.

Then

$$\begin{aligned}
m_{k+1}(1) &= E[R_{k+1}|X_0 = 1] \\
&= E[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1] \\
&= \sum_{i=1}^{n} p_{1i} E[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1 = i] \\
&= \sum_{i=1}^{n} p_{1i} E[g(1) + g(X_1) + ... + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} E[g(X_1) + ... + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} m_k(i)
\end{aligned}$$

and thus in general $m_{k+1}(c) = g(c) + \sum_{i=1}^{n} p_{ci} m_k(i)$ when $c \in \{1, ..., n\}$.

Note that the third equality simply uses the total expectation theorem.

b)

$$\begin{aligned}
v_{k+1}(1) &= Var[R_{k+1}|X_0 = 1] \\
&= Var[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1] \\
&= Var[E[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad E[Var[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[g(1) + E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad E[Var[g(1) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[E[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= Var[E[g(X_1) + ... + g(X_{k+1})|X_1]] + E[Var[g(X_1) + ... + g(X_{k+1})|X_1]] \\
&= Var[m_k(X_1)] + E[v_k(X_1)] \\
&= E[(m_k(X_1))^2] - E[m_k(X_1)]^2 + \sum_{i=1}^{n} p_{1i} v_k(i) \\
&= \sum_{i=1}^{n} p_{1i} m_k^2(i) - (\sum_{i=1}^{n} p_{1i} m_k(i))^2 + \sum_{i=1}^{n} p_{1i} v_k(i)
\end{aligned}$$

so in general $v_{k+1}(c) = \sum_{i=1}^{n} p_{ci} m_k^2(i) - (\sum_{i=1}^{n} p_{ci} m_k(i))^2 + \sum_{i=1}^{n} p_{ci} v_k(i)$ when $c \in \{1, ..., n\}$.

[†]Required for 6.431; optional for 6.041