Quiz 1 Solutions:
October 12, 2011

Problem 1.

1. **(4 points)** There are $\binom{n}{k}$ ways to select the blue team members. Of the remaining $n-k$ members, there are $\binom{n-k}{n-k}$ ways to select the red team members. Therefore, there are $\binom{n}{k} \cdot \binom{n-k}{n-k} = \binom{n}{k}$ ways to select the teams.

2. **(4 points)** Using counting principles, each member can be placed in one of three possible teams. Multiplying these three possibilities for each member yields 3^n possible ways to split the teams.

3. **(6 points)**

   (i) **(3 points)** Let A be the event that persons 1 and 2 belong to the blue team. Given the event A, we need to consider persons 3, 4 and 5. The number of members assigned to the blue team among these three persons is binomial with parameters $n = 3$ and $p = p_B$. Therefore, the conditional PMF of N_B is a shifted binomial as such:

   $$p_{N_B|A}(k) = \binom{3}{k-2}(p_B)^{k-2}(1-p_B)^{5-k} \quad \text{for } k = 2, 3, 4, 5.$$

   (ii) **(3 points)** No. N_B and N_R are not independent. $\mathbf{P}(N_B = 5 \mid A) = (p_B)^3 > 0$. However, $\mathbf{P}(N_B = 5 \mid A \cap N_R = 3) = 0$.

4. **(5 points)** Define $N_{BR} = N_B + N_R$ as the number of persons in the blue and red teams. We can also define N_{BR} as $T_1 + T_2 + \ldots + T_n$ where T_i is a Bernoulli random variable with parameter $p = p_B + p_R$.

   As the $T_i's$ are independent and identically distributed, N_{BR} is a binomial random variable with parameters n and $p = p_B + p_R$.

   Therefore,

   $$\begin{aligned} p_{N_{BR}}(k) &= \binom{n}{k}(p_B + p_R)^k(1 - p_B - p_R)^{n-k} \quad \text{for } k = 0, 1, \ldots n \\ \mathbf{E}[N_{BR}] &= n(p_B + p_R). \end{aligned}$$

5. **(6 points)** The random variable N_B is binomial with parameters n and $p = p_B$.

   $$\begin{aligned} \mathbf{E}[N_B^2] &= \text{var}(N_B) + (\mathbf{E}[N_B])^2 \\ &= np_B(1-p_B) + n^2p_B^2. \end{aligned}$$

6. **(6 points)** Let B be the event that $N_B + N_R = k$. Given B, the conditional probability that a member is on the blue team is $\frac{p_B}{p_B+p_R}$. The conditional PMF of N_B is binomial with parameters $n = k$ and $p = \frac{p_B}{p_B+p_R}$:

   $$p_{N_B|B}(m) = \binom{k}{m}\left(\frac{p_B}{p_B+p_R}\right)^m\left(\frac{p_R}{p_B+p_R}\right)^{k-m} \quad \text{for } m = 0, 1, \ldots k.$$

7. **(6 points)** The joint PMF $p_{N_R,Z}(k,z) = p_{N_R}(k)p_{Z|N_R}(z \mid k)$. N_R is binomial with parameters n and $p = p_R$ and

$$p_{N_R}(k) = \binom{n}{k}(p_R)^k(1-p_R)^{n-k} \quad \text{for } k = 0, 1, \ldots n.$$

The conditional PMF of Z given $N_R = k$ takes on values between $z = k$ and $z = 2k$. The number of members receiving \$2 is $z - k$ and the number of members receiving \$1 is $k - (z-k) = 2k - z$. The conditional PMF of Z is also binomial and

$$\begin{aligned} p_{Z|N_R}(z \mid k) &= \binom{k}{z-k}\left(\frac{1}{2}\right)^{z-k}\left(\frac{1}{2}\right)^{2k-z} \\ &= \binom{k}{z-k}\left(\frac{1}{2}\right)^{k} \quad \text{for } z = k, k+1, \ldots 2k. \end{aligned}$$

Therefore,

$$p_{N_R,Z}(k,z) = \binom{n}{k}(p_R)^k(1-p_R)^{n-k}\binom{k}{z-k}\left(\frac{1}{2}\right)^{k} \quad \text{for } k = 0, 1, \ldots n \text{ and } z = k, k+1, \ldots 2k.$$

8. **(7 points)** Let H_i be 1 if person i is happy and be 0 if person i is not happy and $\mathbf{P}(H_i = 1) = p_B^3 + p_R^3 + p_W^3$. Let H be the total number of happy people and so $H = \sum_{i=2}^{n-1} H_i$. Using linearity of expectation,

$$\begin{aligned} \mathbf{E}[H] &= \mathbf{E}\left[\sum_{i=2}^{n-1} H_i\right] \\ &= \sum_{i=2}^{n-1} \mathbf{E}[H_i] \\ &= \sum_{i=2}^{n-1} p_B^3 + p_R^3 + p_W^3 \\ &= (n-2)(p_B^3 + p_R^3 + p_W^3). \end{aligned}$$

9. **(6 points)** Using the notation in the previous part, we wish to find $\mathbf{P}(H_3 = 1 \mid H_2 = 1)$.

$$\mathbf{P}(H_3 = 1 \mid H_2 = 1) = \frac{\mathbf{P}(\{H_3 = 1\} \cap \{H_2 = 1\})}{\mathbf{P}(H_2 = 1)}.$$

The denominator is $\mathbf{P}(H_2 = 1) = p_B^3 + p_R^3 + p_W^3$.

The event $\{H_3 = 1\} \cap \{H_2 = 1\}$ occurs if persons 1, 2, 3 and 4 are all on the same team and $\mathbf{P}(\{H_3 = 1\} \cap \{H_2 = 1\}) = p_B^4 + p_R^4 + p_W^4$.

Therefore,

$$\mathbf{P}(H_3 = 1 \mid H_2 = 1) = \frac{p_B^4 + p_R^4 + p_W^4}{p_B^3 + p_R^3 + p_W^3}.$$