Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

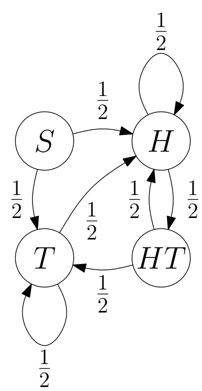
6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 9

Due Friday, December 2, 2011 in Tutorial

1. Assume that a fair coin is tossed repeatedly, with the tosses being independent. We want to determine the expected number of tosses necessary to first observe a head directly followed by a tail. To do so, we define a Markov chain with states S, H, T, HT, where S is a starting state, H indicates a head on the current toss, T indicates a tail on the current toss (without heads on the previous toss), and HT indicates heads followed by tails over the last two tosses. This Markov chain is illustrated below:



We can find the expected number of tosses necessary to first observe a heads directly followed by tails by solving a mean first passage time problem for this Markov chain.

(a) What is the expected number of tosses necessary to first observe a head directly followed by tails?

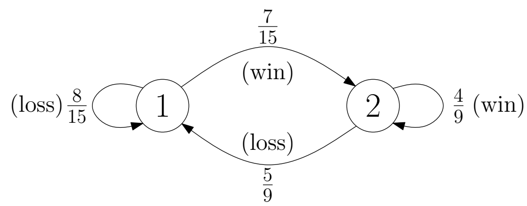
(b) Assuming we have just observed a head followed by a tail, what is the expected number of additional tosses until we again observe a head followed directly by a tail?

Next, we want to answer the same questions for the event tails directly followed by tails. Set up a different Markov chain from which we could calculate the expected number of tosses necessary to first observe tails directly followed by tails.

(c) What is the expected number of tosses necessary to first observe a tail directly followed by a tail?

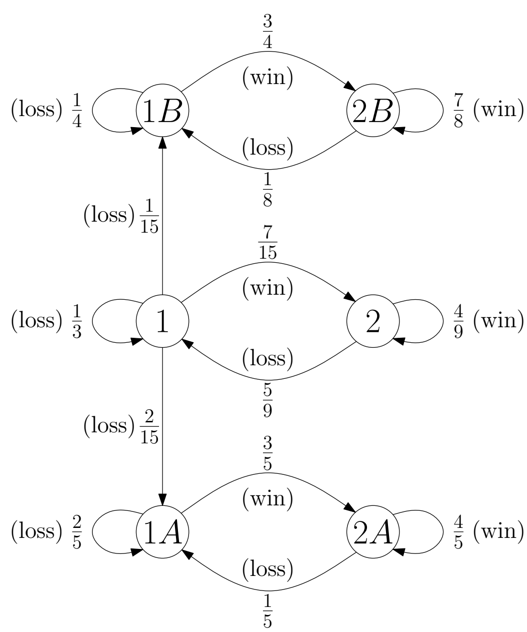
(d) Assuming we have just observed a tail followed by a tail, what is the expected number of additional tosses until we again observe a tail followed directly by a tail? Note that the number of additional tosses could be as little as one, if tails were to come up again.

2. Jack is a gambler who pays for his MIT tuition by spending weekends in Las Vegas. Lately he's been playing 21 at a table that returns cards to the deck and reshuffles them all before each hand. As he has a fixed policy in how he plays, his probability of winning a particular hand remains constant, and is independent of all other hands. There is a wrinkle, however; the dealer switches between two decks (deck #2 is more unfair to Jack than deck #1), depending on whether or not Jack wins. Jack's wins and losses can be modeled via the transitions of the following Markov chain, whose states correspond to the particular deck being used.



(a) What is Jack's long term probability of winning?

Given that Jack loses and the dealer is not occupied with switching decks, with probability $\frac{2}{15}$ the dealer looks away for one second and with probability $\frac{1}{15}$ the dealer looks away for two seconds, independently of everything else. When this happens, Jack secretly inserts additional cards into both of the dealer's decks, transforming the decks into types $1A$ & $2A$ (when he has 1 second) or $1B$ & $2B$ (when he has 2 seconds). Jack slips cards into the decks at most once. The process can be described by the modified Markov chain in the picture. Assume in all future problems that play begins with the dealer using deck #1.



(b) What is the probability of Jack eventually playing with decks $1A$ and $2A$?

(c) What is Jack's long-term probability of winning?

3. Joe the Plumber has been having trouble keeping the income at his plumbing business steady lately. He notices that his profit each week is uniformly distributed between \$1,800 and \$8,600. Being a small business owner, he doesn't have the stomach for these fluctuations, especially because they introduce additional uncertainty as to whether or not he'll soon face higher taxes. Assume that each week's profit is independent of the profits of all other weeks. Find a good approximation to the probability that over the course of a year (52 weeks), Joe's plumbing business will have a yearly income of less than \$250,000.

4. On any given flight, an airline tries to sell as many tickets as possible. Suppose, on average, 10% of ticket holders fail to show up, all independent of each other. Knowing this, an airline will sell more tickets than available seats (overbook the flight) and hope that there are sufficient numbers of passengers who do not show up to compensate for its overbooking. Using the central limit theorem, determine n, the maximum number of tickets an airline should sell on a flight with 300 seats so that it can be approximately 99% confident that it need not deny boarding to any of the n passengers holding tickets.

5. Random variable X is uniformly distributed between -1.0 and 1.0. Let $X_1, X_2, \ldots,$ be independent identically distributed random variables with the same distribution as X. Determine which, if any, of the following sequences (all with $i = 1, 2, \ldots$) are convergent in probability. Fully justify your answers. Include the limits if they exist.

   (a) $U_i = \dfrac{X_1 + X_2 + \ldots + X_i}{i}$

   (b) $W_i = \max(X_1, \ldots, X_i)$

   (c) $V_i = X_1 \cdot X_2 \cdot \ldots \cdot X_i$

6. You have two nearly identical coins in your pocket, a weighted coin that comes up heads with probability 0.55, along with a fair coin. With no obvious way to tell the two apart in appearance, you take out one coin and decide to flip it 1000 times. If it comes up heads more than 525 times, you'll presume that you've been flipping the biased coin. Assume that in reality you have the fair coin.

   (a) By using the de Moivre-Laplace normal approximation to the binomial, approximate the probability that you will presume it is the biased coin.

   (b) Find an upper bound on the probability that you will presume it is the biased coin by using the Markov inequality.

   (c) Find an upper bound on the probability that you will presume it is the biased coin by using the Chebyshev inequality.

G1†. You will probably recall the Double or Quarter game, in which a fair coin is tossed at each round. The player begins with a total wealth of \$1.00. On the first round, the player receives back twice his bet if the coin comes up heads (i.e., he gets \$2.00 back) or else a fourth of his bet if it comes up tails (i.e., he gets \$0.25 back). On the second round the coin is tossed again, he bets his entire wealth (now either \$2.00 or \$0.25), and he receives back twice his bet (either \$4.00 or \$0.50) for heads or else one fourth of his bet (either \$0.50 or \$0.0625) for tails. The game then continues in the same fashion, with the various tosses being independent and the player betting his entire remaining wealth at each toss.

   (a) Calculate the following quantities:

      i. The minimum number of heads required in the first 100 tosses so that the player's wealth after 100 tosses is greater than or equal to \$1.00.

      ii. The probability mass function, the mean, and the standard deviation of the number of heads in 100 tosses of a fair coin.

   (b) Write down (but you need not evaluate) an exact expression for the probability that the player's wealth after 100 tosses is greater than or equal to \$1.00.

(c) Find an upper bound for the value of the expression in part (b) using the Markov bound.

(d) Find an upper bound for the value of the expression in part (b) using the Chebyshev bound. *Note:* A useful fact is that the probability distribution of the number of heads in 100 tosses is symmetric about its mean.

(e) Find an estimate for the value of the expression in part (b) using the central limit theorem, with and without the 1/2 correction.

(f) Compare your answers to parts (c)-(e) and briefly explain what conclusions you can draw about the methods for this example. You may use the fact that the actual value of the probability you were asked to find in part (b) is approximately 0.00043686.

†Required for 6.431; optional challenge problem for 6.041