Quiz 2 Solutions:
November 3, 2009

Problem 2. (49 points)

(a) (7 points)
We start by recognizing that $f_X(x) = e^{-x}$ for $x \geq 0$ and $f_{Y|X}(y \mid x) = xe^{-xy}$ for $y \geq 0$. Furthermore, $f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y \mid x)$. Substituting for $f_x(x)$ and $f_{Y|X}(y \mid x)$ yields,

$$f_{X,Y}(x,y) \quad = \quad \begin{cases} xe^{-(1+y)x}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) (7 points)
The marginal PDF of Y can be found by integrating the joint PDF of X and Y.

$$\begin{aligned} f_Y(y) &= \int_X f_{X,Y}(x,y)dx \\ &= \int_0^\infty xe^{-(1+y)x}dx \end{aligned}$$

$$f_Y(y) \quad = \quad \begin{cases} \frac{1}{(1+y)^2}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(c) (7 points)
We are asked to compute the PDF of the random variable X while conditioning on another random variable Y. The conditional PDF of X given that $Y = 2$ is

$$f_{X|Y}(x \mid 2) \quad = \quad \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{xe^{-3x}}{\frac{1}{3^2}}$$

$$f_{X|Y}(x \mid 2) \quad = \quad \begin{cases} 9xe^{-3x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(d) (7 points)

$$\begin{aligned} \mathbf{E}[X \mid Y = 2] &= \int_X x \cdot f_{X|Y}(x \mid 2)dx \\ &= 9\int_0^\infty x^2 e^{-3x}dx \\ &= 9 \cdot \frac{2}{3^3} \\ &= \frac{2}{3}. \end{aligned}$$

(e) (7 points)
In the new universe in which $X = 2$, we are asked to compute the conditional PDF of Y given the event $Y \geq 3$.

$$f_{Y|X,Y\geq 3}(y \mid 2) = \frac{f_{Y|X}(y \mid 2)}{\mathbf{P}(Y \geq 3 \mid X = 2)}.$$

We first calculate the $\mathbf{P}(Y \geq 3 \mid X = 2)$.

$$\begin{aligned}
\mathbf{P}(Y \geq 3 \mid X = 2) &= \int_3^\infty f_{Y|X}(y \mid 2)dy \\
&= \int_3^\infty 2e^{-2y}dy \\
&= 1 - F_{Y|X}(3|2) \\
&= 1 - (1 - e^{-2\cdot 3}) \\
&= e^{-6},
\end{aligned}$$

where $F_{Y|X}(3|2)$ is the CDF of an exponential random variable with $\lambda = 2$ evaluated at $y = 3$. Substituting the values of $f_{Y|X}(y \mid 2)$ and $\mathbf{P}(Y \geq 3 \mid X = 2)$ yields

$$f_{Y|X,Y\geq 3}(y \mid 2) = \begin{cases} 2e^{-2(y-3)}, & y \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, $f_{Y|X}(y \mid 2)$ is an exponential random variable with $\lambda = 2$. To compute the conditional PMF $f_{Y|X,Y\geq 3}(y \mid 2)$, we can apply the memorylessness property of an exponential variable. Therefore, this conditional PMF is also an exponential random variable with $\lambda = 2$, but it is shifted by 3.

(f) (7 points)
Let's define $Z = e^{2X}$. Since $X \geq 0$, it follows that $Z \geq 1$. We find the PDF of Z by first computing its CDF.

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(e^{2X} \leq z) \\
&= P(2X \leq \ln z) \\
&= P(X \leq \frac{\ln z}{2}) \\
&= 1 - e^{-\frac{\ln z}{2}} \\
&= 1 - e^{\ln z^{-\frac{1}{2}}}
\end{aligned}$$

The CDF of Z is:

$$F_Z(z) = \begin{cases} 1 - z^{-\frac{1}{2}}, & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Differentiating the CDF of Z yields the PDF

$$f_Z(z) = \begin{cases} \frac{1}{2}z^{-\frac{3}{2}}, & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Alternatively, you can apply the PDF formula for a strictly monotonic function of a continuous random variable. Recall if $z = g(x)$ and $x = h(z)$, then

$$f_Z(z) = f_X\left(h(z)\right)\left|\frac{dh}{dy}(z)\right|.$$

In this problem, $z = e^{2x}$ and $x = \frac{1}{2}\ln z$. Note that $f_Z(z)$ is nonzero for $z > 1$. Since X is an exponential random variable with $\lambda = 1$, $f_x(x) = e^{-x}$.

Thus,

$$\begin{aligned} f_z(z) &= e^{-\frac{1}{2}\ln z}\left|\frac{1}{2z}\right| \\ &= e^{\ln z^{-\frac{1}{2}}}\frac{1}{2z} \\ &= \frac{1}{2}z^{-\frac{3}{2}} \quad z \geq 1, \end{aligned}$$

where the second equality holds since the expression inside the absolute value is always positive for $z \geq 1$.

Problem 3. (10 points)

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is:

(i) A number.
(ii) A discrete random variable.
(iii) A continuous random variable.
(iv) $\boxed{\text{Not enough information to choose between (i)-(iii).}}$

If X and Y are not independent, then $\mathbf{E}[X \mid Y]$ is a function of Y and is therefore a continuous random variable. However if X and Y are independent, then $\mathbf{E}[X \mid Y] = \mathbf{E}[X]$ which is a number.

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is:

(i) A number.
(ii) $\boxed{\text{A discrete random variable.}}$
(iii) A continuous random variable.
(iv) Not enough information to choose between (i)-(iii).

If X, Y and N are not independent, then the inner expectation $G(Y, N) = \mathbf{E}[X \mid Y, N]$ is a function of Y and N. Furthermore $\mathbf{E}[G(Y, N) \mid N]$ is a function of N, a discrete random variable. If X, Y and N are independent, then the inner expectation $\mathbf{E}[X \mid Y, N] = \mathbf{E}[X]$, which is a number. The expectation of a number given N is still a number, which is a special case of a discrete random variable.

Problem 4. (25 points)

(a) (i) (5 points)
Using the Law of Iterated Expectations, we have

$$\mathbf{E}[X] = \mathbf{E}\left[\mathbf{E}[X \mid Q]\right] = \mathbf{E}[Q] = \frac{1}{2}.$$

(ii) (5 points)
X is a Bernoulli random variable with a mean $p = \frac{1}{2}$ and its variance is $\text{var}(X) = p(1-p) = 1/4$.

(b) (7 points)
We know that $\text{cov}(X, Q) = \mathbf{E}[XQ] - \mathbf{E}[X]\mathbf{E}[Q]$, so first let's calculate $\mathbf{E}[XQ]$:

$$\mathbf{E}[XQ] = \mathbf{E}[\mathbf{E}[XQ \mid Q]] = \mathbf{E}[Q\mathbf{E}[X \mid Q]] = \mathbf{E}[Q^2] = \frac{1}{3}.$$

Therefore, we have

$$\text{cov}(X, Q) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

(c) (8 points)
Using Bayes' Rule, we have

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{p_X(1)} = \frac{f_Q(q)\mathbf{P}(X = 1 \mid Q = q)}{\mathbf{P}(X = 1)}, \quad 0 \leq q \leq 1.$$

Additionally, we know that

$$\mathbf{P}(X = 1 \mid Q = q) = q,$$

and that for Bernoulli random variables

$$\mathbf{P}(X = 1) = \mathbf{E}[X] = \frac{1}{2}.$$

Thus, the conditional PDF of Q given $X = 1$ is

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{1 \cdot q}{1/2} \\ &= \begin{cases} 2q, & 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Problem 5. (21 points)

(a) (7 points)

$$\begin{aligned} \mathbf{P}(S \geq 1) &= \mathbf{P}(\min\{X, Y\} \geq 1) = \mathbf{P}(X \geq 1 \text{ and } Y \geq 1) = \mathbf{P}(X \geq 1)\mathbf{P}(Y \geq 1) \\ &= (1 - F_X(1))(1 - F_Y(1)) = (1 - \Phi(1))^2 \approx (1 - 0.8413)^2 \approx 0.0252. \end{aligned}$$

(b) (7 points)
Recalling Problem 2 of Problem Set 6, we have

$$\begin{aligned}
\mathbf{P}(s \le S \text{ and } L \le \ell) &= \mathbf{P}(s \le \min\{X, Y\} \text{ and } \max\{X, Y\} \le \ell) \\
&= \mathbf{P}(s \le X \text{ and } s \le Y \text{ and } X \le \ell \text{ and } Y \le \ell) \\
&= \mathbf{P}(s \le X \le \ell)\mathbf{P}(s \le Y \le \ell) \\
&= (F_X(\ell) - F_X(s))(F_Y(\ell) - F_Y(s)).
\end{aligned}$$

(c) (7 points)
Given that $s \le s + \delta \le \ell$, the event $\{s \le S \le s + \delta, \ell \le L \le \ell + \delta\}$ is made up of the union of two disjoint possible events:

$$\{s \le X \le s + \delta, \ell \le Y \le \ell + \delta\} \cup \{s \le Y \le s + \delta, \ell \le X \le \ell + \delta\}.$$

In other words, either $S = X$ and $L = Y$, or $S = Y$ and $L = X$. Because the two events are disjoint, the probability of their union is equal to the sum of their individual probabilities. Using also the independence of X and Y, we have

$$\begin{aligned}
\mathbf{P}(s \le S \le s + \delta, \ell \le L \le \ell + \delta) &= \mathbf{P}(s \le X \le s + \delta, \ell \le Y \le \ell + \delta) \\
&\quad + \mathbf{P}(s \le Y \le s + \delta, \ell \le X \le \ell + \delta) \\
&= \mathbf{P}(s \le X \le s + \delta)\mathbf{P}(\ell \le Y \le \ell + \delta) \\
&\quad + \mathbf{P}(s \le Y \le s + \delta)\mathbf{P}(\ell \le X \le \ell + \delta) \\
&= \int_s^{s+\delta} f_X(x)\, dx \int_\ell^{\ell+\delta} f_Y(y)\, dy \\
&\quad + \int_s^{s+\delta} f_Y(y)\, dy \int_\ell^{\ell+\delta} f_X(x)\, dx
\end{aligned}$$

(d) If δ is small, then $\int_a^{a+\delta} f_X(x)\, dx \approx \delta f_X(a)$. Using this fact, we obtain

$$\begin{aligned}
\mathbf{P}(s \le S \le s + \delta, \ell \le L \le \ell + \delta) &\approx \delta^2 f_{S,L}(s, \ell) \\
&\approx \delta f_X(s) \cdot \delta f_Y(\ell) + \delta f_Y(s) \cdot \delta f_X(\ell) \\
&= \delta^2 \left[f_X(s) f_Y(\ell) + f_Y(s) f_X(\ell)\right],
\end{aligned}$$

and thus,

$$f_{S,L}(s, \ell) = f_X(s) f_Y(\ell) + f_Y(s) f_X(\ell), \quad s < \ell.$$