LECTURE 3

- **Readings:** Section 1.5

Lecture outline

- Comments on use of Stellar
- Review
- Independence of two events
- Independence of a collection of events

Review

- Conditional probability:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \qquad \text{assuming } \mathbf{P}(B) > 0$$

- Multiplication rule:

$$\mathbf{P}(A \cap B) = \mathbf{P}(B) \cdot \mathbf{P}(A \mid B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$$

- Total probability theorem:

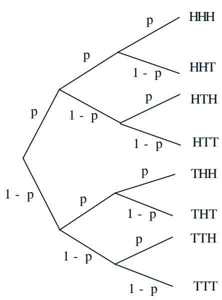
$$\mathbf{P}(B) = \mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)$$

- Bayes rule:

$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)}$$

Models based on conditional probabilities

- 3 tosses of a biased coin: $\mathbf{P}(\mathrm{H}) = p$, $\mathbf{P}(\mathrm{T}) = 1 - p$



$\mathbf{P}(\mathrm{THT}) =$

$\mathbf{P}(1\ \mathrm{H}) =$

$\mathbf{P}(\text{first toss is H} \mid 1\ \mathrm{H}) =$

Independence of two events

- *A provides no information about B:* $\mathbf{P}(B \mid A) = \mathbf{P}(B)$
- Recall that $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A)$
- **Definition:** $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$
- Symmetric with respect to A and B
  - applies even if $\mathbf{P}(A) = 0$
  - implies $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

Independence: Basic examples with fair dice

		first roll X			
		1	2	3	4
	1	•	•	•	•
second	2	•	•	•	•
roll Y	3	•	•	•	•
	4	•	•	•	•

- Let $A_i = \{X = i\}$, $\quad B_j = \{Y = j\}$, $\quad C_k = \{X + Y = k\}$
  - Are A_1 and A_2 independent?
  - Are A_1 and B_3 independent?
  - Are A_1 and C_4 independent?

Independence: Basic properties

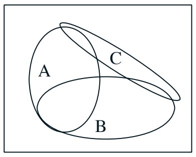
- Can disjoint events A and B be independent?

- Can event A be independent of itself?

Conditional independence

- Conditional independence given C defined as independence under probability law $\mathbf{P}(\cdot \mid C)$

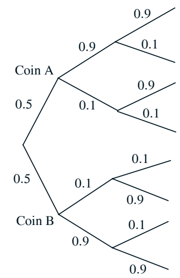
- Suppose A and B are independent. Are they conditionally independent given C?



Conditioning may affect independence

- Two unfair coins, A and B, chosen with equal probability:

$$\mathbf{P}(\text{H} \mid \text{coin A}) = 0.9 \qquad \mathbf{P}(\text{H} \mid \text{coin B}) = 0.1$$



- Knowing coin A is chosen, are tosses independent?

- If we do not know which coin is chosen, are tosses independent?

Independence of a collection of events

- Intuitive definition:
  Information on some of the events tells us nothing about probabilities related to the remaining events
  - E.g., $\mathbf{P}(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c) = \mathbf{P}(A_1 \cap (A_2^c \cup A_3))$
- Mathematical definition:
  Events $A_1, A_2, \ldots, A_n$ are called **independent** when

$$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_q) = \mathbf{P}(A_i)\mathbf{P}(A_j)\cdots\mathbf{P}(A_q)$$

for **every** subset $\{i, j, \ldots, q\} \subset \{1, 2, \ldots, n\}$.

Independence vs. pairwise independence

- Two independent fair coin tosses

HH	HT
TH	TT

  - Let $A = \{\text{first toss is H}\}$, $\quad B = \{\text{second toss is H}\}$, and $C = \{\text{first and second tosses are equal}\}$
  - $\mathbf{P}(C) =$
  - $\mathbf{P}(C \cap A) =$
  - $\mathbf{P}(A \cap B \cap C) =$
  - $\mathbf{P}(C \mid A \cap B) =$