Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem Set 3
Due: September 28, 2011

1. We select a pair of integers in the range $1, \ldots, n$, at random, with each pair being equally likely. What is the probability that the two integers differ by exactly 2?

2. A bank has a push-button combination lock that works as follows. There are 20 numbered buttons, and one pushes 8 of them (in any order) to enter a combination. More than one combination is correct. In fact, 10 buttons are correct; one needs to push 8 of the 10 correct buttons for the vault to open.

   Assume a burglar knows how the lock works, but doesn't know any of the numbers in the combination. Find the probability that the burglar will open the vault on the first try.

3. Count the number of different letter arrangements you can make by changing the order of the letters in the word **aardvark** (count the original word, too).

4. You are lost in the campus of MIT, where the population is entirely composed of brilliant students and absent-minded professors. The students comprise two-thirds of the population, and any one student gives a correct answer to a request for directions with probability $\frac{3}{4}$. (Assume answers to repeated questions are independent, even if the question and the person asked are the same.) If you ask a professor for directions, the answer is always false.

   (a) You ask a passer-by whether the exit from campus is East or West. The answer is East. What is the probability this is correct?
   (b) You ask the same person again, and receive the same reply. Show that the probability that this second reply is correct is $\frac{1}{2}$.
   (c) You ask the same person again, and receive the same reply. What is the probability that this third reply is correct?
   (d) You ask for the fourth time, and receive the answer East again. Show that the probability it is correct is $\frac{27}{70}$.
   (e) Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is $\frac{9}{10}$.
   (f) Reviewing your answers from (a) to (d), do you see a trend? If so, can you explain it mathematically? Can you explain it intuitively?

   Your friend, Ima Nerd, happens to be in the same position as you are, only she has reason to believe *a priori* that, with probability ϵ, East is the correct answer.

   (g) Show that whatever answer is first received, Ima continues to believe that East is correct with probability ϵ.
   (h) Show that if the first two replies are the same (that is, either WW or EE), Ima continues to believe that East is correct with probability ϵ.
   (i) Show that after three like answers, Ima will calculate as follows (in the obvious notation):

   $$\mathbf{P}(\text{East correct}|EEE) = \frac{9\epsilon}{11 - 2\epsilon}, \qquad \mathbf{P}(\text{East correct}|WWW) = \frac{11\epsilon}{9 + 2\epsilon}.$$

   Evaluate these when $\epsilon = \frac{9}{20}$.

5. **Lottery.** Suppose you choose r distinct integer numbers between 1 and n. A lottery chooses a random sequence L of the same size, from the same set of integer numbers between 1 and n. Assume that the order of the chosen numbers does not matter and that $n > 2r$. Find the probability of the following events:

   (a) The numbers in L are drawn in increasing order.

   (b) The set of numbers that you chose is the same as L.

   (c) There are exactly k of your numbers matching members of L.

   (d) L includes no consecutive integers.

   (e) L includes exactly one pair of consecutive integers.

G1†. Let $\{X_n, n \geq 1\}$ be a sequence of continuous random numbers. Each X_n is uniformly distributed on the interval $[0, 1]$ (i.e., $\mathbf{P}(a \leq X_n \leq b) = b - a$ if $0 \leq a \leq b \leq 1$), and all X_n's are independent. In this case, we say that the X_n's are *independent and identically distributed (i.i.d.)*.

   (a) For any $n \geq 2$, find the probability that any value in the sequence is repeated, i.e., find the probability that $X_j = X_k$ for any two values of j and k, with $j \neq k$ and $1 \leq j, k \leq n$.

   We say that X_n is a **record** if its value is larger than that of all preceding $n - 1$ values in the sequence (i.e., if $X_n > X_k$ for all $k < n$).

   (b) Find the probability that X_n is a record, for each $n \geq 1$.

   (c) Are the events $\{X_n \text{ is a record}\}$ and $\{X_k \text{ is a record}\}$, for $k \neq n$, independent?

   (d) For any $m \geq 2$ and any set of indices $j_1 < j_2 < \cdots < j_m$, are the events $\{X_{j_1} \text{ is a record}\}$, $\{X_{j_2} \text{ is a record}\}, \ldots, \{X_{j_m} \text{ is a record}\}$ independent?

   The following part will require use of the fact that, for real values of a,

$$\sum_{k=1}^{\infty} (1/k)^a < \infty \Leftrightarrow a > 1, \qquad \sum_{k=1}^{\infty} (1/k)^a = \infty \Leftrightarrow a \leq 1.$$

   (e) **Extra credit question (Optional):** Show rigorously why the above fact is true.
   Hint: It is easy to show that

$$\int_1^{\infty} (1/x)^a dx = \lim_{y \to \infty} \int_1^y (1/x)^a dx < \infty \Leftrightarrow a > 1, \qquad \int_1^{\infty} (1/x)^a dx = \lim_{y \to \infty} \int_1^y (1/x)^a dx = \infty \Leftrightarrow a \leq 1.$$

   Parts (f) and (i) below require that you use the Borel-Cantelli Lemma, which Prof. Wyatt taught in recitation on Tuesday 9/20 and Thursday 9/22. Two handouts on the topic are available on Stellar.

   (f) Find the probability that the sequence $\{X_n\}$ has an infinite number of records. (Use Borel-Cantelli Lemma.)

   We say that (X_{n-1}, X_n) form a **double-record**[1] if X_{n-1} and X_n are both records.

[1] To clarify, a double-record occurs at time k if X_{k-1} and X_k are both records. For example, if X_{k-1} is not a record, and X_k, X_{k+1} and X_{k+2} are all three records, and X_{k+3} is not a record, then (X_k, X_{k+1}) and (X_{k+1}, X_{k+2}) are both double-records and there are exactly 2 double-records between k and $k + 3$.

(g) Find $\mathbf{P}\{(X_{n-1}, X_n) \text{ is a double-record}\}$, for $n \geq 2$.

(h) Are the events $\{(X_{n-1}, X_n) \text{ is a double-record}\}$ and $\{(X_n, X_{n+1}) \text{ is a double-record}\}$ independent?

(i) Find the probability that the sequence $\{X_n\}$ has an infinite number of double-records. (Use Borel-Cantelli Lemma.)

(j) Continue to assume that the X_n's are independent and identically distributed. Which of the results you derived, if any, depend on the fact the values of each X_n are uniformly distributed between 0 and 1? Which of your results apply to *every* continuous distribution?