Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Recitation 12 Solutions
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1. We need to apply the version of Bayes rule for a continuous random variable conditioned on a discrete random variable:

$$f_{Q|X}(q \mid x) = \frac{f_Q(q)p_{X|Q}(x \mid q)}{p_X(x)} = \frac{f_Q(q)p_{X|Q}(x \mid q)}{\int_0^1 f_Q(q)p_{X|Q}(x \mid q)\,dq}.$$

For $x = 0$ and $q \in [0,1]$,

$$\begin{aligned} f_{Q|X}(q \mid 0) &= \frac{f_Q(q)p_{X|Q}(0 \mid q)}{\int_0^1 f_Q(q)p_{X|Q}(0 \mid q)\,dq} = \frac{6q(1-q)\cdot(1-q)}{\int_0^1 6q(1-q)(1-q)\,dq} \\ &= \frac{6q(1-q)\cdot(1-q)}{1/2} = 12q(1-q)^2. \end{aligned}$$

For $x = 1$ and $q \in [0,1]$,

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{f_Q(q)p_{X|Q}(1 \mid q)}{\int_0^1 f_Q(q)p_{X|Q}(1 \mid q)\,dq} = \frac{6q(1-q)\cdot q}{\int_0^1 6q(1-q)q\,dq} \\ &= \frac{6q(1-q)\cdot q}{1/2} = 12q^2(1-q). \end{aligned}$$

The distributions $f_Q(q)$, $f_{Q|X}(q \mid 0)$, and $f_{Q|X}(q \mid 1)$ are all in the family of *beta distributions*, which arise again in Chapter 8.

2. See Example 3.20 on page 180 of text.

3. Because of the definition of g, the random variable Y takes on only nonnegative values. Thus $f_Y(y) = 0$ for any negative y. For $y > 0$,

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}\left(X \in [-y,0]\right) + \mathbf{P}\left(X \in (0,y^2]\right) \\ &= \left(F_X(0) - F_X(-y)\right) + \left(F_X(y^2) - F_X(0)\right) \\ &= F_X(y^2) - F_X(-y). \end{aligned}$$

Taking the derivative of $F_Y(y)$ (and using the chain rule),

$$\begin{aligned} f_Y(y) &= 2yf_X(y^2) + f_X(-y) \\ &= \frac{1}{\sqrt{2\pi}}\left(2ye^{-y^4/2} + e^{-y^2/2}\right). \end{aligned}$$

4. We will find the PDF of Z by first finding its CDF and then differentiating.

$$\begin{aligned} F_Z(z) = \mathbf{P}(Z \le z) &= \mathbf{P}(\frac{X}{1+Y} \le z) \\ &= \int_0^1 \int_0^{\min\{1,(1+y)z\}} dx\ dy \\ &= \int_0^1 \min\{1,(1+y)z\} dy \\ &= \begin{cases} \dfrac{3}{2}z\,, & 0 \le z \le 1/2 \\ 2 - \dfrac{1}{2z} - \dfrac{z}{2}\,, & 1/2 \le z \le 1 \end{cases} . \end{aligned}$$

By differentiating, we obtain

$$f_Z(z) = \begin{cases} 3/2\,, & 0 \le z \le 1/2 \\ \dfrac{1}{2z^2} - \dfrac{1}{2}\,, & 1/2 \le z \le 1 \end{cases} .$$