LECTURE 17

- **Readings:** Section 7.3

Lecture outline

- Review
- Steady-state behavior
  - Steady-state convergence theorem
  - Balance equations
- Birth-death processes

Review: Discrete-time Markov chain X_n

- Discrete-time random process
- Takes values in a finite set, usually $\{1, \ldots, m\}$
- **Markov property:**

$$\mathbf{P}(X_{n+1} = j \mid X_n = i,\, X_{n-1} = i_{n-1},\, \ldots,\, X_0 = i_0) \\ = \mathbf{P}(X_{n+1} = j \mid X_n = i) = p_{ij}$$

- n-step transition probabilities: $r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$
- Chapman–Kolmogorov equation:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}$$

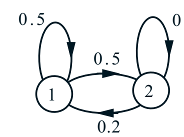
Warmup

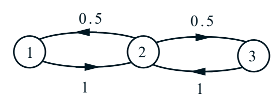
$\mathbf{P}(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$

$\mathbf{P}(X_3 = 7 \mid X_0 = 2) =$

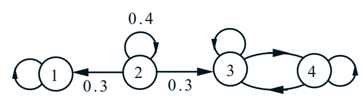
Review: Limits of n-step transition probabilities

- Does $r_{ij}(n)$ converge as $n \to \infty$?
- Does the limit depend on i?





$$r_{22}(n) = \begin{cases} 0, & \text{for odd } n; \\ 1, & \text{for even } n \end{cases}$$



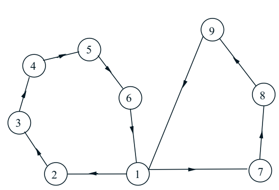
$$r_{11}(n) = 1$$
$$r_{31}(n) = 0$$
$$r_{21}(n) = \tfrac{1}{2}(1 - (0.4)^n)$$

Review: Classification of states/classes

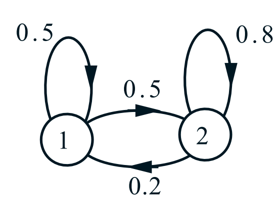
- State i is **recurrent** when:

  for every j accessible from i, i is accessible from j

- When a state is not recurrent, it is **transient**
- A **recurrent class** is a set of states accessible from each other, with no other state accessible from them
- A recurrent class is **periodic** when its states can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group



Steady-state probabilities: Example



Steady-state convergence theorem

- Markov chain with a **single recurrent class**, which is **aperiodic**, converges to a **steady-state** PMF on the states

$$\pi_j \;=\; \lim_{n\to\infty} \mathbf{P}(X_n = j), \qquad j = 1, 2, \ldots, m.$$

"Convergence" includes lack of dependence on initial state:

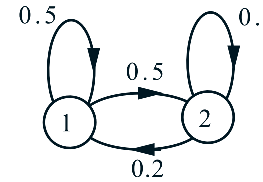
$$\lim_{n\to\infty} r_{ij}(n) \;=\; \pi_j, \qquad \text{for all } i.$$

- Constraint derived from $\quad r_{ij}(n) \;=\; \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}$

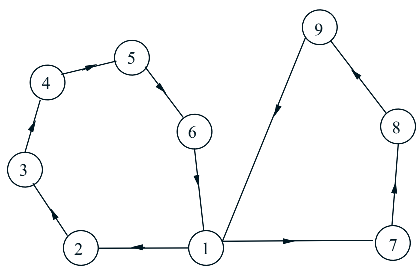
– Take limit $n \to \infty$: $\quad \pi_j \;=\; \sum_{k=1}^{m} \pi_k\, p_{kj}, \quad$ for all j

– Additional equation: $\quad \sum_{j=1}^{m} \pi_j = 1$

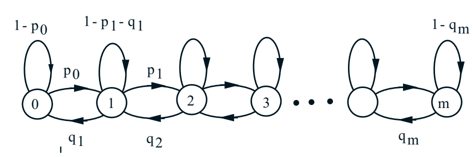
Balance equations: Example

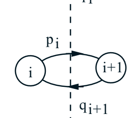


Alternate balance equations



Birth–death processes





$$\pi_i\, p_i \;=\; \pi_{i+1}\, q_{i+1}$$

- Special case: $p_i = p$ and $q_i = q$ for all i, $\quad \rho = \dfrac{p}{q} =$ load factor

$$\pi_{i+1} = \pi_i \frac{p}{q} = \pi_i\, \rho \qquad \text{so} \qquad \pi_i = \pi_0\, \rho^i, \qquad i = 0, 1, \ldots, m$$

Long-term frequency interpretations

- Under the convergence conditions,

$$\lim_{n\to\infty} \frac{v_{ij}(n)}{n} \;=\; \pi_j$$

where $v_{ij}(n)$ is the expected number of **visits** to j in n transitions starting from i

- Under the convergence conditions,

$$\lim_{n\to\infty} \frac{q_{jk}(n)}{n} \;=\; \pi_j\, p_{jk}$$

where $q_{jk}(n)$ is the expected number of $j \to k$ **transitions** in n transitions starting from i

Long-term frequencies: Example