Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 6
Due October 28, 2009

1. The random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B be the event $\{Y > X\}$.

(a) Calculate $\mathbf{P}(B \mid A)$.

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

(c) Calculate $f_{X|B}(x)$.

(d) Calculate $\mathbf{E}[XY]$.

(e) Calculate the PDF of Y/X.

2. Let $X_1, \ldots, X_n$ be independent random variables that are uniformly distributed in the interval $[0,1]$. Let S and L be the smallest and largest, respectively, of $X_1, \ldots, X_n$. Determine the joint PDF of S and L. *Hint:* Use the formula for finding a joint PDF from the joint CDF on page 162 of the text.

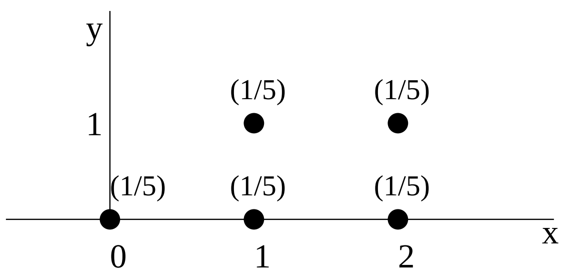
3. Let X be a random variable with PDF f_X. Find the PDF of the random variable $Y = |X|$

(a) when $f_X(x) = 1/3, \quad -2 < x \leq 1$,

(b) when $f_X(x) = 2e^{-2x}, \quad x > 0$,

(c) for general $f_X(x)$.

4. X and Y are random variables and have the following joint PMF.



(a) Find the PMF of X, $p_X(x)$.

(b) Find the PMF of Y, $p_Y(y)$.

(c) Let $Z = X + Y$ and find the PMF of Z, $p_Z(z)$.

(d) Convolve $p_X(x)$ and $p_Y(y)$ and compare your answer to the PMF from part (c). Explain any discrepancy.

5. Let Y and Z be independent random variables, with Y exponentially distributed with parameter 1 and Z uniformly distributed over the interval $[0,1]$. Use convolution to find the PDF of $|Y - Z|$. *Hint:* Find first the PDF of $Y - Z$.

6. Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i. Show that X_1 and X_2 are negatively correlated (intuitively, a large number of ones suggests a smaller number of twos).

7. Let X be a random variable with the following moments:

$$\mathbf{E}[X] = 0, \quad \mathbf{E}[X^2] = 1, \quad \mathbf{E}[X^3] = 0, \quad \mathbf{E}[X^4] = 3, \quad \mathbf{E}[X^5] = 0, \quad \mathbf{E}[X^6] = 15.$$

   Define a new random variable $Y = a + bX^2 + cX^3$, where a, b, c are nonzero constants. Find the correlation coefficient $\rho(X, Y)$ between X and Y.

8. **A financial parable.** An investment bank is managing \$1 billion, which it invests in various financial instruments ("assets") related to the housing market (e.g., the infamous "mortgage backed securities"). Because the bank is investing with borrowed money, its actual assets are only \$50 million (5%). Accordingly, if the bank loses more than 5%, it becomes insolvent. (Which means that it will have to be bailed out, and the bankers may need to forgo any huge bonuses for a few months.)

   (a) The bank considers investing in a single asset, whose gain (over a 1-year period, and measured in percentage points) is modeled as a normal random variable R, with mean 7 and standard deviation 10. (That is, the asset is expected to yield a 7% profit.) What is the probability that the bank will become insolvent? Would you be accepting this level of risk?

   (b) In order to safeguard its position, the bank decides to diversify its risk. It considers investing \$50 million in each of twenty different assets, with the ith one having a gain R_i, which is again normal with mean 7 and standard deviation 10; the bank's gain will be $(R_1 + \cdots + R_{20})/20$. These twenty assets are chosen to reflect the housing sectors at different states or even countries, and the bank's rocket scientists choose to model the R_i as independent random variables. According to this model, what is the probability that the bank becomes insolvent?

   (c) Based on the calculations in part (b), the bank goes ahead with the diversified investment strategy. It turns out that a global economic phenomenon can affect the housing sectors in different states and countries simultaneously, and therefore the gains R_i are in fact positively correlated. Suppose that for every i and j where $i \neq j$, the correlation coefficient $\rho(R_i, R_j)$ is equal to 1/2. What is the probability that the bank becomes insolvent? You can assume that $(R_1 + \cdots + R_{20})/20$ is normal.