Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Tutorial 4 Solutions
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1. In general we have that $\mathbf{E}[aX + bY + c] = a\mathbf{E}[X] + b\mathbf{E}[Y] + c$. Therefore,

$$\mathbf{E}[Z] = 2 \cdot \mathbf{E}[X] - 3 \cdot \mathbf{E}[Y].$$

For the case of independent random variables, we have that if $Z = a \cdot X + b \cdot Y$, then

$$\mathrm{var}(Z) = a^2 \cdot \mathrm{var}(X) + b^2 \cdot \mathrm{var}(Y).$$

Therefore, $\mathrm{var}(Z) = 4 \cdot \mathrm{var}(X) + 9 \cdot \mathrm{var}(Y)$.

2. (a) We can find c knowing that the probability of the entire sample space must equal 1.

$$\begin{aligned} 1 &= \sum_{x=1}^{3}\sum_{y=1}^{3} p_{X,Y}(x,y) \\ &= c + c + 2c + 2c + 4c + 3c + c + 6c \\ &= 20c \end{aligned}$$

Therefore, $c = \frac{1}{20}$.

(b) $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x,2) = 2c + 0 + 4c = 6c = \frac{3}{10}$.

(c) $Z = YX^2$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= \mathbf{E}[YX^2 \mid Y = 2] \\ &= \mathbf{E}[2X^2 \mid Y = 2] \\ &= 2\mathbf{E}[X^2 \mid Y = 2] \end{aligned}$$

$p_{X|Y}(x \mid 2) = \frac{p_{X,Y}(x,2)}{p_Y(2)}$.
Therefore,

$$p_{X|Y}(x \mid 2) = \begin{cases} \frac{1/10}{3/10} = \frac{1}{3} & \text{if } x = 1 \\ \frac{1/5}{3/10} = \frac{2}{3} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= 2\sum_{x=1}^{3} x^2 p_{X|Y}(x \mid 2) \\ &= 2\left((1^2) \cdot \frac{1}{3} + (3^2) \cdot \frac{2}{3}\right) \\ &= \frac{38}{3} \end{aligned}$$

(d) Yes. Given $X \neq 2$, the distribution of X is the same given $Y = y$.
$\mathbf{P}(X = x \mid Y = y, X \neq 2) = \mathbf{P}(X = x \mid X \neq 2)$.
For example,

$$\mathbf{P}(X = 1 \mid Y = 1, X \neq 2) = \mathbf{P}(X = 1 \mid Y = 3, X \neq 2) = \mathbf{P}(X = 1 \mid X \neq 2) = \frac{1}{3}$$

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(e) $p_{Y|X}(y \mid 2) = \frac{p_{X,Y}(2,y)}{p_X(2)}$.
$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2, y) = c + 0 + c = 2c = \frac{1}{10}$.
Therefore,

$$p_{Y|X}(y \mid 2) \;=\; \begin{cases} \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 1 \\ \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 3 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{E}[Y^2 \mid X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y \mid 2) = (1^2) \cdot \frac{1}{2} + (3^2) \cdot \frac{1}{2} = 5$.
$\mathbf{E}[Y \mid X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 2) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2$.
$\text{var}(Y \mid X = 2) = \mathbf{E}[Y^2 \mid X = 2] - \mathbf{E}[Y \mid X = 2]^2 = 5 - 2^2 = 1$.

3. (a) Since $\mathbf{P}(A) > 0$, we can show independence through $\mathbf{P}(B) = \mathbf{P}(B \mid A)$:

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(B \cap A)}{\mathbf{P}(A)} = \frac{\binom{8}{6} p^6 (1-p)^2 p}{\binom{8}{6} p^6 (1-p)^2} = p = \mathbf{P}(B).$$

Therefore, A and B are independent.

(b) Let C be the event "3 heads in the first 4 tosses" and let D be the event "2 heads in the last 3 tosses". Since there are no overlap in tosses in C and D, they are independent:

$$\begin{aligned} \mathbf{P}(C \cap D) &= \mathbf{P}(C)\mathbf{P}(D) \\ &= \binom{4}{3} p^3 (1-p) \cdot \binom{3}{2} p^2 (1-p) \\ &= 12 p^5 (1-p)^2. \end{aligned}$$

(c) Let E be the event "4 heads in the first 7 tosses" and let F be the event "2nd head occurred during 4th trial". We are asked to find $\mathbf{P}(F \mid E) = \mathbf{P}(F \cap E)/\mathbf{P}(E)$. The event $F \cap E$ occurs if there is 1 head in the first 3 trials, 1 head on the 4th trial, and 2 heads in the last 3 trials. Thus, we have

$$\begin{aligned} \mathbf{P}(F \mid E) = \frac{\mathbf{P}(F \cap E)}{\mathbf{P}(E)} &= \frac{\binom{3}{1} p (1-p)^2 \cdot p \cdot \binom{3}{2} p^2 (1-p)}{\binom{7}{4} p^4 (1-p)^3} \\ &= \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} = \frac{9}{35}. \end{aligned}$$

Alternatively, we can solve this by counting. We are given that 4 heads occurred in the first 7 tosses. Each sequence of 7 trials with 4 heads is equally probable, the discrete uniform probability law can be used here. There are $\binom{7}{4}$ outcomes in E. For the event $E \cap F$, there are $\binom{3}{1}$ ways to arrange 1 head in the first 3 trials, 1 way to arrange the 2nd head in the 4th trial and $\binom{3}{2}$ ways to arrange 2 heads in the first 3 trials. Therefore,

$$\mathbf{P}(F \mid E) = \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} = \frac{9}{35}.$$

(d) Let G be the event "5 heads in the first 8 tosses" and let H be the event "3 heads in the last 5 tosses". These two events are not independent as there is some overlap in the tosses (the 6th, 7th, and 8th tosses). To compute the probability of interest, we carefully count all the disjoint, possible outcomes in the set $G \cap H$ by conditioning on the number of heads in the 6th, 7th, and the 8th tosses. We have

$$\begin{aligned}
\mathbf{P}(G \cap H) = {} & \mathbf{P}(G \cap H \mid 1 \text{ head in tosses 6–8})\mathbf{P}(1 \text{ head in tosses 6–8}) \\
& + \mathbf{P}(G \cap H \mid 2 \text{ heads in tosses 6–8})\mathbf{P}(2 \text{ heads in tosses 6–8}) \\
& + \mathbf{P}(G \cap H \mid 3 \text{ heads in tosses 6–8})\mathbf{P}(3 \text{ heads in tosses 6–8})
\end{aligned}$$

$$\begin{aligned}
&= \binom{5}{4} p^4(1-p) \cdot p^2 \cdot \binom{3}{1} p(1-p)^2 \\
&\quad + \binom{5}{3} p^3(1-p)^2 \cdot \binom{2}{1} p(1-p) \cdot \binom{3}{2} p^2(1-p) \\
&\quad + \binom{5}{2} p^2(1-p)^3 \cdot (1-p)^2 \cdot p^3.
\end{aligned}$$

$$= 15p^7(1-p)^3 + 60p^6(1-p)^4 + 10p^5(1-p)^5.$$