MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2010)

Problem Set 5
Due: March 15, 2010

1. Any computer chip will eventually fail, creating a random lifetime. Suppose that a manufacturing process produces a mix of "good" and "bad" chips. For some positive number α, the lifetimes of good chips have the exponential distribution with parameter α and the lifetimes of bad chips have the exponential distribution with parameter 1000α. Assume that the fraction of good chips is p and the fraction of bad chips $1 - p$.

   (a) Find the probability that a randomly selected chip is still functioning after t time units of operation.

   (b) Find the PDF of the time of failure of a randomly selected chip.

   (c) To weed out bad chips, each chip is tested for t time units, and only chips that do not fail during the testing period are shipped to customers. Find a formula for the probability that a customer receives a bad chip (as a function of the constants α, p, and t). If $p = 0.9$, how long should the testing be to make the probability of shipping bad product be below 1%?

2. To help find a pile of gold buried in a swamp, a 6.041 graduate uses a satellite scan to compute the following a priori probability density function for the X and Y coordinates of the treasure:

$$f_{X,Y}(x, y) = \begin{cases} Qx^2y & \text{if } 0 \le x \le 10,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   where Q is a constant. Just then, a librarian rushes in with a fragment of a treasure map. The map says that the X coordinate of the gold is either 6 or 8, but the number is so smudged that the two treasure hunters decides that both possibilities are equally likely.

   Using this information, calculate and sketch the probability density function for the Y coordinate of the gold's location.

3. Suppose N is a geometric random variable with parameter p, where p is a random variable, uniformly distributed from 0 to $\frac{n-1}{n}$. Find the PMF for N, and then find the limit of the PMF as $n \to \infty$.

4. A 6.041 graduate opens a new casino in Las Vegas and decides to make the games more challenging from a probabilistic point of view. In a new version of roulette, each contestant spins the following kind of roulette wheel. The wheel has radius r and its perimeter is divided into 20 intervals, alternating red and black. The red intervals (along the perimeter) are twice the width of the black intervals (also along the perimeter). The red intervals all have the same length and the black intervals all have the same length. After the wheel is spun, the center of the ball is equally likely to settle in any position on the edge of the wheel; in other words, the angle of the final ball position (marked at the ball's center) along the wheel's perimeter is distributed uniformly between 0 and 2π radians.

   (a) What is the probability that the center of the ball settles in a red interval?

   (b) Let B denote the event that the center of the ball settles in a black interval. Find the conditional PDF $f_{Z|B}(z)$, where Z is the distance, along the perimeter of the roulette wheel, between the center of the ball and the edge of the interval immediately clockwise from the center of the ball?

(c) What is the unconditional PDF $f_Z(z)$?

Another attraction of the casino is the Gaussian slot machine. In this game, the machine produces independent identically distributed (IID) numbers $X_1, X_2, \ldots$ that have normal distribution $\mathcal{N}(0, \sigma^2)$. For every i, when the number X_i is positive, the player receives from the casino a sum of money equal to X_i. When X_i is negative, the player pays the casino a sum of money equal to $|X_i|$.

(d) What is the standard deviation of the net total gain of a player after n plays of the Gaussian slot machine?

(e) What is the probability that the absolute value of the net total gain after n plays is greater than $2\sqrt{n}\sigma$?

5. Let Q be a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability Q. Furthermore, given the value of Q, the status of the machine on different days is independent.

   (a) Find the probability that the machine is functional on a particular day.

   (b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of Q. You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}$$

   (c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

G1†. Random variables $X_1, \ldots, X_n$ are independent and identically distributed.
Find $\mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x]$.

†Required for 6.431; optional for 6.041