Quiz 1 Solutions:
October 12, 2010

Problem 1.

1. **(10 points)** Let R_i be the amount of time Stephen spends at the ith red light. R_i is a Bernoulli random variable with $p = 1/3$. The PMF for R_i is:

$$\mathbf{P}_{R_i}(r) = \begin{cases} 2/3, & \text{if } r = 0, \\ 1/3, & \text{if } r = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The expectation and variance for R_i are:

$$\begin{aligned} \mathbf{E}[R_i] &= p = \frac{1}{3}, \\ \text{var}(R_i) &= p(1-p) = \frac{1}{3}\frac{2}{3} = \frac{2}{9}. \end{aligned}$$

Let T_S be the total length of time of Stephen's commute in minutes. Then,

$$T_S = 18 + \sum_{i=1}^{5} R_i.$$

T_S is a shifted binomial with $n = 5$ trials and $p = 1/3$. The PMF for T_S is then:

$$\mathbf{P}_{T_S}(k) = \begin{cases} \binom{5}{k-18}\left(\frac{1}{3}\right)^{k-18}\left(\frac{2}{3}\right)^{23-k}, & \text{if } k \in \{18, 19, 20, 21, 22, 23\}, \\ 0, & \text{otherwise.} \end{cases}$$

The expectation and variance for T_S are:

$$\begin{aligned} \mathbf{E}[T_S] &= \mathbf{E}\left[18 + \sum_{i=1}^{5} R_i\right] \\ &= \frac{59}{3}. \\ \text{var}(T_S) &= \text{var}\left(18 + \sum_{i=1}^{5} R_i\right) \\ &= \frac{10}{9}. \end{aligned}$$

2. **(10 points)** Let N be the number of red lights Stephen encountered on his commute. Given that $T_S \leq 19$, then $N = 0$ or $N = 1$. The unconditional probability of $N = 0$ is $\mathbf{P}(N = 0) = (\frac{2}{3})^5$. The unconditional probability of $N = 1$ is $\mathbf{P}(N = 1) = \binom{5}{1}(\frac{2}{3})^4(\frac{1}{3})^1$.

To find the conditional expectation, the following conditional PDF is calculated:

$$\mathbf{P}_{N|T_S\le 19}(n \mid T_S \le 19) = \begin{cases} \dfrac{(\frac{2}{3})^5}{(\frac{2}{3})^5 + \binom{5}{1}(\frac{2}{3})^4(\frac{1}{3})^1}, & \text{if } n = 0, \\ \dfrac{\binom{5}{1}(\frac{2}{3})^4(\frac{1}{3})^1}{(\frac{2}{3})^5 + \binom{5}{1}(\frac{2}{3})^4(\frac{1}{3})^1}, & \text{if } n = 1, \\ 0, & \text{otherwise}, \end{cases} = \begin{cases} 2/7, & \text{if } n = 0, \\ 5/7, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathbf{E}[N \mid T_S \le 19] = \frac{5}{7}.$$

3. **(10 points)** Given that the last red light encountered by Stephen was the fourth light, $R_4 = 1$ and $R_5 = 0$.

   We are asked to compute $\text{var}(N \mid \{R_4 = 1\} \cap \{R_5 = 0\})$. Therefore,

$$\begin{aligned} \text{var}(N \mid \{R_4 = 1\} \cap \{R_5 = 0\}) &= \text{var}(R_1 + R_2 + R_3 + R_4 + R_5 \mid \{R_4 = 1\} \cap \{R_5 = 0\}) \\ &= \text{var}(R_1 + R_2 + R_3 + 1 + 0 \mid \{R_4 = 1\} \cap \{R_5 = 0\}) \\ &= \text{var}(R_1 + R_2 + R_3 + 1) \\ &= \text{var}(R_1 + R_2 + R_3) \\ &= 3\text{var}(R_1) \\ &= \frac{6}{9}. \end{aligned}$$

4. **(10 points)** Under the given condition, the discrete uniform law can be used to compute the probability of interest. There are $\binom{5}{3}$ ways that Stephen can encounter a total of three red lights. There are $\binom{3}{2}$ ways that two out of the first three lights were red. This leaves one additional red light out of the last two lights and there are $\binom{2}{1}$ possible ways that this event can occur. Putting it all together,

$$\mathbf{P}(\text{two of first three lights were red} \mid \text{total of three red lights}) = \frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}} = \frac{3}{5}.$$

5. **(5 points)** Let T_J be the total length of time of Jon's commute in minutes. The PMF of Jon's commute is:

$$\mathbf{P}_{T_J}(\ell) = \begin{cases} \frac{1}{4}, & \text{if } \ell \in \{20, 21, 22, 23\}, \\ 0, & \text{otherwise.} \end{cases}$$

6. **(10 points)** Let A be the event that Jon arrives at work in 20 minutes and let B be the event that exactly one person arrives in 20 minutes.

$$\begin{aligned} \mathbf{P}(A \mid B) &= \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(\{T_J = 20\} \cap \{T_S \ne 20\})}{\mathbf{P}(\{T_J = 20\} \cap \{T_S \ne 20\}) + \mathbf{P}(\{T_J \ne 20\} \cap \{T_S = 20\})} \\ &= \frac{\mathbf{P}(T_J = 20)\mathbf{P}(T_S \ne 20)}{\mathbf{P}(T_J = 20)\mathbf{P}(T_S \ne 20) + \mathbf{P}(T_J \ne 20)\mathbf{P}(T_S = 20)}. \end{aligned}$$

Jon arrives at work in 20 minutes (or $T_J = 20$) if he does not have to wait for the train at the station (or $X = 0$). The probability of this event occurring is:

$$\mathbf{P}(T_J = 20) = \mathbf{P}(X = 0) = \frac{1}{4}.$$

Stephen arrives at work in 20 minutes if he encounters 2 red lights. The probability of this event is a binomial probability:

$$\mathbf{P}(T_S = 20) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3.$$

Thus,

$$\mathbf{P}(A \mid B) \quad = \quad \frac{\frac{1}{4}\left(1 - \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right)}{\frac{1}{4}\left(1 - \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right) + \frac{3}{4}\left(\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3\right)}.$$

7. **(10 points)** The probability of interest is $\mathbf{P}(T_S \leq T_J)$. This can be calculated using the total probability theorem by conditioning on the length of Jon's commute or Jon's wait at the station. If Jon's commute is 20 minutes (or $X = 0$), then Stephen can encounter up to 2 red lights to satisfy $T_S \leq T_J$. Similarly if Jon's commute is 21 minutes (or $X = 1$), Stephen can encounter up to 3 red lights and so on.

$$\begin{aligned}
\mathbf{P}(T_S \leq T_J) &= \sum_{x=0}^{3} \mathbf{P}(T_S \leq T_J \mid X = x)\mathbf{P}(X = x) \\
&= \frac{1}{4}\sum_{x=0}^{3}\sum_{k=0}^{2+x}\binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{5-k} \\
&= 0.9352.
\end{aligned}$$

An alternative approach follows. We first compute the joint PMF of the commute times of Stephen and Jon $\mathbf{P}_{T_S,T_J}(k,\ell)$. Because of independence, $\mathbf{P}_{T_S,T_J}(k,\ell) = \mathbf{P}_{T_S}(k)\mathbf{P}_{T_J}(\ell)$.

Therefore,

$$\begin{aligned}
\mathbf{P}(T_S \leq T_J) &= \mathbf{P}(T_S = 18) + \mathbf{P}(T_S = 19) + \mathbf{P}(T_S = 20) + \mathbf{P}(\{T_S = 21\} \cap \{T_J \geq 21\}) \\
&\quad + \mathbf{P}(\{T_S = 22\} \cap \{T_J \geq 22\}) + \mathbf{P}(\{T_S = 23\} \cap \{T_J = 23\}) \\
&= \left(\frac{2}{3}\right)^5 + \binom{5}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4 + \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 + \binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{4}\right) \\
&\quad + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 \cdot \left(\frac{2}{4}\right) + \left(\frac{1}{3}\right)^5 \cdot \left(\frac{1}{4}\right) \\
&= 0.9352.
\end{aligned}$$

8. **(10 points)** We express the conditional probability as such:

$$\mathbf{P}(X = 3 \mid T_S \leq T_J) = \frac{\mathbf{P}(\{X = 3\} \cap \{T_S \leq T_J\})}{\mathbf{P}(T_S \leq T_J)}.$$

If Jon waited 3 minutes at the train, his commute was 23 minutes and Stephen's commute takes at most as long as Jon's commute since the longest possible commute for Stephen is 23 minutes. Therefore, the numerator in the previous expression is equal to $\mathbf{P}(X=3)=\frac{1}{4}$. The denominator was computed in the previous part.

$$\begin{aligned}\mathbf{P}(X=3 \mid T_S \le T_J) &= \frac{1}{\displaystyle\sum_{x=0}^{3}\sum_{k=0}^{2+x}\binom{5}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{5-k}} \\ &= 0.2673.\end{aligned}$$

Problem 2.

1. **(10 points) Always True.** We need to show that

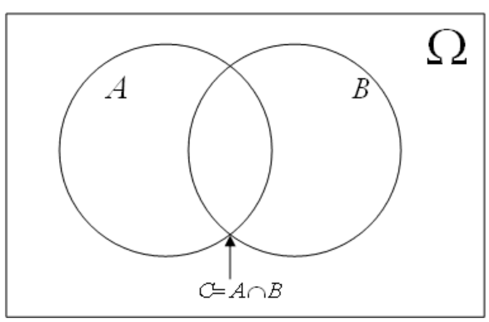
$$\mathbf{P}(A\cap B^c)=\mathbf{P}(A)\mathbf{P}(B^c).$$

We start with expressing $\mathbf{P}(A)$ as $\mathbf{P}(A\cap B)+\mathbf{P}(A\cap B^c)$. Therefore,

$$\begin{aligned}\mathbf{P}(A\cap B^c) &= \mathbf{P}(A)-\mathbf{P}(A\cap B)\\ &= \mathbf{P}(A)-\mathbf{P}(A)\mathbf{P}(B)\\ &= \mathbf{P}(A)(1-\mathbf{P}(B))\\ &= \mathbf{P}(A)\mathbf{P}(B^c),\end{aligned}$$

which shows that A and B^c are independent.

2. **(10 points) Not Always True.** Using the diagram below, let $C=A\cap B$ and let $\mathbf{P}(A)>\mathbf{P}(C)$ and let $\mathbf{P}(B)>\mathbf{P}(C)$. The conditional probability $\mathbf{P}(A\cap B\mid C)=1$. Furthermore, $\mathbf{P}(A\mid C)=1$ and $\mathbf{P}(B\mid C)=1$. Since $\mathbf{P}(A\cap B\mid C)=\mathbf{P}(A\mid C)\mathbf{P}(B\mid C)$, A and B are conditionally independent given a third event C. Given C^c, A and B are disjoint which means that A and B are not independent.



The following is an alternative counterexample. Imagine having 3 coins with the following probability of heads: $p=1/5$, $p=1/3$ and $p=2/3$, respectively. Each coin has equal probability of being selected. Let C be the event that you select the coin with $p=1/5$. Let C^c be the event that you choose one of the other two coins. Let A be the event that the first coin toss results in heads. Let B be the event that the second coin toss results in heads. For a given coin, the tosses are independent such that:

$$\mathbf{P}(B\mid A\cap C)=\mathbf{P}(B\mid C).$$

Given C^c, A and B are not independent since we can have either the $p = 1/3$ coin or the $p = 2/3$ coin. Knowing A changes our beliefs of the result of the second coin toss.

$$\begin{aligned}
\mathbf{P}(B \mid A \cap C^c) &= \frac{B \cap A \cap C^c}{A \cap C^c} \\
&= \frac{\frac{1}{3}\left(\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right)}{\frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\right)} \\
&= \frac{5}{9}.
\end{aligned}$$

However,

$$\begin{aligned}
\mathbf{P}(B \mid C^c) &= \frac{\mathbf{P}(B \cap C^c)}{\mathbf{P}(C^c)} \\
&= \frac{\frac{1}{3}\left(\frac{1}{3} + \frac{2}{3}\right)}{\frac{2}{3}} \\
&= \frac{1}{2}.
\end{aligned}$$

As shown, $\mathbf{P}(B \mid A \cap C^c) \neq \mathbf{P}(B \mid C^c)$.

3. **(10 points) Always True.** Using independence of X and Y, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$. Since variance is always non-negative, $\text{var}(X) + \text{var}(Y) \geq \text{var}(X)$.