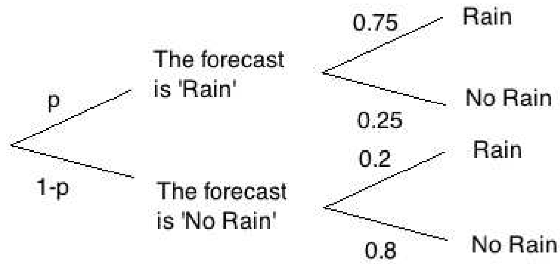
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem Set 2: Solutions
Due: September 22, 2011

1. (a) The tree representation during the winter can be drawn as the following:



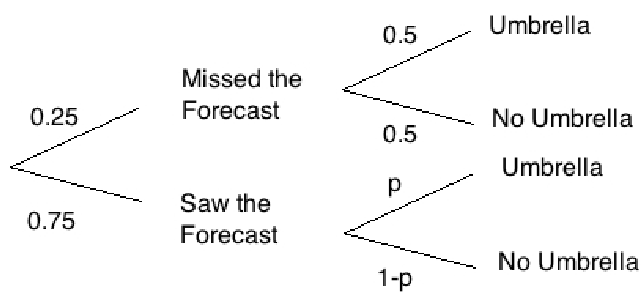
Let A be the event that the forecast was "Rain,"
let B be the event that it rained, and
let p be the probability that the forecast says "Rain." If it is in the winter, $p = 0.7$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.75)(0.7)}{(0.75)(0.7) + (0.2)(0.3)} = \frac{105}{117} \approx 0.8974.$$

Similarly, if it is in the summer, $p = 0.2$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.75)(0.2)}{(0.75)(0.2) + (0.2)(0.8)} = 0.484.$$

(b) Let C be the event that Paul is carrying an umbrella.
Let D be the event that the forecast is no rain.
The tree diagram in this case is:



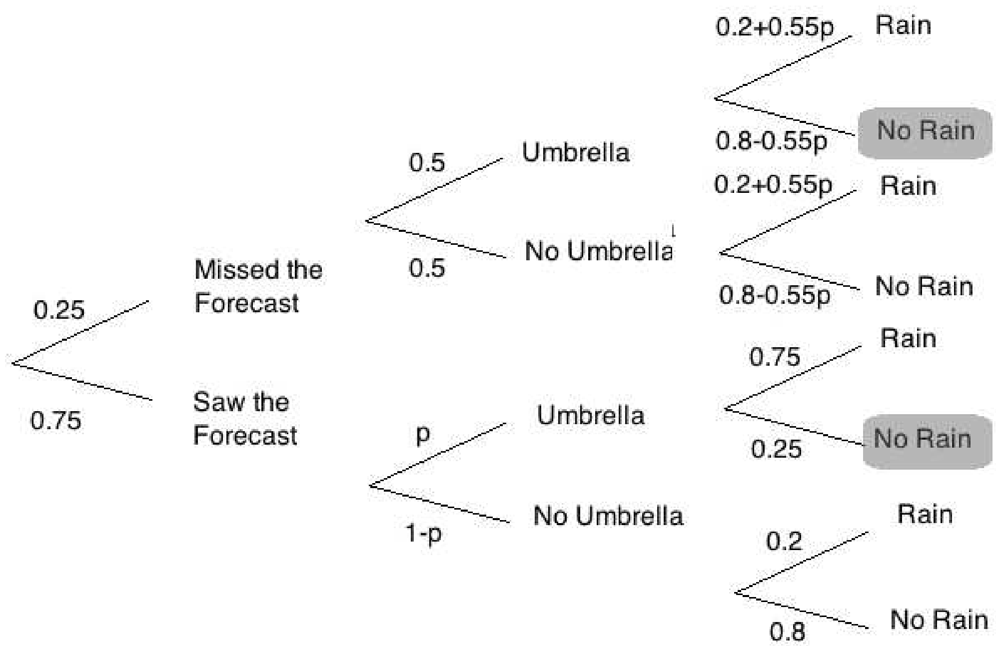
$$\begin{aligned}
\mathbf{P}(D) &= 1 - p \\
\mathbf{P}(C) &= (0.75)p + (0.25)(0.5) = 0.75p + 0.125 \\
\mathbf{P}(C \mid D) &= (0.75)(0) + (0.25)(0.5) = 0.125
\end{aligned}$$

Therefore, $\mathbf{P}(C) = \mathbf{P}(C \mid D)$ if and only if $p = 0$. However, p can only be 0.7 or 0.2, which implies the events C and D can never be independent, and this result does not depend on the season.

(c) Let us first find the probability of rain if Paul missed the forecast.

$$\mathbf{P}(\text{rain} \mid \text{missed forecast}) = (0.75)p + (0.2)(1-p) = 0.2 + 0.55p.$$

Then, we can extend the tree in part (b) as follows:

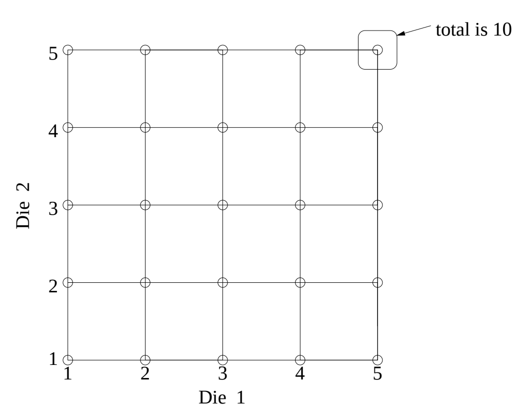


Therefore, given that Paul is carrying an umbrella and it is not raining, we are looking at the two shaded cases.

$$\mathbf{P}(\text{saw forecast} \mid \text{umbrella and not raining}) = \frac{(0.75)p(0.25)}{(0.75)p(0.25) + (0.25)(0.5)(0.8 - 0.55p)}$$

In fall and winter, $p = 0.7$, so the probability is 0.7167.
In summer and spring, $p = 0.2$, so the probability is 0.303.

2. (a) i. $\boxed{\text{No}}$



Overall, there are 25 different outcomes in the sample space. For a total of 10, we should get a 5 on both rolls. Therefore $A \subset B$, and

$$\mathbf{P}(B|A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A)}{\mathbf{P}(A)} = 1$$

We observe that to get at least one 5 showing, we can have 5 on the first roll, 5 on the second roll, or 5 on both rolls, which corresponds to 9 distinct outcomes in the sample space. Therefore

$$\mathbf{P}(B) = \frac{9}{25} \neq \mathbf{P}(B|A)$$

ii. $\boxed{\text{No}}$ Given event A, we know that both roll outcomes must be 5. Therefore, we could not have event C occur, which would require at least one 1 showing. Formally, there are 9 outcomes in C, and

$$\mathbf{P}(C) = \frac{9}{25}$$

But

$$\mathbf{P}(C|A) = 0 \neq \mathbf{P}(C)$$

(b) i. $\boxed{\text{No}}$ Out of the total 25 outcomes, 5 outcomes correspond to equal numbers in the two rolls. In half of the remaining 20 outcomes, the second number is higher than the first one. In the other half, the first number is higher than the second. Therefore,

$$\mathbf{P}(F) = \frac{10}{25}$$

There are eight outcomes that belong to event E:

$$E = \{(1, 2), (2, 3), (3, 4), (4, 5), (2, 1), (3, 2), (4, 3), (5, 4)\}.$$

To find $\mathbf{P}(F|E)$, we need to compute the proportion of outcomes in E for which the second number is higher than the first one:

$$\mathbf{P}(F|E) = \frac{1}{2} \neq \mathbf{P}(F)$$

ii. $\boxed{\text{Yes}}$ Conditioning on event D reduces the sample space to just four outcomes

$$\{(2, 5), (3, 4), (4, 3), (5, 2)\}$$

which are all equally likely. It is easy to see that

$$\mathbf{P}(E|D) = \frac{2}{4} = \frac{1}{2}, \qquad \mathbf{P}(F|D) = \frac{2}{4} = \frac{1}{2}, \qquad \mathbf{P}(E \cap F|D) = \frac{1}{4} = \mathbf{P}(E|D)\mathbf{P}(F|D)$$

3. (a) Suppose the child chooses red apples. Before selecting any apples there are $400 \cdot 0.1 = 40$ rotten red apples. The probability that he chooses three rotten apples is

$$\begin{aligned}
&\mathbf{P}(\text{three rotten}|\text{red}) \\
&= \mathbf{P}(\text{1st is rotten}|\text{red}) \cdot \mathbf{P}(\text{2nd rotten}|\text{1st rotten, red}) \cdot \mathbf{P}(\text{3rd rotten}|\text{1st rotten, 2nd rotten, red}) \\
&= \frac{40}{400}\frac{39}{399}\frac{38}{398} = 0.00093
\end{aligned}$$

Now let's consider the green apples. Before taking any apples, there are $600 \cdot 0.3 = 180$ green rotten apples. Similar to the calculations above,

$$\begin{aligned}&\mathbf{P}(\text{three rotten}|\text{green})\\ &= \mathbf{P}(\text{1st rotten}|\text{green}) \cdot \mathbf{P}(\text{2nd rotten}|\text{1st rotten, green}) \cdot \mathbf{P}(\text{3rd rotten}|\text{1st rotten, 2nd rotten, green})\\ &= \frac{180}{600}\frac{179}{599}\frac{178}{598} = 0.0267\end{aligned}$$

By the total probability theorem,

$$\begin{aligned}\mathbf{P}(\text{three rotten}) &= \mathbf{P}(\text{red}) \cdot \mathbf{P}(\text{three rotten}|\text{red})\\ &\quad +\mathbf{P}(\text{green}) \cdot \mathbf{P}(\text{three rotten}|\text{green})\\ &= 0.5 \cdot 0.00093 + 0.5 \cdot 0.0267 = 0.01382\end{aligned}$$

(b) Using Bayes' rule,

$$\begin{aligned}\mathbf{P}(\text{red}|\text{three rotten}) &= \frac{\mathbf{P}(\text{red}) \cdot \mathbf{P}(\text{three rotten}|\text{red})}{\mathbf{P}(\text{red}) \cdot \mathbf{P}(\text{three rotten}|\text{red}) + \mathbf{P}(\text{green}) \cdot \mathbf{P}(\text{three rotten}|\text{green})}\\ &= \frac{0.5 \cdot 0.00093}{0.5 \cdot 0.00093 + 0.5 \cdot 0.0276} = 0.0326\end{aligned}$$

4. (a)

$$\mathbf{P}(\text{find in A and in A}) = \mathbf{P}(\text{in A}) \cdot \mathbf{P}(\text{find in A}|\text{in A}) = 0.4 \cdot 0.25 = 0.1$$

$$\mathbf{P}(\text{find in B and in B}) = \mathbf{P}(\text{in B}) \cdot \mathbf{P}(\text{find in B}|\text{in B}) = 0.6 \cdot 0.15 = 0.09$$

Oscar should search in Forest A first.

(b) Using Bayes' Rule,

$$\begin{aligned}\mathbf{P}(\text{in A}|\text{not find in A}) &= \frac{\mathbf{P}(\text{not find in A}|\text{in A}) \cdot \mathbf{P}(\text{in A})}{\mathbf{P}(\text{not find in A}|\text{in A}) \cdot \mathbf{P}(\text{in A}) + \mathbf{P}(\text{not find in A}|\text{in B}) \cdot \mathbf{P}(\text{in B})}\\ &= \frac{(0.75) \cdot (0.4)}{(0.4) \cdot (0.75) + (1) \cdot (0.6)} = \frac{1}{3}\end{aligned}$$

(c) Again, using Bayes' Rule,

$$\begin{aligned}\mathbf{P}(\text{looked in A}|\text{find dog}) &= \frac{\mathbf{P}(\text{find dog}|\text{looked in A}) \cdot \mathbf{P}(\text{looked in A})}{\mathbf{P}(\text{find dog})}\\ &= \frac{(0.25) \cdot (0.4) \cdot (0.5)}{(0.25) \cdot (0.4) \cdot (0.5) + (0.15) \cdot (0.6) \cdot (0.5)} = \frac{10}{19}\end{aligned}$$

(d) In order for Oscar to find the dog, it must be in Forest A, not found on the first day, alive, and found on the second day. Note that this calculation requires conditional independence of not finding the dog on different days and the dog staying alive.

$$\begin{aligned}\mathbf{P}(\text{find live dog in A day 2}) &= \mathbf{P}(\text{in A}) \cdot \mathbf{P}(\text{not find in A day 1}|\text{in A})\\ &\quad \cdot \mathbf{P}(\text{alive day 2}) \cdot \mathbf{P}(\text{find day 2}|\text{in A})\\ &= 0.4 \cdot 0.75 \cdot (1 - \frac{1}{3}) \cdot 0.25 = 0.05\end{aligned}$$

5. (a) The statement is true.
First, we prove that if A and B are independent, then A and B^c are independent:

$$\begin{aligned}
A &= (A \cap B) \cup (A \cap B^c) \\
\mathbf{P}(A) &= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B^c) && \text{(additivity axiom)} \\
\mathbf{P}(A \cap B^c) &= \mathbf{P}(A) - \mathbf{P}(A \cap B) \\
&= \mathbf{P}(A) - \mathbf{P}(A)\mathbf{P}(B) && \text{(A and B are independent)} \\
&= \mathbf{P}(A)(1 - \mathbf{P}(B)) \\
&= \mathbf{P}(A)\mathbf{P}(B^c)
\end{aligned}$$

Similarly, we prove that if A, B and C are independent, then $A \cap B^c$ is independent of C:

$$\begin{aligned}
A \cap C &= (A \cap B \cap C) \cup (A \cap B^c \cap C) \\
\mathbf{P}(A \cap B^c \cap C) &= \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C) && \text{(additivity axiom)} \\
&= \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C) && \text{(A and B and C are independent)} \\
&= \mathbf{P}(A)\mathbf{P}(C)(1 - \mathbf{P}(B)) \\
&= \mathbf{P}(A)P(B^c)\mathbf{P}(C) \\
&= \mathbf{P}(A \cap B^c)\mathbf{P}(C) && (A \text{ and } B \text{ indep.} \implies A \text{ and } B^c \text{ indep.})
\end{aligned}$$

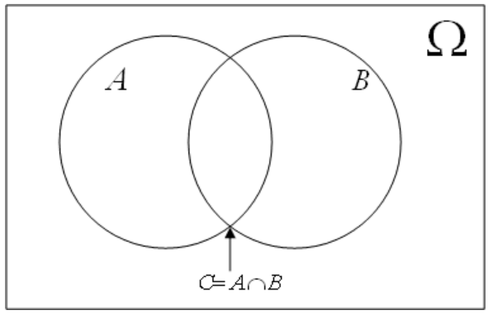
This completes the proof.

(b) The statement is true. Assuming that $\mathbf{P}(C) > 0$:

$$\begin{aligned}
\mathbf{P}(A \cap B \mid C) &= \frac{\mathbf{P}(A \cap B \cap C)}{\mathbf{P}(C)} && \text{(definition of conditional probability)} \\
&= \frac{\mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)}{\mathbf{P}(C)} && \text{(since } A, B \text{ and } C \text{ are mutually independent)} \\
&= \mathbf{P}(A)\mathbf{P}(B) \\
&= \mathbf{P}(A \mid C)\mathbf{P}(B \mid C)
\end{aligned}$$

(c) The statement is false.
Counterexample:
Consider the sample space $\Omega = \{1, 2, 3, 4\}$, where all outcomes are equally likely with probability $1/4$.
Let A, B and C be events such that $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$
A, B and C are pairwise independent since $\mathbf{P}(A \cap B) = \mathbf{P}(\{1\}) = 1/4 = \mathbf{P}(A)\mathbf{P}(B)$, etc.
However, $\mathbf{P}(A \cap B \mid C) = 0 \neq \mathbf{P}(A \mid C)\mathbf{P}(B \mid C) = (1/2)^2 = 1/4$.

(d) The statement is false. Using the diagram below, let $C = A \cap B$ and let $\mathbf{P}(A) > \mathbf{P}(C)$ and let $\mathbf{P}(B) > \mathbf{P}(C)$. The conditional probability $\mathbf{P}(A \cap B \mid C) = 1$. Furthermore, $\mathbf{P}(A \mid C) = 1$ and $\mathbf{P}(B \mid C) = 1$. Since $\mathbf{P}(A \cap B \mid C) = \mathbf{P}(A \mid C)\mathbf{P}(B \mid C)$, A and B are conditionally independent given a third event C. Given C^c, A and B are disjoint which means that A and B are not independent.



6. The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$\mathbf{P}(A_{12}) = \mathbf{P}(A_{13}) = \mathbf{P}(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

Similarly, we also have

$$\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

so

$$\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(A_{12}) \cdot \mathbf{P}(A_{13}).$$

Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{23} also occurs.

G1†. The statement is tantalizing but false. It might be even more tantalizing in words than in symbols. Say that "A suggests B" when $\mathbf{P}(B|A)$ is greater than $\mathbf{P}(B)$. The question asks whether "A_2 suggests A_1" and "A_3 suggests A_2" together imply "A_3 suggests A_1." The answer is no.

As one possible counterexample, consider the sample space generated by a fair 4-side die and define the events $A_1 = \{1\}$, $A_2 = \{1, 2\}$, and $A_3 = \{2\}$. One can easily verify that (a) A_2 suggests A_1; and (b) A_3 suggests A_2. However, $\mathbf{P}(A_1|A_3) = 0 < \frac{1}{4} = \mathbf{P}(A_1)$, so A_3 does not suggest A_1.

G2†. (a) We proceed as follows:

$$\begin{aligned}
\mathbf{P}(A \cap (B \cup C)) &= \mathbf{P}((A \cap B) \cup (A \cap C)) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C) \\
&\overset{*}{=} \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C) \\
&= \mathbf{P}(A)\left[\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B)\mathbf{P}(C)\right] \\
&= \mathbf{P}(A)\mathbf{P}(B \cup C),
\end{aligned}$$

where the equality marked with $*$ follows from the independence of A, B, and C.

(b) Proof 1: If A and B are independent, then A^c and B^c are also independent (see Problem 1.43, page 63 for the proof).

For any two independent events U and V, DeMorgan's Law implies

$$\begin{aligned}
\mathbf{P}(U \cup V) &= \mathbf{P}((U^c \cap V^c)^c) = 1 - \mathbf{P}(U^c \cap V^c) = 1 - \mathbf{P}(U^c) \cdot \mathbf{P}(V^c) \\
&= 1 - (1 - \mathbf{P}(U))(1 - \mathbf{P}(V)).
\end{aligned}$$

†Required for 6.431; optional for 6.041

We proceed to prove the statement by induction. Letting $U = A_1$ and $V = A_2$, the base case is proven above. Now we assume that the result holds for any n and show that it holds for $n+1$. For independent $\{A_1, \ldots, A_n, A_{n+1}\}$, let $B = \cup_{i=1}^{n} A_i$. It is easy to show that B and A_{n+1} are independent. Therefore,

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) &= 1 - (1 - \mathbf{P}(B)) \cdot (1 - \mathbf{P}(A_{n+1})) \\ &= 1 - \prod_{i=1}^{n+1} (1 - \mathbf{P}(A_i)),\end{aligned}$$

which completes the proof.

Proof 2: Alternatively, we can use the version of the DeMorgan's Law for n events:

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) &= \mathbf{P}((A_1^c \cap A_2^c \cap \ldots \cap A_n^c)^c) \\ &= 1 - \mathbf{P}(A_1^c \cap A_2^c \cap \ldots \cap A_n^c).\end{aligned}$$

But we know that $A_1^c, A_2^c, \ldots, A_n^c$ are independent. Therefore

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - \mathbf{P}(A_1^c)\mathbf{P}(A_2^c) \ldots \mathbf{P}(A_n^c) \\ &= 1 - \prod_{i=1}^{n} (1 - \mathbf{P}(A_i)).\end{aligned}$$