## LECTURE 11

- **Readings:** Finish Section 4.1; Section 4.2

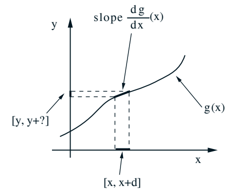
### Lecture outline

- Derived distributions
- Convolution
- Covariance and correlation



### Review: Distribution derived with monotonic function

- Consider $Y = g(X)$, where $g$ is strictly monotonic



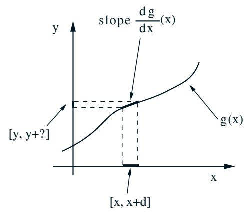

- Event $x \le X \le x+\delta$ is the same as $g(x) \le Y \le g(x+\delta)$
- Approximately:

$$g(x) \;\le\; Y \;\le\; g(x) + \delta \left|\frac{dg}{dx}(x)\right|$$



### Distribution derived with monotonic function

- Consider $Y = g(X)$, where $g = h^{-1}$ is strictly increasing




$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}(g(X) \le y) \\ &= \mathbf{P}(X \le h(y)) \\ &= F_X(h(y)) \end{aligned}$$

Differentiating:

$$f_Y(y) = f_X(h(y))\frac{dh}{dy}(y)$$

- Covering both strictly increasing and strictly decreasing:

$$f_Y(y) = f_X(h(y))\left|\frac{dh}{dy}(y)\right|$$



### Example (from L10): Two light bulbs

- Suppose light bulbs have lifetimes that are independent and identically exponentially distributed

$$f_X(t) = f_Y(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \ge 0; \\ 0, & \text{otherwise} \end{cases}$$

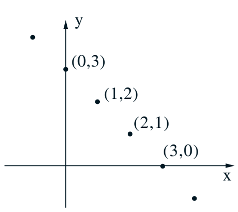
- One is installed at noon, burns out, and is replaced immediately. The replacement burns out at 2pm.
- What is the distribution of the time at which the first bulb burns out? $f_{X|X+Y}(x \mid z) = \begin{cases} 1/z, & \text{if } 0 \le x \le z; \\ 0, & \text{otherwise} \end{cases}$
- By-product: $f_{X+Y}(z) = \begin{cases} \lambda^2 z e^{-\lambda z}, & \text{if } z \ge 0; \\ 0, & \text{otherwise} \end{cases}$



### The distribution of $X+Y$: Discrete case

- $Z = X + Y$; $X, Y$ independent

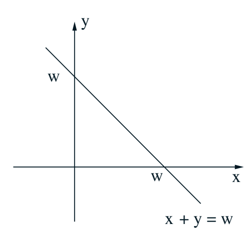
$$\begin{aligned} p_Z(z) &= \mathbf{P}(X+Y=z) \\ &= \sum_x \mathbf{P}(X=x,\, Y=z-x) \\ &= \sum_x \mathbf{P}(X=x)\,\mathbf{P}(Y=z-x) \\ &= \sum_x p_X(x)\,p_Y(z-x) \end{aligned}$$






### The distribution of $X+Y$: Continuous case

- $W = X + Y$; $X, Y$ independent




$$\begin{aligned} f_{W|X}(w \mid x) &= f_Y(w-x) \\ f_{W,X}(w,x) &= f_X(x)\, f_{W|X}(w \mid x) \\ &= f_X(x)\, f_Y(w-x) \end{aligned}$$

- $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$

## Two independent normal r.v.s

- $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, independent

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\} \end{aligned}$$

- PDF is constant on the ellipse where

$$\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2}$$

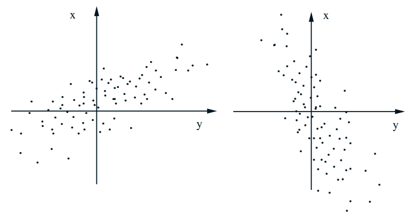
  is constant

- Ellipse is a circle when $\sigma_x = \sigma_y$
- $X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$ [see Example 4.11]



## Covariance

- $\text{cov}(X,Y) = \mathbf{E}\big[(X - \mathbf{E}[X]) \cdot (Y - \mathbf{E}[Y])\big]$
- $\text{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$

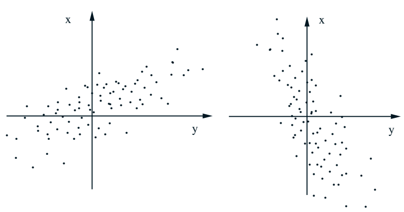



- Zero-mean case: $\text{cov}(X,Y) = \mathbf{E}[XY]$
- independent $\Rightarrow$ $\text{cov}(X,Y) = 0$ (converse is not true)



## Linear least mean squares estimation

- [Preview of §4.3, §8.4, . . . , in simplified setting]
- Let $X$ and $Y$ be jointly distributed with $\mathbf{E}[X] = \mathbf{E}[Y] = 0$




- What linear function $g(X) = aX$ minimizes $\mathbf{E}[(Y - g(X))^2]$?



## Correlation coefficient

- Dimensionless version of covariance:

$$\rho = \mathbf{E}\left[ \frac{(X - \mathbf{E}[X])}{\sigma_X} \cdot \frac{(Y - \mathbf{E}[Y])}{\sigma_Y} \right] = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

- $-1 \le \rho \le 1$
- $|\rho| = 1 \Leftrightarrow X = cY$ (linearly related)
- Independent $\Rightarrow$ $\rho = 0$ (converse is not true)

## Variance of sum of random variables

- $\text{var}\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{var}(X_i) + \sum_{i,j:\ i \neq j} \text{cov}(X_i, X_j)$