LECTURE 8

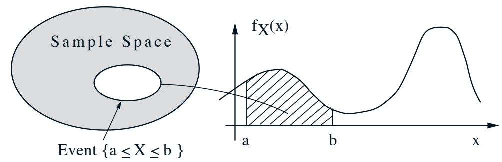
- **Readings:** Sections 3.1-3.3

Lecture outline

- Probability density functions
- Cumulative distribution functions
- Normal random variables

Continuous r.v.'s and pdf's

- **A continuous r.v.** is described by a **probability density function** f_X



$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

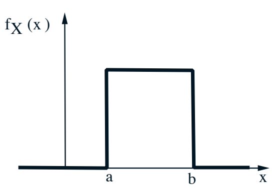
$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\mathbf{P}(x \le X \le x+\delta) = \int_x^{x+\delta} f_X(s)\,ds \approx f_X(x)\cdot\delta$$

$$\mathbf{P}(X \in B) = \int_B f_X(x)\,dx, \quad \text{for "nice" sets } B$$

Means and variances

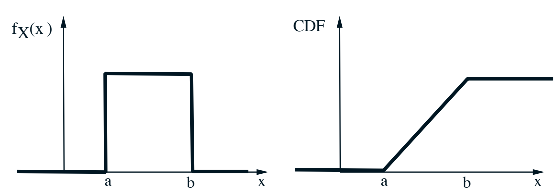
- $\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$
- $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$
- $\text{var}(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2 f_X(x)\,dx$
- **Continuous Uniform r.v.**



- $f_X(x) = \qquad a \le x \le b$
- $\mathbf{E}[X] =$
- $\sigma_X^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx = \frac{(b-a)^2}{12}$

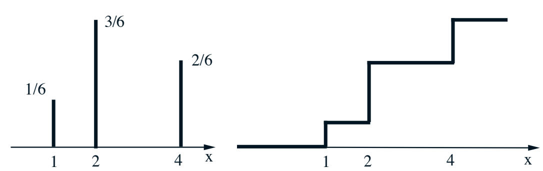
Cumulative distribution function (CDF)

$$F_X(x) = \mathbf{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$



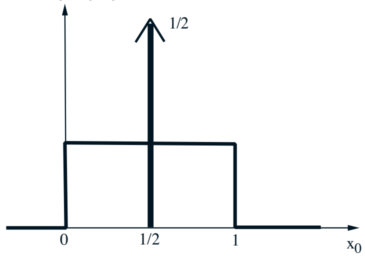
- Also for discrete r.v.'s:

$$F_X(x) = \mathbf{P}(X \le x) = \sum_{k \le x} p_X(k)$$



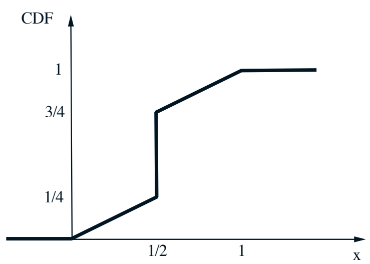
Mixed distributions

- Schematic drawing of a combination of a PDF and a PMF



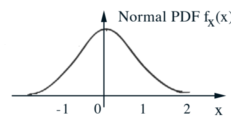
- The corresponding CDF:

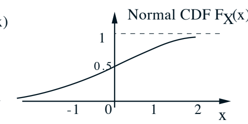
$$F_X(x) = \mathbf{P}(X \le x)$$



Gaussian (normal) PDF

- Standard normal $N(0,1)$: $f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$





- $\mathbf{E}[X] =$ $\qquad$ var$(X) = 1$

- General normal $N(\mu, \sigma^2)$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- It turns out that:
  $\mathbf{E}[X] = \mu$ and Var$(X) = \sigma^2$.

- Let $Y = aX + b$
  - Then: $\mathbf{E}[Y] =$ $\qquad$ Var$(Y) =$
  - Fact: $Y \sim N(a\mu + b, a^2\sigma^2)$

Calculating normal probabilities

- No closed form available for CDF
  - but there are tables (for standard normal)

- If $X \sim N(\mu, \sigma^2)$, then $\dfrac{X - \mu}{\sigma} \sim N($ $\qquad$ $)$

- If $X \sim N(2, 16)$:

$$\mathbf{P}(X \le 3) = \mathbf{P}\left(\frac{X-2}{4} \le \frac{3-2}{4}\right) = \text{CDF}(0.25)$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The constellation of concepts

$p_X(x)$ $\qquad$ $f_X(x)$

$F_X(x)$

$\mathbf{E}[X]$, var(X)

$p_{X,Y}(x,y)$ $\qquad$ $f_{X,Y}(x,y)$

$p_{X|Y}(x \mid y)$ $\qquad$ $f_{X|Y}(x \mid y)$