

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 2
Due September 22, 2010

1. Most mornings, Victor checks the weather report before deciding whether to carry an umbrella. If the forecast is “rain,” the probability of actually having rain that day is 80%. On the other hand, if the forecast is “no rain,” the probability of it actually raining is equal to 10%. During fall and winter the forecast is “rain” 70% of the time and during summer and spring it is 20%.
 - (a) One day, Victor missed the forecast and it rained. What is the probability that the forecast was “rain” if it was during the winter? What is the probability that the forecast was “rain” if it was during the summer?
 - (b) The probability of Victor missing the morning forecast is equal to 0.2 on any day in the year. If he misses the forecast, Victor will flip a fair coin to decide whether to carry an umbrella. On any day of a given season he sees the forecast, if it says “rain” he will always carry an umbrella, and if it says “no rain,” he will not carry an umbrella. Are the events “Victor is carrying an umbrella,” and “The forecast is no rain” independent? Does your answer depend on the season?
 - (c) Victor is carrying an umbrella and it is not raining. What is the probability that he saw the forecast? Does it depend on the season?
2. You have a fair five-sided die. The sides of the die are numbered from 1 to 5. Each die roll is independent of all others, and all faces are equally likely to come out on top when the die is rolled. Suppose you roll the die twice.
 - (a) Let event A be “the total of two rolls is 10”, event B be “at least one roll resulted in 5”, and event C be “at least one roll resulted in 1”.
 - i. Is event A independent of event B ?
 - ii. Is event A independent of event C ?
 - (b) Let event D be “the total of two rolls is 7”, event E be “the difference between the two roll outcomes is exactly 1”, and event F be “the second roll resulted in a higher number than the first roll”.
 - i. Are events E and F independent?
 - ii. Are events E and F independent given event D ?
3. The local widget factory is having a blowout widget sale. Everything must go, old and new. The factory has 500 old widgets, and 1500 new widgets in stock. The problem is that 15% of the old widgets are defective, and 5% of the new ones are defective as well. You can assume that widgets are selected at random when an order comes in. You are the first customer since the sale was announced.
 - (a) You flip a fair coin once to decide whether to buy old or new widgets. You order two widgets of the same type, chosen based on the outcome of the coin toss. What is the probability that they will both be defective?
 - (b) Given that both widgets turn out to be defective, what is the probability that they were old widgets?

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4. Oscar has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6).

On any given day, if the dog is in A and Oscar spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Oscar spends a day looking for it there, the conditional probability that he will find the dog that day is 0.15.

The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only at night.

- (a) In which forest should Oscar look to maximize the probability he finds his dog on the first day of the search?
 - (b) Given that Oscar looked in A on the first day but didn't find his dog, what is the probability that the dog is in A?
 - (c) If Oscar flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?
 - (d) If the dog is alive and not found by the N th day of the search, it will die that evening with probability $\frac{N}{N+2}$. Oscar has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?
5. In solving this problem, feel free to browse problems 43-45 in Chapter 1 of the text for ideas. If you need to, you may quote the results of these problems.
- (a) Suppose that A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.
 - (b) Prove that if A_1, \dots, A_n are independent events, then

$$\mathbf{P}(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - \prod_{i=1}^n (1 - \mathbf{P}(A_i)).$$

- G1[†]. Alice, Bob, and Carroll play a chess tournament. The first game is played between Alice and Bob. The player who sits out a given game plays next the winner of that game. The tournament ends when some player wins two successive games. Let a tournament history be the list of game winners, so for example ACBAA corresponds to the tournament where Alice won games 1, 4, and 5, Carroll won game 2, and Bob won game 3.

- (a) Provide a tree-based sequential description of a sample space where the outcomes are the possible tournament histories.
- (b) We are told that every possible tournament history that consists of k games has probability $1/2^k$, and that a tournament history consisting of an infinite number of games has zero probability. Demonstrate that this assignment of probabilities defines a legitimate probability law.
- (c) Assuming the probability law from part (b) to be correct, find the probability that the tournament lasts no more than 5 games, and the probability for each of Alice, Bob, and Carroll winning the tournament.

[†]Required for 6.431; optional for 6.041