Tutorial 2: Solutions
September 22/23, 2011

1. *A player is randomly dealt 13 cards from a standard 52-card deck.*

   (a) *What is the probability the 13th card dealt is a king?*
   Answer: $\frac{4}{52}$.
   Solution: Since we are not told anything about the first 12 cards that are dealt, the probability that the 13th card dealt is a King, is the same as the probability that the first card dealt, or in fact any particular card dealt is a King, and this equals: $\frac{4}{52}$.

   (b) *What is the probability the 13th card dealt is the first king dealt?*
   Answer: $\frac{1}{13} \cdot 4\binom{48}{12}/\binom{52}{13}$.
   Solution: The probability that the 13th card is the first king to be dealt is the probability that out of the first 13 cards to be dealt, exactly one was a king, and that the king was dealt last. Now, given that exactly one king was dealt in the first 13 cards, the probability that the king was dealt last is just 1/13, since each "position" is equally likely. Thus, it remains to calculate the probability that there was exactly one king in the first 13 cards dealt. To calculate this probability we count the "favorable" outcomes and divide by the total number of possible outcomes. We first count the favorable outcomes, namely those with exactly one king in the first 13 cards dealt. We can choose a particular king in 4 ways, and we can choose the other 12 cards in $\binom{48}{12}$ ways, therefore there are $4 \cdot \binom{48}{12}$ favorable outcomes. There are $\binom{52}{13}$ total outcomes, so the desired probability is

   $$\frac{1}{13} \cdot \frac{4\binom{48}{12}}{\binom{52}{13}}.$$

   For an alternative solution, we argue as in Example 1.10. The probability that the first card is not a king is 48/52. Given that, the probability that the second is not a king is 47/51. We continue similarly until the 12th card. The probability that the 12th card is not a king, given that none of the preceding 11 was a king, is 37/41. (There are $52 - 11 = 41$ cards left, and $48 - 11 = 37$ of them are not kings.) Finally, the conditional probability that the 13th card is a king is 4/40. The desired probability is

   $$\frac{48 \cdot 47 \cdots 37 \cdot 4}{52 \cdot 51 \cdots 41 \cdot 40}.$$

2. (a) There are two different ways to count possible clubs. First, we can choose the club leader from the population. There are n ways to do that. Then for each of the remaining $n-1$ people, we need to decide if that person is in the club or not. There are 2^{n-1} ways to do so. Therefore, the total number of possible clubs in a group of n people is $\boxed{n\,2^{n-1}}$.
   Alternatively, for a club of size k, there are $\binom{n}{k}$ ways to choose the members of the club. From those, there are k ways to choose the leader. Therefore, the total number of possible clubs is $\boxed{\sum_{k=1}^{n} k\binom{n}{k}}$.

   (b) The equality follows immediately from the previous part.

3. (a) This is simply the number of ways of ordering 20 elements, i.e. 20!.

(b) Let us count the number of ways that we can get a sequence of the form (American, foreign, American, foreign, ...). There are 10 fixed slots that the 10 American cars can appear in. The number of ways in which we can order these 10 elements is 10!. Likewise, there are 10 fixed slots that the 10 foreign cars can appear in, and 10! ways in which these 10 elements can be ordered. Hence the total number of ways in which we can get a sequence of this form is $10! \cdot 10!$.

Similarly, the number of ways in which we can get a sequence of the form (foreign, American, foreign, American, ...) is also $10! \cdot 10!$.

The desired probability is thus

$$\frac{2 \cdot 10! \cdot 10!}{20!}.$$

4. A deck of 52 cards is distributed between 4 players.

(a) Find the probability that player 1 gets all spades. There are a total of $\binom{52}{13}$ equally likely hands that player 1 could get. Only one of these hands consists of cards that are all spades. Therefore the probability of getting all spades is $\boxed{\frac{1}{\binom{52}{13}}}$.

(b) Find the probability that some player gets all 13 spades. The four events, player 1 gets all 13 spades, player 2 gets all 13 spades, player 3 gets all 13 spades and player 4 gets all 13 spades are all mutually exclusive. Therefore we can add the probabilities of each to get our answer. Each event has probability $\frac{1}{\binom{52}{13}}$ and so our answer is $\boxed{\frac{4}{\binom{52}{13}}}$.

(c) Consider the following two events:

i. A = Player 1 gets all 13 spades.

ii. B = Player 1 gets the king of hearts.

Events A and B are not independent. If we know B, that is we know that player 1 has the king of hearts, then we know that player 1 CANNOT have all 13 spades. In other words $P(A|B) = 0$ which is not equal to $P(A) = \frac{1}{\binom{52}{13}}$.

Moreover events A and B are mutually exclusive. In general when two events of nonzero probability are mutually exclusive, they CANNOT be independent.

(d) Consider the following two events

i. A = All of player 1's cards have the same suit.

ii. B = Player 1 gets the king of hearts.

We can show that these two events are independent by showing that $P(B|A) = P(B)$. Note that $P(B|A) = 1/4$ since given that player 1 has all 13 cards of a single suit, we know this suit must be one of hearts, diamonds, clubs or spades. Each suit is equally likely with probabiity 1/4, and thus player 1 will have the king of hearts with probability 1/4. Now note that $P(B) = 1/4$, since each of the 4 players can get the king of hearts with equal probability, and so the probability of player 1 getting the king of hearts is 1/4. Thus $P(B|A) = P(B)$ and we conclude the events are independent. Also note that these events are not mutually exclusive since their intersection is not empty – it is the event that Player 1 gets all hearts.