Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Tutorial 7: Solutions

1. Following the usual technique for finding the density of a function of a random variable, we first find the distribution, and then differentiate to find the density.
a) $Y = X^2$.

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}(X^2 \le y) \\ &= \mathbf{P}(X \le \sqrt{y}) \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ \sqrt{y} & 0 \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$

and therefore we have:

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

b) $Y = e^X$.

$$\begin{aligned} F_Y(y) &= \mathbf{P}(Y \le y) \\ &= \mathbf{P}(e^X \le y) \\ &= \mathbf{P}(X \le \ln y) \end{aligned}$$

Thus,

$$F_Y(y) = \begin{cases} 0 & y \le 1 \\ \ln y & 1 \le y \le e \\ 1 & y \ge e \end{cases}$$

and thus we have:

$$f_Y(y) = \begin{cases} \frac{1}{y} & 1 \le y \le e \\ 0 & \text{otherwise} \end{cases}$$

2. Let $Z = X + Y$. Using the 2 step CDF method,

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) \\ &= \mathbf{P}(X + Y \le z) \end{aligned}$$

Using the Total Probability Theorem, we have

$$\begin{aligned} F_Z(z) &= \sum_x p_X(x) p(x+Y \leq z) \\ &= \sum_x p_X(x) p(Y \leq z-x) \\ &= \sum_x p_X(x) F_Y(z-x) \end{aligned}$$

Differentiating both sides with respect to z, we obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \sum_x p_X(x) f_Y(z-x) \end{aligned}$$

3. We will condition on X and use the law of total variance

$$\mathrm{var}(X+Y) = \mathbf{E}\left[\mathrm{var}(X+Y|X)\right] + \mathrm{var}\left(\mathbf{E}[X+Y|X]\right).$$

Given a value x of X, the random variable Y is uniformly distributed in the interval $[x, x+1]$, and the random variable $X+Y$ is uniformly distributed in the interval $[2x, 2x+1]$. Therefore, $\mathbf{E}[X+Y|X] = 0.5 + 2X$ and $\mathrm{var}(X+Y|X) = 1/12$. Thus,

$$\mathrm{var}(X+Y) = \mathrm{var}(0.5+2X) + \mathbf{E}[1/12] = 4\mathrm{var}(X) + \mathbf{E}[1/12] = \frac{5}{12}.$$

4. (a) Let X_i be independent Bernoulli random variables that are equal to 1 if the ith flip results in heads. Let N be the number of coin flips. We have $\mathbf{E}[X_i] = 1/2$, $\mathrm{var}(X_i) = 1/4$, $\mathbf{E}[N] = 7/2$, and $\mathrm{var}(N) = 35/12$. (The last equality is obtained from the formula for the variance of a discrete uniform random variable.) Therefore, the expected number of heads is

$$\mathbf{E}[X_i]\mathbf{E}[N] = \frac{7}{4},$$

and the variance is

$$\mathrm{var}(X_i)\mathbf{E}[N] + (\mathbf{E}[X_i])^2 \mathrm{var}(N) = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot \frac{35}{12} = \frac{77}{48}.$$

(b) The experiment in part (b) can be viewed as consisting of two independent repetitions fo the experiment in part (a). Thus, both the mean and the variance are doubled and become 7/2 and 77/24, respectively.