Problem Set 10
Due December 4, 2009

1. Joe the Plumber has been having trouble keeping the income at his plumbing business steady lately. He notices that his profit each week is uniformly distributed between \$1,800 and \$8,600. Being a small business owner, he doesn't have the stomach for these fluctuations, especially because they introduce additional uncertainty as to whether or not he'll soon face higher taxes. Assume that each week's profit is independent of the profits of all other weeks. Find a good approximation to the probability that over the course of a year (52 weeks), Joe's plumbing business will have a yearly income of less than \$250,000.

2. On any given flight, an airline tries to sell as many tickets as possible. Suppose, on average, 10% of ticket holders fail to show up, all independent of each other. Knowing this, an airline will sell more tickets than available seats (overbook the flight) and hope that there are sufficient numbers of passengers who do not show up to compensate for its overbooking. Using the central limit theorem, determine n, the maximum number of tickets an airline should sell on a flight with 300 seats so that it can be approximately 99% confident that it need not deny boarding to any of the n passengers holding tickets.

3. Random variable X is uniformly distributed between -1.0 and 1.0. Let $X_1, X_2, \ldots,$ be independent identically distributed random variables with the same distribution as X. Determine which, if any, of the following sequences (all with $i = 1, 2, \ldots$) are convergent in probability. Give reasons for your answers. Include the limits if they exist.

   (a) $T_i = X_1 + X_2 + \ldots + X_i$
   (b) $U_i = \dfrac{X_1 + X_2 + \ldots + X_i}{i}$
   (c) $W_i = \max(X_1, \ldots, X_i)$
   (d) $V_i = X_1 \cdot X_2 \cdot \ldots \cdot X_i$

4. You have two nearly identical coins in your pocket, a weighted coin that comes up heads with probability 0.55, along with a fair coin. With no obvious way to tell the two apart in appearance, you take out one coin and decide to flip it 1000 times. If it comes up heads more than 525 times, you'll presume that you've been flipping the biased coin. Assume that in reality you have the fair coin.

   (a) By using the de Moivre-Laplace normal approximation to the binomial, approximate the probability that you will presume it is the biased coin.
   (b) Find an upper bound on the probability that you will presume it is the biased coin by using the Markov inequality.
   (c) Find an upper bound on the probability that you will presume it is the biased coin by using the Chebyshev inequality.

5. Demonstrate that the Chebyshev inequality is tight. That is, for every μ, $\sigma > 0$, and $c \geq \sigma$, construct a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}$$

Hint: You should be able to do this with a discrete random variable that takes on only 3 distinct values with nonzero probability.

6. Let $\{X_i,\, i \geq 1\}$ be i.i.d. Bernoulli random variables. Let $\mathbf{P}(X_i = 1) = \lambda$, $\mathbf{P}(X_i = 0) = 1 - \lambda$. Let $S_n = X_1 + ... + X_n$. Let m be an arbitrary but fixed positive integer. Think! Then evaluate the following and explain your answers:

   (a) $\lim_{n\to\infty} \sum\limits_{n\lambda - m \leq i \leq n\lambda + m} \mathbf{P}(S_n = i)$

   (b) $\lim_{n\to\infty} \sum\limits_{0 \leq i \leq n\lambda + m} \mathbf{P}(S_n = i)$

   (c) $\lim_{n\to\infty} \sum\limits_{n(\lambda - 1/m) \leq i \leq n(\lambda + 1/m)} \mathbf{P}(S_n = i)$

7. A fair coin is tossed on each round of the ***Double or Quarter Game***. The player begins with a total wealth of dollar 1.00. On the first round he gets back twice his bet (i.e., \$2.00) if the coin comes up heads, but only a quarter of his bet (i.e., 25 cents) if it comes up tails. On the second round he bets his entire wealth (now either \$2.00 or 25 cents) and receives back twice his bet if the coin comes up heads, or a quarter of his bet if it comes up tails. The game then continues in the same fashion, with his remaining wealth being doubled or else divided by 4 at each toss. The various tosses are independent, and the player bets his entire remaining wealth at each toss.

   (a) How many heads does he need to get on the first 100 tosses to break even or come out ahead (i.e., so that his remaining wealth after 100 tosses is at least \$1.00)?

   Parts (b)-(e) are best handled in terms of the number of heads required for his wealth after 100 tosses to be at least 1 dollar.

   (b) Write down (but do not evaluate) an expression for the probability p_w that his wealth is at least \$1.00 after 100 tosses.
   The above expression can be difficult to evaluate numerically, since terms like 100! exceed a googol(10^{100}!!). For that reason certain estimates and bounds can be helpful.

   (c) Find an upper bound on p_w by using the Markov inequality.

   (d) Find a better upper bond on p_w by using the Chebyshev inequality.

   (e) Find a good estimate for p_w by using the central limit theorem. Compare the results you obtained in parts (c), (d) and (e).

G1†. The ***Chernoff Bound*** is a variant on the Markov bound that provides an exponential bound on the probability of a certain range of values for a random variable X. Let $Y = e^{rX}$. Then $\mathbf{E}[Y] = \mathbf{E}[e^{rX}] = g_x(r)$ is the moment generating function for X. Since $Y \geq 0$, the Markov inequality guarantees that $\mathbf{P}\left(Y = e^{rX}\right) \geq y \leq \frac{g_X(r)}{y}$ or, letting $y = e^{ra}$, we have the two inequalities:

$$\mathbf{P}\left(e^{rX} \geq e^{ra}\right) = \mathbf{P}\left(X \geq a\right) \leq g_X(r)e^{-ra}, \text{ for all } r > 0,$$
$$\mathbf{P}\left(e^{rX} \geq e^{ra}\right) = \mathbf{P}(X \leq a) \leq g_X(r)e^{-ra}, \text{ for all } r < 0.$$

These bounds are exponential in a (or $-a$ respectively) and can be optimized over r to get the strongest bound. Using the Chernoff bound with the best value of r, find an upper bound on p_w. Briefly compare the value you obtain to those you found in the 6.041 problem above.

G2†. **Optimizing the Long-Term Outcome from Gambling or Investments**

This problem deals with the long-term outcome from multiplicative or investment models. We take the **Double or Quarter game** as an example. Let X_k be the return on the kth trial, i.e., the ratio by which the amount the player bets is multiplied on the kth game. For example, in the Double-or-Quarter game;

$$\mathbf{P}(X_k = 2) = \mathbf{P}(X_k = \frac{1}{4}) = \frac{1}{2}$$

Assume the player initially bets \$1. If she reinvests all her holdings on each subsequent trial, her wealth after n trials is:

$$W_n = X_1 \cdot X_2 \cdots X_n$$

A commonly used notation in investment is the *effective return* $R_n \geq 0$, a random variable given by

$$(R_n)^n = W_n = X_1 \cdot X_2 \cdots X_n$$

i.e.,

$$(R_n) = (X_1 \cdot X_2 \cdots X_n)^{1/n}$$

The return R_n summarizes the outcome in the sense that her wealth after n trials would remain W_n if all individual returns had been exactly $X_k = R_n$ for each trial X_k, k=1,2,...,n. For example, let each trial represent the 1 year return on an investment, and suppose an investment company claims that one of their mutual funds had an effective return over the past 10 years of 9%. In our notation, the company is claiming that R_{10}=1.09, and therefore for each dollar invested 10 years ago, a shareholder would now have $(1.09)^{10} = \$2.37$. It turns out that the effective return is a much better indicator of the performance of such a scheme than the expected wealth after n trials, which is biased by the possibility of huge but improbable wins.

We would like to know the value to which R_n converges (in probability) for a given game and strategy. Since we have much more powerful tools for dealing with averages of independent random variables than we do for dealing with products, we use the logarithm to convert the product above into an average:

$$\log_b(R_n) = \frac{1}{n}\sum_{k=1}^{n}\log_b(X_k) \tag{1}$$

$$R_n = b^{\frac{1}{n}\sum_{k=1}^{n}\log_b(X_k)}, \tag{2}$$

where b is the base for the logarithm, $b > 1$.

You will need the following Lemma, which you are encouraged to prove, if you wish, for substantial extra credit.

Lemma

Let Z_n, $n \geq 1$, be a sequence of random variables. Let c be any constant, and let f be any function that is continuous at c.

$$\text{If } Z_n \xrightarrow{prob.} \text{c as } n \to \infty, \text{(i.e., if } Z_n \text{ converges in probability to } c \text{ as } n \to \infty),$$

then

$$f(Z_n) \xrightarrow{prob.} f(c) \text{ as } n \to \infty, \text{(i.e., then } f(Z_n) \text{ converges in probability to } f(c) \text{ as } n \to \infty.)$$

Assume for the remainder of this problem that $\log_b(X)$ has a finite mean and variance

a) Show that as $n \to \infty, R_n \to r$ in probability as $n \to \infty$, for some constant r. Find a general expression for r and evaluate it numerically for the Double or Quarter game with P{heads}=1/2. (The choice $b = 2$ for the base of the logarithm will make this easier.) Also express your answer as a certain long-term percentage loss or gain per toss.

b) Find the asymptotic value of her wealth W_n in the double or Quarter Game as n becomes large, i.e., the value to which W_n converges in probability. Explain your methods and answer.

A General Problem for Gamblers

In parts a) and b) above, we found that $R_n \to r$ in probability as $n \to \infty$, where, unfortunately for the gambler, $r \leq \mathbf{E}[X]$.

c) Show that, sadly, this is always the case. In what special case(s) (if any), is it true that $r = \mathbf{E}[X]$?

(Hint: It is easy to show that $r \leq \mathbf{E}[X]$ once you have shown that $\mathbf{E}[\log(X)] \leq \log(\mathbf{E}[X])$. To show that this latter inequality is true, compare the curve $\log(x)$ to its tangent at $x = \mathbf{E}[X]$, (e.g., if we use e as the base, the tangent to $\ln(x)$ at $x = \mathbf{E}[X]$ is the linear function $f_L(x) = \log(E[X]) + (x - \mathbf{E}[X])/\mathbf{E}[X]$. Then take expectations and compare $\mathbf{E}[\log(X)]$ to $\mathbf{E}[f_L(X)]$. Pictures are very helpful here.)
Congratulations! You have just derived an important instance of the Jensen Inequality (page 287 in the text).

The Good News

The **Kelly strategy for gambling** tells you the optimal method for spreading risk (or, in investment language, for diversifying your investments). In the next part of this problem you will derive a version of the Kelly strategy for the Double or Quarter game.

Specifically, suppose you bet a fixed fraction of your wealth on each toss and put the remaining fraction $(1-f)$ in reserve. After the nth toss, your wealth is W_n, and you set aside $(1-f)W_n$ dollars and bet $f\ W_n$ dollars on the next toss. Your wealth after the $(n+1)$st toss will then be $W_{n+1} = fW_nX_{n+1} + (1-f)W_n$.

d) For the Double or Quarter Game, with P(heads)=1/2, find the range of fixed fractions f of your wealth that you can bet and be guaranteed than R_n converges in probability to a number greater than 1 and therefore that your wealth grows to infinity as $n \to \infty$. Find the maximum value of r to which R_n converges in probability and the value of f at which this maximum is achieved.