LECTURE 14

- **Readings:** Start Section 6.2

Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

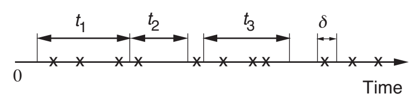
Review: Bernoulli process

- $X_1, X_2, \ldots$: independent Bernoulli RVs with success prob. p
- Number of arrivals in n time slots: binomial PMF
- Interarrival times: independent with geometric PMF
- Time of kth arrival: Pascal PMF of order k
- Independence and memorylessness

Limitation of the Bernoulli arrival model

- Consider cars entering Stata Center parking garage
  - Mark each 10-second interval with an entry a "success"
  - Good model? (captures what is happening?)
- ... entering any parking garage in the world
  - Mark each 10-second interval with an entry a "success"
  - Good model? (captures what is happening?)

Definition of the Poisson process

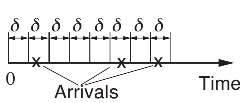


- Defining characteristics:
  - **Time homogeneity**: Probability of k arrivals in an interval of duration τ is some function $P(k,\tau)$
  - **Independence**: Numbers of arrivals in disjoint time intervals are independent
  - **Small interval probabilities**: For very small δ,

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta, & \text{if } k=0; \\ \lambda\delta, & \text{if } k=1; \\ 0, & \text{if } k>1, \end{cases}$$

where λ is called the **rate**

PMF of number of arrivals N_t



$n = t/\delta$ $\quad$ $np = \lambda t$

$p = \lambda\delta$

- Finely discretizing $[0,t]$, process is approximately Bernoulli
- N_t (of discrete approximation) is binomial
- Taking $\delta \to 0^+$ (or $n \to \infty$) allows approximation from L13:

$$P(k,t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \qquad k = 0, 1, \ldots$$

  - (consider this exact)
- $\mathbf{E}[N_t] = \lambda t, \qquad \text{var}(N_t) = \lambda t$

Examples

Email arrives as Poisson process with rate $\lambda = 0.4$ per hour.

Recall general form $\quad P(k,t) = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, \ldots$

- $\mathbf{P}$(0 new msgs between noon and 12:30pm) =
- $\mathbf{P}$(1 new msg between noon and 12:30pm) =
- number of msgs between noon and 1pm $\sim$
- number of msgs between 1pm and 3pm $\sim$
- number of msgs between noon and 3pm $\sim$

Time of first arrival T

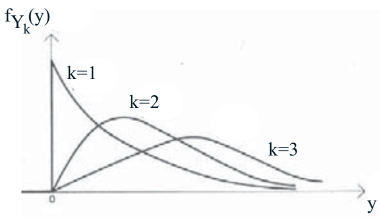
$F_T(t) \ =$

$f_T(t) \ =$

$\mathbf{P}(T > t + s \mid T > t) \ =$

Time of kth arrival Y_k

- All interarrival times independent with exponential distribution: $f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$
- $Y_k = T_1 + T_2 + \cdots + T_k$
- Erlang PDF of order k: $f_{Y_k}(y) \ = \ \dfrac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$

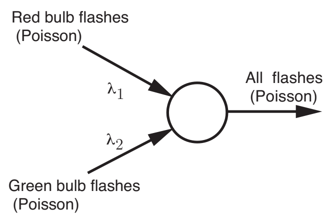


Bernoulli/Poisson correspondences

	Poisson	**Bernoulli**
times of arrival	continuous	discrete
arrival rate	λ per unit time	p per trial
number of arrivals	Poisson PMF	binomial PMF
interarrival times	exponential PDF	geometric PMF
kth arrival time	Erlang PDF of order k	Pascal PMF of order k

Adding (merging) Poisson processes

- Merging indep. Poisson **processes** gives a Poisson process



  - What is the probability that the next arrival comes from the first process?
- Sum of independent Poisson **random variables** is Poisson