Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 4: Solutions
Due October 7, 2009

1. (a) From the joint PMF, there are nine (x, y) coordinate pairs with nonzero probabilities of occurring. These pairs are (1, 1), (1, 2), (1, 3), (4, 1), (4, 2), (4, 3), (6, 1), (6, 2), and (6, 3). The probability of a pair is proportional to the quotient of the y and x coordinate of the pair. Because the probability of the entire sample space must equal 1, we have:

$$\frac{1}{1}c + \frac{2}{1}c + \frac{3}{1}c + \frac{1}{4}c + \frac{2}{4}c + \frac{3}{4}c + \frac{1}{6}c + \frac{2}{6}c + \frac{3}{6}c = 1.$$

Solving for c, we get $c = \boxed{\frac{2}{17}}$

(b) There are three sample points for which $2Y < X$.

$$\mathbf{P}(2Y < X) = \mathbf{P}(\{(4,1)\}) + \mathbf{P}(\{(6,1)\}) + \mathbf{P}(\{(6,2)\}) = \frac{2}{17}\left(\frac{1}{4} + \frac{1}{6} + \frac{2}{6}\right) = \boxed{\frac{3}{34}}$$

(c) There are four sample points for which $2Y > X$.

$$\mathbf{P}(2Y > X) = \mathbf{P}(\{(1,1)\}) + \mathbf{P}(\{(1,2)\}) + \mathbf{P}(\{(1,3)\}) + \mathbf{P}(\{(4,3)\}) = \frac{2}{17}\left(\frac{1}{1} + \frac{2}{1} + \frac{3}{1} + \frac{3}{4}\right) = \boxed{\frac{27}{34}}$$

(d) There are two sample points for which $2Y = X$.

$$\mathbf{P}(2Y = X) = \mathbf{P}(\{(4,2)\}) + \mathbf{P}(\{(6,3)\} = \frac{2}{17}\left(\frac{1}{1} + \frac{2}{1} + \frac{3}{1} + \frac{3}{4}\right) = \boxed{\frac{4}{34}}$$

Notice that, using the above two parts:

$$\mathbf{P}(2Y < X) + \mathbf{P}(2Y > X) + \mathbf{P}(2Y = X) = \frac{3}{34} + \frac{27}{34} + \frac{4}{34} = 1$$

as expected.

(e) In general, for two discrete random variables X and Y for which a joint PMF is defined, we have

$$p_X(x) = \sum_{y=-\infty}^{\infty} p_{X,Y}(x,y) \qquad \text{and} \qquad p_Y(y) = \sum_{x=-\infty}^{\infty} p_{X,Y}(x,y).$$

In this problem the number of possible (X, Y) pairs is quite small, so we can determine the marginal PMFs by enumeration. For example,

$$p_X(4) = \mathbf{P}(\{(4,1)\}) + \mathbf{P}(\{(4,2)\}) + \mathbf{P}(\{(4,3)\}) = \frac{6}{34}.$$

Overall, we get:

$$p_X(x) = \begin{cases} 12/17, & x = 1; \\ 3/17, & x = 4; \\ 2/17, & x = 6; \\ 0, & \text{otherwise} \end{cases}$$

and

$$p_Y(y) = \begin{cases} 1/6, & y = 1; \\ 1/3, & y = 2; \\ 1/2, & y = 3; \\ 0, & \text{otherwise} \end{cases}$$

(f) In general, the expected value of any discrete random variable X is given by

$$\mathbf{E}[X] = \sum_{x=-\infty}^{\infty} x p_X(x).$$

For this problem,

$$\mathbf{E}[X] = 1 \cdot \frac{12}{17} + 4 \cdot \frac{3}{17} + 6 \cdot \frac{2}{17} = \boxed{\frac{36}{17}}$$

and

$$\mathbf{E}[Y] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{2} = \boxed{\frac{7}{3}}$$

(g) The variance of a random variable X can be computed as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$ or as $\mathbf{E}[(X - \mathbf{E}[X])^2]$. Here we use the second approach.

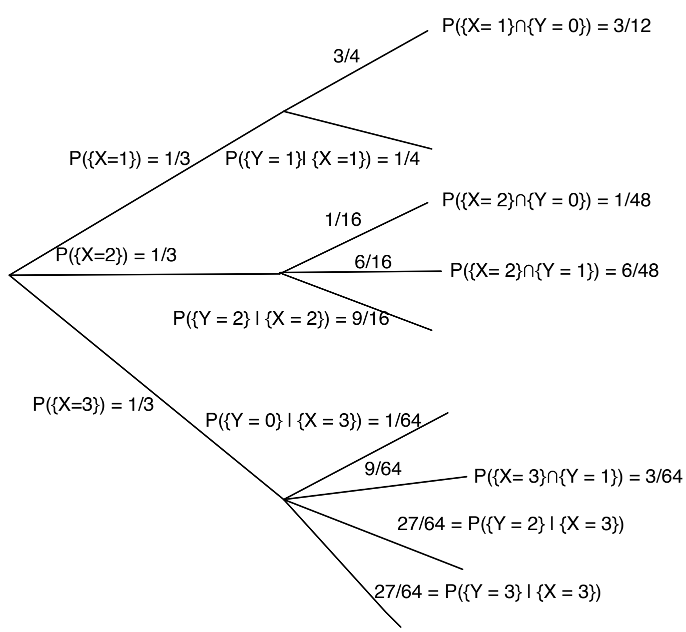
$$\text{var}(X) = \left(1 - \frac{36}{17}\right)^2 \cdot \frac{12}{17} + \left(4 - \frac{36}{17}\right)^2 \cdot \frac{3}{17} + \left(6 - \frac{36}{17}\right)^2 \cdot \frac{2}{17} = \boxed{\frac{948}{289}}$$

$$\text{var}(Y) = \left(1 - \frac{7}{3}\right)^2 \frac{1}{6} + \left(2 - \frac{7}{3}\right)^2 \frac{1}{3} + \left(3 - \frac{7}{3}\right)^2 \frac{1}{2} = \boxed{\frac{5}{9}}$$

2. There are several ways that we can define events and random varibles for this problem, but since we look ahead and see questions defined in terms of the random variable X, the number of questions May is asked, and random variable Y, the number of questions she answers correctly, we decide to stick with these definitions. The random variable X is uniformly distributed over $\{1, 2, 3\}$. And if we condition on the random variable X, Y has a binomial distribution:

$$p_{Y|X}(y|x) = \binom{x}{y} \left(\frac{3}{4}\right)^y \left(\frac{1}{4}\right)^{x-y} \qquad y = 0, \ldots, x$$

We can illustrate this problem via a probability tree as shown in the diagram below:



(a) If we define A as the event that May gives wrong answers to all questions she is asked, this is the event $\{Y = 0\}$. Using the total probability theorem $P(A)$ is equivalently expressed as:

$$\begin{aligned} P(A) &= P(\{X=1\} \cap \{Y=0\}) + P(\{X=2\} \cap \{Y=0\}) + P(\{X=3\} \cap \{Y=0\}) \\ &= P(\{Y=0\}|\{X=1\})P(\{X=1\}) + P(\{Y=0\}|\{X=2\})P(\{X=2\}) \\ &\quad + P(\{Y=0\}|\{X=3\})P(\{X=3\}) \\ &= 1/4 \cdot 1/3 + 1/16 \cdot 1/3 + 1/64 \cdot 1/3 \\ &= 21/192 \end{aligned}$$

This is equivalent to:

$$p_Y(0) = \sum_{x=1}^{3} p_{Y|X}(0|x) p_X(x)$$

(b) We define B to be the event that May was asked 3 questions, and we seek to find $P(B|A) = P(A \cap B)/P(A)$ where event A is defined above. Event B is the event $\{X = 3\}$, and probability of the intersection of events A and B is $P(\{X = 3\} \cap \{Y = 0\}) = p_{X,Y}(3, 0)$. The $P(B|A)$ is given by:

$$\begin{aligned} P(B|A) &= P(\{X=3\} \cap \{Y=0\})/P(A) \\ &= P(\{Y=0\}|\{X=3\})P(\{X=3\})/P(A) \\ &= \frac{1/3 \cdot 1/64}{21/192} \\ &= 1/21 \end{aligned}$$

The probability $P(B|A)$ is equvalently expressed as:

$$P(B|A) = \frac{p_{X,Y}(3,0)}{p_Y(0)} = \frac{p_{Y|X}(0|3)p_X(3)}{p_Y(0)}$$

(c) The random variable X is the number of questions May is asked and the random variable Y is the number of questions she answers correctly in a lecture. The PMF for X is:

$$p_X(x) = \begin{cases} 1/3 & \text{if } x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

The mean of X is:

$$\mathbf{E}[X] = 1 \times 1/3 + 2 \times 1/3 + 3 \times 1/3 = 2$$

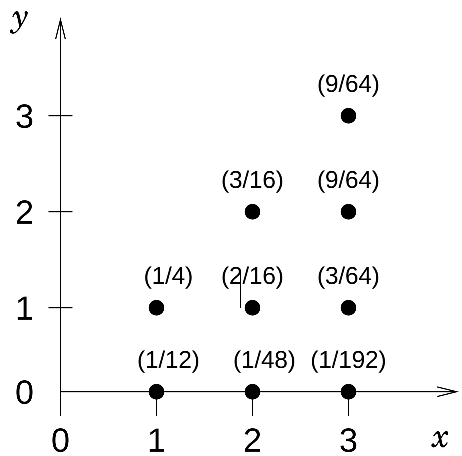
The variance of X is computed using the formula $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$.

$$\mathbf{E}[X^2] = 1 \times 1/3 + 4 \times 1/3 + 9 \times 1/3 = 14/3$$

$$\text{var}(X) = 14/3 - 4 = 2/3$$

(d) The sketch of the joint PMF, $p_{X,Y}(x,y)$ is shown. The values, $p_{X,Y}(x,y)$, are found from $p_{X,Y}(x,y) = p_{Y|X}(y|x)p_X(x)$.

$$p_{X,Y}(x,y) = \begin{cases} \binom{x}{y}\left(\frac{3}{4}\right)^y \left(\frac{1}{4}\right)^{x-y} \frac{1}{3} & x = 1,2,3 \text{ and } 0 \le y \le x \\ 0 & o.w. \end{cases} \quad (1)$$



(e) The bonus will be defined as the random variable $B = 10X + 20Y$. The expected value of B can be computed easily since the expected value of a linear combination of two random variables is the linear combination of the expected values of the two random variables.

First, we find $E[Y]$. From part (d), we can find the PMF for Y using the formula:

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

This gives:

$$p_Y(y) = \begin{cases} 7/64 & \text{if } y = 0 \\ 27/64 & \text{if } y = 1 \\ 21/64 & \text{if } y = 2 \\ 9/64 & \text{if } y = 3 \\ 0 & \text{otherwise} \end{cases}$$

Thus:

$$E[Y] = 0 \times 7/64 + 1 \times 27/64 + 2 \times 21/64 + 3 \times 9/64 = 96/64 = 3/2$$

$$\mathbf{E}[B] = \mathbf{E}[10X + 20Y] = 10\mathbf{E}[X] + 20\mathbf{E}[Y] = 10 \times 2 + 20 \times 3/2 = 50$$

To compute the variance of B, we use the formula $\text{var}(B) = \mathbf{E}[B^2] - (\mathbf{E}[B])^2$. $\mathbf{E}[B^2]$ is calculated by noting that since $B = g(X,Y)$,

$$\begin{aligned} \mathbf{E}[B^2] &= \sum_x \sum_y (g(x,y))^2 p_{X,Y}(x,y) = \sum_x \sum_y (10X + 20Y)^2 p_{X,Y}(x,y) \\ &= 100 \times \frac{1}{12} + 900 \times \frac{1}{4} + 400 \times \frac{1}{48} + 1600 \times \frac{2}{16} + 3600 \times \frac{3}{16} + 900 \times \frac{1}{192} + \\ &\quad 2500 \times \frac{3}{64} + 4900 \times \frac{9}{64} + 8100 \times \frac{9}{64} = 588800/192 \end{aligned}$$

$$\text{var}(B) = \mathbf{E}[B^2] - (\mathbf{E}[B])^2 = 108800/192$$

(f) The random variable Z is defined as the total number of wrong answers May gives in a 20 lecture series. Let us define the random variable Q_n as the number of questions she answers wrong in the nth lecture. Then $Z = Q_1 + Q_2 + Q_3 + \ldots + Q_{20}$.

$$\mathbf{E}[Z] = \mathbf{E}[Q_1 + Q_2 + Q_3 + \ldots + Q_{20}]$$

Since the number of questions she answers wrong in the first lecture is distributed the same as the number she answers wrong in any subsequent lecture,

$$\mathbf{E}[Z] = 20\mathbf{E}[Q_1].$$

To determine the expected value of the number of questions she answered wrong in lecture 1, we define random variable Q_1 as a function of the random variables X and Y defined in parts (c) and (d). Random variable X is the number of questions May is asked in a given lecture and the random variable Y is the number of questions she answers correctly in a a given lecture, and so random variable $Q_1 = X - Y$. Using the results in parts (c) and (e),

$$\mathbf{E}[Q_1] = \mathbf{E}[X - Y] = \mathbf{E}[X] - \mathbf{E}[Y] = 2 - 3/2 = 1/2$$

and

$$\mathbf{E}[Z] = 20\mathbf{E}[Q_1] = 10$$

As a consequence of the independence of the RVs $Q_1, \ldots Q_{20}$, we know that the var(Z) is the sum of the variances of $Q_1, \ldots Q_{20}$:

$$\text{var}(Z) = \text{var}(Q_1 + Q_2 + \ldots + Q_{20}) = \text{var}(Q_1) + \text{var}(Q_2) + \ldots + \text{var}(Q_{20}) = 20\text{var}(Q_1)$$

The last equality is valid because $Q_1, \ldots, Q_{20}$ are identically distributed.

To use the formula $\text{var}(Q_1) = \mathbf{E}[Q_1^2] - (\mathbf{E}[Q_1])^2$, we need to find $\mathbf{E}[Q_1^2]$:

$$\mathbf{E}[Q_1^2] = \sum_x \sum_y (f(x,y))^2 p_{X,Y}(x,y) = \sum_x \sum_y (X - Y)^2 p_{X,Y}(x,y)$$

$$\mathbf{E}[Q_1^2] = 1/12 + 2/16 + 9/64 + 4 \times 1/48 + 4 \times 3/64 + 9 \times 1/192 = 2/3$$

$$\text{var}(Q_1) = 2/3 - 1/4 = 5/12$$

$$\text{var}(Z) = 20 \times 5/12 = 100/12$$

Alternatively, we could find the PMF for Q_1 and then compute

$$\mathbf{E}[Q_1^2] = \sum_q q^2 p_{Q_1}(q)$$

The PMF for Q_1 is found by the formula,

$$p_{Q_1}(q) = \sum_{\{x,y|x-y=q\}} p_{X,Y}(x,y)$$

The PMF for Q_1 is

$$p_{Q_1}(q) = \begin{cases} 101/192 & \text{if } q=0 \\ 67/192 & \text{if } q=1 \\ 13/192 & \text{if } q=2 \\ 1/192 & \text{if } q=3 \\ 0 & \text{otherwise} \end{cases}$$

3. We are given the following information:

$$p_K(k) = \begin{cases} 1/4, & \text{if } k=1,2,3,4; \\ 0, & \text{otherwise} \end{cases}$$

$$p_{N|K}(n \mid k) = \begin{cases} 1/k, & \text{if } n=1,\ldots,k; \\ 0, & \text{otherwise} \end{cases}$$

(a) We use the fact that $p_{N,K}(n,k) = p_{N|K}(n \mid k)p_K(k)$ to arrive at the following joint PMF:

$$p_{N,K}(n,k) = \begin{cases} 1/(4k), & \text{if } k=1,2,3,4 \text{ and } n=1,\ldots,k; \\ 0, & \text{otherwise} \end{cases}$$

(b) The marginal PMF $p_N(n)$ is given by the following formula:

$$p_N(n) = \sum_k p_{N,K}(n,k) = \sum_{k=n}^{4} \frac{1}{4k}$$

On simplification this yields

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48, & n=1; \\ 1/8 + 1/12 + 1/16 = 13/48, & n=2; \\ 1/12 + 1/16 = 7/48, & n=3; \\ 1/16 = 3/48, & n=4; \\ 0, & \text{otherwise.} \end{cases}$$

(c) The conditional PMF is

$$p_{K|N}(k \mid 3) = \frac{p_{N,K}(3,k)}{p_N(3)} = \begin{cases} 4/7, & k=3; \\ 3/7, & k=4; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let A be the event $2 \le N \le 3$. We first find the conditional PMF of K given A.

$$p_{K|A}(k) \quad = \quad \frac{\mathbf{P}(\{K=k\} \cap A)}{\mathbf{P}(A)}$$

$$\begin{aligned}
\mathbf{P}(A) &= p_N(2) + p_N(3) = \frac{5}{12} \\
\mathbf{P}(\{K = k\} \cap A) &= \begin{cases} \frac{1}{8}, & k = 2; \\ \frac{1}{12} + \frac{1}{12}, & k = 3; \\ \frac{1}{16} + \frac{1}{16}, & k = 4; \\ 0, & \text{otherwise} \end{cases} \\
p_{K|A}(k) &= \begin{cases} \frac{3}{10}, & k = 2; \\ \frac{2}{5}, & k = 3; \\ \frac{3}{10}, & k = 4; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Because the conditional PMF of K given A is symmetric around $k = 3$, we know $\boxed{\mathbf{E}[K \mid A] = 3}$. We now find the conditional variance of K given A.

$$\begin{aligned}
\text{var}(K \mid A) &= \mathbf{E}[(K - \mathbf{E}[K \mid A])^2 \mid A] \\
&= \frac{3}{10} \cdot (2-3)^2 + \frac{2}{5} \cdot 0 + \frac{3}{10} \cdot (4-3)^2 \\
&= \boxed{\tfrac{3}{5}}
\end{aligned}$$

(e) Let C_i be the cost of book i and $\mathbf{E}[C_i] = 3$. Let T be the total cost, so $T = C_1 + \ldots + C_N$. We now find $\mathbf{E}[T]$ using the total expectation theorem.

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}[T \mid N = 1]p_N(1) + \mathbf{E}[T \mid N = 2]p_N(2) + \mathbf{E}[T \mid N = 3]p_N(3) + \mathbf{E}[T \mid N = 4]p_N(4) \\
&= \mathbf{E}[C_1]p_N(1) + \mathbf{E}[C_1 + C_2]p_N(2) + \mathbf{E}[C_1 + C_2 + C_3]p_N(3) + \mathbf{E}[C_1 + C_2 + C_3 + C_4]p_N(4) \\
&= \mathbf{E}[C_i]p_N(1) + 2\mathbf{E}[C_i]p_N(2) + 3\mathbf{E}[C_i]p_N(3) + 4\mathbf{E}[C_i]p_N(4) \\
&= 3 \cdot \frac{25}{48} + 6 \cdot \frac{13}{48} + 9 \cdot \frac{7}{48} + 12 \cdot \frac{1}{16} \\
&= \boxed{\tfrac{21}{4}}
\end{aligned}$$

4. (a) Since the X_is are identically distributed,

$$\text{(a)} \qquad \mathbf{E}[X_1] = \mathbf{E}[X_2] = \cdots = \mathbf{E}[X_n]$$

and

$$\text{(b)} \qquad \mathbf{E}[X_1^2] = \mathbf{E}[X_2^2] = \cdots = \mathbf{E}[X_n^2].$$

Furthermore, using (a) and the independence of the X_is,

$$\text{(c)} \qquad \mathbf{E}[X_i X_j] = \mathbf{E}[X_i]\mathbf{E}[X_j] = (\mathbf{E}[X_1])^2 \quad \text{when} \quad i \neq j.$$

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] &= \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right)\right] \quad \text{where separate dummy variables are for clarity} \\
&= \mathbf{E}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right)\right] \quad \text{by the distributive law}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{1\le i,j\le n,\, i\ne j} X_i X_j\right] \quad \text{by separating the } i=j \text{ and } i\ne j \text{ terms} \\
&= \sum_{i=1}^{n} \mathbf{E}\left[X_i^2\right] + \sum_{1\le i,j\le n,\, i\ne j} \mathbf{E}\left[X_i X_j\right] \quad \text{by linearity of expectation} \\
&= \sum_{i=1}^{n} \mathbf{E}\left[X_1^2\right] + \sum_{1\le i,j\le n,\, i\ne j} \left(\mathbf{E}\left[X_1\right]\right)^2 \quad \text{using (b) and (c)} \\
&= n\mathbf{E}\left[X_1^2\right] + n(n-1)\left(\mathbf{E}\left[X_1\right]\right)^2 \quad \text{by counting the numbers of terms}
\end{aligned}$$

Thus $c = n$ and $d = n(n-1)$.

(b) Since $X_1, \ldots, X_n$ are identically distributed, the given expression can be equivalently expressed as:

$$E\left[(X_1 + \ldots + X_n - nE[X_1])^2\right]$$

This simplifies to:

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - 2nE[X_1]E\left[\sum_{i=1}^{n} X_i\right] + (nE[X_1])^2$$

Using the result from part (a) to simplify the first term, we conclude:

$$\begin{aligned}
&= nE\left[X_1^2\right] + n(n-1)(E[X_1])^2 - 2n^2(E[X_1])^2 + n^2(E[X_1])^2 \\
&= nE\left[X_1^2\right] - n(E[X_1])^2
\end{aligned}$$

Thus, variable $g = n$, and variable $h = -n$. Please note that we've shown that the variance of the sum of n independent and identically distributed random variables is n times the variance of one of the random variables.

5. (a) An easy way to derive $p_{X,Y,Z}(x,y,z)$ is in sequential terms as $p_X(x) \cdot p_{Y,Z|X}(y,z|x)$. Note $p_X(x)$ is geometric with parameter p. Conditioned on X even, $(Y,Z) = (0,0)$ with probability 1. Conditioned on X odd, $p_{Y,Z|X}(y,z) = \frac{1}{4}$ for $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$.

$$p_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{4}p(1-p)^{x-1}, & \text{if } x \text{ is odd and } (y,z) \in \{(0,0),(0,2),(2,0),(2,2)\} \\ p(1-p)^{x-1}, & \text{if } x \text{ is even and } (y,z) = (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

(b) (i) No. Notice that even though conditional on X (i.e. given a realization, x, of random variable X), the random variables Y and Z are independent (that's why they look "regular"), Y and Z are not independent. Given Y, the distribution over Z changes (i.e. if Y is 2, Z is equally likely to be 0 or 2; however if Y is 0, Z is more likely to be 0).

(ii) Yes. Given $Z = 2$, if we are further given $X = x$, Y is equally likely to take on the value 0 or 2.

(iii) No. Given $Z = 0$, if we are further given $X = x$, then if x is even, Y must be 0, whereas if x is odd, Y is equally likely to take on 0 or 2.

(iv) Yes. Given $Z = 2$, if we are further given $X = x$, $Z = 2$ still holds (i.e. with probability 1)! Double conditioning has no effect.

(c) If $X = 5$, then Y and Z are uniformly distributed on the set S specified in the problem statement, so $Y + Z$ takes the values 0 and 4 with probability $\frac{1}{4}$, and takes the value 2 with probability $\frac{1}{2}$. This PMF is symmetric about 2, so the mean value of $Y + Z$ is evidently 2. Hence the variance is

$$(0-2)^2\frac{1}{4} + (4-2)^2\frac{1}{4} = 2.$$