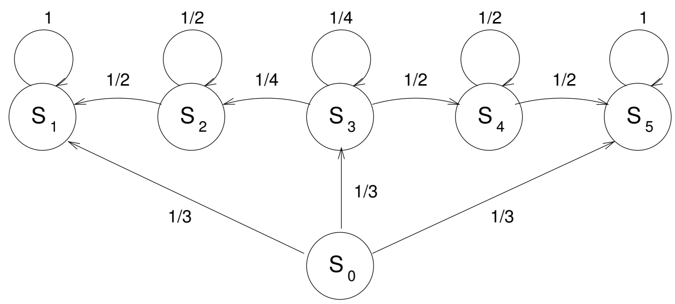
Problem Set 8
Due: April 20, 2011

1. Consider the following Markov chain, with states labelled from $s_0, s_1, \ldots, s_5$:



Given that the above process is in state s_0 just before the first trial, determine by inspection the probability that:

(a) The process enters s_2 for the first time as the result of the kth trial.

(b) The process never enters s_4.

(c) The process enters s_2 and then leaves s_2 on the next trial.

(d) The process enters s_1 for the first time on the third trial.

(e) The process is in state s_3 immediately after the nth trial.

2. There are n classes offered by a particular department, and each year, the students rank each class from 1 to n, in order of difficulty. Unfortunately, the ranking is completely arbitrary. In fact, any given class is equally likely to receive any given rank on a given year (two classes may not receive the same rank). A certain professor chooses to remember only the highest ranking his class has ever gotten.

(a) Show that the system described by the ranking that the professor remembers is a Markov chain.

(b) Find the transition probabilities for the above Markov chain.

(c) List all recurrent and transient states.

3. A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5 & , \quad (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1 & , \quad (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0 & , \quad \text{otherwise} \end{cases}$$

In the questions below, we let X_k be the state of the Markov process at time k.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of n is the probability $r_{15}(n) = \mathbf{P}(X_n = 5 \mid X_0 = 1) > 0$?

(c) What is the set of states $A(i)$ that is accessible from state i, for each $i = 1, 2, \ldots, 7$?

(d) Identify which states are transient and which states are recurrent. For each recurrent class, state whether it is periodic (and give the period) or aperiodic.

(e) What is the minimum number of transitions with nonzero probability that must be added so that all seven states form a single recurrent class?

4. The MIT football team's performance in any given game is very much correlated to its morale. In fact, if the team has won the past two games, then it has a .7 probability of winning the next game. If it lost the last game but won before that, it has a .4 probability of winning. If it won its last game but lost before that it has a .5 probability of winning, and finally if it lost the last two games it has only a .2 probability of winning the next game. Assume that the above details the complete correlation between the history of victories and defeats, and the future performance.

   (a) Define with a Markov chain that models the above process. Remember that the process must have the Markov property.

   (b) Find the long run probability that the MIT football team will win its next game.

5. Mr. Mean Variance has the only key which locks or unlocks the door to building 59, the Probability Building. He visits the door once each hour on the hour. When he arrives:

   If the door is open, he locks it with probability 0.3.
   If the door is locked, he unlocks it with probability 0.8.

   (a) After he has been on the job several months, is he more likely to lock the door or to unlock it on a randomly selected visit?

   (b) With the process in the steady state, Joe arrived at Building 59 two hours ahead of Harry. What is the probability that each of them found the door in the same condition?

   (c) Given the door was open at the time Mr. Variance was hired, determine the expected value of the number of visits up to and including the one on which he unlocks the door himself for the first time.

G1†. Let P be the transition probability for a particular Markov chain, with p_{ij} denoting the individual transition probabilities. Suppose that each time a transition occurs out of state i, a reward r_i is collected.

   (a) Show that P^n_{ij}, the (i, j)th element of the nth power of the matrix P, is the probability of getting to state j from state i in n steps.

   (b) The random variable V_i is the total reward collected when the initial state is i. Use the Total Probability Law to show that the expected total reward $v_i = E(V_i)$ satisfies the following set of equations:

$$v_i = r_i + \sum_j p_{ij} v_j.$$

   In matrix notation, the equation above becomes:

$$V = R + PV$$

   where $V = [v_1, \cdots, v_m]^T$ and $R = [r_1, \cdots, r_m]^T$ for the case of an m-state chain.

For the remainder of this problem, consider a chain that has $(m-1)$ transient states and 1 absorbing state. Without loss of generality, denote the absorbing state as state m, and denote the remaining states as $1, \cdots, m-1$. Also assume that as soon as the absorbing state is entered, no rewards are collected. i.e. $r_m = 0$.

(c) Show that $P_{im}^{m} > 0$ for all states i.