Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2010)

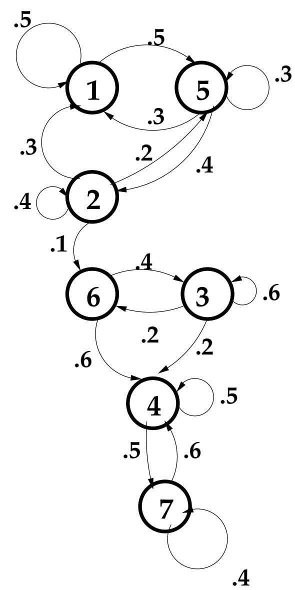
Problem Set 8 Solutions

1. The dot location of the yarn, as related to the size of the pieces of the yarn cut for any particular customer, can be viewed in light of the random incident paradox.

   (a) Here, the length of each piece of yarn is exponentially distributed. Due to the memorylessness of the exponential, the distribution of the length of the piece of yarn containing the red dot is a second order Erlang. Thus, the $\mathbf{E}[R] = 2\mathbf{E}[L] = \frac{2}{\lambda}$.

   (b) Think of exponentially-spaced marks being made on the yarn, so the length requested by the customers each involve *three* such sections of exponentially distributed lengths (since the PDF of L is third-order Erlang). The piece of yarn with the dot will have the dot in any one of these three sections, and the expected length of that section, by (a), will be $2/\lambda$, while the expected lengths of the other two sections will be $1/\lambda$. Thus, the total expected length containing the dot is $4/\lambda$.

2. Let i ($i = 1, \dots, 7$) be the states of the Markov chain. From the graphical representation of the transition matrix it is easy to see the following:



   (a) $\{4, 7\}$ are recurrent and the rest are transient. All states are aperiodic.

   (b) There is only one class formed by the recurrent states.

3. (a) i. Since the state X_k is the largest number rolled in k rolls, the set of states $S = \{1, 2, 3, 4, 5, 6\}$. The probability of the largest number rolled in the first $(k+1)$ trials is only dependent to the what the largest number that was rolled in the first$k$ trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} 0 & , \quad j < i \\ \frac{i}{6} & , \quad j = i \\ \frac{1}{6} & , \quad j > i \end{cases}$$

ii. Since the state X_k is the number of sixes in the first k rolls, the set of states $S = \{0, 1, 2, ...\}$. The probability of getting a six in a given trial is 1/6. The number of sixes rolled in the first $(k + 1)$ trials is only dependent to the number of sixes rolled in the first k trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = i + 1 \\ \frac{5}{6} & , \quad j = i \\ 0 & , \quad \text{otherwise} \end{cases}$$

iii. Since the state X_k is the number of rolls since the most recent six, the set of states $S = \{0, 1, 2, ...\}$. If the roll of the die is 6 on the next trial the chain goes to state 0. If not, the state goes to the next higher state. Therefore, the probability of the next state depends on the past only through the present state. Clearly, this satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = 0 \\ \frac{5}{6} & , \quad j = i + 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(b) i. For $X_k = Y_{r+k}$, and by the Markov property of Y

$$\begin{aligned} \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i, ...Y_r = i_r) \\ &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i) \\ \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. Also we can see that, X_k is a delayed process by r of Y_k. Therefore, they should have the same transition probability p_{ij}, So, we have:

$$p_{ij} = q_{ij} \ .$$

ii. For $X_k = Y_{2k}$, and by the Markov property of Y

$$\begin{aligned} &\mathbf{P}(X_{k+1} = j | X_k = i, X_{k-1} = i_{k-1}.....X_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i, Y_{2k-2} = i_{2k-2}.....Y_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. The transition probabilities p_{ij} are given by:

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{k+1} = j | X_k = i) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= r_{ij}^{y}(2) \end{aligned}$$

where $r_{ij}^{y}(n)$ is the n step transition probability of Y.

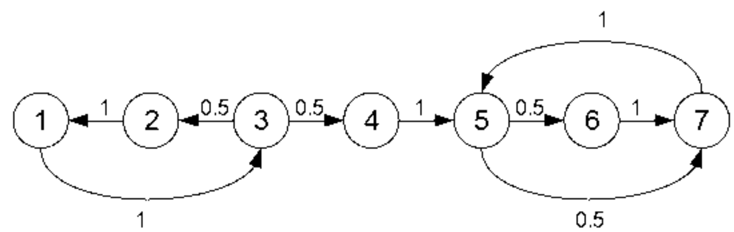
iii.

$$\begin{aligned} &\mathbf{P}(X_{k+1} = (n, l) | X_0 = (i_0, i_1), X_1 = (i_1, i_2), ..., X_k = (i_k, n)) \\ &= \mathbf{P}(X_{k+1} = (n, l) | Y_0 = i_0, Y_1 = i_1, Y_2 = i_2........., Y_k = i_k, Y_{k+1} = n) \\ &= \mathbf{P}(X_{k+1} = (n, l) | Y_{k+1} = n) \\ &= \mathbf{P}(X_{k+1} = (n, l) | X_k = (i_k, n)) \end{aligned}$$

Letting $i = (i_k, i_{k+1})$ and $j = (n, l)$, the transition probabilities p_{ij} are given by:

$$p_{ij} = \mathbf{P}(X_{k+1} = (n,l) | X_k = (i_k, i_{k+1})) = \begin{cases} q_{nl} & , \quad i_{k+1} = n \\ 0 & , \quad i_{k+1} \neq n \end{cases}$$

4. (a) The state diagram of the Markov chain is:



(b) State 5 is reachable from state 1 in a minimum of three transitions. Paths from state 1 to state 5 also include paths with a loop from 1 back to 1 (of length 3) and/or a loop from 5 back to 5 by way of state 7 (either length 2 or length 3). Therefore potential path lengths are $3 + 2m + 3n$, for $m, n \geq 0$. Therefore, $r_{15}(n) > 0$ for $n = 3$ or $n \geq 5$.

(c) From states 1, 2, and 3, all states are accessible because there is a non-zero probability path from these states by way of state 3 to any other state. From states, 4, 5, 6, and 7, paths only exist to states 5, 6, and 7.

(d) States 5-7 are recurrent because by the logic in (c), they can be reached from any other state. States 1-4 are transient; once the system has transitioned out of state 4, it cannot return to any state other than states 5, 6, or 7.

States 5, 6, and 7 form a recurrent class. Because it can be traversed from state 5 back to 5 in either 2 or 3 steps (as discussed in (b)), the system can return to state 5 after n steps for any $n \geq 2$; therefore it is aperiodic.

(e) One transition must be added to create a single recurrent class: for example, adding a transition from state 5 to state 1 would allow every state to be reached from every other state. Any transition from the recurrent class states 5,6, or 7 to any of the states 1, 2, or 3 would work.

5. (a) Given L_{n-1}, the history of the process (i.e., $L_{n-2}, L_{n-3}, \ldots$) is irrelevant for determining the probability distribution of L_n, the number of remaining unlocked doors at time n. Therefore, L_n is Markov. More precisely,

$$\mathbf{P}(L_n = j | L_{n-1} = i, L_{n-2} = l, \ldots, L_1 = m) = \mathbf{P}(L_n = j | L_{n-1} = i) = p_{ij}.$$

Clearly, at one step the number of unlocked doors can only decrease by one or stay constant. So, for $1 \leq i \leq k$, if $j = i - 1$, then $p_{ij} = \mathbf{P}(\text{selecting an unlocked door on day } n+1 | L_n = i) = \frac{i}{d}$. For $0 \leq i \leq k$, if $j = i$, then $p_{ij} = \mathbf{P}(\text{selecting an locked door on day } n+1 | L_n = i) = \frac{d-i}{d}$. Otherwise, $p_{ij} = 0$. To summarize, for $0 \leq i, j \leq k$, we have the following:

$$p_{ij} = \begin{cases} \frac{d-i}{d} & j = i \\ \frac{i}{d} & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

(b) The state with 0 unlocked doors is the only recurrent state. All other states are then transient, because from each, there is a positive probability of going to state 0, from which it is not possible to return.

(c) Note that once all the doors are locked, none will ever be unlocked again. So the state 0 is an absorbing state: there is a positive probability that the system will enter it, and once it does, it will remain there forever. Then, clearly, $\lim_{n\to\infty} r_{i0}(n) = 1$ and $\lim_{n\to\infty} r_{ij}(n) = 0$ for all $j \neq 0$ and all i.

(d) In this case L_n is not a Markov process. To see this, note that $\mathbf{P}(L_n = i+1 | L_{n-1} = i, L_{n-2} = i-1) = 0$ since according to my strategy I do not unlock doors two days in a row. But clearly, $\mathbf{P}(L_n = i+1 | L_{n-1} = i) > 0$ for $i < d$ since it is possible to go from a state of i unlocked doors to a state of $i+1$ unlocked doors in general. Thus $\mathbf{P}(L_n = i+1 | L_{n-1} = i, L_{n-2} = i-1) \neq \mathbf{P}(L_n = i+1 | L_{n-1} = i)$, which shows that L_n does not have the Markov property.

G1†. (a) First let the p_{ij}'s be the transition probabilities of the Markov chain. Then

$$\begin{aligned}
m_{k+1}(1) &= \mathbf{E}[R_{k+1}|X_0 = 1] \\
&= \mathbf{E}[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1] \\
&= \sum_{i=1}^{n} p_{1i}\mathbf{E}[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1 = i] \\
&= \sum_{i=1}^{n} p_{1i}\mathbf{E}[g(1) + g(X_1) + ... + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i}\mathbf{E}[g(X_1) + ... + g(X_{k+1})|X_1 = i] \\
&= g(1) + \sum_{i=1}^{n} p_{1i} m_k(i)
\end{aligned}$$

and thus in general $m_{k+1}(c) = g(c) + \sum_{i=1}^{n} p_{ci} m_k(i)$ when $c \in \{1, ..., n\}$.

Note that the third equality simply uses the total expectation theorem.

(b)

$$\begin{aligned}
v_{k+1}(1) &= \mathrm{var}[R_{k+1}|X_0 = 1] \\
&= \mathrm{var}[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1] \\
&= \mathrm{var}[\mathbf{E}[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad\ \mathbf{E}[\mathrm{var}[g(X_0) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= \mathrm{var}[g(1) + \mathbf{E}[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \\
&\quad\ \mathbf{E}[\mathrm{var}[g(1) + g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= \mathrm{var}[\mathbf{E}[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] + \mathbf{E}[\mathrm{var}[g(X_1) + ... + g(X_{k+1})|X_0 = 1, X_1]] \\
&= \mathrm{var}[\mathbf{E}[g(X_1) + ... + g(X_{k+1})|X_1]] + \mathbf{E}[\mathrm{var}[g(X_1) + ... + g(X_{k+1})|X_1]] \\
&= \mathrm{var}[m_k(X_1)] + \mathbf{E}[v_k(X_1)]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}[(m_k(X_1))^2] - \mathbf{E}[m_k(X_1)]^2 + \sum_{i=1}^{n} p_{1i} v_k(i) \\
&= \sum_{i=1}^{n} p_{1i} m_k^2(i) - (\sum_{i=1}^{n} p_{1i} m_k(i))^2 + \sum_{i=1}^{n} p_{1i} v_k(i)
\end{aligned}
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so in general $v_{k+1}(c) = \sum_{i=1}^{n} p_{ci} m_k^2(i) - (\sum_{i=1}^{n} p_{ci} m_k(i))^2 + \sum_{i=1}^{n} p_{ci} v_k(i)$ when $c \in \{1, ..., n\}$.