Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

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1. Problem 6.22, page 335 in text. See solution in text.

2. Problem 6.23, page 335 in text. See solution in text.

3. The PMF describing K, the number of children in any randomly selected family, is

$$p_K(k) = \begin{cases} 1/10 & , \quad k = 0 \\ 4/10 & , \quad k = 1 \\ 3/10 & , \quad k = 2 \\ 1/10 & , \quad k = 3 \\ 1/10 & , \quad k = 4 \end{cases}$$

   (a) $\mathbf{E}[K] = 0 \cdot \frac{1}{10} + 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} + 4 \cdot \frac{1}{10} = \frac{17}{10}$

   (b) Note that there are a total of 170 children in the village; 40 of them come from a family with only one child, 60 of them from a family with two children, 30 of them from a family with three children and 40 of them from a family of four children. Picking any one child equally-likely, the PMF for W, the number of children in the family of any randomly selected *child*, is

$$p_W(w) = \begin{cases} 4/17 & , \quad w = 1 \\ 6/17 & , \quad w = 2 \\ 3/17 & , \quad w = 3 \\ 4/17 & , \quad w = 4 \end{cases} \qquad \Rightarrow \mathbf{E}[W] = 1 \cdot \frac{4}{17} + 2 \cdot \frac{6}{17} + 3 \cdot \frac{3}{17} + 4 \cdot \frac{4}{17} = \frac{41}{17} \quad .$$

   (c) Parts (a) and (b) both deal with a random variable that describes the number of children in a particular family; the distinction is, of course, in the manner with which that particular family is selected. By selecting any child at random, we immediately remove the possibility of selecting a family with no children and in general induce a bias towards families with many children. It is a clear illustration of the random incidence paradox; it is only until we appreciate the differences in the underlying experiments that the paradox is resolved.

   There is a neat relationship between K, the number of members in any randomly selected set, and W, the number of members in the set associated with any randomly selected member. Generalizing the logic in part (b), the PMF for W is merely the PMF for K but weighted by the number of members at each k; mathematically, letting c denote a normalizing constant,

$$p_W(k) = c \cdot k p_K(k) \quad \Rightarrow c = \frac{1}{\mathbf{E}[K]} \qquad \Rightarrow p_W(k) = \frac{k p_K(k)}{\mathbf{E}[K]}, k = 0, 1, \ldots$$

   From this, it follows that

$$\mathbf{E}[W] = \sum_k k p_W(k) = \sum_k \frac{k^2 p_K(k)}{\mathbf{E}[K]} = \frac{\mathbf{E}[K^2]}{\mathbf{E}[K]} \qquad .$$

4. Problem 6.19, page 332 in text. See solution in text.