Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2010)

Problem set 6

Due March 29, 2010

1. X and Y are jointly Gaussian random variables with zero mean and unit variance, such that $\mathbf{E}[XY] = 0.5$. Find the PDF of $|X+Y|$.

   Answer: for $Z = |X+Y|$,

$$f_Z(z) = \begin{cases} \frac{2}{\sqrt{6\pi}} e^{-z^2/6}, & z > 0, \\ 0, & z \le 0. \end{cases}$$

   Derivation: since X, Y are jointly Gaussian, $W = X + Y$ is Gaussian, with

$$\mathbf{E}[W] = \mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] = 0,$$

$$\text{var}(W) = \mathbf{E}[W^2] = \mathbf{E}[X^2 + 2XY + Y^2] = 3.$$

   Hence

$$f_W(w) = \frac{1}{\sqrt{6\pi}} e^{-w^2/6}.$$

   Since $Z = |W|$, we have $f_Z(z) = 2f_W(z)$ for $z > 0$, $f_Z(z) = 0$ for $z < 0$.

2.

6.041: Random variables X, Y are independent and uniformly distributed over the interval $[0,1]$. Find the CDF of $W = \max\{X,Y\} - X$.

Answer:

$$F_W(w) = \begin{cases} 1 - 0.5(1-w)^2, & w \in [0,1], \\ 0, & w < 0, \\ 1, & w > 1. \end{cases}$$

Derivation: since $0 \le \max\{x,y\} - x \le 1$ whenever $x, y \in [0,1]$, we have $\mathbf{P}(W \le w) = 0$ for $w < 0$, and $\mathbf{P}(W \le w) = 1$ for $w > 1$. For $w \in [0,1]$,

$$\mathbf{P}(W \le w) = \mathbf{P}(X \le Y \le X + w) + \mathbf{P}(X \ge Y),$$

where $\mathbf{P}(X \ge Y) = 0.5$ is the area of the triangle with vertices at $(0,0)$, $(1,1)$, and $(1,0)$, while $\mathbf{P}(X \le Y \le X + w) = 0.5 - 0.5(1-w)^2$ is the area of the trapezoid with vertices at $(0,0)$, $(w,0)$, $(1,1-w)$, and $(1,1)$.

6.431: Random variables X, Y, Z are independent and uniformly distributed over the interval $[0,1]$. Find the CDF of $W = \max\{X,Y\} - \min\{Y,Z\}$.

Answer:

$$F_W(w) = \begin{cases} 0, & w < 0, \\ 1/6 + w + w^2/2 - 2w^3/3, & w \in [0,1], \\ 1, & w > 1. \end{cases}$$

Derivation: since $0 \le \max\{x,y\} - \min\{y,z\} \le 1$ whenever $x, y, z \in [0,1]$, we have $\mathbf{P}(W \le w) = 0$ for $w < 0$, and $\mathbf{P}(W \le w) = 1$ for $w > 1$. For $w \in [0,1]$,

$$\mathbf{P}(W \le w) = \mathbf{P}(X \le Y \le Z) + \mathbf{P}(A_w) + \mathbf{P}(B_w) + \mathbf{P}(C_w) + \mathbf{P}(D_w) + \mathbf{P}(E_w),$$

where A_w, B_w, C_w, D_w, E_w are the events defined by

$$\begin{aligned} A_w &= \{Z < Y < X,\ X - Z < w\} \\ B_w &= \{X < Z < Y,\ Y - Z < w\} \\ C_w &= \{Z < X < Y,\ Y - Z < w\} \\ D_w &= \{Y < X < Z,\ X - Y < w\} \\ E_w &= \{Y < Z < X,\ X - Y < w\}. \end{aligned}$$

By symmetry, $\mathbf{P}(A_w) = \mathbf{P}(C_w) = \mathbf{P}(E_w)$, and $\mathbf{P}(B_w) = \mathbf{P}(D_w)$. Hence, it is sufficient to calculate $\mathbf{P}(X \le Y \le Z)$, $\mathbf{P}(A_w)$, and $\mathbf{P}(B_w)$. We have

$$\mathbf{P}(X \le Y \le Z) = \int_0^1 dy \int_0^y dx \int_y^1 dz = \int_0^1 y(1-y)dy = \frac{1}{6}.$$

Also,

$$\mathbf{P}(A_w) = \mathbf{P}(Z < 1-w, 0 < Z < Y < X < Z + w) + \mathbf{P}(Z > 1-w, 0 < Z < Y < X < 1),$$

where

$$\begin{aligned} \mathbf{P}(Z < 1-w, 0 < Z < Y < X < Z + w) &= \int_0^{1-w} dz \int_z^{z+w} dx \int_z^x dy \\ &= \int_0^{1-w} dz \int_z^{z+w} (x-z)dx \\ &= \int_0^{1-w} dz \int_0^w x dx \\ &= \int_0^{1-w} \frac{w^2}{2} dz \\ &= \frac{(1-w)w^2}{2}, \end{aligned}$$

$$\begin{aligned} \mathbf{P}(Z > 1-w, 0 < Z < Y < X < 1) &= \int_{1-w}^1 dz \int_z^1 dx \int_z^x dy \\ &= \int_{1-w}^1 dz \int_z^1 (x-z)dx \\ &= \int_{1-w}^1 dz \int_0^{1-z} x dx \\ &= \int_{1-w}^1 \frac{(1-z)^2}{2} dz \\ &= \int_0^w \frac{z^2}{2} dz \\ &= \frac{w^3}{6}. \end{aligned}$$

Similarly,

$$\mathbf{P}(B_w) = \mathbf{P}(Z < 1-w, 0 < X < Z < Y < Z + w) + \mathbf{P}(Z > 1-w, 0 < X < Z < Y < 1),$$

where

$$\begin{aligned}
\mathbf{P}(Z < 1-w, 0 < X < Z < Y < Z+w) &= \int_0^{1-w} dz \int_0^z dx \int_z^{z+w} dy \\
&= \int_0^{1-w} zw dz \\
&= \frac{w(1-w)^2}{2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(Z > 1-w, 0 < X < Z < Y < 1) &= \int_{1-w}^{1} dz \int_0^z dx \int_z^{1} dy \\
&= \int_{1-w}^{1} z(1-z) dz \\
&= \frac{w^2}{2} - \frac{w^3}{3}.
\end{aligned}$$

Finally, we conclude that, for $w \in [0,1]$,

$$\begin{aligned}
\mathbf{P}(W \le w) &= \frac{1}{6} + 3\left(\frac{(1-w)w^2}{2} + \frac{w^3}{6}\right) + 2\left(\frac{w(1-w)^2}{2} + \frac{w^2}{2} - \frac{w^3}{3}\right) \\
&= \frac{1}{6} + w + \frac{w^2}{2} - \frac{2w^3}{3}.
\end{aligned}$$

3. Find a function h such that $\mathbf{E}[X \mid (X-1)^2] = h(X)$, where $X \sim N(0,1)$.

   Answer:

$$h(x) = \frac{1 - |x-1| + (1+|x-1|)e^{-|x-1|}}{1 + e^{-|x-1|}}.$$

   Derivation: let $\psi(x) = (x-1)^2$ and $Z = \psi(X)$. We begin by finding a function g such that $\mathbf{E}[X \mid Z] = g(Z)$. We have $Z = z$ for a given $z > 0$ when $X = g_+(z)$ or $X = g_-(z)$, where $g_\pm(z) = 1 \pm \sqrt{z}$. Hence, for $z > 0$,

$$g(z) = \frac{g_+(z) f_X(g_+(z))/|\dot{\psi}(g_+(z))| + g_-(z) f_X(g_-(z))/|\dot{\psi}(g_-(z))|}{f_X(g_+(z))/|\dot{\psi}(g_+(z))| + f_X(g_-(z))/|\dot{\psi}(g_-(z))|} = \frac{1 - \sqrt{z} + (1+\sqrt{z})e^{-\sqrt{z}}}{1 + e^{-\sqrt{z}}}.$$

   Substituting $z = (x-1)^2$ (i.e. $\sqrt{z} = |x-1|$) into $g(z)$ yields $h(x)$.

4. Z is a random point on the plane which is distributed uniformly over the rectangle with vertices $(1,1)$, $(0,2)$, $(2,4)$, and $(3,3)$. Let X, Y be the random variables defined as the coordinates of Z. We are interested in finding functions $h: \mathbf{R} \mapsto \mathbf{R}$ which produce good estimates $\hat{X} = h(Y)$ of X given Y. Find a function $h_{NL}: \mathbf{R} \mapsto \mathbf{R}$ which minimizes $J(h(\cdot)) = \mathbf{E}[|X - h(Y)|^2]$ over the set of all continuous functions $h: \mathbf{R} \mapsto \mathbf{R}$. What is the value of $J(h_{NL})$?

   Answer: one such function is $h_{NL} = h_*$, where

$$h_*(y) = \begin{cases} 1, & y < 2, \\ y - 1, & y \in [2,3], \\ 2, & y > 3 \end{cases}$$

(the values $h_{NL}(\cdot)$ takes outside the interval $(1,4)$ are irrelevant), with $J(h_{NL}) = 1/4$.

Derivation: for every $y \in (1,4)$ the conditional distribution $f_{X|Y}(x|y)$ is uniform, with center at $h_*(y)$ and width

$$d(y) = \begin{cases} 2(y-1), & y \in (1,2), \\ 2, & y \in [2,3], \\ 2(4-y), & y \in (3,4), \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathbf{E}[X \mid Y] = h_*(Y)$ (i.e. $y \mapsto h_{NL}(y)$ is an optimal estimator), and $\text{var}(X|Y) = d(Y)^2/12$. Since the probability density of Y is proportional to $d(y)$, normalization yields $f_Y(y) = d(y)/4$, hence

$$J(h_{NL}) = \mathbf{E}[\text{var}(X|Y)] = \int_1^4 \frac{d(y)^2}{12} f_Y(y)dy = \frac{1}{4}.$$

5. John is participating in a 6.041 magic ritual. He is given an unfair coin with random probability Q of "tail" distributed uniformly between 0 and 1. John tosses the coin two times. For every "tail" toss, he is given a light bulb with exponentially distributed lifetime (parameter $\lambda = 1$). In addition, he is given one such bulb for just participating in the ritual (so he ends up with one, two, or three light bulbs). John turns all light bulbs on simultaneously. Let T be the time until the first of these bulbs burns out. Find $\mathbf{E}[T]$ and $\text{var}(T)$.

   Answer: $\mathbf{E}[T] = \frac{11}{18}$, $\text{var}(T) = \frac{173}{324}$.

   Derivation: let N be the random variable representing the number of light bulbs given to John (by the problem description, $N \in \{1,2,3\}$).

   For a given N, T is exponential with parameter N, hence

$$\mathbf{E}[T \mid N] = 1/N, \quad \text{var}(T \mid N) = 1/N^2.$$

   Therefore

$$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}[\mathbf{E}[T \mid N]] = \mathbf{E}[1/N], \\ \text{var}(T) &= \mathbf{E}[\text{var}(T|N)] + \text{var}(\mathbf{E}[T|N]) = \mathbf{E}[1/N^2] + \text{var}(1/N) = 2\mathbf{E}[1/N^2] - \mathbf{E}[1/N]^2. \end{aligned}$$

   For a given Q, N takes values 1,2, and 3, with probabilities $(1-Q)^2$, $2Q(1-Q)$, and Q^2 respectively. Hence

$$\begin{aligned} \mathbf{E}[1/N|Q] &= (1-Q)^2 + \frac{1}{2}2Q(1-Q) + \frac{1}{3}Q^2 = 1 - Q + \frac{Q^2}{3}, \\ \mathbf{E}[1/N^2|Q] &= (1-Q)^2 + \frac{1}{4}2Q(1-Q) + \frac{1}{9}Q^2 = 1 - \frac{3Q}{2} + \frac{11Q^2}{18}. \end{aligned}$$

   Therefore

$$\begin{aligned} \mathbf{E}[1/N] &= \mathbf{E}[\mathbf{E}[1/N|Q]] = \int_0^1 \left\{1 - q + \frac{q^2}{3}\right\} dq = \frac{11}{18}, \\ \mathbf{E}[1/N^2] &= \mathbf{E}[\mathbf{E}[1/N^2|Q]] = \int_0^1 \left\{1 - \frac{3q}{2} + \frac{11q^2}{18}\right\} dq = \frac{49}{108}. \end{aligned}$$

Finally,

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}[1/N] = \frac{11}{18}, \\
\text{var}(T) &= 2\mathbf{E}[1/N^2] - \mathbf{E}[1/N]^2 = \frac{49}{54} - \frac{11^2}{18^2} = \frac{173}{324}.
\end{aligned}$$