Tutorial 6: Solutions

1. See Example 3.19, p. 179 of the text

2. Note first that, since X only takes on two values, we need only find $p_{X|Z}(1|z)$, since

$$p_{X|Z}(0|z) = 1 - p_{X|Z}(1|z).$$

Using Bayes' rule,

$$p_{X|Z}(1|z) = \frac{p_X(1)p_{Z|X}(z|1)}{p_Z(z)},$$

and

$$p_Z(z) = p_X(1)p_{Z|X}(z|1) + p_X(-1)p_{Z|X}(z|-1).$$

We know that $p_X(1) = p_X(-1) = \frac{1}{2}$. Also,

$$p_{Z|X}(z|1) = \frac{1}{2}e^{-|z-1|},$$

$$p_{Z|X}(z|-1) = \frac{1}{2}e^{-|z+1|}.$$

So,

$$p_{X|Z}(1|z) = \frac{e^{-|z-1|}}{e^{-|z-1|} + e^{-|z+1|}}.$$

3. Since $Y = |X|$ you can visualize the PDF for any given y as

$$f_Y(y) \;=\; \begin{cases} f_X(y) + f_X(-y), & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Also note that since $Y = |X|$, $Y \geq 0$.

(a) $f_X(x) = \begin{cases} \frac{1}{3}, & \text{if } -2 < x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

So, $f_X(x)$ for $-1 \leq x \leq 0$ gets added to $f_X(x)$ for $0 \leq x \leq 1$:

$$f_Y(y) \;=\; \begin{cases} 2/3, & \text{if } 0 \leq y \leq 1, \\ 1/3, & \text{if } 1 < y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Here we are told $X > 0$. So there are no negative values of X that need to be considered. Thus,

$$f_Y(y) \;=\; f_X(y) \;=\; \begin{cases} 2e^{-2y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) As explained in the beginning, $f_Y(y) = f_X(y) + f_X(-y)$.

4. We are given $Z = g(X) = 3X^2 - 1$. Noting that g is a monotonic function of $|X|$, let's consider a random variable $h(Z) = \sqrt{\frac{(Z+1)}{3}} = |X|$. Now use the PDF formula to find f_Z given $f_{|X|}$ for $Z \geq -1$:

$$\begin{aligned} \frac{dh(z)}{dz} &= \frac{1}{6}\sqrt{\frac{3}{z+1}} \\ \Longrightarrow f_Z(z) &= f_{|X|}(h(z))\left|\frac{dh(z)}{dz}\right| \quad \text{for } z > -1 \end{aligned}$$

Since X is a standard normal random variable, its PDF is symmetric about the origin and so

$$f_{|X|}(x) = \begin{cases} 2f_X(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Note that $h(z) \geq 0$ for $Z \geq -1$. Thus we obtain the PDF of Z by just combining the formulae above:

$$\begin{aligned} f_Z(z) &= \begin{cases} 2\frac{1}{\sqrt{2\pi}}e^{-\frac{h(z)^2}{2}}\frac{1}{6}\sqrt{\frac{3}{z+1}}, & z \geq -1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{3\sqrt{2\pi}}e^{-\frac{z+1}{6}}\sqrt{\frac{3}{z+1}}, & z \geq -1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$