LECTURE 9

- **Readings:** Sections 3.4–3.5

Lecture outline

- Review
- Multiple random variables
  - Conditioning
  - Independence
- Examples

Review/preview

discrete		continuous
$p_X(x)$		$f_X(x)$
	$F_X(x)$	
$\sum_x x p_X(x)$	$\mathbf{E}[X]$	$\int_x x f_X(x)\,dx$
	$\text{var}(X)$	
$p_{X,Y}(x,y)$		$f_{X,Y}(x,y)$
	$F_{X,Y}(x,y)$	
$p_{X\|A}(x)$		$f_{X\|A}(x)$
$p_{X\|Y}(x \mid y)$		$f_{X\|Y}(x \mid y)$
$\vdots$	$\vdots$	$\vdots$

Joint PDF $f_{X,Y}(x,y)$

- X and Y are jointly continuous r.v.s with joint probability density function $f_{X,Y}(x,y)$ when

$$\mathbf{P}((X,Y) \in S) \;=\; \iint_S f_{X,Y}(x,y)\,dx\,dy$$

- Interpretation: When $\delta > 0$ is very small

$$\mathbf{P}(x \le X \le x+\delta,\; y \le Y \le y+\delta) \;\approx\; f_{X,Y}(x,y)\cdot\delta^2$$

$$f_{X,Y}(x,y) \;\approx\; \frac{\mathbf{P}(x \le X \le x+\delta,\; y \le Y \le y+\delta)}{\delta^2}$$

- Expectations:

$$\mathbf{E}[g(X,Y)] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

Derivation of marginal PDF

- From the joint to the marginal:

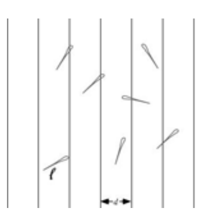
$$f_X(x)\cdot\delta \;\approx\; \mathbf{P}(x \le X \le x+\delta)$$
$$=$$

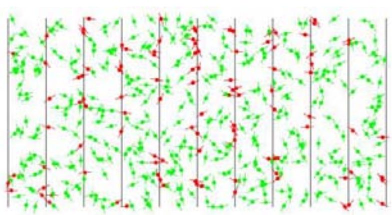
- X and Y are called independent when

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Buffon's needle

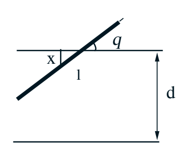
Georges-Louis Leclerc, Compte de Buffon (1707–1788)





Can dropping needles on a hardwood floor perform an interesting computation?

Buffon's needle



- Parallel lines at distance d

  Needle of length ℓ (assume $\ell < d$)
- Find $\mathbf{P}$(needle intersects one of the lines)
- $X \in [0, d/2]$: distance of needle midpoint to nearest line
- **Model:** X, Θ uniform, independent

$$f_{X,\Theta}(x,\theta) \;=\; \qquad\qquad 0 \le x \le d/2,\; 0 \le \theta \le \pi/2$$

- Intersect if $X \le \frac{\ell}{2}\sin\Theta$

Conditioning

- Conditioning on an event A with $\mathbf{P}(A) > 0$: conditional PDF defined to satisfy

$$\mathbf{P}(X \in S \mid A) \;=\; \int_S f_{X|A}(x)\,dx$$

- $\mathbf{E}[X \mid A] \;=\; \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$
- If $A_1, A_2, \ldots, A_n$ partitions the sample space:

$$f_X(x) \;=\; \sum_{i=1}^{n} \mathbf{P}(A_i) f_{X|A_i}(x)$$

$$\mathbf{E}[X] \;=\; \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{E}[X \mid A_i]$$

- (many natural variations)

Conditioning on a continuous random variable

- Recall

$$\mathbf{P}(x \le X \le x+\delta) \;\approx\; f_X(x)\cdot\delta$$

- By analogy, would like:

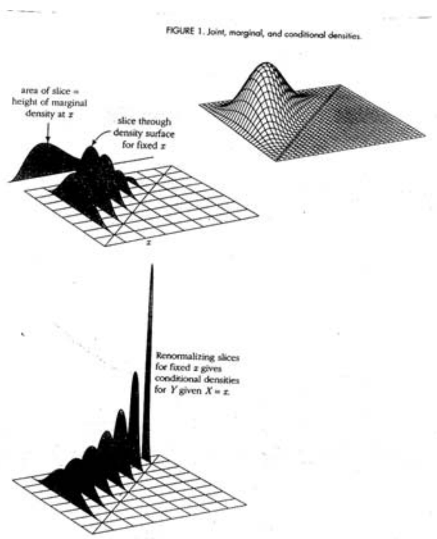
$$\mathbf{P}(x \le X \le x+\delta \mid Y \approx y) \;\approx\; f_{X|Y}(x \mid y)\cdot\delta$$

- This leads us to the **definition**

$$f_{X|Y}(x \mid y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{(where } f_Y(y) \neq 0\text{)}$$

- If X and Y are independent, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, so

$$f_{X|Y}(x \mid y) \;=\; f_X(x)$$

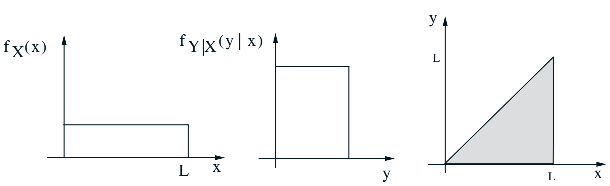


[*Probability*, J. Pittman, 1999]

Stick-breaking example

- Break a stick of length ℓ (an interval $[0,\ell]$) twice:
  - break at a uniformly chosen random point X
  - break remaining stick $[0, X]$ at a uniformly chosen point Y
- Find $\mathbf{E}[Y]$

Stick-breaking example (2)



$$f_{X,Y}(x,y) \;=\; f_X(x) f_{Y|X}(y \mid x) \;=$$