Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 1
Due: September 14, 2011

1. Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

   (a) at least two of the events A, B, C occurs;

   (b) at most two of the events A, B, C occurs;

   (c) none of the events A, B, C occurs;

   (d) all three events A, B, C occur;

   (e) exactly one of the events A, B, C occurs;

   (f) events A and C occur, but not B;

   (g) either event A occurs or, if not, then C also does not occur.

   In each case draw the corresponding Venn diagrams.

2. Anne and Bob each have a deck of playing cards. Each flips over a randomly selected card. Assume that all pairs of cards are equally likely to be drawn. Determine the following probabilities:

   (a) the probability that at least one card is an ace,

   (b) the probability that the two cards are of the same suit,

   (c) the probability that neither card is an ace,

   (d) the probability that neither card is a diamond or club.

3. Mary and Tom park their cars in a parking lot with n consecutive parking spaces, i.e. n spaces in a row, where only one car fits in each space. Mary and Tom each pick a parking space at random, i.e. all pairs of parking spaces are equally likely. Find the probability that they park next to each other.

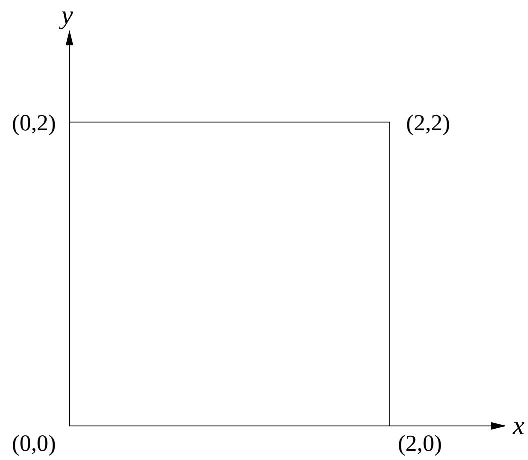
4. Let A, B and C be three events. Use the axioms of probability to prove the following:

   (a) $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$

   (b) $\mathbf{P}(A \cup B \cup C) \leq \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$

   Note: Your proof should be a step-by-step derivation, where each step appeals to an axiom or a logical rule.

5. A baseball pitcher, Bill, has good control of his pitches. He always throws his pitches inside the "box" which we consider to be a 2 by 2 square. He throws the pitches uniformly over the square (i.e. the probability of a pitch falling within an area of the square is proportional to this area.) Let $(0, 0)$ and $(2, 2)$ be the coordinates of the lower-left corner and the upper-right corner of the square, respectively as shown below.



Two groups A and B of fans are betting on where Bill's next pitch will fall. Among group A,

- person 1 bets that the pitch is going to be in the left half part of the square, i.e. $0 \le x \le 1$.
- person 2 bets that it will be in one third of the square from the left, i.e. $0 \le x \le \frac{2}{3}$.
- and in general, person n makes the bet that the pitch will fall in the area $0 \le x \le 2/(n+1)$.

(a) What is the probability that individual n from group A wins his bet?

(b) What is the probability that individual n wins but not individual $n + 1$?

Among group B, that fans bet in a similar fashion, but on the height of the pitch, i.e. individual n bets that the next pitch will fall in the area $0 \le y \le 2/(n + 1)$.

(c) What is the probability that all individuals (1 through n of *both* groups) win their bets?

(d) When n goes to infinity, what is the probability that all fans of both groups win their bets? Note: Be precise in your derivation.

G1†. Consider an experiment whose sample space is the real line.

(a) Let $\{a_n\}$ be an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence that converges to b. Show that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

Here, the notation $[a, b]$ stands for the closed interval $\{x \mid a \le x \le b\}$. *Note:* This result seems intuitively obvious. The issue is to derive it using the axioms of probability theory.

†Required for 6.431; optional for 6.041

(b) Let $\{a_n\}$ be a decreasing sequence that converges to a and $\{b_n\}$ an increasing sequence that converges to b. Is it true that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b])?$$

Note: You may use freely the results from the problems in the text in your proofs.

[†]Required for 6.431; optional for 6.041