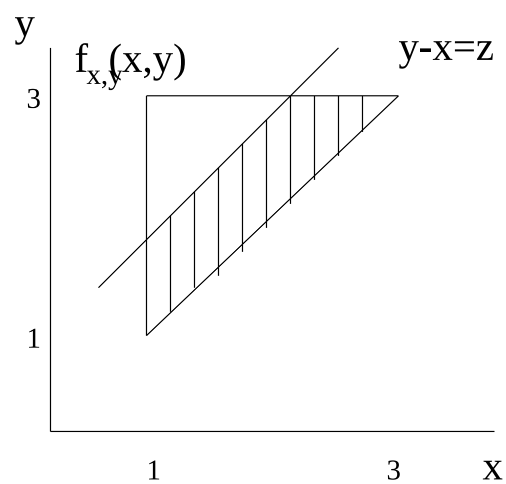
Problem Set 6: Solutions

1. First, let's draw the joint PDF on a 2D plot,



   (a) The joint PDF must integrate to 1. From $\int_{x=1}^{x=3}\int_{y=x}^{y=3} axdydx = \frac{10}{3}a = 1$, we get $\boxed{a = \frac{3}{10}}$.

   (b) $f_Y(y) = \int f_{X,Y}(x,y)dx = \begin{cases} \int_1^y \frac{3}{10}xdx & 1 \le y \le 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{20}(y^2-1) & 1 < y \le 3 \\ 0 & \text{otherwise} \end{cases}$.

   (c) $f_{X|Y}(x|\frac{3}{2}) = \frac{f_{X,Y}(x,y)}{f_Y(\frac{3}{2})} = \frac{8}{5}x,\ 1 \le x \le \frac{3}{2}$. Then, $E[\frac{1}{X}|Y=\frac{3}{2}] = \int_1^{\frac{3}{2}} \frac{1}{x}\frac{8}{5}xdx = \boxed{\frac{4}{5}}$.

   (d) We calculate the CDF of Z,

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(Y - X \le z) \\ &= \begin{cases} 0 & z < 0 \\ 1 - \int_{x=1}^{x=3-z}\int_{y=x+z}^{y=3} \frac{3}{10}xdydx = \frac{9}{10} + \frac{3}{20}(3-z) - \frac{1}{20}(3-z)^3 & 0 \le z \le 2 \\ 1 & 2 < z \end{cases}. \end{aligned}$$

   Then, we get PDF of Z by taking the derivative of CDF,

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{3}{20}z^2 - \frac{9}{10}z + \frac{6}{5} & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

2. The PDF of Z, $f_Z(z)$, can be readily computed using the convolution integral:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)f_Y(z-t)\,dt.$$

   For $z \in [-1, 0]$,

$$f_Z(z) = \int_{-1}^{z} \frac{1}{3}\cdot\frac{3}{4}(1-t^2)\,dt = \frac{1}{4}\left(z - \frac{z^3}{3} + \frac{2}{3}\right).$$

   For $z \in [0, 1]$,

$$f_Z(z) = \int_{z-1}^{z} \frac{1}{3}\cdot\frac{3}{4}(1-t^2)\,dt = \frac{1}{4}\left(1 - \frac{z^3}{3} + \frac{(z-1)^3}{3}\right).$$

For $z \in [1,2]$,

$$f_Z(z) = \int_{z-1}^{1} \frac{1}{3} \cdot \frac{3}{4}(1-t^2)\,dt + \int_{-1}^{z-2} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\,dt = \frac{1}{4}\left(z + \frac{(z-1)^3}{3} - \frac{2(z-2)^3}{3} - 1\right).$$

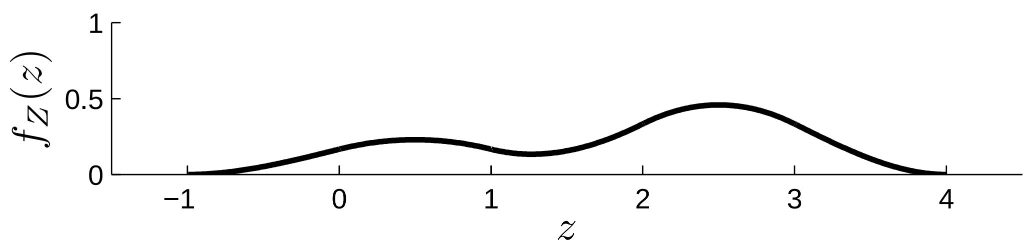
For $z \in [2,3]$,

$$f_Z(z) = \int_{z-3}^{z-2} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\,dt = \frac{1}{6}\left(3 + (z-3)^3 - (z-2)^3\right).$$

For $z \in [3,4]$,

$$f_Z(z) = \int_{z-3}^{1} \frac{2}{3} \cdot \frac{3}{4}(1-t^2)\,dt = \frac{1}{6}\left(11 - 3z + (z-3)^3\right).$$

A sketch of $f_Z(z)$ is provided below.



3. (a) X_1 and X_2 are negatively correlated. Intuitively, a large number of tosses that result in a 1 suggests a smaller number of tosses that result in a 2.

   (b) Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = \frac{1}{k} \cdot \frac{1}{k} \qquad \text{for} \quad s \neq t.$$

   Thus,

$$\begin{aligned} \mathbf{E}[X_1 X_2] &= \mathbf{E}\left[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\right] \\ &= n\mathbf{E}\left[A_1(B_1 + \cdots + B_n)\right] = n(n-1) \cdot \frac{1}{k} \cdot \frac{1}{k} \end{aligned}$$

   and

$$\begin{aligned} \mathrm{cov}(X_1, X_2) &= \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]E[X_2] \\ &= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}. \end{aligned}$$

   The covariance of X_1 and X_2 is negative as expected.

4. **A financial parable.**

(a) The bank becomes insolvent if the asset's gain $R \leq -5$ (i.e., it loses more than 5%). This probability is the CDF of R evaluated at -5. Since R is normally distributed, we can convert this CDF to be in terms of a standard normal random variable by subtracting away the mean and dividing by the standard deviation, and then look up the value in a standard normal CDF table.

$$\begin{aligned}
\mathbf{E}[R] &= 7, \\
\mathrm{var}(R) &= 10^2 = 100, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{10} \leq \frac{-5-7}{10}\right) = \Phi(-1.2) \approx 0.115.
\end{aligned}$$

Thus, by investing in just this one asset, the bank has a 11.5% chance of becoming insolvent.

(b) If we model the R_i's as **independent** normal random variables, then their sum $R = (R_1 + \cdots + R_{20})/20$ is also a normal random variable (see Example 4.11 on page 214 of the text). Thus, we can calculate the mean and variance of this new R and proceed as in part (a). Note that since the random variables are assumed to be independent, the variance of their sum is just the sum of their individual variances.

$$\begin{aligned}
\mathbf{E}[R] &= (\mathbf{E}[R_1] + \cdots + \mathbf{E}[R_{20}])/20 = 7, \\
\mathrm{var}(R) &= \frac{1}{20^2}(\mathrm{var}(R_1) + \cdots + \mathrm{var}(R_{20})) = \frac{20 \cdot 100}{400} = 5, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{\sqrt{5}} \leq \frac{-5-7}{\sqrt{5}}\right) = \Phi(-5.367) \approx 0.0000000439 = 4.39 \cdot 10^{-8}.
\end{aligned}$$

Thus, by diversifying and assuming that the 20 assets have **independent** gains, the bank has seemingly decreased its probability of becoming insolvent to a palatable value.

(c) Now, if the gains R_i are positively correlated, then we can no longer sum up the individual variances; we need to account for the covariance between pairs of random variables. The covariance is given by

$$\mathrm{cov}(R_i, R_j) = \rho(R_i, R_j)\sqrt{\mathrm{var}(R_i)\mathrm{var}(R_j)} = \frac{1}{2}\sqrt{10^2 \cdot 10^2} = 50.$$

From page 220 in the text, we know that the variance in this case is

$$\begin{aligned}
\mathrm{var}(R) &= \mathrm{var}\left(\frac{1}{20}\sum_{i=1}^{20} R_i\right) = \frac{1}{400}\left(\sum_{i=1}^{20}\mathrm{var}(R_i) + \sum_{\{(i,j)|i\neq j\}}\mathrm{cov}(R_i, R_j)\right) \\
&= \frac{1}{400}(20 \cdot 100 + 380 \cdot 50) = 52.5.
\end{aligned}$$

Since we assume that $R = (R_1 + \cdots + R_{20})/20$ is still normal, we can again apply the same steps as in parts (a) and (b):

$$\begin{aligned}
\mathbf{E}[R] &= (\mathbf{E}[R_1] + \cdots + \mathbf{E}[R_{20}])/20 = 7, \\
\mathrm{var}(R) &= 52.5, \\
\mathbf{P}(R \leq -5) &= \mathbf{P}\left(\frac{R-7}{\sqrt{52.5}} \leq \frac{-5-7}{\sqrt{52.5}}\right) = \Phi(-1.656) \approx 0.0488.
\end{aligned}$$

Thus, by taking into account the positive correlation between the assets' gains, we are no longer as comfortable with the probability of insolvency as we thought we were in part (b).

5. (a) (i) Using the Law of Iterated Expectations, we have

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Q]] = \mathbf{E}[Q] = \frac{1}{2}.$$

(ii) X is a Bernoulli random variable with a mean $p = \frac{1}{2}$ and its variance is $\text{var}(X) = p(1-p) = 1/4$.

(b) We know that $\text{cov}(X, Q) = \mathbf{E}[XQ] - \mathbf{E}[X]\mathbf{E}[Q]$, so first let's calculate $\mathbf{E}[XQ]$:

$$\mathbf{E}[XQ] = \mathbf{E}[\mathbf{E}[XQ \mid Q]] = \mathbf{E}[Q\mathbf{E}[X \mid Q]] = \mathbf{E}[Q^2] = \frac{1}{3}.$$

Therefore, we have

$$\text{cov}(X, Q) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

(c) Using Bayes' Rule, we have

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{p_X(1)} = \frac{f_Q(q)\mathbf{P}(X = 1 \mid Q = q)}{\mathbf{P}(X = 1)}, \quad 0 \le q \le 1.$$

Additionally, we know that

$$\mathbf{P}(X = 1 \mid Q = q) = q,$$

and that for Bernoulli random variables

$$\mathbf{P}(X = 1) = \mathbf{E}[X] = \frac{1}{2}.$$

Thus, the conditional PDF of Q given $X = 1$ is

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{1 \cdot q}{1/2} \\ &= \begin{cases} 2q, & 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

6. (a) If X takes a value x between -1 and 1, the conditional PDF of Y is uniform between -2 and 2. If X takes a value x between 1 and 2, the conditional PDF of Y is uniform between -1 and 1.

Similarly, if Y takes a value y between -1 and 1, the conditional PDF of X is uniform between -1 and 2. If Y takes a value y between 1 and 2, or between -2 and -1, the conditional PDF of X is uniform between -1 and 1.

(b) We have

$$\mathbf{E}[X \mid Y = y] = \begin{cases} 0, & \text{if } -2 \le y \le -1, \\ 1/2, & \text{if } -1 < y \le 1, \\ 0, & \text{if } 1 \le y \le 2, \end{cases}$$

and

$$\text{var}(X \mid Y = y) = \begin{cases} 1/3, & \text{if } -2 \le y \le -1, \\ 3/4, & \text{if } -1 < y \le 1, \\ 1/3, & \text{if } 1 \le y \le 2. \end{cases}$$

It follows that $\mathbf{E}[X] = 3/10$ and $\text{var}(X) = 193/300$.

(c) By symmetry, we have $\mathbf{E}[Y \mid X] = 0$ and $\mathbf{E}[Y] = 0$. Furthermore, $\text{var}(Y \mid X = x)$ is the variance of a uniform PDF (whose range depends on x), and

$$\text{var}(Y \mid X = x) \;=\; \begin{cases} 4/3, & \text{if } -1 \le x \le 1, \\ 1/3, & \text{if } 1 < x \le 2. \end{cases}$$

Using the law of total variance, we obtain

$$\text{var}(Y) \;=\; \mathbf{E}[\text{var}(Y \mid X)] \;=\; \frac{4}{5}\cdot\frac{4}{3} + \frac{1}{5}\cdot\frac{1}{3} = 17/15.$$

7. First let us write out the properties of all of our random variables. Let us also define K to be the number of members attending a meeting and B to be the Bernoulli random variable describing whether or not a member attends a meeting.

$$\begin{aligned} \mathbf{E}[N] &= \frac{1}{1-p}, & \text{var}(N) &= \frac{p}{(1-p)^2}, \\ \mathbf{E}[M] &= \frac{1}{\lambda}, & \text{var}(M) &= \frac{1}{\lambda^2}, \\ \mathbf{E}[B] &= q, & \text{var}(B) &= q(1-q). \end{aligned}$$

(a) Since $K = B_1 + B_2 + \cdots B_N$,

$$\begin{aligned} \mathbf{E}[K] &= \mathbf{E}[N]\cdot\mathbf{E}[B] = \frac{q}{1-p}, \\ \text{var}(K) &= \mathbf{E}[N]\cdot\text{var}(B) + (\mathbf{E}(B))^2\cdot\text{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}. \end{aligned}$$

(b) Let G be the total money brought to the meeting. Then $G = M_1 + M_2 + \cdots + M_K$.

$$\begin{aligned} \mathbf{E}[G] &= \mathbf{E}[M]\cdot\mathbf{E}[K] = \frac{q}{\lambda(1-p)}, \\ \text{var}(G) &= \text{var}(M)\cdot\mathbf{E}[K] + (\mathbf{E}[M])^2\text{var}(K) \\ &= \frac{q}{\lambda^2(1-p)} + \frac{1}{\lambda^2}\left(\frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}\right). \end{aligned}$$

G1†. (a) We first find $E[X_n|X_{n-1} = k]$. Using the total expectation theorem,

$$\begin{aligned} E[X_n|X_{n-1} = k] = \;& E[X_n|X_{n-1} = k, (k+1)^{st} \text{ toss is a H}]\cdot P((k+1)^{st} \text{ toss is a H}) \\ & + E[X_n|X_{n-1} = k, (k+1)^{st} \text{ toss is a T}]\cdot P((k+1)^{st} \text{ toss is a T}) \end{aligned}$$

Now, if we are given that $X_{n-1} = k$, then this means that the first time $(n-1)$ heads occurred in succession was on the k^{th} toss.
If in addition we are given that the $(k+1)^{st}$ toss is a H, then this means that the first time n heads occur in succes sion is on the $(k+1)^{st}$ toss, *i.e.* $X_n = k+1$. Hence,

$$E[X_n|X_{n-1} = k, (k+1)^{st} \text{ toss is a H}] = k+1.$$

However, if the $(k+1)^{st}$ toss is given to be a T, then the first time n heads occur in succession in the part of the sequence starting from the $(k+2)^{nd}$ toss is also the first time that n heads

occur in succession in the entire sequence. Since the toss es are independent, the additional number of tosses after the $(k+1)^{st}$ toss for this to happen, has the same distribution as X_n without any conditioning.
This gives:

$$E[X_n|X_{n-1} = k, (k+1)^{st} \text{ toss is a T}] = k + 1 + E[X_n].$$

Substituting in the above,

$$\begin{aligned} E[X_n|X_{n-1} = k] &= p \cdot (k+1) + (1-p) \cdot (k + 1 + E[X_n]) \\ &= k + 1 + (1-p) \cdot E[X_n] \\ \text{Hence,} \qquad E[X_n|X_{n-1}] &= X_{n-1} + 1 + (1-p) \cdot E[X_n] \end{aligned}$$

Taking expectation throughout,

$$\begin{aligned} E\Big[E[X_n|X_{n-1}]\Big] = E[X_n] &= E[X_{n-1}] + 1 + (1-p) \cdot E[X_n] \\ \Rightarrow E[X_n] &= \frac{1}{p} + \frac{1}{p} E[X_{n-1}] \end{aligned}$$

Now, X_1 is the number of tosses till the first head. Hence, X_1 is a geometric random variable with parameter p, and its mean is $E[X_1] = \frac{1}{p}$. Using this as the basis step, we can prove by induction that for all $n \geq 1$,

$$E[X_n] = \sum_{k=1}^{n} \frac{1}{p^k}$$

(b) Using the law of iterated expectations, $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]]$. Conditioned on Y, X is a geometric random variable, and therefore $\mathbf{E}[\mathbf{E}[X \mid Y]] = 1/Y$. Therefore,

$$\mathbf{E}[X] = \mathbf{E}[1/Y] = \int_0^1 \frac{1}{y}\, dy = +\infty.$$