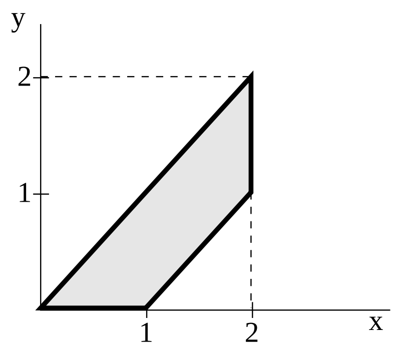
Recitation 24
December 6, 2011

1. Continuous random variables X and Y have a joint PDF given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \text{ belongs to the closed shaded region} \\ 0 & \text{otherwise} \end{cases}$$



(a) Find constant value c.

(b) The value of X will be revealed to us; we have to design an estimator $g(X)$ of Y that minimizes the conditional expectation $\mathbf{E}[(Y - g(X))^2 | X = x]$, for all x, over all possible estimators. Provide a plot of the optimal estimator as a function of its argument.

(c) Let $g^*(X)$ be the optimal estimator of part (a). Find the numerical value of $\mathbf{E}[g^*(X)]$ and $\text{var}(g^*(X))$?

(d) Find the least mean squared estimation error $\mathbf{E}[(Y - g^*(X))^2]$. Is that the same as $\mathbf{E}[\text{var}(Y \mid X)]$?

(e) Find $\text{var}(Y)$.

(f) Let $l^*(X)$ be the optimal linear LMS estimator. Plot $l^*(X)$ and find the numerical value of $\mathbf{E}[l^*(X)]$ and $\text{var}(l^*(X))$?

(g) The mean squared error of the linear LMS estimator is defined as $\mathbf{E}[(Y - l^*(X))^2]$. Which do you think will be larger, $\mathbf{E}[(Y - g^*(X))^2]$ or $\mathbf{E}[(Y - l^*(X))^2]$. Calculate $\mathbf{E}[(Y - l^*(X))^2]$ and verify your answer.

2. **Problem 8.19, page 450 of textbook**
Consider a photodetector in an optical communications system that counts the number of photons that arrive in a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is p. If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable Θ with mean λ. If the transmitter is off, the number of photons transmitted is zero.

   Unfortunately, regardless of whether or not the transmitter is on or off, photons may still be detected due to a phenomenon called "shot noise". The number N of detected shot noise photons is a Poisson random variable with mean μ. Thus, the total number X of detected photons is equal to $\Theta + N$ if the transmitter is on, and is equal to N otherwise. We assume that N and Θ are independent, so that $\Theta + N$ is also Poisson with mean $\lambda + \mu$.

   (a) What is the probability that the transmitter was on, given that the photodetector detected k photons?

   (b) Describe the MAP rule for deciding whether the transmitter was on.

   (c) Find the linear LMS estimator of the number of transmitted photons, based on the number of detected photons.