Tutorial 11
December 9, 2011

1. We wish to estimate the probability of heads of a biased coin, which we denote θ. We model θ as a value of a random variable Θ, which is uniformly distributed over the interval $[0, 1]$. Let X be the number of heads observed in n independent tosses of the coin.

   In your computations, you might find the following equality useful:

   $$\int_0^1 t^a (1-t)^b \, dt = \frac{a!b!}{(a+b+1)!},$$

   for $a, b = 0, 1, 2, \ldots$.

   (a) Find the MAP estimator of θ as a function of the number of observed heads, $\hat{\theta}_{\text{MAP}}(x) = \arg\max_\theta p_{\Theta|X}(\theta|x)$.

   (b) Find the linear LMS estimator of θ as a function of X.

   (c) Find the LMS estimator for the value of θ as a function of X, $\hat{\theta}_{\text{LMS}}(x) = E[\Theta \mid X = x]$.

   (d) How do you expect the mean square error of the MAP estimator to compare to that of the LMS estimator?

   (e) Find the ML estimator of θ as a function of X. Hint: give a 1 line answer.

2. Consider a noisy channel over which you send messages consisting of 0s and 1s to your friend. It is known that the channel independently flips each bit sent with some fixed probability p; however the value of p is unknown. You decide to conduct some experiments to estimate p and seek your friend's help. Your friend, cheeky as she is, insists that you send her messages consisting of three bits each (which you will both agree upon in advance); for each message, she will only tell you the total number of bits in that message that were flipped. Let X denote the number of bits flipped in a particular three-bit message.

   (a) Find the PMF of X.

   (b) Derive the ML estimator for p based on $X_1, \ldots, X_n$, the numbers of bits flipped in the first n three-bit messages.

   (c) Is the ML estimator unbiased?

   (d) Is the ML estimator consistent?

   (e) You send $n = 100$ three-bit messages and find that the total number of bits flipped is 20. Construct a 95% confidence interval for p. If necessary, you may use a conservative bound on the variance of the number of bits flipped.

   (f) What are some other ways to estimate the variance. How do you expect your confidence interval to change with different estimates of the variance.