Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 9
Due November 25, 2009

1. Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to go out either the front or back door; and similarly, when he returns, he is equally likely to go to either the front or back door. Oscar owns only five pairs of running shoes which he takes off immediately after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted. We are interested in determining the long-term proportion of time that he runs barefooted.

   (a) Set the scenario up as a Markov chain, specifying the states and transition probabilities.

   (b) Determine the long-run proportion of time Oscar runs barefooted.

2. The outcomes of successive flips of a particular coin are dependent and are found to be described fully by the conditional probabilities

$$P(H_{n+1}|H_n) = 3/4 \quad P(T_{n+1}|T_n) = 2/3$$

   where we have used the notation: Event H_k: Heads on kth toss; Event T_k: Tails on kth toss. We know that the first toss came up heads.

   (a) Determine the probability that the *first* tail will occur on the kth toss ($k = 2, 3, 4, \ldots$).

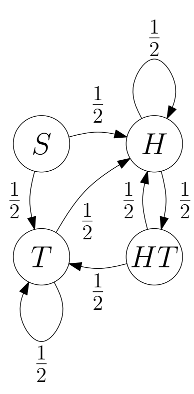
   (b) What is the approximate probability that flip 5000 will come up heads?

   (c) What is the approximate probability that flip 5000 will come up heads and flip 5002 will also come up heads?

   (d) Given that flips $5001, 5002, \ldots, 5000+m$ all have the same result, what is the approximate probability that all of these m outcomes are heads? Simplify your answer as much as possible, and interpret your result for large values of m.

   (e) We are told that the 375th head just occurred on the 500th toss. Determine the expected value of the number of additional tosses required until we observe the 379th head.

3. Assume that a fair coin is tossed repeatedly, with the tosses being independent. We want to determine the expected number of tosses necessary to first observe a heads directly followed by tails. To do so, we define a Markov chain with states S, H, T, HT, where S is a starting state, H indicates heads on the current toss, T indicates tails on the current toss (without heads on the previous toss), and HT indicates heads followed by tails over the last two tosses. This Markov chain is illustrated below:



We can find the expected number of tosses necessary to first observe a heads directly followed by tails by solving a mean first passage time problem for this Markov chain.

(a) What is the expected number of flips necessary to first observe heads directly followed by tails?

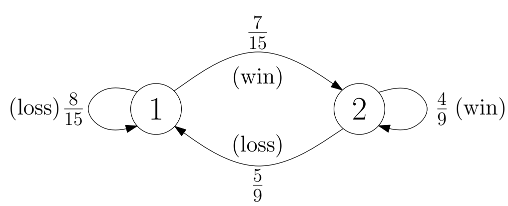
(b) Assuming we have just observed heads followed by tails, what is the expected number of additional tosses until we again observe heads followed directly by tails?

Next, we want to answer the same questions for the event tails directly followed by tails. Set up a different Markov chain from which we could calculate the expected number of tosses necessary to first observe tails directly followed by tails.

(c) What is the expected number of flips necessary to first observe tails directly followed by tails?

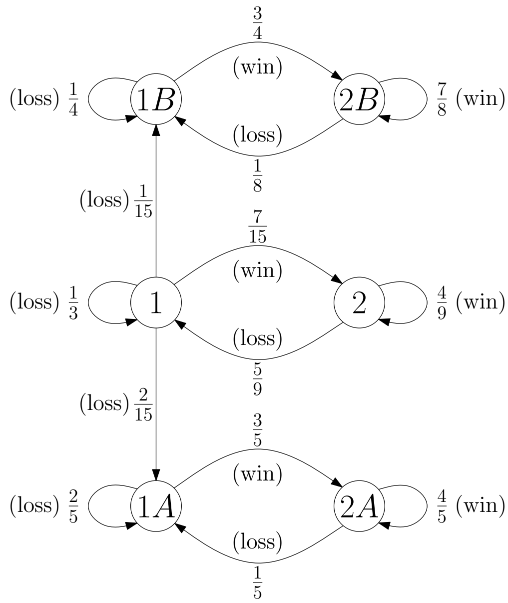
(d) Assuming we have just observed tails followed by tails, what is the expected number of additional tosses until we again observe tails followed directly by tails? Note that the number of additional tosses could be as little as one, if tails were to come up again.

4. Jack is a gambler who pays for his MIT tuition by spending weekends in Las Vegas. Lately he's been playing 21 at a table that returns cards to the deck and reshuffles them all before each hand. As he has a fixed policy in how he plays, his probability of winning a particular hand remains constant, and is independent of all other hands. There is a wrinkle, however; the dealer switches between two decks (deck #2 is more unfair to Jack than deck #1), depending on whether or not Jack wins. Jack's wins and losses can be modeled via the transitions of the following Markov chain, whose states correspond to the particular deck being used.



(a) What is Jack's long term probability of winning?

Given that Jack loses and the dealer is not occupied with switching decks, with probability $\frac{3}{8}$ the dealer looks away for one or two seconds, independently of everything else. When this happens, Jack secretly inserts additional cards into both of the dealer's decks, transforming the decks into types $1A$ & $2A$ (when he has 1 second) or $1B$ & $2B$ (when he has 2 seconds). Jack slips cards into the decks at most once. Note the illustration of the modified Markov chain. Assume in all future problems that play begins with the dealer using deck #1.



(b) What is the probability of Jack eventually playing with decks $1A$ and $2A$?

(c) What is Jack's long-term probability of winning?

(d) What is the expected time (as in number of hands) until Jack slips additional cards into the deck?

(e) What is the distribution of the number of times that the dealer switches from deck 2 to deck 1?

(f) What is the distribution of the number of wins that Jack has before he slips extra cards into the deck? *Hint*: Note that after some conditioning, we have a geometric number of geometric random variables, all of which are independent.

(g) What is the average net losses (number of losses minus the number of wins, sometimes negative) prior to Jack slipping additional cards into the deck?

(h) Given that after a very long period of time Jack is playing a hand with deck $1A$, what is the approximate probability that his previous hand was played with deck $2A$?

5. Mary loves gambling. She starts out with \$200. She can bet either \$100 or \$200 (assuming she has sufficient funds) and wins with probability p. If she wins, she receives back double what she bet, and if she loses, she receives back nothing.

   (a) Assume that Mary stops when she runs out of money or has reached \$400 or more, whichever comes first. What is her optimal betting strategy? Here, "optimal" means the strategy that gives her the greatest probability of reaching \$400, and "strategy" means a rule saying how much she should bet when she has \$100, \$200, and \$300 (the amount she bets need not be the same in these three cases).

   (b) What is the expected number of transitions until she either runs out of money or reaches \$400 for $p = 0.75$ under the optimal strategy?