Problem Set 7: Solutions

1. (a) The event of the ith success occuring before the jth failure is equivalent to the ith success occurring within the first $(i+j-1)$ trials (since the ith success must occur no later than the trial right before the jth failure). This is equivalent to event that i or more successes occur in the first $(i+j-1)$ trials (where we can have, at most, $(i+j-1)$ successes). Let S_i be the time of the ith success, F_j be the time of the jth failure, and N_k be the number of successes in the first k trials (so N_k is a binomial random variable over k trials). So we have:

$$\mathbf{P}(S_i < F_j) = \mathbf{P}(N_{i+j-1} \geq i) = \sum_{k=i}^{i+j-1} \binom{i+j-1}{k} p^k (1-p)^{i+j-1-k}$$

(b) Let K be the number of successes which occur before the jth failure, and L be the number of trials to get to the jth failure. L is simply a jth order Pascal, with probability of $1-p$ (since we are now interested in the failures, not the successes.) Plugging into the formula for jth order Pascal random variable,

$$\mathbf{E}[L] = \frac{j}{1-p}, \sigma_K^2 = \frac{p}{(1-p)^2} j$$

Since $K = L - j$,

$$\mathbf{E}[K] = \frac{p}{1-p} j, \sigma_K^2 = \frac{p}{(1-p)^2} j$$

(c) This expression is the same as saying we need at least 42 trials to get the 17th success. Therefore, it can be rephrased as having a maximum of 16 successes in the first 41 trials. Hence $b = 41$, $a = 16$.

2. (a) Since the result of each quiz is independent of others, the probability that Iwana fails exactly two of the next six quizzes is obtained from a binomial distribution.

$$\mathbf{P}(\text{2 failures out of 6 quizzes}) = \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 = \frac{5 \cdot 3^5}{4^6}.$$

(b) Note that since the result of each quiz is a bernoulli random variable, the number of quizzes until the first failure is a geometric random variable. Now consider a random variable X indicating the number of quizzes until the third failure. Because of the memoryless property of geometric random variables, X is a sum of three geometric random variables. Hence X is a Pascal random variable with the following PMF.

$$p_X(x) = p_{L_3}(x) = \binom{x-1}{3-1} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{x-3}$$

The expected value of X is $\mathbf{E}[X] = \frac{3}{1/4} = 12$. Therefore the expected number of quizzes that Iwana will pass before she fails three times is

$$\mathbf{E}[X] - 3 = 9$$

(c) Let A be the event that her second and third failures happen on the 8th and the 9th quizzes, respectively. This event requires that she fail exactly one quiz out of first seven quizzes. Remembering that each quiz is independent of others, we have

$$\begin{aligned}
\mathbf{P}(A) &= \mathbf{P}(1 \text{ failure out of 7 quizzes})\mathbf{P}(\text{failures on both 8th and 9th quizzes}) \\
&= \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{7\cdot 3^6}{4^9}
\end{aligned}$$

(d) Let Q be the event corresponding to the union of events R and S where:

$$\begin{aligned}
&\text{event } R \text{ is} \qquad (F(SF)^kF) \text{ for } k = 0,1,2,\ldots \\
&\text{event } S \text{ is} \qquad ((SF)^kF) \text{ for } k = 1,2,3,\ldots
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(Q) &= \mathbf{P}(R) + \mathbf{P}(S) \\
&= \frac{1}{16}\sum_{k=0}^{\infty}\left(\frac{3}{16}\right)^k + \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{3}{16}\right)^k \\
&= \left(\frac{1}{16}+\frac{3}{64}\right)\sum_{k=0}^{\infty}\left(\frac{3}{16}\right)^k \\
&= \frac{7}{64}\cdot\frac{1}{1-3/16} = \frac{7\times 16}{64\times 13} = \frac{7}{52}
\end{aligned}$$

3. (a) K has a Poisson distribution with average arrival time $\mu = \lambda_c T$

$$p_K(k) = \frac{(\lambda_c T)^k e^{-\lambda_c T}}{k!}, \quad k = 0,1,2,\ldots; T \geq 0.$$

(b) i. $\mathbf{P}(\text{conscious response}) = \left(\frac{\lambda_c}{\lambda_c+\lambda_s}\right)$.

ii. $\mathbf{P}(\text{conscious correct response}) = \mathbf{P}(\text{conscious resp})\ \mathbf{P}(\text{correct resp}|\text{ conscious resp}) = \left(\frac{\lambda_c}{\lambda_c+\lambda_s}p_c\right)$.

(c) Since the conscious and subconscious responses are generated independently,

$$\begin{aligned}
&\mathbf{P}(r \text{ conscious responses and } s \text{ subconscious responses in interval } T) \\
&= \mathbf{P}(r \text{ conscious responses in } T)\mathbf{P}(s \text{ unconscious responses in } T) \\
&= \frac{(\lambda_c T)^r e^{-\lambda_c T}}{r!}\cdot\frac{(\lambda_s T)^s e^{-\lambda_s T}}{s!}
\end{aligned}$$

(d) i. $\mathbf{E}(\text{correct answers}) = N\cdot\mathbf{E}(\text{correct on each question}) = N[\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s]$.

ii. We can view this as a binomial experiment where p, the probability of a success, is simply

$$p = \frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s$$

Therefore,

$$p_L(l) = \binom{N}{l}\left[\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s\right]^l \left[1-\left(\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s\right)\right]^{N-l},$$

where $l = 0, 1, \ldots, N$.

(e) i. $\mathbf{E}(\text{correct answers}) = (\lambda_c p_c + \lambda_s p_s)T$.

ii. Denoting $\lambda = \lambda_c p_c + \lambda_s p_s$ as the average rate of correct responses,

$$p_L(l) = \frac{(\lambda T)^l e^{-\lambda T}}{l!}, \quad l = 0, 1, \ldots$$

4. We view the random variables T_1 and T_2 as interarrival times in two independent Poisson processes both with rate λ S as the interarrival time in a third Poisson process (independent from the first two) with rate μ. We are interested in the expected value of the time Z until either the first process has had two arrivals or the second process has had an arrival.

   Given that the first arrival was from the second process, the expected wait time for that arrival would be $\frac{1}{\mu+\lambda}$. The probability of an arrival from the second process is $\frac{\mu}{\mu+\lambda}$. Given that the first arrival time was from the first process, the expected wait time would be that for first arrival, $\frac{1}{\mu+\lambda}$, plus the expected wait time for another arrival from the merged process. Similarily, the probability of an arrival from the first process is $\frac{\lambda}{\mu+\lambda}$. Thus,

$$\begin{aligned}\mathbf{E}[Z] = \mathbf{P}(\text{Arrival from second process})\mathbf{E}[\text{wait time}|\text{Arrival from second process}] +\\ \mathbf{P}(\text{Arrival from first process})\mathbf{E}[\text{wait time}|\text{Arrival from first process}]\\ = \frac{\mu}{\mu+\lambda}\cdot\frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\cdot\left(\frac{1}{\mu+\lambda}+\frac{1}{\mu+\lambda}\right).\end{aligned}$$

   After some simplifications, we see that

$$\mathbf{E}[Z] = \frac{1}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}\cdot\frac{1}{\mu+\lambda}$$

5. For simplicity, introduce the notation $N_i = N(G_i)$ for $i = 1, ..., n$ and $N_G = N(G)$. Then

$$\begin{aligned}\mathbf{P}(N_1 = k_1, ..., N_n = k_n | N_G = k) &= \frac{\mathbf{P}(N_1 = k_1, ..., N_n = k_n, N_G = k)}{\mathbf{P}(N_G = k)}\\ &= \frac{\mathbf{P}(N_1 = k_1)\cdots\mathbf{P}(N_n = k_n)}{\mathbf{P}(N_G = k)}\\ &= \frac{\left(\frac{(c_1\lambda)^{k_1}e^{-c_1\lambda}}{k_1!}\right)\cdots\left(\frac{(c_n\lambda)^{k_n}e^{-c_n\lambda}}{k_n!}\right)}{\left(\frac{(c\lambda)^k e^{-c\lambda}}{k!}\right)}\\ &= \frac{k!}{k_1!\ldots k_n!}\left(\frac{c_1}{c}\right)^{k_1}\cdots\left(\frac{c_n}{c}\right)^{k_n}\\ &= \binom{k}{k_1 \;\cdots\; k_n}\left(\frac{c_1}{c}\right)^{k_1}\cdots\left(\frac{c_n}{c}\right)^{k_n}\end{aligned}$$

The result can be interpreted as a *multinomial distribution*. Imagine we throw an n-sided die k times, where Side i comes up with probability $p_i = c_i/c$. The probability that side i comes up k_i times is given by the expression above. Now relating it back to the Poisson process that we have, each side corresponds to an interval that we sample, and the probability that we sample it depends directly on its relative length. This is consistent with the intuition that, given a number of Poisson arrivals in a specified interval, the arrivals are uniformly distributed.

G1†. (a) The event $\{M = m \cap N = n\}$ occurs when $\{N = n\}$ and $\{M - N = m - n\}$. That is, from $(0, t]$ there have to be n arrivals, and after t but prior to $t + s$ there have to be $m - n$ arrivals. Since the increment $(0, t]$ is disjoint from the increment $(t, t + s]$, the number of arrivals in each are independent and have a poisson distribution with rate λ. Symbolically,

$$\begin{aligned} p_{N,M}(n, m) &= p_N(n) p_{M|N}(m|n) \\ &= \left[\frac{(\lambda t)^n \exp^{-\lambda t}}{n!}\right] \left[\frac{(\lambda s)^{m-n} \exp^{-\lambda s}}{(m-n)!}\right] \end{aligned}$$

(b) A similar principle is helpful here as well. We can rewrite $E[NM]$ as

$$\begin{aligned} E[NM] &= E[N(M - N) + N^2] \\ &= E[N]E[M - N] + E[N^2] \\ &= (\lambda t)(\lambda s) + \left[var(N) + E[N]^2\right] \\ &= (\lambda t)(\lambda s) + \lambda t + (\lambda t)^2 \end{aligned}$$

where the second equality is obtained via the independent increment property of the poisson process.

†Required for 6.431; optional for 6.041