Recitation 22 Solutions: November 29, 2011

1. (a) Let X_i be a random variable indicating the quality of the ith bulb ("1" for good bulbs, "0" for bad ones). X_i's are independent Bernoulli random variables. Let Z_n be

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

$$\mathbf{E}[Z_n] = p \qquad \text{var}(Z_n) = \frac{n\text{var}(X_i)}{n^2} = \frac{\sigma^2}{n},$$

where σ^2 is the variance of X_i.
Applying Chebyshev's inequality yields,

$$\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

As $n \to \infty$, $\frac{\sigma^2}{n\epsilon^2} \to 0$ and $\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \to 0$.
Hence, Z_n converges to p in probability.

(b) By Chebychev's inequality,

$$\mathbf{P}\left(|Z_{50} - p| \geq 0.1\right) \leq \frac{\sigma^2}{50(0.1)^2},$$

Since X_i is a Bernoulli random variable, its variance σ^2 is $p(1-p)$, which is less than or equal to $\frac{1}{4}$. Thus,

$$\mathbf{P}\left(|Z_{50} - p| \geq 0.1\right) \leq \frac{1/4}{50(0.1)^2} = 0.5.$$

This means

$$\mathbf{P}\left(|Z_{50} - p| < 0.1\right) \geq 0.5.$$

(c) By Chebychev's inequality,

$$\mathbf{P}\left(|Z_n - p| \geq 0.1\right) \leq \frac{\sigma^2}{n\epsilon^2} \leq \frac{1/4}{n(0.1^2)}$$

To guarantee a probability 0.95 of falling in the desired range,

$$\frac{1/4}{n(0.1)^2} < 0.05,$$

which yields $n \geq 500$. Note that $n \geq 500$ guarantees the accuracy specification even for the highest variance, namely 1/4. For smaller variances, we need smaller values of n to guarantee the desired accuracy. For example, if $\sigma^2 = 1/16$, $n \geq 125$ would suffice.

2. (a) $\mathbf{E}[X_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + 1 \cdot \frac{1}{n} = \frac{1}{n}$
$\text{var}(X_n) = \left(0 - \frac{1}{n}\right)^2 \cdot \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 \cdot \left(\frac{1}{n}\right) = \frac{n-1}{n^2}$
$\mathbf{E}[Y_n] = 0 \cdot \left(1 - \frac{1}{n}\right) + n \cdot \frac{1}{n} = 1$
$\text{var}(Y_n) = (0-1)^2 \cdot \left(1 - \frac{1}{n}\right) + (n-1)^2 \cdot \left(\frac{1}{n}\right) = n - 1$

(b) Using Chebyshev's inequality, we have
$\lim_{n\to\infty} \mathbf{P}(|X_n - \frac{1}{n}| \geq \epsilon) \leq \lim_{n\to\infty} \frac{n-1}{n^2\epsilon^2} = 0$
Moreover, $\lim_{n\to\infty} \frac{1}{n} = 0$.
It follows that X_n converges to 0 in probability. For Y_n, Chebyshev suggests that,

$$\lim_{n\to\infty} \mathbf{P}(|Y_n - 1| \geq \epsilon) \leq \lim_{n\to\infty} \frac{n-1}{\epsilon^2} = \infty,$$

Thus, we cannot conclude anything about the convergence of Y_n through Chebychev's inequality.

(c) For every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}(|Y_n| \geq \epsilon) \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

Thus, Y_n converges to zero in probability.

(d) The statement is false. A counter example is Y_n. It converges in probability to 0 yet its expected value is 1 for all n.

(e) Using the Markov bound, we have

$$\mathbf{P}\left(|X_n - c| \geq \epsilon\right) = P\left(|X_n - c|^2 \geq \epsilon^2\right) \leq \frac{\mathbf{E}\left[(X_n - c)^2\right]}{\epsilon^2}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - c| \geq \epsilon\right) = 0,$$

which establishes convergence in probability.

(f) A counter example is Y_n. Y_n converges to 0 in probability, but

$$\mathbf{E}\left[(Y_n - 0)^2\right] = 0 \cdot \left(1 - \frac{1}{n}\right) + (n^2).\frac{1}{n} = n$$

Thus,

$$\lim_{n\to\infty} \mathbf{E}\left[(Y_n - 0)^2\right] = \infty,$$

and Y_n does not converge to 0 in the mean square.

3. (a) To use the Markov inequality, let $X = \sum_{i=1}^{10} X_i$. Then,

$$\mathbf{E}[X] = 10\mathbf{E}[X_i] = 5,$$

and the Markov inequality yields

$$\mathbf{P}(X \geq 7) \leq \frac{5}{7} = 0.7142.$$

(b) Using the Chebyshev inequality, we find that

$$\begin{aligned} 2\mathbf{P}(X - 5 \geq 2) &= \mathbf{P}(|X - 5| \geq 2) \\ &\leq \frac{\text{var}(X)}{4} = \frac{10/12}{4} \\ \mathbf{P}(X - 5 \geq 2) &\leq \frac{5}{48} = 0.1042. \end{aligned}$$

(c) Finally, using the Central Limit Theorem, we find that

$$\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{10} X_i \geq 7\right) &= 1 - \mathbf{P}\left(\sum_{i=1}^{10} X_i \leq 7\right) \\
&= 1 - \mathbf{P}\left(\frac{\sum_{i=1}^{10} X_i - 5}{\sqrt{10/12}} \leq \frac{7-5}{\sqrt{10/12}}\right) \\
&\approx 1 - \Phi(2.19) \\
&\approx 0.0143.
\end{aligned}$$

4. (a) Let $S_n = X_1 + \cdots + X_n$ be the total number of gadgets produced in n days. Note that the mean, variance, and standard deviation of S_n is $5n$, $9n$, and $3\sqrt{n}$, respectively. Thus,

$$\begin{aligned}
\mathbf{P}(S_{100} < 440) &= \mathbf{P}(S_{100} \leq 439.5) \\
&= \mathbf{P}\left(\frac{S_{100} - 500}{30} < \frac{439.5 - 500}{30}\right) \\
&\approx \Phi\left(\frac{439.5 - 500}{30}\right) \\
&= \Phi(-2.02) \\
&= 1 - \Phi(2.02) \\
&= 1 - 0.9783 \\
&= 0.0217.
\end{aligned}$$

(b) The requirement $\mathbf{P}(S_n \geq 200 + 5n) \leq 0.05$ translates to

$$\mathbf{P}\left(\frac{S_n - 5n}{3\sqrt{n}} \geq \frac{200}{3\sqrt{n}}\right) \leq 0.05,$$

or, using a normal approximation,

$$1 - \Phi\left(\frac{200}{3\sqrt{n}}\right) \leq 0.05,$$

and

$$\Phi\left(\frac{200}{3\sqrt{n}}\right) \geq 0.95.$$

From the normal tables, we obtain $\Phi(1.65)0.95$, and therefore,

$$\frac{200}{3\sqrt{n}} \geq 1.65,$$

which nally yields $n \leq 1632$.

(c) The event $N \geq 220$ (it takes at least 220 days to exceed 1000 gadgets) is the same as the event $S_{219} \leq 1000$ (no more than 1000 gadgets produced in the rst 219 days). Thus,

$$\begin{aligned}
\mathbf{P}(N \geq 220) &= \mathbf{P}(S_{219} \leq 1000) \\
&= \mathbf{P}\left(\frac{S_{219} - 5 \cdot 219}{3\sqrt{219}} \leq \frac{1000 - 5 \cdot 219}{3\sqrt{219}}\right) \\
&= 1 - \Phi(2.14) \\
&= 1 - 0.9838 \\
&= 0.0162.
\end{aligned}$$