**Problem Set 2: Solutions**
**Due: February 17, 2010**

1. (a) Reusing "$A$" and "$B$" in a natural way, let $A$ be the event that we get heads and $B$ be the event that we get tails. Also, let $O_n$ be the event that the $n$th roll is olive. Using total probability:

$$\mathbf{P}(O_n) = \mathbf{P}(O_n \mid A)\mathbf{P}(A) + \mathbf{P}(O_n \mid B)\mathbf{P}(B) = (5/6)(1/2) + (1/2)(1/2) = 2/3$$

(b) Again, we can use total probability:

$$\begin{aligned}\mathbf{P}(O_n \cap O_{n+1}) &= \frac{1}{2}\mathbf{P}((O_n \cap O_{n+1}) \mid A) + \frac{1}{2}\mathbf{P}((O_n \cap O_{n+1}) \mid B) \\ &= \frac{1}{2}\left(\left(\frac{5}{6}\right)^2 + \left(\frac{1}{2}\right)^2\right) \\ &= 17/36\end{aligned}$$

(c) Let $F_n$ be the event that all the first $n$ rolls are olive.

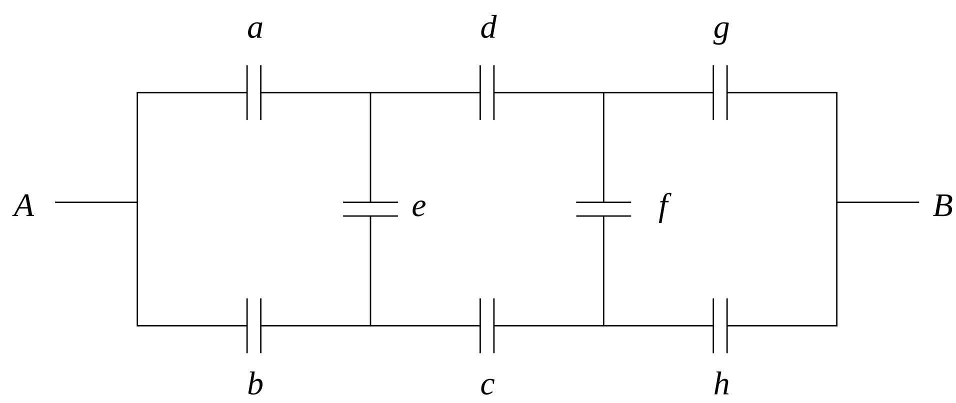
$$\begin{aligned}\mathbf{P}(O_{n+1} \mid F_n) &= \frac{\mathbf{P}(F_{n+1})}{\mathbf{P}(F_n)} \\ &= \frac{(1/2)\mathbf{P}(F_{n+1} \mid A) + (1/2)\mathbf{P}(F_{n+1} \mid B)}{(1/2)\mathbf{P}(F_n \mid A) + (1/2)\mathbf{P}(F_n \mid B)} \\ &= \frac{(5/6)^{n+1} + (1/2)^{n+1}}{(5/6)^n + (1/2)^n}\end{aligned}$$

When $n$ is very large $(5/6)^n$ is much larger than $(1/2)^n$, so $\mathbf{P}(O_{n+1} \mid F_n)$ approaches $\frac{(5/6)^{n+1}}{(5/6)^n} = 5/6$. This makes sense: the more times we see olive come up, the more likely it is that we chose die $A$ (which has probability 5/6 of coming up olive).
Notice the behavior of $\mathbf{P}(A \mid F_n)$ as $n \to \infty$: using Bayes' rule and total probability give us:

$$\begin{aligned}\mathbf{P}(A \mid F_n) = \frac{\mathbf{P}(F_n \mid A)\mathbf{P}(A)}{\mathbf{P}(F_n)} &= \frac{\mathbf{P}(F_n \mid A)\mathbf{P}(A)}{\mathbf{P}(F_n \mid A)\mathbf{P}(A) + \mathbf{P}(F_n \mid B)\mathbf{P}(B)} \\ &= \frac{(5/6)^n(1/2)}{(5/6)^n(1/2) + (1/2)^n(1/2)}\end{aligned}$$

As $n$ becomes large, the $(1/2)^n$ term in the denominator becomes small, and this probability goes to 1: we are more likely to have chosen die $A$.

2. Please refer to this drawing in the solutions below.




(a) Define $M$ as the event that exactly five links have failed. Define $C$ as the event that A communicates with B.

Now, given that exactly five links have failed, exactly three still work. For $A$ to still communicate with $B$ given that there are five failures, the links ($a$, $d$, and $g$) or ($b$, $c$, and $h$) must operate. Remembering that each link fails independently of the other links, we have:

$$\begin{aligned}\mathbf{P}(C \mid M) &= \mathbf{P}(\{(a,\ d,\ g) \text{ works}\} \cup \{(b,\ c,\ h) \text{ works}\} \mid M) \\ &= \frac{2(1-p)^3p^5}{56(1-p)^3p^5} = \frac{1}{28}\end{aligned}$$

(b) Again let $M$ be the event that exactly five links have failed. The probabilities that $g$ works given $M$, that $h$ works given $M$, and that both work given $M$ are computed as follows:

$$\begin{aligned}\mathbf{P}(\{g \text{ works}\} \mid M) &= \frac{\mathbf{P}(\{g \text{ works}\} \cap M)}{\mathbf{P}(M)} = \frac{21}{56} \\ \mathbf{P}(\{h \text{ works}\} \mid M) &= \frac{\mathbf{P}(\{h \text{ works}\} \cap M)}{\mathbf{P}(M)} = \frac{21}{56} \\ \mathbf{P}(\{\text{both } g \text{ and } h \text{ work}\} \mid M) &= \frac{\mathbf{P}(\{\text{both } g \text{ and } h \text{ work}\} \cap M)}{\mathbf{P}(M)} = \frac{6}{56}\end{aligned}$$

These can now be combined to obtained the desired result:

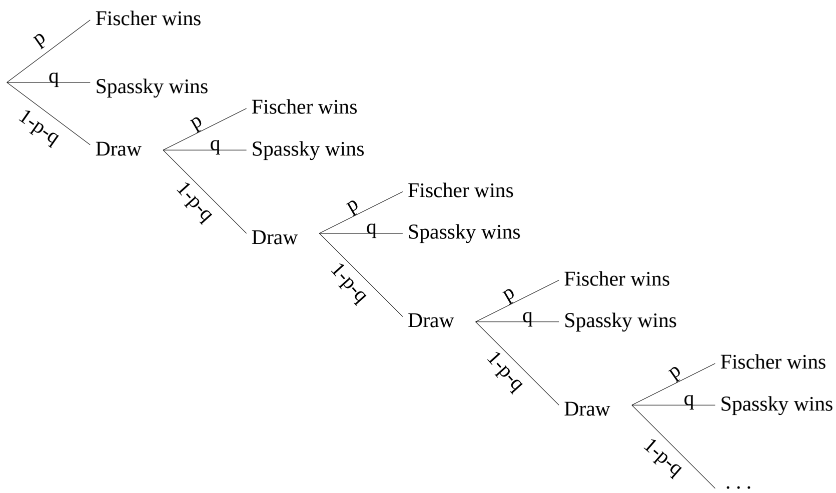
$$\begin{aligned}&\mathbf{P}(\{g \text{ or } h \text{ works but not both}\} \mid M) \\ &= \mathbf{P}(\{g \text{ works} \mid M) + \mathbf{P}(\{h \text{ works} \mid M) - 2\mathbf{P}(\{g \text{ and } h \text{ both work}\} \mid M) \\ &= \frac{21}{56} + \frac{21}{56} - 2 \cdot \frac{6}{56} = \frac{15}{28}\end{aligned}$$

(c) Define $Q$ as the event that $a$, $d$, and $h$ failed. Now given this event occurred, the only way for $A$ to communicate with $B$ is if $b$, $c$, $f$, and $g$ work. Again, since each link fails independently of the others, we have:

$$\begin{aligned}\mathbf{P}(C \mid Q) &= \mathbf{P}(b \text{ works}) \cdot \mathbf{P}(c \text{ works}) \cdot \mathbf{P}(f \text{ works}) \cdot \mathbf{P}(g \text{ works}) \\ &= (1-p)^4\end{aligned}$$

3. (a)

$$\begin{aligned}\mathbf{P}(\text{Fischer wins}) &= p + p(1-p-q) + p(1-p-q)^2 + \cdots \\ &= \frac{p}{1-(1-p-q)} = \boxed{\tfrac{p}{p+q}}\end{aligned}$$




We may also find the solution through a simpler method:

$$\mathbf{P}(\text{Fischer wins} \mid \text{someone wins}) = \frac{\mathbf{P}(\text{Fischer wins})}{\mathbf{P}(\text{someone wins})} = \boxed{\tfrac{p}{p+q}}$$

(b)

$$\begin{aligned}&\mathbf{P}(\text{the match lasted no more than 5 games}) \\ &\quad= (p+q) + (p+q)(1-p-q) + (p+q)(1-p-q)^2 + (p+q)(1-p-q)^3 + (p+q)(1-p-q)^4 \\ &\quad= \frac{(p+q)[1-(1-p-q)^5]}{1-(1-p-q)} \\ &\quad= 1-(1-p-q)^5.\end{aligned}$$

Also,

$$\mathbf{P}(\text{Fischer wins in the first game} \cap \text{ the match lasted no more than 5 games}) = p.$$

Therefore,

$$\begin{aligned}&\mathbf{P}(\text{Fischer wins} \mid \text{the match lasted no more than 5 games}) \\ &\quad= \frac{\mathbf{P}(\text{Fischer wins} \cap \text{ the match lasted no more than 5 games})}{\mathbf{P}(\text{the match lasted no more than 5 games})} = \boxed{\tfrac{p}{1-(1-p-q)^5}}\end{aligned}$$

(c) $\mathbf{P}(\text{the match lasted no more than 5 games}) = 1-(1-p-q)^5$. Also,

$$\begin{aligned}&\mathbf{P}(\text{Fischer wins} \cap \text{ the match lasted no more than 5 games}) \\ &\quad= p + p(1-p-q) + p(1-p-q)^2 + p(1-p-q)^3 + p(1-p-q)^4 \\ &\quad= \frac{p[1-(1-p-q)^5]}{1-(1-p-q)} = \frac{p[1-(1-p-q)^5]}{p+q}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{P}(\text{Fischer wins} \mid \text{the match lasted no more than 5 games})\\ &= \frac{\mathbf{P}(\text{Fischer wins} \cap \text{ the match lasted no more than 5 games})}{\mathbf{P}(\text{the match lasted no more than 5 games})} = \boxed{\frac{p}{p+q}}\end{aligned}$$

(d)

$$\begin{aligned}&\mathbf{P}(\text{Fischer wins at or before the 5th game} \mid \text{Fischer wins})\\ &= \frac{\mathbf{P}(\text{Fischer wins at or before the 5th game} \cap \text{ Fischer wins})}{\mathbf{P}(\text{Fischer wins})}\\ &= \left(\frac{p[1-(1-p-q)^5]}{p+q}\right) / \left(\frac{p}{p+q}\right)\\ &= \boxed{1-(1-p-q)^5}\end{aligned}$$

This part may be solved by observing that the events {Fischer wins} and {the match lasted no more than 5 games} are independent (we know this from parts (a) and (c)):

$$\begin{aligned}&\mathbf{P}(\text{the match lasted no more than 5 games} \mid \text{ Fischer wins})\\ &= \mathbf{P}(\text{the match lasted no more than 5 games})\\ &= \boxed{1-(1-p-q)^5}\end{aligned}$$

4. The key to this problem is to recall the identity

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(A \cap E) + \mathbf{P}(A \cap E^c)\\ &= \mathbf{P}(E) \cdot \mathbf{P}(A \mid E) + \mathbf{P}(E^c) \cdot \mathbf{P}(A \mid E^c),\end{aligned}$$

where the second equality following from a manipulation of the definition of conditional probability. Let $E$ be the event that the first race George bets on is won by the horse Heads, and thus $E^c$ is the event that Tails wins that race. Now, since the outcome of future races is given to be independent of the outcome of past races, given that Heads won the first race, we have no more information about how many more races Heads will win before Tails wins one, and hence we can imagine that George and Bob just started their betting game, with the difference that now George only needs to win $n-1$ more times before Bob wins $m$ in order to win the private bet. Thus, given that George wins the first race, the probability that he will win the private bet is

$$\mathbf{P}(\text{George wins} \mid \text{Heads wins the first race}) = P_{n-1,m}.$$

Now if George loses the first race, then the probability that he wins the private bet is

$$\mathbf{P}(\text{George wins} \mid \text{Heads loses the first race}) = 1-\mathbf{P}(\text{Bob wins } m \text{ games before George wins } n \text{ games}).$$

By symmetry, when it is Bob's turn to bet, the probability of winning $m$ races before George wins $n$ races will be $P_{m,n}$. Therefore,

$$\mathbf{P}(\text{George wins} \mid \text{Heads loses the first race}) = 1 - P_{m,n}.$$

Combining these two results we find

$$P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{m,n})$$

as required.

5. Many possible examples can be generated. We present one simple one. Let

$$\begin{aligned} \Omega &= \{1,2,3,4\}, \\ A &= \{1,2\}, \qquad \text{and} \\ B &= \{2,4\}, \qquad \text{so} \\ A \cap B &= \{2\}. \end{aligned}$$

Let $\mathbf{P}_1$ be the discrete uniform probability law on $\Omega$, i.e.

$$\mathbf{P}_1(\{1\}) = \mathbf{P}_1(\{2\}) = \mathbf{P}_1(\{3\}) = \mathbf{P}_1(\{4\}) = \frac{1}{4}.$$

Define another probability law $\mathbf{P}_2$ to be

$$\mathbf{P}_2(\{1\}) = \frac{1}{2},\ \mathbf{P}_2(\{2\}) = \frac{1}{4},\ \mathbf{P}_2(\{3\}) = \frac{1}{8},\ \mathbf{P}_2(\{4\}) = \frac{1}{8}.$$

From the two probability laws, we can see that

$$\mathbf{P}_1(A)\mathbf{P}_1(B) = (\tfrac{1}{4}+\tfrac{1}{4})(\tfrac{1}{4}+\tfrac{1}{4}) = \tfrac{1}{4} = \mathbf{P}_1(A\cap B)$$

and

$$\mathbf{P}_2(A)\mathbf{P}_2(B) = (\tfrac{1}{2}+\tfrac{1}{4})(\tfrac{1}{4}+\tfrac{1}{8}) = \tfrac{9}{32} \neq \tfrac{1}{4} = \mathbf{P}_2(A\cap B).$$

Thus, $A$ and $B$ are independent under $\mathbf{P}_1$ but not under $\mathbf{P}_2$.

G1$^\dagger$. Answer: $D = [0,2) = \{x \in \mathbb{R} :\ 0 \leq x < 2\}$.

Proof: Let us begin with two elementary but far reaching observations.

(a) *If* $\{\Delta_n\}_{n=1}^{\infty}$ *is a countably infinite sequence of mutually disjoint events* $\Delta_n \subset \Omega$ *then*

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbf{P}(\Delta_k) = 0.$$

To see this, note that by the countable additivity axiom of probability,

$$\sum_{n=1}^{\infty} \mathbf{P}(\Delta_n) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} \Delta_n\right) \leq 1 < \infty,$$

and

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} x_k = 0 \quad \text{whenever} \quad \sum_{n=1}^{\infty} |x_n| < \infty.$$

(b) *A sequence* $\{F_n\}_{n=1}^{\infty}$ *of events* $F_n \subset \Omega$ *which is* monotonically decreasing*, in the sense that* $F_n \subset F_m$ *for* $n > m$*, has probabilities converging to the probability of their intersection:*

$$\lim_{n\to\infty} \mathbf{P}(F_n) = \mathbf{P}(F_\infty), \quad \text{where} \quad F_\infty = \bigcap_{n=1}^{\infty} F_n.$$

For each $n$, the event

$$F_n = F_\infty \cup \Delta_n \cup \Delta_{n+1} \cup \Delta_{n+2} \cup \ldots$$

is a union of disjoint events, where $\{\Delta_k\}_{k=1}^{\infty}$ are the "increment sets"

$$\Delta_k = F_k \cap F_{k+1}^c = \{x \in F_k : \ x \notin F_{k+1}\}.$$

By the countable additivity axiom of probability,

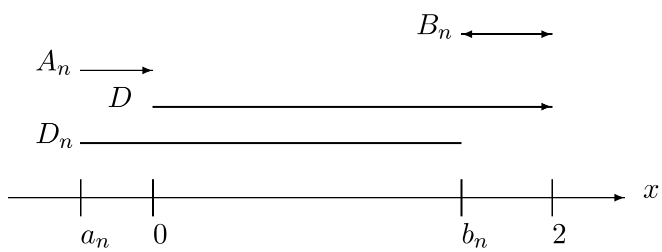
$$\mathbf{P}(F_n) = \mathbf{P}(F_\infty) + \sum_{k=n}^{\infty} \mathbf{P}(\Delta_n),$$

which, due to the previous observation (a), implies that $\mathbf{P}(F_n)$ converges to $\mathbf{P}(F_\infty)$ as $n \to \infty$.

To prove the answer, note first that the set $D = [0, 2)$ is an event (i.e., $\mathbf{P}(D)$ is defined) because

$$D = \bigcap_{n=1}^{\infty} C_n, \quad \text{where} \quad C_n = \bigcup_{k=n}^{\infty} D_n,$$

and, according to the axioms of probability, countable unions and intersections of events are events.




The next step is to define the "mismatch" events

$$A_n = D_n \cap D^c = [a_n, 0), \qquad B_n = D \cap D_n^c = (b_n, 2).$$

By construction,

$$A_n \cup D = [a_n, 2) = B_n \cup D_n, \qquad A_n \cap D = B_n \cap D_n = \emptyset,$$

which implies

$$\mathbf{P}(D) + \mathbf{P}(A_n) = \mathbf{P}(D \cup A_n) = \mathbf{P}(D_n \cup B_n) = \mathbf{P}(D_n) + \mathbf{P}(B_n).$$

On the other hand, $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are monotonic sequences of events, hence

$$\lim_{n\to\infty} \mathbf{P}(A_n) = \mathbf{P}(A_\infty) \qquad \text{for } A_\infty = \bigcap_{n=1}^{\infty} A_n$$

and

$$\lim_{n\to\infty} \mathbf{P}(B_n) = \mathbf{P}(B_\infty) \qquad \text{for } B_\infty = \bigcap_{n=1}^{\infty} B_n.$$

Together with the previous identity, these yield

$$\mathbf{P}(D) + \mathbf{P}(A_\infty) = \lim_{n\to\infty} \mathbf{P}(D_n) + \mathbf{P}(B_\infty).$$

Since both sets $A_\infty$ and $B_\infty$ are empty, $\mathbf{P}(A_\infty) = \mathbf{P}(B_\infty) = 0$ and the desired limit is proven.

$^\dagger$Optional (not graded for anyone).