Problem Set 1
Due: February 10, 2010

1. Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

   (a) at least one of the events A, B, C occurs;

   (b) at most one of the events A, B, C occurs;

   (c) none of the events A, B, C occurs;

   (d) all three events A, B, C occur;

   (e) exactly one of the events A, B, C occurs;

   (f) events A and B occur, but not C;

   (g) either event A occurs or, if not, then B also does not occur.

   In each case draw the corresponding Venn diagrams.

2. Twenty distinct cars park in a parking lot with 20 spaces every day. Ten are US-made cars and 10 are foreign-made cars. The drivers arrive in any order and thus the position any car might take on a certain day is random.

   (a) In how many different ways can the cars line up?

   (b) What is the probability that on a given day, the cars are parked to alternate in origin (e.g., US-made, foreign-made, US-made, foreign-made, etc.)?

3. Alice and Bob each choose at random a number in the interval $[0, 2]$. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

   A : The magnitude of the difference of the two numbers is greater than $1/3$.

   B : At least one of the numbers is greater than $1/3$.

   C : The two numbers are equal.

   D : Alice's number is greater than $1/3$.

   Find the probabilities $\mathbf{P}(B)$, $\mathbf{P}(C)$, and $\mathbf{P}(A \cap D)$.

4. Consider n people who are attending a party. We assume that each person has an equal probability of being born on each day of the year, independently of everyone else, and we ignore the additional complication presented by leap years (i.e., nobody is born on February 29). What is the probability that each person has a distinct birthday?

5. Consider the circle and equilateral triangle of Figure 1.7.

   (a) Consider a random point inside the circle, with all points being equally likely (i.e., the probability that the point lies inside a particular subset S of the circle is proportional to the area of S). Consider the chord that has this point as the midpoint. Show that the probability that the length of this chord is greater than the side of the triangle is $1/4$.

(b) Fix a point on the circumference of the circle, and choose another point on the circumference of the circle, with all points being equally likely (i.e., the probability that the point lies inside a particular subset S of the circumference is proportional to the length of S). Find the probability that the length of the chord is greater than the side of the triangle.

G1†. Let $A, B, C, A_1, \ldots, A_n$ be some events. Show the following identities. A mathematical derivation is required, but you can use Venn diagrams to guide your thinking.

(a) $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap B) - \mathbf{P}(B \cap C) - \mathbf{P}(A \cap C) + \mathbf{P}(A \cap B \cap C)$

(b) $\mathbf{P}(\cup_{k=1}^{n} A_k) = \mathbf{P}(A_1) + \mathbf{P}(A_1^c \cap A_2) + \mathbf{P}(A_1^c \cap A_2^c \cap A_3) + \cdots + \mathbf{P}(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n)$

†Optional (not graded for anyone); recommended for 6.431 students.