Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Recitation 8 Solutions
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1. The solution is on page 131 in the textbook.

2. (a)

$$\begin{aligned} p_X(1) &= \mathbf{P}(X=1,Y=1)+\mathbf{P}(X=1,Y=2)+\mathbf{P}(X=1,Y=3) \\ &= 1/12+2/12+1/12 = 1/3 \end{aligned}$$

(b) The solution is a sketch of the following conditional PMF:

$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & \text{if } y=1, \\ 1/2, & \text{if } y=2, \\ 1/4, & \text{if } y=3, \\ 0, & \text{otherwise.} \end{cases}$$

(c) $\mathbf{E}[Y \mid X=1] = \sum_{y=1}^{3} y\, p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

(d) Assume that X and Y are independent. Because $p_{X,Y}(3,1) = 0$ and $p_Y(1) = 1/4$, $p_X(3)$ must equal zero. This further implies $p_{X,Y}(3,2) = 0$ and $p_{X,Y}(3,3) = 0$. All the remaining probability mass must go to $(X,Y) = (2,2)$, making $p_{X,Y}(2,2) = 5/12$, $p_X(2) = 8/12$, and $p_Y(2) = 7/12$. However, $p_{X,Y}(2,2) \neq p_X(2) \cdot p_Y(2)$, contradicting the assumption; thus X and Y are not independent.

A simpler explanation uses only two X values and two Y values for which all four (X,Y) pairs have specified probabilities. Note that if X and Y are independent, then $p_{X,Y}(1,3)/p_{X,Y}(1,1)$ and $p_{X,Y}(2,3)/p_{X,Y}(2,1)$ must be equal because they must both equal $p_Y(3)/p_Y(1)$. This necessary equality does not hold, so X and Y are not independent.

(e) Knowing that X and Y are conditionally independent given B, we must have

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}.$$

(f) Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.

3. (a) We want to find the probability that $\{X_1, X_2, X_3\} = \{1,2,3\}$. The total number of ways this can happen is $3!(n-3)!$, and the total number of arrangments is $n!$, therefore we have:

$$\mathbf{P}(\{X_1, X_2, X_3\} = \{1,2,3\}) = \frac{3!(n-3)!}{n!}.$$

(b) These events are indeed independent.

(c) These events are not independent. Notice simply that given that $X_1 = i_1, X_2 = i_2$, then for any $i_3 \neq i_1, i_2$, we have $\mathbf{P}(X_4 = i_3 | X_1 = i_1, X_2 = i_2) = 1/(n-2)$, where as $\mathbf{P}(X_4 = i_3 | X_1 = i_1, X_2 = i_2, X_3 = i_3) = 0$.

(d) There are $\binom{10}{5}$ ways of choosing 5 people out of 10, and $\binom{n-10}{3}$ ways of picking the other 3 people who will sit in the first eight seats. Then, there are 8! ways to arrange these eight people, and $(n-8)!$ ways to arrange the remaining people. Meanwhile, the total number of ways to arrange n people is $n!$. Therefore, the probability of the given event is:

$$\frac{\binom{10}{5}\binom{n-10}{3}8!(n-8)!}{n!}.$$

(e) We will use the method of indicator functions. For all $i < j$, define variables E_{ij} to be equal to 1 if $X_i > X_j$, and zero otherwise. Then we have

$$N = \sum_{i<j} E_{ij}.$$

By the linearity of expectation, we have:

$$\begin{aligned}
\mathbf{E}[N] &= \mathbf{E}\left[\sum_{i<j} E_{ij}\right] \\
&= \sum_{i<j} \mathbf{E}[E_{ij}] \\
&= \sum_{i<j} \mathbf{P}(X_i > X_j) \\
&= \sum_{i<j} \frac{1}{2} \\
&= \frac{1}{2}\frac{n(n-1}{2} = \frac{1}{2}\binom{n}{2}.
\end{aligned}$$

Another way to see this is to observe that there are $\binom{n}{2}$ pairs total, and by symmetry, the expected number of pairs in reverse order will be half.