MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Spring 2009)

**Problem Set 8 Solutions**
**Due: April 22, 2009**

1. A successful call occurs with probability $p = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

   (a) Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply

   $$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (b) The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials and again our answer is

   $$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (c) We desire the probability that $L_2$, the second-order interarrival time is equal to five trials. We know that $p_{L_2}(l)$ is a Pascal PMF of order 2, and we have

   $$p_{L_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}$$

   (d) Here we require the conditional probability that the experimental value of $L_2$ is equal to 5, given that it is greater than 2.

   $$\begin{aligned} p_{L_2|L_2>2}(L_2 = 5 | L_2 > 2) &= \frac{p_{L_2}(5)}{P(L_2 > 2)} = \frac{p_{L_2}(5)}{1 - p_{L_2}(2)} \\ &= \frac{\binom{5-1}{2-1} p^2 (1-p)^{5-2}}{1 - \binom{2-1}{2-1} p^2 (1-p)^0} = \frac{4 \cdot \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{6} \end{aligned}$$

   (e) The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of $L_2$ is greater than or equal to 5.

   $$\begin{aligned} P(L_2 \geq 5) = 1 - P(L_2 \leq 4) = 1 - \sum_{l=2}^{4} \binom{l-1}{2-1} p^2 (1-p)^{l-2} \\ = 1 - (\frac{1}{2})^2 - \binom{2}{1}(\frac{1}{2})^3 - \binom{3}{1}(\frac{1}{2})^4 = \frac{5}{16} \end{aligned}$$

   (f) Let discrete random variable $F$ represent the number of failures before Fred runs out of samples on his $m$th successful call. Since $L_m$ is the number of trials up to and including the $m$th success, we have $F = L_m - m$. Given that Fred makes $L_m$ calls before he needs a new supply, we can regard each of the $F$ unsuccessful calls as trials in another Bernoulli process with parameter $r$, where $r$ is the probability of a success (a disappointed dog) obtained by

   $$\begin{aligned} r &= \text{P(dog lives there | Fred did not leave a sample)} \\ &= \frac{\text{P(dog lives there AND door not answered)}}{1 - \text{P(giving away a sample)}} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{1 - \frac{1}{2}} = \frac{1}{3} \end{aligned}$$

We define $X$ to be a Bernoulli random variable with parameter $r$. Then, the number of dogs passed up before Fred runs out, $D_m$, is equal to the sum of $F$ Bernoulli random variables each with parameter $r = \frac{1}{3}$, where $F$ is a random variable. In other words,

$$D_m \quad = \quad X_1 + X_2 + X_3 + \cdots + X_F.$$

Note that $D_m$ is a sum of a random number of independent random variables. Further, $F$ is independent of the $X_i$'s since the $X_i$'s are defined in the conditional universe where the door is not answered, in which case, whether there is a dog or not does not affect the probability of that trial being a failed trial or not. From our results in class, we can calculate its expectation and variance by

$$\begin{aligned} E[D_m] &= E[F]E[X] \\ \text{var}(D_m) &= E[F]\text{var}(X) + (E[X])^2\text{var}(F), \end{aligned}$$

where we make the following substitutions.

$$\begin{aligned} E[F] &= E[L_m - m] = \frac{m}{p} - m = m. \\ \text{var}(F) &= \text{var}(L_m - m) = \text{var}(L_m) = \frac{m(1-p)}{p^2} = 2m. \\ E[X] &= r = \frac{1}{3}. \\ \text{var}(X) &= r(1-r) = \frac{2}{9}. \end{aligned}$$

Finally, substituting these values, we have

$$\begin{aligned} E[D_m] &= m \cdot \frac{1}{3} = \frac{m}{3} \\ \text{var}(D_m) &= m \cdot \frac{2}{9} + \left(\frac{1}{3}\right)^2 \cdot 2m = \frac{4m}{9} \end{aligned}$$

2. (a) For each round, the probability that both Alice and Bob have a loss is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Let random variable $X$ represent the total number of rounds played until the first time where they both have a loss. Then $X$ is a geometric random variable with parameter $p = 1/9$ and has the following PMF.

$$p_X(x) = (1-p)^{x-1}p = \left(\frac{8}{9}\right)^{x-1}\left(\frac{1}{9}\right), \qquad x = 1, 2, \ldots$$

(b) Denote by $Y_i$ the gain from Alice's $i$th game. Let $M$ be a random variable representing the total gain of Alice up to the time of the first loss by Bob. Then we have

$$M = Y_1 + \cdots + Y_K$$

where random variable $K$ indicates the number of games Bob played up to and including his first loss (Alice will play exactly $K$ games because she plays before Bob in each round). The transform of $M$ is obtained by

$$M_M(s) = M_K(s)\Big|_{e^s \leftarrow M_Y(s)}$$

Note that $K$ is a geometric random variable with $p = \frac{1}{3}$. Therefore the transform of $K$ is $M_K(s) = \frac{pe^s}{1-(1-p)e^s} = \frac{\frac{1}{3}e^s}{1-\frac{2}{3}e^s}$. The transform of $Y$ is $M_Y(s) = M_G(s) = \frac{1}{3}e^{-2s} + \frac{1}{2}e^s + \frac{1}{6}e^{3s}$. Hence, $M_M(s)$ is

$$M_M(s) \;=\; \frac{\frac{1}{3}e^s}{1-\frac{2}{3}e^s}\Big|_{e^s \leftarrow \frac{1}{3}e^{-2s}+\frac{1}{2}e^s+\frac{1}{6}e^{3s}} = \frac{\frac{1}{3}(\frac{1}{3}e^{-2s}+\frac{1}{2}e^s+\frac{1}{6}e^{3s})}{1-\frac{2}{3}(\frac{1}{3}e^{-2s}+\frac{1}{2}e^s+\frac{1}{6}e^{3s})}$$

(c) First, consider the number of games, $K_3$ Bob played until his third loss. Random variable $K_3$ is a Pascal random variable and has the following PMF.

$$p_{K_3}(k) = \binom{k-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{k-3} \qquad k = 3, 4, 5, \cdots$$

In this question, we are interested in another random variable $Z$ defined as the time at which Bob has his third loss. Note that $Z = 2K_3$. By changing variables, we obtain

$$p_Z(z) = \binom{\frac{z}{2}-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{\frac{z}{2}-3} \qquad z = 6, 8, 10, \cdots$$

(d) Let A be the event that Alice wins, and Let B be the event that Bob wins. The event $A \cup B$ is then the event that either A wins or B wins or both A and B win, and the event $A \cap B$ is the event that both A and B win. Suppose we observe this gambling process, and let $U$ be a random variable indicating the number of rounds we see until at least one of them wins. Random variable $U$ is a geometric random variable with parameter $p = P(A \cup B) = 1 - \frac{1}{3} \cdot \frac{1}{3}$.

Consider another random variable $V$ representing the number of additional rounds we have to observe until the other wins. If both Alice and Bob win at the $U$th round, then $V = 0$. This occurs with probability $P(A \cap B | A \cup B) = \frac{\frac{2}{3}\frac{2}{3}}{\frac{8}{9}}$. If Alice wins the $U$th round, then the time $V$ until Bob wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(A|A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$. Likewise, if Bob wins the $U$th round, then the time $V$ until Alice wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(B|A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$.The number of rounds until each one of them has won at least once, $N$ is

$$N = U + V$$

The expectation of $N$ is then:

$$\begin{aligned}
E[N] &= E[U] + E[V] \\
&= \frac{1}{\frac{8}{9}} + 0 \cdot P(A \cap B | A \cup B) + \frac{1}{\frac{2}{3}} P(A|A \cup B) + \frac{1}{\frac{2}{3}} P(B|A \cup B) \\
&= 9/8 + \frac{3}{2}\frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} + \frac{3}{2}\frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} \\
&= 15/8
\end{aligned}$$

There is another approach to this problem. Consider the following partition.

$A_1$: both win first round
$A_2$:Alice wins first round
$A_3$: Bob wins first round
$A_4$: both lose first round

Event $A_1$ occurs with probability $\frac{2}{3} \cdot \frac{2}{3}$. Event $A_2$ occurs with probability $\frac{2}{3} \cdot \frac{1}{3}$. Event $A_3$ occurs with probability $\frac{1}{3} \cdot \frac{2}{3}$. Event $A_4$ occurs with probability $\frac{1}{3} \cdot \frac{1}{3}$. When event $A_2$ ($A_3$) occurs, the distribution on the time until Bob (Alice) wins is a geometric random variable with mean $\frac{1}{\frac{2}{3}}$. When event $A_4$ occurs, the additional time until Alice and Bob win is distributed identically to that at time 0 by the fresh-start property. By the total expectation theorem,

$$\begin{aligned}
E[N] &= E[N|A_1]P(A_1) + E[N|A_2]P(A_2) + E[N|A_3]P(A_3)E[N] + E[N|A_4]P(A_4) \\
&= 1 \cdot \left(\frac{2}{3} \cdot \frac{2}{3}\right) + \left(1 + \frac{1}{\frac{2}{3}}\right) \cdot + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + (1 + E[N]) \cdot \left(\frac{1}{3} \cdot \frac{1}{3}\right)
\end{aligned}$$

Solving for $E[N]$, we get $E[N] = \frac{15}{8}$.

3. (a) We may view the time until a particular player is injured as the time until the first arrival in a Poisson process of rate $\lambda$. Since each player is independent, and since we have 8 players, we have 8 independent Poisson processes of rate $\lambda$. Thus, we may view the time until any player is injured as the time until the first arrival in the merged Poisson process, which has rate $8\lambda$. The expected time till the first arrival is therefore

$$\frac{1}{8\lambda}$$

(b) If we assume that you can only be injured if you are playing, then the time till the first injury is exponential with rate $8\lambda$. The team will play until the time of the $(n-7)$th injury, which leaves only 7 players. The time until the 7th injury is an Erlang random variable of order $n-7$, rate $8\lambda$.

$$f_{T_{n-7}}(t) = \begin{cases} \frac{(8\lambda)^{n-7} t^{(n-7)-1} e^{-8\lambda t}}{((n-7)-1)!} & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

4. (a) Since the shuttles depart exactly every hour on the hour, the number of passengers that arrive in a one hour interval is the number of passengers on a shuttle. So, the arrivals are described by a Poisson process, and the expected number of arrivals (and therefore the expected number of passengers on a shuttle) is the mean of a Poisson random variable, or $\lambda$.

$$E[\text{number of passengers on a shuttle}] = \lambda$$

(b) Recall that in continuous time, each inter-arrival time in a Poisson process is described by the exponential distribution. Here, we consider the times in between shuttle arrivals with an exponential distribution, rate $\mu$ per hour. Then each shuttle arrival is a Poisson process. Let $A$ be the number of shuttles arriving in one hour with parameter $\mu$ and the following distribution,

$$p_A(a) = \begin{cases} \frac{e^{-\mu}\mu^a}{a!} & a = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) In the terminal, there is a Poisson process describing the arrival of passengers and another Poisson process describes the departures of shuttles. The "event" described includes either process (or both), so the event is a merged process (still Poisson). The two processes are independent from one another, so the merged process has an arrival rate of: $\lambda + \mu$.

$$E[\text{number of events per hour}] = \lambda + \mu$$

(d) The wait time until the next shuttle is the inter-arrival time of the shuttles, which is exponential, with parameter $\mu$. Recall that the exponential distribution is memoryless, so seeing $2\lambda$ people waiting around does not affect the expected wait time for a shuttle. So from the time the passenger arrives at the gate, the wait time is exponential with parameter $\mu$, with mean $1/\mu$.

$$E[\text{wait time} \mid 2\lambda \text{ people waiting}] = \frac{1}{\mu}$$

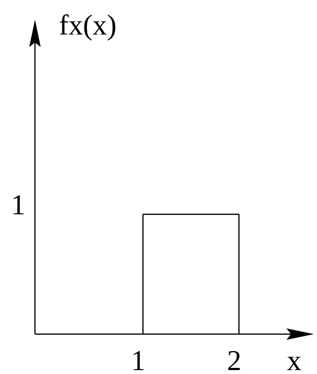
(e) To find the PMF for the number of passengers in a shuttle, we go back to part c, where we determined that the event of either a passenger arrival or shuttle departure is a merged Poisson process, with paramter $\lambda + \mu$. In the merged Poisson, the probability that the arrival was a passenger arrival is $\frac{\lambda}{\lambda+\mu}$, and the probability that the "arrival" was a shuttle departure is $\frac{\mu}{\lambda+\mu}$.
Let $N$ be the number of people on a shuttle. There must be $n$ successive passenger arrivals before a shuttle departure. Therefore, the PMF for $N$ is:

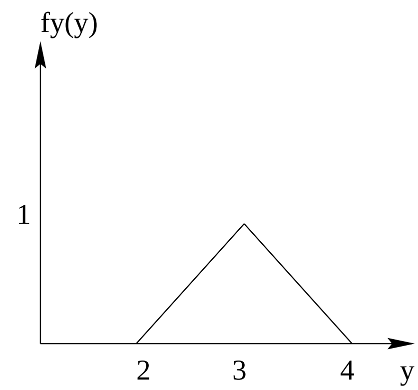
$$p_N(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

One can also think of the PMF of $N$ as number of "failures" (passenger arrivals) until the first "success" (shuttle departure), but shifted to start at 0 rather than 1 in a standard geometric distribution.

5. Let $X_i$ be the $i$th interarrival interval and let $Y_i$ be the arrival time of the $i$th bus.

   (a) Because the interarrival times $X_1$ and $X_2$ are independent, we can convolve to find the PDF of $Y_2 = X_1 + X_2$.




$$f_Y(y) = \int_{-\infty}^{\infty} f_X(t) f_X(y-t)dt = \begin{cases} y-2, & 2 \le y \le 3 \\ -y+4, & 3 < y \le 4 \\ 0, & o.w. \end{cases}$$

   (b) $Y_4 = X_1 + X_2 + X_3 + X_4$.
   Thus $\mathbf{E}[Y_4] = \mathbf{E}[X_1 + X_2 + X_3 + X_4] = 4\mathbf{E}[X] = 4(1.5) = 6$ hours.
   Therefore, the expected arrival time of the fourth bus is 6pm.

   (c) The time T (in hours) till the next Greyhound bus can be written as

$$T = X_1 + X_2 + ... + X_N$$

   where $X_i$ is the time between the $(i-1)$th bus from now and the ith bus, and $G$ is the number of buses till the next Greyhound bus. This is a sum of a random number of independent random variables: the $X_i$ are iid random variables, each uniform between 1 and 2; $N$ is a geometric random variable with parameter 3/4; and $N$ and the $X_i$ are all independent. Therefore we have

$$E[T] = E[N]E[X] = (4/3) * (3/2) = 2.$$

   Therefore, we expect the next Greyhound bus to arrive after 2 hours, i.e. at 2pm.

G1$^\dagger$. For simplicity, introduce the notation $N_i = N(G_i)$ for $i = 1, ..., n$ and $N_G = N(G)$. Then

$$\begin{aligned} & P(N_1 = k_1, ..., N_n = k_n | N_G = k) \\ = & \frac{P(N_1 = k_1, ..., N_n = k_n, N_G = k)}{P(N_G = k)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{P(N_1 = k_1)\cdots P(N_n = k_N)}{P(N_G = k)} \\
&= \frac{\frac{(c_1\lambda)^{k_1} e^{-c_1\lambda}}{k_1!}\cdots\frac{(c_n\lambda)^{k_n} e^{-c_n\lambda}}{k_n!}}{\frac{(c\lambda)^k e^{-c\lambda}}{k!}} \\
&= \begin{pmatrix} & k! & \\ k_1! & \cdots & k_n! \end{pmatrix} (\frac{c_1}{c})^{k_1}\cdots(\frac{c_n}{c})^{k_n}
\end{aligned}
$$

The result can be interpreted as a multinomial distribution. Imagine we throw an n-sided die $k$ times, where Side $i$ comes up with probability $p_i = \frac{c_i}{c}$. The probability that Side $i$ comes up $k_i$ times is given by the expression above. Now relating it back to the Poisson process that we have, each side corresponds to an interval that we sample, and the probability that we sample it depends directly on its relative length. This is consistent with the intuition that, given a number of Poisson arrivals in a specified interval, the arrivals are uniformly distributed.

† Required for 6.431; optional for 6.041