MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem Set 10

Due: Friday, December 9, 2011

1. Alice has two coins. The probability of heads for the first coin is 1/3; the probability of heads for the second coin is 2/3. Other than this difference in their bias, the coins are indistinguishable through any measurement known to man. Alice chooses one of the coins randomly and sends it to Bob. Let p be the probability that Alice chose the first coin. Bob tries to guess which of the two coins he received by flipping it 3 times in a row and observing the outcome. Assume that all coin flips are independent. Let Y be the number of heads Bob observed.

   (a) Given that Bob observed k heads, what is the probability that he received the first coin?

   (b) Find values of k for which the probability that Alice sent the first coin increases after Bob observes k heads out of 3 tosses. In other words, for what values of k is the probability that Alice sent the first coin given that Bob observed k heads greater than p? If we increase p, how does your answer change (goes up, goes down, or stays unchanged)?

   (c) Help Bob develop the rule for deciding which coin he received based on the number of heads k he observed in 3 tosses if his goal is to minimize the probability of error.

   (d) For this part, assume $p = 2/3$.

      i. Find the probability that Bob will guess the coin correctly using the rule above.

      ii. How does this compare to the probability of guessing correctly if Bob tried to guess which coin he received before flipping it?

   (e) If we increase p, how does that affect the decision rule?

   (f) Find the values of p for which Bob will never guess he received the first coin, regardless of the outcome of the tosses.

   (g) Find the values of p for which Bob will always guess he received the first coin, regardless of the outcome of the tosses.

2. The joint PDF of X and Y is defined as follows:

$$f_{X,Y}(x,y) = \begin{cases} cxy & \text{if } 0 < x \le 1,\ 0 < y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the normalization constant c.

   (b) Compute the conditional expectation estimator of X based on the observed value $Y = y$.

   (c) Is this estimate different from what you would have guessed before you saw the value $Y = y$? Explain.

   (d) Repeat (b) and (c) for the MAP estimator.

3. Suppose that the joint distribution of random variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 1/4 & \text{if } (x,y) \in A \\ 0 & \text{if } (x,y) \notin A \end{cases}$$

   where A is the shaded area in Figure 1.

(a) You have the choice of estimating Y based on observations of X or to estimate X based on observations of Y. Which option should you pick to minimize the conditional mean squared error of your estimation for the worst choice of your measured variable?

(b) Compute and plot the Bayesian Least Mean Square estimate of X based on observation of Y.

(c) Compute and plot the Bayesian Least Mean Square estimate of Y based on observation of X.

(d) Compute the conditional mean squared errors of your answers in parts (b) and (c). For what observation value(s) of Y does the LMS estimate of X have the worst error? Similarly, for what observation value(s) of X does the LMS estimate of Y have the worst error?

(e) Compute the mean squared errors of your answers in parts (b) and (c) and compare the results?

(f) What problems do you expect to encounter, if any, if you repeat parts (b) and (c) using the MAP rule for estimation instead of the LMS.

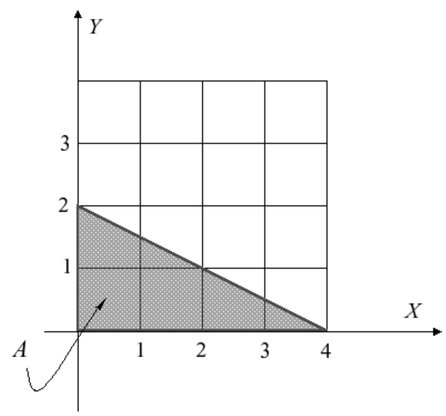

Figure 1: The joint distribution of random variables X and Y.

4. Consider a Bernoulli process X_1, X_2, X_3, ... with unknown probability of success q. As usual, define the kth inter-arrival time T_k as

$$T_1 = Y_1, \qquad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

where Y_k is the time of the kth success. This problem explores estimation of q from observed inter-arrival times $\{t_1, t_2, t_3, \ldots\}$.

You may find the following integral useful: For any non-negative integers k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!m!}{(k+m+1)!}$$

Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$.

(a) Compute the PMF of T_1, $p_{T_1}(t_1)$

(b) Compute the least squares estimate (LSE) of Q from the first recording $T_1 = t_1$.

(c) Compute the maximum a posteriori (MAP) estimate of Q given the k recordings, $T_1 = t_1, \ldots, T_k = t_k$.

For this part only assume q is sampled from the random variable Q which is now uniformly distributed over $[0.5, 1]$

(d) Find the linear least squares estimate (LLSE) of the second inter-arrival time (T_2), from the observed first arrival time ($T_1 = t_1$).

5. Let W_1, W_2, and W_3 be independent, continuous random variables each uniformly distributed over $[0, 1]$. Let $X = W_1 + W_2$ and $Y = X + W_3$.

(a) Find $\text{cov}(X, Y)$.

(b) Find the linear least mean squares (LLMS) estimator of X from Y. What choices of numbers a and b minimizes $\mathbf{E}[(X - (aY + b))^2]$?

(c) Find the maximum a posteriori probability (MAP) estimator of X from Y.

G1†. **Optimizing the Long-Term Outcome from Gambling or Investments**

This problem deals with the long-term outcome from multiplicative investment models. We take the **Double or Quarter game** as an example. Let X_k be the return on the kth trial, i.e., the ratio by which the amount the player bets is multiplied on the kth game. For example, in the Double-or-Quarter game;

$$\mathbf{P}(X_k = 2) = \mathbf{P}\left(X_k = \frac{1}{4}\right) = \frac{1}{2}$$

Assume the player initially bets \$1. If she reinvests all her holdings on each subsequent trial, her wealth after n trials is:

$$W_n = X_1 \cdot X_2 \cdots X_n$$

A commonly used notation in investment is the *effective return* $R_n \geq 0$, a random variable given by

$$(R_n)^n = W_n = X_1 \cdot X_2 \cdots X_n$$

i.e.,

$$(R_n) = (X_1 \cdot X_2 \cdots X_n)^{1/n}$$

The return R_n summarizes the outcome in the sense that her wealth after n trials would remain W_n if all individual returns had been exactly $X_k = R_n$ for each trial X_k, $k = 1, 2, \ldots, n$. For example, let each trial represent the 1 year return on an investment, and suppose an investment company claims that one of their mutual funds had an effective return over the past 10 years of 9%. In our notation, the company is claiming that R_{10}=1.09, and therefore for each dollar invested 10 years ago, a shareholder would now have $(1.09)^{10}$ = \$2.37. It turns out that the effective return is a much better indicator of the performance of such a scheme than the expected wealth after n trials, which is biased by the possibility of huge but improbable wins.

We would like to know the value to which R_n converges (in probability) for a given game and strategy. Since we have much more powerful tools for dealing with averages of independent random variables than we do for dealing with products, we use the logarithm to convert the product above into an average:

$$\log_b(R_n) = \frac{1}{n}\sum_{k=1}^{n}\log_b(X_k)$$

or, equivalently,

$$R_n = b^{\frac{1}{n}\sum_{k=1}^{n}\log_b(X_k)},$$

where b is the base for the logarithm, $b > 1$.

Assume for the remainder of this problem that $\log_b(X)$ has a finite mean and variance.

(a) Does $\log_b(R_n)$ converge to a specific value in probability as $n \to \infty$? If so, to what value does it converge?

For the next parts you will need the following Lemma, which you are encouraged to prove.

Lemma

Let Z_n, $n \geq 1$, be a sequence of random variables. Let c be any constant, and let f be any function that is continuous at c.

$$\text{If } Z_n \xrightarrow{prob.} \text{c as } n \to \infty, \text{(i.e., if } Z_n \text{ converges in probability to } c \text{ as } n \to \infty),$$

then

$$f(Z_n) \xrightarrow{prob.} f(c) \text{ as } n \to \infty, \text{(i.e., then } f(Z_n) \text{ converges in probability to } f(c) \text{ as } n \to \infty.)$$

(b) Show that as $n \to \infty, R_n \to r$ in probability as $n \to \infty$, for some constant r. Find a general expression for r and evaluate it numerically for the Double or Quarter game with $\mathbf{P}\{\text{heads}\}$=1/2. (The choice $b = 2$ for the base of the logarithm will make this easier.) Also express your answer as a certain long-term percentage loss or gain per toss.

(c) Find the asymptotic value of her wealth W_n in the double or Quarter Game as n becomes large, i.e., the value to which W_n converges in probability. Explain your methods and answer.

A General Problem for Gamblers

In part (b) above, we found that $R_n \to r$ in probability as $n \to \infty$, where, unfortunately for the gambler, $r \leq \mathbf{E}[X]$.

(d) Show that, sadly, this is always the case. In what special case(s) (if any), is it true that $r = \mathbf{E}[X]$?

(Hint: It is easy to show that $r \leq \mathbf{E}[X]$ once you have shown that $\mathbf{E}[\log(X)] \leq \log(\mathbf{E}[X])$. To show that this latter inequality is true, compare the curve $\log(x)$ to its tangent at $x = \mathbf{E}[X]$, (e.g., if we use e as the base, the tangent to $\ln(x)$ at $x = \mathbf{E}[X]$ is the linear function $f_L(x) = \log(E[X]) + (x - \mathbf{E}[X])/\mathbf{E}[X]$. Then take expectations and compare $\mathbf{E}[\log(X)]$ to $\mathbf{E}[f_L(X)]$. Pictures are very helpful here.)
Congratulations! You have just derived an important instance of the Jensen Inequality (pg. 287 in the text).

The Good News
The **Kelly strategy for gambling** tells you the optimal method for spreading risk (or, in investment language, for diversifying your investments). In the next part of this problem you will derive a version of the Kelly strategy for the Double or Quarter game.

Specifically, suppose you bet a fixed fraction, f, of your wealth on each toss and put the remaining fraction $(1 - f)$ in reserve. After the nth toss, your wealth is W_n, and you set aside $(1 - f)W_n$ dollars and bet $f\ W_n$ dollars on the next toss. Your wealth after the $(n + 1)$st toss will then be $W_{n+1} = fW_nX_{n+1} + (1 - f)W_n = W_n[(1 - f) + fX_{n+1}]$.

(e) For the Double or Quarter Game, with $\mathbf{P}$(heads)=1/2, find the range of fixed fractions f of your wealth that you can bet and be guaranteed than R_n converges in probability to a number greater than 1 and therefore that your wealth grows to infinity as $n \to \infty$. Find the maximum value of r to which R_n converges in probability and the value of f at which this maximum is achieved.