Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 7 Solutions

Due November 16, 2011

1. A successful call occurs with probability $p = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

   (a) Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply

   $$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (b) The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials and again our answer is

   $$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (c) We desire the probability that L_2, the time to the second arrival is equal to five trials. We know that $p_{L_2}(\ell)$ is a Pascal PMF of order 2, and we have

   $$p_{L_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}$$

   (d) Here we require the conditional probability that the experimental value of L_2 is equal to 5, given that it is greater than 2.

   $$\begin{aligned} \mathbf{P}(L_2 = 5 | L_2 > 2) &= \frac{p_{L_2}(5)}{P(L_2 > 2)} = \frac{p_{L_2}(5)}{1 - p_{L_2}(2)} \\ &= \frac{\binom{5-1}{2-1} p^2 (1-p)^{5-2}}{1 - \binom{2-1}{2-1} p^2 (1-p)^0} = \frac{4 \cdot \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{6} \end{aligned}$$

   (e) The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of L_2 is greater than or equal to 5.

   $$\begin{aligned} \mathbf{P}(L_2 \geq 5) = 1 - P(L_2 \leq 4) &= 1 - \sum_{\ell=2}^{4} \binom{\ell-1}{2-1} p^2 (1-p)^{\ell-2} \\ &= 1 - \left(\frac{1}{2}\right)^2 - \binom{2}{1}\left(\frac{1}{2}\right)^3 - \binom{3}{1}\left(\frac{1}{2}\right)^4 = \frac{5}{16} \end{aligned}$$

   (f) Let discrete random variable F represent the number of failures before Fred runs out of samples on his mth successful call. Since L_m is the number of trials up to and including the mth success, we have $F = L_m - m$. Given that Fred makes L_m calls before he needs a new supply, we can regard each of the F unsuccessful calls as trials in another Bernoulli process with parameter r, where r is the probability of a success (a disappointed dog) obtained by

   $$\begin{aligned} r &= \mathbf{P}(\text{dog lives there} \mid \text{Fred did not leave a sample}) \\ &= \frac{\mathbf{P}(\text{dog lives there AND door not answered})}{1 - \mathbf{P}(\text{giving away a sample})} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{1 - \frac{1}{2}} = \frac{1}{3} \end{aligned}$$

We define X to be a Bernoulli random variable with parameter r. Then, the number of dogs passed up before Fred runs out, D_m, is equal to the sum of F Bernoulli random variables each with parameter $r = \frac{1}{3}$, where F is a random variable. In other words,

$$D_m = X_1 + X_2 + X_3 + \cdots + X_F.$$

Note that D_m is a sum of a random number of independent random variables. Further, F is independent of the X_i's since the X_i's are defined in the conditional universe where the door is not answered, in which case, whether there is a dog or not does not affect the probability of that trial being a failed trial or not. From our results in class, we can calculate its expectation and variance by

$$\begin{aligned} \mathbf{E}[D_m] &= \mathbf{E}[F]\mathbf{E}[X] \\ \text{var}(D_m) &= \mathbf{E}[F]\text{var}(X) + (\mathbf{E}[X])^2\text{var}(F), \end{aligned}$$

where we make the following substitutions.

$$\begin{aligned} \mathbf{E}[F] &= \mathbf{E}[L_m - m] = \frac{m}{p} - m = m. \\ \text{var}(F) &= \text{var}(L_m - m) = \text{var}(L_m) = \frac{m(1-p)}{p^2} = 2m. \\ \mathbf{E}[X] &= r = \frac{1}{3}. \\ \text{var}(X) &= r(1-r) = \frac{2}{9}. \end{aligned}$$

Finally, substituting these values, we have

$$\begin{aligned} \mathbf{E}[D_m] &= m \cdot \frac{1}{3} = \frac{m}{3} \\ \text{var}(D_m) &= m \cdot \frac{2}{9} + \left(\frac{1}{3}\right)^2 \cdot 2m = \frac{4m}{9} \end{aligned}$$

2. Note that we can view N as the number of arrivals of a Poisson process with rate λ in the interval $[0, 1]$.

   (a) For any $i \in \{1, 2, \ldots, k\}$, we can split the original process into two processes: the first process is composed of all arrivals corresponding to outcome a_i, and the second process is composed of arrivals corresponding to all other outcomes. N_i is then the number of arrivals in the first split process during the interval $[0, 1]$. The first split process has rate λp_i, and therefore N_i has a Poisson distribution with parameter λp_i:

   $$\mathbf{P}(N_i = n) = \frac{(\lambda p_i)^n e^{-\lambda p_i}}{n!},$$

   for $n = 0, 1, 2, \ldots$.

   (b) We can split the original process into three processes, corresponding to a_1, a_2, and $\{a_3, \ldots, a_k\}$, respectively. These are independent Poisson processes, and we can merge the first two into a new Poisson process with rate $\lambda p_1 + \lambda p_2$. $N_1 + N_2$ is then the number of arrivals in this merged process during the interval $[0, 1]$, and its PMF is given by

   $$\mathbf{P}(N_1 + N_2 = n) = \frac{(\lambda p_1 + \lambda p_2)^n e^{-(\lambda p_1 + \lambda p_2)}}{n!},$$

   for $n = 0, 1, 2, \ldots$.

(c) Each individual experiment has a probability p_1 of giving outcome a_1. Given that there were n total experiments, the number of experiments resulting in a_1 is binomially distributed with a PMF given by

$$p_{N_1|N}(m \mid n) = \binom{n}{m} p_1^m (1-p_1)^{n-m},$$

for $m \in \{0, 1, \ldots, n\}$.

(d) Similarly, the PMF of $N_1 + N_2$ given $N = n$ is

$$p_{N_1+N_2|N}(m \mid n) = \binom{n}{m} (p_1+p_2)^m (1-p_1-p_2)^{n-m},$$

for $m \in \{0, 1, \ldots, n\}$.

(e) We wish to find $\mathbf{P}(N = n \mid N_1 = n_1)$. One approach is to use Bayes' rule, which is straightforward given that we know $\mathbf{P}(N = n)$ from the problem statement, $\mathbf{P}(N_1 = n_1 \mid N = n)$ from part (c), and $\mathbf{P}(N_1 = n_1)$ from part (a). For an alternative approach, let us think in terms of Poisson processes again. If we split the original process into a_1 and $\{a_2, \ldots, a_k\}$, then we can let N_1 denote the number of arrivals in the first split process and N_1^c denote the number of arrivals in the second split process. Given that $N_1 = n_1$, we can have $N = n$ if and only if $N_1^c = n - n_1$. Since the two split processes are independent, we conclude that

$$p_{N|N_1}(n \mid n_1) = p_{N_1^c}(n - n_1) = \frac{(\lambda(1-p_1))^{n-m} e^{-\lambda(1-p_1)}}{(n-m)!},$$

for $n \geq n_1$.

3. (a) We are given that the previous ship to pass the pointer was traveling westward.

i. The direction of the next ship is independent of those of any previous ships. Therefore, we are simply looking for the probability that a westbound arrival occurs before an eastbound arrival, or

$$\mathbf{P}(\text{next} = \text{westbound}) = \frac{\lambda_W}{\lambda_E + \lambda_W}$$

ii. The pointer will change directions on the next arrival of an east-bound ship. By definition of the Poisson process, the remaining time until this arrival, denote it by X, is exponential with parameter λ_E, or

$$f_X(x) = \lambda_E e^{-\lambda_E x}, x \geq 0 \quad .$$

(b) For this to happen, no westbound ship can enter the channel from the moment the eastbound ship enters until the moment it exits, which consumes an amount of time t after the eastward ship enters the channel. In addition, no westbound ships may already be in the channel prior to the eastward ship entering the channel, which requires that no westbound ships enter for an amount of time t *before* the eastbound ship enters. Together, we require no westbound ships to arrive during an interval of time $2t$, which occurs with probability

$$\frac{(\lambda_W 2t)^0 e^{-\lambda_W 2t}}{0!} = e^{-\lambda_W 2t} \quad .$$

(c) Letting X be the first-order interarrival time for eastward ships, we can express the quantity $V = X_1 + X_2 + \ldots + X_7$, and thus the PDF for V is equivalent to the 7th order Erlang distribution

$$f_V(v) = \frac{\lambda_E^7 v^6 e^{-\lambda_E v}}{6!}, v \geq 0 \quad .$$

4. (a) Since the shuttles depart exactly every hour on the hour, the number of passengers that arrive in a one hour interval is the number of passengers on a shuttle. So, the arrivals are described by a Poisson process, and the expected number of arrivals (and therefore the expected number of passengers on a shuttle) is the mean of a Poisson random variable, or λ.

$$E[\text{number of passengers on a shuttle}] = \lambda$$

(b) Recall that in continuous time, each inter-arrival time in a Poisson process is described by the exponential distribution. Here, we consider the times in between shuttle arrivals with an exponential distribution, rate μ per hour. Then each shuttle arrival is a Poisson process. Let A be the number of shuttles arriving in one hour with parameter μ and the following distribution,

$$p_A(a) = \begin{cases} \frac{e^{-\mu}\mu^a}{a!} & a = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(c) In the terminal, there is a Poisson process describing the arrival of passengers and another Poisson process describes the departures of shuttles. The "event" described includes either process (or both), so the event is a merged process (still Poisson). The two processes are independent from one another, so the merged process has an arrival rate of: $\lambda + \mu$.

$$E[\text{number of events per hour}] = \lambda + \mu$$

(d) The wait time until the next shuttle is the inter-arrival time of the shuttles, which is exponential, with parameter μ. Recall that the exponential distribution is memoryless, so seeing 2λ people waiting around does not affect the expected wait time for a shuttle. So from the time the passenger arrives at the gate, the wait time is exponential with parameter μ, with mean $1/\mu$.

$$E[\text{wait time} \mid 2\lambda \text{ people waiting}] = \frac{1}{\mu}$$

(e) To find the PMF for the number of passengers in a shuttle, we go back to part c, where we determined that the event of either a passenger arrival or shuttle departure is a merged Poisson process, with paramter $\lambda + \mu$. In the merged Poisson, the probability that the arrival was a passenger arrival is $\frac{\lambda}{\lambda+\mu}$, and the probability that the "arrival" was a shuttle departure is $\frac{\mu}{\lambda+\mu}$.

Let N be the number of people on a shuttle. There must be n successive passenger arrivals before a shuttle departure. Therefore, the PMF for N is:

$$p_N(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

One can also think of the PMF of N as number of "failures" (passenger arrivals) until the first "success" (shuttle departure), but shifted to start at 0 rather than 1 in a standard geometric distribution.

5. We view the random variables T_1, T_2, and S as interarrival times in two independent Poisson processes with rates λ and μ, respectively. We are interested in the expected value of the time Z until either the first process ($T_1 + T_2$) has had two arrivals or the second process (S) has had one arrival.
The expected time until the first arrival is $1/(\lambda+\mu)$. With probability $\mu/(\lambda+\mu)$ this arrival comes from the second process and we are done. If it comes from the first process, we have to wait until an arrival comes from either process. The expected additional waiting time is $1/(\lambda+\mu)$. Using the total expectation theorem, we obtain

$$\mathbf{E}[Z] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot \frac{1}{\lambda+\mu}.$$

6. (a) The event $\{M = m \cap N = n\}$ occurs when $\{N = n\}$ and $\{M - N = m - n\}$. That is, from $(0, t]$ there have to be n arrivals, and after t but prior to $t + s$ there have to be $m - n$ arrivals. Since the increment $(0, t]$ is disjoint from the increment $(t, t + s]$, the number of arrivals in each are independent and have a poisson distribution with rate λ. Symbolically,

$$\begin{aligned} p_{N,M}(n, m) &= p_N(n)p_{M|N}(m|n) \\ &= \left[\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right] \left[\frac{(\lambda s)^{m-n} e^{-\lambda s}}{(m-n)!}\right]. \end{aligned}$$

   (b) A similar principle is helpful here as well. We can rewrite $\mathbf{E}[NM]$ as

$$\begin{aligned} \mathbf{E}[NM] &= \mathbf{E}[N(M-N) + N^2] \\ &= \mathbf{E}[N]\mathbf{E}[M-N] + \mathbf{E}[N^2] \\ &= (\lambda t)(\lambda s) + \left[\text{var}(N) + \mathbf{E}[N]^2\right] \\ &= (\lambda t)(\lambda s) + \lambda t + (\lambda t)^2, \end{aligned}$$

   where the second equality is obtained via the independent increment property of the Poisson process.

G1†. (a) Let t_d be the time at which the tumor size is d_{det}; then $T_{\text{det}} = t_d - t_b$. Also, $d_{\text{det}} = d_{\min} e^{(t_d - t_b)/t_g}$. Thus,

$$T_{\text{det}} = t_g \ln\left(\frac{d_{\text{det}}}{d_{\min}}\right).$$

   (b) Similarly as in (a) above, $d_{\text{met}} = d_{\min} e^{T_{\text{met}}/t_g}$; thus,

$$T_{\text{met}} = t_g \ln\left(\frac{d_{\text{met}}}{d_{\min}}\right).$$

(c) When the diameter doubles, the volume increases by a factor of 8. Since the volume doubles every 130 days, it will take 390 days to increase by a factor of 8, and therefore the diameter will double every 390 days. Hence, we can use the given exponential growth formula to calculate t_g:

$$2 = e^{390/t_g} \Longrightarrow t_g = 390/\ln 2 \approx 563 \text{ days.}$$

(d) In this simple model, a tumor born at time t_b becomes detectable at time $t_b + T_{\text{det}}$, and, if not found and removed, metastasizes at a later time $t_b + T_{\text{met}}$. This cannot happen if a mammogram is given every T years with $T < T_{\text{met}} - T_{\text{det}}$, since in this case at least one mammogram will be given in the interval $(t_b + T_{\text{det}}, t_b + T_{\text{met}})$.

But for $T \geq T_{\text{met}} - T_{\text{det}}$, there is either one or two intervals of time in each interexamination period in which the tumor becomes detectable and then metastasizes with no mammogram taking place between these two events. The length of time within one interexamination period when this will occur is $T - (T_{\text{met}} - T_{\text{det}})$.

Therefore, the probability that one or more tumors will be born in a single interexamination period and eventually metastasize is given by

$$P_{\text{met},T} = \begin{cases} 0, & T < T_{\text{met}} - T_{\text{det}}, \\ 1 - e^{-\lambda[T-(T_{\text{met}}-T_{\text{det}})]}, & T \geq T_{\text{met}} - T_{\text{det}}. \end{cases}$$

(e) For $x \approx 0$, we can use the approximation $e^x \approx 1 + x$. Therefore, the previous answer can be approximated by

$$P_{\text{met},T} \approx \begin{cases} 0, & T < T_{\text{met}} - T_{\text{det}}, \\ \lambda[T - (T_{\text{met}} - T_{\text{det}})], & T \geq T_{\text{met}} - T_{\text{det}}. \end{cases}$$

(f) We first calculate T_{det} and T_{met}:

$$T_{\text{det}} = t_g \ln\left(\frac{d_{\text{det}}}{d_{\text{min}}}\right) \approx 563 \ln(7) \approx 1096 \text{ days,}$$

$$T_{\text{met}} = t_g \ln\left(\frac{d_{\text{met}}}{d_{\text{min}}}\right) \approx 563 \ln(50) \approx 2202 \text{ days.}$$

Therefore, our simple model predicts that there is no possibility of a breast cancer metastasis provided a woman is examined every $T_{\text{met}} - T_{\text{det}} \approx 1106$ days, or more often.

i. Since 3 years is 1095 days, which is less than 1106, our model predicts that there will be no metastasis in this case.

ii. If a woman is examined every 5 years, then in each interexamination period there is a period of $5 * 365 - (2202 - 1096) = 719$ days during which a tumor can arise and eventually metastasize. Over the course of 30 years, this amounts to $719 * 6 = 4314$ days, or 11.8 years. From our Poisson model, this yields a probability of

$$1 - e^{-\frac{1}{250}(11.8)} \approx 0.046$$

of developing a cancer that eventually metastasizes.