Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Recitation 13 Solutions
October 25, 2011

1. If $a = 0$, the equation is trivial. If $a \neq 0$, then

$$\begin{aligned}
\rho(aX+b,Y) &= \frac{\text{cov}(aX+b,Y)}{\sqrt{\text{var}(aX+b)(\text{var}(Y))}} \\
&= \frac{\mathbf{E}[(aX+b-\mathbf{E}[aX+b])(Y-\mathbf{E}[Y])]}{\sqrt{a^2\text{var}(X)\text{var}(Y)}} \\
&= \frac{\mathbf{E}[(aX+b-a\mathbf{E}[X]-b)(Y-\mathbf{E}[Y])]}{a\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \frac{a\mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]}{|a|\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \text{sgn}(a)\frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \text{sgn}(a)\rho(X,Y)
\end{aligned}$$

As an example where this property of the correlation coefficient is relevant, consider the homework and exam scores of students in a class. We expect the homework and exam scores to be positively correlated and thus have a positive correlation coefficient. Note that, in this example, the above property will mean that the correlation coefficient will not change whether the exam is out of 105 points, 10 points, or any other number of points.

2.

(a) When $z \geq 0$:

$$\begin{aligned}
F_Z(z) = \mathbf{P}(X - Y \leq z) &= \mathbf{P}(X \leq Y + z) \\
&= \int_0^\infty \int_0^{y+z} f_{X,Y}(x,y')dxdy \\
&= \int_0^\infty \lambda e^{-\lambda y} \int_0^{y+z} \lambda e^{-\lambda x} dxdy \\
&= \int_0^\infty \lambda e^{-\lambda y}\left(1 - e^{-\lambda(y+z)}\right) dy \\
&= 1 + \frac{e^{-\lambda z}}{2} e^{-2\lambda y}\Big|_{y=0}^{y=\infty} \\
&= 1 - \frac{1}{2}e^{-\lambda z} \qquad z \geq 0
\end{aligned}$$

When $z < 0$:

$$\begin{aligned}
\mathbf{P}(X \leq Y + z) &= \int_0^\infty \int_{x-z}^\infty f_{X,Y}(x,y)dydx \\
&= \int_0^\infty \lambda e^{-\lambda x} \int_{x-z}^\infty \lambda e^{-\lambda y} dydx \\
&= \int_0^\infty \lambda e^{-\lambda x} e^{-\lambda(x-z)} dx \\
&= e^{\lambda z} \int_0^\infty \lambda e^{-2\lambda x} dx \\
&= \frac{1}{2} e^{\lambda z} \qquad z \leq 0
\end{aligned}$$

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda z} & z \geq 0 \\ \frac{1}{2}e^{\lambda z} & z < 0 \end{cases}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{\lambda}{2}e^{-\lambda z} & z \geq 0 \\ \frac{\lambda}{2}e^{\lambda z} & z < 0 \end{cases}$$

Hence,

$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda |z|}$$

(b) Solving using the convolution formula, we have:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^\infty f_X(x) f_{Z|X}(z|x)dx \\
&= \int_{-\infty}^\infty f_X(x) f_{Y|X}(x-z|x)dx \\
&= \int_{-\infty}^\infty f_X(x) f_Y(x-z)dx
\end{aligned}$$

First when $z < 0$, we have:

$$\begin{aligned}
\int_{-\infty}^\infty f_X(x) f_Y(x-z)dx &= \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)} dx \\
&= \lambda e^{\lambda z} \int_0^\infty \lambda e^{-2\lambda x} dx \\
&= \frac{\lambda}{2} e^{\lambda z}
\end{aligned}$$

Then, when $z \geq 0$ we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x) f_Y(x-z)dx &= \int_z^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)} dx \\
&= \lambda e^{\lambda z} \int_z^{\infty} \lambda e^{-2\lambda x} dx \\
&= \frac{\lambda}{2} e^{\lambda z} e^{-2\lambda z} \\
&= \frac{\lambda}{2} e^{-\lambda z} \\
f_Z(z) &= \frac{\lambda}{2} e^{-\lambda |z|} \qquad \forall z
\end{aligned}$$

3. (a) We have $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Recall that in polar coordinates, the differential area is $dA = dxdy = rdrd\theta$. So

$$\begin{aligned}
F_R(r) = \mathbf{P}(R \leq r) &= \int_0^r \int_0^{2\pi} f_X(r'\cos\theta) f_Y(r'\sin\theta) d\theta\, r'dr' \\
&= \int_0^r \int_0^{2\pi} \frac{1}{2\pi} e^{-(r')^2/2} d\theta\, r'dr' \\
&= \int_0^r r' e^{-(r')^2/2} dr' \int_0^{2\pi} \frac{d\theta}{2\pi} \\
&= \int_0^{r^2/2} e^{-u} du \qquad (u = (r')^2/2) \\
F_R(r) &= \begin{cases} 1 - e^{-r^2/2} & r \geq 0 \\ 0 & r < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
f_R(r) = \frac{d}{dr} F_R(r) &= (-1/2)(2r)(-e^{-r^2/2}) \\
&= r e^{-r^2/2}, \qquad r \geq 0
\end{aligned}$$

(b)

$$\begin{aligned}
F_\Theta(\theta) = \mathbf{P}(\Theta \leq \theta) &= \int_0^\theta \int_0^\infty f_X(r\cos\theta') f_Y(r\sin\theta') r dr\, d\theta' \\
&= \int_0^\theta \int_0^\infty \frac{1}{2\pi} e^{-r^2/2} r dr\, d\theta' \\
&= \int_0^\infty r e^{-r^2/2} dr \int_0^\theta \frac{d\theta'}{2\pi} \\
&= \frac{\theta}{2\pi} \qquad 0 \leq \theta \leq 2\pi
\end{aligned}$$

where the last equality follows from the fact that $f(r) = re^{-r^2/2}, r \geq 0$ is a PDF shown in part (a), i.e. $\int_0^\infty re^{-r^2/2}dr = 1$.

$$F_\Theta(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{\theta}{2\pi} & 0 \leq \theta \leq 2\pi \\ 1 & \theta \geq 2\pi \end{cases}$$

$$f_\Theta(\theta) = \frac{d}{d\theta}F_\Theta(\theta) = \frac{1}{2\pi} \qquad 0 \leq \theta \leq 2\pi$$

(c)

$$\begin{aligned} F_{R,\Theta}(r,\theta) &= P(R \leq r, \Theta \leq \theta) \\ &= \int_0^\theta \int_0^r \frac{1}{2\pi} r' e^{-(r')^2/2} dr'\, d\theta' \\ &= \int_0^\theta \int_0^{r^2/2} \frac{1}{2\pi} e^{-u} du\, d\theta' \qquad (u = (r')^2/2) \\ &= \int_0^\theta \frac{1}{2\pi}\left(1 - e^{-r^2/2}\right) d\theta' \\ &= \frac{\theta}{2\pi}\left(1 - e^{-r^2/2}\right) \qquad r \geq 0, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

$$F_{R,\Theta}(r,\theta) = \begin{cases} \frac{\theta}{2\pi}\left(1 - e^{-r^2/2}\right) & r \geq 0, \quad \theta > 2\pi \\ 1 - e^{-r^2/2} & r \geq 0, \quad 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$f_{R,\Theta}(r,\theta) = \frac{\partial}{\partial r}\frac{\partial}{\partial \theta}F_{R,\Theta}(r,\theta) = \frac{1}{2\pi} re^{-r^2/2} \qquad r \geq 0, \quad 0 \leq \theta \leq 2\pi$$

Note: The PDF of R^2 is exponentially distributed with parameter $\lambda = 1/2$. This is a very convenient way to generate normal random variables from independent uniform and exponential random variables. We can generate an arbitrary random variable X with CDF F_X by first generating a uniform random variable and then passing the samples from the uniform distribution through the function F_X^{-1}. But since we don't have a closed-form expression for the CDF of a normal random variable, this method doesn't work. However, we do have a closed-form expression for the exponential distribution. Therefore, we can generate an exponential distribution with parameter $1/2$ and we can generate a uniform distribution in $[0, 2\pi]$, and with these two distributions we can generate standard normal distributions.

4. Problem 4.20, page 250 in text. See text for the proof.