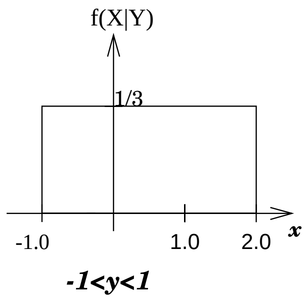
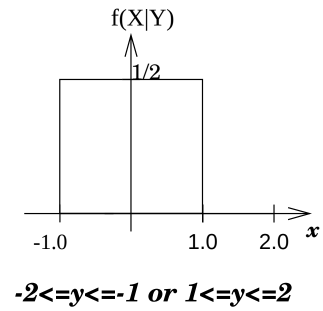
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 7: Solutions
Due: April 8, 2009

1. (a)





(b) $\mathbf{E}[X|Y=y] = \begin{cases} 0, & -2 \le y \le -1 \\ \frac{1}{2}, & -1 < y \le 1 \\ 0, & 1 \le y \le 2 \end{cases}$ $\quad \text{var}(X|Y=y) = \begin{cases} \frac{1}{3}, & -2 \le y \le -1 \\ \frac{3}{4}, & -1 < y \le 1 \\ \frac{1}{3}, & 1 \le y \le 2 \end{cases}$

(c) $\mathbf{E}[X] = \frac{3}{10}$

(d) $\text{var}(X) = \frac{193}{300}$

(e) Let $Z = \mathbf{E}[X|Y]$ and it is a discrete random variable with PMF

$$P(Z=z) = \begin{cases} 0.6, & z = \frac{1}{2}, \\ 0.4, & z = 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. (a)

$$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}\left[\mathbf{E}[T|N]\right] \\ &= \mathbf{E}\left[N \cdot \mathbf{E}[X]\right] \\ &= \mathbf{E}\left[3N\right] = 9. \end{aligned}$$

(b)

$$\begin{aligned} \text{var}(T) &= \mathbf{E}\left[\text{var}(T|N)\right] + \text{var}(\mathbf{E}[T|N]) \\ &= \mathbf{E}\left[N\text{var}(X)\right] + \text{var}(N\mathbf{E}[X]) \\ &= (9-3)\,\mathbf{E}\left[N\right] + 9\,\text{var}(N) \\ &= 18 + 9 \cdot (9-3) = 72. \end{aligned}$$

(c)

$$\begin{aligned} M_T(s) = M_N\left(\log M_X(s)\right) &= \frac{pM_X(s)}{1-(1-p)M_X(s)} \\ &= \frac{p\,\frac{pe^s}{1-(1-p)e^s}}{1-(1-p)\frac{pe^s}{1-(1-p)e^s}} \\ &= \frac{p^2e^s}{1-(1-p^2)e^s}, \end{aligned}$$

where $p = 1/3$.

(d) From its transform, T is a geometric random variable with $p = 1/9$.

(e) Least-square estimator $\hat{T} = \mathbf{E}\left[T|N = n\right] = n\mathbf{E}[X] = 3n$.

3. (a) Let N be the outcome of die. N is a discrete random variable taking values of 1,2, or 3 equally likely. So,

$$\begin{aligned}
\mathbf{E}[N] &= \frac{1}{3}1 + \frac{1}{3}2 + \frac{1}{3}3 \\
&= 2 \\
\mathrm{var}(N) &= \frac{1}{3}(1 - \mathbf{E}[N])^2 + \frac{1}{3}(2 - \mathbf{E}[N])^2 + \frac{1}{3}(3 - \mathbf{E}[N])^2 \\
&= \frac{2}{3}
\end{aligned}$$

Let X be the result of spinning the wheel of fortune. It is a uniform random varible between $(0, 1)$. Thus:

$$\begin{aligned}
\mathbf{E}[X] &= \frac{1}{2} \\
\mathrm{var}(X) &= \frac{1}{12}
\end{aligned}$$

$Y = X_1 + \cdots + X_N$ where N is the outcome of die. Using properties listed on pg. 234 of the text, we get:

$$\begin{aligned}
\mathbf{E}[Y] &= \mathbf{E}[X]\mathbf{E}[N] \\
&= 2\left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
\mathrm{var}(Y) &= \mathrm{var}(X)\mathbf{E}[N] + \mathbf{E}[X]^2\mathrm{var}(N) \\
&= \left(\frac{1}{12}\right)2 + \left(\frac{1}{2}\right)^2\frac{2}{3} \\
&= \frac{1}{6} + \frac{2}{12} \\
&= \frac{1}{3}
\end{aligned}$$

4. We are given that Y and Z are zero mean and uncorrelated. For any choice of a, b, the mean squared error is equal to

$$\min_{(a,b)}\left[\mathbf{E}(X - aY - bZ)^2]\right] = \min_{(a,b)}\left[[\mathbf{E}[X^2] + a^2\mathbf{E}[Y^2] + b^2\mathbf{E}[Z^2] - 2a\mathbf{E}[XY] - 2b\mathbf{E}[XZ]\right].$$

We differentiate with respect to a and b, and set the derivatives to zero, to obtain

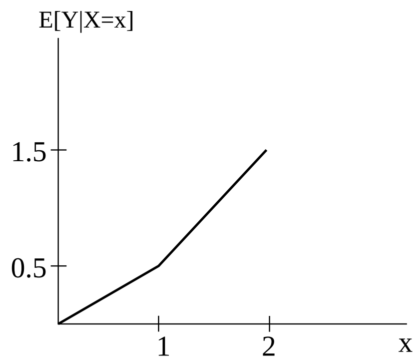
$$a = \frac{\mathbf{E}[XY]}{\mathbf{E}[Y^2]} = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(Y)}, \qquad b = \frac{\mathbf{E}[XZ]}{\mathbf{E}[Z^2]} = \frac{\mathrm{cov}(X,Z)}{\mathrm{var}(Z)}.$$

5. (a) $c = \frac{2}{3}$.

   (b) Here we are trying to choose a $g(X)$ that minimizes the conditional mean squared error $\mathbf{E}[(Y - g(X))^2|X]$. As shown in Section 4.6 in the text, this estimator is $g(X) = \mathbf{E}[Y|X]$.

$$g(x) = \mathbf{E}[Y|X = x] \quad = \quad \begin{cases} \frac{1}{2}x & 0 \le x < 1 \\ x - \frac{1}{2} & 1 \le x \le 2 \\ \text{Undefined} & \text{Otherwise} \end{cases}$$

   A plot of $g(x)$:



   (c) $\mathbf{E}[g^*(X)] = \mathbf{E}[\mathbf{E}[Y|X]] = \mathbf{E}[Y]$, and note that $f_{X,Y}(x, y) = c = \frac{2}{3}$.

$$\begin{aligned} \mathbf{E}[Y] &= \int_x \int_y y f_{X,Y}(x, y) dy dx \\ &= \int_0^1 \int_0^x y \, \frac{2}{3} dy dx + \int_1^2 \int_{x-1}^x y \, \frac{2}{3} dy dx \\ &= \frac{7}{9} \end{aligned}$$

$$\begin{aligned} \text{var}(g^*(X)) &= \text{var}(\mathbf{E}[Y|X]) = \mathbf{E}[\mathbf{E}[Y|X]^2] - (\mathbf{E}[\mathbf{E}[Y|X]])^2 \\ &= \int_0^2 \mathbf{E}[Y|X = x]^2 f_X(x) dx - (\mathbf{E}[Y])^2 \\ &= \int_0^1 (\frac{1}{2}x)^2 \cdot \frac{2}{3} x dx + \int_1^2 (x - \frac{1}{2})^2 \cdot \frac{2}{3} dx - (\frac{7}{9})^2 \\ &= \frac{103}{648} \\ &= 0.159 \end{aligned}$$

   where $f_X(x) = \frac{2}{3}x$ for $0 \le x \le 1$, $f_X(x) = \frac{2}{3}$ for $1 \le x \le 2$, and 0 otherwise.

   (d) $\mathbf{E}[(Y - g^*(X))^2]$ and $\mathbf{E}[\text{var}(Y|X)]$ are the same thing.

$$\mathbf{E}[(Y - g^*(X))^2] \quad = \quad \int_x \int_y (y - \mathbf{E}[Y|X = x])^2 f_{Y|X}(y|x) f_X(x) dy dx$$

$$\begin{aligned} &= \int_x \mathrm{var}(Y|X=x) f_X(x)dx \\ &= \mathbf{E}[\mathrm{var}(Y|X)] \end{aligned}$$

For any given value of X, $f_{Y|X}(y|x)$ is uniform. When $0 \leq x \leq 1$, $f_{Y|X}(y|x)$ is uniform over $0 \leq y \leq x$. When $1 \leq x \leq 2$, $f_{Y|X}(y|x)$ is uniform over $x-1 \leq y \leq x$. Thus,

$$\mathrm{var}(Y|X=x) = \begin{cases} \frac{(x-0)^2}{12} = \frac{x^2}{12} & 0 \leq x < 1 \\ \frac{(x-(x-1))^2}{12} = \frac{1}{12} & 1 \leq x \leq 2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{E}[\mathrm{var}(Y|X)] &= \int_x \mathrm{var}(Y|X=x) f_X(x)dx \\ &= \int_0^1 \frac{x^2}{12} \cdot \frac{2}{3} x dx + \int_1^2 \frac{1}{12} \cdot \frac{2}{3} dx \\ &= \frac{5}{72} \end{aligned}$$

(e) By the law of total variance, we have $\mathrm{var}(Y) = \mathbf{E}[\mathrm{var}(Y|X)] + \mathrm{var}(\mathbf{E}[Y|X])$. Using the answers to (b) and (c),

$$\begin{aligned} \mathrm{var}(Y) &= \mathbf{E}[\mathrm{var}(Y|X)] + \mathrm{var}(\mathbf{E}[Y|X]) \\ &= \frac{5}{72} + \frac{103}{648} = \frac{37}{162} \end{aligned}$$

Of course, you can always find $f_Y(y)$ first and then calculate the variance in the usual way; it's just that in this problem we happen to have both components in the sum above.

(f) The optimal linear estimate is given by,

$$l^*(X) = \mathbf{E}[Y] + \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}[X - \mathbf{E}[X]].$$

In part(b) we calculated, $\mathbf{E}[Y] = \frac{7}{9}$. In order to calculate $\mathrm{var}(X)$ we first calculate $\mathbf{E}[X^2]$ and $\mathbf{E}[X]^2$.

$$\begin{aligned} \mathbf{E}[X^2] &= \int_0^2 x^3 \frac{2}{3} dx + \int_1^2 x^2 \frac{2}{3} dx, \\ &= \frac{31}{18}, \\ \mathbf{E}[X] &= \int_0^2 x^2 \frac{2}{3} dx + \int_1^2 x \frac{2}{3} dx, \\ &= \frac{11}{9} \end{aligned}$$

The $\text{var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{37}{162}$. To determine $\text{cov}(X,Y)$ we need to evaluate $\mathbf{E}[XY]$.

$$\begin{aligned}
\mathbf{E}[YX] &= \int_x \int_y xy f_{X,Y}(x,y) dy dx \\
&= \int_0^1 \int_0^x yx\, \frac{2}{3} dy dx + \int_1^2 \int_{x-1}^x yx\, \frac{2}{3} dy dx \\
&= \frac{41}{36}
\end{aligned}$$

Therefore $\text{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = \frac{61}{324}$. Therefore,

$$l^*(X) = \frac{7}{9} + \frac{61}{74}[X - \frac{11}{9}].$$

The linear LMS estimator is unbiased, therefore expectation is given by,

$$\mathbf{E}[l^*(X)] = \mathbf{E}[Y] = \frac{7}{9}.$$

The $\text{var}(l^*(X)) = \frac{\text{cov}(X,Y)^2}{\text{var}(X)} = 0.155$

(g) The LMS estimator is the optimal estimator among all classes of estimators that minimize the mean squared error. The linear LMS estimator therefore performs worse or equal to the LMS estimator, i.e., we expect $\mathbf{E}[(Y - l^*(X))^2] > \mathbf{E}[(Y - g^*(X))^2]$.

$$\begin{aligned}
\mathbf{E}[(Y - l^*(X))^2] &= \sigma_Y^2(1-\rho^2), \\
&= \sigma_Y^2(1 - \frac{\text{cov}(X,Y)^2}{\sigma_X^2\sigma_Y^2}), \\
&= \frac{37}{162}\left(1 - (\frac{61}{74})^2\right), \\
&= 0.073.
\end{aligned}$$

This value is larger than $\frac{5}{72} = 0.06$.

G1†. Notice that

$$M_X(s) = \mathbf{E}[e^{sX}] = M_{X,Y}(s,0) = \exp\{(\alpha+\gamma)(e^s - 1)\},$$

which can be recognized as the transform function of a Poisson random variable with parameter $\lambda = \alpha + \gamma$. Similarly, one can find out Y is a Poisson random variable with parameter $\lambda = \beta + \gamma$. Furthermore, the transform function of $Z = X + Y$ is

$$\begin{aligned}
M_Z(v) = \mathbf{E}[e^{v(X+Y)}] = M_{X,Y}(v,v) &= \exp\left\{\alpha(e^v - 1) + \beta(e^v - 1) + \gamma(e^{2v} - 1)\right\} \\
&= \exp\{(\alpha+\beta)(e^v - 1)\} \cdot \exp\left\{\gamma(e^{2v} - 1)\right\}.
\end{aligned}$$

Observe that the first part corresponds to a Poisson random variable K_1 with $\lambda = \alpha + \beta$ and the second part corresponds to a random variable K_2 with transform function

$$P(K_2 = 2k) = \begin{cases} \frac{\gamma^k}{k!} e^{-\gamma}, & k = 0, 1, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $Z = K_1 + K_2$, where K_1 and K_2 are independent, and its distribution is given by

$$P(Z = k) = \begin{cases} \frac{(\alpha+\beta)^k}{k!} e^{-(\alpha+\beta)}, & k = 1, 3, 5, \cdots \\ \sum_{m=0,2,\cdots,k} \frac{(\alpha+\beta)^m \gamma^{k-m}}{m!(k-m)!} e^{-(\alpha+\beta+\gamma)}, & k = 0, 2, 6, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

And it is clear that Z is not Poisson unless $\gamma = 0$.