Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Tutorial 8 Solutions
November 3/4, 2011

1. See the online solution for 6.3, page 326, of the text.

2. (a) For each round, the probability that both Alice and Bob have a loss is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Let random variable X represent the total number of rounds played until the first time where they both have a loss. Then X is a geometric random variable with parameter $p = 1/9$ and has the following PMF.

$$p_X(x) = (1-p)^{x-1}p = \left(\frac{8}{9}\right)^{x-1}\left(\frac{1}{9}\right), \qquad x = 1, 2, \ldots$$

(b) First, consider the number of games, K_3 Bob played until his third loss. Random variable K_3 is a Pascal random variable and has the following PMF.

$$p_{K_3}(k) = \binom{k-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{k-3} \qquad k = 3, 4, 5, \cdots$$

In this question, we are interested in another random variable Z defined as the time at which Bob has his third loss. Note that $Z = 2K_3$. By changing variables, we obtain

$$p_Z(z) = \binom{\frac{z}{2}-1}{3-1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{\frac{z}{2}-3} \qquad z = 6, 8, 10, \cdots$$

(c) Let A be the event that Alice wins, and Let B be the event that Bob wins. The event $A \cup B$ is then the event that either A wins or B wins or both A and B win, and the event $A \cap B$ is the event that both A and B win. Suppose we observe this gambling process, and let U be a random variable indicating the number of rounds we see until at least one of them wins. Random variable U is a geometric random variable with parameter $p = P(A \cup B) = 1 - \frac{1}{3} \cdot \frac{1}{3}$.

Consider another random variable V representing the number of additional rounds we have to observe until the other wins. If both Alice and Bob win at the Uth round, then $V = 0$. This occurs with probability $P(A \cap B | A \cup B) = \frac{\frac{2}{3}\frac{2}{3}}{\frac{8}{9}}$. If Alice wins the Uth round, then the time V until Bob wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(A \cap B^c | A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$. Likewise, if Bob wins the Uth round, then the time V until Alice wins is a geometric random variable with parameter $p = 1/2 + 1/6 = 2/3$. This occurs with probability $P(B \cap A^c | A \cup B) = \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}}$.The number of rounds until each one of them has won at least once, N is

$$N = U + V$$

The expectation of N is then:

$$\begin{aligned}
E[N] &= E[U] + E[V] \\
&= \frac{1}{\frac{8}{9}} + 0 \cdot P(A \cap B | A \cup B) + \frac{1}{\frac{2}{3}} P(A | A \cup B) + \frac{1}{\frac{2}{3}} P(B | A \cup B) \\
&= 9/8 + \frac{3}{2} \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} + \frac{3}{2} \frac{\frac{1}{3}\frac{2}{3}}{\frac{8}{9}} \\
&= 15/8
\end{aligned}$$

There is another approach to this problem. Consider the following partition.

A_1: both win first round
A_2:Only Alice wins first round
A_3: Only Bob wins first round
A_4: both lose first round

Event A_1 occurs with probability $\frac{2}{3} \cdot \frac{2}{3}$. Event A_2 occurs with probability $\frac{2}{3} \cdot \frac{1}{3}$. Event A_3 occurs with probability $\frac{1}{3} \cdot \frac{2}{3}$. Event A_4 occurs with probability $\frac{1}{3} \cdot \frac{1}{3}$. When event A_2 (A_3) occurs, the distribution on the time until Bob (Alice) wins is a geometric random variable with mean $\frac{1}{\frac{2}{3}}$. When event A_4 occurs, the additional time until Alice and Bob win is distributed identically to that at time 0 by the fresh-start property. By the total expectation theorem,

$$\begin{aligned}
E[N] &= E[N|A_1]P(A_1) + E[N|A_2]P(A_2) + E[N|A_3]P(A_3)E[N] + E[N|A_4]P(A_4) \\
&= 1 \cdot \left(\frac{2}{3} \cdot \frac{2}{3}\right) + \left(1 + \frac{1}{\frac{2}{3}}\right) \cdot + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + \left(\frac{1}{3} \cdot \frac{2}{3}\right) + (1 + E[N]) \cdot \left(\frac{1}{3} \cdot \frac{1}{3}\right)
\end{aligned}$$

Solving for $E[N]$, we get $E[N] = \frac{15}{8}$.

3. Let M be the total number of draws you make until you have signed all n papers. Let T_i be the number of draws you make until drawing the next unsigned paper after having signed i papers. Then $M = T_0 + \cdots + T_{n-1}$.

   We can view the process of selecting the next unsigned paper after having signed i papers as a sequence of independent Bernoulli trials with probability of success $p_i = \frac{n-i}{n}$, since there are $n-i$ unsigned papers out of a total of n papers and receiving any paper is equally likely in a particular draw. The PMF governing the number of attempts we make until we succeed in drawing the next unsigned paper after having signed i papers is geometric. More concretely, the probability that it takes k tries to draw the next unsigned paper after having signed i papers is

$$\mathbf{P}(T_i = k) = (1 - p_i)^{k-1} p_i.$$

   With this model, the expected value of M, the number of draws you make until you sign all n papers is:

$$\mathbf{E}[M] = \mathbf{E}\left[\sum_{i=0}^{n-1} T_i\right] = \sum_{i=0}^{n-1} \mathbf{E}\left[T_i\right] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{k=1}^{n} \frac{1}{k}.$$

   For large n, this is on the order of: $n \int_1^n \frac{1}{x}\, dx = n \log n$.