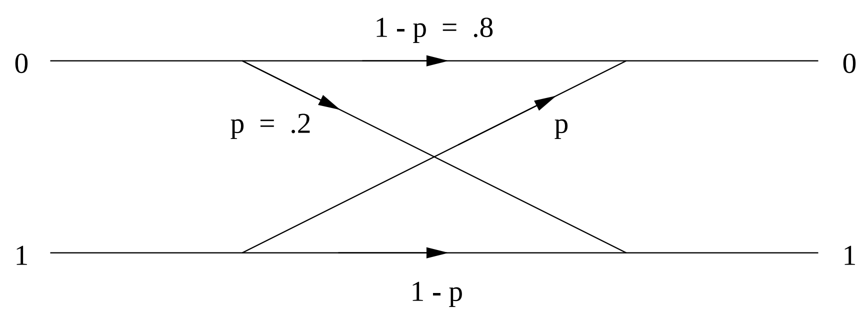
Problem Set 3
Due September 30, 2009

1. Mary and Tom park their cars in an empty parking lot that consists of N parking spaces in a row. Assume that each possible pair of parking locations is equally likely. Calculate the probability that the parking spaces they select are adjacent.

2. Consider a backgammon match with 25 games, each of which can have one of two results: player A wins, or player B wins. There can be no tie in a game.
   (a) Find the number of all possible distinct result sequences.
   (b) Now assume that the match is stopped as soon as one player wins 13 games. Find the number of all possible distinct result sequences.

3. This problem deals (no pun intended) with a well-shuffled deck of 52 distinct cards. The deck has four aces and four kings. In each question below, assume you start with the full deck.
   (a) Find the probability that the top card on the deck is an ace.
   (b) Suppose you draw a card from the deck, put it aside without looking at it, and draw another card. What is the probability that the second card is an ace?
   (c) Now suppose you draw a card from the deck, and it's a king. You put it aside and draw another card. What is the probability that the second card is an ace?
   (d) Now let's assume you draw 7 cards from the deck.
      i. Find the probability that the 7 cards include exactly 3 aces.
      ii. Find the probability that the 7 cards include exactly 2 kings.
      iii. Find the probability that the 7 cards include exactly 3 aces or exactly 2 kings or both.

4. Consider a so-called binary symmetric channel shown below. The input consists of a binary sequence of 0's and 1's. Each bit is flipped with probability $p = 0.2$ in transmission, independently of all other bits.



Let X be a random variable equal to the number of errors made in the transmission of five eight-bit words, i.e., a total of 40 bits.

   (a) Find the PMF of X.

(b) What is the probability that at least 38 bits were transmitted without error?

(c) Let $p = 5 \cdot 10^{-8}$ and assume that 10^6 binary digits are transmitted per second.

  i. What is the expected number of errors in a minute?

  ii. What is the probability of at least one error in a minute?

5. Two fair three-sided dice are rolled simultaneously. Let X be the sum of the two rolls.

   (a) Calculate the PMF, the expectation, and the variance of X.

   (b) Let $Z = X^2$. Find the PMF and the expectation of Z.

   (c) Let $Y = 0.5X^2$ and $W = (X - 1)^2$. Which of the two random variables has higher expectation?

6. (a) Let X be a random variable that takes nonnegative integer values. Show that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k).$$

   Hint: Start with the above formula and interchange the order of the summation.

   (b) Use the formula in the previous part to find the expectation of a random variable Y whose PMF is defined as follows:

$$p_Y(y) = p(1-p)^{y-1}, \qquad y = 1, 2, \ldots$$

   where p is a constant between 0 and 1. This distribution is called the Geometric distribution.

   Hint: You might find the following equality useful:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \qquad \text{for } 0 < \alpha < 1.$$