MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2010)

Problem Set 7: Solutions
Due: April 5, 2010

1. See the online solution for 6.3, page 326, of the text.

2. (a) Since the result of each quiz is independent of others, the probability that Iwana fails exactly two of the next six quizzes is obtained from a binomial distribution.

$$\mathbf{P}(\text{2 failures out of 6 quizzes}) = \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 = \frac{5\cdot 3^5}{4^6}.$$

(b) Note that since the result of each quiz is a bernoulli random variable, the number of quizzes until the first failure is a geometric random variable. Now consider a random variable X indicating the number of quizzes until the third failure. Because of the memoryless property of geometric random variables, X is a sum of three geometric random variables. Hence X is a Pascal random variable with the following PMF.

$$p_X(x) = p_{L_3}(x) = \binom{x-1}{3-1}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{x-3}$$

The expected value of X is $\mathbf{E}[X] = \frac{3}{1/4} = 12$. Therefore the expected number of quizzes that Iwana will pass before she fails three times is

$$\mathbf{E}[X] - 3 = 9$$

(c) Let A be the event that her second and third failures happen on the 8th and the 9th quizzes, respectively. This event requires that she fail exactly one quiz out of first seven quizzes. Remembering that each quiz is independent of others, we have

$$\begin{aligned}\mathbf{P}(A) &= \mathbf{P}(\text{1 failure out of 7 quizzes})\mathbf{P}(\text{failures on both 8th and 9th quizzes})\\ &= \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{7\cdot 3^6}{4^9}\end{aligned}$$

(d) Let Q be the event corresponding to the union of events R and S where:

$$\begin{aligned}&\text{event } R \text{ is} && (F(SF)^kF) \text{ for } k = 0,1,2,\ldots\\ &\text{event } S \text{ is} && ((SF)^kF) \text{ for } k = 1,2,3,\ldots\end{aligned}$$

$$\begin{aligned}\mathbf{P}(Q) &= \mathbf{P}(R) + \mathbf{P}(S)\\ &= \frac{1}{16}\sum_{k=0}^{\infty}\left(\frac{3}{16}\right)^k + \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{3}{16}\right)^k\\ &= \left(\frac{1}{16}+\frac{3}{64}\right)\sum_{k=0}^{\infty}\left(\frac{3}{16}\right)^k\\ &= \frac{7}{64}\cdot\frac{1}{1-3/16} = \frac{7\times 16}{64\times 13} = \frac{7}{52}\end{aligned}$$

3. Let $N_s(t)$ be the counting process for the sports cars stopped, and $N_o(t)$ be the counting process for the ordinary cars. The joint counting process, $N(t)$ is a poisson process of rate 6. Each time the police officer stops a car, the probability that it will be a red sports car is:

$$\frac{4}{2+4} = \frac{2}{3}$$

Therefore,

$$\begin{aligned}
&\mathbf{P}(\text{she stops at least 2 ordinary cars before stopping 3 sports cars}) \\
&= 1 - \mathbf{P}(\text{she stops 0 ordinary cars before stopping 3 sports cars}) \\
&\quad -\mathbf{P}(\text{she stops 1 ordinary car before stopping 3 sports cars}) \\
&= 1 - \left(\frac{2}{3}\right)^3 - 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{33}{81} = \frac{11}{27}.
\end{aligned}$$

4. (a) We know that looking at a very small time interval δ, then

$$\begin{aligned}
\mathbf{P}(A) &= \mathbf{P}(\{\text{Request is from A}\}|\{\text{a request arrives}\}) \\
&= \frac{\mathbf{P}(\{\text{a request from A arrives}\})}{\mathbf{P}(\{\text{a request arrives}\})} \\
&= \frac{a\delta}{(a+b)\delta} \\
&= \frac{a}{a+b}.
\end{aligned}$$

Therefore $\mathbf{P}(\text{8 of 12 requests are from customer A}) = \binom{12}{8}\left(\frac{a}{a+b}\right)^8\left(\frac{b}{a+b}\right)^4$

(b) First note that this is asking about the combined process of A and B (requests). The combined process has rate of $a+b$.

Note that the question asks for $P(7,t)$ with rate $\lambda = a+b$. Thus the answer is:

$$\frac{((a+b)t)^7 e^{-(a+b)t}}{7!}$$

(c) Restricting our attention to the combined process:
$\{M = m\} = \{m$ requests from A in $m+5$ total requests; $m+6$th request is from B$\}$

Using the memoryless property of poisson processes:

$$\begin{aligned}
\mathbf{P}(M=m) &= \mathbf{P}(m \text{ requests from A in } m+5 \text{ requests})\mathbf{P}(m+6\text{th request is type B}) \\
p_M(m) &= \binom{m+5}{5}\left(\frac{b}{a+b}\right)^6\left(\frac{a}{a+b}\right)^m, \; m = 0,1,2,\cdots
\end{aligned}$$

To find the expectation and variance of M: we may either use the direct application of the PMF or simply note that

$$M = X_1 + X_2 + \cdots + X_6$$

where X_i is a random variable for the number of requests after the $(i-1)th$ and before the ith request from B. It is simple to see that the X_i are IID random variables with a geometric distribution shifted by 1 with parameter $p = 1 - \mathbf{P}(A) = \left(\frac{b}{a+b}\right)$.

Therefore,

$$\mathbf{E}[M] = 6\mathbf{E}[X] = 6(\left(\frac{a+b}{b} - 1\right)) = \left(\frac{6a}{b}\right) \qquad \text{var}(M) = 6\text{var}(X) = \left(\frac{6a(a+b)}{b^2}\right)$$

(d) $\mathbf{P}$(2 items requested)$=$ $\mathbf{P}$(3 items requested)$=$ $\mathbf{P}$(4 items requested)$=$ 1/3

$$\begin{aligned} \mathbf{P}(\text{first 4 requests are for the same number of items}) &= \mathbf{P}(\text{first 4 requests are for 2 items}) \\ &+ \mathbf{P}(\text{first 4 requests are for 3 items}) \\ &+ \mathbf{P}(\text{first 4 requests are for 4 items}) \\ &= 3(\left(\frac{1}{3}\right))^4 = \left(\frac{1}{27}\right) \end{aligned}$$

(e) Define the random variable N as $N = Y_1 + Y_2 + \cdot + Y_K$ where Y_i is a set of random variables for the number of items in request i, and where K is a random variable for the number of request arrivals in period t. Note that $\mathbf{E}[Y_i] = 3$, $\text{var}(Y_i) = 2/3$. K is a Poisson random variable with parameter $(a+b)t$. Therefore, $\mathbf{E}[K] = \text{var}(K) = (a+b)t$. From the question we know that Y_i and K are independent.

Using results relating to sums of random number of random variables

$$\mathbf{E}[N] = \mathbf{E}[K]\mathbf{E}[Y] = 3(a+b)t$$

$$\text{var}(N) = \mathbf{E}[K]\text{var}(Y) + \text{var}(K)\mathbf{E}[Y]^2 = \frac{29}{3}(a+b)t$$

(f) The easiest way to arrive at this answer is to think about splitting of poisson processes. We can create a new process split from the original process A with arrivals being three-item requests from B.

Note that new process has rate $=$ $\mathbf{P}$(three-item request)*(original request rate) $= \left(\frac{1}{3}a\right)$

Therefore X is simply a fifth order Erlang random variable with parameter $\left(\frac{1}{3}a\right)$

$$f_X(x) = \frac{(\frac{1}{3}a)^5 x^4 e^{-\frac{1}{3}ax}}{4!}, \quad x \geq 0$$

5. (a) K has a Poisson distribution with average arrival time $\mu = \lambda_c T$

$$p_K(k) = \frac{(\lambda_c T)^k e^{-\lambda_c T}}{k!}, \quad k = 0, 1, 2, \ldots; T \geq 0.$$

(b) i. $\mathbf{P}$(conscious response) $= \left(\frac{\lambda_c}{\lambda_c + \lambda_s}\right)$.

ii. $\mathbf{P}$(conscious correct response) $=$ $\mathbf{P}$(conscious resp) $\mathbf{P}$(correct resp| conscious resp) $= \left(\frac{\lambda_c}{\lambda_c+\lambda_s} p_c\right)$.

(c) Since the conscious and subconscious responses are generated independently,

$$\begin{aligned}
&\mathbf{P}(r \text{ conscious responses and } s \text{ subconscious responses in interval } T) \\
&= \mathbf{P}(r \text{ conscious responses in } T)\mathbf{P}(s \text{ unconscious responses in } T) \\
&= \frac{(\lambda_c T)^r e^{-\lambda_c T}}{r!} \cdot \frac{(\lambda_s T)^s e^{-\lambda_s T}}{s!}
\end{aligned}$$

(d) Let X_s = the time from the start of the exam to the time of the 1st subconscious response, and X_c = the time from the 1st subconscious response to the time of the next conscious response.

Note that X_s and X_c are independent exponentially distributed random variables with parameters λ_s and λ_c , respectively.

$$\begin{aligned}
f_{X_s}(x_s) &= \lambda_s e^{-\lambda_s x_s} \text{ when } x_s \geq 0 \\
&= 0 \,\text{otherwise} \\
f_{X_c}(x_c) &= \lambda_c e^{-\lambda_c x_c} \text{ when } x_c \geq 0 \\
&= 0 \,\text{otherwise}
\end{aligned}$$

$X = X_s + X_c$. So its PDF is the convolution of the two exponential distributions. For $x \geq 0$

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} \lambda_s e^{-\lambda_s(x-x_c)} \lambda_c e^{-\lambda_c x_c}\, dx_c \\
&= \int_0^x \lambda_s \lambda_c e^{-\lambda_s x} e^{(\lambda_s-\lambda_c)x_c}\, dx_c \quad \because x - x_c > 0 \\
&= \lambda_s \lambda_c e^{-\lambda_s x} \int_0^x e^{(\lambda_s-\lambda_c)x_c}\, dx_c \\
&= \frac{\lambda_s \lambda_c}{\lambda_s - \lambda_c} e^{-\lambda_s x}(e^{(\lambda_s-\lambda_c)x} - 1) \quad \because \lambda_s \neq \lambda_c \\
&= \frac{\lambda_s \lambda_c}{\lambda_s - \lambda_c}(e^{-\lambda_c x} - e^{-\lambda_s x})
\end{aligned}$$

(e) Let *success* indicate a conscious response and *failure* indicate a subconscious response. Then

$$\mathbf{P}(\textit{success}) = \frac{\lambda_c}{\lambda_c + \lambda_s}, \quad \mathbf{P}(\textit{failure}) = \frac{\lambda_s}{\lambda_c + \lambda_s}.$$

The total number of responses N up to and including the third success is then the third-order inter-arrival time for a Bernoulli process (third-order Pascal):

$$p_N(n) = \binom{n-1}{2} \left(\frac{\lambda_c}{\lambda_c + \lambda_s}\right)^3 \left(\frac{\lambda_s}{\lambda_c + \lambda_s}\right)^{n-3}, \quad n = 3, 4, \ldots.$$

(f) i. $\mathbf{E}(\text{correct answers}) = N \cdot \mathbf{E}(\text{correct on each question}) = N[\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s]$.

ii. We can view this as a binomial experiment where p, the probability of a success, is simply

$$p = \frac{\lambda_c}{\lambda_c + \lambda_s}p_c + \frac{\lambda_s}{\lambda_c + \lambda_s}p_s$$

Therefore,

$$p_L(l) = \binom{N}{l}\left[\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s\right]^l \left[1 - \left(\frac{\lambda_c}{\lambda_c+\lambda_s}p_c + \frac{\lambda_s}{\lambda_c+\lambda_s}p_s\right)\right]^{N-l},$$

where $l = 0, 1, \ldots, N$.

(g) i. $\mathbf{E}(\text{correct answers}) = (\lambda_c p_c + \lambda_s p_s)T$.

ii. Denoting $\lambda = \lambda_c p_c + \lambda_s p_s$ as the average rate of correct responses,

$$p_L(l) = \frac{(\lambda T)^l e^{-\lambda T}}{l!}, \quad l = 0, 1, \ldots$$

G1†. For simplicity, introduce the notation $N_i = N(G_i)$ for $i = 1, ..., n$ and $N_G = N(G)$. Then

$$\begin{aligned}
\mathbf{P}(N_1 = k_1, ..., N_n = k_n | N_G = k) &= \frac{\mathbf{P}(N_1 = k_1, ..., N_n = k_n, N_G = k)}{\mathbf{P}(N_G = k)} \\
&= \frac{\mathbf{P}(N_1 = k_1)\cdots\mathbf{P}(N_n = k_n)}{\mathbf{P}(N_G = k)} \\
&= \frac{\left(\frac{(c_1\lambda)^{k_1}e^{-c_1\lambda}}{k_1!}\right)\cdots\left(\frac{(c_n\lambda)^{k_n}e^{-c_n\lambda}}{k_n!}\right)}{\left(\frac{(c\lambda)^k e^{-c\lambda}}{k!}\right)} \\
&= \frac{k!}{k_1!\ldots k_n!}\left(\frac{c_1}{c}\right)^{k_1}\cdots\left(\frac{c_n}{c}\right)^{k_n} \\
&= \begin{pmatrix} & k & \\ k_1 & \cdots & k_n \end{pmatrix}\left(\frac{c_1}{c}\right)^{k_1}\cdots\left(\frac{c_n}{c}\right)^{k_n}
\end{aligned}$$

The result can be interpreted as a *multinomial distribution.* Imagine we throw an n-sided die k times, where Side i comes up with probability $p_i = c_i/c$. The probability that side i comes up k_i times is given by the expression above. Now relating it back to the Poisson process that we have, each side corresponds to an interval that we sample, and the probability that we sample it depends directly on its relative length. This is consistent with the intuition that, given a number of Poisson arrivals in a specified interval, the arrivals are uniformly distributed.