Recitation 15 Solutions
November 1, 2011

1. (a) Let X_i be the outcome of a coin toss on the i^{th} trial, where $X_i = 1$ if the coin lands 'heads', and $X_i = 0$ if the coin lands 'tails.' By the total probability theorem:

$$\begin{aligned}\mathbf{P}(X_1 = 0) &= \int_0^1 \mathbf{P}(X_1 = 0 \mid Q = q) f(q)\, dq \\ &= \int_0^1 (1-q) 2q\, dq \\ &= \frac{1}{3}.\end{aligned}$$

(b) Let Y_i be a Bernoulli random variable describing the outcome of a coin tossed on morning i. Then, $Y_i = 1$ corresponds to the event that on morning i, Saif goes to the local shelter; $Y_i = 0$ corresponds to the event that on morning i, Saif goes to the mall. Assuming that the coin lands heads with probability q, i.e. that $Q = q$, we have that $\mathbf{P}(Y_i = 1) = q$, and $\mathbf{P}(Y_i = 0) = 1 - q$ for $i = 1, \dots, 30$.
Saif's payout for next 30 days is described by random variable $X = 4(Y_1 + Y_2 + \cdots + Y_{30})$.

$$\begin{aligned}\text{var}(X) &= 16\text{var}(Y_1 + Y_2 + \cdots + Y_{30}) \\ &= 16\text{var}(\mathbf{E}[Y_1 + Y_2 + \cdots + Y_{30} \mid Q]) + \mathbf{E}[\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q)].\end{aligned}$$

Now note that, conditioned on $Q = q$, $Y_1, \dots, Y_{30}$ are independent. Thus, $\text{var}(Y_1 + Y_2 + \cdots + Y_{30} \mid Q) = \text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)$. So,

$$\begin{aligned}\text{var}(X) &= 16\text{var}(30Q) + 16\mathbf{E}[\text{var}(Y_1 \mid Q) + \ldots + \text{var}(Y_{30} \mid Q)] \\ &= 16 \cdot 30^2 \text{var}(Q) + 16 \cdot 30\mathbf{E}[Q(1-Q)] \\ &= 16 \cdot 30^2 (\mathbf{E}[Q^2] - (\mathbf{E}[Q])^2) + 16 \cdot 30(\mathbf{E}[Q] - \mathbf{E}[Q^2]) \\ &= 16 \cdot 30^2(1/2 - 4/9) + 16 \cdot 30(2/3 - 1/2) = 880\end{aligned}$$

since $\mathbf{E}[Q] = \int_0^1 2q^2\, dq = 2/3$ and $\mathbf{E}[Q^2] = \int_0^1 2q^3\, dq = 1/2$.

(c) By Bayes Rule:

$$\begin{aligned}f_{Q|B}(q) &= \frac{\mathbf{P}(B \mid Q = q) f_Q(q)}{\int \mathbf{P}(B \mid Q = q) f_Q(q)\, dq} \\ &= \frac{(1-q^k)2q}{\int_0^1 (1-q^k) 2q\, dq} \\ &= \frac{2q(1-q^k)}{1 - 2/(k+2)} \qquad 0 \le q \le 1.\end{aligned}$$

(d)

$$\begin{aligned}
\mathbf{E}[R] &= \int_0^{10} e^{z+2} f_Z(z)\, dz \\
&= \frac{e^2}{10} \int_0^{10} e^z \, dz \\
&= \frac{e^{12} - e^2}{10}.
\end{aligned}$$

2. (a) Let M_1 be the life time of mac book 1 and M_2 the lifetime of mac book 2, where M_1 and M_2 are iid exponential random variables with CDF $F_M(m) = 1 - e^{-\lambda m}$. T_1, the time of the first mac book failure, is the minimum of M_1 and M_2 . To derive the distribution of T_1, we first find the CDF $F_{T_1}(t)$, and then differentiate to find the PDF $f_{T_1}(t)$.

$$\begin{aligned}
F_{T_1}(t) &= \mathbf{P}(\min(M_1, M_2) < t) \\
&= 1 - \mathbf{P}(\min(M_1, M_2) \geq t) \\
&= 1 - \mathbf{P}(M_1 \geq t)\mathbf{P}(M_2 \geq t) \\
&= 1 - (1 - F_M(t))^2 \\
&= 1 - e^{-2\lambda t} \qquad t \geq 0.
\end{aligned}$$

Differentiating $F_{T_1}(t)$ with respect to t yields:

$$f_{T_1}(t) = 2\lambda e^{-2\lambda t} \qquad t \geq 0.$$

(b) Conditioned on the time of the first mac book failure, the time until the other mac book fails is an exponential random variable by the memoryless property. The memoryless property tells us that regardless of the elapsed life time of the mac book, the time until failure has the same exponential CDF. Consequently,

$$f_{X|T_1}(x) = \lambda e^{-\lambda x} \qquad x \geq 0.$$

(c) Since we have shown in (b) that $f_{X|T_1}(x \mid t)$ does not depend on t, X and T_1 are independent.

(d) The time of the second laptop failure T_2 is equal to $T_1 + X$. Since X and T_1 were shown to be independent in (b), we convolve the densities found in (a) and (b) to determine $f_{T_2}(t)$.

$$\begin{aligned}
f_{T_2}(t) &= \int_0^{\infty} f_{T_1}(\tau) f_X(t - \tau)\, d\tau \\
&= \int_0^t 2(\lambda)^2 e^{-2\lambda\tau} e^{-\lambda(t-\tau)}\, d\tau \\
&= 2\lambda e^{-\lambda t} \int_0^t \lambda e^{-\lambda\tau}\, d\tau \\
&= 2\lambda e^{-\lambda t}(1 - e^{-\lambda t}) \qquad t \geq 0.
\end{aligned}$$

Also, by the linearity of expectation, we have that $\mathbf{E}[T_2] = \mathbf{E}[T_1] + \mathbf{E}[X] = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}$.

An equivalent method for solving this problem is to note that T_2 is the maximum of M_1 and M_2, and deriving the distribution of T_2 in our standard CDF to PDF method:

$$\begin{aligned} F_{T_2}(t) &= \mathbf{P}(\max(M_1, M_2) < t) \\ &= \mathbf{P}(M_2 \leq t)\mathbf{P}(M_2 \leq t) \\ &= (F_M(t))^2 \\ &= 1 - 2e^{-\lambda t} + e^{-2\lambda t} \qquad t \geq 0. \end{aligned}$$

Differentiating $F_{T_2}(t)$ with respect to t yields:

$$f_{T_2}(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t} \qquad t \geq 0$$

which is equivalent to our solution by convolution above.
Finally, from the above density we obtain that $\mathbf{E}[T_2] = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$, which matches our earlier solution.