## LECTURE 4

- **Readings:** Section 1.6

### Lecture outline

- Principles of counting
- Many examples
  - permutations
  - $k$-permutations
  - combinations
  - partitions
- Binomial probabilities

### Discrete uniform law

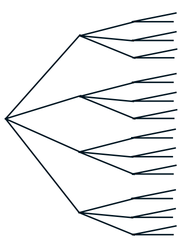
- Let all sample points be equally likely
- Then,

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}} = \frac{|A|}{|\Omega|}$$

- Just count...

### Basic counting principle

- $r$ stages
- $n_i$ choices at stage $i$



- Number of choices is: $n_1 n_2 \cdots n_r$
- Number of license plates with 3 letters and 4 digits $=$
- ... if repetition is prohibited $=$
- **Permutations:** Number of ways of ordering $n$ elements is:
- Number of subsets of $\{1, \ldots, n\}$ $=$

### Example

- Probability that six rolls of a six-sided die all give different numbers?
  - Number of outcomes that make the event happen:
  - Number of elements in the sample space:
  - Answer:

## Combinations

- $\binom{n}{k}$: number of $k$-element subsets of a given $n$-element set
- Two ways of constructing an ordered sequence of $k$ **distinct** items:
  - Choose the $k$ items one at a time: $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ choices
  - Choose $k$ items, then order them ($k!$ possible orders)
- Hence:

$$\binom{n}{k}\cdot k! = \frac{n!}{(n-k)!}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\sum_{k=0}^{n}\binom{n}{k} =$$

## Binomial probabilities

- $n$ independent coin tosses
  - $\mathbf{P}(H) = p$
- $\mathbf{P}(HTTHHH) =$
- $\mathbf{P}(\text{sequence}) = p^{\#\text{ heads}}(1-p)^{\#\text{ tails}}$

$$\mathbf{P}(k\text{ heads}) = \sum_{k-\text{head seq.}} \mathbf{P}(\text{seq.})$$

$$= (\#\text{ of } k-\text{head seqs.})\cdot p^k(1-p)^{n-k}$$

$$= \binom{n}{k}p^k(1-p)^{n-k}$$

## Coin tossing problem

- event $B$: 3 out of 10 tosses were "heads".
  - Given that $B$ occurred, what is the (conditional) probability that the first 2 tosses were heads?
- All outcomes in set $B$ are equally likely: probability $p^3(1-p)^7$
  - Conditional probability law is uniform
- Number of outcomes in $B$:
- Out of the outcomes in $B$, how many start with HH?

## Partitions

- 52-card deck, dealt to 4 players
- Find $\mathbf{P}(\text{each gets an ace})$
- Outcome: a partition of the 52 cards
  - number of outcomes:

$$\frac{52!}{13!\,13!\,13!\,13!}$$

- Count number of ways of distributing the four aces: $4\cdot 3\cdot 2$
- Count number of ways of dealing the remaining 48 cards

$$\frac{48!}{12!\,12!\,12!\,12!}$$

- Answer:

$$\frac{4\cdot 3\cdot 2\,\frac{48!}{12!\,12!\,12!\,12!}}{\frac{52!}{13!\,13!\,13!\,13!}}$$