MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

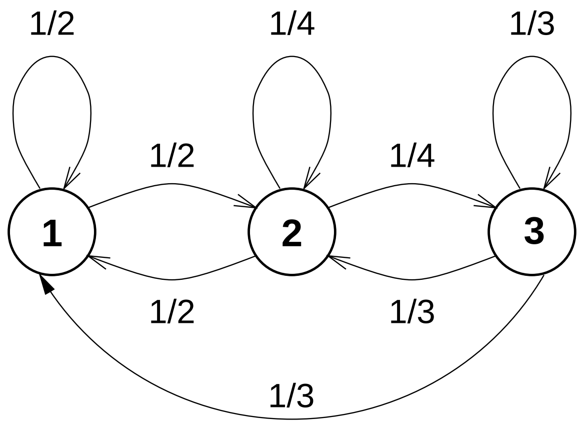
6.041/6.431: Probabilistic Systems Analysis

(Spring 2009)

Problem Set 9

Due: April 29, 2009

1. Consider the following Markov chain:



   (a) Identify the recurrent and transient states.

   (b) Either find the steady state probabilities in this chain, or explain why they do not exist.

   (c) Given the process is presently in state 1, find the expectation and variance for N, the number of transitions up to and including the next transitions on which the process makes a transition out of state 2.

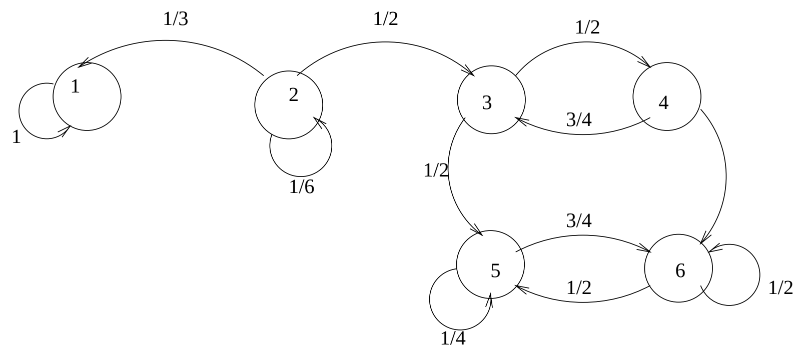
2. Two thimbles (like tiny cups) are under a dripping roof. At the end of each second, thimble A receives 1 drop of water with probability 1, and thimble B receives 1 drop with probability 2/3 and 0 drops otherwise.

   By a complicated automatic mechanism, right before a 4th drop lands in thimble A, both thimbles are emptied. While the thimbles are being emptied, they miss catching the drops that would have otherwise landed inside the thimbles.

   (a) Set up a Markov chain model: identify the states, draw the state transition diagram, and indicate the transition probabilities.

   (b) If both thimbles were empty when you started watching, what is the approximate probability that both thimbles contain exactly 1 drop after exactly 10,001 seconds?

3. Consider the following Markov chain (and note, $p_{46} = 1/4$):



(a) What are the recurrent classes? Are they aperiodic?

(b) For $X_0 = 2$, compute the probabilities that the Markov chain eventually enters each of the recurrent classes.

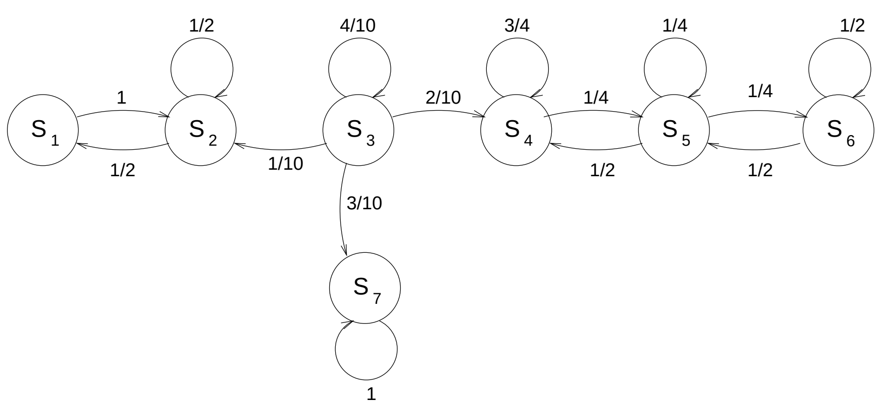
(c) Repeat (b) for $X_0 = 1, 3, 4, 5, 6$.

(d) For all pairs of states (i, j) compute $\lim_{n\to\infty} \mathrm{P}(X_n = j | X_0 = i)$.

(e) Find the expected value and variance of the number of transitions N up to and including the last transition out of state 2 given that the Markov chain starts out in state 2.

(f) Conditional on eventually entering the recurrent class $\{5, 6\}$, find the expected value of the number of transitions M up to and including the transition into the recurrent class given that the Markov chain starts out in state 2.

4. Consider the Markov chain below. For all parts of this problem, the process is in state 3 immediately before the first transition. Be sure to comment on any unusual results.



(a) Find the variance for J, the number of transitions up to and including the transition on which the process leaves state S_3 for the last time.

(b) Find the expectation for K, the number of transitions up to and including the transition on which the process enters state S_4 for the first time.

(c) Find π_{S_i} for $i = 1, 2, \ldots, 7$, the approximate probability that the process is in state i after 10^{10} transitions.

G1†. Consider a Markov chain $\{X_k\}$ on the state space $\{1, \ldots, n\}$, and suppose that whenever the state is i, a reward $g(i)$ is obtained. Let R_k be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state i, let

$$m_k(i) = E[R_k \mid X_0 = i],$$

and

$$v_k(i) = \text{var}(R_k \mid X_0 = i)$$

respectively be the conditional mean and conditional variance of R_k, conditioned on the initial state being i.

(a) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$, allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$ and $v_k(1), \ldots, v_k(n)$, allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint*: Use the law of total variance.

†Required for 6.431; optional for 6.041