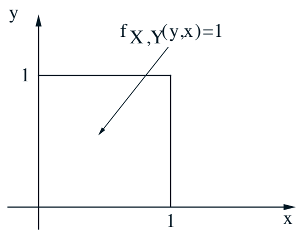
LECTURE 11

Derived distributions; convolution; covariance and correlation

- **Readings:**
  Finish Section 4.1;
  Section 4.2

Example



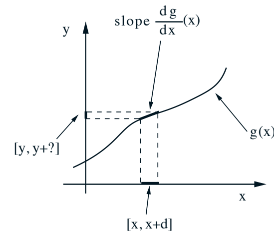
Find the PDF of $Z = g(X, Y) = Y/X$

$F_Z(z) =$ $\qquad\qquad\qquad z \leq 1$

$F_Z(z) =$ $\qquad\qquad\qquad z \geq 1$

A general formula

- Let $Y = g(X)$
  g strictly monotonic.



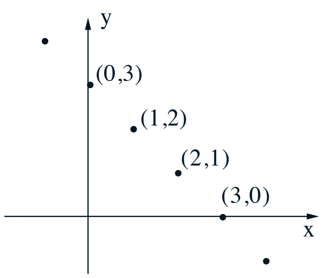
- Event $x \leq X \leq x + \delta$ is the same as
  $g(x) \leq Y \leq g(x + \delta)$
  or (approximately)
  $g(x) \leq Y \leq g(x) + \delta|(dg/dx)(x)|$
- Hence,

$$\delta f_X(x) = \delta f_Y(y) \left|\frac{dg}{dx}(x)\right|$$

where $y = g(x)$

The distribution of $X + Y$

- $W = X + Y$; X, Y independent

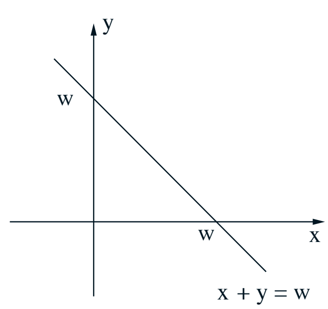


$$\begin{aligned} p_W(w) &= \mathbf{P}(X + Y = w) \\ &= \sum_x \mathbf{P}(X = x)\mathbf{P}(Y = w - x) \\ &= \sum_x p_X(x) p_Y(w - x) \end{aligned}$$

- Mechanics:
  - Put the pmf's on top of each other
  - Flip the pmf of Y
  - Shift the flipped pmf by w (to the right if $w > 0$)
  - Cross-multiply and add

The continuous case

- $W = X + Y$; X, Y independent



- $f_{W|X}(w \mid x) = f_Y(w - x)$
- $f_{W,X}(w, x) = f_X(x) f_{W|X}(w \mid x)$
  $\qquad = f_X(x) f_Y(w - x)$
- $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx$

Two independent normal r.v.s

- $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, independent

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\} \end{aligned}$$

- PDF is constant on the ellipse where
$$\frac{(x-\mu_x)^2}{2\sigma_x^2} + \frac{(y-\mu_y)^2}{2\sigma_y^2}$$
is constant

- Ellipse is a circle when $\sigma_x = \sigma_y$

The sum of independent normal r.v.'s

- $X \sim N(0, \sigma_x^2)$, $Y \sim N(0, \sigma_y^2)$, independent

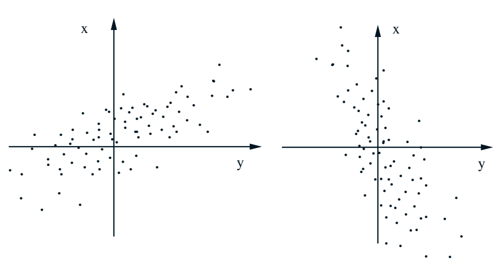
- Let $W = X + Y$

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-(w-x)^2/2\sigma_y^2}\,dx \\ \text{(algebra)} &= ce^{-\gamma w^2} \end{aligned}$$

- Conclusion: W is normal
  - mean=0, variance=$\sigma_x^2 + \sigma_y^2$
  - same argument for nonzero mean case

Covariance

- $\text{cov}(X,Y) = \mathbf{E}\Big[(X - \mathbf{E}[X]) \cdot (Y - \mathbf{E}[Y])\Big]$
- Zero-mean case: $\text{cov}(X,Y) = \mathbf{E}[XY]$



- $\text{cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$
- $\text{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{var}(X_i) + 2 \sum_{(i,j):i \neq j} \text{cov}(X_i, X_j)$
- independent $\Rightarrow$ $\text{cov}(X,Y) = 0$
(converse is not true)

Correlation coefficient

- Dimensionless version of covariance:
$$\begin{aligned} \rho &= \mathbf{E}\left[\frac{(X - \mathbf{E}[X])}{\sigma_X} \cdot \frac{(Y - \mathbf{E}[Y])}{\sigma_Y}\right] \\ &= \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} \end{aligned}$$

- $-1 \leq \rho \leq 1$
- $|\rho| = 1 \;\Leftrightarrow\; (X - \mathbf{E}[X]) = c(Y - \mathbf{E}[Y])$
(linearly related)
- Independent $\Rightarrow$ $\rho = 0$
(converse is not true)