Problem Set 4
Due: March 2, 2011

1. Professor May B. Right often has her science facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture Professor Right is asked either 1 or 2 questions with equal probability.

   (a) What is the probability that Professor Right gives wrong answers to all the questions she gets in a given lecture?

   (b) Given that Professor Right gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she got two questions?

   (c) Let X and Y be the number of questions asked and the number of questions answered correctly in a lecture, respectively. What are the mean and variance of X and the mean and the variance of Y?

   (d) Give a neatly labeled sketch of the joint PMF $p_{X,Y}(x, y)$.

   (e) Let $Z = X + 2Y$. What are the expectation and variance of Z?

   For the remaining parts of this problem, assume that Professor Right has 20 lectures each semester and each lecture is independent of any other lecture.

   (f) The university where Professor Right works has a peculiar compensation plan. For each lecture, she gets paid a base salary of \$1,000 plus \$40 for each question she answers and an additional \$80 for each of these she answers correctly. In terms of random variable Z (defined in part (e)), she gets paid $\$1000 + \$40Z$ per lecture. What are the expected value and variance of her *semesterly* salary?

   (g) Determined to improve her reputation, Professor Right decides to teach an additional 20-lecture class in her specialty (math), where she answers questions incorrectly with probability 1/10 rather than 1/4. What is the expected number of questions that she will answer wrong in a randomly chosen lecture (math or science).

2. Oscar, ecstatic that you've helped him find his lost dog, now seeks your assistance on another problem. At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up either one, two, or three pens, with each of these possibilities being equally likely. If he receives three pens, he does not return to the supply room again that day. If he receives only one or two pens, he will make one additional trip to the supply room, where he again picks up one, two, or three pens, with equal probability.

   Note: The number of pens taken in one trip will not affect the number of pens taken in any other trip. Evaluate:

   (a) $\mathbf{P}(A)$, where A is the event that Oscar picks up a total of three pens on a given day.

   (b) $\mathbf{P}(B|A)$, where B is the event that he visited the supply room twice on the day in question.

   (c) $\mathbf{E}[N]$ and $\mathbf{E}[N|C]$, where N is the total number of pens Oscar picks up any given day, and C is the event that $(N > 3)$).

   (d) $\sigma_{N|C}$, the conditional standard deviation of N given C.

(e) $\mathbf{P}(D)$, where D is the event that he receives more than three pens on *each* of the next 16 days.

3. The president of a company discovers that one of her two vice presidents, A and B is embezzling money from the company. In order to determine who is guilty, she decides to hire a private detective to investigate. If she chooses to investigate VP A she will have to pay D_A to the detective, and if A turns out to be guilty, the president will have to pay R_A to replace A. Similarly, investigating B has costs D_B and R_B. Furthermore, if the detective decides that one of the VP's is innocent, the president will have to pay the detective to investigate the other VP. If the a priori probability that A is guilty is p, and that B is guilty is $1-p$, find the conditions on p, D_A, D_B, R_A, R_B for which investigating A first would minimize the expected cost of the procedure.

4. At Tony's pizza, the following four toppings are available: (1) mushroom, (2) sausage, (3) pepperoni and (4) onion. A random pizza has topping i with probability $p_i = 2^{-i}$ independent of whether that pizza has any other topping and each pizza is ordered independently of every other pizza. On a day in which the number of pizzas sold is n, let N_i equal the number of pizzas sold with topping i. What is the joint PMF $p_{N_1,N_2,N_3,N_4}(n_1, n_2, n_3, n_4)$?

G1†. Prove the following for random variables X and Y.

(a) **Cauchy-Schwarz inequality.** $(\mathbf{E}[XY])^2 \leq \mathbf{E}[X^2]\mathbf{E}[Y^2]$. (Hint: Consider the random variable $Z = X - tY$ where t is a constant.)

(b) If X and Y are positive, independent and identically distributed,

$$\mathbf{E}\left[\frac{X}{Y}\right] \geq 1 \quad .$$

Hint: Use Jensen's Inequality.
(This statement may seem paradoxical: X and Y are identically distributed and therefore, in some sense, of the "same size." But, the result $\mathbf{E}[X/Y] \geq 1$ suggests that X tends to be "larger" than Y; note also that $\mathbf{E}[Y/X] \geq 1$ follows from your proof!)

(c) Define the "indicator function of a set A, $I_A(\omega)$," in the following manner

$$I_A(\omega) = \begin{cases} 1, \omega \in A, \\ 0, \omega \notin A. \end{cases}$$

Show that, if $X = I_A$ then $E[X] = P(A)$.

(d) Show that $\inf_{-\infty<a<\infty} E[(X-a)^2]$ is attained at $a = E[X]$ and consequently

$$\inf_{-\infty<a<\infty} E[(X-a)^2] = var(X).$$

†Required for 6.431; optional for 6.041