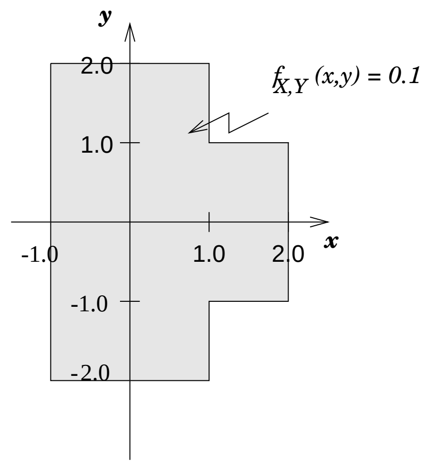
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Spring 2009)

**Problem Set 7**
**Due: April 8, 2009**

1. Random variables $X$ and $Y$ have the joint PDF shown below:




   (a) Prepare neat, fully labeled sketches of $f_{X|Y}(x|y)$.
   (b) Find $\mathbf{E}[X|Y=y]$ and $\text{var}(X|Y=y)$.
   (c) Find $\mathbf{E}[X]$.
   (d) Find $\text{var}(X)$ using the law of total variances.
   (e) Find the distribution $\mathbf{E}[X|Y]$? Is it continuous or discrete?

2. Sambuca bottles are placed into boxes, and boxes are packed into a crate.
   Let $X$ be the number of bottles in any particular box.
   Let $N$ be the number of boxes in a crate.
   $X$ and $N$ are independent identically distributed geometric random variables with the PMF:

$$p_X(u) = p_N(u) = \begin{cases} (\frac{1}{3})(\frac{2}{3})^{u-1} & u = 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases}$$

   Let $T$ be the number of bottles in a crate.

   (a) Find $\mathbf{E}[T]$.
   (b) Find $\text{var}(T)$.
   (c) Find the transform of $T$, $M_T(s)$.
   (d) Find the PMF of $T$, $p_T(t)$.
   (e) Suppose we count the number of boxes in a crate, and we know that $N = n$. Find the least-squares estimate of $T$ given $N = n$.

3. Using a fair three-sided die (construct one, if you dare), we will decide how many times to spin a fair wheel of fortune. The wheel of fortune is calibrated infinitely finely and has numbers between 0 and 1. The die has the numbers 1,2 and 3 on its faces. Whichever number results from our throw of the die, we will spin the wheel of fortune that many times and add the results to obtain random variable $Y$.

(a) Determine the expected value of $Y$.

(b) Determine the variance of $Y$.

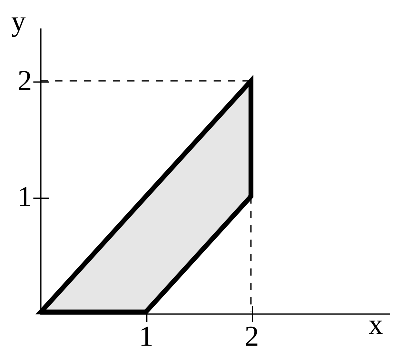
4. Consider three zero-mean random variables $X$, $Y$, and $Z$, with known variances and covariances. Give a formula for the linear least squares estimator of $X$ based on $Y$ and $Z$, that is, find $a$ and $b$ that minimize

$$\mathbf{E}[(X - aY - bZ)^2].$$

For simplicity, assume that $Y$ and $Z$ are uncorrelated.
Hint: Expand the quadratical form and take the partial derivative with respect to $a$ and $b$.

5. Continuous random variables $X$ and $Y$ have a joint PDF given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \text{ belongs to the closed shaded region} \\ 0 & \text{otherwise} \end{cases}$$




(a) Find constant value $c$.

(b) The value of $X$ will be revealed to us; we have to design an estimator $g(X)$ of $Y$ that minimizes the conditional expectation $\mathbf{E}[(Y - g(X))^2 | X = x]$, for all $x$, over all possible estimators. Provide a plot of the optimal estimator as a function of its argument.

(c) Let $g^*(X)$ be the optimal estimator of part (a). Find the numerical value of $\mathbf{E}[g^*(X)]$ and $\text{var}(g^*(X))$?

(d) Find the least mean squared estimation error $\mathbf{E}[(Y - g^*(X))^2]$. Is that the same as $\mathbf{E}[\text{var}(Y \mid X)]$?

(e) Find $\text{var}(Y)$.

(f) Let $l^*(X)$ be the optimal linear LMS estimator. Plot $l^*(X)$ and find the numerical value of $\mathbf{E}[l^*(X)]$ and $\text{var}(l^*(X))$?

(g) The mean squared error of the linear LMS estimator is defined as $\mathbf{E}[(Y - l^*(X))^2]$. Which do you think will be larger, $\mathbf{E}[(Y - g^*(X))^2]$ or $\mathbf{E}[(Y - l^*(X))^2]$. Calculate $\mathbf{E}[(Y - l^*(X))^2]$ and verify your answer.

G1†. If $X$ and $Y$ have joint probability transform function

$$M_{X,Y}(s,t) = \mathbf{E}[e^{sX+tY}] = \exp\left\{\alpha(e^s - 1) + \beta(e^t - 1) + \gamma(e^{s+t} - 1)\right\},$$

find the marginal distributions of $X$, $Y$, and the distribution of $X + Y$, showing that $X$ and $Y$ have the Poisson distribution, but that $X + Y$ does not unless $\gamma = 0$.

†Required for 6.431; optional for 6.041