# 6.041/6.431 Spring 2007 Quiz 2 Solutions
# Wednesday, April 18, 7:30 - 9:30 PM

# DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

**Name:** ____________________

**Recitation Instructor:** ____________________

**TA:** ____________________

| **Question** | | Score | Out of |
|---|---|---|---|
| **0** | | | 3 |
| **1a** | (i) | | 5 |
| | (ii) | | 5 |
| | (iii) | | 6 |
| | (iv) | | 6 |
| **1b** | | | 9 |
| **1c** | | | 9 |
| **1d** | | | 9 |
| **1e** | | | 9 |
| **2a** | (i) | | 6 |
| | (ii) | | 6 |
| | (iii) | | 6 |
| **2b** | (i) | | 7 |
| | (ii) | | 7 |
| **2c** | | | 7 |
| **Your Grade** | | 100 | 100 |

**Problem 1:** Xavier and Wasima are participating in the 6.041 MIT marathon, where race times are defined by random variables[1]. Let $X$ and $W$ denote the race time of Xavier and Wasima respectively. All race times are in hours. Assume the race times for Xavier and Wasima are independent (i.e. $X$ and $W$ are independent). Xavier's race time, $X$, is defined by the following density

$$f_X(x) = \begin{cases} 2c, & \text{if } 2 \le x < 3, \\ c, & \text{if } 3 \le x \le 4, \\ 0, & \text{otherwise,} \end{cases}$$

where $c$ is an unknown constant. Wasima's race time, $W$, is uniformly distributed between 2 and 4 hours. The density of $W$ is then

$$f_W(w) = \begin{cases} \frac{1}{2}, & \text{if } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (i) (5 pts) Find the constant $c$

*Soln:* The plot for the PDF of $X$ is shown in Figure 1. The PDF has to integrate to 1, so the area under $f_X(x)$ is 2c+c, which must equal 1. Therefore c = 1/3.

Integration of the PDF:

$$\begin{aligned} \int_2^4 f_X(x)dx &= 1 \\ \text{which breaks up to } \int_2^3 2cdx + \int_3^4 cdx &= 1 \\ = 2c + c &= 1 \\ \text{and } c &= 1/3. \end{aligned}$$

(ii) (5 pts) Compute $\mathbf{E}[X]$

*Soln:*

$$\begin{aligned} \mathbf{E}[X] &= \int_2^4 x f_X(x)\, dx = \int_2^3 x \cdot 2/3\, dx + \int_3^4 x \cdot 1/3\, dx \\ &= 1/3 \cdot (3^2 - 2^2) + 1/6 \cdot (4^2 - 3^2) = 5/3 + 17/6 \\ &= 17/6. \end{aligned}$$

(iii) (6 pts) Compute $\mathbf{E}[X^2]$

*Soln:*

$$\begin{aligned} \mathbf{E}[X^2] &= \int_2^4 x^2 f_X(x)\, dx = \int_2^3 x^2 \cdot 2/3\, dx + \int_3^4 x^2 \cdot 1/3\, dx \\ &= 2/9 \cdot (3^3 - 2^3) + 1/9 \cdot (4^3 - 3^3) = 38/9 + 37/9 \\ &= 25/3. \end{aligned}$$

[1]A runner's race time is defined as the time required for a given runner to complete the marathon.

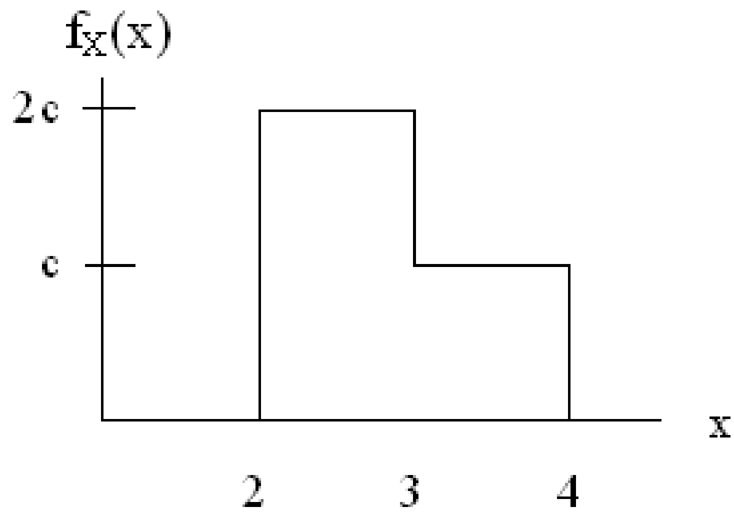


Figure 1: PDF of X

(iv) (6 pts) Provide a fully labeled sketch of the PDF of $2X + 1$
*Soln:* Let $Y = 2X + 1$. The range of $Y$ is not from 2 to 4, but now $5 \leq y \leq 9$. The shape of the PDF of $Y$ should look like the PDF of $X$, but scaled by a factor such that it normalizes to 1. The range of $Y$ is double the range of $X$, so the density is half. Plot shown below in Figure 2.

Since $Y = g(X)$ is a linear function of $X$, we can use the formula for the derived distribution for a linear function. $Y = 2X + 1$, so $f_Y(y) = \frac{1}{2} f_X(\frac{y-1}{2})$ for $5 \leq y \leq 9$. Figure 2 matches this distribution.

(b) (9 pts) Compute $\mathbf{P}(X \leq W)$.
*Soln:*

First we calculate the joint PDF. It should have a non-zero joint density for the region, $2 \leq x \leq 4$ and $2 \leq w \leq 4$. However, it is not uniform within this entire square, as we have seen often in class. Due to the piece-wise uniform density of $X$, the square is partitioned into two rectangles of uniform joint densities. $X$ and $W$ are independent, so the joint density is just the product of the marginals.

$$
\begin{aligned}
f_{X,W}(x, w) &= f_X(x) f_W(w) \\
&= f_X(x) \cdot 1/2 \\
&= \begin{cases} c1 = 2/3 \cdot 1/2 = 1/3 & , \quad 2 \leq x \leq 3, 2 \leq w \leq 4. \\ c2 = 1/3 \cdot 1/2 = 1/6 & , \quad 3 \leq x \leq 4, 2 \leq w \leq 4. \end{cases}
\end{aligned}
$$

Variables $c1$ and $c2$ are used to denote the different joint densities, and are shown in the joint plot.

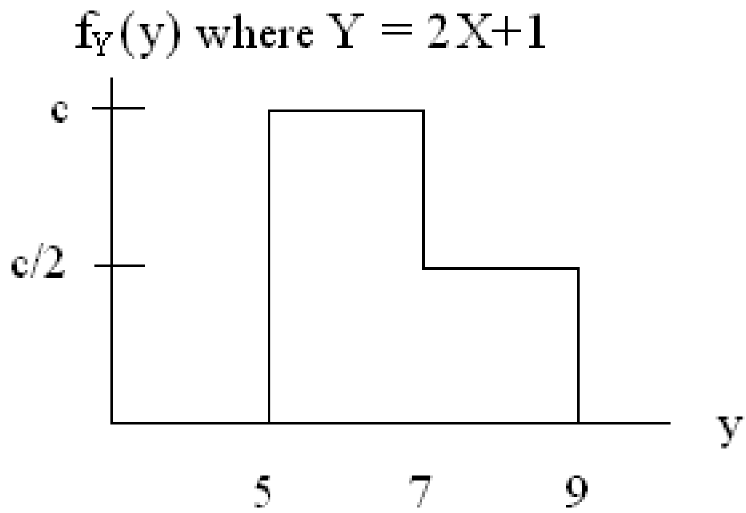


Figure 2: PDF of $Y = 2X + 1$

As a check, the joint PDF should be normalized to 1, which it is.
The joint PDF for $X$ and $W$ is shown in Figure 3

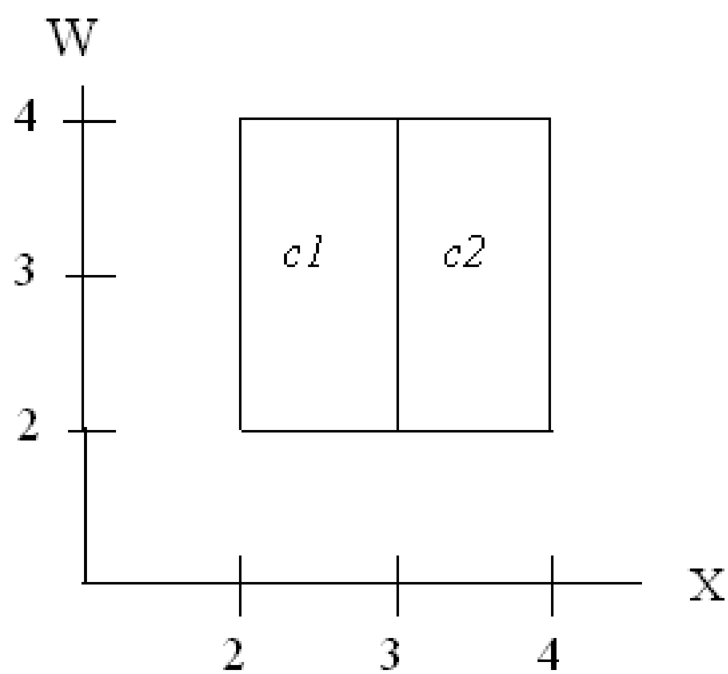


Figure 3: Joint PDF of X and W

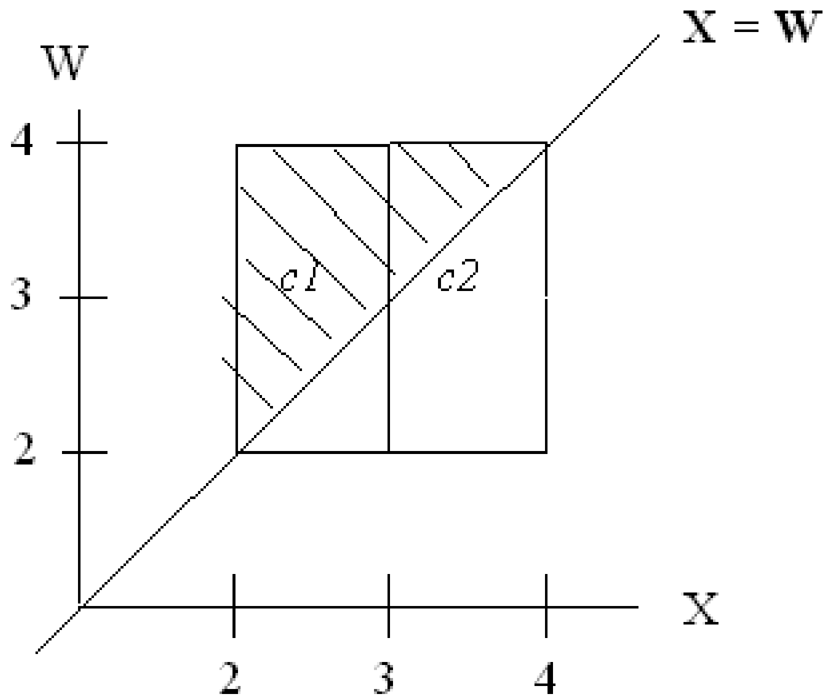


Figure 4: $\mathbf{P}(X \leq W)$

Looking at the plot of the joint PDF, $\mathbf{P}(X \leq W)$ is the region above the $X = W$ line. See Figure 4.

We calculate the weighted area of the shaded region to be:

$$\begin{aligned} \mathbf{P}(X \leq W) &= 1/2 \cdot 1/6 + 3/2 \cdot 1/3 = 1/12 + 1/2 \\ &= 7/12. \end{aligned}$$

The graphical way is the easy solution. Of course, one can integrate:

$$\begin{aligned} \mathbf{P}(X \leq W) &= \int_2^3 \int_x^4 1/3 \; dwdx + \int_3^4 \int_x^4 1/6 \; dwdx \\ &= \frac{1}{3} \int_2^3 (4 - x) \; dx + \frac{1}{6} \int_3^4 (4 - x) \; dx \\ &= 7/12 \end{aligned}$$

(c) (9 pts) Wasima is using a stopwatch to time herself. However, the stopwatch is faulty; it over-estimates her race time by an amount that is uniformly distributed between 0 and $\frac{1}{10}$

hours, which is independent of the actual race time. Thus, if $T$ is the time measured by the stopwatch, then we have

$$f_{T|W}(t|w) = \begin{cases} 10, & \text{if } w \le t \le w + \frac{1}{10} \text{ and } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{W|T}(w|t)$, when $t = 3$.
*Soln:* Be careful here, that $T$ is the race time measured by the stopwatch, not just the over-estimated race time.

$$\begin{aligned}
f_{W|T}(w|3) &= \frac{f_{W,T}(w,3)}{f_T(3)} \\
\text{where } f_{W,T}(w,3) &= f_{T|W}(3|w) f_W(w) = 10 \cdot 1/2 = 5 \text{ for } (3 - 1/10) \le w \le 3. \\
\text{and } f_T(3) &= \int_{3-1/10}^{3} f_{W,T}(w,3)\, dw = 5 \cdot (1/10) = 1/2. \\
\text{Therefore,} & \\
f_{W|T}(w|3) &= \begin{cases} 10, & \text{if } (3 - 1/10) \le w \le 3 \text{ and } t = 3, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(d) (9 pts) Wasima realizes her stopwatch is faulty and buys a new stopwatch. Unfortunately, the new stopwatch is also faulty; this time, the watch adds random noise $N$ that is normally distributed with mean $\mu = \frac{1}{60}$ hours and variance $\sigma^2 = \frac{4}{3600}$. Find the probability that the watch over-estimates the actual race time by more than 5 minutes, $\mathbf{P}(N > \frac{5}{60})$. For full credit express your final answer as a number.
*Soln:* $N$ is Normal(1/60, 4/3600). We standardize $N$ to have mean 1 and standard deviation 1 to utilize the Normal table.

$$\begin{aligned}
\mathbf{P}(N > \frac{5}{60}) &= 1 - \mathbf{P}(N < \frac{5}{60}) \\
&= 1 - \mathbf{P}(\frac{N - 1/60}{2/60} < \frac{5/60 - 1/60}{2/60}) \\
&= 1 - \Phi(2).
\end{aligned}$$

If we looked it up, $\Phi(2) = 0.9772$, so $\mathbf{P}(N > \frac{5}{60}) = 1$ - $0.9772 = 0.0028$.

(e) (9 pts) Wasima has a sponsor for the marathon! If Wasima finishes the marathon in $w$ hours, the sponsor pays her $\frac{24}{w}$ thousand dollars. Define

$$S = \frac{24}{W}$$

Find the PDF of $S$.
*Soln:* Use derived distributions to find the CDF of $S$, then differentiate with respect to $s$ to find the PDF of $S$.

The range of $S$ is determined from the range of $W$. Since $2 \leq w \leq 4$ for a nonzero PDF of $W$, $24/4 \leq s \leq 24/2$ for a nonzero PDF of $S$.

$$\begin{aligned}
\mathbf{P}(S \leq s) &= \mathbf{P}(24/W \leq s) = \mathbf{P}(W \geq 24/s) \\
&= 1 - F_W(24/s) = 1 - \int_2^{24/s} f_W(w)dw \\
&= 1 - (12/s - 1) = 2 - 12/s
\end{aligned}$$

Taking the derivative with respect to s,

$$\begin{aligned}
f_S(s) &= \frac{d}{ds}(2 - 12/s) \\
&= \begin{cases} 12/s^2, & \text{if } 6 \leq s \leq 12 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

**Problem 2.** Consider the following family of **independent** random variables $N, A_1, B_1, A_2, B_2, \ldots$, where $N$ is a nonnegative discrete random variable and each $A_i$ or $B_i$ is normal with mean 1 and variance 1. Let $A = \sum_{i=1}^{N} A_i$ and $B = \sum_{i=1}^{N} B_i$. Recall that the sum of a fixed number of independent normal random variables is normal.

(a) Assume $N$ is geometrically distributed with a mean of $1/p$.

(i) (6 pts) Find the mean, $\mu_a$, and the variance, $\sigma_a^2$, of $A$.
*Soln:* This is a random sums problem so the mean and variance of $A$ is found using the laws of iterated expectations and total variance.

$$\begin{aligned}
\mu_a &= \mathbf{E}[A] = \mathbf{E}[\mathbf{E}[A|N]] = \mathbf{E}[N\mathbf{E}[A_i]] = \mathbf{E}[A_i]\mathbf{E}[N] \\
&= 1/p. \\
\sigma_a^2 &= Var(A) = \mathbf{E}[Var(A|N)] + Var(\mathbf{E}[A|N]) = \mathbf{E}[N\ Var(A_i)] + Var(N\mathbf{E}[A_i]) \\
&= Var(A_i)\mathbf{E}[N] + \mathbf{E}[A_i]^2 Var(N) = 1/p + (1-p)/p^2 \\
&= 1/p^2.
\end{aligned}$$

(ii) (6 pts) Find $c_{ab}$, defined by $c_{ab} = \mathbf{E}[AB]$.
*Soln:* It is important that although $A_i$ and $B_i$ are both Normal, they are NOT the same. Having the same distribution does not make two random variables *equal* to one another. For those who set $A = B$, it is incorrect.
Also, $A$ and $B$ are not independent... they BOTH depend on a random variable $N$! However, if $N$ is known, then $A$ and $B$ become independent, which is what we make use of in this problem. The trick here is to use iterated expectations, because we are working with random sums.

$$\begin{aligned}
c_{ab} = \mathbf{E}[AB] &= \mathbf{E}[(A_1 + A_2 + A_3 + \ldots A_N)(B_1 + B_2 + B_3 + \ldots B_N)] \\
&= \mathbf{E}[\mathbf{E}[(A_1 + A_2 + A_3 + \ldots A_N)(B_1 + B_2 + B_3 + \ldots B_N)|N]] \\
&= \mathbf{E}[N\mathbf{E}[A_i]N\mathbf{E}[B_i]] = \mathbf{E}[N^2\mathbf{E}[A_i]\mathbf{E}[B_i]] = \mathbf{E}[A_i]\mathbf{E}[B_i]\mathbf{E}[N^2] \\
&= 1 \cdot 1 \cdot (Var(N) + \mathbf{E}[N]^2) = (1-p)/p^2 + 1/p^2 \\
&= (2-p)/p^2.
\end{aligned}$$

Many of you wrote that $\mathbf{E}[AB] = cov(A, B) + \mathbf{E}[A]\mathbf{E}[B]$. This is technically correct, but the calculation was usually incorrect. $cov(A, B)$ is not 1 or 0 (common assumption, not to be confused with $cov(A_i, B_j) = 0$). It falls out that the covariance between $A$ and $B$ is $(1-p)/p^2 = Var(N)$. You can derive this from the law of total variance.

(iii) (6 pts) We observe $B$ (but not $N$) and wish to estimate $A$, using a linear estimator of the form $c + dB$, where $c$ and $d$ are constants. Find values for $c$ and $d$ that result in the smallest possible mean squared error. Express your answer in terms of constants such as $\mu_a$, $c_{ab}$, etc., without plugging in the values found in parts (i) and (ii).
*Soln:* Given $B$, the linear estimator for $A$ is

$$\hat{A} = \mathbf{E}[A] + \rho_{A,B}\frac{\sigma_A}{\sigma_B}(B - \mathbf{E}[B]).$$

Therefore the constants $c$ and $d$ are:

$$
\begin{aligned}
c &= \mathbf{E}[A] - \rho_{A,B}\frac{\sigma_A}{\sigma_B}\mathbf{E}[B], \text{ and} \\
d &= \rho_{A,B}\frac{\sigma_A}{\sigma_B}. \\
\text{where } \rho_{A,B} &= \frac{cov(A,B)}{\sigma_A\sigma_B} \\
&= \frac{\mathbf{E}[AB] - \mathbf{E}[A]\mathbf{E}[B]}{\sigma_A\sigma_B}.
\end{aligned}
$$

Plugging everything in,

$$
\begin{aligned}
c &= \mu_a - \frac{(c_{ab} - \mu_a\mu_b)}{\sigma_b^2}\mu_b. \\
d &= (c_{ab} - \mu_a\mu_b)/\sigma_b^2.
\end{aligned}
$$

(b) Now assume that $N$ can take only the values 1 (with probability 1/3) and 2 (with probability 2/3).

(i) (7 pts) Give a formula for the PDF of $A$.
*Soln:* If $N = 1$, $A = A_1$, which has a Normal distribution with mean 1 and variance 1.
If $N = 2$, $A = A_1 + A_2$, which is the sum of two Normals. Therefore the distribution of $A$ is Normal$(1+1{,}1+1)$ or Normal(2,2).
Using total probability theorem, we find:

$$
\begin{aligned}
f_A(a) &= f_{A|N=1}(a)P_N(1) + f_{A|N=2}(a)P_N(2) \\
&= \text{Normal(1,1)} \cdot 1/3 + \text{Normal(2,2)} \cdot 2/3 \\
&= \frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2} + \frac{2}{3\sqrt{4\pi}}e^{-(a-2)^2/4}.
\end{aligned}
$$

(ii) (7 pts) Find the conditional probability $\mathbf{P}(N = 1 \mid A = a)$.
*Soln:* This is simply:

$$
\begin{aligned}
\mathbf{P}(N = 1 \mid A = a) &= \frac{\mathbf{P}(A = a, N = 1)\delta}{\mathbf{P}(A = a)\delta}. \\
\text{where } \mathbf{P}(A = a)\delta &= f_A(a) \text{ was found in part (a)} \\
\text{and the joint is } P(A = a)P(N = 1)\delta &= f_A(a)P_N(1). \\
\text{Then, } \mathbf{P}(N = 1 \mid A = a) &= \frac{\frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2}}{\frac{1}{3\sqrt{2\pi}}e^{-(a-1)^2/2} + \frac{2}{3\sqrt{4\pi}}e^{-(a-2)^2/4}}.
\end{aligned}
$$

(c) (7 pts) Is it true that $\mathbf{E}[A \mid N] = \mathbf{E}[A \mid B, N]$ ? Either provide a proof, or an explanation why the equality does not hold.
*Soln:* Yes they are equal.

As a first check, they are both random variables. $A$ and $B$ are not independent from one another because they both depend on the RV $N$ for the random sum. But, if we condition on $N$, then $A$ and $B$ are independent (hence they are conditionally independent). Is that what the right side of the equation states?
These expectations are equal if the PDFs of $A|N$ and $A|(B,N)$ are equal. Once $N$ is known, knowing $B$ doesn't change what one knows about $A$, so this not only shows that $A$ and $B$ are conditionally independent, given $N$, but $A|N$ has the same information as $A, B|N$.

Conditional independence of events $X$ and $Y$ on $Z$ is defined as:

$$\begin{aligned} \mathbf{P}(X \cap Y|Z) = \mathbf{P}(X|Z)\mathbf{P}(Y|Z) \\ \text{or, equivalently} \\ \mathbf{P}(X|Y \cap Z) = \mathbf{P}(X|Z) \end{aligned}$$

Therefore, we show that the equality holds here.

$$\begin{aligned} \mathbf{E}[A \mid N] &= \mathbf{E}[A \mid B, N] \\ \int a f_{A|N}(a|n)\ da &= \int a f_{A|B,N}(a|b,n)\ da \end{aligned}$$

The above statement can be shown to be equal if the probabilities are shown to be the same or the PDFs are derived to be equal:

$$\begin{aligned} f_{A|N}(a|n) &= f_{A|B,N}(a|b,n) = \frac{f_{A,B,N}(a,b,n)}{f_{B,N}(b,n)} \\ &= \frac{f_{A,B|N}(a,b|n)P_N(n)}{f_{B|N}(b|n)P_N(n)} = \frac{f_{A|N}(a|n)f_{B|N}(b|n)}{f_{B|N}(b|n)} \\ &= f_{A|N}(a|n). \end{aligned}$$

So $\mathbf{E}[A \mid N] = \mathbf{E}[A \mid B, N]$ is true.