6.041/6.431 Fall 2010 Quiz 1
Tuesday, October 12, 7:30 - 9:00 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
1.1		10
1.2		10
1.3		10
1.4		10
1.5		5
1.6		10
1.7		10
1.8		10
2.1		10
2.2		10
2.3		10
Your Grade		105

- This quiz has 2 problems, worth a total of 105 points.
- You may tear apart pages 3, 4 and 5, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- **You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of** numbers **that could be evaluated using a calculator.** Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 90 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/14.

Problem 0: (0 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Time
Vivek Goyal	Uzoma Orji	10 & 11 AM
Peter Hagelstein	Ahmad Zamanian	12 & 1 PM
Ali Shoeb	Shashank Dwivedi	2 PM
Dimitri Bertsekas (6.431)	Aliaa Atwi	2 & 3 PM

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Problem 1: (75 points)

Note: All parts can be done independently, with the exception of the last part. Just in case you made a mistake in the previous part, you can use a symbol for the expression you found there, and use that symbol in the formulas for the last part.

Note: **Algebraic or numerical expressions do not need to be simplified in your answers.**

Jon and Stephen cannot help but think about their commutes using probabilistic modeling. Both of the them start promptly at 8am.

Stephen drives and thus is at the mercy of traffic lights. When all traffic lights on his route are green, the entire trip takes 18 minutes. Stephen's route includes 5 traffic lights, each of which is red with probability 1/3, independent of every other light. Each red traffic light that he encounters adds 1 minute to his commute (for slowing, stopping, and returning to speed).

1. **(10 points)** Find the PMF, expectation, and variance of the length (in minutes) of Stephen's commute.
2. **(10 points)** Given that Stephen's commute took him at most 19 minutes, what is the expected number of red lights that he encountered?
3. **(10 points)** Given that the last red light encountered by Stephen was the fourth light, what is the conditional variance of the total number of red lights he encountered?
4. **(10 points)** Given that Stephen encountered a total of three red lights, what is the probability that exactly two out of the first three lights were red?

Jon's commuting behavior is rather simple to model. Jon walks a total of 20 minutes from his home to a station and from a station to his office. He also waits for X minutes for a subway train, where X has the discrete uniform distribution on $\{0, 1, 2, 3\}$. (All four values are equally likely, and independent of the traffic lights encountered by Stephen.)

5. **(5 points)** What is the PMF of the length of Jon's commute in minutes?
6. **(10 points)** Given that there was exactly one person arriving at **exactly** 8:20am, what is the probability that this person was Jon?
7. **(10 points)** What is the probability that Stephen's commute takes at most as long as Jon's commute?
8. **(10 points)** Given that Stephen's commute took at most as long as Jon's, what is the conditional probability that Jon waited 3 minutes for his train?

Problem 2. (30 points) For each one of the statements below, give either a proof or a counterexample showing that the statement is not always true.

1. **(10 points)** If events A and B are independent, then the events A and B^c are also independent.
2. **(10 points)** Let A, B, and C be events associated with a common probabilistic model, and assume that $0 < \mathbf{P}(C) < 1$. Suppose that A and B are conditionally independent given C. Then, A and B are conditionally independent given C^c.

3. **(10 points)** Let X and Y be independent random variables. Then, $\mathrm{var}(X+Y) \geq \mathrm{var}(X)$.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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(Additional space for Problem 2.2)

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