LECTURE 4

- **Readings:** Section 1.6

Lecture outline

- Review
- Counting
  - Permutations
  - k-permutations
  - Combinations
  - Partitions
- Binomial probabilities

Review

- Events A_1 and A_2 are called **independent** when

$$\mathbf{P}(A_1 \cap A_2) \;=\; \mathbf{P}(A_1)\mathbf{P}(A_2)$$

- Events $A_1, A_2, \ldots, A_n$ are called **independent** when

$$\mathbf{P}(A_i \cap A_j \cap \cdots \cap A_q) \;=\; \mathbf{P}(A_i)\mathbf{P}(A_j)\cdots\mathbf{P}(A_q)$$

  for **every** subset $\{i, j, \ldots, q\} \subset \{1, 2, \ldots, n\}$.
  - No shortcuts: Checking subsets of size 2 not enough

Discrete uniform law

- Often all outcomes are equally likely
- Then,

$$\mathbf{P}(A) \;=\; \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} \;=\; \frac{|A|}{|\Omega|}$$

- Problem is "merely" to count

Basic counting principle

- Analyze as a sequence of stages:
  - r stages
  - n_i choices at stage i [same number at each stage]
  - Number of choices is $n_1 n_2 \cdots n_r$

Example

- Number of license plates with 3 letters and 4 digits
  - letters first, then digits:
  - letters first, then digits; no repetitions:
  - any order; no repetitions:

Number of subsets

- What are the subsets of $\{a, b, c\}$?
- How many subsets does $\{1, 2, \ldots, n\}$ have?

Permutations

- What are the orderings of $\{a, b, c\}$?
- How many ways can $\{1, 2, \ldots, n\}$ be ordered?

Example

- Probability that six rolls of a six-sided die all give different numbers?
  - Number of outcomes that make the event happen:
  - Number of elements in the sample space:
  - Answer:

k-permutations

- What are the ordered 2-tuples formed from distinct elements of $\{a, b, c, d\}$?
- Starting with n objects, how many ordered lists of k distinct objects can be formed?

Combinations

- What are the 2-element subsets of $\{a, b, c, d\}$?
- Denote the number of k-element subsets of an n-element set by $\binom{n}{k}$. What is this number?

Binomial probabilities

- n independent coin tosses with $\mathbf{P}(\text{H}) = p$
  - $\mathbf{P}(\text{HTTHHH}) =$
  - $\mathbf{P}(\text{sequence}) = p^{\#\text{ heads}}(1-p)^{\#\text{ tails}}$
  - $\mathbf{P}(k \text{ Hs}) = \sum_{k-\text{H seq.}} \mathbf{P}(\text{seq.})$

$$= (\#\text{ of } k\text{−H seqs.}) \cdot p^k (1-p)^{n-k}$$
$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Coin tossing problem

- Event B: 3 out of 10 tosses H
  - Given B, what is the conditional probability that the first 2 tosses are HH?

$$\frac{\mathbf{P}(B \cap \{\text{first two HH}\})}{\mathbf{P}(B)} =$$

- All outcomes in B are equally likely: probability $p^3(1-p)^7$
  - Conditional probability law is uniform
- Number of outcomes in B:
- Out of the outcomes in B, how many start with HH?

Partitions

- How many ways can a 52-card deck be dealt to four players?
- Find $\mathbf{P}(\text{each gets an ace})$