Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
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Recitation 5: Solutions
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1. Arrange the original set of n objects in a sequence, and arrange the subsets as r distinct slots in the sequence. Then there are $n!$ sequences in which the objects can be arranged, but the various permutations within each subset yield the same partition. Thus (by the "Counting Principle" on page 45 of the text) we divide the number of sequences, $n!$, by the number of distinct sequences which yield the same collection of subsets, $n_1!n_2!\cdots n_r!$.

2. The probability of each sequence of rolls where face k comes up y_k times is $p_1^{y_1}p_2^{y_2}\cdots p_r^{y_r}$. Every such sequence determines a partition of the set of n rolls into r subsets, with the kth subset having cardinality y_k. The number of these partitions is the multinomial coefficient
$$\binom{n}{y_1, y_2, \ldots, y_r}.$$
Therefore the probability is
$$\binom{n}{y_1, y_2, \ldots, y_r} p_1^{y_1}p_2^{y_2}\cdots p_r^{y_r}.$$

3. *The hats of n persons are thrown into a box. The persons then pick up their hats at random (i.e., so that every assignment of the hats to the persons is equally likely). What is the probability that*

   (a) *every person gets his or her hat back?*
   Answer: $\frac{1}{n!}$.
   Solution: consider the sample space of all possible hat assignments. It has $n!$ elements (n hat selections for the first person, after that $n-1$ for the second, etc.), with every single-element event equally likely (hence having probability $1/n!$). The question is to calculate the probability of a single-element event, so the answer is $1/n!$

   (b) *the first m persons who picked hats get their own hats back?*
   Answer: $\frac{(n-m)!}{n!}$.
   Solution: consider the same sample space and probability as in the solution of (a). The probability of an event with $(n-m)!$ elements (this is how many ways there are to disribute the remaining $n-m$ hats after the first m are assigned to their owners) is $(n-m)!/n!$

   (c) *everyone among the first m persons to pick up the hats gets back a hat belonging to one of the last m persons to pick up the hats?*
   Answer: $\frac{m!(n-m)!}{n!} = \frac{1}{\binom{n}{m}} = \frac{1}{\binom{n}{n-m}}.$.
   Solution: there are $m!$ ways to distribute m hats among the first m persons, and $(n-m)!$ ways to distribute the remaining $n-m$ hats. The probability of an event with $m!(n-m)!$ elements is $m!(n-m)!/n!$.

   Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). What is the probability that

(d) *the first m persons will pick up clean hats?*

Answer: $(1-p)^m$.

Solution: the probability of a given person picking up a clean hat is $1-p$. By the independence assumption, the probability of m selected persons picking up clean hats is $(1-p)^m$.

(e) *exactly m persons will pick up clean hats?*

Answer: $(1-p)^m p^{n-m} \binom{n}{m}$.

Solution: every group G of m persons defines the event "everyone from G picks up a clean hat, everyone not from G picks up a dirty hat". The events are disjoint. Each has probability $(1-p)^m p^{n-m}$. Since there are $\binom{n}{m}$ such events, the answer follows.

4. Think of lining up the jelly beans, by first placing the red ones, then the orange ones, etc. We also place 5 dividers to indicate where one color ends and another starts. (Note that two dividers can be adjacent if there are no jelly beans of some color.) By considering both jelly beans and dividers, we see that there is a total of 105 positions. Choosing the number of jelly beans of each color is the same as choosing the positions of the dividers. Thus, there are $\binom{105}{5}$ possibilities, and this is the number of possible jars.