Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Recitation 19: Solutions
November 15, 2011

1. a) The number of remaining green fish at time n completely determines all the relevant information of the system's entire history (relevant to predicting the future state.) Therefore it is immediate that the number of green fish is the state of the system and the process has the Markov property:

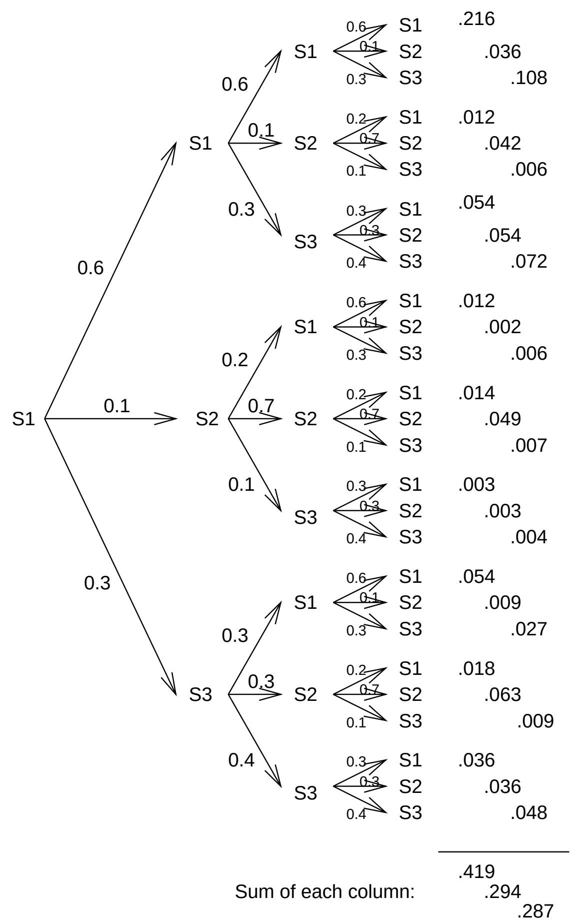
$$\mathbf{P}(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \ldots, X_1 = i_1) = \mathbf{P}(X_{m+1} = j | X_m = i).$$

b) For $j > i$ clearly $p_{ij} = 0$, since a blue fish will never be painted green. For $0 \leq i, j \leq k$, we have the following:

$$p_{ij} = \mathbf{P}(i \to j \text{ green fish are caught} | \text{current state} = i) = \begin{cases} \frac{n-i}{n} & j = i \\ \frac{i}{n} & j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

c) The state 0 is an absorbing state since there is a positive probability that the system will enter it, and once it does, it will remain there forever. Therefore the state with 0 green fish is the only recurrent state, and all other states are then transient.

2. Starting in state 1, we can draw a tree:



The events with numbers in the first column correspond to events in $r_{11}(3)$; the events with numbers in the second column correspond to events in $r_{12}(3)$; and the events with numbers in the third column correspond to events in $r_{13}(3)$. Therefore, $r_{11}(3) = .419$, $r_{12}(3) = .294$, and $r_{13}(3) = .287$.

Now using $r_{ij}(n+1) = \sum_{k=1}^{k=3} r_{ik}(n)p_{kj}$, where $r_{11}(1) = p_{11} = .6$, $r_{12}(1) = p_{12} = .1$, and $r_{13}(1) = p_{13} = .3$, we have

$$\begin{aligned}
r_{11}(2) &= \sum_{k=1}^{k=3} r_{1k}(1)p_{k1} = (.6)(.6) + (.1)(.2) + (.3)(.3) = .47 \\
r_{12}(2) &= \sum_{k=1}^{k=3} r_{1k}(1)p_{k2} = (.6)(.1) + (.1)(.7) + (.3)(.3) = .22 \\
r_{13}(2) &= \sum_{k=1}^{k=3} r_{1k}(1)p_{k3} = (.6)(.3) + (.1)(.1) + (.3)(.4) = .31
\end{aligned}$$

$$\begin{aligned}
r_{11}(3) &= \sum_{k=1}^{k=3} r_{1k}(2)p_{k1} = (.47)(.6) + (.22)(.2) + (.31)(.3) = .419 \\
r_{12}(3) &= \sum_{k=1}^{k=3} r_{1k}(2)p_{k2} = (.47)(.1) + (.22)(.7) + (.31)(.3) = .294 \\
r_{13}(3) &= \sum_{k=1}^{k=3} r_{1k}(2)p_{k3} = (.47)(.3) + (.22)(.1) + (.31)(.4) = .287
\end{aligned}$$

These numbers agree with our previous results.

3. (a) Let A_k be the event that the process enters s_2 for first time on trial k. The only way to enter state s_2 for the first time on the kth trial is to enter state s_3 on the first trial, remain in s_3 for the next $k-2$ trials, and finally enter s_2 on the last trial. Thus,

$$\mathbf{P}(A_k) = p_{03} \cdot p_{33}^{k-2} \cdot p_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1} \qquad \text{for} \qquad k = 2, 3, \ldots$$

(b) Let A be the event that the process never enters s_4.

There are three possible ways for A to occur. The first two are if the first transition is either from s_0 to s_1 or s_0 to s_5. This occurs with probability $\frac{2}{3}$. The other is if The first transition is from s_0 to s_3, and that the next change of state *after* that is to the state s_2. We know that the probability of going from s_0 to s_3 is $\frac{1}{3}$. Given this has occurred, and given a change of state occurs from state s_3, we know that the probability that the state transitioned to is the state s_2 is simply $\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$. Thus, the probability of transitioning from s_0 to s_3 and then eventually transitioning to s_2 is $\frac{1}{9}$. Thus, the probability of never entering s_4 is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$.

(c) $\mathbf{P}(\{\text{process enters } s_2 \text{ and then leaves } s_2 \text{ on next trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{process enters } s_2\})\mathbf{P}(\{\text{leaves } s_2 \text{ on next trial }\}|\{\text{ in } s_2\}) \\
&= \left[\sum_{k=2}^{\infty}\mathbf{P}(A_k)\right]\cdot\frac{1}{2} \\
&= \left[\sum_{k=2}^{\infty}\frac{1}{3}(\frac{1}{4})^{k-1}\right]\cdot\frac{1}{2} \\
&= \frac{1}{6}\cdot\frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{18}.
\end{aligned}$$

(d) This event can only happen if the sequence of state transitions is as follows:

$$s_0 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1.$$

Thus, $\mathbf{P}(\{\text{process enters } s_1 \text{ for first time on third trial}\}) = p_{03}\cdot p_{32}\cdot p_{21} = \frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{24}$.

(e) $\mathbf{P}(\{\text{process in } s_3 \text{ immediately after the } N\text{th trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{moves to } s_3 \text{ in first trial and stays in } s_3 \text{ for next } N-1 \text{ trials}\}) \\
&= \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

(f) The transient state for the chain are s_0, s_2, s_3, and s_4. Note that for $n \geq 1$

$$\begin{aligned}
q_2(n) &= q_4(n) \\
&= \left(\frac{1}{2}\right)^n
\end{aligned}$$

$$\begin{aligned}
q_0(n) &= \mathbf{P}\left(X_n \text{ transient}|X_0 = s_0\right) \\
&= \frac{1}{3}\mathbf{P}\left(X_n \text{ transient}|X_1 = s_3\right) \\
&= \frac{1}{3}q_3(n-1)
\end{aligned}$$

$$\begin{aligned}
q_3(n) &= \mathbf{P}\left(X_n \text{ transient}|X_0 = s_3\right) \\
&= \frac{1}{4}\mathbf{P}\left(X_n \text{ transient}|X_1 = s_3\right) + \frac{1}{4}\mathbf{P}\left(X_n \text{ transient}|X_1 = s_2\right) + \frac{1}{2}\mathbf{P}\left(X_n \text{ transient}|X_1 = s_4\right),
\end{aligned}$$

i.e.,

$$q_3(n) = \frac{1}{4}q_3(n-1) + \frac{3}{4}\left(\frac{1}{2}\right)^{n-1}.$$

Note that the recurrent states s_1 and s_5 can be reached in a maximum of two steps from any transient states. That means that $q_0(2)$, $q_2(2)$, $q_3(2)$, $q_4(2)$ are all less than one. In fact, using the fact that $q_3(0) = 1$, we can calculate that

$$q_0(2) = \frac{1}{3}, \quad q_2(2) = \frac{1}{4}, \quad q_3(2) = \frac{5}{8}, \quad q_4(2) = \frac{1}{4}.$$

Hence,

$$\max_{j \text{ transient}} q_j(2) = \frac{5}{8}.$$

Now for each transient state s_i

$$\begin{aligned}
q_i(4) &= \mathbf{P}\left(X_4 \text{ transient} | X_0 = i\right) \\
&= \sum_{j \text{transient}} \mathbf{P}\left(X_4 \text{ transient} | X_2 = j\right) \mathbf{P}\left(X_2 = j | X_0 = i\right) \\
&= \sum_{j \text{transient}} q_j(2) \mathbf{P}\left(X_2 = j | X_0 = i\right) \\
&\leq \frac{5}{8} \sum_{j \text{transient}} \mathbf{P}\left(X_2 = j | X_0 = i\right) \\
&= \frac{5}{8} q_i(2) \\
&\leq \left(\frac{5}{8}\right)^2.
\end{aligned}$$

Continuing in this manner, we have

$$q_i(2k) \leq \left(\frac{5}{8}\right)^k.$$

So, for any integer n, let k be the largest integer $\leq \frac{n}{2}$. Then,

$$\begin{aligned}
q_i(n) \leq q_i(2k) \leq \left(\frac{5}{8}\right)^k &= \left(\frac{8}{5}\right)\left(\frac{5}{8}\right)^{k+1} \\
&\leq \left(\frac{8}{5}\right)\left(\frac{5}{8}\right)^{\frac{n}{2}}.
\end{aligned}$$

So, we can take $c = \frac{8}{5}$ and $\gamma = \sqrt{\frac{5}{8}}$.