# Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

**6.041/6.431: Probabilistic Systems Analysis**

(Spring 2010)

**Problem Set 8**

**Due: April 21, 2010**

1. A "perpetual" yarn mill in the fairy land of fantasia made an infinite long yarn before the beginning of time. From time immemorial it has been selling pieces of this yarn to residents of fantasia. The probability density function of $L$, the length of yarn purchased by any particular customer, is given by
$$f_L(l) = \lambda e^{-\lambda l}, \qquad l \geq 0.$$
A single dot was placed by a fairy on the yarn when it was created.
   (a) Determine the expected value of $R$, where $R$ is the length of yarn purchased by that customer whose purchase included the dot.
   (b) Redo above, given the probability density function for $L$ is
$$f_L(l) = \frac{\lambda^3 l^2 e^{-\lambda l}}{2}, \qquad l \geq 0.$$

2. (a) Identify the transient, recurrent, and periodic states of the Markov chain described by
$$[p_{ij}] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$
   (b) How many classes are formed by the recurrent states of this process?

3. For each of the following definitions of state $X_k$ at time $k$ ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities $p_{ij}$:
   (a) A six-sided die is rolled repeatedly.
      i. Let $X_k$ denote the largest number rolled in the first $k$ rolls.
      ii. Let $X_k$ denote the number of sixes in the first $k$ rolls.
      iii. At time $k$, let $X_k$ be the number of rolls since the most recent six.
   (b) Let $Y_k$ be the state of some discrete-time Markov process at time $k$ (i.e., it is known $Y_k$ satisfies the Markov property) with known transition probabilities $q_{ij}$.
      i. For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.
      ii. Let $X_k = Y_{2k}$.
      iii. Let $X_k = (Y_k, Y_{k+1})$; that is, the state $X_k$ is defined by the sequence of state *pairs* in a given Markov process.

4. A discrete-time Markov chain with seven states has the following transition probabilities:
$$p_{ij} = \begin{cases} 0.5 & , \quad (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1 & , \quad (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0 & , \quad \text{otherwise} \end{cases}$$
In the questions below, we let $X_k$ be the state of the Markov process at time $k$.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of $n$ is the probability $r_{15}(n) = \mathbf{P}(X_n = 5 \mid X_0 = 1) > 0$?

(c) What is the set of states $A(i)$ that is accessible from state $i$, for each $i = 1, 2, \ldots, 7$?

(d) Identify which states are transient and which states are recurrent. For each recurrent class, state whether it is periodic (and give the period) or aperiodic.

(e) What is the minimum number of transitions with nonzero probability that must be added so that all seven states form a single recurrent class?

5. Out of the $d$ doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the $d$ doors. At the end of the day, I leave the door I used that day locked.

(a) Show that the number of unlocked doors at the end of day $n$, $L_n$, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities $p_{ij}$.

(b) List transient and recurrent states.

(c) Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

(d) My second strategy is to alternate between leaving the door I use locked one day and unlocked the next day (regardless of the initial condition of the door.) In this case, does the number of unlocked doors evolve as a Markov chain, why/why not?

G1[†]. Consider a Markov chain $\{X_k\}$ on the state space $\{1, \ldots, n\}$, and suppose that whenever the state is $i$, a reward $g(i)$ is obtained. Let $R_k$ be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state $i$, let

$$m_k(i) = \mathbf{E}[R_k \mid X_0 = i],$$

and

$$v_k(i) = \mathrm{var}(R_k \mid X_0 = i)$$

respectively be the conditional mean and conditional variance of $R_k$, conditioned on the initial state being $i$.

(a) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$, allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$ and $v_k(1), \ldots, v_k(n)$, allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint*: Use the law of total variance.

[†]Required for 6.431 students and optional (not graded) for 6.041.