Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 8
Due: April 22, 2009

1. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

   (a) Determine the probability that Fred gives away his first sample on his third call.

   (b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

   (c) Determine the probability that he gives away his second sample on his fifth call.

   (d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

   (e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

   (f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

2. Alice and Bob alternate playing at the casino table. (Alice starts and plays at odd times $i = 1, 3, \ldots$; Bob plays at even times $i = 2, 4, \ldots$.) At each time i, the net gain of whoever is playing is a random variable G_i with the following PMF:

$$p_G(g) = \begin{cases} \frac{1}{3} & g = -2, \\ \frac{1}{2} & g = 1, \\ \frac{1}{6} & g = 3, \\ 0 & \text{otherwise} \end{cases}$$

   Assume that the net gains at different times are independent. We refer to an outcome of -2 as a "loss."

   (a) They keep gambling until the first time where a loss by Bob immediately follows a loss by Alice. Write down the PMF of the total number of rounds played. (A round consists of two plays, one by Alice and then one by Bob.)

   (b) Write down an expression for the transform of the net gain of Alice up to the time of the first loss by Bob.

   (c) Write down the PMF for Z, defined as the time at which Bob has his third loss.

   (d) Let N be the number of rounds until each one of them has won at least once. Find $\mathbf{E}[N]$.

3. The MIT soccer team needs at least 8 players to avoid forfeiting a game. Assume that each player has some chance of being eliminated for the remainder of the season owing to injury, and that her playing lifetime for a given season is exponentially distributed with parameter λ. For simplicity, assume that the coach insists on only playing 8 players at a time, and then replaces a player as soon as she gets hurt. Find

(a) The expected time until the first substitution.

(b) The distribution of total time the team can play in a season, given that there are n women on the team.

4. (a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

   (b) Now suppose that the shuttles are no longer operating on a deterministic schedule, but rather their interdeparture times are independent and exponentially distributed with rate μ per hour. Find the PMF for the number of shuttles arriving in one hour.

   (c) Let us define an "event" to be either the arrival of a passenger, or the departure of a shuttle (or both simultaneously). With the same assumptions as in (b) above, find the expected number of "events" that occur in one hour.

   (d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

   (e) Find the PMF for the number of people on a shuttle.

5. The amount of time between arrivals in an arrival process is the interarrival time. If the interarrival times are independent and identically distributed positive random variables, we call the arrival process a renewal process. (Notice that a Poisson process of rate λ is a renewal process where the interarrival times are exponentially distributed with parameter λ.)

   Suppose bus arrivals at a bus stop form a renewal process where the interarrival times are uniformly distributed between 1 and 2 hours.

   (a) Find the PDF for the second-order interarrival time (i.e. the interarrival time between every other arrival).

   (b) Given that a bus arrived at 12pm, at what time do we expect the fourth bus which comes after 12pm to arrive?

   (c) Suppose that each bus that arrives has probability 3/4 of being a Greyhound bus and 1/4 of being a Peter Pan bus, independently of all other buses. Given that a Greyhound bus arrived at 12pm, at what time do we expect the next Greyhound bus to arrive?

G1†. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geq 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

$$P\Big(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \,|\, N(G) = k\Big) .$$

Interpret your answer as a multinomial distribution.

†Required for 6.431; optional for 6.041