Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2011)

Problem Set 8: Solutions

1. (a) Let A_k be the event that the process enters s_2 for first time on trial k. The only way to enter state s_2 for the first time on the kth trial is to enter state s_3 on the first trial, remain in s_3 for the next $k-2$ trials, and finally enter s_2 on the last trial. Thus,

$$\mathbf{P}(A_k) = p_{03} \cdot p_{33}^{k-2} \cdot p_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1} \qquad \text{for} \quad k = 2, 3, \ldots$$

   (b) Let A be the event that the process never enters s_4.

   There are three possible ways for A to occur. The first two are if the first transition is either from s_0 to s_1 or s_0 to s_5. This occurs with probability $\frac{2}{3}$. The other is if The first transition is from s_0 to s_3, and that the next change of state *after* that is to the state s_2. We know that the probability of going from s_0 to s_3 is $\frac{1}{3}$. Given this has occurred, and given a change of state occurs from state s_3, we know that the probability that the state transitioned to is the state s_2 is simply $\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$. Thus, the probability of transitioning from s_0 to s_3 and then eventually transitioning to s_2 is $\frac{1}{9}$. Thus, the probability of never entering s_4 is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$.

   (c) $\mathbf{P}(\{\text{process enters } s_2 \text{ and then leaves } s_2 \text{ on next trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{process enters } s_2\})\mathbf{P}(\{\text{leaves } s_2 \text{ on next trial }\}|\{\text{ in } s_2\}) \\
&= \left[\sum_{k=2}^{\infty} \mathbf{P}(A_k)\right] \cdot \frac{1}{2} \\
&= \left[\sum_{k=2}^{\infty} \frac{1}{3}\left(\frac{1}{4}\right)^{k-1}\right] \cdot \frac{1}{2} \\
&= \frac{1}{6} \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{18}.
\end{aligned}$$

   (d) This event can only happen if the sequence of state transitions is as follows:

$$s_0 \longrightarrow s_3 \longrightarrow s_2 \longrightarrow s_1.$$

   Thus, $\mathbf{P}(\{\text{process enters } s_1 \text{ for first time on third trial}\}) = p_{03} \cdot p_{32} \cdot p_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$.

   (e) $\mathbf{P}(\{\text{process in } s_3 \text{ immediately after the } N\text{th trial}\})$

$$\begin{aligned}
&= \mathbf{P}(\{\text{moves to } s_3 \text{ in first trial and stays in } s_3 \text{ for next } N-1 \text{ trials}\}) \\
&= \frac{1}{3}\left(\frac{1}{4}\right)^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

2. (a) If we can show that this process has the Markov property, then it is a Markov chain since it clearly satisfies the other conditions. Thus we need only show:

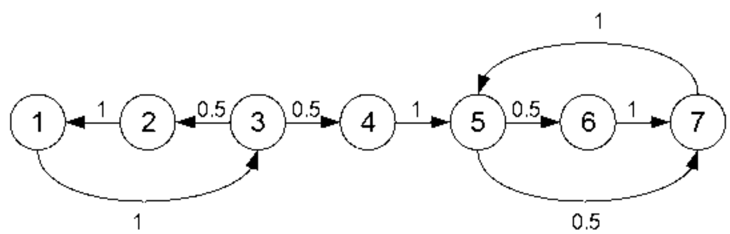
$$\mathbf{P}(X_{m+1} = j | X_m = i, X_{m-1} = k, \ldots, X_1 = q) = \mathbf{P}(X_{m+1} = j | X_m = i).$$

But this is clear from the statement of the problem, since the only relevant information is the highest ranking ever attained, and this information is precisely the information contained by X_m.

(b) For $i > j$, $p_{ij} = 0$. Since the professor will continue to remember the highest ranking, even if he gets a lower ranking in a subsequent year, we have: $p_{ii} = \frac{i}{n}$. Finally, for $j > i$, $p_{ij} = \frac{1}{n}$ since the class is equally likely to receive any given rating.

(c) Since the professor only remembers the *highest* ranking received by his class, only the highest possible rank is recurrent (i.e. the state { Professor recalls rank n } is recurrent). All of the lower-rank states are transient, since the class will eventually receive a higher rank, and then the professor will never recall that lower ranking again.

3. (a) The state diagram of the Markov chain is:



(b) State 5 is reachable from state 1 in a minimum of three transitions. Paths from state 1 to state 5 also include paths with a loop from 1 back to 1 (of length 3) and/or a loop from 5 back to 5 by way of state 7 (either length 2 or length 3). Therefore potential path lengths are $3 + 2m + 3n$, for $m, n \geq 0$. Therefore, $r_{15}(n) > 0$ for $n = 3$ or $n \geq 5$.

(c) From states 1, 2, and 3, all states are accessible because there is a non-zero probability path from these states by way of state 3 to any other state. From states, 4, 5, 6, and 7, paths only exist to states 5, 6, and 7.

(d) States 5-7 are recurrent because by the logic in (c), they can be reached from any other state. States 1-4 are transient; once the system has transitioned out of state 4, it cannot return to any state other than states 5, 6, or 7.

States 5, 6, and 7 form a recurrent class. Because it can be traversed from state 5 back to 5 in either 2 or 3 steps (as discussed in (b)), the system can return to state 5 after n steps for any $n \geq 2$; therefore it is aperiodic.

(e) One transition must be added to create a single recurrent class: for example, adding a transition from state 5 to state 1 would allow every state to be reached from every other state. Any transition from the recurrent class states 5,6, or 7 to any of the states 1, 2, or 3 would work.

4. The outcome of the next game depends on the outcome of the past two games, thus we need a 4 state Markov chain to model the process. The states will be all the ordered pairs of outcomes of the past two games, where the second entry marks the outcome of the most recent game:

$$\{S_1 = (W, W); S_2 = (W, L); S_3 = (L, W); S_4 = (L, L)\}$$

Therefore the transition probability matrix will be:

$$[P] = \begin{pmatrix} .7 & .3 & 0 & 0 \\ 0 & 0 & .4 & .6 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 0.2 & .8 \end{pmatrix}$$

We see from the chain that it has one recurrent, aperiodic class, and therefore is ergodic. Thus we can apply the fundamental theorem. The theorem tells us that if we can find positive numbers $\{\pi_i\}$ that satisfy:

$$\pi_j = \sum_i \pi_i p_{ij}, \text{ and } \sum_i \pi_i = 1$$

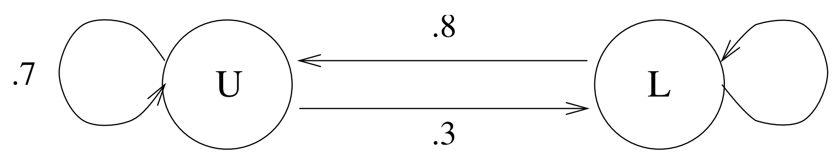
then these $\{\pi_i\}$ are in fact the steady state probabilities. Solving the linear system, we find that:

$$\pi_1 = \frac{5}{20}, \pi_2 = \frac{3}{20}, \pi_3 = \frac{3}{20}, \pi_4 = \frac{9}{20}$$

and the desired probability is $\pi_1 + \pi_3 = \frac{8}{20} = \frac{2}{5}$.

The long run probability that the team will win its next game, i.e., the sum probability of states (W,W) and (L,W), is $\frac{2}{5}$. This conclusion can be reached via $\mathbf{P}(\text{winning next game}) = 0.7 * \pi_1 + 0.4 * \pi_2 + 0.5 * \pi_3 + 0.2 * \pi_4$.

5. The state-transition diagram is the following:



(a) We are interested in finding the steady-state probabilites of the states in this Markov chain. Since this is a birth-death process, we use the local balance equations based on the frequency of transitions between two successive states and the normalization equation to solve for π_U and π_L.

$$\begin{aligned} \pi_L &= \frac{\pi_U \cdot 3/10}{8/10} = \frac{3}{8}\pi_U \\ 1 &= \pi_L + \pi_U. \end{aligned}$$

Solving this system of equations, we get,

$$\pi_U = \frac{8}{11} \qquad \pi_L = \frac{3}{11}.$$

Thus,

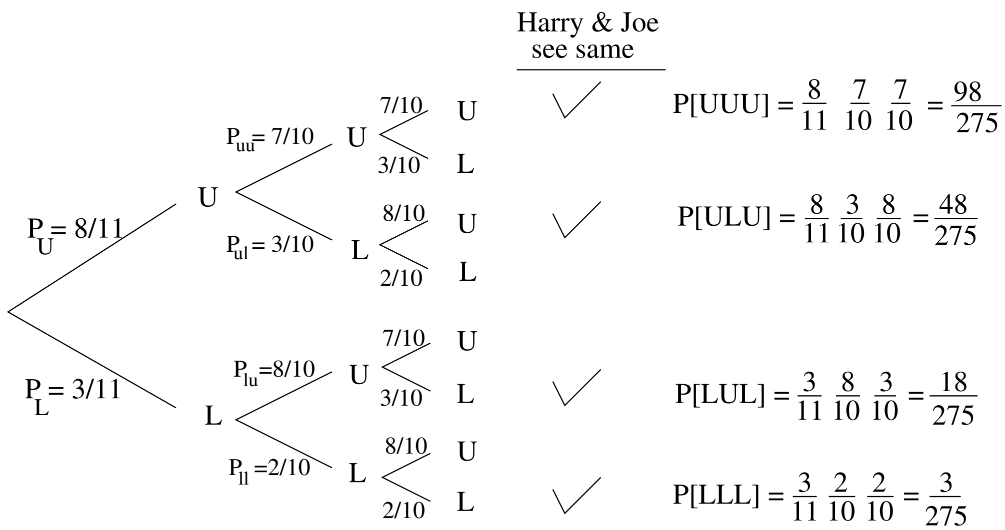
$$P(\text{he unlocks the door}) = \pi_L \cdot p_{LU} = \frac{3}{11} \cdot \frac{8}{10} = \frac{12}{55}$$

and

$$P(\text{he locks the door}) = \pi_U \cdot p_{UL} = \frac{8}{11} \cdot \frac{3}{10} = \frac{12}{55}.$$

So, the two events are equally likely.

(b) We can draw a tree of the possible outcomes of Mean Variance's two visits between Joe's arrival and Harry's.



$$P(\text{both Joe and Harry see the same condition}) = \frac{98}{275} + \frac{48}{275} + \frac{18}{275} + \frac{3}{275} = \boxed{\frac{167}{275}.}$$

(c) Define

X = number of visits from hiring to locking

Y = number of visits from locking to unlocking.

W = number of visits from hiring to unlocking (this is the random variable of interest)

Note that $W = X + Y$. X is a geometric random variable with success probability equal to 0.3 and Y is a geometric random variable with success probability equal to 0.8:

$$\begin{aligned} p_X(x) &= \frac{3}{10}\left(\frac{7}{10}\right)^{x-1}, x = 1, 2, 3, \ldots \\ p_Y(y) &= \frac{8}{10}\left(\frac{2}{10}\right)^{y-1}, y = 1, 2, 3, \ldots \end{aligned}$$

Using the linearity property of expectation and the expected value of a geometric random variable we obtain,

$$\begin{aligned} E[W] &= E[X] + E[Y] \\ &= \frac{10}{3} + \frac{10}{8} \\ &\approx 4.583 \end{aligned}$$

G1†. (a) This is just the re-writing of the Chapman-Kolmogorov Equation for the n-step transition probabilities $r_{ij}(n)$ in matrix form. Let X_n be the state of the Markov chain at time n. We will show that $P_{ij}^n = P(X_n = j|X_0 = i) \equiv r_{ij}(n)$ by using induction. First, note that $P_{ij} =$

$P(X_1 = j|X_0 = i) = r_{ij}(1)$. Now, we simply need to show that if $P^n_{ij} = P(X_n = j|X_0 = i)$, then $P^{n+1}_{ij} = P(X_{n+1} = j|X_0 = i)$.

Induction Step: As $P^{n+1} = P^n P$, it follows that

$$\begin{aligned} P^{n+1}_{ij} &= \sum_k P^n_{ik} p_{kj} \\ &= \sum_k P(X_n = k|X_0 = i)P(X_{n+1} = j|X_n = k) \\ &= \sum_k P(X_n = k|X_0 = i)P(X_{n+1} = j|X_n = k, X_0 = i) \\ &= P(X_{n+1} = j|X_0 = i). \end{aligned}$$

The equalities above follow from the Markov property and the total probability theorem.

(b) Using the total expectation theorem, we get

$$E(V_i) = r_i + \sum_j E(V_j|X_1 = j, X_0 = i)P(X_1 = j|X_0 = i).$$

Note that what the randomness in V_j is due to the states visited starting at state j and does not depend on the reward/state previously collected/visited, and by the Markov property it follows that $E(V_j|X_1 = j, X_0 = i) = E(V_j)$. Therefore,

$$v_i = r_i + \sum_j p_{ij} v_j$$

.

In addition, the total reward starting at the mth state is $v_m = 0$, as it's an absorbing state with zero-reward.

(c) From each transient states i, the absorbing state is accessible as it is the only recurrent state, i.e. there is a path $(X_0 = i, X_1, \cdots, X_n = m)$ with positive probability. Note that it's always possible to find a path with distinct elements, whose length n is at most m. Hence $P^n_{im} > 0$ for some $n \le m$ and consequently $P^m_{im} > 0$.

†Required for 6.431; optional for 6.041