

LECTURE 16

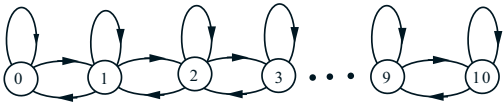
- **Readings:** Sections 7.1–7.2

Lecture outline

- Checkout counter example
- Markov process definition
- $n$ -step transition probabilities
- Classification of states

Checkout counter model

- Discrete time  $n = 1, 2, \dots$
- Customer arrivals: Bernoulli( $p$ ) process
  - geometric interarrival times
- Customer service times: geometric( $q$ )
- “State”  $X_n$ : number of customers at time  $n$



Discrete-time Markov chains

- $X_n$ : state after  $n$  transitions (discrete-time random process)
  - belongs to a finite set, usually  $\{1, \dots, m\}$
  - $X_0$  is either given or random
- **Markov property/assumption:**
$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ = P(X_{n+1} = j \mid X_n = i) = p_{ij}$$
(given current state, the past does not matter)
- Model specification:
  - identify set of states  $S = \{1, \dots, m\}$
  - identify possible transitions:  $(i, j)$  pairs such that  $p_{ij} > 0$
  - specify the transition probabilities

Example: Spiders and fly

- A fly moves among  $\{1, 2, 3, 4\}$ . At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independent of the past history of movements. Spiders lurk at positions 1 and 4: if the fly lands there, it is captured (does not move again).

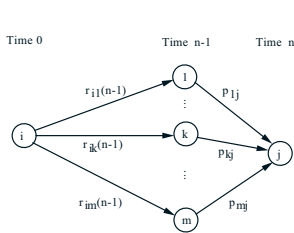
transition  
probability graph

transition  
probability matrix

$n$ -step transition probabilities

- Given initial state  $i$ , probability of state  $j$  after  $n$  steps:

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$



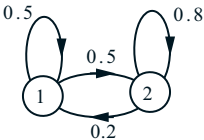
- Key recursion (Chapman–Kolmogorov equation):

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1)p_{kj}$$

- With random initial state:

$$P(X_n = j) = \sum_{i=1}^m P(X_0 = i)r_{ij}(n)$$

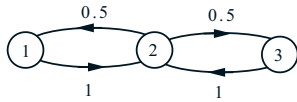
Example



	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 100$	$n = 101$
$r_{11}(n)$						
$r_{12}(n)$						
$r_{21}(n)$						
$r_{22}(n)$						

### Generic convergence questions (1)

- Does  $r_{ij}(n)$  converge to something (as  $n \rightarrow \infty$ )?



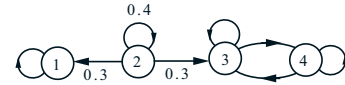
n odd:  $r_{22}(n) =$

n even:  $r_{22}(n) =$

### Generic convergence questions (2)

- Does the limit depend on initial state?

(Does  $\lim_{n \rightarrow \infty} r_{ij}(n)$  depend on  $i$ ?)



$r_{11}(n) =$

$r_{31}(n) =$

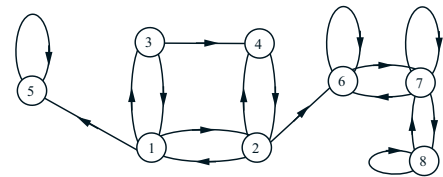
$r_{21}(n) =$

### Classification of states

- State  $j$  is **accessible** from state  $i$  when there is a positive probability of transitioning  $i \rightarrow j$  in some number of steps:  
 $r_{ij}(n) > 0$  for some  $n > 0$
- State  $i$  is **recurrent** when, wherever the state reaches starting from  $i$ , the state can return to  $i$ :  
 for every  $j$  accessible from  $i$ ,  $i$  is accessible from  $j$
- When a state is not recurrent, it is **transient**
- A **recurrent class** is a set of states accessible from each other, with no other state accessible from them

### Example

- Classification of states depends only on transition probability graph nodes and edges (not labels)



### Periodic recurrent class

- A recurrent class is **periodic** if its states can be grouped into  $d > 1$  groups so that all transitions from one group lead to the next group

