Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Fall 2011)

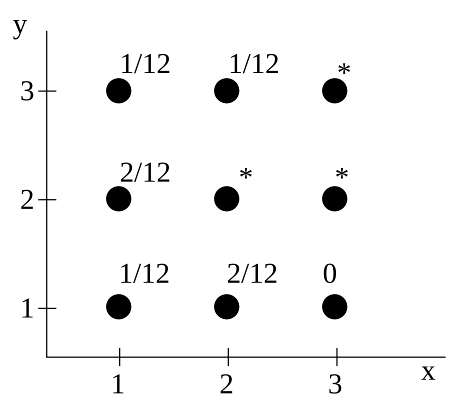
**Recitation 8**
**October 4, 2011**

1. **Problem 2.35, page 130 in the text.** Verify the expected value rule

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y),$$

using the expected value rule for a function of a single random variable.

2. Random variables $X$ and $Y$ can take any value in the set $\{1, 2, 3\}$. We are given the following information about their joint PMF, where the entries indicated by a * are left unspecified:




(a) What is $p_X(1)$?
(b) Provide a clearly labeled sketch of the conditional PMF of $Y$ given that $X = 1$.
(c) What is $\mathbf{E}[Y \mid X = 1]$?
(d) Is there a choice for the unspecified entries that would make $X$ and $Y$ independent?

Let $B$ be the event that $X \le 2$ and $Y \le 2$. We are told that conditioned on $B$, the random variables $X$ and $Y$ are independent.

(e) What is $p_{X,Y}(2, 2)$?
(If there is not enough information to determine the answer, say so.)
(f) What is $p_{X,Y|B}(2, 2 \mid B)$?
(If there is not enough information to determine the answer, say so.)

3. There are $n$ persons, numbered 1 to $n$. Each person $i$ is assigned a seat number $X_i$. The seat numbers are distinct integers in the range $1, \ldots, n$. We assume that the seating is "completely random", that is, the sequence $(X_1, \ldots, X_n)$ is a permutation of the numbers $1, \ldots, n$, and all permutations are equally likely.

   (a) Find the probability that the first three persons are seated in the first three seats. (Mathematically, this is the event that the set $\{X_1, X_2, X_3\}$ is the set $\{1, 2, 3\}$.)

   (b) Are the events $\{X_1 < X_2\}$ and $\{X_3 < X_4\}$ independent? Provide a brief justification (1-4 lines).

   (c) Are $X_1$ and $X_2$ conditionally independent, given the random variables $X_3$ and $X_4$? Provide a brief justification (1-4 lines).

   (d) Consider the first 10 people, $i = 1, \ldots, 10$. Find the probability that exactly 5 of the first 10 people are seated in seats with numbers in the range $1, \ldots, 8$. (You do not need to simplify your answer.)

   (e) For any $i$ and $j$, with $1 \le i < j \le n$, we say that we have an inversion if $X_i > X_j$. Let $N$ be the number of inversions. Find $\mathbf{E}[N]$.

   The following formula may prove useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$