Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2011)

Problem Set 10
Due: May 4, 2011

1. A storm warning model is 95% accurate. We want to find the probability that out of 50 predictions, at least 45 will be correct.
   (a) Find the above probability by using the normal approximation to the binomial.
   (b) Repeat part (a) this time using the Poisson approximation to the binomial and briefly discuss which one of the answers above you feel to be more accurate, and why.

2. Let X_1, X_2, ... be independent, identically distributed, continuous random variables with $\mathbf{E}[X] = 2$ and $\text{var}(X) = 9$. Define $Y_i = (0.5)^i X_i$, $i = 1, 2, \ldots$. Also define T_n and A_n to be the sum and the average, respectively, of the terms $Y_1, Y_2, \ldots, Y_n$.
   (a) Is Y_n convergent in probability? If so, to what value? Explain.
   (b) Is T_n convergent in probability? If so, to what value? Explain.
   (c) Is A_n convergent in probability? If so, to what value? Explain.

3. Let $X_1, \cdots, X_{10}$ be independent random variables, uniformly distributed over the unit interval $[0, 1]$.
   (a) Bound $\mathbf{P}(X_1 + \cdots + X_{10} \geq 7)$ using the Markov inequality.
   (b) Bound the expression in part (a) using the Chebyshev inequality.
   (c) Estimate the expression in part (a) using the Central Limit Theorem.

4. Based on Example 8.2, page 414 in the text.

   Sasha just got a new puppy when they moved into a new home. As it turns out, the puppy destroys a random amount X square meters of the lawn every day. X is uniformly distributed over the interval $[0, \theta]$. At the end of each day, annoyed gardeners fix the damage wrought by the puppy. Sasha, a budding Bayesian statistician, chooses to model the destruction of the lawn. She treats the parameter θ as an unknown value of a random variable Θ, which is uniformly distributed between zero and one square meter. She further assumes that given Θ, the amount of lawn destroyed by the puppy on any given day is independent of what happened on all other days.
   (a) On one day, Sasha observes that the puppy destroyed x m^2. How should Sasha use this information to update the distribution of Θ?
   (b) Let X_1, ..., X_n be the amounts of lawn destroyed by the puppy on n consecutive days. Sasha keeps careful measurements over the n days, and records $X_1 = x_1$, ..., $X_n = x_n$. How should Sasha use this information to update the distribution of Θ?

G1†. In this problem you will use the central limit theorem to "derive" (but not rigorously prove) the Stirling approximation for the factorial for large n:

$$n! \approx \sqrt{2\pi n}(n/e)^n$$

(a) **Gaussian approximation to the Poisson distribution.** Recall that in Problem G1) of problem Set 8 you found that the family of Poisson distributions is stable, i.e., closed under addition of independent random variables. Use this fact to show that, if $N^{(m)}$ is a discrete random variable having a Poisson distribution with integer-valued mean $\lambda = m \geq 1$, then

$$\lim_{m\to\infty} \mathbf{P}(N^{(m)} \leq m + a\sqrt{m}) = \Phi(a), \text{ for all a,}$$

where Φ is the CDF of the standard $N(0,1)$ normal random variable.

(b) Use the conclusion above to show that, for n sufficiently large,

$$\mathbf{P}(N^{(n)} = n) = e^{-n}\frac{n^n}{n!} \approx \frac{1}{\sqrt{2\pi n}}.$$

(c) Use the result of part b) to derive the Stirling approximation above.

†Required for 6.431 students and optional (not graded) for 6.041.