Problem Set 11[1] (never due)

1. A sequence of n i.i.d. Bernoulli trials with unknown p (probability of success in the first trial) yields K_n successes.

   (a) Let M_n be the number of the trials *before* the *first* success has occured (for example, $M_n = 0$ when the first trial was a success, and $M_n = n$ when $K_n = 0$). Find, in terms of K_n, n, and p, a minimal probability of error estimator $\hat{M}_n$ for M_n given K_n. Given K_n, n, and p, is $\hat{M}_n$ uniquely defined?

   (*b) Given n and K_n, what is the maximal likelihood estimate of p^2? Is it unbiased? Is consistent as $n \to \infty$?

   (*c) Consider the case when $n = 9$ and the probability of success in the first trial p is either 1/2 or 2/3. Among all functions $g : \{0, 1, \ldots, n\} \mapsto \{0, 1\}$ such that $\mathbf{P}(1 = g(K_n))$ is not larger than 0.5 when $p = 1/2$, find the one that achieves the smallest $\beta = \mathbf{P}(0 = g(K_n))$ when $p = 2/3$. Also find the corresponding value of β.

2. Real parameter Θ is measured by a device which introduces multiplicative measurement noise W, modeled as a random variable which is uniformly distributed over the interval $[0, 1]$, so that the measurement is $Y = \Theta W$.

   In questions (a)-(d) assume that Θ is a random variable which is independent of W and is uniformly distributed over the interval $[-1, 1]$.

   (a) the MAP estimator for Θ given Y;

   (b) the least mean squares estimator for Θ given Y;

   (c) the linear least mean squares estimator for Θ given Y;

   (d) the least mean absolute error estimator for Θ given Y.

   In question (e),(f) no a-priori distribution information about Θ is assumed, apart from Θ being a *positive* number.

   (*e) Find the ML estimate $\hat{\Theta}$ of Θ given Y.

   (*f) What is the minimal real number ρ for which $(Y, \rho Y)$ is a 95 percent confidence interval for Θ?

3. A factory produces light bulbs for which the life time T (measured in hours) has exponential distribution with parameter $\lambda = 1$. One such light bulb is turned on, then checked in exactly 1 hour, 2 hours, etc. Let the Kth check (i.e. exactly $K \in \{1, 2, 3, \ldots\}$ hours after being turned on) be the first one when the light bulb was found not working.

   (a) Find function $g_0 : \{1, 2, 3, \ldots\} \mapsto [0, \infty)$ which produces the MAP estimate $\hat{T} = g_0(K)$ of T.

   (b) Find function $g_1 : \{1, 2, 3, \ldots\} \mapsto [0, \infty)$ which produces least mean absolute error estimate $\hat{T} = g_1(K)$ i.e. the one minimizing $\mathbf{E}[|g(K) - T|]$.

   (c) Find function $g_2 : \{1, 2, 3, \ldots\} \mapsto [0, \infty)$ which produces least mean squares error estimate $\hat{T} = g_2(K)$ i.e. the one minimizing $\mathbf{E}[|g(K) - T|^2]$.

[1] Questions marked * rely on the material to be presented during the week of May 10-14

4. John tests rubber ducks produced at the plant where he works. Squeezing and then releasing a grade A duck produces a squeak of sound intensity (measured in dB) which can be modeled as a sum of two independent random variables uniformly distributed in the range 25 to 55. Accordingly, the noise generated by a grade B duck can be modeled as a sum of two independent random variables uniformly distributed in the range 10 to 40.[2]

   (a) Assuming that one third of the rubber ducks are grade A, and the rest are grade B, find a function $g = g_{mpe} : [20, 110] \mapsto \{A, B\}$ defining a minimal probability of error estimator $\hat{U} = g(Y)$ for a rubber duck's grade $U \in \{A, B\}$, based on the intensity $Y \in [20, 110]$ of the sound it produces.

   (*b) Assuming that no a-priori information is given about the relative frequency of grade A and grade B ducks, find a function $g = g_{ML} : [20, 110] \mapsto \{A, B\}$ defining the maximal likelihood estimator $\hat{U} = g(Y)$ of U given Y.

   (*c) Assuming that no a-priori information is given about the relative frequency of grade A and grade B ducks, among all functions $g : [20, 110] \mapsto \{A, B\}$ such that $\mathbf{P}(B = g(Y))$ is not larger than 1/18 for a grade A duck, find the one ($g = g_*$) that achieves the smallest $\mathbf{P}(A = g(Y))$ for a grade B duck. What is the corresponding optimal value of $\mathbf{P}(A = g_*(Y))$ for a grade B duck?

5. Random variables X, N are independent: N is a geometric with parameter $p = 1 - e^{-1}$, and X is uniformly distributed on $[0, 1]$. Knowing that $X^N = e^{-12}$, find the most likely value of N.

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