Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

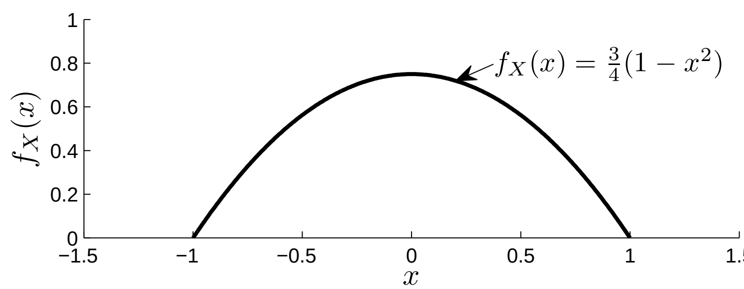
Problem Set 6
Due October 31, 2011

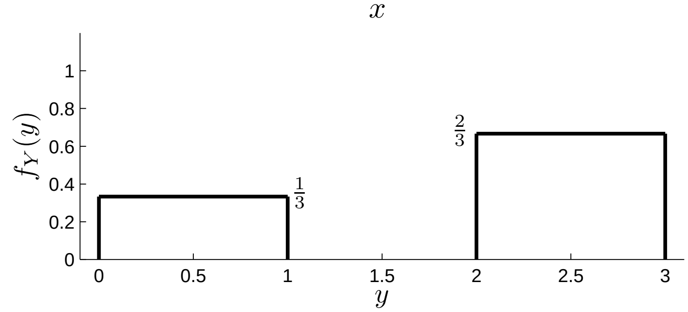
1. Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le y \le 3 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Evaluate the constant a.
   (b) Determine the marginal PDF $f_Y(y)$.
   (c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.
   (d) Random variable Z is defined by $Z = Y - X$. Determine the PDF $f_Z(z)$.

2. Let X and Y be two independent random variables. Their probability densities functions are shown below.





   Let $Z = X + Y$. Determine $f_Z(z)$.

3. Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i.

   (a) Are X_1 and X_2 uncorrelated, positively correlated, or negatively correlated? Give a one-line justification.
   (b) Compute the covariance $\text{cov}(X_1, X_2)$ of X_1 and X_2.

4. **A financial parable.** An investment bank is managing \$1 billion, which it invests in various financial instruments ("assets") related to the housing market (e.g., the infamous "mortgage backed securities"). Because the bank is investing with borrowed money, its actual assets are only \$50 million (5%). Accordingly, if the bank loses more than 5%, it becomes insolvent. (Which means that it will have to be bailed out, and the bankers may need to forgo any huge bonuses for a few months.)

   (a) The bank considers investing in a single asset, whose gain (over a 1-year period, and measured in percentage points) is modeled as a normal random variable R, with mean 7 and standard deviation 10. (That is, the asset is expected to yield a 7% profit.) What is the probability that the bank will become insolvent? Would you accept this level of risk?

   (b) In order to safeguard its position, the bank decides to diversify its investments. It considers investing \$50 million in each of twenty different assets, with the ith one having a gain R_i, which is again normal with mean 7 and standard deviation 10; the bank's gain will be $(R_1 + \cdots + R_{20})/20$. These twenty assets are chosen to reflect the housing sectors at different states or even countries, and the bank's rocket scientists choose to model the R_i as independent random variables. According to this model, what is the probability that the bank becomes insolvent?

   (c) Based on the calculations in part (b), the bank goes ahead with the diversified investment strategy. It turns out that a global economic phenomenon can affect the housing sectors in different states and countries simultaneously, and therefore the gains R_i are in fact positively correlated. Suppose that for every i and j where $i \neq j$, the correlation coefficient $\rho(R_i, R_j)$ is equal to 1/2. What is the probability that the bank becomes insolvent? You can assume that $(R_1 + \cdots + R_{20})/20$ is normal.

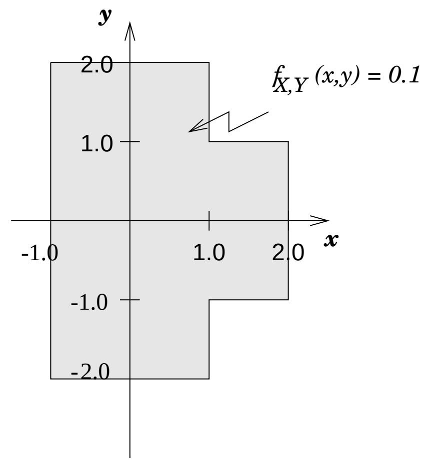
5. The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by Q, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

   (a) (i) Find the mean of X.
       (ii) Find the variance of X.

   (b) Find the covariance of X and Q.

   (c) Find the conditional PDF of Q given that $X = 1$.

6. Random variables X and Y have the joint PDF shown below:



   (a) Find the conditional PDFs $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$, for various values of x and y, respectively.

   (b) Find $\mathbf{E}[X \mid Y = y]$, $\mathbf{E}[X]$, and $\text{var}(X \mid Y = y)$. Use these to calculate $\text{var}(X)$.

   (c) Find $\mathbf{E}[Y \mid X = x]$, $\mathbf{E}[Y]$, and $\text{var}(Y \mid X = x)$. Use these to calculate $\text{var}(Y)$.

7. The wombat club has N members, where N is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p) \quad \text{for } n = 1, 2, 3, \ldots.$$

On the second Tuesday night of every month, the club holds a meeting. Each wombat member attends the meeting with probability q, independently of all the other members. If a wombat attends the meeting, then it brings an amount of money, M, which is a continuous random variable with PDF

$$f_M(m) = \lambda e^{-\lambda m} \quad \text{for } m \geq 0.$$

N, M, and whether each wombat member attends are all independent. Determine:

   (a) The expectation and variance of the number of wombats showing up to the meeting.

   (b) The expectation and variance for the total amount of money brought to the meeting.

G1†. A coin shows heads with probability p. Let X_n be the number of flips until a sequence of n consecutive heads is obtained. For example, in the sequence $\{H, T, T, H, H, T, H, H, H\}$, $X_1 = 1, X_2 = 5, X_3 = 9$.

   (a) Find $\mathbf{E}[X_n]$ for every $n \geq 1$. *Hint:* Condition on X_{n-1} and use the Total Expectation Theorem to derive a recursive formula in terms of $\mathbf{E}[X_{n-1}]$.

   (b) A coin is chosen randomly, and it shows heads with a probability described by a random variable Y, which is uniformly distributed between 0 and 1. Let X be the number of flips until the first head is obtained. Find $\mathbf{E}[X]$.

†Required for 6.431; optional for 6.041