Massachusetts Institute of Technology

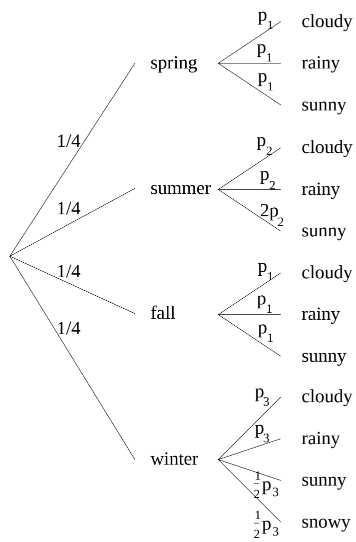
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2009)

Problem Set 2: Solutions
Due: February 18, 2009

1. The sample space is described in the figure below.



The probabilities on the branches extending from each season must add to 1. Therefore we have

$$p_1 = \frac{1}{3}, \qquad p_2 = \frac{1}{4}, \qquad p_3 = \frac{1}{3}.$$

The probability that a randomly chosen day is sunny is then

$$\begin{aligned} P(\text{sunny}) &= P(\text{sunny}|\text{spring})P(\text{spring}) + P(\text{sunny}|\text{summer})P(\text{summer}) + \\ &\quad P(\text{sunny}|\text{fall})P(\text{fall}) + P(\text{sunny}|\text{winter})P(\text{winter}) \\ &= \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{6}\cdot\frac{1}{4} \\ &= \boxed{1/3}. \end{aligned}$$

2. **The Chess Problem.**

(a) i. $\mathbf{P}(\text{2nd Rnd Req}) = (0.6)^2 + (0.4)^2 = 0.52$

ii. $\mathbf{P}(\text{Bo Wins 1st Rnd}) = (0.6)^2 = 0.36$

iii. $\mathbf{P}(\text{Al Champ}) = 1 - \mathbf{P}(\text{Bo Champ}) - \mathbf{P}(\text{Ci Champ})$
$= 1 - (0.6)^2 * (0.5)^2 - (0.4)^2 * (0.3)^2 = 0.8956$

(b) i. $\mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req}) = \frac{(0.6)^2}{0.52} = \frac{0.36}{0.52} = 0.6923$

ii. $\mathbf{P}(\text{Al Champ}|\text{2nd Rnd Req})$
$= \mathbf{P}(\text{Al Champ}|\text{Bo Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req})$
$+ \ \mathbf{P}(\text{Al Champ}|\text{Ci Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Ci Challenger}|\text{2nd Rnd Req})$
$= (1 - (0.5)^2) \times 0.6923 + (1 - (0.3)^2) \times 0.3077$
$= 0.7992$

(c) $\mathbf{P}((\text{Bo Challenger})|\{(\text{2nd Rnd Req}) \cap (\text{One Game})\}) = \frac{(0.6)^2*(0.5)}{(0.6)^2*(0.5)+(0.4)^2*(0.7)}$
$= \frac{(0.6)^2(0.5)}{0.2920} = 0.6164$

3. (a) No. A and B are not independent. To see this, note that $A \subset B$, hence $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. This is equal to $\mathbf{P}(A) \cdot \mathbf{P}(B)$ only when $\mathbf{P}(B) = 1$ or $\mathbf{P}(A) = 0$. But in our example, clearly $\mathbf{P}(B) < 1$ and $\mathbf{P}(A) > 0$. Hence $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$, and thus A and B are not independent.

(b) Yes. Conditioned on C, A will happen if and only if Imno meets 5 people during the second week. Hence $\mathbf{P}(A|C) = 1/5$.

If Imno made 5 friends in the first week, she is certain to make more than 5 friends in total. Hence $\mathbf{P}(B|C) = 1$.

If A happens, B will also happen, so clearly $\mathbf{P}(A \cap B|C) = \mathbf{P}(A|C) = \mathbf{P}(A|C) \cdot \mathbf{P}(B|C)$, therefore A and B are conditionally independent. Note that A and B were not independent prior to the conditioning.

(c) No. We found in part (b) that $\mathbf{P}(A|C) = 1/5$, whereas $\mathbf{P}(A) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$. Hence A and C are not independent. (Note: $\mathbf{P}(A|C) = \frac{\mathbf{P}(A \cap C)}{\mathbf{P}(C)}$ by definition, and independence implies that $\mathbf{P}(A|C) = \frac{\mathbf{P}(A) \cdot \mathbf{P}(C)}{\mathbf{P}(C)} = \mathbf{P}(A)$. Hence $\mathbf{P}(A|C) = \mathbf{P}(A)$ is a necessary and sufficient condition for checking indepence, as long as $\mathbf{P}(C) > 0$.) $\mathbf{P}(B|C) = 1$, as we found above, but clearly $\mathbf{P}(B) < 1$, hence B and C are not independent.

(d) Let F_i where $(i = 1, ..., 5)$ denote the event that in the first week i friends were made. Similarly let S_i denote the event that in the second week i friends were made. Let T_j where $(j = 2, ..., 10)$ denote the event that the total number of friends made in the two weeks is j.

$$\begin{aligned}
\mathbf{P}(\text{2 friends in first week}|\text{6 friends total}) &= \mathbf{P}(F_2|T_6) \\
&= \frac{\mathbf{P}(T_6|F_2) \cdot \mathbf{P}(F_2)}{\mathbf{P}(T_6)} \\
&= \frac{\mathbf{P}(S_4 \cdot)\mathbf{P}(F_2)}{\sum_{i=1}^{5} \mathbf{P}(F_i \cap S_{(6-i)})} \\
&= \frac{\frac{1}{5} \cdot \frac{1}{5}}{5 \cdot \frac{1}{5}\frac{1}{5}} = \frac{1}{5},
\end{aligned}$$

where the second equality uses Bayes' Rule, the third equality uses the Total Probability Theorem, and the fourth equality uses the fact that the numbers of friends made in each week are independent.

Similarly, $\mathbf{P}(F_3|T_6) = 1/5$.

(e)

$$\begin{aligned}
\mathbf{P}(F_2 \cup S_2|T_6) &= \mathbf{P}(F_2|T_6) + \mathbf{P}(S_2|T_6) - \mathbf{P}(F_2 \cap S_2|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - 0 = \frac{2}{5}.
\end{aligned}$$

In the first equality we used the fact that conditional probabilities satisfy all the probability axioms. $\mathbf{P}(F_2|T_6)$ was found above to be $1/5$. Since weeks are identically distributed, $\mathbf{P}(S_2|T_6)$ is also $1/5$. The second equality follows.

$$\begin{aligned}
\mathbf{P}(F_3 \cup S_3 | T_6) &= \mathbf{P}(F_3|T_6) + \mathbf{P}(S_3|T_6) - \mathbf{P}(F_3 \cap S_3|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - \mathbf{P}(S_3|F_3 \cap T_6) \cdot \mathbf{P}(F_3|T_6) \\
&= \frac{1}{5} + \frac{1}{5} - 1 \cdot \frac{1}{5} \; = \; \frac{1}{5}.
\end{aligned}$$

Note that this result makes sense, since there is only one way to meet 6 people by meeting three in at least one week, whereas there are two ways of meeting 6 people by meeting two in at least one. The above probability is one half of the probability of the latter event, which is 2/5.

4. Initially, there are 10 forks and no knives in the left drawer, which we denote by L(10F,0K). Similarly, the right drawer has R(0F,10K). After the roommate takes two forks out of the left drawer and places them in the right drawer, the composition of the two drawers becomes L(8F,0K) and R(2F,10K). For the roommate's second action from the right drawer, the probability of pulling a fork is $2/12 = 1/6$, and the probability of pulling a knife is $10/12 = 5/6$.

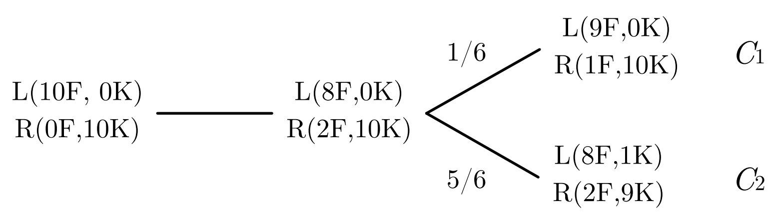


Figure 1: Tree diagram for problem 2

From the initial configuration, the system may have evolved in two possible ways (Please see Fig. 1.) We call these cases C_1 and C_2. Case C_1 refers to a fork being transferred from the right drawer to the left, leaving the two drawers as L(9F,0K) and R(1F,10K). C_2 refers to a knife transferred to the left, resulting in L(8F,1K) and R(2F,9K).

From the above argument, $\mathbf{P}(C_1) = 1/6$, and $\mathbf{P}(C_2) = 5/6$.

From Bayes's Rule,

$$\mathbf{P}(\text{left chosen}|\text{knife pulled}) = \frac{\mathbf{P}(\text{knife pulled}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife pulled})}$$

Using the Total Probability Theorem,

$$\begin{aligned}
\mathbf{P}(\text{knife}|\text{left chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{left chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{left chosen})\mathbf{P}(C_2) \\
&= 0 \cdot \frac{1}{6} + \frac{1}{9} \cdot \frac{5}{6} = \frac{5}{54}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{P}(\text{knife}|\text{right chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{right chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{right chosen})\mathbf{P}(C_2) \\
&= \frac{10}{11} \cdot \frac{1}{6} + \frac{9}{11} \cdot \frac{5}{6} = \frac{5}{6}
\end{aligned}$$

Using Bayes's Theorem,

$$\begin{aligned}
\mathbf{P}(\text{left chosen}|\text{knife}) &= \frac{\mathbf{P}(\text{knife}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife})} \\
&= \frac{5/54 \cdot 1/2}{5/54 \cdot 1/2 + 5/6 \cdot 1/2} \\
&= \frac{1}{10}
\end{aligned}$$

5. The problem statement provides $\mathbf{P}(s_i \text{ sent}) = 1/3$ as well as a table listing the conditional probabilities $\mathbf{P}(s_j \text{ received}|s_i \text{ sent})$.

   (a) Via the Total Probability Theorem, we can calculate the unconditional probability that s_j is received, for each $j = 1, 2, 3$, according to

$$\mathbf{P}(s_j \text{ received}) = \sum_i \mathbf{P}(s_j \text{ received}|s_i \text{ sent})\mathbf{P}(s_i \text{ sent}) = \boxed{\begin{cases} 0.3633 & , \quad j = 1 \\ 0.5167 & , \quad j = 2 \\ 0.12 & , \quad j = 3 \end{cases}} \quad .$$

   (b) For each unique pair (i, j), Bayes's Rule yields

$$\mathbf{P}(s_i \text{ sent}|s_j \text{ received}) = \frac{\mathbf{P}(s_j \text{ received}|s_i \text{ sent})\mathbf{P}(s_i \text{ sent})}{\mathbf{P}(s_j \text{ received})}$$

   where the denominator for each j has already been computed in part (a). Performing this calculation for each pair (i, j), the desired inverse conditional probabilities can be arranged in a table similar to the one given in the problem statement:

		Sent, i		
	$\mathbf{P}(s_i\|s_j)$	s_1	s_2	s_3
	s_1	0.229	0.037	0.735
Received, j	s_2	0.323	0.581	0.097
	s_3	0.694	0.167	0.139

   Note that we have rounded all calculations to the third decimal place. Also, note that (subject to rounding errors) the conditional distribution $\mathbf{P}(s_i \text{ sent}|s_j \text{ received})$ sums to unity for each $j = 1, 2, 3$.

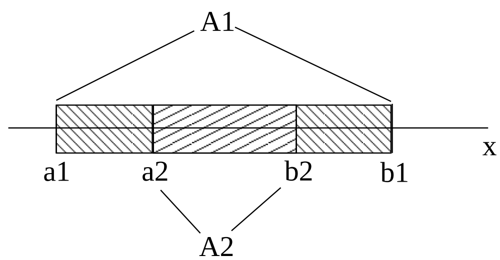
G1†. Since A, B, and C are independent, $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$, $\mathbf{P}(B \cap C) = \mathbf{P}(B)\mathbf{P}(C)$, $\mathbf{P}(A \cap C) = \mathbf{P}(A)\mathbf{P}(C)$, and $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$. To prove that A and $B \cup C$ are independent, we must show that $\mathbf{P}(A \cap (B \cup C)) = \mathbf{P}(A)\mathbf{P}(B \cup C)$

$$\begin{aligned}
\mathbf{P}(A \cap (B \cup C)) &= \mathbf{P}((A \cap B) \cup (A \cap C)) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}((A \cap B) \cap (A \cap C)) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C) \\
&= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)
\end{aligned}$$

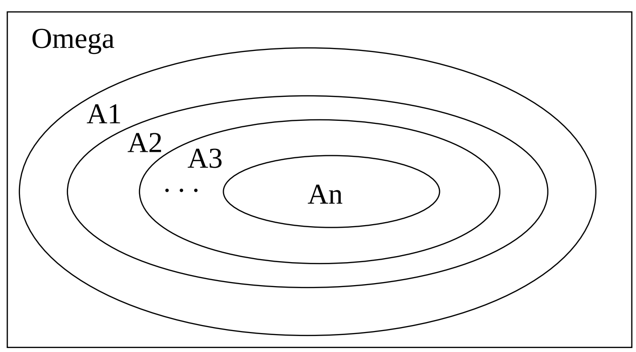
and

$$\begin{aligned}
\mathbf{P}(A)\mathbf{P}(B \cup C) &= \mathbf{P}(A)(\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B \cap C)) \\
&= \mathbf{P}(A)(\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B)\mathbf{P}(C)) \\
&= \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)
\end{aligned}$$

G2†. First, consider the set $A_1 = \{x \mid a_1 \leq x \leq b_1\}$ and the set $A_2 = \{x \mid a_2 \leq x \leq b_2\}$. Since a_n is an increasing sequence, $a_1 \leq a_2$ and since b_n is a decreasing sequence, $b_1 \geq b_2$. As we see in the following diagram, we have $A_2 \subset A_1$.



Continuing this argument up to A_n, we get the following picture:



Finally, $A_\infty = \{x \mid a \leq x \leq b\}$. So, by the above picture, $A_\infty = \lim_{n\to\infty} A_n = (\bigcup_{i=1}^{\infty} A_i^c)^c$.

Observe that $A_n, n \geq 1$ is a decreasing sequence and that $A_n^c, n \geq 1$ is an increasing sequence.

Now, we define the events $B_n, C_n, n \geq 1$ as follows:

$$C_n = A_n^c$$

$$\begin{aligned}
B_1 &= C_1 \\
B_2 &= C_2 \cap C_1^c \\
B_n &= C_n \cap C_{n-1}^c
\end{aligned}$$

Thus, each B_n consists of elements that are not in the previous events and are consequently mutually exclusive. Furthermore:

$$\begin{aligned}
\bigcup_{i=1}^{n} B_i &= C_1 \bigcup (C_2 \cap C_1^c) \bigcup \ldots \bigcup (C_n \cap C_{n-1}^c) \\
\bigcup_{i=1}^{n} B_i &= \bigcup_{i=1}^{n} C_i \\
\bigcup_{i=1}^{\infty} B_i &= \bigcup_{i=1}^{\infty} C_i
\end{aligned}$$

So, by additivity,

$$\begin{aligned}
P(\bigcup_{i=1}^{\infty} C_i) &= P(\bigcup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty} P(B_i) \\
&= \lim_{n\to\infty} \Sigma_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} B_i) \\
&= \lim_{n\to\infty} P(\bigcup_{i=1}^{n} C_i) = \lim_{n\to\infty} P(C_n) \\
&= \lim_{n\to\infty} P(A_n^c)
\end{aligned}$$

Also:

$$P(\bigcup_{i=1}^{\infty} C_i) = P(\bigcup_{i=1}^{\infty} A_i^c) = P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c]$$

$$\begin{aligned}
P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c] &= 1 - P[(\bigcup_{i=1}^{\infty} A_i^c)^c] = 1 - P(A_\infty) \\
\lim_{n\to\infty} P(A_n^c) &= \lim_{n\to\infty} (1 - P(A_n)) = 1 - \lim_{n\to\infty} P(A_n)
\end{aligned}$$

Therefore:

$$\begin{aligned}
1 - P(A_\infty) &= 1 - \lim_{n\to\infty} P(A_n) \\
P(A_\infty) &= \lim_{n\to\infty} P(A_n) \\
P(\{x \mid a \le x \le b\}) &= \lim_{n\to\infty} P(\{x \mid a_n \le x \le b_n\}) \\
P([a,b]) &= \lim_{n\to\infty} P([a_n, b_n])
\end{aligned}$$