Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 6 Solutions
Due: April 1st, 2009

- **Text Sections: 4.1, 4.2, 4.4, 8.3, 8.4**

1. (a) For $t \geq 0$,

$$\mathbf{P}(Y \leq t) = P(\sqrt{|X|} \leq t) = P(-t^2 \leq X \leq t^2) = \Phi(t^2) - \Phi(-t^2) = 2\Phi(t^2) - 1.$$

Therefore, the PDF of Y is $4tf_X(t^2)$ for $t \geq 0$, and equal to 0 for $t < 0$.

(b) For any t,

$$\mathbf{P}(Y \leq t) = P(-\ln|X| \leq t) = P(|X| \geq e^{-t}) = P(X \leq -e^{-t}) + P(X \geq e^{-t}) = 2 - 2\Phi(e^{-t})$$

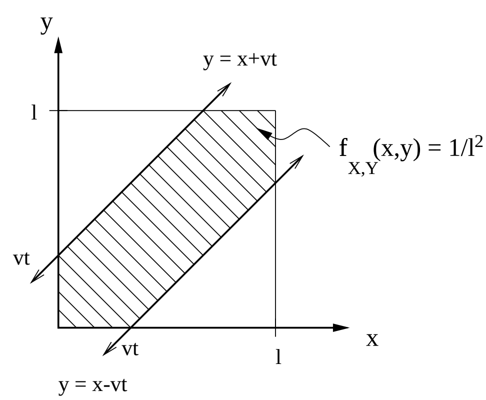
Therefore, the PDF of Y is $2e^{-t}f_X(e^{-t})$ for any t.

2. We want to compute the CDF of the ambulance's travel time T, $\mathbf{P}(T \leq t) = \mathbf{P}(|X - Y| \leq vt)$, where X and Y are the locations of the ambulance and accident (uniform over $[0, \ell]$). Since X and Y are independent, we know:

$$f_{X,Y}(x,y) = \begin{cases} 1/\ell^2 & , \text{ if } 0 \leq x, y \leq \ell \\ 0 & , \text{ otherwise.} \end{cases}$$

$$\begin{aligned} P(T \leq t) &= P(|X - Y| \leq vt) = P(-vt \leq Y - X \leq vt) \\ &= P(X - vt \leq Y \leq X + vt) \end{aligned}$$

We can see that $P(X - vt \leq Y \leq X + vt)$ corresponds to the shaded region in the figure below, which area is equal to $(\ell^2 - (\ell - vt)^2) = 2vt\ell - (vt)^2$.



Therefore we have:

$$F_T(t) = P(X - vt \leq Y \leq X + vt) = \begin{cases} 0 & , \text{ if } t < 0; \\ \frac{1}{\ell^2}(2vt\ell - (vt)^2) = \frac{2vt}{\ell} - \frac{(vt)^2}{\ell^2} & , \text{ if } 0 \leq t < \frac{\ell}{v}; \\ 1 & , \text{ if } t \geq \frac{\ell}{v}. \end{cases}$$

This leads to the result:

$$f_T(t) = \begin{cases} \frac{2v}{\ell} - \frac{2v^2 t}{\ell^2} & , \text{ if } 0 \leq t \leq \frac{\ell}{v}; \\ 0 & , \text{ otherwise.} \end{cases}$$

3. For this problem we will need to compute the 1st, 2nd, 3rd, and 4th moments of the standard normal distribution. To facilitate this, we will find the moment generating function:

$$\begin{aligned}
\mathbf{E}[e^{rx}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rx} \cdot e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{1}{2}r^2} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-r)^2} dx \\
&= e^{\frac{1}{2}r^2}
\end{aligned}$$

the second equality above following from completing the square in the exponent, and the third following because the Gaussian density function integrates to 1. Therefore we now take derivatives w.r.t. r, and easily find the first four moments:

$$\mathbf{E}[X] = 0, \qquad \mathbf{E}[X^2] = 1, \qquad \mathbf{E}[X^3] = 0, \qquad \mathbf{E}[X^4] = 3.$$

We know that the correlation coefficient is given by:

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}.$$

We first compute the covariance:

$$\begin{aligned}
\text{cov}(X, Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[aX + bX^2 + cX^3] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= a\mathbf{E}[X] + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] \\
&= b.
\end{aligned}$$

Now $\text{var}(X) = 1$ therefore $\sigma_X = 1$ so we have left to find $\sigma_Y = \sqrt{\text{var}(Y)}$.

$$\begin{aligned}
\text{var}(Y) &= \text{var}(a + bX + cX^2) \\
&= \mathbf{E}[(a + bX + cX^2)^2] - \mathbf{E}[a + bX + cX^2]^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2
\end{aligned}$$

and therefore we find:

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

To find the best linear estimator, note that $E[Y] = a + c$. Use this and $\rho(X, Y)$ found above in the standard linear estimation formula to get:

$$\begin{aligned}
\hat{Y}_{Linear}(X) &= E[Y] + \rho(X, Y)\frac{\sigma_Y}{\sigma_X}(X - E[X]) \\
&= a + c + \frac{b}{\sqrt{b^2 + 2c^2}}\sqrt{b^2 + 2c^2}X \\
&= a + c + bX
\end{aligned}$$

4. a) X, Y cannot be independent, since given X we know the value of Y to within two values, and hence it is easy to show that:

$$f(x|y) \neq f(x).$$

b) Y, Z are independent because X is symmetric about around the ordinate (i.e. what we typically call the Y-axis).
c)

$$\begin{aligned} f_{YZ}(y,z) &= f_{Y|Z}(y|z) \cdot f_Z(z) \\ &= f_X(x) \cdot f_Z(z) \end{aligned}$$

and therefore:

$$f_Y(y) = \sum_I f_X(x) \cdot f_Z(z) = f_X(x)$$

and therefore $Y \sim N(0,1)$ as desired.
d) We want to show that $\text{cov}(X, Y) = 0$. Since $E[X] = E[Y] = 0$, we have:

$$\begin{aligned} \text{cov}(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{Y|X}(y|x) \cdot f_X(x)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\frac{1}{2}(\delta(x) + \delta(-x) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} y[xf_X(x) - xf_X(x)] \\ &= 0 \end{aligned}$$

as required. The last equality follows from the fact that since X is a standard normal random variable, $f_X(x) = f_X(-x)$. Note that we have two dependent normal random variables X, Y that have zero correlation. There is a small subtlety here. We know that if two random variables have bivariate joint distribution, and are uncorrelated, then they are independent. However in this case, we have two dependent normal random variables, whose correlation is zero. The difference here is that the joint distribution is not bivariate normal.

5. (a) The definition of the transform of X is

$$M(s) = \mathbf{E}[e^{sX}]$$

Therefore, we know the following must be true:

$$M(0) = \mathbf{E}[e^{0X}] = \mathbf{E}[1] = 1.$$

So in our case

$$M(0) = \frac{c}{\sqrt{0+1}} = 1$$

and

$$c = 1.$$

(b) $\mathbf{E}[X] = \frac{d}{ds}M(s)\Big|_{s=0} = \frac{d}{ds}(\frac{1}{\sqrt{s+1}})\Big|_{s=0} = \frac{-1}{2}(s+1)^{\frac{-3}{2}}\Big|_{s=0} = -\frac{1}{2}$.

$\mathbf{E}[X^2] = \frac{d^2}{ds^2}M(s)\Big|_{s=0} = \frac{d^2}{ds^2}(\frac{1}{\sqrt{s+1}})\Big|_{s=0} = \frac{3}{4}(s+1)^{\frac{-5}{2}}\Big|_{s=0} = \frac{3}{4}$.

$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{3}{4} - (\frac{-1}{2})^2 = \frac{1}{2}$.

6. (a) The area under $f_Y(y)$ should sum to 1, which gives us $c = \frac{1}{5}$.

(b)

$$\begin{aligned} M_Y(s) &= \mathbf{E}[e^{sY}] \\ &= \frac{3}{5}\int_{-2}^{-1} e^{sy}dy + \frac{1}{5}\int_0^2 e^{sy}dy \\ &= \frac{3}{5s}[e^{-s} - e^{-2s}] + \frac{1}{5s}[e^{2s} - 1]. \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)dy \\ &= \frac{3}{5}\int_{-2}^{-1} ydy + \frac{1}{5}\int_0^2 ydy \\ &= \frac{3}{5}\frac{y^2}{2}\Big|_{-2}^{-1} + \frac{1}{5}\frac{y^2}{2}\Big|_0^2 \\ &= -\frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \mathbf{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)dy \\ &= \frac{3}{5}\int_{-2}^{-1} y^2dy + \frac{1}{5}\int_0^2 y^2dy \\ &= \frac{3}{5}\frac{y^3}{3}\Big|_{-2}^{-1} + \frac{1}{5}\frac{y^3}{3}\Big|_0^2 \\ &= \frac{29}{15} \end{aligned}$$

Therefore $\text{var}[Y] = E[Y^2] - (E[Y])^2 = \frac{101}{60}$.

(d) Using the independence of X and Y, we can write:

$$\begin{aligned} M_W(s) &= \mathbf{E}[e^{s(\alpha X + \beta Y + \gamma)}] \\ &= \mathbf{E}[e^{\alpha s X}e^{\beta s Y}e^{\gamma s}] \\ &= e^{s\gamma}M_X(\alpha s)M_Y(\beta s) \end{aligned}$$

(e) Looking at $M_X(s)$, we can conclude that X is uniform between 3 and 4, i.e.,

$$f_X(x) = \begin{cases} 1, & 3 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Because X and Y are independent and $W = X + Y$, the pdf of W, $f_W(w)$, can be written as the convolution of $f_X(x)$ and $f_Y(y)$:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx$$

Flipping $f_Y(y)$ and shifting by w, we can carry out the convolution as follows,

$$f_W(w) = \begin{cases} \int_3^{w+2} f_X(x)f_Y(w-x)dx, & 1 \le w \le 2 \\ \int_{w+1}^{4} f_X(x)f_Y(w-x)dx, & 2 \le w \le 3 \\ \int_3^{w} f_X(x)f_Y(w-x)dx, & 3 \le w \le 4 \\ \int_3^{4} f_X(x)f_Y(w-x)dx, & 4 \le w \le 5 \\ \int_{w-2}^{4} f_X(x)f_Y(w-x)dx, & 5 \le w \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_W(w) = \begin{cases} \frac{3}{5}(w-1), & 1 \le w \le 2 \\ \frac{3}{5}(3-w), & 2 \le w \le 3 \\ \frac{1}{5}(w-3), & 3 \le w \le 4 \\ \frac{1}{5}, & 4 \le w \le 5 \\ \frac{1}{5}(6-w), & 5 \le w \le 6 \\ 0, & \text{otherwise.} \end{cases}$$

G1†. (a) Using the definition of the transform of X, we obtain:

$$M_X(s) = e^{sa}.$$

(b) $Y \in [0, L]$, therefore $\mathbf{E}[Y^k] \in [0, L^k]$. For any $s \in \mathbb{R}$,

$$\begin{aligned} \mathbf{E}\left[|e^{sY}|\right] &= \mathbf{E}\left[\left|\sum_{k=0}^{\infty} \frac{1}{k!} s^k Y^k\right|\right] \\ &\le \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!} |s|^k Y^k\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} |s|^k \mathbf{E}[Y^k] \qquad \text{(by Fubini's)} \\ &\le \sum_{k=0}^{\infty} \frac{1}{k!} |s|^k L^k \\ &= e^{|s|L} \\ &< \infty \ \text{ for all } s \in (-\delta, \delta), \text{ for any finite } \delta > 0. \end{aligned}$$

(c) We have $S_n = \frac{1}{n}(Y_1 + \cdots + Y_n) = \frac{Y_1}{n} + \cdots + \frac{Y_n}{n}$.

$$\begin{aligned}
M_{\frac{Y_n}{n}}(s) &= \mathbf{E}\left[e^{\frac{Y_1}{n}s}\right] \\
&= \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!} s^k \frac{Y_1^k}{n^k}\right] \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} s^k \frac{1}{n^k} \mathbf{E}[Y_1^k] \qquad \text{(use results of (b) and Fubini's)} \\
&= 1 + s\frac{\mathbf{E}[Y_1]}{n} + \sum_{k=2}^{\infty} \frac{1}{n^k} s^k \frac{\mathbf{E}[Y^k]}{k!}
\end{aligned}$$

Therefore,

$$\left[M_{\frac{Y_n}{n}}(s) - 1 - s\frac{\mathbf{E}[Y_1]}{n}\right] = \sum_{k=2}^{\infty} \frac{1}{n^k} s^k \frac{\mathbf{E}[Y^k]}{k!} \tag{1}$$

Now, the right handside in equation (1) can be capped as follows,

$$\begin{aligned}
\left|\sum_{k=2}^{\infty} \frac{1}{n^k} s^k \frac{\mathbf{E}[Y^k]}{k!}\right| &\leq \sum_{k=2}^{\infty} \frac{1}{n^k} |s|^k \frac{\mathbf{E}[Y^k]}{k!} \\
&\leq \frac{1}{n^2}\left[\sum_{k=2}^{\infty} \frac{1}{n^{k-2}} |s|^k \frac{L^k}{k!}\right] \\
&\leq \frac{1}{n^2}\left[\sum_{k=0}^{\infty} |s|^k \frac{L^k}{k!}\right] = \frac{1}{n^2} e^{|s|L}.
\end{aligned}$$

Now,

$$\begin{aligned}
M_{S_n}(s) &= M_{\frac{Y_1}{n}}(s) \dot{M}_{\frac{Y_2}{n}}(s) \cdots M_{\frac{Y_n}{n}}(s) \\
&= (M_{\frac{Y_1}{n}}(s))^n \\
&= (1 + s\frac{\mathbf{E}[Y_1]}{n} + O(\frac{1}{n^2}))^n \\
&= (1 + q_n)^n \qquad \text{where } q_n := s\frac{\mathbf{E}[Y_1]}{n} + O(\frac{1}{n^2}) \\
&= ((1 + q_n)^{\frac{1}{q_n}})^{nq_n}.
\end{aligned}$$

We know that

$$(1 + q_n)^{\frac{1}{q_n}} \to e \quad \text{as} \quad n \to \infty.$$

Also,

$$nq_n = s\mathbf{E}[Y_1] + O(\frac{1}{n}) \to s\mathbf{E}[Y_1] \quad \text{as} \quad n \to \infty.$$

Thus,

$$M_{S_n}(s) \to e^{s\mathbf{E}[Y]} \quad \text{as} \quad n \to \infty.$$

From (a), we find that S_n converges to $\mathbf{E}[Y_1]$.