MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Tutorial 9: Solutions

1. Problem 7.1, page 380 in textbook. See online solutions.

2. (a) Recurrent: 1, 2, 4, 5 , 6; Transient: 3; Periodic: 4,5,6.
   (b) 0.2^n
   (c) This is a geometric random variable with parameter $p = 0.5 + 0.3$. Hence, the expected number of trials up to and includ ing the trial on which the process leaves state 3 is $\mathbf{E}[X] = 1/p = 5/4$.
   (d) 3/8
   (e) $\mathbf{P}(A) = 0.3 + 0.2^3 0.3 + 0.2^6 0.3 + 0.2^9 0.3 = 0.3024$.
   (f) $0.3/\mathbf{P}(A) = 0.992$.

3. The outcome of the next game depends on the outcome of the past two games, thus we need a 4 state Markov chain to model the process. The states will be all the ordered pairs of outcomes of the past two games, where the second entry marks the outcome of the most recent game:

$$\{S_1 = (W, W); S_2 = (W, L); S_3 = (L, W); S_4 = (L, L)\}$$

Therefore the transition probability matrix will be:

$$[P] = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

We see from the chain that it has one recurrent, aperiodic class, and therefore is ergodic. Thus we can apply the fundamental theorem. The theorem tells us that if we can find positive numbers $\{\pi_i\}$ that satisfy:

$$\pi_j = \sum_i \pi_i p_{ij}, \text{ and } \sum_i \pi_i = 1$$

then these $\{\pi_i\}$ are in fact the steady state probabilities. Solving the linear system, we find that:

$$\pi_1 = \frac{5}{20}, \pi_2 = \frac{3}{20}, \pi_3 = \frac{3}{20}, \pi_4 = \frac{9}{20}$$

and the desired probability is $\pi_1 + \pi_3 = \frac{8}{20} = \frac{2}{5}$.

The long run probability that the team will win its next game, i.e., the sum probability of states (W,W) and (L,W), is $\frac{2}{5}$. This conclusion can be reached via $\mathbf{P}(\text{winning next game}) = 0.7 * \pi_1 + 0.4 * \pi_2 + 0.5 * \pi_3 + 0.2 * \pi_4$.