LECTURE 5

- **Readings:** Sections 2.1–2.3, start 2.4

Lecture outline

- Review
- Random variables
- Probability mass function (PMF)
- Expectation

Review: Binomial probabilities

- n independent coin tosses with $\mathbf{P}(\text{H}) = p$

 – $\mathbf{P}(\text{a particular sequence}) = p^{\#\text{ heads}}(1-p)^{\#\text{ tails}}$

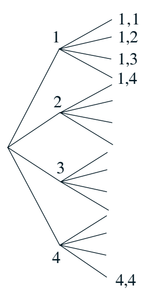
 – $\mathbf{P}(k\text{ Hs}) = \sum_{k-\text{H seq.}} \mathbf{P}(\text{seq.})$

$$= (\#\text{ of } k\text{–H seqs.}) \cdot p^k(1-p)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Outcomes are not numbers

Early dice found in Williamsburg, Virginia.
Photo by Joe Fudge/Daily Press.



Random variables

- Assignment of a value (number) to each possible outcome
- Mathematically: A real-valued function on Ω
 – range can discrete or continuous
- In Chapter 2:
 – range is discrete, **discrete random variable**
 – almost exclusively, range is a subset of the integers

- Notation:
 – random variable X
 – numerical value x

Visualization of discrete random variables

- Can have several random variables defined on the same sample space

Probability mass function (PMF)

- Also "probability law" or "probability distribution"
- Definition and notation for PMF of X:

$$\begin{aligned} p_X(x) &= \mathbf{P}(X = x) \\ &= \mathbf{P}(\{\omega \in \Omega \text{ s.t. } X(\omega) = x\}) \qquad \text{[more carefully]} \end{aligned}$$

 – Defined for all values that X can take

Basic properties of any PMF

- For any x where $p_X(x)$ is defined, $p_X(x) \geq 0$
- $\sum_x p_X(x) = 1$

How to compute a PMF $p_X(x)$

- Collect all possible outcomes for which X is equal to x
- Add their probabilities
- Repeat for all x

- **Example:** Two independent rolls of a fair 4-sided die

$X = \min(F, S)$

		first roll F 1	2	3	4
	1	•	•	•	•
second	2	•	•	•	•
roll S	3	•	•	•	•
	4	•	•	•	•

$p_X(x) =$

Deriving the geometric PMF

- Consider sequences of coin tosses ending with the first H
- Let outcomes be the sequences
- Let probabilities be assigned according to $\mathbf{P}(\text{H}) = p > 0$
- Let X be the length of a sequence (number of tosses until first H)
- Derive the PMF:

$$\begin{aligned} p_X(x) &= \mathbf{P}(X = x) \\ &= \mathbf{P}(\underbrace{\text{T T} \cdots \text{T}}_{x-1 \text{ Ts}}\text{H}) \end{aligned}$$

Binomial PMF

- X: number of heads in n independent coin tosses
- $\mathbf{P}(H) = p$
- Let $n = 4$

$$\begin{aligned} p_X(2) &= \mathbf{P}(\text{HHTT}) + \mathbf{P}(\text{HTHT}) + \mathbf{P}(\text{HTTH}) \\ &\quad + \mathbf{P}(\text{THHT}) + \mathbf{P}(\text{THTH}) + \mathbf{P}(\text{TTHH}) \\ &= 6p^2(1-p)^2 \\ &= \binom{4}{2} p^2 (1-p)^2 \end{aligned}$$

In general:

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

Expectation

- Definition:

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

- Interpretations:
  - Center of gravity of PMF
  - Average in large number of repetitions of the experiment (to be substantiated later in this course)

- Example: Uniform on $0, 1, \ldots, n$

Properties of expectations

- Let X be a r.v. and let $Y = g(X)$
  - Hard: $\mathbf{E}[Y] = \sum_y y p_Y(y)$
  - Easy: $\mathbf{E}[Y] = \sum_x g(x) p_X(x)$

- If α and β are constants:
  - $\mathbf{E}[\alpha] =$
  - $\mathbf{E}[\alpha X] =$
  - $\mathbf{E}[\alpha X + \beta] =$