6.041/6.431: Probabilistic Systems Analysis

Department of Electrical Engineering and Computer Science — MIDTERM 2

MASSACHUSETTS INSTITUTE OF TECHNOLOGY — 18 April 2013

LAST Name ____________________ FIRST Name ____________________

Recitation Time (Circle One): 10 11 12

- **(10 Points)** Print your name and recitation time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may or may not find the following information useful:

Integration by Parts (Indefinite Form):

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x) dx.$$

Let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\, dx$ and $dv = g'(x)\, dx$.
Substitute to obtain

$$\int u\, dv = uv - \int v\, du.$$

Integration by Parts (Definite Form):

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x)\Big]_a^b - \int_a^b f'(x)\, g(x) dx.$$

Let $u = f(x)$ and $v = g(x)$, so $du = f'(x)\, dx$ and $dv = g'(x)\, dx$.
Substitute to obtain

$$\int_a^b u\, dv = \Big[uv\Big]_a^b - \int_a^b v\, du.$$

Some Integrals of Potential Use:

$$\begin{aligned}
\int e^{-\lambda x} dx &= -\frac{e^{-\lambda x}}{\lambda} \\
\int_a^b e^{-\lambda x} dx &= -\frac{e^{-\lambda x}}{\lambda}\Bigg]_a^b = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda} \\
\int_a^b x\, e^{-\lambda x} dx &= -\frac{x\, e^{-\lambda x}}{\lambda}\Bigg]_a^b + \frac{1}{\lambda}\int_a^b e^{-\lambda x} dx.
\end{aligned}$$

Transform Associated With a Random Variable X:

$$M_X(s) \triangleq \mathbf{E}\left[e^{sX}\right].$$

MT2.1 (20 Points) Consider a random variable X whose transform (i.e., moment generating function) is $M_X(s)$. For each candidate transform expression in parts (a) to (d), select the strongest correct statement from the choices below:

(I) The expression *is* a valid transform associated with a random variable.

(II) The expression *can* be a valid transform associated with a random variable, but more information is needed to reach a definitive conclusion.

(III) The expression *cannot* be a valid transform associated with a random variable.

If you choose option (I), express the new random variable (whose transform is given) in terms of X. If you choose option (II), find one random variable having the given transform expression, and establish its relationship with X. Whatever your choice, provide a succinct, but clear and convincing, explanation.

(a) (5 Points) $M_V(s) = 3\, M_X(2s)$.

(b) (5 Points) $M_W(s) = M_X(s) M_X(-s)$.

(c) (5 Points) $M_Y(s) = M_Q(s)M_X(s)$, where $M_Q(s) = \exp[2(e^s - 1)]$.

(d) (5 Points) $M_Z(s) = M_R(s)M_X(s)$, where $M_R(s) = \frac{1}{6}e^{-3s} + \frac{1}{2}e^{-s} + \frac{1}{3}e^{5s}$.

MT2.2 (30 Points) Consider a random quadratic polynomial $Q(x) = Ax^2 + Bx + C$, where A, B, and C are mutually independent random variables uniformly distributed over the interval $[1, 2]$. Let $\widehat{X}$ denote the value of x corresponding to the extremum (global minimum or maximum) of the polynomial Q.

(a) (15 Points) Determine, and provide a well-labeled plot of, $f_{\widehat{X}}(\widehat{x})$, the PDF of $\widehat{X}$.

(b) (10 Points) Determine $\mathbf{E}[\widehat{X}]$.

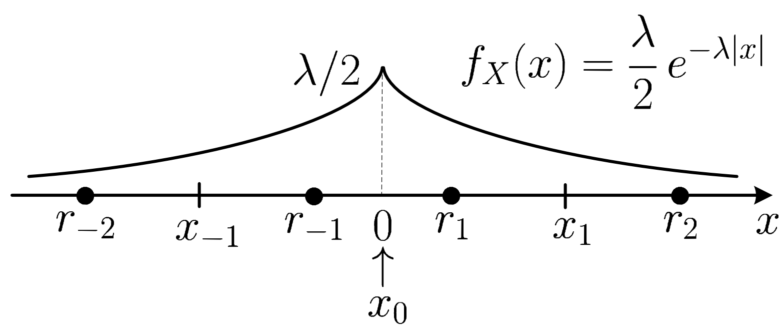
(c) (5 Points) Determine $\text{cov}(\widehat{X}, A)$. Explain why your answer makes sense.

MT2.3 (40 Points)

(a) (10 Points) Consider a random variable X that has a finite mean $\mathbf{E}[X]$, finite variance σ_X^2, and PDF $f_X(x)$. Suppose we want to estimate X with a constant parameter α. Then the quantity $X - \alpha$ denotes the *estimation error*. Show that the value of α that minimizes the *mean squared error* $\mathbf{E}[(X-\alpha)^2]$ is given by $\alpha = \mathbf{E}[X]$.

A similar result holds if we condition on an event A. In particular, the value of α that minimizes the mean squared error $\mathbf{E}\left[(X-\alpha)^2|A\right]$ is given by $\alpha = \mathbf{E}[X|A]$. You need not show the result for the conditional case here; however, feel free to use it if you need to.

For the remainder of this problem, let X be a random variable whose PDF is the double-sided exponential shown below:



We want to encode X onto a two-bit binary number, an example of a discretization scheme known as *quantization*. We divide the x axis into a set of *quantization intervals* demarcated by the *decision boundaries* x_{-1}, x_0, and x_1, as shown in the figure above.

The quantized value Y is then defined as follows:

$$Y = \begin{cases} r_{-2} & \text{if } X < x_{-1} \\ r_{-1} & \text{if } x_{-1} \le X < x_0 \\ r_1 & \text{if } x_0 \le X < x_1 \\ r_2 & \text{if } x_1 \le X. \end{cases}$$

Each of the values r_k is called a *reconstruction level*. We want to design our quantizer so that X is *equally likely* to be mapped to any of the four reconstruction levels. In other words, we want to design our decision boundaries x_{-1} and x_1 so that Y is equally likely to take on any of the values r_{-2}, r_{-1}, r_1, and r_2.

(b) (3 Points) Explain why x_{-1} must be equal to $-x_1$, and $r_{-k} = -r_k$, for $k = 1, 2$.

(c) (10 Points) Determine the decision boundary x_1.

(d) (5 Points) Determine, and provide a well-labeled plot of, the PMF $p_Y(y)$. Also determine the mean $\mathbf{E}[Y]$, and the variance σ_Y^2. Your answers should be in terms of the reconstruction levels r_1 and r_2.

(e) (12 Points) Determine r_1 and r_2 to minimize the total distortion, defined as follows:

$$\mathcal{D} = \mathbf{E}\left[(X-Y)^2\right] = \sum_k \mathbf{E}\left[(X-Y)^2 | Y = r_k\right] \mathbf{P}(Y = r_k).$$

MT2.4 (15 Points) Consider a pair of IID Gaussian random variables X and Y each having a mean of zero and a variance equal to σ^2. Let Z be a third Gaussian random variable defined by $Z = X + Y$.

(a) (5 Points) Determine $f_{Z|X}(z|x)$.

(b) (10 Points) Determine $f_{X|Z}(x|z)$.

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Recitation Time (Circle One): 10 11 12

Problem	Points	Your Score
Name	10	
1	20	
2	30	
3	40	
4	15	
Total	**115**	