Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 5
Due October 24, 2011

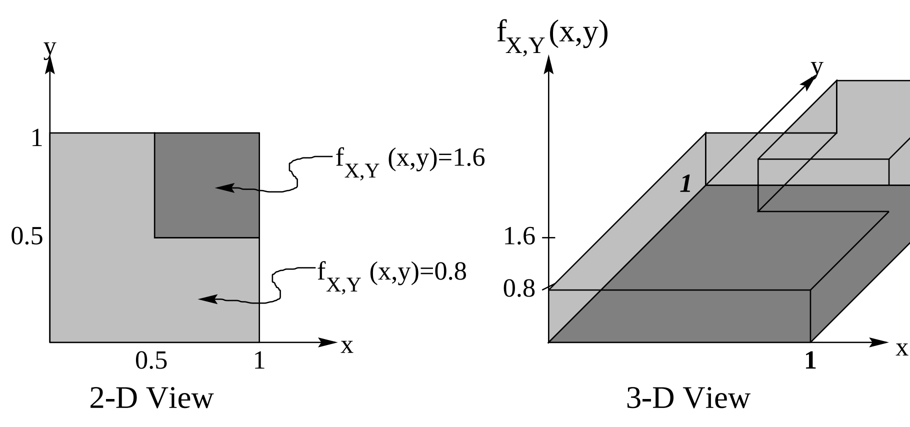
1. Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ay, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.
   (b) Determine the marginal PDF $f_X(x)$.
   (c) Determine the marginal PDF $f_Y(y)$.
   (d) Determine the expected value of X, given that $Y = y$, where $1 \le y \le 2$.
   (e) Consider the random variable $W = X^2$. Find the PDF of W.

2. Aisha has a collection of 10 indistinguishable and unused light bulbs. Three of the bulbs are energy saving and long lasting bulbs with a mean lifetime of 2100 hours. The rest of the bulbs are the more typical incandescent variety with a mean lifetime of 700 hours. The lifetimes of all the light bulbs are exponentially distributed. If she chooses one of the light bulbs from her collection at random for a lamp that she leaves on at all times, what is the PDF, CDF, mean and variance of the time until the bulb burns out? What is the probability the bulb lasts for longer than 1400 hours?

3. Continuous random variables X and Y each take on experimental values between zero and one, with the joint pdf indicated below (the cutoff between probability density 0.8 and 1.6 occurs at $x = 0.5$ and $y = 0.5$):



   (a) Are X and Y independent? Present a convincing argument for your answer.
   (b) Prepare neat, fully labelled plots for $f_X(x)$ and $f_{Y|X}(y \mid 0.75)$.
   (c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.
   (d) Let $W = \min\{X, Y\}$ and determine the cumulative distribution function (CDF) of W. You should be able to reason out this part without doing any formal integrals.

4. A signal $s = 2$ is transmitted from a satellite but is corrupted by noise, so that the received signal is $X = s + W$. When the weather is good, which happens with probability 2/3, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 9. In the absence of any weather information, find the PDF of X and calculate the probability that X is between 1 and 3. (Express the probability using the standard normal CDF Φ.)

5. Consider a first random variable X that is uniformly distributed between 0 and 1, and a second random variable Y, which, conditioned on $X = x$ is uniformly distributed between x and $x + 1$.

   (a) Sketch the region in the (x, y)-plane in which the joint PDF $f_{X,Y}(x, y)$ is nonzero.

   (b) Determine and sketch the marginal PDF of Y.

   (c) Sketch $f_{X|Y}(x|y)$ and determine $\mathbf{E}[X|Y = y]$ for all feasible values of y.

   (d) Find the expected value of $X + Y$.

6. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF
$$f_P(p) = \begin{cases} 2(1-p), & \text{if } p \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$
In essence, a specific coin produced by this machine will have a fixed probability $P = p$ of giving heads, but you do not know initially what that probability is. A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

   (a) Find the probability that the first coin toss results in heads.

   (b) Given that the first coin toss resulted in heads, find the conditional PDF of P.

   (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the second toss.

G1†. **<u>The "Double or Quarter" Game</u>**

In this problem set we will only ask elementary questions about this game, but we will return to it in several subsequent problem sets to develop a coherent theory of gambling (or "investment") known as the Kelly theory of gambling.

In the Double or Quarter game a fair coin is tossed at each round. The player begins with a total wealth of \$1.00. On the first round the player receives back twice his bet if the coin comes up heads (i.e., he gets \$2.00 back) or else a fourth of his bet if it comes up tails (i.e., he gets 25¢ back). On the second round the coin is tossed again, he bets his entire wealth (now either \$2.00 or 25¢), and he receives back twice his bet (either \$4.00 or 50¢) for heads or else one fourth of his bet (either 50¢ or 6.25¢) for tails. The game then continues in the same fashion, with the various tosses being independent and the player betting his entire remaining wealth at each toss.

(a) The player's fortune W_n after n tosses can be written as the product of n independent, identically distributed random variables $X_1, X_2, ..., X_n$. Find the probability mass function of each X_k.

†Required for 6.431; optional for 6.041

(b) Find an expression for the player's expected wealth after the nth toss (assuming his initial bet of $1 was his entire wealth before the game began). Give numerical values for his expected wealth for n=3, 6 and 12.

(c) Find an expression for the probability that his wealth after n tosses is greater than or equal to his starting wealth of $1, and give explicit numerical values for n=3, 6, and 12.

(d) The correct answer to (b) can be understood to imply that this would be an attractive game from the player's point of view, especially as n becomes large. But the correct answer to (c) suggests that the game is rather unattractive from the player's point of view, especially as n becomes large. Please explain and resolve this apparent contradiction.

(e) Find an expression for the standard deviation of the player's wealth after n tosses.