Problem Set 2
Due: February 17, 2010

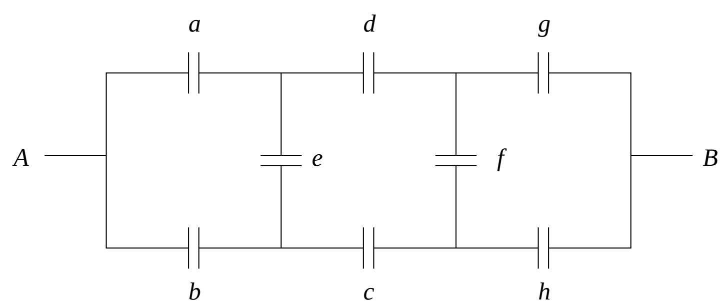
1. Fair die A has five olive faces and one lavender face; fair die B has three faces of each of these colors. A fair coin is tossed once. If it falls heads, the game continues by rolling die A alone; if it falls tails, die B alone is used to continue the game.

   (a) Find the probability that the nth roll of the die results in olive.

   (b) Find the probability that both the nth and $(n+1)$st rolls of the die result in olive.

   (c) If olive readings result from all the first n rolls, find the conditional probability of an olive outcome on the $(n+1)$st roll. What happens as n grows large? Explain this intuitively.

2. In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links. In the network shown below, $\{a,\ b,\ c,\ d,\ e,\ f,\ g,\ h\}$ is the set of links.



   (a) Given that exactly five links have failed, determine the probability that A can still communicate with B.

   (b) Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

   (c) Given that a, d and h have failed (but no information about the state of the other links), determine the probability that A can communicate with B.

3. Fischer and Spassky play a sudden-death chess match. Each game ends up with either a win by Fischer with probability p, a win for Spassky with probability q, or a draw with probability $1-p-q$. The match continues until one of the players wins a game (and hence the match).

   (a) What is the probability that Fischer will win the last game of the match?

   (b) Given that the match lasted no more than 5 games, what is the probability that Fischer won in the first game?

   (c) Given that the match lasted no more than 5 games, what is the probability that Fischer won the match?

   (d) Given that Fischer won the match, what is the probability that he won at or before the 5th game?

4. George and Bob love going to the racetrack. This is a very peculiar track because the same two horses, Heads and Tails, are the only ones that ever race. Heads wins with probability p and Tails with probability $(1-p)$. Bob and George are also somewhat peculiar, because they always insist on betting on the same horse, Heads, and they also never bet on the same race. Bob will let George bet on Heads again and again until Heads loses, at which point Bob will bet on Heads again and again until he loses, and then George bets again and so on. George and Bob have a private bet amongst themselves: If George wins n races before Bob can win m races, then George wins; otherwise Bob wins. Let $P_{n,m}$ be the probability that George, who always bets first, wins n times before Bob wins m times. Find $P_{n,m}$ as a function of $P_{n-1,m}$, $P_{m,n}$ and p.

5. Find a sample space Ω, two probability laws $\mathbf{P}_1$ and $\mathbf{P}_2$, and two events A and B (subsets of Ω) that are independent for $\mathbf{P}_1$ but not for $\mathbf{P}_2$.

G1†. Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers that converges to 0, and let $\{b_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive real numbers that converges to 2. Consider a probabilistic model with sample space equal to the real line (i.e., $\Omega = \mathbb{R}$) and such that every interval

$$D_n = [a_n, b_n] = \{x \in \mathbb{R} : \ a_n \leq x \leq b_n\}$$

is an event (i.e., $\mathbf{P}(D_n)$ is defined for every $n \in \{1, 2, \ldots\}$).

Find an event D (i.e., a set $D \subset \mathbb{R}$ for which $\mathbf{P}(D)$ is defined) for which

$$\lim_{n\to\infty} \mathbf{P}(D_n) = \mathbf{P}(D)$$

is guaranteed to be true.

†Optional (not graded for anyone).