Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 1: Solutions

Due: September 14, 2011

1. (a) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$

   (b) $(A \cap B \cap C)^c = (A^c \cup B^c \cup C^c)$

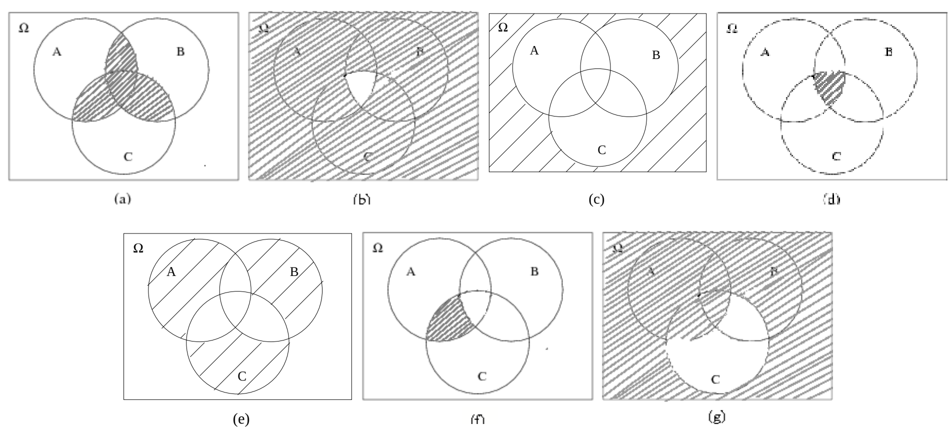
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

   (d) $A \cap B \cap C$

   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

   (f) $A \cap B^c \cap C$

   (g) $A \cup (A^c \cap C^c)$



2. We could have a two-dimensional sample space containing 52^2 points, where each axis represents a particular card. However, this sample space would be finer grain than necessary to determine the desired probabilities.

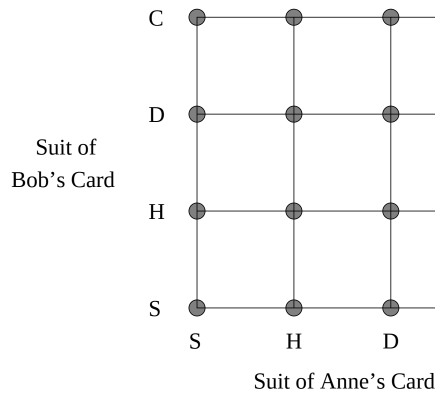
   For parts a) and c), we have a sample space of 169 points representing the 169 possible outcomes.

   Define event B to be when Bob draws an ace, event A to be when Anne draws an ace. Then we know that

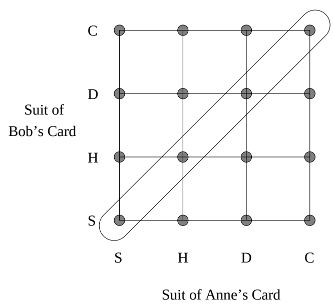
   $\mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{13}$
   $\mathbf{P}(A \cap B) = \frac{1}{169}$
   $\mathbf{P}(\text{at least one card is an ace}) = \mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \boxed{\frac{25}{169}}$

   $\mathbf{P}(\text{neither card is an ace}) = 1\text{- } \mathbf{P}(\text{at least one card is an ace}) = \boxed{\frac{144}{169}}$

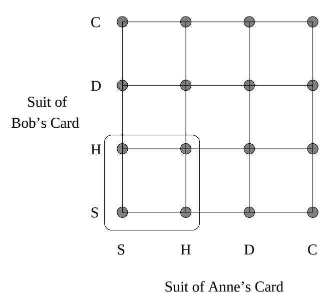
   For parts b) and d), since we are only interested in the suits of the cards, we represent the sample space as the following 16 points. The horizontal axis represents the suit of Anne's card, and the vertical axis represents the suit of Bob's card. Each of the points is equally likely; therefore, the probability of any particular point occurring is $\frac{1}{16}$.



The probabilities requested can be determined by counting the number of points satisfying each condition and dividing the total by 16, as shown in the figures below.



b) 4/16



d) 4/16

3. We employ the discrete probability to find the required probability. Define event A to be

$$A : \{\text{Mary and Tom park next to each other}\}$$

and let k be the total number of ways that Mary and Tom can park their cars in n consecutive spaces. The probability we seek is

$$\mathbf{P}(A) = \frac{\text{number of elements in A}}{k}$$

To find k we note there are n spaces for the first placement and $(n-1)$ for the second placement, resulting in $k = n(n-1)$. To find the numerator we note there are $(n-1)$ pairs for parking spaces which are next to each other. Finally note that for each pair there are 2 ways to order Mary and Tom. Putting everything together we find

$$\mathbf{P}(A) = \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$

4. (a) Using $\mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B)$ and the nonnegativity axiom (the probability of any event, in particular the event $A \cup B$, is nonnegative, i.e. $\mathbf{P}(A \cup B) \geq 0$), we obtain

$$\mathbf{P}(A \cap B) \leq \mathbf{P}(A) + \mathbf{P}(B).$$

(b) Let D denote the event $A \cap B$. From (b) we have $\mathbf{P}(D) = \mathbf{P}(A \cap B) \leq \mathbf{P}(A) + \mathbf{P}(B)$ and $\mathbf{P}(D \cap C) \leq \mathbf{P}(D) + \mathbf{P}(C)$. Therefore $\mathbf{P}(D \cap C) \leq \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$, i.e.

$$\mathbf{P}(A \cap B \cap C) \leq \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C).$$

An alternative way to solve this problem is the following:
For any A, B, $(A \cap B) \subseteq A$. Therefore, $\mathbf{P}(A \cap B) \leq \mathbf{P}(A)$. Using $(B \cap C) \subseteq B$, we get $\mathbf{P}(A \cap B \cap C) \leq \mathbf{P}(A \cap B) \leq \mathbf{P}(A)$. By the nonnegativity axiom, $\mathbf{P}(B) \geq 0$ and $\mathbf{P}(C) \geq 0$. Therefore,

$$\mathbf{P}(A \cap B \cap C) \leq \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C).$$

5. (a) The probability that individual n from group A wins his bet is equal to the ratio of the desired target area to the whole area of the square. Thus $\mathbf{P}(\text{individual } n \text{ wins}) = \frac{\frac{4}{n+1}}{4} = \frac{1}{n+1}$.

(b) The probability that individual n from group A wins but not individual $n+1 =$

$$\frac{\textit{desired target area of ind. } n - \textit{desired target area of ind. } n+1}{\textit{total area of square}} = \frac{\frac{4}{n+1} - \frac{4}{n+2}}{4} = \frac{1}{(n+1)(n+2)}.$$

(c) In order for individuals 1 through n of both groups to win, we must have $x \leq \frac{2}{n+1}$ and $y \leq \frac{2}{n+1}$. For instance, for individuals 1 and 2 from both groups to win their bets, Bill's pitch must land inside the square with vertices $(0,0)$, $(\frac{2}{3}, 0)$, $(\frac{2}{3}, \frac{2}{3})$, and $(0, \frac{2}{3})$.
So, the probability that individuals 1 through n win their bets is the ratio of the area of the square with vertices $(0,0)$, $(\frac{2}{n+1}, 0)$, $(\frac{2}{n+1}, \frac{2}{n+1})$, and $(0, \frac{2}{n+1})$ to the area of the whole square. This ratio equals $\frac{\frac{4}{(n+1)^2}}{4} = \frac{1}{(n+1)^2}$.

(d) Let A_i represent the event that individual i of group A wins his bet. Similarly, let B_i represent the event that individual i of group B wins his bet. For finitely many individuals, the probability that individuals 1 through n from both groups win their bets, $\mathbf{P}(A_1 \cap A_2 ... \cap A_n \cap B_1 \cap B_2 ... \cap B_n)$, equals $\frac{1}{(n+1)^2}$.
As the number of individuals goes to infinity,

$$\lim_{n\to\infty} \mathbf{P}(A_1 \cap A_2 ... \cap A_n \cap B_1 \cap B_2 ... \cap B_n) = \lim_{n\to\infty} \frac{1}{(n+1)^2} = 0$$

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G1†. (a) Let $A_n = [a_n, b_n]$ and $A = [a, b]$. Because the sequence a_n is increasing and the sequence b_n is decreasing, we have $A_n \supset A$, for all n. Furthermore, we have $\cap_{n=1}^{\infty} A_n = A$. The result then follows from the continuity property of probabilities (part (b) of Problem 13 in the text).

(b) The answer is negative. Think of a probability law that assigns unit probability to the single point a. Then, $\mathbf{P}(A_n) = 0$ for all n, but $\mathbf{P}(A) = 1$. The catch here is that the union of the sets A_n is the open interval (a, b), not the set $A = [a, b]$.