Problem Set 7
Due: April 5, 2010

1. Problem 6.3, page 326, of the text.

   A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $p_I = 1/6$, and busy with probability $p_B = 5/6$. During a busy slot, there is probability $p_{1|B} = 2/5$ (respectively, $p_{2|B} = 3/5$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.

   (a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

   (b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is 12.

   (c) Find the expected number of slots up to and including the 5th task from user 1.

   (d) Find the expected number of busy slots up to and including the 5th task from user 1.

   (e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

2. The probability that Iwana Passe fails any quiz is $\frac{1}{4}$. Iwana's performance on each quiz is independent of her performance on all other quizzes.

   (a) Determine the probability that Iwana fails exactly two of the next six quizzes.

   (b) Find the expected number of quizzes that Iwana will pass before she has failed three times.

   (c) Find the probability that the second and third time Iwana fails a quiz will occur when she takes her 8^{th} and 9^{th} quizzes, respectively.

   (d) Determine the probability that Iwana fails two quizzes in a row before she passes two quizzes in a row. *Hint: Write the event of interest as the union of two events, depending on whether or not she passes or fails the first quiz.*

3. A certain police officers stops cars for speeding. The number of red sports cars she stops in one hour follows a Poisson distribution of rate 4, while the number of other cars she stops follows a Poisson process of rate 2.

   Assume that these two processes are independent of each other. Find the probability that this police officer stops at least 2 ordinary cars before she stops 3 red sports cars.

4. Beginning at $t = 0$, requests from customers A and B are received according to independent Poisson processes with rates, respectively, of a and b orders per hour.

   (a) Find the probability that exactly eight of the next twelve requests come from customer A.

   (b) Find the probability that, during an interval of duration t hours, a total of exactly seven requests will be received.

   (c) Find the PMF, expectation, and variance for M, the number of requests from customer A that arrive before the sixth request from customer B.

The number of items in any particular request is either two, three, or four (each with equal probability). The number of items in any particular request is independent of all other requests and of whether the request is from customer A or customer B.

(d) What is the probability that the four earliest requests received are all for exactly the same number of items?

(e) Let N be the total number of items in all requests received during an interval of t hours. Determine the expectation and the variance for N.

(f) Determine the PDF for X, the time from $t = 0$ until a total of exactly five three-item requests from customer A have been received. *Hint: This is easiest if you think of splitting the Poisson process, which you can read about in Section 6.2*

5. Iwana Passe is taking a multiple-choice exam. You may assume that the number of questions is infinite. *Simultaneously, but independently,* her conscious and subconscious faculties are generating answers for her, each in a Poisson manner. (Her conscious and subconscious are always working on different questions.) Conscious responses are generated at the rate λ_c responses per minute. Subconscious responses are generated at the rate λ_s responses per minute. Assume $\lambda_c \neq \lambda_s$. Each conscious response is an independent Bernoulli trial with probability p_c of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability p_s of being correct. Iwana responds only once to each question, and you can assume that her time for recording these conscious and subconscious responses is negligible.

   (a) Determine $p_K(k)$, the probability mass function for the number of *conscious responses* Iwana makes in an interval of T minutes.

   (b) If we pick any question to which Iwana has responded, what is the probability that her answer to that question:

      i. Represents a conscious response

      ii. Represents a conscious correct response

   (c) If we pick an interval of T minutes, what is the probability that in that interval Iwana will make exactly r conscious responses *and* s subconscious responses?

   (d) Determine the probability density function for random variable X, where X is the time from the start of the exam until Iwana makes her first conscious response which is preceded by at least one subconscious response.

   (e) Determine the probability mass function for the total number of responses up to and including her third conscious response.

   (f) The papers are to be collected as soon as Iwana has completed exactly N responses. Determine:

      i. The expected number of questions she will answer correctly

      ii. The probability mass function for L, the number of questions she answers correctly.

   (g) Repeat part (f) for the case in which the exam papers are to be collected at the end of a fixed interval of T minutes.

G1†. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geq 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

$$\mathbf{P}\Big(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \,|\, N(G) = k\Big) .$$

† Required for 6.431 students and optional (not graded) for 6.041.