Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2011)

Problem Set 2: Solutions
Due: February 16, 2011

1. This follows from the definition of conditional probability and simple rearrangement. Suppose $\mathbf{P}(A \mid B) > \mathbf{P}(A)$. Then
$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \mathbf{P}(A \mid B) > \mathbf{P}(A),$$
so
$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} > \mathbf{P}(B), \qquad \text{or} \qquad \mathbf{P}(B \mid A) > \mathbf{P}(B).$$

2. **The Chess Problem.**

   (a) i. $\mathbf{P}(\text{2nd Rnd Req}) = (0.6)^2 + (0.4)^2 = 0.52$

   ii. $\mathbf{P}(\text{Bo Wins 1st Rnd}) = (0.6)^2 = 0.36$

   iii. $\mathbf{P}(\text{Al Champ}) = 1 - \mathbf{P}(\text{Bo Champ}) - \mathbf{P}(\text{Ci Champ})$
   $= 1 - (0.6)^2 * (0.5)^2 - (0.4)^2 * (0.3)^2 = 0.8956$

   (b) i. $\mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req}) = \frac{(0.6)^2}{0.52} = \frac{0.36}{0.52} = 0.6923$

   ii. $\mathbf{P}(\text{Al Champ}|\text{2nd Rnd Req})$
   $= \mathbf{P}(\text{Al Champ}|\text{Bo Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Bo Challenger}|\text{2nd Rnd Req})$
   $+ \ \mathbf{P}(\text{Al Champ}|\text{Ci Challenger, 2nd Rnd Req}) \times \mathbf{P}(\text{Ci Challenger}|\text{2nd Rnd Req})$
   $= (1 - (0.5)^2) \times 0.6923 + (1 - (0.3)^2) \times 0.3077$
   $= 0.7992$

   (c) $\mathbf{P}((\text{Bo Challenger})|\{(\text{2nd Rnd Req}) \cap (\text{One Game})\}) = \frac{(0.6)^2*(0.5)}{(0.6)^2*(0.5)+(0.4)^2*(0.7)}$
   $= \frac{(0.6)^2(0.5)}{0.2920} = 0.6164$

3. It turns out that both of the proposed strategies are equal. Of course, if the hunter lets one dog choose the path, the hunter will go down the correct path with probability p. Now let us consider the event space for the hunter's strategy. The points in the event space which lead to the correct path are:

   (a) Both dogs agree on the correct path.

   (b) Dogs disagree, dog 1 chooses right path, and hunter follows dog 1.

   (c) Dogs disagree, dog 2 chooses right path, and hunter follows dog 2.

   The above events are mutually exclusive, so we can add the probabilities to find that under the hunter's strategy, the probability that he chooses the correct path is:
$$p^2 + \frac{1}{2} \cdot p(1-p) + \frac{1}{2} \cdot p(1-p) = p$$
   as claimed.

4. This is a straightforward application of Bayes's Theorem:
$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$$

Let A be the event that you read in a 1, and let event B_1 be the event that there was in fact a 1, and B_0 be the event that there was in fact a 0. The result follows immediately from this, yielding:

$$P(B_1|A) = .895$$

5. The network of friendship is best represented as a square with diagonals, with the corners labelled A (Anne), B (Betty), C (Chloe), and D (Daisy). Each link of the network is absent with probability p, independent of the status of any other link. We write XY for the event that the direct link XY is present, and XY^c when that link is absent. We write $X \leftrightarrow Y$ for the event that person X's rumour is (possibly indirectly) heard by person Y. Also note that all communication in this problem is bidirectional; that is, if person X's rumour can be heard by person Y, then person Y's rumour can also be heard by person X.

   (a) If the link between A and D is present (event AD), Daisy hears the rumour from Anne directly. If the direct link between A and D is absent (event AD^c, i.e. Anne and Dasiy have quarrelled), Daisy might still hear the rumour depending on the status of the links involving Betty or Chloe. There are $2^6 = 64$ possible network states, implying enumeration of the sample space and summing up outcomes that correspond to $A \leftrightarrow D$ is cumbersome. Rather, we repeatedly apply the Total Probability Theorem and exploit independence between links:

$$\begin{aligned} P(A \leftrightarrow D) &= \underbrace{P(A \leftrightarrow D|AD)}_{1}\underbrace{P(AD)}_{1-p} + P(A \leftrightarrow D|AD^c)\underbrace{P(AD^c)}_{p} \quad , \\ P(A \leftrightarrow D|AD^c) &= P(A \leftrightarrow D|AD^c \cap BC)\overbrace{P(BC|AD^c)}^{1-p} + \\ & \quad P(A \leftrightarrow D|AD^c \cap BC^c)\underbrace{P(BC^c|AD^c)}_{p} \quad , \end{aligned}$$

   Given $AD^c \cap BC$, note that the event $A \leftrightarrow D$ is still true provided that at least one of the links AB and AC is present and at least one of the links BD and CD is present. In terms of the events and set operations, we require $(AB \cup AC) \cap (BD \cup CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC$. Therefore, again relying on independence,

$$\begin{aligned} P(A \leftrightarrow D|AD^c \cap BC) &= P((AB \cup AC) \cap (BD \cup CD)) = P(AB \cup AC)P(BD \cup CD) \\ &= (1 - P(AB^c \cap AC^c))(1 - P(BD^c \cap CD^c)) \\ &= (1 - \underbrace{P(AB^c)P(AC^c)}_{p^2})(1 - \underbrace{P(BD^c)P(CD^c)}_{p^2}) = \left(1 - p^2\right)^2 \quad . \end{aligned}$$

   Given $AD^c \cap BC^c$, note that the event $A \leftrightarrow D$ is still true provided that both of the links AB and BD are present or both of the links AC and CD are present. In terms of the events and set operations, we require $(AB \cap BD) \cup (AC \cap CD)$ for $A \leftrightarrow D$ to still be true given $AD^c \cap BC^c$. Thus,

$$\begin{aligned} P(A \leftrightarrow D|AD^c \cap BC^c) &= P((AB \cap BD) \cup (AC \cap CD)) \\ &= 1 - P((AB \cap BD)^c \cap (AC \cap CD)^c) \\ &= 1 - P((AB \cap BD)^c)P((AC \cap CD)^c) \end{aligned}$$

$$
\begin{aligned}
&= 1-(1-P(AB\cap BD))(1-P(AC\cap CD)) \\
&= 1-(1-\underbrace{P(AB)P(BD)}_{(1-p)^2})(1-\underbrace{P(AC)P(CD)}_{(1-p)^2}) \\
&= 1-\left(1-(1-p)^2\right)^2 \quad .
\end{aligned}
$$

Finally, substituting these previous two answers into the above equations yields

$$\boxed{P(A\leftrightarrow D)=1-p+\left\{\left(1-p^2\right)^2(1-p)+\left[1-\left(1-(1-p)^2\right)^2\right]p\right\}p}$$

(b) Using all the similar reasonings discussed in part (a),

$$
\begin{aligned}
P(A\leftrightarrow D|AB^c) &= \overbrace{P(A\leftrightarrow D|AB^c\cap AD)}^{1}\overbrace{P(AD|AB^c)}^{1-p}+ \\
&\quad P(A\leftrightarrow D|AB^c\cap AD^c)\underbrace{P(AD^c|AB^c)}_{p}
\end{aligned}
$$

$$
\begin{aligned}
P(A\leftrightarrow D|AB^c\cap AD^c) &= P(AC\cap(CD\cup(BC\cap BD))) \\
&= P(AC)\,(1-P(CD^c\cap(BC\cap BD)^c)) \\
&= (1-p)\,(1-P(CD^c)(1-P(BC\cap BD))) \\
&= (1-p)\left(1-p\left(1-(1-p)^2\right)\right)
\end{aligned}
$$

$$\boxed{P(A\leftrightarrow D|AB^c)=1-p+(1-p)\left\{1-p\left[1-(1-p)^2\right]\right\}p} \quad .$$

(c) Again reasoning as in part (a), where we already computed $P(A\leftrightarrow D|AD^c\cap BC^c)$,

$$
\begin{aligned}
P(A\leftrightarrow D|BC^c) &= \underbrace{P(A\leftrightarrow D|BC^c\cap AD)}_{1}\underbrace{P(AD)}_{1-p}+\underbrace{P(A\leftrightarrow D|AD^c\cap BC^c)}_{1-(1-(1-p)^2)^2}\underbrace{P(AD^c)}_{p} \\
&= \boxed{1-p+\left\{1-\left[1-(1-p)^2\right]^2\right\}p} \quad .
\end{aligned}
$$

(d) All the required calculations were done in part (a):

$$\boxed{P(A\leftrightarrow D|AD^c)=\left(1-p^2\right)^2(1-p)+\left[1-\left(1-(1-p)^2\right)^2\right]p} \quad .$$

G1†. (a) Let A denote the event that the man is eventually bankrupted. Let B denote the event that the first toss of the coin shows heads. Using the total probability theorem,

$$P_k(A)=P_k(A|B)P(B)+P_k(A|B^c)P(B^c), \tag{1}$$

where P_k denotes probabilities calculated relative to the starting point k. (Thus, $p_k = P_k(A)$.) Now, $P(B)=P(B^c)=0.5$.

Consider $P_k(A|B)$. For $0 < k < N$, if the first toss is a head, then the man's capital increases to $(k+1)$ units, and the situation can be viewed as though the game starts afresh from a different starting point, namely $(k+1)$. Thus, $P_k(A|B) = P_{k+1}(A) = p_{k+1}$. By similar reasoning, $P_k(A|B^c) = P_{k-1}(A) = p_{k-1}$. Substituting in (1),

$$p_k = \frac{1}{2}(p_{k+1} + p_{k-1}) \quad \text{if } 0 < k < N. \tag{2}$$

(b) The boundary conditions are $p_0 = 1$ (the man is initially bankrupt) and $p_N = 0$ (he already has enough money, so he does not toss the coin even once).

(c) Put $b_k = p_k - p_{k-1}$. Then, (2) implies that $b_{k+1} = b_k$ for $0 < k < N$. Hence, $b_k = b_1$ for $0 < k < N$, and we get:

$$p_k = b_1 + p_{k-1} = 2b_1 + p_{k-2} = \ldots = kb_1 + p_0.$$

This is the general solution to (2). Using the boundary conditions, we obtain $b_1 = -\frac{1}{N}$. Thus, the probability that the man is ultimately bankrupted, is

$$p_k = 1 - \frac{k}{N}.$$

Note that, as the price N rises or his initial capital k falls, ultimate bankruptcy becomes more and more likely.