Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 4
Due October 7, 2009

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} \frac{cy}{x}, & x \in \{1,4,6\} \quad \text{and} \quad y \in \{1,2,3\} \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(2Y < X)$?
   (c) What is $\mathbf{P}(2Y > X)$?
   (d) What is $\mathbf{P}(2Y = X)$?
   (e) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (f) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.
   (g) Find the variances $\text{var}(X)$ and $\text{var}(Y)$.

2. Professor May B. Right often makes mistakes in her science class. She answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture May is asked 1, 2, or 3 questions with equal probability 1/3.

   (a) What is the probability that May gives wrong answers to all the questions she is asked in a given lecture?
   (b) Given that May gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she was asked three questions?
   (c) Let X and Y be the number of questions May is asked and the number of questions she answers correctly in a lecture, respectively. What is the mean and variance of X?
   (d) Give a clearly labeled sketch of the joint PMF $p_{X,Y}(x,y)$.
   (e) To encourage questions in May's class, May's college adopts an unusual incentive policy and offers a bonus of $10X + 20Y$ dollars to May. What are the expected value and the variance of the bonus.
   (f) May's semester has 20 lectures. Let Z be the total number of questions she answers wrong in a semester. What is the mean and variance of Z?

3. Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is a random variable that depends on how long he shops.

   We are told that

$$p_{N|K}(n \mid k) = \frac{1}{k}, \qquad \text{for } n = 1, \dots, k.$$

   (a) Find the joint PMF of K and N.
   (b) Find the mean and variance of N.
   (c) Find the conditional PMF of K given that $N = 3$.
   (d) We are now told that he bought at least 2 but no more than 3 books. Find the conditional mean and variance of K, given this piece of information.

(e) The cost of each book is a random variable with mean 3. What is the expected value of his total expenditure? *Hint:* Condition on events $N = 1, \ldots, N = 4$ and use the total expectation theorem.

4. Let X_1, X_2, $\ldots$, X_n be independent, identically distributed random variables.

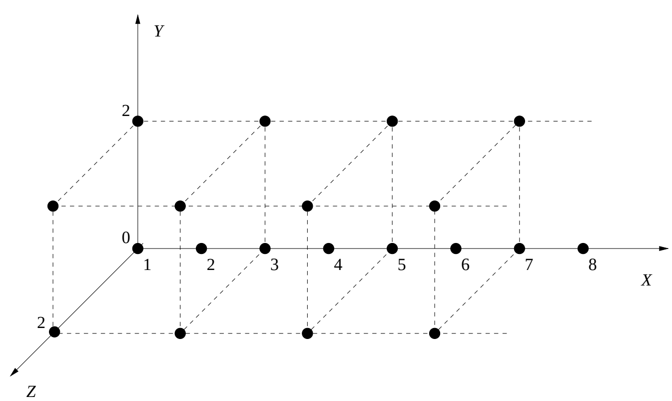
(a) Find c and d in terms of n that will make the following formula true:

$$\mathbf{E}\left[(X_1 + \cdots + X_n)^2\right] = c\mathbf{E}[X_1^2] + d\left(\mathbf{E}[X_1]\right)^2.$$

(b) Find g and h in terms of n that will make the following formula true:

$$\mathbf{E}\left[(X_1 + \cdots + X_n - E[X_1] - \cdots - E[X_n])^2\right] = g\mathbf{E}[X_1^2] + h\left(\mathbf{E}[X_1]\right)^2.$$

5. Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter p. If X is even, then Y and Z are equal to zero. If X is odd, (Y, Z) is uniformly distributed on the set $S = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$. The figure below shows all the possible values for the triple (X, Y, Z) that have $X \leq 8$. (Note that the X axis starts at 1 and that a complete figure would extend indefinitely to the right.)



(a) Find the joint PMF $p_{X,Y,Z}(x, y, z)$.

(b) Answer with "yes" or "no" and one sentence of explanation:

(i) Are Y and Z independent?

(ii) Given that $Z = 2$, are X and Y independent?

(iii) Given that $Z = 0$, are X and Y independent?

(iv) Given that $Z = 2$, are X and Z independent?

(c) Find $\text{var}(Y + Z \mid X = 5)$.