Problem Set 8
Due November 23, 2011

1. For a series of dependent trials, the probability of success on any given trial is given by $(k+1)/(k+3)$, where k is the number of successes in the previous three trials. Define a state description and a set of transition probabilities which allow this process to be described as a Markov chain. Draw the state transition diagram. Try to use the smallest possible number of states.

2. For each of the following definitions of state X_k at time k ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities p_{ij}:

   (a) A six-sided die is rolled repeatedly.

   i. Let X_k denote the largest number rolled in the first k rolls.
   ii. Let X_k denote the number of sixes in the first k rolls.
   iii. At time k, let X_k be the number of rolls since the most recent six.

   (b) Let Y_k be the state of some discrete-time Markov process at time k (i.e., it is known Y_k satisfies the Markov property) with known transition probabilities q_{ij}.

   i. For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.
   ii. Let $X_k = Y_{2k}$.
   iii. Let $X_k = (Y_k, Y_{k+1})$; that is, the state X_k is defined by the sequence of state *pairs* in a given Markov process.

3. As flu season is upon us, we wish to have a Markov chain that models the spread of a flu virus. Assume a population of n individuals. At the beginning of each day, each individual is either infected or susceptible (capable of contracting the flu). Suppose that each pair (i, j), $i \neq j$, independently comes into contact with one another during the daytime with probability p. Whenever an infected individual comes into contact with a susceptible individual, he/she infects him/her. In addition, assume that overnight, any individual who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one restless night battling the flu).

   (a) Suppose that there are m infected individuals at daybreak. What is the distribution of the number of new infections by day end?

   (b) Draw a Markov chain with as few states as possible to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIS (Susceptible-Infected-Susceptible) model.

   (c) Identify all recurrent states.

   Due to the nature of the flu virus, individuals almost always develop immunity after contracting the virus. Consequently, we improve our model and assume that individuals become infected at most one time. Thus, we consider individuals as either infected, susceptible, or recovered.

   (d) Draw a Markov chain to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIR (Susceptible-Infected-Recovered) model.

   (e) Identify all recurrent states.

4. The outcomes of successive flips of a particular coin are dependent and are found to be described fully by the conditional probabilities

$$P(H_{n+1}|H_n) = 3/4 \quad P(T_{n+1}|T_n) = 2/3$$

where we have used the notation: Event H_k: Heads on kth toss; Event T_k: Tails on kth toss. We know that the first toss came up heads.

   (a) Determine the probability that the *first* tail will occur on the kth toss ($k = 2, 3, 4, \ldots$).

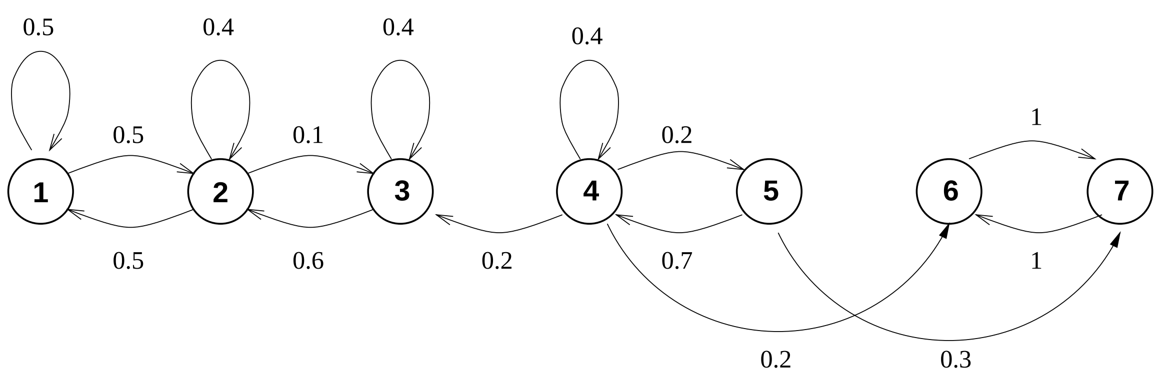
   (b) What is the approximate probability that flip 5000 will come up heads?

   (c) What is the approximate probability that flip 5000 will come up heads and flip 5002 will also come up heads?

   (d) Given that flips $5001, 5002, \ldots, 5000 + m$ all have the same result, what is the approximate probability that all of these m outcomes are heads? Simplify your answer as much as possible, and interpret your result for large values of m.

   (e) We are told that the 375th head just occurred on the 500th toss. Determine the expected value of the number of additional tosses required until we observe the 379th head.

5. Consider the Markov chain below:



   (a) Identify all transient, recurrent, and periodic states.

   (b) Find $r_{12}(3) = \mathbf{P}(X_3 = 2|X_0 = 1)$.

   (c) Find the steady-state probabilities given that we started in state 1, or explain why they do not exist.

   (d) Find the steady-state probabilities given that we started in state 6, or explain why they do not exist.

   (e) Assume that we started in state 1 and that we begin observing the process 10,000,000 transitions from now.

      i. Find the probability there is a birth on the first transition we observe.

      ii. Find the conditional probability the process was in state 2 when we arrived, given a birth occurred on the first transition we observed.

      iii. Find the probability there is a birth on the first change of state we observe.

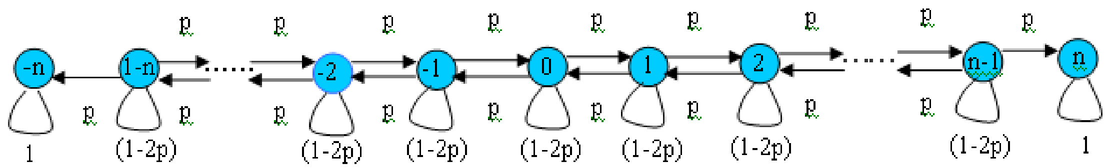
      iv. Suppose the process is in state 2 when we begin observing. What is the probability that the chain was in state 3 at the preceding step?

†Required for 6.431; optional challenge problem for 6.041

G1†. This problem involves first-passage times, which will be covered in Monday's lecture on 11/21. You could read section 7.4 in advance if you want to do this earlier.

An important phenomena in physics and other fields is *diffusion*. The random motion of a particle (e.g., the thermally-driven motion of a single molecule of ink in a bottle of still water) is called diffusion when there is no average direction of motion. An important finding is that the average time t required to diffuse a distance x (in any direction) grows not as the first power of distance but rather as x^2/D, where D is a "diffusion coefficient."

The simplest model for these phenomena is a birth-death Markov chain in which the state represents the x value of position, and the probability the particle moves one step to the left and to the right is $p \leq \frac{1}{2}$:



(a) Let the initial state be 0 and find the expected time for the particle to first reach either 1 or -1.

(b) Find the expected time for it to first reach either 2 or -2 from the initial state 0.

(c) Now find the expected number of steps for the particle to first reach state n or $-n$ from state 0, and use this to determine a useful notion of diffusion coefficient for this system.

Hint: It might be helpful to combine the two terminal states (e.g., states n and $-n$ for part c) into a single absorbing state.

†Required for 6.431; optional challenge problem for 6.041