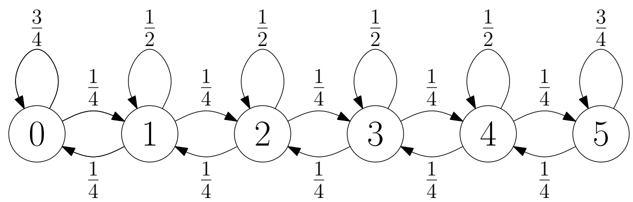
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 9 Solutions
Due November 25, 2009

1. (a) We consider a Markov chain with states $0, 1, 2, 3, 4, 5$, where state i indicates that there are i shoes available at the front door in the morning before Oscar leaves on his run.

Now we can determine the transition probabilities. Assuming i shoes are at the front door before Oscar sets out on his run, with probability $\frac{1}{2}$ Oscar will return to the same door from which he set out, and thus before his next run there will still be i shoes at the front door. Alternatively, with probability $\frac{1}{2}$ Oscar returns to a different door, and in this case, with equal probability there will be $\min\{i+1, 5\}$ or $\max\{i-1, 0\}$ shoes at the front door before his next run. These transition probabilities are illustrated in the following Markov chain:



(b) When there are either 0 or 5 shoes at the front door, with probability $\frac{1}{2}$ Oscar will leave on his run from the door with 0 shoes and hence run barefooted. To find the long-term probability of Oscar running barefooted, we must find the steady-state probabilities of being in states 0 and 5, π_0 and π_5, respectively. Note that the steady-state probabilities exist because the chain is recurrent and aperiodic.

Since this is a birth-death process, we can use the local balance equations. We have

$$\pi_0 p_{01} = \pi_1 p_{10} \ ,$$

implying that

$$\pi_1 = \pi_0$$

and similarly,

$$\pi_5 = \ldots = \pi_1 = \pi_0 \ .$$

As

$$\sum_{i=0}^{5} \pi_i = 1 \ ,$$

it follows that $\pi_i = \frac{1}{6}$ for $i = 0, 1, \ldots, 5$. Hence,

$$\mathbf{P}(\text{Oscar runs barefooted in the long-term}) = \frac{1}{2}\left(\pi_0 + \pi_5\right) = \frac{1}{6} \ .$$

2. The outcomes of successive flips can be viewed as a Markov chain with two states, T and H. The transition probabilities are

$$\begin{aligned} p_{TH} &= 1/3 \\ p_{TT} &= 2/3 \\ p_{HH} &= 3/4 \\ p_{HT} &= 1/4. \end{aligned}$$

Let $X_k, k = 1, \ldots$ denote the outcomes of the flips.

(a) For $k \geq 2$,

$$\begin{aligned}
&P(\text{1st tail occurs on kth toss}|X_1 = H)\\
&= P(\text{first k-2 transitions are } H \to H \text{ and the last transition is } H \to T)\\
&= \left(\frac{3}{4}\right)^{k-2}\frac{1}{4}.
\end{aligned}$$

(b) Irrespective of the starting state, $P(X_{5000} = H) \approx \pi_H$ where π_H, π_T are steady state probabilities. These probabilities $\pi_H = 4/7$ and $\pi_T = 3/7$ are obtained by solving equations

$$\begin{aligned}
\pi_T p_{TH} + \pi_H p_{HH} &= \pi_H\\
\pi_T + \pi_H &= 1
\end{aligned}$$

(c)

$$\begin{aligned}
P(X_{5000} = H, X_{5002} = H) &= P(X_{5000} = H)P(X_{5002} = H|X_{5000} = H)\\
&\approx \pi_H P(X_{5002} = H|X_{5000} = H)\\
&= \pi_H(p_{HT}p_{TH} + p_{HH}p_{HH})\\
&= \frac{4}{7}\left(\frac{1}{4}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{3}{4}\right)\\
&= \frac{124}{336}
\end{aligned}$$

(d)

$$\begin{aligned}
&P(X_{5001} = \cdots = X_{5000+m} = H|X_{5001} = \cdots = X_{5000+m})\\
&= \frac{P(X_{5001} = \cdots = X_{5000+m} = H)}{P(X_{5001} = \cdots = X_{5000+m} = H) + P(X_{5001} = \cdots = X_{5000+m} = T)}\\
&= \frac{P(X_{5001} = H)p_{HH}^{m-1}}{P(X_{5001} = H)p_{HH}^{m-1} + P(X_{5001} = T)p_{TT}^{m-1}} \approx \frac{\pi_H p_{HH}^{m-1}}{\pi_H p_{HH}^{m-1} + \pi_T p_{TT}^{m-1}}\\
&\to 1 \text{ as } m \to \infty
\end{aligned}$$

(e) Let's examine the expected number of additional tosses until the next heads, given heads currently. This can be found by total expectation, by conditioning on what happens on the next toss. Given that the next toss is tails, the number of additional tosses until we observe the next heads is geometric with parameter $\frac{1}{3}$. Therefore, given tails, the expected number of additional tosses required until we observe the next heads is 3. Hence, the expected number of additional flips required until we observe the next heads, given heads on the current toss is

$$p_{HH} \cdot 1 + p_{HT} \cdot (1 + 3) = \frac{7}{4}\,.$$

Given that the 375th heads occurs on the 500th toss, the number of additional flips until the 379th heads can be expressed as the sum of four random variables, each with an expectation equal to 7/4. Thus by linearity of expectation, the required answer is $4 \cdot \frac{7}{4} = 7$.

3. (a) Let t_i be the expected time until the state HT is reached, starting in state i, i.e., the mean first passage time to reach state HT starting in state i. Note that t_S is the expected number of tosses until first observing heads directly followed by tails. We have,

$$\begin{aligned} t_S &= 1 + \frac{1}{2}t_H + \frac{1}{2}t_T \\ t_T &= 1 + \frac{1}{2}t_H + \frac{1}{2}t_T \\ t_H &= 1 + \frac{1}{2}t_H \end{aligned}$$

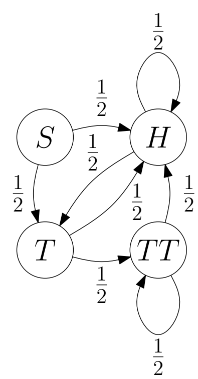
and by solving these equations, we find that the expected number of tosses until first observing heads directly followed by tails is

$$t_S = 4 \ .$$

(b) To find the expected number of additional tosses necessary to again observe heads followed by tails, we recognize that this is the mean recurrence time t^*_{HT} of state HT. This can be determined as

$$\begin{aligned} t^*_{HT} &= 1 + p_{HT,H}t_H + p_{HT,T}t_T \\ &= 1 + \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 4 \\ &= 4 \ . \end{aligned}$$

(c) Let's consider a Markov chain with states S, H, T, TT, where S is a starting state, H indicates heads on the current toss, T indicates tails on the current toss (without tails on the previous toss), and TT indicates tails over the last two tosses. The transition probabilities for this Markov chain are illustrated below in the state transition diagram:



Let t_i be the expected time until the state TT is reached, starting in state i, i.e., the mean first passage time to reach state TT starting in state i. Note that t_S is the expected number of tosses until first observing tails directly followed by tails. We have,

$$\begin{aligned} t_S &= 1 + \frac{1}{2}t_H + \frac{1}{2}t_T \\ t_T &= 1 + \frac{1}{2}t_H \\ t_H &= 1 + \frac{1}{2}t_H + \frac{1}{2}t_T \end{aligned}$$

and by solving these equations, we find that the expected number of tosses until first observing two consecutive tails is

$$t_S = 6 \ .$$

(d) To find the expected number of additional tosses necessary to again observe heads followed by tails, we recognize that this is the mean recurrence time t^*_{TT} of state TT. This can be determined as

$$\begin{aligned} t^*_{TT} &= 1 + p_{TT,H} t_H + p_{TT,TT} t_T T \\ &= 1 + \frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 0 \\ &= 4 \ . \end{aligned}$$

It may be surprising that the average number of tosses until the first two consecutive tails is greater than the average number of tosses until heads is directly followed by tails, considering that the mean recurrence time between pairs of tosses with heads directly followed by tails equals the mean recurrence time between pairs of tosses that are both tails (or equivalently, the long-term frequency of pairs of tosses with heads followed by tails equals the long-term frequency of pairs of tosses with two consecutive tails †). This is a start-up artifact. Note that the distribution of the first passage time to reach state HT (or TT) starting in state S is the same as the conditional distribution of the recurrence time of state HT (or TT), given that it is greater than 1. Although in both cases the *expected values* of the recurrence times are equal (this is what parts (b) and (d) tell us), the conditional expected values of the recurrence time given that it is greater than 1 is not the same in both cases (possible, because the unconditional distributions are not equal).

4. (a) The long-term frequency of winning can be found as sum of the long-term frequency of transitions from 1 to 2 and 2 to 2. These can be found from the steady-state probabilities π_1 and π_2, which are known to exist as the chain is aperiodic and recurrent. The local balance and normalization equations are as follows:

$$\begin{aligned} \frac{7}{15}\pi_1 &= \frac{5}{9}\pi_2 \ , \\ \pi_1 + \pi_2 &= 1 \ . \end{aligned}$$

Solving these we obtain,

$$\pi_1 = \frac{25}{46} \approx 0.54, \ \pi_2 = \frac{21}{46} \approx 0.46 \ .$$

The probability of winning, which is the long-term frequency of the transitions from 1 to 2 and 2 to 2, can now be found as

$$\mathbf{P}(\text{winning}) = \pi_1 p_{12} + \pi_2 p_{22} = \frac{25}{46}\frac{7}{15} + \frac{21}{46}\frac{4}{9} = \frac{21}{46} \approx 0.46 \ .$$

Note that from the balance equation for state 2,

$$\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} \ ,$$

the long-term probability of winning always equals π_2.

† See problem 7.34 on page 399 of the text for a detailed explanation of this correspondence between mean recurrence times and steady-state probabilities.

(b) This question is one of determining the probability of absorption into the recurrent class $\{1A, 2A\}$. This probability of absorption can be found by recognizing that it will be the ratio of probabilities

$$\frac{p_{1,1A}}{p_{1,1A} + p_{1,1B}} = \frac{\frac{2}{15}}{\frac{2}{15} + \frac{1}{15}} = \frac{2}{3} .$$

More methodically, if we define a_i as the probability of being absorbed into the class $\{1A, 2A\}$, starting in state i, we can solve for the a_i by solving the system of equations

$$\begin{aligned} a_1 &= p_{1,1A} + p_{11}a_1 + p_{12}a_2 \\ &= \frac{2}{15} + \frac{1}{3}a_1 + \frac{7}{15}a_2 \\ a_2 &= p_{21}a_1 + p_{22}a_2 \\ &= \frac{5}{9}a_1 + \frac{4}{9}a_2 , \end{aligned}$$

from which we determine that $a_1 = \frac{p_{1,1A}}{p_{1,1A}+p_{1,1B}} = \frac{2}{3}$.

(c) Let A, B be the events that Jack eventually plays with decks $1A$ & $2A$, $1B$ & $2B$, respectively, when starting in state 1. From part (b), we know that $\mathbf{P}(A) = a_1 = \frac{2}{3}$ and $\mathbf{P}(B) = 1 - a_1 = \frac{1}{3}$. The probability of winning can be determined as

$$\mathbf{P}(\text{winning}) = \mathbf{P}(\text{winning}|A)\mathbf{P}(A) + \mathbf{P}(\text{winning}|B)\mathbf{P}(B) .$$

By considering the corresponding the appropriate recurrent class and solving a problem similar to part (a), $\mathbf{P}(\text{winning}|A)$ and $\mathbf{P}(\text{winning}|B)$ can be determined; in these cases, the steady-state probabilities of each recurrent class are defined under the assumption of being absorbed into that particular recurrent class. Let's begin with $\mathbf{P}(\text{winning}|A)$. The local balance and normalization equations for the recurrent class $\{1A, 2A\}$ are

$$\begin{aligned} \frac{3}{5}\pi_{1A} &= \frac{1}{5}\pi_{2A} , \\ \pi_{1A} + \pi_{2A} &= 1 . \end{aligned}$$

Solving these we obtain,

$$\pi_{1A} = \frac{1}{4},\ \pi_{2A} = \frac{3}{4} ,$$

and hence conclude that

$$\mathbf{P}(\text{winning}|A) = p_{1A,2A}\pi_{1A} + p_{2A,2A}\pi_{2A} = \pi_{2A} = \frac{3}{4} .$$

Similarly, the local balance and normalization equations for the recurrent class $\{1B, 2B\}$ are

$$\begin{aligned} \frac{3}{4}\pi_{1B} &= \frac{1}{8}\pi_{2B} , \\ \pi_{1B} + \pi_{2B} &= 1 . \end{aligned}$$

Solving these we obtain,

$$\pi_{1B} = \frac{1}{7},\ \pi_{2B} = \frac{6}{7} ,$$

and hence conclude that

$$\mathbf{P}(\text{winning}|B) = p_{1B,2B}\pi_{1B} + p_{2B,2B}\pi_{2B} = \pi_{2B} = \frac{6}{7} \, .$$

Putting these pieces together, we have that

$$\begin{aligned}\mathbf{P}(\text{winning}) &= \mathbf{P}(\text{winning}|A)\mathbf{P}(A) + \mathbf{P}(\text{winning}|B)\mathbf{P}(B) \\ &= \frac{3}{4}\cdot\frac{2}{3} + \frac{6}{7}\cdot\frac{1}{3} \\ &= \frac{11}{14} \approx 0.79 \, ,\end{aligned}$$

meaning that Jack substantially increases the odds to his favor by slipping additional cards into the decks.

(d) The expected time until Jack slips cards into the deck is the same as the expected time until the Markov chain enters a recurrent state. Let μ_i be the expected amount of time until a recurrent state is reached from state i. We have the equations

$$\begin{aligned}\mu_1 &= 1 + p_{11}\mu_1 + p_{12}\mu_2 = 1 + \frac{1}{3}\mu_1 + \frac{7}{15}\mu_2 \\ \mu_2 &= 1 + p_{21}\mu_1 + p_{22}\mu_2 = 1 + \frac{5}{9}\mu_1 + \frac{4}{9}\mu_2 \, ,\end{aligned}$$

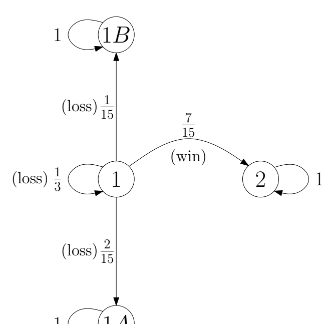
which when solved, yields the expected time until Jack slips cards into the deck,

$$\mu_1 = 9.2 \, .$$

(e) Let S be the number of times that the dealer switches from deck #2 to deck #1, which equals the number of times that he/she switches from deck #1 to deck #2. Let p be the probability that $S = 0$, which is the sum of the probability of all ways for the first change of state to be from state 1 to state 2,

$$p = \left(\frac{2}{15}+\frac{1}{15}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{15}+\frac{1}{15}\right) + \left(\frac{1}{3}\right)^2\left(\frac{2}{15}+\frac{1}{15}\right) + \ldots = \frac{1}{1-1/3}\cdot\frac{3}{15} = \frac{3}{10} \, .$$

Alternatively, p is the probability of absorption of the following modified chain into the recurrent class $\{2\}$, when started in state 1:



As $\mathbf{P}(S > 0) = 1 - p$, and similarly, $\mathbf{P}(S > k+1|S > k) = 1 - p$, it should be clear that S will be a shifted geometric, and thus

$$p_S(k) = \left(\frac{7}{10}\right)^k \frac{3}{10} \quad k = 0, 1, 2, \ldots \, .$$

(f) Note that S from part (e) is the total number of cycles from 1 to 2 and back to 1. During the ith cycle, the number of wins, W_i, is a geometric random variable with parameter $q = \frac{5}{9}$. Thus the total number of wins by Jack before he slips extra cards into the deck is

$$W = W_1 + W_2 + \ldots + W_S ,$$

which is a random number of random variables, all of which are independent. Conditioned on $S > 0$, W is a geometric (with parameter p) number of geometric (with parameter q) random variables, all conditionally independent, and thus from the theory of splitting Bernoulli processes,

$$p_{W|S>0}(k) = (1 - pq)^{k-1} pq \quad k = 1, 2, \ldots ,$$

where $pq = \frac{3}{10} \cdot \frac{5}{9} = \frac{1}{6}$. When $S = 0$, it follows that $W = 0$, and thus by total probability,

$$p_W(k) = \begin{cases} \frac{3}{10} & k = 0 \\ (\frac{7}{10})(\frac{5}{6})^{k-1}\frac{1}{6} & k = 1, 2, \ldots . \end{cases} .$$

(g) Let W be the total number of wins before slipping cards into the deck (as in part (f)), and similarly let L be the total number of losses before absorption. We know from part (d) that $\mathbf{E}[W + L] = \mu_1 = 9.2$. From part (f) we can find $\mathbf{E}[W]$ by total expectation,

$$\mathbf{E}[W] = E[W|S = 0]\mathbf{P}(S = 0) + E[W|S > 0]\mathbf{P}(S > 0) = \frac{7/10}{1/6} = \frac{42}{10} = 4.2 ,$$

because when conditioned on $S > 0$, the number of wins, W, is a geometric random variable with parameter $pq = \frac{1}{6}$. From linearity of expectation, we find

$$\mathbf{E}[L - W] = \mathbf{E}[W + L] - 2\mathbf{E}[W] = 9.2 - 2 \cdot 4.2 = 0.8 .$$

(h) Using A to again denote the probability of being absorbed into the recurrent class $\{1A, 2A\}$, starting in state 1,

$$\begin{aligned} \mathbf{P}(X_n = 2A|X_{n+1} = 1A) &= \frac{\mathbf{P}(X_{n+1} = 1A|X_n = 2A)\mathbf{P}(X_n = 2A)}{\mathbf{P}(X_{n+1} = 1A)} \\ &= \frac{\mathbf{P}(X_{n+1} = 1A|X_n = 2A)\mathbf{P}(X_n = 2A|A)\mathbf{P}(A)}{\mathbf{P}(X_{n+1} = 1A|A)\mathbf{P}(A)} \\ &\approx \frac{p_{2A,1A}\pi_{2A}}{\pi_{1A}} \\ &= \frac{\frac{1}{5} \cdot \frac{3}{4}}{\frac{1}{4}} \\ &= \frac{3}{5} . \end{aligned}$$

Note that the right hand side above equals $p_{1A,2A}$, as clear from the local balance equation $\pi_{1A}p_{1A,2A} = \pi_{2A}p_{2A,1A}$.

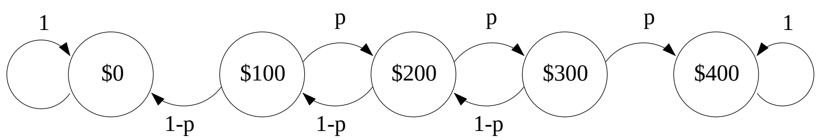
5. (a) We calculate the simple cases first. The easiest decision is when Mary has \$100. She must bet \$100 because she can't bet \$200.

When Mary has \$300, she should bet \$100. Whether she bets \$100 or \$200, she will meet her goal if she wins the next game. If she bets \$100 and loses, she will then have \$200. If she bets \$200 and loses, she will then have \$100. Everything else being equal, it's more advantageous to have \$200 than to have \$100.

The more difficult decision is how much to bet when Mary has \$200. We'll investigate both possible strategies and decide which is preferable.

First, Mary can bet \$200. This case is easy to analyze. She will either win the desired amount or go "bust" on the next game. The probability that she will win is p.

Second, Mary can bet \$100. In this case, we have the following state transition diagram:



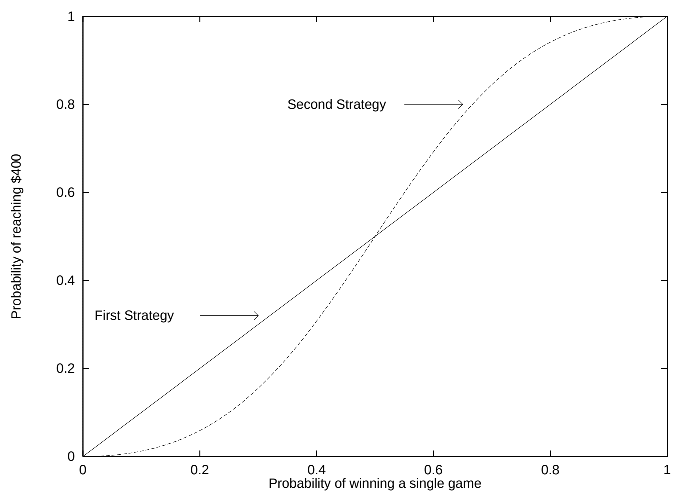
Winning and going "bust" are the two absorption states, and we want to find the probability of eventually winning, given that she starts with \$200. We will denote the probability that Mary wins given that she starts with j hundred dollars by a_j.

$$\begin{aligned} a_1 &= a_2 p \\ a_2 &= a_1(1-p) + a_3 p \\ a_3 &= a_2(1-p) + p \end{aligned}$$

A few simple substitutions yield the following.

$$a_2 = \frac{p^2}{1 - 2p + 2p^2}$$

We need to compare p and a_2 to determine which strategy is optimal. Solving for the condition such that $p > a_2 = \frac{p^2}{1-2p+2p^2}$ yields that betting \$200 is advantageous when $p < 1/2$ and betting \$100 is advantageous when $p > 1/2$. When $p = 1/2$ neither strategy is better than the other. The following figure shows the graph of p against the absorption probability to state 400 for each strategy.



(b) With $p = .75$, the optimal strategy is for Mary to bet \$100 when she has \$200. We need to find the expected time until absorption, given that Mary started with \$200.

Let $\mu_i = \mathbf{E}$[number of transitions to absorption starting with i hundred dollars]. We know that $\mu_0 = 0$ and $\mu_4 = 0$ because these are absorption states. We have the following relationship to determine the other μ_is.

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$$

So, we get the following three equations.

$$\begin{aligned} \mu_1 &= 1 + p\mu_2 \\ \mu_2 &= 1 + (1-p)\mu_1 + p\mu_3 \\ \mu_3 &= 1 + (1-p)\mu_2 \end{aligned}$$

We need to solve for μ_2. Inserting $p = .75$, we get the following values for μ_i:

$$\begin{aligned} \mu_1 &= 3.4 \\ \mu_2 &= 3.2 \\ \mu_3 &= 1.8 \end{aligned}$$

Therefore, the answer is $\boxed{3.2}$.