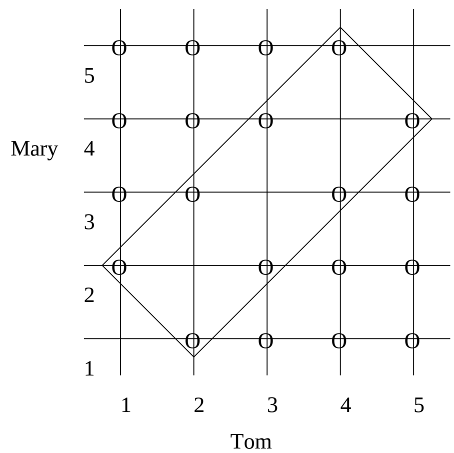
Problem Set 3: Solutions
Due September 30, 2009

1. The problem did not explicitly state that two cars cannot share a parking space, but it was expected that you would assume this when doing the required counting.

   The figure below depicts the sample space for the case of $N = 5$. The 8 outcomes in the box (out of the total of 20 outcomes) are those for which Mary and Tom are parked adjacently.



   Extending this idea to a parking lot with N spaces, the desired probability is given by

$$\begin{aligned}\mathbf{P}(\text{parked adjacently}) &= \frac{\text{number of outcomes with adjacent parking}}{\text{total number of outcomes}} \\ &= \frac{2(N-1)}{N^2 - N} = \boxed{\frac{2}{N}}\end{aligned}$$

2. (a) Each game has two possible outcomes. Thus, the number of distinct score sequences must be $2^{25} = 33,554,432$.

   (b) We define L to be the length of the match. Since the match is stopped when one player wins 13 games, we know that $13 \leq L \leq 25$. Let's assume for a moment that Player A wins. Obviously Player A must win the final game. We can count the number of distinct score sequences by choosing the 12 games which Player A wins out of the first $(L-1)$ games in the match. We also need to double this result to properly account for sequences where Player B wins:

$$2\sum_{L=13}^{25}\binom{L-1}{12} = 2\sum_{k=0}^{12}\binom{12+k}{12} = 10,400,600.$$

   The second formula above uses k to count the number of games that the losing player won before the winner won 13 games.

   Alternatively, we can derive the same answer by padding the sequence with wins for the losing player, up to the length of 25. Now we need to choose the 13 games out of 25 which the winning player wins:

$$2\binom{25}{13} = 10,400,600.$$

This is a bit subtle as the length of the match becomes embedded in our choice of which games the winner wins.

3. (a) Since any card has an equally likely chance of being on top (the deck is well-shuffled), and there are 4 aces out of 52 cards, the probability that the top card is an ace is $\frac{4}{52} = \boxed{\frac{1}{13}}$.

   (b) This probability is again $\frac{4}{52} = \boxed{\frac{1}{13}}$ since any card is equally likely to be second in the deck, and there are 4 aces out of 52 cards. There is also a conditional argument for obtaining this same answer. Note that

$$\begin{aligned}\mathbf{P}(\text{2nd} = \text{ace}) &= \mathbf{P}(\text{2nd} = \text{ace}|\text{1st} = \text{ace})\mathbf{P}(\text{1st} = \text{ace}) \\ &\quad + \mathbf{P}(\text{2nd} = \text{ace}|\text{1st} \neq \text{ace})\mathbf{P}(\text{1st} \neq \text{ace}) = \frac{3}{51}\frac{1}{13} + \frac{4}{51}\frac{12}{13} = \frac{1}{13}.\end{aligned}$$

   (c) Once we draw a king from the deck, there are 51 cards remaining in the deck, 4 of which are aces. Since each card is still equally likely to be anywhere in the deck, the probability that the second card is an ace is $\boxed{4/51}$.

   (d) We first note that there are $\binom{52}{7}$ ways to draw 7 cards from the deck of 52 cards.

      i. There are $\binom{4}{3}$ ways to pick 3 aces from the 4 aces in the deck. There are $\binom{48}{4}$ ways to pick the rest of the 7 cards from the remaining 48 cards that are not aces. Thus there are $\binom{4}{3}\binom{48}{4}$ ways to pick 7 cards with exactly 3 aces. We divide this by the total number of ways we can choose 7 cards to obtain the probability that the 7 cards contain exactly 3 aces: $\boxed{\frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}}$.

      ii. Using arguments identical to those in the previous part, the probability that the 7 cards contain exactly 2 kings is equal to $\boxed{\frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}}$.

      iii. Let event A correspond to getting exactly 3 aces in the 7 cards and event B correspond to getting exactly 2 kings in the 7 cards. Then we are looking for $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. We know $\mathbf{P}(A)$ and $\mathbf{P}(B)$ from the previous two parts. We need to find the $\mathbf{P}(A \cap B)$.
      There are $\binom{4}{3}\binom{4}{2}\binom{44}{2}$ ways to choose 7 cards with exactly 3 aces and 2 kings. This is a straightforward application of partitions. Therefore,

$$\mathbf{P}(A \cap B) = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}$$

      and

$$\mathbf{P}(A \cup B) = \boxed{\frac{\binom{4}{3}\binom{48}{4}+\binom{4}{2}\binom{48}{5}-\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}}$$

4. (a) It is easy to see that X is a Binomial random variable with $n = 40$ and $p = 0.2$:

$$p_X(k) = \binom{40}{k} 0.2^k \, 0.8^{40-k}, \qquad k = 0, \ldots, 40.$$

   and $p_X(k) = 0$ for all other values of k.

(b)

$$\begin{aligned}\mathbf{P}(\text{At least 38 errorless bits}) &= \mathbf{P}(X \le 2) = p_X(0) + p_X(1) + p_X(2) \\ &= \binom{40}{0} 0.8^{40} + \binom{40}{1} 0.2^1\, 0.8^{39} + \binom{40}{2} 0.2^2\, 0.8^{38} \\ &= \boxed{0.00794}.\end{aligned}$$

(c) We let Y be the number of errors in a minute. Y is a Binomial random variable with $n = 6 \cdot 10^7$ and $p = 5 \cdot 10^{-8}$.

$$\begin{aligned}\mathbf{P}(\text{At least 1 error in a minute}) &= \mathbf{P}(Y > 0) = 1 - p_Y(0) \\ &= 1 - (1 - 5 \cdot 10^{-8})^{6 \cdot 10^7} = \boxed{0.9502}\end{aligned}$$

5. (a) For each value of X, we count the number of outcomes that sum to that value:

$$p_X(x) = \begin{cases} 1/9 & x = 2, 6 \\ 2/9 & x = 3, 5 \\ 3/9 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[X] = \sum_{x=2}^{6} x p_X(x) = 2\,\frac{1}{9} + 3\,\frac{2}{9} + 4\,\frac{3}{9} + 5\,\frac{2}{9} + 6\,\frac{1}{9} = \boxed{4}.$$

We can also see that $\mathbf{E}[X] = 4$ because the PMF is symmetric around 4.
To find the variance of X, we first compute

$$\mathbf{E}[X^2] = \sum_{x=2}^{6} x^2 p_X(x) = 4\,\frac{1}{9} + 9\,\frac{2}{9} + 16\,\frac{3}{9} + 25\,\frac{2}{9} + 36\,\frac{1}{9} = \boxed{17\tfrac{1}{3}}.$$

and

$$\mathrm{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \boxed{\tfrac{4}{3}}.$$

(b) By matching the possible values of X and their probabilities to the possible values of Z, we obtain

$$p_Z(z) = \begin{cases} 1/9 & z = 4, 36 \\ 2/9 & z = 9, 25 \\ 3/9 & z = 16 \\ 0 & \text{otherwise.} \end{cases}$$

To find the expectation of Z, we use the results of the previous part:

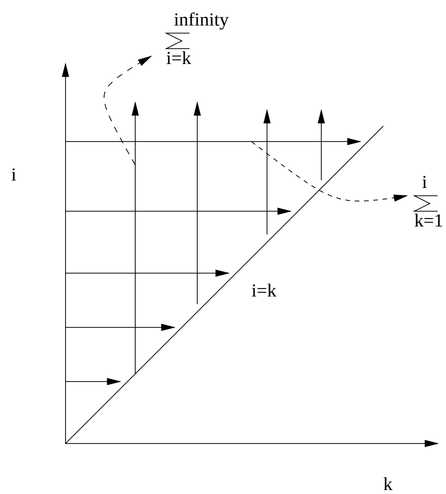
$$\mathbf{E}[Z] = \mathbf{E}[X^2] = \boxed{17\tfrac{1}{3}}.$$

(c) Here again we can use the linearity of expectation and the results above to quickly compute the expectations of Y and W:

$$\begin{aligned} \mathbf{E}[Y] &= 0.5\,\mathbf{E}[X^2] = 8\frac{2}{3} \\ \mathbf{E}[W] &= \mathbf{E}[X^2 - 2X + 1] = \mathbf{E}[X^2] - 2\,\mathbf{E}[X] + 1 = 10\frac{1}{3} \end{aligned}$$

and therefore, the expectation of W is higher than the expectation of Y.

6. (a) The picture below illustrates the double sum needed to prove the statement of this problem:



We first note that

$$\mathbf{P}(X \geq k) = \sum_{i=k}^{\infty} p_X(i)$$

and proceed as follows:

$$\sum_{k=1}^{\infty} \mathbf{P}(X \geq k) = \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} p_X(i) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} p_X(i) = \sum_{i=1}^{\infty} i\, p_X(i) = \mathbf{E}[X].$$

(b) We first compute

$$\mathbf{P}(Y \geq k) = \sum_{i=k}^{\infty} p(1-p)^{i-1} = p(1-p)^{k-1}\sum_{i=0}^{\infty}(1-p)^i = (1-p)^{k-1} \qquad \text{for } k = 1, 2, \ldots$$

Combining part (a) and the above equation, we obtain

$$\mathbf{E}[Y] = \sum_{k=1}^{\infty} \mathbf{P}(Y \geq k) = \sum_{k=1}^{\infty}(1-p)^{k-1} = \boxed{\tfrac{1}{p}}.$$