

**Recitation 26**  
**December 13, 2011**

1. A blackbody at temperature  $\theta$  radiates photons of all wavelengths, described by its characteristic spectrum. This problem will have you estimate  $\theta$ , which is fixed but unknown. The PMF for the number of photons  $K$  in a given wavelength range and a fixed very short time interval is given by,

$$p_K(k; \theta) = \frac{1}{Z(\theta)} e^{-k/\theta}, k = 0, 1, 2, \dots$$

$Z(\theta)$  is a normalization factor for the probability distribution (the physicists call it the partition function). You are given the task of determining the temperature of the body to two significant digits by photon counting in non-overlapping time intervals of duration one second. The photon emissions in non-overlapping time intervals are statistically independent from each other.

- (a) Determine the normalization factor  $Z(\theta)$ .
- (b) Compute the expected value of the photon number measured in any 1 second time interval,  $\mu_K = \mathbf{E}_\theta[K]$ , and its variance,  $\text{var}_\theta(K) = \sigma_K^2$ .
- (c) You count the number  $k_i$  of photons detected in  $n$  non-overlapping 1 second time intervals. Find the maximum likelihood estimator,  $\hat{\theta}_n$ , for temperature  $\theta$ . Note, it might be useful to introduce the average photon number  $s_n = \frac{1}{n} \sum_{i=1}^n k_i$ . In order to keep the analysis simple we assume that the body is hot, i.e.  $\theta \gg 1$ .  
You may use the approximation:  $\frac{1}{e^{1/\theta} - 1} \approx \theta$  for  $\theta \gg 1$ .

In the following questions we wish to estimate the mean of the photon count in a one second time interval using the estimator  $\hat{K}$ , which is given by,

$$\hat{K} = \frac{1}{n} \sum_{i=1}^n K_i.$$

- (d) Find the number of samples  $n$  for which the noise to signal ratio for  $\hat{K}$ , (i.e.,  $\frac{\sigma_{\hat{K}}}{\mu_{\hat{K}}}$ ), is 0.01.
  - (e) Find a 95% confidence interval for the mean photon count estimate for the situation in part (d). (You may use the central limit theorem.)
2. (Problem 18, see page 514.) We want to estimate  $Y$  given  $x$ , assuming a linear relation  $Y_i = \theta_0 + \theta_1 x_i + W_i$ , where  $W_1, \dots, W_n$  are i.i.d. normal random variables with mean zero and variance  $\sigma^2$ . Then the maximum likelihood estimates of  $\theta_0$  and  $\theta_1$  are given by the linear regression formulas, as discussed in Section 9.2. Find the bias of the estimators  $\hat{\Theta}_0$  and  $\hat{\Theta}_1$ .