MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2010)

Problem Set 4
Due: March 3, 2010

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} cxy, & x \in \{1,2,4\} \quad \text{and} \quad y \in \{1,3\} \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(Y < X)$?
   (c) What is $\mathbf{P}(Y > X)$?
   (d) What is $\mathbf{P}(Y = X)$?
   (e) What is $\mathbf{P}(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.
   (h) Find the variances $\text{var}(X)$ and $\text{var}(Y)$.

2. In a certain 2-stage gambling game, equal numbers of cards with the numbers 1, 2 and 3 are shuffled and then placed in a jar. The dealer draws one card at random. Let M =1, 2 or 3 be the number that was drawn. The dealer then flips a fair coin M times, and the number N of heads is the number of dollars the player wins.

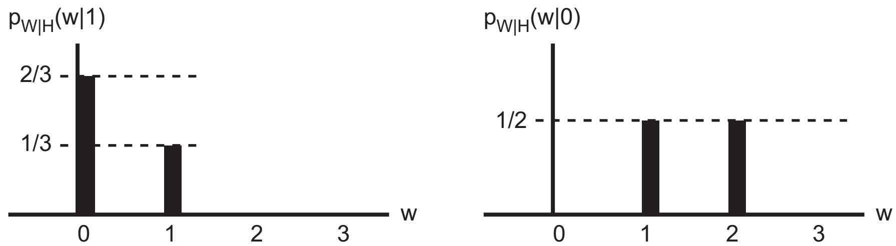
   (a) Find the probability a player wins \$1.00 in a single game.
   (b) Find the expected amount of money a player will win in a single game.
   (c) Find the conditional probability distribution of M, given that the player wins \$1.00. In other words, find $P(M = m|N = 1)$ for m =1,2 and 3.
   (d) Find the probability all the tosses in a single game come up heads.
   (e) The last question concerns multiple games. At the end of each game, the drawn card is replaced in the jar and all the cards are shuffled. Find the expected number of games the player needs to play up to and including the first game in which all the tossed coins come up heads.

3. n balls are thrown into n bins uniformly at random, i.e. each of the n balls is thrown independently into a bin with equal probability of landing in any of the n bins.

   (a) Find the probability that bin 1 is empty.
   (b) Find the expected number of empty bins.

4. Random variable H takes on the value 1 with probability 1/3, and the value of 0 with probability 2/3. Random variable W is described by the conditional probabilities as follows:

$$p_{W|H}(w|1) = \begin{cases} 2/3, & w = 0 \\ 1/3, & w = 1 \\ 0, & \text{otherwise} \end{cases} \qquad p_{W|H}(w|0) = \begin{cases} 1/2, & w = 1 \\ 1/2, & w = 2 \\ 0, & \text{otherwise} \end{cases}$$



(a) Determine the joint PMF of H and W, i.e., $p_{H,W}(h, w)$.

(b) Determine $p_{H|W}(0|1)$.

(c) Are W and H independent? Explain why or why not.

5. Let X and Y be independent random variables. Random variable X has a discrete uniform distribution over the set $\{1, 2, 3\}$, and Y has a discrete uniform distribution over the set $\{1, 3\}$. Let $V = X + Y$, and $W = X - Y$.

(a) Are V and W independent? Explain without calculations.

(b) Find $p_V(v)$. Also, determine $E[V]$ and $var(V)$.

(c) Find $p_{V,W}(v, w)$.

(d) Find $E[V \mid W > 0]$.

(e) Find the conditional variance of W given the event $V = 4$.

(f) Find the conditional PMF $p_{X|V}(x \mid v)$, for all values.

G1†. A spider and a fly move along a straight line. At each second, the fly moves a unit step to the right or to the left with equal probability $p < 1/2$, and stays where it is with probability $1 - 2p$. At each second, the spider takes a unit step in the direction of the fly. The spider and the fly start d units apart, where d is a uniform random variable taking on integer values $1, 2, 3, 4, 5$. If the spider lands on top of the fly, it's the end. Find an expression for the expected value of T, the time it takes for this to happen.