LECTURE 17

Markov Processes – II

- **Readings:** Section 7.3

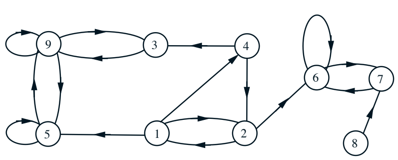
Lecture outline

- Review
- Steady-State behavior
  - Steady-state convergence theorem
  - Balance equations
- Birth-death processes

Review

- Discrete state, discrete time, time-homogeneous
  - Transition probabilities p_{ij}
  - Markov property
- $r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$
- Key recursion:
  $r_{ij}(n) = \sum_k r_{ik}(n-1) p_{kj}$

Warmup



$\mathbf{P}(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$

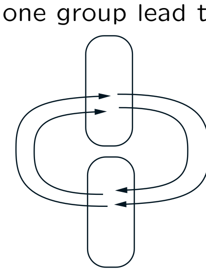
$\mathbf{P}(X_4 = 7 \mid X_0 = 2) =$

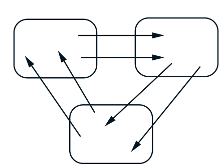
Recurrent and transient states

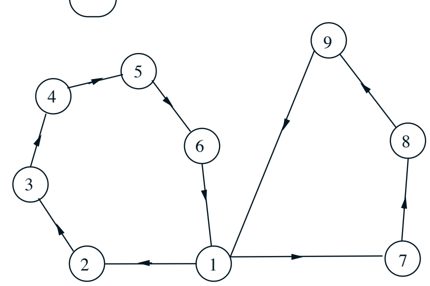
- State i is **recurrent** if:
  starting from i,
  and from wherever you can go,
  there is a way of returning to i
- If not recurrent, called **transient**
- **Recurrent class**:
  collection of recurrent states that
  "communicate" to each other
  and to no other state

Periodic states

- The states in a recurrent class are **periodic** if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group







Steady-State Probabilities

- Do the $r_{ij}(n)$ converge to some π_j? (independent of the initial state i)
- Yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic
- Assuming "yes," start from key recursion

$$r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$$

  - take the limit as $n \to \infty$

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad \text{for all } j$$

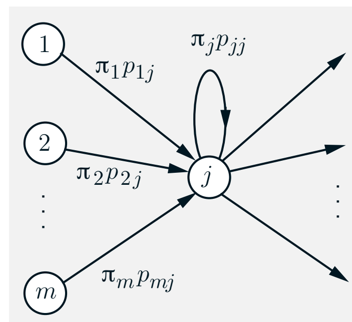
  - Additional equation:

$$\sum_j \pi_j = 1$$

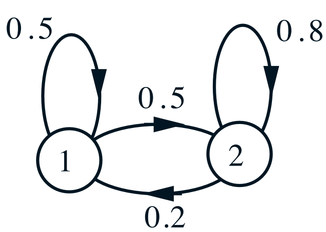
Visit frequency interpretation

$$\pi_j = \sum_k \pi_k p_{kj}$$

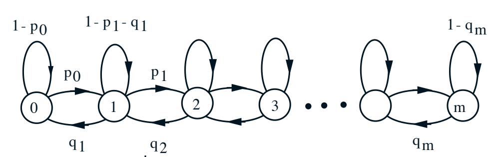
- (Long run) frequency of being in j: π_j
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$
- Frequency of transitions into j: $\sum_k \pi_k p_{kj}$

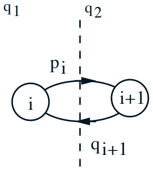


Example



Birth-death processes





$$\pi_i p_i = \pi_{i+1} q_{i+1}$$

- Special case: $p_i = p$ and $q_i = q$ for all i
  $\rho = p/q$ =load factor

$$\pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho$$

$$\pi_i = \pi_0 \rho^i, \qquad i = 0, 1, \ldots, m$$

- Assume $p < q$ and $m \approx \infty$

$\pi_0 = 1 - \rho$

$\mathbf{E}[X_n] = \dfrac{\rho}{1-\rho}$ (in steady-state)