**Tutorial 10 Solutions**
**December 2, 2011**

1. (a) No. Since $X_i$ for any $i \geq 1$ is uniformly distributed between -1.0 and 1.0.

   (b) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}\left(|Y_i - 0| > \epsilon\right) &= \lim_{i\to\infty} \mathbf{P}\left(|\frac{X_i}{i} - 0| > \epsilon\right)\\ &= \lim_{i\to\infty} \left[\mathbf{P}\left(X_i > i\epsilon\right) + \mathbf{P}\left(X_i < -i\epsilon\right)\right] = 0.\end{aligned}$$

   (c) Yes, to 0. Since for $\epsilon > 0$,

$$\begin{aligned}\lim_{i\to\infty} \mathbf{P}\left(|Z_i - 0| > \epsilon\right) &= \lim_{i\to\infty} \mathbf{P}\left(|(X_i)^i - 0| > \epsilon\right)\\ &= \lim_{i\to\infty} \left[\mathbf{P}(X_i > \epsilon^{\frac{1}{i}}) + \mathbf{P}(X_i < -(\epsilon)^{\frac{1}{i}})\right]\\ &= \lim_{i\to\infty} \left[\frac{1}{2}(1-\epsilon^{\frac{1}{i}}) + \frac{1}{2}(1-\epsilon^{\frac{1}{i}})\right] = \lim_{i\to\infty}\left(1 - \sqrt[i]{\epsilon}\right)\\ &= 0.\end{aligned}$$

2. Note that $n$ is deterministic and $H$ is a random variable.

   (a) Use $X_1$, $X_2$, ... to denote the (random) measured heights.

$$\begin{aligned}H &= \frac{X_1 + X_2 + \cdots + X_n}{n}\\ \mathbf{E}[H] &= \frac{\mathbf{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{n\mathbf{E}[X]}{n} = h\\ \sigma_H &= \sqrt{\mathrm{var}(H)} = \sqrt{\frac{n\,\mathrm{var}(X)}{n^2}} \quad \text{(var of sum of \textit{independent} r.v.s is sum of vars)}\\ &= \frac{1.5}{\sqrt{n}}\end{aligned}$$

   (b) We solve $\frac{1.5}{\sqrt{n}} < 0.01$ for $n$ to obtain $n > 22500$.

   (c) Apply the Chebyshev inequality to $H$ with $\mathbf{E}[H]$ and $\mathrm{var}(H)$ from part (a):

$$\begin{aligned}\mathbf{P}(|H-h| \geq t) &\leq \left(\frac{\sigma_H}{t}\right)^2\\ \mathbf{P}(|H-h| < t) &\geq 1 - \left(\frac{\sigma_H}{t}\right)^2\end{aligned}$$

   To be "99% sure" we require the latter probability to be at least 0.99. Thus we solve

$$1 - \left(\frac{\sigma_H}{t}\right)^2 \geq 0.99$$

   with $t = 0.05$ and $\sigma_H = \frac{1.5}{\sqrt{n}}$ to obtain

$$n \geq \left(\frac{1.5}{0.05}\right)^2 \frac{1}{0.01} = 90000.$$

(d) Since $H = \frac{X_1+X_2+...+X_n}{n}$, where $X_i$ is the measured height of the $i$th Canadian,

$$\begin{aligned}\mathbf{P}\left(|H-h| < 0.05\right) &= \mathbf{P}\left(\frac{-0.05}{1.5/\sqrt{n}} \le \frac{H-h}{1.5/\sqrt{n}} \le \frac{0.05}{1.5/\sqrt{n}}\right)\\ &\approx \Phi\left(\frac{0.05}{1.5/\sqrt{n}}\right) - \Phi\left(\frac{-0.05}{1.5/\sqrt{n}}\right).\end{aligned}$$

The normal distribution is symmetric and hence, $\Phi(x)-\Phi(-x) \ge 0.99$ implies $\Phi(x) \ge 0.995$. From the table, we see that this means $x \ge 2.575$. Now,

$$\frac{0.05}{1.5/\sqrt{n}} \ge 2.575 \qquad \Longrightarrow \qquad n \ge 5968.$$

(e) Intuitively, the variance of a random variable $X$ that takes values in the range $[0, b]$ is maximum when $X$ takes the value 0 with probability 0.5 and the value $b$ with probability 0.5, in which case the variance of $X$ is $b^2/4$ and its standard deviation is $b/2$.

More formally, we have for any random variable $X$ taking values in $[0, b]$,

$$\begin{aligned}\text{var}(X) &= \text{var}\left(\mathrm{X} - \frac{\mathrm{b}}{2}\right)\\ &= \mathbf{E}\left[\left(X-\frac{b}{2}\right)^2\right] - \mathbf{E}\left[X-\frac{b}{2}\right]^2\\ &\le \mathbf{E}[(X-\frac{b}{2})^2]\\ &= \mathbf{E}[X^2] - b\mathbf{E}[X] + \frac{b^2}{4}\\ &= \mathbf{E}[X(X-b)] + \frac{b^2}{4}\\ &\le 0 + \frac{b^2}{4},\end{aligned}$$

since $0 \le X \le b \Rightarrow X(X-b) \le 0$. Thus $\sigma_X \le b/2$.

In our example, we have $b = 3$, so $\sigma_X \le 3/2$.

3. (a) The MAP estimate is type $A$ if:

$$\begin{aligned}\mathbf{P}[A|T_1 = t_1] &\ge \mathbf{P}[B|T_1 = t_1]\\ \frac{f_{T_1|A}(t_1)\cdot\mathbf{P}(A)}{f_{T_1}(t_1)} &\ge \frac{f_{T_1|A}(t_1)\cdot\mathbf{P}(B)}{f_{T_1}(t_1)}\end{aligned}$$

Equivalently, we decide that the bulb is of type A if:

$$\begin{aligned}f_{T_1|A}(t_1)\cdot\mathbf{P}(A) &\ge f_{T_1|A}(t_1)\cdot\mathbf{P}(B)\\ \lambda e^{-\lambda t_1}\cdot\frac{2}{3} &\ge \mu e^{-\mu t_1}\cdot\frac{1}{3}\\ \frac{\lambda}{\mu}e^{(\mu-\lambda)t_1} &\ge \frac{1}{2}\\ (\mu-\lambda)t_1 &\ge \ln\left(\frac{\mu}{2\lambda}\right)\end{aligned}$$

Thus, since $(\mu - \lambda) > 0$, the MAP estimate is A if

$$t_1 \geq \ln\left(\frac{\mu}{2\lambda}\right) \cdot \frac{1}{\mu - \lambda},$$

and B otherwise.

(b) Let $\hat{A}$ be the event that the MAP estimate is A, and $\hat{B}$ be the event that the MAP estimate is B. An error occurs whenever the MAP estimate is different from the actual type of the bulb.

$$\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(\hat{A} \cap B) + \mathbf{P}(\hat{B} \cap A) \\
&= \mathbf{P}(\hat{A}|B) \cdot \mathbf{P}(B) + \mathbf{P}(\hat{B}|A) \cdot \mathbf{P}(A) \\
&= \mathbf{P}\left(T_1 \geq \ln\left(\frac{\mu}{2\lambda}\right) \cdot \frac{1}{\mu - \lambda} \mid B\right) \cdot \frac{1}{3} + \mathbf{P}\left(T_1 < \ln\left(\frac{\mu}{2\lambda}\right) \cdot \frac{1}{\mu - \lambda} \mid A\right) \cdot \frac{2}{3} \\
&= e^{-\mu\left(\ln\left(\frac{\mu}{2\lambda}\right) \cdot \frac{1}{\mu - \lambda}\right)} \cdot \frac{1}{3} + \left(1 - e^{-\lambda\left(\ln\left(\frac{\mu}{2\lambda}\right) \cdot \frac{1}{\mu - \lambda}\right)}\right) \cdot \frac{2}{3}
\end{aligned}$$

(c) The LMS estimator of $T_2$ based on observing $T_1$ is $\mathbf{E}[T_2|T_1]$.

$$\begin{aligned}
\mathbf{E}[T_2|T_1 = t_1] &= \mathbf{E}[T_2|T_1 = t_1, A] \cdot \mathbf{P}(A|T_1 = t_1) + \mathbf{E}[T_2|T_1 = t_1, B] \cdot \mathbf{P}(B|T_1 = t_1) \\
&= \mathbf{E}[T_2|A] \cdot \mathbf{P}(A|T_1 = t_1) + \mathbf{E}[T_2|B] \cdot \mathbf{P}(B|T_1 = t_1) \\
&= \frac{1}{\lambda} \cdot \left(\frac{f_{T_1|A}(t_1) \cdot \mathbf{P}(A)}{f_{T_1}(t_1)}\right) + \frac{1}{\mu} \cdot \left(\frac{f_{T_1|B}(t_1) \cdot \mathbf{P}(B)}{f_{T_1}(t_1)}\right) \\
&= \frac{\frac{1}{\lambda}\frac{2}{3}\lambda e^{-\lambda t_1} + \frac{1}{\mu}\frac{1}{3}\mu e^{-\mu t_1}}{\frac{2}{3}\lambda e^{-\lambda t_1} + \frac{1}{3}\mu e^{-\mu t_1}}
\end{aligned}$$

So,

$$\mathbf{E}[T_2|T_1] = \frac{\frac{1}{\lambda}\frac{2}{3}\lambda e^{-\lambda T_1} + \frac{1}{\mu}\frac{1}{3}\mu e^{-\mu T_1}}{\frac{2}{3}\lambda e^{-\lambda T_1} + \frac{1}{3}\mu e^{-\mu T_1}}$$