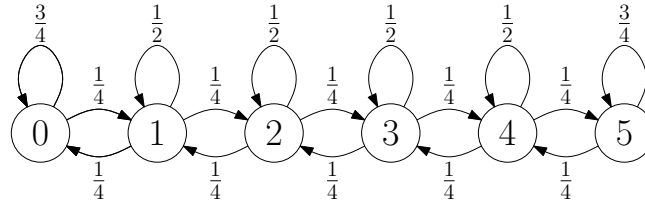


Problem Set 8: Solutions

1. (a) We consider a Markov chain with states 0, 1, 2, 3, 4, 5, where state i indicates that there are i shoes available at the front door in the morning before Oscar leaves on his run.

Now we can determine the transition probabilities. Assuming i shoes are at the front door before Oscar sets out on his run, with probability $\frac{1}{2}$ Oscar will return to the same door from which he set out, and thus before his next run there will still be i shoes at the front door. Alternatively, with probability $\frac{1}{2}$ Oscar returns to a different door, and in this case, with equal probability there will be $\min\{i+1, 5\}$ or $\max\{i-1, 0\}$ shoes at the front door before his next run. These transition probabilities are illustrated in the following Markov chain:



- (b) When there are either 0 or 5 shoes at the front door, with probability $\frac{1}{2}$ Oscar will leave on his run from the door with 0 shoes and hence run barefooted. To find the long-term probability of Oscar running barefooted, we must find the steady-state probabilities of being in states 0 and 5, π_0 and π_5 , respectively. Note that the steady-state probabilities exist because the chain is recurrent and aperiodic.

Since this is a birth-death process, we can use the local balance equations. We have

$$\pi_0 p_{01} = \pi_1 p_{10} ,$$

implying that

$$\pi_1 = \pi_0$$

and similarly,

$$\pi_5 = \dots = \pi_1 = \pi_0 .$$

As

$$\sum_{i=0}^5 \pi_i = 1 ,$$

it follows that $\pi_i = \frac{1}{6}$ for $i = 0, 1, \dots, 5$. Hence,

$$\mathbf{P}(\text{Oscar runs barefooted in the long-term}) = \frac{1}{2} (\pi_0 + \pi_5) = \frac{1}{6} .$$

2. (a) Consider any possible sequence of values $x_1, x_2, \dots, x_{t-1}, i$ for X_1, X_2, \dots, X_t , and note that

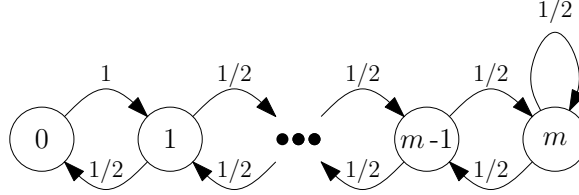
$$\mathbf{P}(|X_{t+1}| = |i| + 1 \mid X_t = i, X_{t-1} = x_{t-1}, \dots, X_1 = x_1) = \begin{cases} \frac{1}{2} & 0 < |i| < m \\ 1 & i = 0 \\ 0 & |i| = m \end{cases} ,$$

$$\mathbf{P}(|X_{t+1}| = |i| \mid X_t = i, X_{t-1} = x_{t-1}, \dots, X_1 = x_1) = \begin{cases} \frac{1}{2} & |i| = m \\ 0 & |i| \neq m \end{cases} ,$$

$$\mathbf{P}(|X_{t+1}| = |i| - 1 | X_t = i, X_{t-1} = x_{t-1}, \dots, X_1 = x_1) = \begin{cases} \frac{1}{2} & 0 < |i| \leq m \\ 0 & i = 0 \end{cases},$$

$$\mathbf{P}(|X_{t+1}| = j | X_t = i, X_{t-1} = x_{t-1}, \dots, X_1 = x_1) = 0, \quad ||i| - j| > 1.$$

As the conditional probabilities above only depend on $|i|$, where $|X_t| = |i|$, it follows that $|X_1|, |X_2|, \dots$ satisfy the Markov property. The associated Markov chain is illustrated below.



(b) Note that Y_1, Y_2, \dots is not a Markov chain for $m > 1$, because

$$\mathbf{P}(Y_{t+1} = d + 1 | Y_t = d, Y_{t-1} = d - 1) = \frac{1}{2}$$

does not equal

$$\mathbf{P}(Y_{t+1} = d + 1 | Y_t = d, Y_{t-1} = d, Y_{t-2} = d - 1) = 0,$$

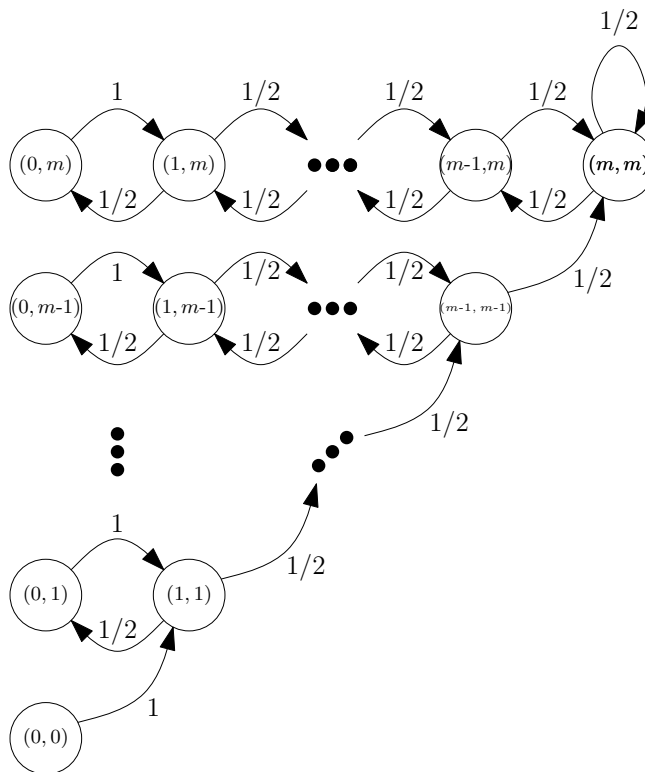
for $0 < d < m$ (the idea is that if $Y_{t-2} = d - 1, Y_{t-1} = d$, and $Y_t = d$, then $|X_t| = d - 1$, while if $Y_{t-1} = d - 1$, and $Y_t = d$, then $|X_t| = d$). If, however, we keep track of $|X_t|$ and Y_t , we do have a Markov chain, because for any possible sequence of pairs of values $(x_1, y_1), \dots, (x_{t-1}, y_{t-1}), (i_1, i_2)$ for $(|X_1|, Y_1), \dots, (|X_{t-1}|, Y_{t-1}), (|X_t|, Y_t)$,

$$\begin{aligned} \mathbf{P}(|X_{t+1}|, Y_{t+1}) = (i_1 + 1, i_2 + 1) \quad & | \quad (|X_t|, Y_t) = (i_1, i_2), \dots, (|X_1|, Y_1) = (x_1, y_1)) \\ & = \begin{cases} \frac{1}{2} & 0 < i_1 = i_2 < m \\ 1 & i_1 = i_2 = 0 \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

$$\begin{aligned} \mathbf{P}(|X_{t+1}|, Y_{t+1}) = (i_1 - 1, i_2) \quad & | \quad (|X_t|, Y_t) = (i_1, i_2), \dots, (|X_1|, Y_1) = (x_1, y_1)) \\ & = \begin{cases} \frac{1}{2} & 0 < i_1 \leq i_2 \leq m \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

$$\begin{aligned} \mathbf{P}(|X_{t+1}|, Y_{t+1}) = (i_1, i_2) \quad & | \quad (|X_t|, Y_t) = (i_1, i_2), \dots, (|X_1|, Y_1) = (x_1, y_1)) \\ & = \begin{cases} \frac{1}{2} & i_1 = i_2 = m \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

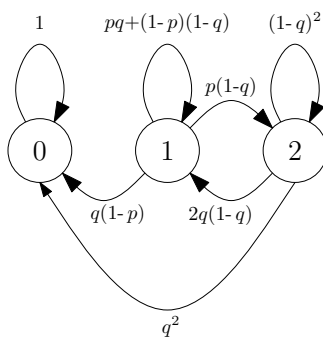
from which it is clear that the conditional probabilities only depend on (i_1, i_2) , the values of $|X_t|$ and Y_t , respectively. The corresponding Markov chain is illustrated below.



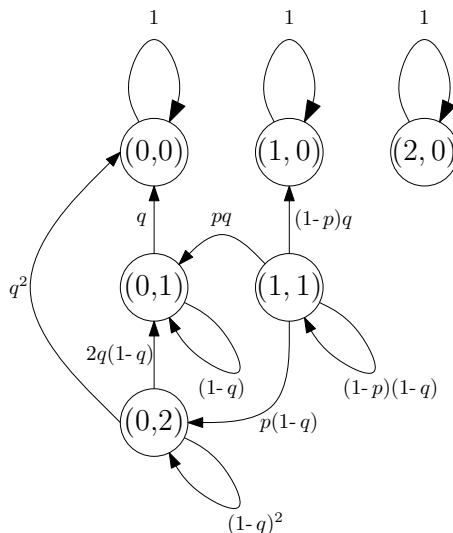
3. (a) If m out of n individuals are infected, then there must be $n - m$ susceptible individuals. Each one of these individuals will be independently infected over the course of the day with probability $\rho = 1 - (1 - p)^m$. Thus the number of new infections, I , will be a binomial random variable with parameters $n - m$ and ρ . That is,

$$p_I(k) = \binom{n-m}{k} \rho^k (1-\rho)^{n-m-k} \quad k = 0, 1, \dots, n-m.$$

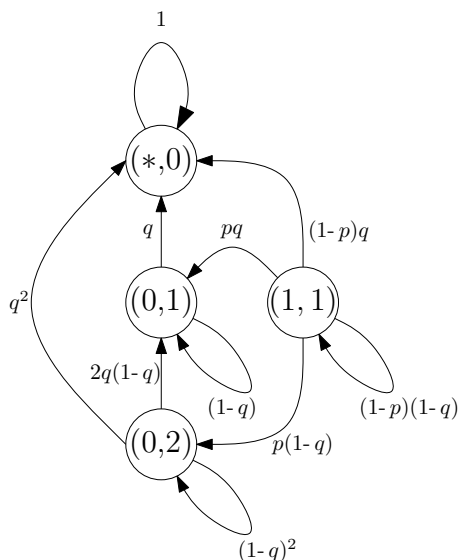
- (b) Let the state of the SIS model be the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.



- (c) The only recurrent state is the state with 0 infected individuals.
- (d) Let the state of the SIR model be (S, I) , where S is the number of susceptible individuals and I is the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.



If one did not wish to keep track of the breakdown of susceptible and recovered individuals when no one was infected, the three states free of infections could be consolidated into a single state as illustrated below.



- (e) Any state where the number of infected individuals equals 0 is a recurrent state. For $n = 2$, there are either one or three recurrent states, depending on the Markov chain drawn in part (d).
4. (a) The process is in state 3 immediately before the first transition. After leaving state 3 for the first time, the process cannot go back to state 3 again. Hence J , which represents the number of transitions up to and including the transition on which the process leaves state 3 for the last time is a geometric random variable with success probability equal to 0.6. The variance for J is given by:

$$\sigma_J^2 = \frac{1-p}{p^2} = \frac{10}{9}$$

- (b) There is a positive probability that we never enter state 4; i.e., $P(K < \infty) < 1$. Hence the expected value of K is ∞ .
- (c) The Markov chain has 3 different recurrent classes. The first recurrent class consists of states $\{1, 2\}$, the second recurrent class consists of state $\{7\}$ and the third recurrent class consists of states $\{4, 5, 6\}$. The probability of getting absorbed into the first recurrent class starting from the transient state 3 is,

$$\frac{1/10}{1/10 + 2/10 + 3/10} = \frac{1}{6}$$

which is the probability of transition to the first recurrent class given there is a change of state. Similarly, probability of absorption into second and third recurrent classes are $\frac{3}{6}$ and $\frac{2}{6}$ respectively.

Now, we solve the balance equations within each recurrent class, which give us the probabilities conditioned on getting absorbed from state 3 to that recurrent class. The unconditional steady-state probabilities are found by weighing the conditional steady-state probabilities by the probability of absorption to the recurrent classes.

The first recurrent class is a birth-death process. We write the following equations and solve for the conditional probabilities, denoted by p_1 and p_2 .

$$p_1 = \frac{p_2}{2}$$

$$p_1 + p_2 = 1$$

Solving these equations, we get $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$. For the second recurrent class, $p_7 = 1$. The third recurrent class is also a birth-death process, we can find the conditional steady-state probabilities as follows,

$$p_4 = 2p_5$$

$$p_5 = 2p_6$$

$$p_4 + p_5 + p_6 = 1$$

and thus, $p_4 = \frac{4}{7}$, $p_5 = \frac{2}{7}$, $p_6 = \frac{1}{7}$.

Using these data, the unconditional steady-state probabilities for all the states are found as follows:

$$\pi_1 = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$

$$\pi_2 = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$$

$$\pi_3 = 0 \text{ (transient state)}$$

$$\pi_7 = 1 \cdot \frac{3}{6} = \frac{1}{2}$$

$$\pi_4 = \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21}$$

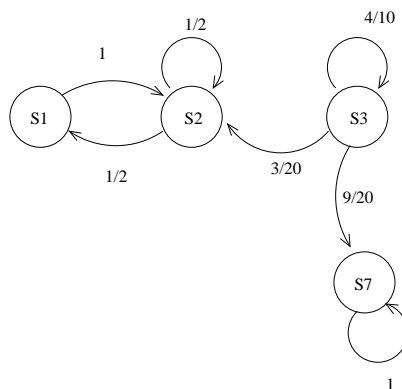
$$\pi_5 = \frac{2}{7} \cdot \frac{2}{6} = \frac{2}{21}$$

$$\pi_6 = \frac{1}{7} \cdot \frac{2}{6} = \frac{1}{21}$$

- (d) The given conditional event, that the process never enters state 4, changes the absorption probabilities to the recurrent classes. The probability of getting absorbed to the first recurrent class is $\frac{1}{4}$, to the second recurrent class is $\frac{3}{4}$, and to the third recurrent class is 0. Hence, the steady state probabilities are given by,

$$\begin{aligned}\pi_1 &= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} \\ \pi_2 &= \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6} \\ \pi_3 &= \pi_4 = \pi_5 = \pi_6 = 0 \\ \pi_7 &= 1 \cdot \frac{3}{4} = \frac{3}{4}\end{aligned}$$

For pedagogical purposes, let us actually draw what the new Markov chain would look like, given the event that the process never enters state 4. The resulting chain is shown below. Let us see how we came up with these transition probabilities.



We need to be careful when rescaling the new transition probabilities. First of all, it is clear that the probabilities within the recurrent classes $\{S1, S2\}$ and $\{S7\}$ don't get affected. We also note that the self loop transition probability of the transient state $S3$ doesn't get changed either. (this would be true for any other transient state)

To see that the self loop probability $p_{3,3}$ doesn't get changed, we condition on the event that we eventually enter $S2$ or $S7$. Let's call the new self loop probability, $q_{3,3}$.

Then,

$$\begin{aligned}q_{3,3} &= P(X_1 = S3 | \text{absorbed into 2 or 7}, X_0 = S3) = \frac{p_{3,3} * P(\text{absorbed into 2 or 7} | X_1 = S3, X_0 = S3)}{P(\text{absorbed into 2 or 7} | X_0 = S3)} \\ &= \frac{p_{3,3} * (a_{3,2} + a_{3,7})}{(a_{3,2} + a_{3,7})} = p_{3,3} = \frac{4}{10}\end{aligned}$$

Now, we calculate $q_{3,7}$ and $q_{3,2}$.

$$\begin{aligned}q_{3,7} &= P(X_1 = S7 | \text{absorbed into 2 or 7}, X_0 = S3) = \frac{p_{3,7} * P(\text{absorbed into 2 or 7} | X_1 = S7, X_0 = S3)}{P(\text{absorbed into 2 or 7} | X_0 = S3)} \\ &= \frac{p_{3,7} * 1}{(a_{3,2} + a_{3,7})} = \frac{\frac{3}{10}}{\frac{1}{6} + \frac{1}{2}} = \frac{9}{20}\end{aligned}$$

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$$q_{3,2} = P(X_1 = S2 | \text{absorbed into 2 or 7}, X_0 = S3) = \frac{p_{3,2} * P(\text{absorbed into 2 or 7} | X_1=S2, X_0=S3)}{P(\text{absorbed into 2 or 7} | X_0=S3)}$$

$$= \frac{p_{3,2} * 1}{(a_{3,2} + a_{3,7})} = \frac{\frac{1}{10}}{\frac{1}{6} + \frac{1}{2}} = \frac{3}{20}$$

Now, we can calculate the absorption probabilities of this new Markov chain.

The probability of getting absorbed into the recurrent class $\{1, 2\}$, starting from $S3$, is $\frac{\frac{3}{20}}{\frac{3}{20} + \frac{9}{20}} = \frac{1}{4}$. The probability of getting absorbed into the recurrent class $\{7\}$, starting from $S3$, is $\frac{\frac{9}{20}}{\frac{3}{20} + \frac{9}{20}} = \frac{3}{4}$. Thus, our calculated absorption probabilities match the probabilities we intuited earlier. The important thing to take away from this example is that, when doing problems of this sort, (i.e given we do/don't enter a particular set of recurrent classes), it is necessary to rescale the transition probabilities of the new chain, coming out of ALL the transient states. In other words, to find each of the new transition probabilities, we condition on the given event, that we do or do not enter particular recurrent classes.

G1[†]. a) First let the p_{ij} 's be the transition probabilities of the Markov chain.

Then

$$\begin{aligned} m_{k+1}(1) &= E[R_{k+1} | X_0 = 1] \\ &= E[g(X_0) + g(X_1) + \dots + g(X_{k+1}) | X_0 = 1] \\ &= \sum_{i=1}^n p_{1i} E[g(X_0) + g(X_1) + \dots + g(X_{k+1}) | X_0 = 1, X_1 = i] \\ &= \sum_{i=1}^n p_{1i} E[g(1) + g(X_1) + \dots + g(X_{k+1}) | X_1 = i] \\ &= g(1) + \sum_{i=1}^n p_{1i} E[g(X_1) + \dots + g(X_{k+1}) | X_1 = i] \\ &= g(1) + \sum_{i=1}^n p_{1i} m_k(i) \end{aligned}$$

and thus in general $m_{k+1}(c) = g(c) + \sum_{i=1}^n p_{ci} m_k(i)$ when $c \in \{1, \dots, n\}$.

Note that the third equality simply uses the total expectation theorem.

b)

$$\begin{aligned} v_{k+1}(1) &= \text{Var}[R_{k+1} | X_0 = 1] \\ &= \text{Var}[g(X_0) + g(X_1) + \dots + g(X_{k+1}) | X_0 = 1] \\ &= \text{Var}[E[g(X_0) + g(X_1) + \dots + g(X_{k+1}) | X_0 = 1, X_1]] + \end{aligned}$$

$$\begin{aligned}
 & E[Var[g(X_0) + g(X_1) + \dots + g(X_{k+1})|X_0 = 1, X_1]] \\
 = & Var[g(1) + E[g(X_1) + \dots + g(X_{k+1})|X_0 = 1, X_1]] + \\
 & E[Var[g(1) + g(X_1) + \dots + g(X_{k+1})|X_0 = 1, X_1]] \\
 = & Var[E[g(X_1) + \dots + g(X_{k+1})|X_0 = 1, X_1]] + E[Var[g(X_1) + \dots + g(X_{k+1})|X_0 = 1, X_1]] \\
 = & Var[E[g(X_1) + \dots + g(X_{k+1})|X_1]] + E[Var[g(X_1) + \dots + g(X_{k+1})|X_1]] \\
 = & Var[m_k(X_1)] + E[v_k(X_1)] \\
 = & E[(m_k(X_1))^2] - E[m_k(X_1)]^2 + \sum_{i=1}^n p_{1i} v_k(i) \\
 = & \sum_{i=1}^n p_{1i} m_k^2(i) - \left(\sum_{i=1}^n p_{1i} m_k(i)\right)^2 + \sum_{i=1}^n p_{1i} v_k(i)
 \end{aligned}$$

so in general $v_{k+1}(c) = \sum_{i=1}^n p_{ci} m_k^2(i) - (\sum_{i=1}^n p_{ci} m_k(i))^2 + \sum_{i=1}^n p_{ci} v_k(i)$ when $c \in \{1, \dots, n\}$.