Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

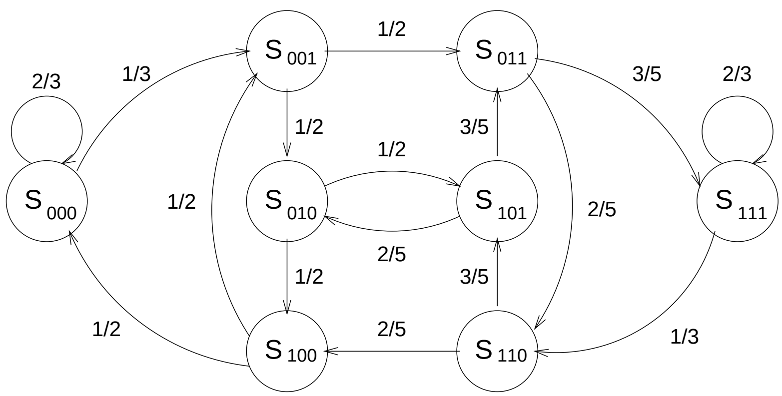
6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 8 Solutions
Due November 23, 2011

1. Since the probability of success depends on the results of the previous three trials, we need a separate state for every possible result in the previous three trials. Therefore we need $2^3 = 8$ states. We label each state S_{ijl}, where the triplet (i, j, l) represents the result of the last three trials, with l being the most recent trial. Each component of the triplet has a value of 1 if the trial was successful, and a value of 0 if the trial was unsuccessful.

   Now we can easily draw the state transition diagram. Note that the transition between any two states is $\frac{k+1}{k+3}$ where k is the number of successes in the last three trials, if the transition is leading to a success, and $\frac{2}{k+3}$ if the transition is leading to a failure.



2. (a) i. Since the state X_k is the largest number rolled in k rolls, the set of states $S = \{1, 2, 3, 4, 5, 6\}$. The probability of the largest number rolled in the first $(k + 1)$ trials is only dependent to the what the largest number that was rolled in the first k trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} 0 & , \quad j < i \\ \frac{i}{6} & , \quad j = i \\ \frac{1}{6} & , \quad j > i \end{cases}$$

   ii. Since the state X_k is the number of sixes in the first k rolls, the set of states $S = \{0, 1, 2, ...\}$. The probability of getting a six in a given trial is 1/6. The number of sixes rolled in the first $(k + 1)$ trials is only dependent to the number of sixes rolled in the first k trials. This satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = i + 1 \\ \frac{5}{6} & , \quad j = i \\ 0 & , \quad \text{otherwise} \end{cases}$$

   iii. Since the state X_k is the number of rolls since the most recent six, the set of states $S = \{0, 1, 2, ...\}$. If the roll of the die is 6 on the next trial the chain goes to state 0. If not, the state goes to the next higher state. Therefore, the probability of the next state depends on the past only through the present state. Clearly, this satisfies the Markov property. The transition probabilities are given by

$$p_{ij} = \begin{cases} \frac{1}{6} & , \quad j = 0 \\ \frac{5}{6} & , \quad j = i+1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(b) i. For $X_k = Y_{r+k}$, and by the Markov property of Y

$$\begin{aligned} \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i, ...Y_r = i_r) \\ &= \mathbf{P}(Y_{r+k+1} = j | Y_{r+k} = i) \\ \mathbf{P}(X_{k+1} = j | X_k = i, ...X_0 = i_0) &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. Also we can see that, X_k is a delayed process by r of Y_k. Therefore, they should have the same transition probability p_{ij}, So, we have:

$$p_{ij} = q_{ij} \ .$$

ii. For $X_k = Y_{2k}$, and by the Markov property of Y

$$\begin{aligned} &\mathbf{P}(X_{k+1} = j | X_k = i, X_{k-1} = i_{k-1}.....X_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i, Y_{2k-2} = i_{2k-2}.....Y_0 = i_0) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= \mathbf{P}(X_{k+1} = j | X_k = i) \end{aligned}$$

This satisfies the Markov property for X. The transition probabilities p_{ij} are given by:

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{k+1} = j | X_k = i) \\ &= \mathbf{P}(Y_{2k+2} = j | Y_{2k} = i) \\ &= r^y_{ij}(2) \end{aligned}$$

where $r^y_{ij}(n)$ is the n step transition probability of Y.

iii.

$$\begin{aligned} &\mathbf{P}(X_{k+1} = (n,l) | X_0 = (i_0, i_1), X_1 = (i_1, i_2), ..., X_k = (i_k, n)) \\ &= \mathbf{P}(X_{k+1} = (n,l) | Y_0 = i_0, Y_1 = i_1, Y_2 = i_2........., Y_k = i_k, Y_{k+1} = n) \\ &= \mathbf{P}(X_{k+1} = (n,l) | Y_{k+1} = n) \\ &= \mathbf{P}(X_{k+1} = (n,l) | X_k = (i_k, n)) \end{aligned}$$

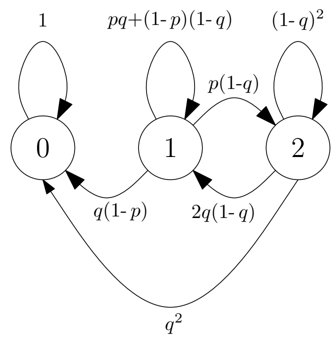
Letting $i = (i_k, i_{k+1})$ and $j = (n, l)$, the transition probabilities p_{ij} are given by:

$$p_{ij} = \mathbf{P}(X_{k+1} = (n,l) | X_k = (i_k, i_{k+1})) = \begin{cases} q_{nl} & , \quad i_{k+1} = n \\ 0 & , \quad i_{k+1} \neq n \end{cases}$$

3. (a) If m out of n individuals are infected, then there must be $n - m$ susceptible individuals. Each one of these individuals will be independently infected over the course of the day with probability $\rho = 1 - (1-p)^m$. Thus the number of new infections, I, will be a binomial random variable with parameters $n - m$ and ρ. That is,

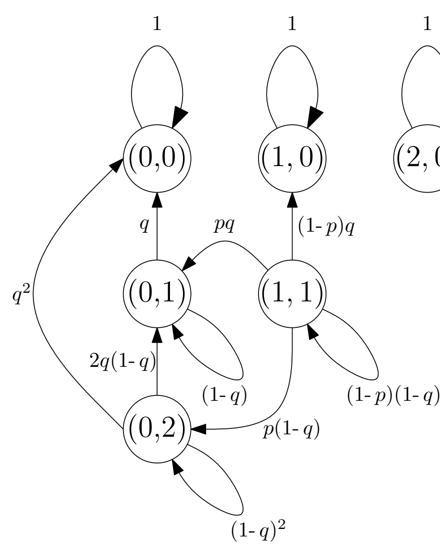
$$p_I(k) = \binom{n-m}{k} \rho^k (1-\rho)^{n-m-k} \qquad k = 0, 1, \ldots, n-m \ .$$

(b) Let the state of the SIS model be the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.

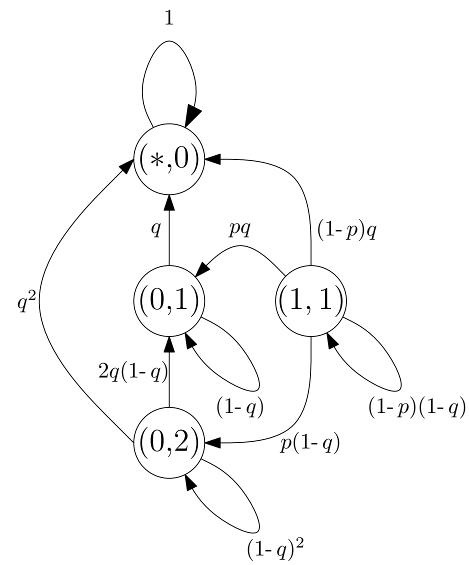


(c) The only recurrent state is the state with 0 infected individuals.

(d) Let the state of the SIR model be (S, I), where S is the number of susceptible individuals and I is the number of infected individuals. For $n = 2$, the corresponding Markov chain is illustrated below.



If one did not wish to keep track of the breakdown of susceptible and recovered individuals when no one was infected, the three states free of infections could be consolidated into a single state as illustrated below.



(e) Any state where the number of infected individuals equals 0 is a recurrent state. For $n = 2$, there are either one or three recurrent states, depending on the Markov chain drawn in part (d).

4. The outcomes of successive flips can be viewed as a Markov chain with two states, T and H. The transition probabilities are

$$\begin{aligned} p_{TH} &= 1/3 \\ p_{TT} &= 2/3 \\ p_{HH} &= 3/4 \\ p_{HT} &= 1/4. \end{aligned}$$

Let $X_k, k = 1, \ldots$ denote the outcomes of the flips.

(a) For $k \geq 2$,

$$\begin{aligned} &P(\text{1st tail occurs on kth toss}|X_1 = H) \\ &= P(\text{first k-2 transitions are } H \to H \text{ and the last transition is } H \to T) \\ &= \left(\frac{3}{4}\right)^{k-2} \frac{1}{4}. \end{aligned}$$

(b) Irrespective of the starting state, $P(X_{5000} = H) \approx \pi_H$ where π_H, π_T are steady state probabilities. These probabilities $\pi_H = 4/7$ and $\pi_T = 3/7$ are obtained by solving equations

$$\begin{aligned} \pi_T p_{TH} + \pi_H p_{HH} &= \pi_H \\ \pi_T + \pi_H &= 1 \end{aligned}$$

(c)

$$\begin{aligned}
P(X_{5000} = H, X_{5002} = H) &= P(X_{5000} = H)P(X_{5002} = H | X_{5000} = H) \\
&\approx \pi_H P(X_{5002} = H | X_{5000} = H) \\
&= \pi_H (p_{HT} p_{TH} + p_{HH} p_{HH}) \\
&= \frac{4}{7}\left(\frac{1}{4}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{3}{4}\right) \\
&= \frac{124}{336}
\end{aligned}$$

(d)

$$\begin{aligned}
&P(X_{5001} = \cdots = X_{5000+m} = H | X_{5001} = \cdots = X_{5000+m}) \\
&= \frac{P(X_{5001} = \cdots = X_{5000+m} = H)}{P(X_{5001} = \cdots = X_{5000+m} = H) + P(X_{5001} = \cdots = X_{5000+m} = T)} \\
&= \frac{P(X_{5001} = H)p_{HH}^{m-1}}{P(X_{5001} = H)p_{HH}^{m-1} + P(X_{5001} = T)p_{TT}^{m-1}} \approx \frac{\pi_H p_{HH}^{m-1}}{\pi_H p_{HH}^{m-1} + \pi_T p_{TT}^{m-1}} \\
&\to 1 \text{ as } m \to \infty
\end{aligned}$$

(e) Let's examine the expected number of additional tosses until the next heads, given heads currently. This can be found by total expectation, by conditioning on what happens on the next toss. Given that the next toss is tails, the number of additional tosses until we observe the next heads is geometric with parameter $\frac{1}{3}$. Therefore, given tails, the expected number of additional tosses required until we observe the next heads is 3. Hence, the expected number of additional flips required until we observe the next heads, given heads on the current toss is

$$p_{HH} \cdot 1 + p_{HT} \cdot (1 + 3) = \frac{7}{4} .$$

Given that the 375th heads occurs on the 500th toss, the number of additional flips until the 379th heads can be expressed as the sum of four random variables, each with an expectation equal to 7/4. Thus by linearity of expectation, the required answer is $4 \cdot \frac{7}{4} = 7$.

5. (a) States 4 and 5 are transient. The class {1,2,3} is recurrent and not periodic. The class {6,7} is recurrent with period 2.

(b) We can calculate this by writing out the complete formula for the 3-step transition probabilities. Alternatively, by examining the state transition diagram, we see that there are four ways to get from state 1 to state 2 in these transitions, namely $1 \to 1 \to 1 \to 2$, $1 \to 1 \to 2 \to 2$, $1 \to 2 \to 2 \to 2$, and $1 \to 2 \to 3 \to 2$. Thus,

$$\begin{aligned}
r_{12}(3) &= p_{11}^2 p_{12} + p_{11} p_{12} p_{22} + p_{12} p_{22}^2 + p_{12} p_{23} p_{32} \\
&= p12(p_{11}^2 + p_{11} p_{22} + p_{22}^2 + p23 p_{32}) \\
&= (0.5)((0.5)^2 + (0.5)(0.4) + (0.4)^2 + (0.1)(0.6)) \\
&= 0.335
\end{aligned}$$

(c) Given that we start in state 1, the process will remain in the recurrent class { 1, 2, 3 }. Local balance equations take the form

$$0.5\pi_1 = 0.5\pi_2$$
$$0.1\pi_2 = 0.6\pi_3,$$

from which we find that

$$\pi_1 = \pi_2 = \frac{6}{13}, \quad \pi_3 = \frac{1}{13}.$$

(d) The recurrent class { 6, 7 } is periodic, so, given that we start in state 6, steady-state probabilities do not exist.

(e) Assuming we start in state 1 and begin observing after 10,000,000 transiti ons, we can safely make the approximation that we are in steady state, with prob abilities π_1, π_2, π_3 from part (c).

i.

$$\begin{aligned}\mathbf{P}(\text{birth on first transition}) &= \pi_1 p_{12} + \pi_2 p_{23} \\ &= \frac{6}{13}(0.5 + 0.1) \\ &= \frac{18}{65}.\end{aligned}$$

ii.

$$\begin{aligned}\mathbf{P}(\text{state 2 arrival} \mid \text{birth first transition}) &= \frac{\mathbf{P}(\text{state 2 on arrival and birth on first transition})}{\mathbf{P}(\text{birth on first transition})} \\ &= \frac{\pi_2 p_{23}}{\pi_1 p_{12} + \pi_2 p_{23}} \\ &= \frac{1}{6}.\end{aligned}$$

iii. If the state is 1, the first change of state is definitely a birth (i.e., birth occurs with probability 1). If the state is 2, the probability that the first change of state is a birth is 0.1/(0.1 + 0.5) = 1/6. Finally, if the state is 3, the first change of state must be a death (i.e., birth occurs with probability 0). Denote BFS = birth on first change of state. We have

$$\begin{aligned}\mathbf{P}(\text{BFS}) &= \mathbf{P}(\text{BFS} \mid \text{state 1})\mathbf{P}(\text{state 1}) + \mathbf{P}(\text{BFS} \mid \text{state 2})\mathbf{P}(\text{state 2}) + \mathbf{P}(\text{BFS} \mid \text{state 3})\mathbf{P}(\text{state 3}) \\ &= 1 \cdot \pi_1 + \frac{1}{6} \cdot \pi_2 + 0 \cdot \pi_3 \\ &= \frac{6}{13} + \frac{1}{6} \cdot \frac{6}{13} \\ &= \frac{7}{13}.\end{aligned}$$

iv.

$$\begin{aligned}
\mathbf{P}(X_{n-1}=3|X_n=2) &= \frac{\mathbf{P}(X_n=2|X_{n-1}=3)\,\mathbf{P}(X_{n-1}=3)}{\mathbf{P}(X_n=2)} \\
&= \frac{p_{32}\pi_3}{\pi_2} \\
&= \frac{(0.6)\frac{1}{13}}{\frac{6}{13}} \\
&= 0.1.
\end{aligned}$$

G1†. (a) Starting from state 0, let a "success" be the particle reaching either state 1 or -1, which has probability $2p$. The time until the first success is therefore a geometric random variable, and so the expected time until the first success is $1/\mathbf{P}(\text{success}) = \frac{1}{2p}$.

(b) Since we care only about reaching states 2 or -2, we can effectively modify the chain by cutting it off at states 2 and -2. Specifically,

- states 2 and -2 are combined into a single state, denoted ± 2;
- states 3, -3, through n, $-n$ are removed;
- there is a probability p of transitioning from state 1 to the new state ± 2, and also a probability p of transitioning from state -1 to state ± 2;
- state ± 2 is absorbing and has only a probability 1 of a self-transition;
- the remaining transitions are carried over.

In the context of this new modified chain, the quantity we are trying to find is just the expected time until absorption into the state ± 2 starting from state 0. Let us define μ_i as the expected time until absorption starting from state i. We are then looking for μ_0.

We can first note that by symmetry, $\mu_1 = \mu_{-1}$. Moreover, $\mu_{\pm 2} = 0$. We can then solve the following system of equations:

$$\begin{aligned}
\mu_0 &= 1 + p\mu_1 + p\mu_{-1} + (1-2p)\mu_0, \\
\mu_{-1} = \mu_1 &= 1 + p\mu_{\pm 2} + p\mu_0 + (1-2p)\mu_1.
\end{aligned}$$

Simplifying, we obtain that

$$\mu_1 = 1 + p\cdot 0 + p\mu_0 + (1-2p)\mu_1 \Longrightarrow \mu_1 = \frac{1+p\mu_0}{2p} = \mu_{-1},$$

$$\mu_0 = 1 + 2p\mu_1 + (1-2p)\mu_0 \Longrightarrow 2p\mu_0 = 1 + 1 + p\mu_0 \Longrightarrow \mu_0 = \frac{2}{p}.$$

Substituting the value of μ_0 back into the expression for μ_1, we also get that $\mu_1 = \mu_{-1} = \frac{3}{2p}$.

(c) Similar to part (b) above, we will collapse the two states n and $-n$ into a single absorbing state, denoted $\pm n$. Define μ_i as the expected time until absorption into state $\pm n$, starting from state i. Again, we have that $\mu_{\pm n} = 0$ and $\mu_i = \mu_{-i}$ by symmetry. As a consequence of this symmetry, we can focus on just the positive states. We can write the following system

†Required for 6.431; optional challenge problem for 6.041

of equations for the unknown μ's:

$$\begin{aligned}
\mu_0 &= 1 + p\mu_1 + p\mu_{-1} + (1-2p)\mu_0 \\
&= 1 + 2p\mu_1 + (1-2p)\mu_0, \\
\mu_i &= 1 + p\mu_{i-1} + p\mu_{i+1} + (1-2p)\mu_i, \qquad i = 1, 2, \ldots, n-2, \\
\mu_{n-1} &= 1 + p\mu_{\pm n} + p\mu_{n-2} + (1-2p)\mu_{n-1} \\
&= 1 + p\mu_{n-2} + (1-2p)\mu_{n-1}.
\end{aligned}$$

By collecting the μ's on one side of the equations and dividing through by p, we obtain the following equivalent system of equations:

$$\begin{aligned}
2\mu_0 - 2\mu_1 &= 1/p, \\
-\mu_{i-1} + 2\mu_i - \mu_{i+1} &= 1/p, \quad i = 1, 2, \ldots, n-2, \\
-\mu_{n-2} + 2\mu_{n-1} &= 1/p.
\end{aligned}$$

It simply remains to solve this system of equation. By writing the equations in matrix form, it becomes obvious that they have a special (tridiagonal) form.

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
\mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \vdots \\ \mu_{n-3} \\ \mu_{n-2} \\ \mu_{n-1}
\end{bmatrix}
= \frac{1}{p}
\begin{bmatrix}
\frac{1}{2} \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1
\end{bmatrix}. \tag{1}$$

To solve, we first simplify the equations by adding the first equation to the second equation, then adding the resulting equation to the third equation, and so on (i.e., performing elementary row operations). The resulting system of equations is

$$\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \vdots \\ \mu_{n-3} \\ \mu_{n-2} \\ \mu_{n-1}
\end{bmatrix}
= \frac{1}{p}
\begin{bmatrix}
\frac{1}{2} \\ \frac{1}{2} + 1 \\ \frac{1}{2} + 2 \\ \frac{1}{2} + 3 \\ \vdots \\ \frac{1}{2} + n - 2 \\ \frac{1}{2} + n - 1
\end{bmatrix}. \tag{2}$$

The last equation immediately gives us the value of $\mu_{n-1} = \frac{1}{p}(\frac{1}{2} + n - 1)$. Note that for the case of $n = 1$, this formula gives $\mu_0 = \frac{1}{2p}$, which agrees with our answer to part (a), and for the case of $n = 2$, it gives $\mu_1 = \frac{3}{2p}$, which also agrees with our answer to part (b).

To solve for the remaining μ's, we simply add the equations from the bottom up, one by one. In particular, the value of μ_0 is simply the sum of all the values on the right-hand side. The general formula is

$$\begin{aligned}
\mu_i &= \frac{1}{p}\left(\frac{1}{2}+i\right)+\frac{1}{p}\left(\frac{1}{2}+i+1\right)+\cdots+\frac{1}{p}\left(\frac{1}{2}+n-1\right) \\
&= \sum_{j=i}^{n-1}\frac{1}{p}\left(\frac{1}{2}+j\right) \\
&= \frac{1}{p}\left[\frac{1}{2}(n-1-i+1)+\frac{(n-1+i)(n-1-i+1)}{2}\right] \\
&= \frac{1}{2p}(n-1-i+1)(n-1+i+1).
\end{aligned}$$

Hence, we conclude that

$$\mu_0 = \frac{n^2}{2p},$$

and so the diffusion coefficient is $D = 2p$. Note that for the cases of $n = 1$ and $n = 2$, the values of $\mu_0 = \frac{1}{2p}$ and $\mu_0 = \frac{2}{p}$ agree with parts (a) and (b), respectively.