LECTURE 10

- **Readings:** Section 3.6; start Section 4.1

Lecture outline

- Review
- Continuous Bayes' rule
- Derived distributions

Review (from L07, emphasis added)

- Multiplication rule for **discrete** random variables:

$$\begin{aligned} p_{X,Y}(x,y) &= p_Y(y)\, p_{X|Y}(x \mid y) \qquad \text{and} \\ p_{X,Y}(x,y) &= p_X(x)\, p_{Y|X}(y \mid x) \end{aligned}$$

- Bayes' rule for **discrete** random variables:

$$p_{X|Y}(x \mid y) = \frac{p_X(x)\, p_{Y|X}(y \mid x)}{p_Y(y)}$$

(where the conditional PMF is defined)

Continuous counterparts

- Multiplication rule for **continuous** random variables:

$$\begin{aligned} f_{X,Y}(x,y) &= f_Y(y)\, f_{X|Y}(x \mid y) \qquad \text{and} \\ f_{X,Y}(x,y) &= f_X(x)\, f_{Y|X}(y \mid x) \end{aligned}$$

- Bayes' rule for **continuous** random variables:

$$f_{X|Y}(x \mid y) = \frac{f_X(x)\, f_{Y|X}(y \mid x)}{f_Y(y)}$$

(where the conditional PDF is defined)

Example: Two light bulbs

- Suppose light bulbs have lifetimes that are independent and identically exponentially distributed.
- One is installed at noon, burns out, and is replaced immediately. The replacement burns out at 2pm.
- What is the distribution of the time at which the first bulb burns out?

Conditioning an event on a continuous random variable

- Suppose $f_{Y|A}(y)$ and $f_{Y|A^c}(y)$ are known
- Defining $\mathbf{P}(A \mid Y = y)$ requires care because $\mathbf{P}(\{Y = y\}) = 0$
  - When $\delta > 0$ is very small and $f_Y(y) > 0$,

$$\begin{aligned} \mathbf{P}(A \mid Y = y) &\approx \mathbf{P}(A \mid \{Y \in [y, y+\delta]\}) \\ &= \frac{\mathbf{P}(A)\, \mathbf{P}(\{Y \in [y, y+\delta]\} \mid A)}{\mathbf{P}(\{Y \in [y, y+\delta]\})} \\ &\approx \frac{\mathbf{P}(A)\, f_{Y|A}(y)\, \delta}{f_Y(y)\, \delta} \\ &= \frac{\mathbf{P}(A)\, f_{Y|A}(y)}{f_Y(y)} \\ &= \frac{\mathbf{P}(A)\, f_{Y|A}(y)}{\mathbf{P}(A) f_{Y|A}(y) + \mathbf{P}(A^c) f_{Y|A^c}(y)} \end{aligned}$$

Discrete X, Continuous Y

$$\mathbf{P}(\{X = x\} \mid \{Y = y\}) = \frac{p_X(x) f_{Y|\{X=x\}}(y)}{f_Y(y)}$$

$$p_{X|Y}(x \mid y) = \frac{p_X(x) f_{Y|X}(y \mid x)}{f_Y(y)}$$

$$f_Y(y) = \sum_x p_X(x) f_{Y|X}(y \mid x)$$

Example:

- X: a discrete signal; "prior" $p_X(x)$
- Y: noisy version of X
- $f_{Y|X}(y \mid x)$: continuous noise model

Continuous X, Discrete Y

$$f_{X|Y}(x \mid y) \;=\; \frac{f_X(x)p_{Y|X}(y \mid x)}{p_Y(y)}$$

$$p_Y(y) \;=\; \int_x f_X(x)p_{Y|X}(y \mid x)\,dx$$

Example:

- X: a continuous signal; "prior" $f_X(x)$ (e.g., intensity of light beam)
- Y: discrete r.v. affected by X (e.g., photon count)
- $p_{Y|X}(y \mid x)$: model of the discrete r.v. (e.g., Poisson with parameter that depends on x)

Derived distributions

- When $Y = g(X)$ and the distribution of X is known, the distribution of Y is **derived** from the distribution of X
- Term and techniques apply to functions of any number of variables $g(X, Y, Z)$, etc.

When not to find them

- Don't need distribution of $g(X)$ to compute $\mathbf{E}[g(X)]$:

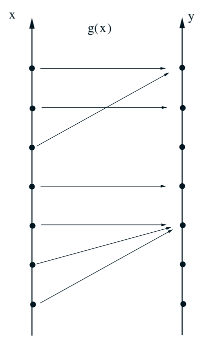
$$\mathbf{E}[g(X)] \;=\; \int g(x)f_X(x)\,dx \qquad \text{(continuous case)}$$

$$\mathbf{E}[g(X)] \;=\; \sum g(x)p_X(x) \qquad \text{(discrete case)}$$

Finding derived distributions: Discrete case

- Obtain probability mass for each possible value of $Y = g(X)$:

$$p_Y(y) \;=\; \mathbf{P}(\{g(X) = y\}) \;=\; \sum_{x:\, g(x)=y} p_X(x)$$



Finding derived distributions: Continuous case

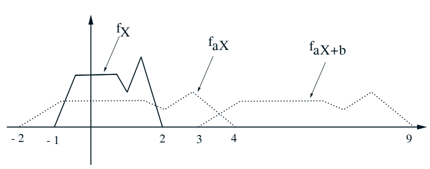
- Two-step procedure:
  - Get CDF of Y: $F_Y(y) = \mathbf{P}(Y \le y)$
  - Differentiate to get

$$f_Y(y) \;=\; \frac{dF_Y}{dy}(y)$$

- Example:
  - X: uniform on [0,2]
  - Find PDF of $Y = X^3$

The PDF of $Y = aX + b$

$Y = 2X + 5$:

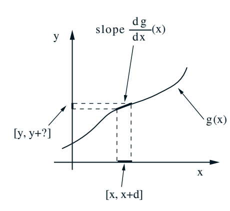


$$f_Y(y) \;=\; \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

- Check: if X is normal, then $Y = aX + b$ is also normal.

Preview: A more general formula

- Consider $Y = g(X)$, where g is strictly monotonic.



- Event $x \le X \le x + \delta$ is the same as $g(x) \le Y \le g(x + \delta)$
- Approximately:

$$g(x) \;\le\; Y \;\le\; g(x) + \delta \left|\frac{dg}{dx}(x)\right|$$