Problem Set 5: Solutions
Due: March 14, 2011

1. We proceed by direct calculation:

$$\begin{aligned}
\mathbf{E}[\mathrm{Int}(X)] &= \int_0^\infty \mathrm{Int}(x)e^{-x}\,dx \\
&= \sum_{N=0}^{\infty} N \int_N^{N+1} e^{-x}\,dx \\
&= \sum_{N=1}^{\infty} N(e^{-N-1} - e^{-N}) \\
&= (e^{-1} - 1)\sum_{N=1}^{\infty} Ne^{-N} \\
&= (1 - e^{-1})\frac{e^{-1}}{(1-e^{-1})^2} \\
&= \frac{e^{-1}}{1-e^{-1}}
\end{aligned}$$

The second to last equality above follows by a clever manipulation:

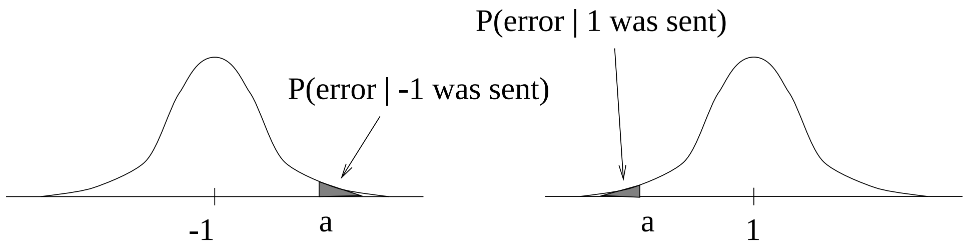
$$\begin{aligned}
\sum_{N=1}^{\infty} Ne^{-N} &= \sum_{N=1}^{\infty} Ne^{-Nx}\Big|_{x=1} \\
&= \left[\frac{d}{dx}\sum_{N=1}^{\infty} e^{-Nx}\right]_{x=1} \\
&= \left[\frac{d}{dx}\frac{e^{-x}}{(1-e^{-x})}\right]_{x=1} \\
&= \left[-\frac{e^{-x}}{1-e^{-x}} - \left(\frac{e^{-x}}{1-e^{-x}}\right)^2\right]_{x=1} \\
&= \left[-\frac{e^{-x}}{(1-e^{-x})^2}\right]_{x=1}
\end{aligned}$$

Thus the step is justified.

2. (a) Let Z be the random variable representing the additive zero-mean Gaussian noise; that is, $Z \sim N(0, \sigma^2)$. Let S_0 be the event that -1 is sent and S_1 be the event that $+1$ is sent. Let R_0 be the event that we conclude that an encoded signal of -1 was sent based on the received value being less than a. Let R_1 be the event that we conclude that an encoded signal of $+1$ was sent based on the received value being greater than a.

   There are two ways for errors to occur. The true encoded signal could be -1 but we could conclude that the encoded signal of $+1$ was sent. Conditioned on the true encoded signal being -1, the received signal is $Z - 1$; we would erroneously conclude that the encoded signal of $+1$ was sent if $Z - 1 > a$. Similarly, the true encoded signal could be $+1$ but we could conclude that the encoded signal of -1 was sent. In this case, conditioned on the true encoded signal being +1, the received signal is $Z+1$ and we would erroneously conclude that the true signal was -1 if $Z+1<a$.

The figure below illustrates the situations under which errors can occur.



Let Φ such that

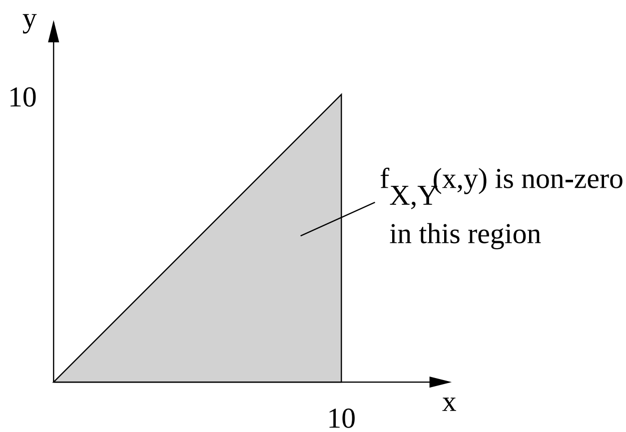
$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt.$$

Therefore

$$\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(R_1|S_0)\mathbf{P}(S_0)+\mathbf{P}(R_0|S_1)\mathbf{P}(S_1)\\
&= \mathbf{P}(Z-1>a)(p)+\mathbf{P}(Z+1<a)(1-p)\\
&= p\cdot\left(1-\Phi\left(\frac{a-(-1)}{\sigma}\right)\right)+(1-p)\cdot\Phi\left(\frac{a-1}{\sigma}\right)\\
&= p-p\cdot\Phi\left(\frac{a+1}{\sigma}\right)+(1-p)\cdot\left(1-\Phi\left(\frac{1-a}{\sigma}\right)\right)\\
&= 1-p\cdot\Phi\left(\frac{a+1}{\sigma}\right)-(1-p)\cdot\Phi\left(\frac{1-a}{\sigma}\right)
\end{aligned}$$

(b) $\mathbf{P}(\text{error})=1-0.4\cdot\Phi(\frac{3/2}{1/2})-0.6\cdot\Phi(\frac{1/2}{1/2})$

3. Let us first find the marginal probability of X and the conditional probability of Y.



$$f_X(x)=\int_0^x f_{X,Y}(x,y)dy=\int_0^x Qx^2y\,dy=Qx^2\left[\frac{y^2}{2}\right]_0^x=\frac{Qx^4}{2},$$

Therefore

$$f_X(x) = \begin{cases} \frac{Qx^4}{2}, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

And for the conditional probability,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2Qx^2y}{Qx^4} = \frac{2y}{x^2},$$

or

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{x^2}, & 0 \le x \le 10, 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$

Let A be the event "Experimental value of X is 6", and B be the event "Experimental value of X is 8". Then

$$\mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{2}.$$

Now to find the density of Y we can condition on whether A or B occurs. Hence, we get

$$f_y(y) = f_{Y|A}(y|A) \cdot \mathbf{P}(A) + f_{Y|B}(Y|B) \cdot \mathbf{P}(B) = f_{Y|X}(y|6) \cdot \frac{1}{2} + f_{Y|X}(y|8) \cdot \frac{1}{2}$$

From the formula derived above,

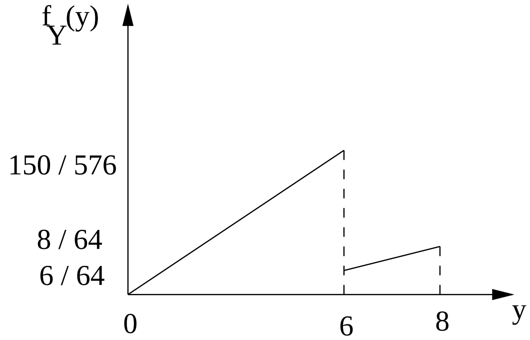
$$f_{Y|X}(y|6) = \begin{cases} \frac{y}{18}, & 0 \le y \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|8) = \begin{cases} \frac{y}{32}, & 0 \le y \le 8 \\ 0, & \text{otherwise} \end{cases}$$

This implies

$$f_Y(y) = \begin{cases} \frac{25y}{576}, & 0 \le y \le 6 \\ \frac{y}{64}, & 6 < y \le 8 \\ 0, & \text{otherwise} \end{cases}$$

Graphical representation:



4. (a) Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} 1 & 0 < x, y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the event R_1 as $X \leq 0.25$ and the event R_2 as $Y \leq 0.25$. A message is *received* 15 minutes after A sent both messages whenever at least one of R_1 and R_2 occurs. The probability we wish to compute is thus

$$\mathbf{P}(R_1 \cup R_2) = \mathbf{P}(R_1) + \mathbf{P}(R_2) - \mathbf{P}(R_1 \cap R_2).$$

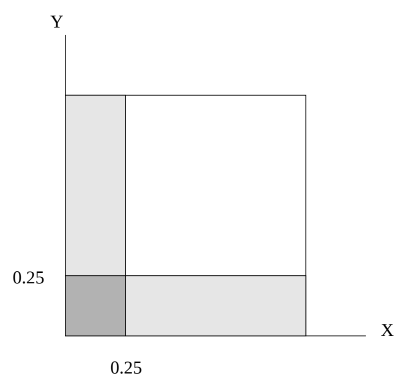
We compute the individual terms as

$$\begin{aligned} \mathbf{P}(R_1) &= \mathbf{P}\left(X \leq \frac{1}{4}\right) = \frac{1}{4}, \\ \mathbf{P}(R_2) &= \mathbf{P}\left(Y \leq \frac{1}{4}\right) = \frac{1}{4}, \\ \mathbf{P}(R_1 \cap R_2) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}, \quad \text{since } X \text{ and } Y \text{ are independent.} \end{aligned}$$

Thus the desired probability is

$$\frac{1}{4} + \frac{1}{4} - \frac{1}{16} = \frac{7}{16}.$$

Note also that the probability is the total area of the shaded regions in the following sketch.



(b) Let B be the event that the message is received but not verified within 15 minutes. Then

$$B = R_1 \cap R_2^c \cup R_1^c \cap R_2$$

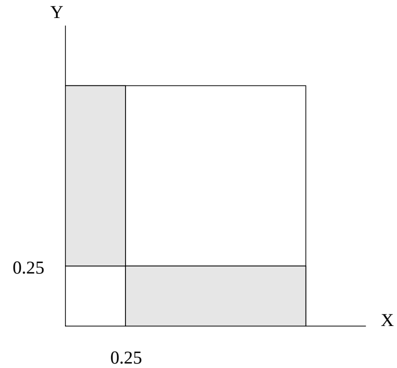
Note that this is a union of disjoint events, so we have

$$\mathbf{P}(B) = \mathbf{P}(R_1 \cap R_2^c) + \mathbf{P}(R_1^c \cap R_2),$$

and the independence of R_1 and R_2 allows the simplification

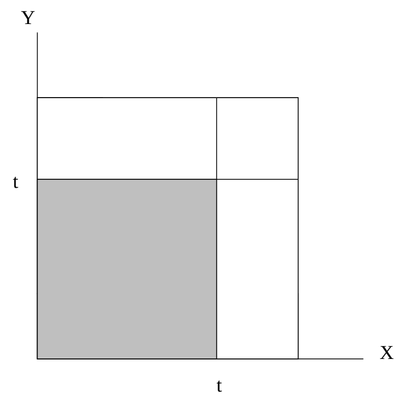
$$\begin{aligned} \mathbf{P}(B) &= \mathbf{P}(R_1) \cdot \mathbf{P}(R_2^c) + \mathbf{P}(R_1^c) \cdot \mathbf{P}(R_2) \\ &= \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

Note also that the probability is the total area of the shaded regions in the following sketch.



(c) Verification occurs when the second of the messages arrives, so for $t \in [0, 1]$

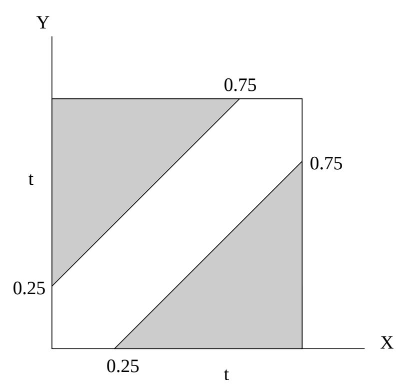
$$\begin{aligned} F_T(t) &= \mathbf{P}(T \leq t) \;=\; \mathbf{P}(X \leq t \cap Y \leq t) \\ &= \mathbf{P}(X \leq t) \cdot \mathbf{P}(Y \leq t) \quad \text{by the independence of } X \text{ and } Y \\ &= t \cdot t = t^2. \end{aligned}$$



From this we can deduce the full CDF and differentiate to determine the PDF:

$$F_T(t) = \begin{cases} 0 & \text{if } -\infty < t \leq 0, \\ t^2 & \text{if } 0 < t \leq 1, \\ 1 & \text{if } 1 < t < \infty \end{cases} \qquad \Rightarrow \qquad f_T(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(d) The event that the clerk will be there to receive the message is $\{|X - Y| > \frac{1}{4}\}$. We can deduce the probability of this event easily from a sketch:



$$\mathbf{P}(|X - Y| > \frac{1}{4}) = 2 \cdot \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}\right) = \frac{9}{16}.$$

(e) We know the strategy from (d) has $\frac{9}{16}$ probability of verification. The other strategy, of sending the employee home after 45 minutes, has probability of verification $\mathbf{P}(T \leq \frac{3}{4}) = \frac{9}{16}$ by evaluating the expression from part (c). Therefore, the two strategies are equally effective.

G1†. (a) In the figure below, the diagram on the left shows the joint PDF for X_1 and X_2.

In the PDF, the normalization constant of 2 comes from geometry, since the triangle only fills half of the two-dimensional cube.

The diagram on the right shows how to compute $P(X_2 > X_1 + t)$ by computing the area of the darker triangle. The normalization constant cancels with the area factor, leaving

$$P(X_2 > X_1 + t) = \boxed{(1-t)^2}.$$

The same diagram can be used to show that the answer is the same for $P(X_1 > 0 + t)$ and $P(1 > X_2 + t)$.

(b) For general n, the PDF is non-zero in an n-dimensional pyramid. As with the two-dimensional case, the normalization constant cancels with the volume factor, leaving

$$P(X_{i+1} > X_i + t) = \boxed{(1-t)^n}.$$

(c) Let $Y_i = X_i - X_{i-1}$. From the previous part, we know that the distribution of Y_i is independent from i. Therefore, for all i, Y_i is identically distributed. Thus

$$\mathbf{E}[Y_1] = \mathbf{E}[Y_2] = \cdots = \mathbf{E}[Y_{n+1}]$$

But,

$$\sum_{i=1}^{n+1} \mathbf{E}[Y_i] = 1$$

Hence, $\forall i = 1, 2, \cdots, n+1, \mathbf{E}[Y_i] = \frac{1}{n+1}$.

†Required for 6.431; optional for 6.041