LECTURE 13

- **Readings:** Section 6.1

Lecture outline

- Random processes
- Bernoulli process
  - Definition and basic properties
  - Interarrival times
  - Distribution of kth arrival
  - Merging and splitting

Random processes

- A **discrete-time random process** is a collection of random variables (defined in the same probabilistic model), e.g.,

  $(X_1, X_2, X_3, \ldots)$, $(X_0, X_1, X_2, \ldots)$, or $(\ldots, X_{-1}, X_0, X_1, \ldots)$
- New:
  - rarely before had infinite collections
  - interpret the index as time
  - focus is often on dependencies and long-term behavior
- Third quarter of course:
  - Ch. 6: memoryless processes, discrete and continuous time
  - Ch. 7: Markov processes, discrete time

The Bernoulli process

- A sequence of independent Bernoulli trials $(X_1, X_2, X_3, \ldots)$
- At each trial (each $i \in \{1, 2, \ldots\}$):
  - $\mathbf{P}(\text{success}) = \mathbf{P}(X_i = 1) = p$
  - $\mathbf{P}(\text{failure}) = \mathbf{P}(X_i = 0) = 1 - p$
- Examples:
  - Sequence of coin tosses
  - Sequence of lottery wins/losses
  - Sequence of ups and downs of the Dow Jones
  - Arrivals of tasks to computer (in time slots)

Basic properties

- $\mathbf{E}[X_i] =$ $\qquad$ $\text{var}(X_i) =$
- Let S be the number of successes/arrivals up to and including time n

  $p_S(k) =$ $\qquad$ $\mathbf{E}[S] =$ $\qquad$ $\text{var}(S) =$
- Let T be the number of trials up to and including the first success/arrival

  $p_T(t) =$ $\qquad$ $\mathbf{E}[T] =$ $\qquad$ $\text{var}(T) =$
- $\mathbf{P}(X_i = 1 \text{ for all } i) =$

Independence and memorylessness

- Trials at disjoint sets of times are independent
  - Example: $X_4, X_5, X_6, \ldots$ is independent of X_1, X_2, X_3
  - From any time, the future is independent of the past
  - Any fixed reindexing gives a Bernoulli process, e.g.,

  $$(X_4, X_5, X_6, \ldots) \quad \text{or} \quad (X_1, X_3, X_5, \ldots)$$
- Memoryless property

  $$\mathbf{P}(T - n = t \,|\, T > n) =$$

Arrival and interarrival times

- Let Y_k be the time of the kth arrival
- Increments

  $$T_1 = Y_1, \qquad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

  are called **interarrival times**
  - Distribution of T_1
  - Distribution of other T_ks
- After a rainy day, the number of days until it rains again is geometrically distributed with parameter p, independent of the past. What is the probability that it rains on April 3 and April 7?

Time of the kth arrival

- Y_k: number of trials to kth success

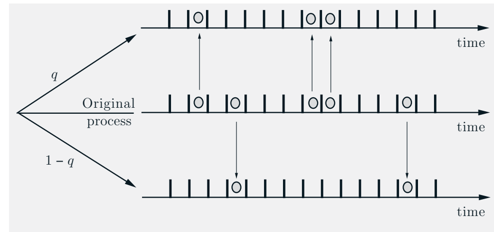
  - $\mathbf{E}[Y_k] \;=$

  - $\mathrm{var}(Y_k) \;=$

  - $\mathbf{P}(Y_k = t) \;=$

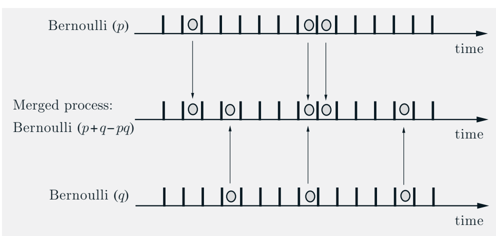
Splitting of a Bernoulli process

(using independent coin flips)



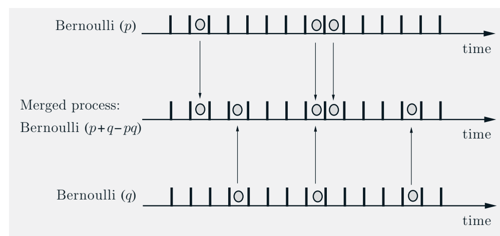
yields Bernoulli processes

Merging of independent Bernoulli processes



yields a Bernoulli process

Refining the time scale



- "Collisions" are a limitation in the model:
  - cannot tell the difference between 1 and 2 arrivals
- Shorter time slots decreases p, can make $p^2 \ll p$

Poisson approximation to binomial

- Number of arrivals in n slots is binomial

$$p_S(k) \;=\; \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k}, \qquad \text{for } k \geq 0$$

- Interesting to think of $n \to \infty$ with $\lambda = np$ constant

$$\begin{aligned} p_S(k) \;&=\; \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k} \\ &=\; \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &=\; \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-k+1}{n} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} \end{aligned}$$

- For any fixed $k \geq 0$, $\lim_{n\to\infty} (1 - \lambda/n)^{n-k} = e^{-\lambda}$ so

$$\lim_{n\to\infty} p_S(k) \;=\; \frac{\lambda^k}{k!} e^{-\lambda}$$