Problem Set 3
Due: February 25, 2009

1. A candy factory has an endless supply of red, orange, yellow, green, blue, and violet jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each. One possible color distribution, for example, is a jar of 56 red, 22 yellow, and 22 green jelly beans. As a marketing gimmick, the factory guarantees that no two jars have the same color distribution. What is the maximum number of jars the factory can produce?

2. You are standing on Mass Ave, with a notepad, recording all the cars approaching MIT from Hahvahd bridge. The cars can either be a Compact or a non-Compact. Suppose you know that the number of non-Compacts in Metro Boston is approximately 2 times that of Compacts. We can model this process as a sequence of independent "coin flips" with $P(\text{Compact}) = 1/3$ and $P(\text{non-Compact}) = 2/3$.

   (a) A non-Compact just passes by. What is the probability that the next car you see is also a non-Compact? What is the probability that you have to wait k more cars to see another non-Compact again? (i.e. You first see $k-1$ Compacts, then followed by a non-Compact.)

   (b) You have just observed a sequence of ten cars.

      i. What is the probability that there were exactly one Compact and nine non-Compacts?

      ii. How many distinct sequences of cars can be formed if the $10th$ car was also the $4th$ Compact you saw? What is the probability of each sequence?

      iii. Using the result in part b ii), find the probability that the $10th$ car is the kth Compact you saw.

   (c) You continue to record and gradually become bored. You decide that you will record 25 more cars. Find the number of all possible distinct car sequences under the following alternative assumptions.

      i. All 25 cars are recorded.

      ii. You stop when the number of either type of car reaches 13.

3. Consider a biased coin. Coin tosses are independent and P(heads) $= p$, P(tails) $= 1 - p$.

   (a) Assume the coin is tossed N times. Let k be some positive integer. Conditioned on the event that at least one coin toss results in "heads," what is the probability that we have exactly k heads?

   (b) The coin is tossed 10 times and we are told that exactly 5 of these tosses resulted in "heads." Derive the probability of the sequence $HTHTHTHTHT$.

4. We consider a sequence of independent tosses of a biased coin. Let $p = P(\text{heads}) = 0.7$.

   (a) Let K be the number of tosses up to and including the toss on which the first head occurs. Write down the PMF $p_K(k)$.

   (b) Let X be the number (zero or one) of heads that occur on any particular toss. Determine $E[X]$ and $\text{var}(X)$.

(c) Let M be the number of heads that occur during the first n tosses. Write down a formula for the PMF of M.

(d) (This is harder!) Given that a total of exactly six heads resulted from the first ten tosses, what is the conditional probability that there were exactly three heads in the first six tosses?

5. The president of a company discovers that exactly one of her two vice presidents, A and B is embezzling money from the company. In order to determine who is guilty, she decides to hire a private detective to investigate. If she chooses to investigate VP A she will have to pay D_A to the detective, and if A turns out to be guilty, the president will have to pay R_A to replace A. Similarly, investigating B has costs D_B and R_B. Furthermore, if the detective decides that one of the VP's is innocent, due process still requires that the president will have to pay the detective to investigate the other VP. If the a priori probability that A is guilty is p, and that B is guilty is $1 - p$, find the conditions on p, D_A, D_B, R_A, R_B for which investigating A first would minimize the expected cost of the procedure.

6. The probability of a royal flush in poker is $p = 1/649740$. Show that at least 649740 hands would have to be dealt in order that the probability of getting at least one royal flush is above $1 - 1/e$.

G1†. (a) A fair coin is tossed successively until two consecutive heads or two consecutive tails appear. Find the PMF, the expected value, and the variance of the number of tosses until this happens.

(b) Assume now that the coin is tossed successively until a head followed by a tail appear. Find the PMF and the expected value of the number of tosses until this happens.

†Required for 6.431; optional for 6.041