MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2011)

Problem Set 3
Due: February 23, 2011

1. A group of MIT entrepreneurs at Beaver Airlines just purchased troubled Air Harvard. Air Harvard currently only offers service to Reykjavik and Auckland. Because Air Harvard is so unorganized, flights occur in an independent manner and the planes often crash. The probability that a flight to Reykjavik or Auckland crashes is 1/5 and 1/10, respectively. Any particular flight goes to Reykjavik with probability 2/3 and Auckland with probability 1/3.

   (a) What is the probability that a randomly chosen flight crashes?

   (b) What is the expected number of flights before the first crash?

   (c) What is the expected number of flights that occur before the first crash and after 3 non-crash flights?

   (d) Air Harvard has 1000 flights per year. What is the probability that there is ≤ 1 crash in a year?

   (e) Sloan school graduates at Beaver Airlines discovered that if 100 new mechanics are hired, the probability of a safe flight on any particular Air Harvard flight will be 0.9999. Using the Poisson approximation, what is the probability that all 1000 flights in a given year arrive safely at their destination?

2. A family has 5 children. Each child has equal a apriori probability of being male or female, independently of the other children. If G is the number of girls out of the 5 children, find the probability mass function for the random variable $G + 2$.

3. Consider another game played with dice. Each of two players rolls a fair, four-sided die. Player A scores the maximum of the two dice minus 1, which is denoted X. Player B scores the minimum of the two dice, which is denoted Y.

   (a) Find the expectations of X, Y, and $X - Y$.

   (b) Find the variances of X, Y, and $X - Y$.

4. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables. Find c and d in terms of n that will make the following formula true:

$$\mathbf{E}\left[(X_1 + \cdots + X_n)^2\right] = c\mathbf{E}[X_1^2] + d\left(\mathbf{E}[X_1]\right)^2.$$

5. Customers arrive one at a time at a bagel shop. A customer is either from New York (with probability $\frac{1}{10}$) or from Boston (with probability $\frac{9}{10}$), independently of other customers. Each customer orders **two** bagels and, for each bagel, chooses one of two types of bagels (*onion* or *garlic*). The choices of types for the two bagels are independent. A New York customer chooses an onion bagel with probability $\frac{3}{4}$, while a Boston customer chooses an onion bagel with probability $\frac{1}{2}$. [Each customer chooses his/her bagels independently of all other customers].

   (a) Given that the first customer ordered an onion bagel and a garlic bagel (not necessarily in that order), what is the probability that he/she is from New York?

   (b) Find the expected number of onion bagels ordered by the first customer.

(c) Consider the first customer. Given that the first bagel he/she orders is an onion bagel, what is the probability that the second bagel he/she orders will also be an onion bagel?

(d) Find the probability that none of the first five customers orders an onion bagel.

(e) Suppose that the bagel shop only has fifteen bagels: ten onion, five garlic. Suppose that a customer from Connecticut arrives, and that the Connecticut customer is equally likely to choose any bagel in the shop (not any *type* of bagel; any physical *bagel*). What is the probability that of the first three bagels the Connecticut customer selects, exactly one is an onion bagel?

G1†. There are $n = 20$ people at a party. The host selects a set of the $n(n-1)/2$ pairs of people and introduces the 2 people in each pair to one another. Show that the host can introduce people in a way such that for every group of 7 people, there are at least 2 people who are introduced to each other and there are at least 2 people who are not.

This problem can also be stated in terms of a complete graph of $n = 20$ vertices, where people are vertices and introductions and no introductions are the edges of the graph. A complete graph means that every pair of two vertices are connected by an edge. Thus, there are $\binom{n}{2} = n(n-1)/2$ total edges. Suppose each edge is colored either red or blue. Show that there exists a coloring method for the $n(n-1)/2$ edges, such that no monochromatic complete subgraph of 7 nodes exists after coloring. In other words, for any subset of 7 vertices, the edges among these vertices are not all the same color.

Hint 1: You are not required to provide a solution (although this is certainly one way to prove the statement). You just need to show that there **exists** *a solution.*

Hint 2: Consider a random coloring of each edge with $1/2$ *probability Red and* $1/2$ *probability Blue. What is the probability that at least one monochromatic complete subgraph of* 7 *vertices appears?*

Hint 3: The Union Bound might be useful:
For events $E_1, E_2, \ldots E_n$*, the inequality holds,* $\Pr\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \Pr(E_i)$

†Required for 6.431; optional for 6.041