6.041 Fall 2011 Quiz 2
Wednesday, November 2, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
1a		9
1b		8
2a		5
2b		5
2c		8
2d		8
2e		10
2f		10
2g		11
2h		11
3		15
Your Grade		100

- This quiz has 3 problem, worth a total of 100 points.
- You may tear apart pages 3-6, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- For full credit, answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.
- You are allowed two two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- You have 120 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Tuesday 11/8.

Problem 0: Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Times
Patrick Jaillet	Aliaa Atwi	10 AM & 11 AM
Alan Willsky	Jagdish Ramakrishnan	1 PM & 2 PM
John Wyatt	Jimmy Li	2 PM & 3 PM

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Note: Most parts can be done independently. If worried about a possible mistake in a previous part, you can use a symbol for the expression you found there, and use that symbol in the formulas for subsequent parts.

Problem 1: (17 points)

Let U, V, W be independent standard normal random variables (i.e., Gaussian random variables with zero mean and unit variance), and let $X = 3U + 4V$, $Y = U + W$.

(a) **(9 points)** What is the probability that $X \geq 8$? (Give a numerical answer.)

(b) **(8 points)** Find $\mathbf{E}[XY]$.

Problem 2: (68 points)

Let X be a random variable that takes non-zero values in $[1, \infty)$, with PDF of the form

$$f_X(x) = \begin{cases} \dfrac{c}{x^3}, & x \geq 1 \\ 0, & x < 1. \end{cases}$$

Let U be a uniform random variable on $[0, 2]$. Assume that X and U are independent.
Possibly useful fact: for an integer $n \geq 2$,

$$\int \frac{1}{x^n}\, dx = \frac{-1}{(n-1)x^{n-1}} + \text{constant}.$$

(a) **(5 points)** What is the value of c?

(b) **(5 points)** Find $\mathbf{P}(2 \leq X \leq 3)$. (Give a numerical answer.)

(c) **(8 points)** Find $\mathbf{E}[X^3 e^{-X}]$. (Give a numerical answer.)

(d) **(8 points)** Provide a fully labeled sketch of the PDF of $2U + 1$.

(e) **(10 points)** Compute $\mathbf{P}(X \leq U)$. (You may leave your answer in integral form.)

(f) **(10 points)** Find the PDF of $1/X$.

(g) **(11 points)** Let K be a random variable independent of U, such that $\mathbf{P}(K = 0) = 2/3$, $\mathbf{P}(K = 1) = 1/3$. Give a fully labeled plot of the distribution (e.g., a PMF or PDF) of the random variable $Y = U + K$.

(h) **(11 points)** With K as in the previous part, find the conditional distribution of K, given that $U + K = y$, for every possible $y \in [0, 3]$. (Your answer can consist of either formulae or fully labeled plots.)

Problem 3: (15 points)

Let $N, X_1, Y_1, X_2, Y_2, \ldots$ be independent random variables. The random variable N takes positive integer values and has mean μ_N and variance σ_N^2. The random variables X_i (respectively, Y_i) are i.i.d. with mean μ_X (respectively, μ_Y) and variance σ_X^2 (respectively, σ_Y^2). Let

$$A = \sum_{i=1}^{N} X_i, \qquad B = \sum_{i=1}^{N} Y_i.$$

Find the covariance $\text{cov}(A, B)$ of A and B in terms of the given means and variances.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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