LECTURE 18

- **Readings:** Section 7.4

Lecture outline

- Review
- Multiple recurrent classes
- Absorption probability
- Expected time to absorption
- Mean first passage and recurrence times

Review

- Markov chain with a **single recurrent class**, which is **aperiodic**, has convergence of n-step transition probabilities

$$\lim_{n\to\infty} r_{ij}(n) \;=\; \pi_j \qquad \text{(with no dependence on } i\text{)}$$

to values that give a **steady-state PMF** on the states

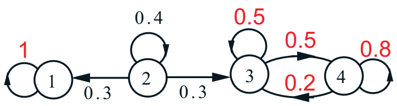
$$\pi_j \;=\; \lim_{n\to\infty} \mathbf{P}(X_n = j), \qquad j = 1, 2, \ldots, m.$$

- $\pi_1, \pi_2, \ldots, \pi_m$ can be found as unique solution to:

$$\pi_j \;=\; \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, 2 \;\ldots, m \qquad \text{(balance equations)}$$

$$\textstyle\sum_{j=1}^{m} \pi_j \;=\; 1 \qquad \text{(normalization)}$$

Multiple recurrent classes



- Is there convergence to a steady-state distribution?
- $\lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = 2)$

$$= \begin{cases} \qquad\qquad\qquad\qquad, & \text{for } j = 1; \\ \qquad\qquad\qquad\qquad, & \text{for } j = 2; \\ \qquad\qquad\qquad\qquad, & \text{for } j = 3; \\ \qquad\qquad\qquad\qquad, & \text{for } j = 4. \end{cases}$$

Absorption probability

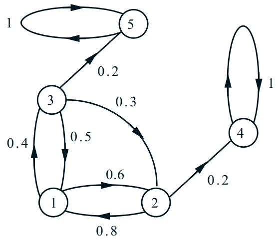
- A state k is **absorbing** when $p_{kk} = 1$
- Fix an absorbing state s. The **absorption probability** (to s) starting from state i is defined as

$$a_i \;=\; \mathbf{P}(\text{state } s \text{ is eventually reached} \mid X_0 = i)$$

- When a Markov chain has only transient and absorbing states [each recurrent class has only a single state], absorption probabilities are the unique solution to

$$\begin{aligned} a_s &= 1 \\ a_i &= 0 \qquad \text{for all absorbing } i \neq s \\ a_i &= \sum_{j=1}^{m} p_{ij}\, a_j \qquad \text{for all transient } i \end{aligned}$$

Absorption probability: Example



- What are the probabilities of absorption to state 5?

$$\begin{aligned} a_1 &= \\ a_2 &= \\ a_3 &= \\ a_4 &= \\ a_5 &= \end{aligned}$$

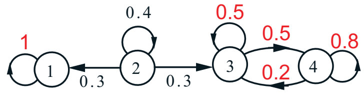
Absorption probability: Slight generalization

- Fix a recurrent class S. The **absorption probability** (to S) starting from state i is defined as

$$a_i \;=\; \mathbf{P}(\text{class } S \text{ is eventually reached} \mid X_0 = i)$$

- Absorption probabilities are the unique solution to

$$\begin{aligned} a_i &= 1 \qquad \text{for all } i \in S \\ a_i &= 0 \qquad \text{for all recurrent } i \notin S \\ a_i &= \sum_{j=1}^{m} p_{ij}\, a_j \qquad \text{for all transient } i \end{aligned}$$



Expected time to absorption

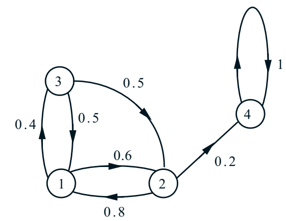
- Entering a recurrent state is called **absorption**
- **Expected time to absorption** starting from state i:

$$\mu_i \;=\; \mathbf{E}[\min\{n \geq 0 \text{ such that } X_n \text{ is recurrent}\} \,|\, X_0 = i]$$

- $\mu_1, \mu_2, \ldots, \mu_m$ are the unique solution to

$$\mu_i \;=\; 0 \qquad \text{for all recurrent states } i$$
$$\mu_i \;=\; 1 + \sum_{j=1}^{m} p_{ij}\,\mu_j \qquad \text{for all transient states } i$$

Expected time to absorption: Example



- What are the expected times to absorption?

$\mu_1 \;=$

$\mu_2 \;=$

$\mu_3 \;=$

$\mu_4 \;=$

$\mu_5 \;=$

Mean first passage and recurrence times

- Consider an MC with one recurrent class S and fix $s \in S$
- **Mean first passage time from** i **to** s is defined as

$$t_i \;=\; \mathbf{E}[\min\{n \geq 0 \text{ such that } X_n = s\} \,|\, X_0 = i]$$

- **Mean recurrence time of** s is defined as

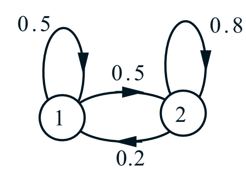
$$t_s^* \;=\; \mathbf{E}[\min\{n \geq 1 \text{ such that } X_n = s\} \,|\, X_0 = s]$$

- $t_1, t_2, \ldots, t_m$ are the unique solution to

$$t_s \;=\; 0$$
$$t_i \;=\; 1 + \sum_{j=1}^{m} p_{ij}\,t_j \qquad \text{for all } i \neq s$$

- Mean recurrence time is $t_s^* \;=\; 1 + \sum_{j=1}^{m} p_{sj}\,t_j$

Mean first passage and recurrence times: Example



- What are the mean first passage times to 1?

$t_1 \;=$

$t_2 \;=$

- What is the mean recurrence time of 1?

$t_1^* \;=$

Generality of Markov chain models

- Most discrete-time, finite-valued processes can be approximated well by a Markov chain, with a suitable state
  - Memoryless: $p_{ij} \;=\; \mathbf{P}(X_n = j)$ (no dependence on i)
  - "basic case": X_{n+1} and X_{n-1} are conditionally independent given X_n
  - Longer memory: handled by increasing m (size of the state space)

Example

- $X_0, X_1, X_2, \ldots$ are Bernoulli random variables with

$$p_{X_{n+1}|X_n,X_{n-1}}(1 \,|\, x_n, x_{n-1}) \;=\; \begin{cases} q_{00}, & \text{when } (x_n, x_{n-1}) = (0,0); \\ q_{01}, & \text{when } (x_n, x_{n-1}) = (0,1); \\ q_{10}, & \text{when } (x_n, x_{n-1}) = (1,0); \\ q_{11}, & \text{when } (x_n, x_{n-1}) = (1,1). \end{cases}$$

- $X_0, X_1, X_2, \ldots$ is not a Markov chain, but we can define one over a larger set $\{1, 2, \ldots, m\}$