Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2011)

Problem Set 3: Solutions

Due: February 23, 2011

1. (a) $\mathbf{P}(\text{Crash}) = \frac{2}{3} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{10} = \frac{1}{6}$

   (b) $\mathbf{E}[\text{Flights before crash}] = \mathbf{E}[\text{Geometeric RV with } p = \frac{1}{6}] - 1 = 5$

   (c) Same as part b: 5

   (d) $\mathbf{P}(\text{0 or 1 crashes}) = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{1000} + \begin{pmatrix} 1000 \\ 1 \end{pmatrix} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{999}$

   (e) Use p = probability of failure = 0.0001 $\quad \lambda = np = 1000 \cdot 0.0001 = 0.1$

   $\mathbf{P}(\text{0 Crashes}) = \frac{e^{-0.1}(0.1)^0}{0!} = e^{-0.1} \approx 0.9045$

   What does this suggest about the probability of a crash on a real airline?

2. The probability of a child being a girl is $\frac{1}{2}$. Then the number of girls G out of five children is a binomial random variable. The probability mass function of G is the binomial PMF, given by

$$\begin{aligned} p_G(G_0) &= \binom{5}{G_0} \left(\frac{1}{2}\right)^{G_0} \left(1 - \frac{1}{2}\right)^{5-G_0} \\ &= \binom{5}{G_0} \left(\frac{1}{2}\right)^5 \end{aligned}$$

for $G_0 = 0, 1, 2, 3, 4, 5$. The random variable $G+2$ takes on values from 2 to 7, with mass function given by

$$\begin{aligned} \mathbf{P}(G+2=2) &= p_G(0) = \left(\frac{1}{2}\right)^5 \\ \mathbf{P}(G+2=3) &= p_G(1) = \binom{5}{1} \left(\frac{1}{2}\right)^5 \\ \mathbf{P}(G+2=4) &= p_G(2) = \binom{5}{2} \left(\frac{1}{2}\right)^5 \\ \mathbf{P}(G+2=5) &= p_G(3) = \binom{5}{3} \left(\frac{1}{2}\right)^5 \\ \mathbf{P}(G+2=6) &= p_G(4) = \binom{5}{4} \left(\frac{1}{2}\right)^5 \\ \mathbf{P}(G+2=7) &= p_G(5) = \binom{5}{5} \left(\frac{1}{2}\right)^5 \end{aligned}$$

3. Denote the die rolls by W and Z. The sixteen equally-likely (W, Z) ordered pairs are depicted below, where the label in each cell is the (X, Y) pair.

	$Z=1$	$Z=2$	$Z=3$	$Z=4$
$W=1$	(0,1)	(1,1)	(2,1)	(3,1)
$W=2$	(1,1)	(1,2)	(2,2)	(3,2)
$W=3$	(2,1)	(2,2)	(2,3)	(3,3)
$W=4$	(3,1)	(3,2)	(3,3)	(3,4)

(a) From the table, we can read off the PMFs

$$p_X(k) = \begin{cases} 1/16, & k=0; \\ 3/16, & k=1; \\ 5/16, & k=2; \\ 7/16, & k=3; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad p_Y(k) = \begin{cases} 7/16, & k=1; \\ 5/16, & k=2; \\ 3/16, & k=3; \\ 1/16, & k=4; \\ 0, & \text{otherwise}, \end{cases}$$

and thus compute the expectations

$$\mathbf{E}[X] = \frac{1}{16}\cdot 0 + \frac{3}{16}\cdot 1 + \frac{5}{16}\cdot 2 + \frac{7}{16}\cdot 3 = \frac{17}{8}$$

and

$$\mathbf{E}[Y] = \frac{7}{16}\cdot 1 + \frac{5}{16}\cdot 2 + \frac{3}{16}\cdot 3 + \frac{1}{16}\cdot 4 = \frac{15}{8}.$$

We get by linearity of the expectation that $\mathbf{E}[X-Y] = \mathbf{E}[X] - \mathbf{E}[Y] = \frac{1}{4}$. (We could also find the PMF of the new random variable $X-Y$. See below.)

(b) Using the PMFs in part (a), we can compute

$$\mathbf{E}[X^2] = \frac{1}{16}\cdot 0^2 + \frac{3}{16}\cdot 1^2 + \frac{5}{16}\cdot 2^2 + \frac{7}{16}\cdot 3^2 = \frac{43}{8}$$

and

$$\mathbf{E}[Y^2] = \frac{7}{16}\cdot 1^2 + \frac{5}{16}\cdot 2^2 + \frac{3}{16}\cdot 3^2 + \frac{1}{16}\cdot 4^2 = \frac{35}{8}.$$

Thus, $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{55}{64}$ and $\text{var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 = \frac{55}{64}$. Since X and Y are *not* independent, the variance of X and Y is not any simple combination of previous results. Instead, let $Z = X - Y$ and find the PMF of Z as

$$p_Z(k) = \begin{cases} 4/16, & k=-1; \\ 6/16, & k=0; \\ 4/16, & k=1; \\ 2/16, & k=2; \\ 0, & \text{otherwise}. \end{cases}$$

Now

$$\mathbf{E}[Z^2] = \frac{4}{16}\cdot(-1)^2 + \frac{6}{16}\cdot 0^2 + \frac{4}{16}\cdot 1^2 + \frac{2}{16}\cdot 2^2 = 1,$$

and $\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 = 1 - (1/4)^2 = \frac{15}{16}$. ($\mathbf{E}[Z]$ was computed in part (a) and can also be double-checked with the PMF above.)

4. Since the X_is are identically distributed,

$$\text{(a)} \qquad \mathbf{E}[X_1] = \mathbf{E}[X_2] = \cdots = \mathbf{E}[X_n]$$

and

$$\text{(b)} \qquad \mathbf{E}[X_1^2] = \mathbf{E}[X_2^2] = \cdots = \mathbf{E}[X_n^2].$$

Furthermore, using (a) and the independence of the X_is,

$$\text{(c)} \qquad \mathbf{E}[X_iX_j] = \mathbf{E}[X_i]\mathbf{E}[X_j] = (\mathbf{E}[X_1])^2 \quad \text{when} \quad i \neq j.$$

$$
\begin{aligned}
\mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] &= \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right)\right] && \text{where separate dummy variables are for clarity} \\
&= \mathbf{E}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right)\right] && \text{by the distributive law} \\
&= \mathbf{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{1\le i,j\le n,\, i\ne j} X_i X_j\right] && \text{by separating the } i=j \text{ and } i\ne j \text{ terms} \\
&= \sum_{i=1}^{n} \mathbf{E}\left[X_i^2\right] + \sum_{1\le i,j\le n,\, i\ne j} \mathbf{E}\left[X_i X_j\right] && \text{by linearity of expectation} \\
&= \sum_{i=1}^{n} \mathbf{E}\left[X_1^2\right] + \sum_{1\le i,j\le n,\, i\ne j} \left(\mathbf{E}\left[X_1\right]\right)^2 && \text{using (b) and (c)} \\
&= n\mathbf{E}\left[X_1^2\right] + n(n-1)\left(\mathbf{E}\left[X_1\right]\right)^2 && \text{by counting the numbers of terms}
\end{aligned}
$$

Thus $c = n$ and $d = n(n-1)$.

5. Let $\mathbf{P}(NY) = \frac{1}{10}$ be the probability that a random customer is from New York.

   Let $\mathbf{P}(B) = \frac{9}{10}$be the probability that a random customer is from Boston.

   Let $\mathbf{P}(Onion|NY) = \frac{3}{4}$ be the probability that a customer buys an onion bagel given that the customer is from New York.

   Let $\mathbf{P}(Onion|B) = \frac{1}{2}$ be the probability that a customer buys an onion bagel given that the customer is from New York.

   Let $\mathbf{P}(Garlic|NY) = \frac{1}{4}$ be the probability that a customer buys a garlic bagel given that the customer is from New York.

   Let $\mathbf{P}(Garlic|B) = \frac{1}{2}$ be the probability that a customer buys a garlic bagel given that the customer is from Boston.

   (a) For this question we apply Bayes Rule. We know

   $$\mathbf{P}(NY|customer\ bought\ Onion\ and\ Garlic) = \frac{\mathbf{P}(NY \cap customer\ bought\ Onion\ and\ Garlic)}{\mathbf{P}(customer\ bought\ Onion\ and\ Garlic)}$$

   $$\frac{\mathbf{P}(NY \cap customer\ bought\ Onion\ and\ Garlic)}{\mathbf{P}(customer\ bought\ Onion\ and\ Garlic)} = \frac{\mathbf{P}(bought\ Onion\ and\ Garlic|NY)P(NY)}{\mathbf{P}(customer\ bought\ Onion\ and\ Garlic)}$$

   $$\frac{\mathbf{P}(bought\ Onion\ and\ Garlic|NY)P(NY)}{\mathbf{P}(customer\ bought\ Onion\ and\ Garlic)} = \frac{(\frac{3}{4}\frac{1}{4} + \frac{1}{4}\frac{3}{4})\frac{1}{10}}{(\frac{3}{4}\frac{1}{4} + \frac{1}{4}\frac{3}{4})\frac{1}{10} + (\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2})\frac{9}{10}}$$

   Note that we used the total probability theorem in the denominator, splitting the possibilities of the customer buying the onion and garlic bagel into two cases. The first case was if the customer was from NY, and the second was if the customer was from Boston.

   (b) First note that the customer's choice of the first bagel is independent of the customer's choice of the second bagel. So, we can treat the random variable of the number of onion bagels bought from the first two bagels as the sum of two independent random variables.

Let O_i be random variable such that $O_i = 1$ if the *ith* bagel ordered is an onion bagel, 0 otherwise. Note that O_1 and O_2 have identical distributions and are independent. Thus,

$$\mathbf{E}[\textit{number of total onion bagels ordered}] = \mathbf{E}[O_1 + O_2]$$

$$\mathbf{E}[O_1 + O_2] = \mathbf{E}[O_1] + \mathbf{E}[O_2]$$

Note that because O_1 and O_2 are identically distributed random variables, $\mathbf{E}[O_1] = \mathbf{E}[O_2]$. Note that we are *not* saying that $O_1 = O_2$.

Thus, $\mathbf{E}[\textit{number of total onion bagels ordered}] = 2\mathbf{E}[O_1]$

By the total expectation theorem, we know

$$2\mathbf{E}[O_1] = 2(\mathbf{E}[O_1|NY]\mathbf{P}(NY) + \mathbf{E}[O_1|B]\mathbf{P}(B))$$

$$2(\mathbf{E}[O_1|NY]\mathbf{P}(NY) + \mathbf{E}[O_1|B]\mathbf{P}(B)) = 2(\frac{3}{4}\frac{1}{10} + \frac{1}{2}\frac{9}{10})$$

(c) Using the same notation as in part b, we are asked to find $\mathbf{P}(O_2|O_1)$ We know

$$\mathbf{P}(O_2|O_1) = \frac{\mathbf{P}(O_2 \cap O_1)}{\mathbf{P}(O_1)}$$

Applying the total probability theorem in both the numerator and denominator, meaning we just take into account for both the numerator and the denominator the individual cases of the customer being from New York and the customer being from Boston:

$$\frac{\mathbf{P}(O_2 \cap O_1)}{\mathbf{P}(O_1)} = \frac{\mathbf{P}(O_1 \cap O_2|NY)\mathbf{P}(NY) + \mathbf{P}(O_1 \cap O_2|B)\mathbf{P}(B)}{\mathbf{P}(O_1|NY)\mathbf{P}(NY) + \mathbf{P}(O_1|B)\mathbf{P}(B)}$$

$$\frac{\mathbf{P}(O_1 \cap O_2|NY)\mathbf{P}(NY) + \mathbf{P}(O_1 \cap O_2|B)\mathbf{P}(B)}{\mathbf{P}(O_1|NY)\mathbf{P}(NY) + \mathbf{P}(O_1|B)\mathbf{P}(B)} = \frac{\frac{1}{10}\frac{3}{4}\frac{3}{4} + \frac{9}{10}\frac{1}{2}\frac{1}{2}}{\frac{1}{10}\frac{3}{4} + \frac{9}{10}\frac{1}{2}}$$

(d) We know that the probability that none of the first five customers orders an onion bagel is just the probability that each of the first five customers order 2 garlic bagels each. Let $P(G)$ be the probability that a random customers orders two garlic bagels. Note that our answer to this part will simply be $P(G)^5$. Once again, using the total probability theorem, we know

$$\mathbf{P}(G) = \mathbf{P}(G|NY)\mathbf{P}(NY) + \mathbf{P}(G|B)\mathbf{P}(B)$$

$$\mathbf{P}(G|NY)\mathbf{P}(NY) + \mathbf{P}(G|B)\mathbf{P}(B) = \frac{1}{10}\frac{1}{4}\frac{1}{4} + \frac{9}{10}\frac{1}{2}\frac{1}{2}$$

Thus, the probability that none of the first five customers orders an onion bagel is $(\frac{1}{10}\frac{1}{4}\frac{1}{4} + \frac{9}{10}\frac{1}{2}\frac{1}{2})^5$

(e) Let us first decide how many possibilities the customer has. We know that the customer is just as likely to select any three bagels. Hence, given that each possible triplet of bagels is equally likely, we know that the customer has $\binom{15}{3}$ total possible ways to select three bagels. Now we must ask ourselves how many ways can the customer select three bagels such that exactly one is an onion bagel, and exactly two (the remaining bagels he must choose) are garlic bagels. From the ten onion bagels, he can select any one of them. That is $\binom{10}{1}$ possibilities. Now for *each* of those onion bagels, he has $\binom{5}{2}$ ways to select the garlic bagels he wants. Thus, the total number of ways the customer can select exactly one onion bagel and two garlic bagels is $\binom{10}{1}\binom{5}{2}$. Thus, the probability of the customer selecting three bagels such that exactly one of them is an onion bagel is $\frac{\binom{10}{1}\binom{5}{2}}{\binom{15}{3}}$

G1†. Label the vertices of the graph by $V_1, V_2, \cdots, V_{20}$, and represent the edge between V_i and $V_j, (i \neq j)$ by (V_i, V_j). Randomly color each edge (V_i, V_j) either red or blue with equal probability, independently of the other edges. Define the set $\mathcal{S}$ to be the set of all subsets of exactly 7 vertices. For any $S \in \mathcal{S}$, let $I(S)$ be the event that the subgraph formed by vertices in S is not a monochromatic complete subgraph. In other words, the set $I(S)$ contains the set of all nodes of 7 vertices such that the edges among them contain both colors. The complement of the set $I(S)$ contains subgraphs of 7 nodes where all the edges are of one color. In this case, one of the two events must occur.

(a) All edges in the subgraph formed by vertices in S are colored Red.

(b) All eges in the subgraph formed by vertices in S are colored Blue.

For any subgraph of 7 nodes, there are $\binom{7}{2}$ edges. For a fixed S, all edges are colored red with probability $\frac{1}{2}^{\binom{7}{2}}$ and all edges are colored blue with probability $\frac{1}{2}^{\binom{7}{2}}$. Thus, for any $S \in \mathcal{S}$, $P(I(S)^c) = 2 \cdot \frac{1}{2}^{\binom{7}{2}}$.

The total number of possible subsets with 7 vertices is $\binom{20}{7}$, which means $|\mathcal{S}| = \binom{20}{7}$. Now consider the event $\bigcup_{S \in \mathcal{S}} I(S)^c$, which is the union of all monochromatic subgraphs of 7 vertices (this is the event that there is at least one monochromatic subgraph of 7 vertices).

$$\begin{aligned} P(\bigcup_{S \in \mathcal{S}} I(S)^c) &\leq \sum_{s \in \mathcal{S}} P(I(S)^c) \\ &= \binom{20}{7} \cdot \frac{1}{2}^{\binom{7}{2}} \cdot 2 \\ &= \frac{4835}{65536} \\ &< 1 \end{aligned}$$

This means that if we randomly color edges with probability 1/2 red and probability 1/2 blue, the probability that there exists at least one monochromatic subgraph of 7 vertices is less than 1. Thus, there exists at least one coloring method that gives no monochromatic complete subgraph of 7 vertices.

Alternatively, we can give a specific solution to this problem. One of the solutions to the party of 20 people example is the following: Randomly divide the guests into 5 groups of 4 guests. All the guests within each group are introduced to each other (6 introductions per group). If we randomly choose 7 guests, there will be at least 2 people not introduced and at least 2 people who are introduced. Another solution can be applied to 6 groups of 3 and a group of 2, where all guests are introduced to each other in each group of 3, and each person in the group of 2 is introduced to everyone at the party.

†Required for 6.431; optional for 6.041