MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 4
Due: March 4, 2009

1. Consider a set of n i.i.d. random variables: $\{X_1, X_2, \ldots, X_n\}$. Find the conditional expectation:
$$E[X_1 | X_1 + X_2 + \cdots + X_n = x],$$
as a function of x and n.

2. Suppose you arrive at a bus stop at time 0 and that at the end of each minute, with probability p a bus arrives or with probability $1-p$, no bus arrives. Whenever a bus arrives, you board that bus with probability q and depart. Let T equal the number of minutes you stand at a bus stop, and let N be the number of buses that arrive while you wait at the bus stop.

   (a) Identify the set of points (n, t) for which $p_{N,T}(n, t) > 0$.
   (b) Find $p_{N,T}(n, t)$.
   (c) Find the marginal PMFs $p_N(n)$ and $p_T(t)$.
   (d) Find the conditional PMFs $p_{N|T}(n|t)$ and $p_{T|N}(t|n)$.

3. Suppose that you have 2 fair six-sided dice with sides numbered from one to six. We define the following random variables:

   - Random variable S is the sum of the numbers showing on the two dice.
   - Random variable T is the product of the numbers showing on the two dice.
   - Random variable M is the maximum of the two numbers showing on the two dice.
   - Random variable N is the minimum of the two numbers showing on the two dice.

   Determine and sketch the following PMFs:

   (a) $p_S(s)$
   (b) $p_T(t)$
   (c) $p_M(m)$
   (d) $p_N(n)$
   (e) $p_{S|M}(s \mid m)$
   (f) $p_{M,N}(m, n)$

4. You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random.

   Find the mean and the variance of the number of trials you will need to open the door, under the following alternative assumptions:

   (a) after an unsuccessful trial, you mark the corresponding key so that you never try it again, or
   (b) at each trial, you are equally likely to choose any key.

5. A discrete random variable X has zero mean and unit variance. It is also known that random variables $Y = X^2$ and $Z = (X+1)^2$ are independent.

   (a) How many values does X take with a non-zero probability?

   (b) Find all possible PMF of X.

G1†. You have n urns, labeled from 1 to n, each containing $n+1$ balls. The k-th urn ($k = 1, \ldots, n$) contains exactly k white balls and $n+1-k$ black balls. You blindly extract one ball from each urn, and mix all the extracted balls together.

   (a) Find the mean and the variance of the number of white balls extracted. Give a simple interpretation of the value of the mean.

   (b) What is the probability of extracting all white balls, except a black one from urn k?

   (c) Given that all the extracted balls except one are white, what is the probability that the black ball was originally in urn k? You may leave the result as an algebraic sum, but try to simplify the answer as much as possible.

†Required for 6.431; optional for 6.041