Problem Set 10: Solutions
Due: May 5, 2010

1. (a) Let X_i be random variables indicating the quality of the ith bulb ("1" for good bulbs, "0" for bad ones). Then X_i are independent Bernoulli random variables. Let Z_n be

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

We apply the Chebyshev inquality and obtain

$$\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

where σ^2 is the variance of the Bernoulli random variable. Hence, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(|Z_n - p| \geq \epsilon\right) = 0,$$

by noticing $\lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$. This means that Z_n converges to p in probability.

(b) **Using Chebyshev's Inequality:**

For any number greater than 500, we know the number of bulbs would be enough for the test by using Chebyshev. Since the variance of a Bernoulli random variable is $p(1-p)$ which is less than or equal to $\frac{1}{4}$, we have $\sigma^2 \leq \frac{1}{4}$. Hence, for $n \geq 500$,

$$\begin{aligned}
\mathbf{P}\left(\left|\frac{X_1 + X_2 + ... + X_n}{n} - p\right| \geq 0.1\right) &\leq \frac{\sigma^2}{n0.1^2} \\
&\leq \frac{\frac{1}{4}}{n \times 0.1^2} \\
&\leq 1 - 0.95 = 0.05.
\end{aligned}$$

However, for a number less than 500, we can not tell by Chebyshev's inequality if the number of bulbs is enough for the test because we don't know the variance. If the variance is very small, which is possible when p is quite small, 27 bulbs could be enough actually.

Thus, the answer is "cannot be decided". In reality, we need to estimate the variance first.

Using central limit theorem:

We know by the CLT that as $n \to \infty$, $Z_n = \frac{X_1+X_2+...+X_n}{n}$ behaves like a normal random variable with mean p and variance σ^2/n. For finite n, this is not accurate, but we can still use the CLT to *approximate* the CDF of Z_n .The quality of our approximation improves as n increases.

We want to find n such that $\mathbf{P}(p - 0.1 \leq Z_n \leq p + 0.1) \geq 0.95$.

$$\begin{aligned}
&\mathbf{P}(p - 0.1 \leq Z_n \leq p + 0.1) \\
&= \mathbf{P}\left(\frac{-0.1}{\sigma/\sqrt{n}} \leq \frac{Z_n - p}{\sigma/\sqrt{n}} \leq \frac{0.1}{\sigma/\sqrt{n}}\right) \\
&\approx \Phi\left(\frac{0.1}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{-0.1}{\sigma/\sqrt{n}}\right) \\
&= 2\Phi\left(\frac{0.1}{\sigma/\sqrt{n}}\right) - 1
\end{aligned}$$

The worst case scenario happens when the variance of X_i is large. In that case, we need to take more samples (larger n), before we can satisfy our criterion. The maximum variance happens at $p = 1/2$. σ is then $p(1-p) = 1/4$.
In the worst case scenario, $\mathbf{P}(p - 0.1 \leq Z_n \leq p + 0.1) \approx 2\Phi\left(0.2\sqrt{n}\right) - 1$.
By solving for n in the equation $2\Phi\left(0.2\sqrt{n}\right) - 1 = 0.95$, we get $0.2\sqrt{n} = 1.96$, and $n \geq 96$ trials.

2. (a) The Chebyshev inequality yields $\mathbf{P}(|X - 7| \geq 3) \leq \frac{9}{3^2} = 1$, which implies the uninformative/useless bound $\mathbf{P}(4 < X < 10) \geq 0$.

   (b) We will show that $\mathbf{P}(4 < X < 10)$ can be as small as 0 and can be arbitrarily close to 1. Consider a random variable that equals 4 with probability 1/2, and 10 with probability 1/2. This random variable has mean 7 and variance 9, and $\mathbf{P}(4 < X < 10) = 0$. Therefore, the lower bound from part (a) is the best possible.

   Let us now fix a small positive number ϵ and another positive number c, and consider a discrete random variable X with PMF

$$p_X(x) = \begin{cases} 0.5 - \epsilon, & \text{if } x = 4 + \epsilon; \\ 0.5 - \epsilon, & \text{if } x = 10 - \epsilon; \\ \epsilon, & \text{if } x = 7 - c; \\ \epsilon, & \text{if } x = 7 + c. \end{cases}$$

   This random variable has a mean of 7. Its variance is

$$(0.5 - \epsilon)(3 - \epsilon)^2 + (0.5 - \epsilon)(3 - \epsilon)^2 + 2\epsilon c^2$$

   and can be made equal to 9 by suitably choosing c. For this random variable, we have $\mathbf{P}(4 < X < 10) = 1 - 2\epsilon$, which can be made arbitrarily close to 1.

   On the other hand, this probability can not be made equal to 1. Indeed, if this probability were equal to 1, then we would have $|X - 7| < 3$, which would imply that the variance is less than 9.

3. This question is perfectly suited to the De Moivre–Laplace approximation to the binomial. (This is an instance of a normal approximation based on the Central Limit Theorem, with a slight

adjustment to account for the fact that the random variable in question is integer valued.)

(a)
$$\begin{aligned}\mathbf{P}(190 \le L \le 210) &\approx \Phi\left(\frac{210+\frac{1}{2}-400\cdot 0.51}{\sqrt{(400)(0.51)(0.49)}}\right) - \Phi\left(\frac{190-\frac{1}{2}-400\cdot 0.51}{\sqrt{(400)(0.51)(0.49)}}\right)\\ &\approx \Phi(0.65)-\Phi(-1.45)\approx 0.6687\end{aligned}$$

(b)
$$\begin{aligned}\mathbf{P}(210 \le L \le 230) &\approx \Phi\left(\frac{230+\frac{1}{2}-400\cdot 0.51}{\sqrt{(400)(0.51)(0.49)}}\right) - \Phi\left(\frac{210-\frac{1}{2}-400\cdot 0.51}{\sqrt{(400)(0.51)(0.49)}}\right)\\ &\approx \Phi(2.65)-\Phi(0.55)\approx 0.2872\end{aligned}$$

4. First, let's calculate the expectation and the variance for Y_n, T_n, and A_n.

$$\begin{aligned}Y_n &= (0.5)^n X_n\\ T_n &= Y_1+Y_2+\cdots+Y_n\\ A_n &= \frac{1}{n}T_n\end{aligned}$$

$$\begin{aligned}\mathbf{E}[Y_n] &= \mathbf{E}\left[(\tfrac{1}{2})^n X_n\right] = (\tfrac{1}{2})^n\mathbf{E}[X_n] = \mathbf{E}[X](\tfrac{1}{2})^n = 2(\tfrac{1}{2})^n\\ \mathrm{var}(Y_n) &= \mathrm{var}\left((\tfrac{1}{2})^n X_n\right) = (\tfrac{1}{2})^{2n}\mathrm{var}(X_n) = \mathrm{var}(X)(\tfrac{1}{2})^{2n} = 9(\tfrac{1}{4})^{2n}\\ \mathbf{E}[T_n] &= \mathbf{E}[Y_1+Y_2+\cdots+Y_n] = \mathbf{E}[Y_1]+\mathbf{E}[Y_2]+\cdots+\mathbf{E}[Y_n]\\ &= 2\sum(\tfrac{1}{2})^i = 2\frac{0.5(1-0.5^n)}{1-0.5} = 2\left(1-(\tfrac{1}{2})^n\right)\\ \mathrm{var}(T_n) &= \mathrm{var}(Y_1+Y_2+\cdots+Y_n) = \sum_{i=1}^{n}(\tfrac{1}{4})^i\mathrm{var}(X_i)\\ &= 9\left(\frac{\frac{1}{4}\left(1-(\frac{1}{4})^n\right)}{1-\frac{1}{4}}\right) = 3\left(1-(\tfrac{1}{4})^n\right)\\ \mathbf{E}[A_n] &= \mathbf{E}\left[\frac{1}{n}T_n\right] = \frac{1}{n}\mathbf{E}[T_n] = \frac{2}{n}\left(1-(\tfrac{1}{2})^n\right)\\ \mathrm{var}(A_n) &= \mathrm{var}\left(\frac{1}{n}T_n\right) = \left(\frac{1}{n}\right)^2\mathrm{var}(T_n) = \frac{3}{n^2}\left(1-\left(\frac{1}{4}\right)^n\right)\end{aligned}$$

(a) Yes. Y_n converges to 0 in probability. As n becomes very large, the expected value of Y_n approaches 0 and the variance of Y_n approaches 0. So, by the Chebychev Inequality, Y_n converges to 0 in probability.

(b) No. Assume that T_n converges in probability to some value a. We also know that:

$$\begin{aligned}T_n &= Y_1+(Y_2+Y_3+.....Y_n)\\ &= Y_1+((0.5)^2X_2+(0.5)^3X_3+\cdots+(0.5)^nX_n)\\ &= Y_1+\frac{1}{2}(0.5X_2+(0.5)^2X_3+\cdots+(0.5)^{n-1}X_n).\end{aligned}$$

Notice that $0.5X_2+(0.5)^2X_3+\cdots+(0.5)^{n-1}X_n$ converges to the same limit as T_n when n goes to infinity. If T_n is to converge to a, Y_1 must converge to $a/2$. But this is clearly false, which presents a contradiction in our original assumption.

(c) Yes. A_n converges to 0 in probability. As n becomes very large, the expected value of A_n approaches 0, and the variance of A_n approaches 0. So, by the Chebychev Inequality, A_n converges to 0 in probability. You could also show this by noting that the A_ns are i.i.d. with finite mean and variance and using the WLLN.

5. a) The random variables, Y_i, are not independent. We can guess this intuitively by observing that consecutive Y_i depend on the same value of X_i; we can also prove it using covariance.

Since $E[Y_i] = \mu$,

$$\begin{aligned}
\text{cov}(Y_i, Y_{i+1}) &= \mathbf{E}[(Y_i - \mathbf{E}[Y_i])(Y_{i+1} - \mathbf{E}[Y_{i+1}])] \\
&= \mathbf{E}[(\frac{1}{3}X_i + \frac{2}{3}X_{i+1} - \mu)(\frac{1}{3}X_{i+1} + \frac{2}{3}X_{i+2} - \mu)] \\
&= \mathbf{E}[(\frac{1}{3}(X_i - \mu) + \frac{2}{3}(X_{i+1} - \mu))(\frac{1}{3}(X_{i+1} - \mu) + \frac{2}{3}(X_{i+2} - \mu))] \\
&= \frac{1}{9}\mathbf{E}[(X_i - \mu)(X_{i+1} - \mu)] + \frac{2}{9}\mathbf{E}[(X_{i+1} - \mu)^2] \\
&+ \frac{2}{9}\mathbf{E}[(X_i - \mu)(X_{i+2} - \mu)] + \frac{4}{9}\mathbf{E}[(X_{i+1} - \mu)(X_{i+2} - \mu)] \\
&= \frac{2}{9}\mathbf{E}[(X_{i+1} - \mu)^2] \\
&= \frac{2}{9}\sigma^2
\end{aligned}$$

Since Y_i and Y_{i+1} are correlated, they are not independent.

b) Yes, they are identically distributed. Each Y_i is the same weighted sum of identical variables.

c) We need to show that

$$\lim_{n\to\infty} P(|M_n - \mu| > \epsilon) = 0$$

for all $\epsilon > 0$. The form of this equation is very similar to that of the Chebyshev inequality, so we will try to apply the Chebyshev inequality to the problem.

The Chebyshev inequality is given below.

$$P(|Y - \text{E}(Y)| > a) \le \frac{\text{var}(Y)}{a^2}$$

If the expected value of M_n were equal to μ, the Chebyshev inequality would be very useful.

$$\begin{aligned}
E[M_n] &= \frac{1}{n}E[Y_1 + Y_2 + \cdots + Y_n] \\
&= E[Y_i] = \mu
\end{aligned}$$

If we compute the variance of M_n, we can plug and chug with the Chebyshev inequality and hope everything works. The variance calculation is shown below.

$$\begin{aligned}
M_n &= \frac{1}{n}\left(\frac{1}{3}X_1 + X_2 + X_3 + \cdots + X_n + \frac{2}{3}X_{n+1}\right) \\
\text{var}(M_n) &= \frac{1}{n^2}\left(\frac{1}{9}\text{var}(X_1) + \text{var}(X_2) + \text{var}(X_3) + \cdots + \text{var}(X_n) + \frac{4}{9}\text{var}(X_{n+1})\right) \\
&= \frac{9n-4}{9n^2}\sigma^2
\end{aligned}$$

Now, we are ready to solve the problem using the Chebyshev inequality.

$$
\begin{aligned}
P(|M_n - \mu| > \epsilon) &\leq \frac{\text{var}(M_n)}{\epsilon^2} \\
P(|M_n - \mu| > \epsilon) &\leq \frac{\frac{9n-4}{9n^2}\sigma^2}{\epsilon^2} \\
\lim_{n\to\infty} P(|M_n - \mu| > \epsilon) &\leq \lim_{n\to\infty} \frac{(9n-4)\sigma^2}{9n^2\epsilon^2} \\
&= 0
\end{aligned}
$$

Therefore, M_n converges in probability to μ.

6. (a) The following property of the absolute value is useful:

$$|U + V| \leq |U| + |V|.$$

Since the event $\{|U + V| \geq \epsilon\}$ is a subset of the event $\{|U| + |V| \geq \epsilon\}$, we have

$$\mathbf{P}(|U + V| \geq \epsilon) \leq \mathbf{P}(|U| + |V| \geq \epsilon). \tag{1}$$

Now, think a moment about the right side of expression (1). The sum of two non-negative random variables can't be greater than ϵ unless one or the other of them is greater than $\frac{\epsilon}{2}$. Symbolically,

$$\mathbf{P}(|U| + |V| \geq \epsilon) \leq \mathbf{P}(|U| \geq \frac{\epsilon}{2}) + \mathbf{P}(|V| \geq \frac{\epsilon}{2}) - \mathbf{P}(|U| \geq \frac{\epsilon}{2} \cap |V| \geq \frac{\epsilon}{2}) \tag{2}$$

where the right side of expression (2) is just the formula for the probability of the union of the two events $\{|U| \geq \frac{\epsilon}{2}\}$ and $\{|V| \geq \frac{\epsilon}{2}\}$.
We can drop the negative term in (2), combine it with expression (1) and obtain the desired result:

$$\mathbf{P}(|U + V| \geq \epsilon) \leq \mathbf{P}(|U| \geq \frac{\epsilon}{2}) + \mathbf{P}(|V| \geq \frac{\epsilon}{2})$$

(b) Consider two random variables $U_n - a$ and $V_n - b$ and the following probability:

$$\mathbf{P}(|U_n - a + V_n - b| \geq \epsilon).$$

By the inequality we obtained in part a), we have

$$\mathbf{P}(|U_n - a + V_n - b| > \epsilon) \leq \mathbf{P}(|U_n - a| \geq \frac{\epsilon}{2}) + \mathbf{P}(|V_n - b| \geq \frac{\epsilon}{2})$$

If U_n and V_n converge in probability to a and b respectively, then both terms on the right side of the equation become 0 in the limit, which implies

$$
\begin{aligned}
\lim_{n\to\infty} \mathbf{P}(|U_n - a + V_n - b| > \epsilon) &= 0 \quad \text{and therefore} \\
\lim_{n\to\infty} \mathbf{P}(|(U_n + V_n) - (a + b)| > \epsilon) &= 0
\end{aligned}
$$

which is the condition that must be met if $U_n + V_n$ converges to $a + b$ in probability.