Problem Set 3 Solutions
Due: February 25, 2009

1. Think of lining up the jelly beans, by first placing the red ones, then the orange ones, etc. We also place 5 dividers to indicate where one color ends and another starts. (Note that two dividers can be adjacent if there are no jelly beans of some color.) By considering both jelly beans and dividers, we see that there is a total of 105 positions. Choosing the number of jelly beans of each color is the same as choosing the positions of the dividers. Thus, there are $\binom{105}{5}$ possibilities, and this is the number of possible jars.

2. We know that the probability of the whole sample space equals to one.

$$\mathbf{P}(\text{non-compact}) + \mathbf{P}(\text{compact}) = 1.$$

And we also know that $\mathbf{P}(\text{non-compact}) = 2\mathbf{P}(\text{compact})$. So,

$$\mathbf{P}(\text{non-compact}) = \frac{2}{3} \qquad \mathbf{P}(\text{compact}) = \frac{1}{3}.$$

(a) Since the fact that the last car is non-Compact does not tell anything about the made of the next car, these two events are independent. Therefore,

$$\mathbf{P}(\text{next is non-compact} \mid \text{last car non-compact}) = \mathbf{P}(\text{next is non-compact}) = \frac{2}{3}$$

Also, the next k cars are also independent of the last one we saw. Then, the probability that we have to wait k more cars to see another non-Compact given that the last car we saw is non-Compact is simply the probability that the next $k-1$ cars are all Compacts and the k^{th} car is non-Compact. And this probability equals to

$$\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right)$$

(b) i. The probability that the i^{th} car is a Compact is $1/3$. The probability that the rest are non-Compact is $(2/3)^9$. Futhermore, there are ten choices for i, i.e. the 1^{st} to the 10^{th}. So, the probability of this event is

$$10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^9.$$

ii. If the 10^{th} car was the 4^{th} Compact, that means 3 of the first 9 cars must be Compacts. So the number of distinct sequences is $\binom{9}{3}$.

In each sequence, 4 Compacts and 6 non-Compacts are obrseved. So, the probability of each sequence is

$$\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6.$$

iii. If the 10^{th} car was the k^{th} Compact, that means $k-1$ of the first 9 cars must be Compacts and the rest of the first 9 cars must be non-Compacts. Each sequence has the probability of $(\frac{1}{3})^k(\frac{2}{3})^{10-k}$. And the number of ways we can arrange the $k-1$ Compacts in the sequence of 9 cars is $\binom{9}{k-1}$. So the probability for this event is

$$\binom{9}{k-1}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{10-k}.$$

(c) i. Each car can be either Compact or non-Compact independent of other cars. Therefore, the number of all possible distinct car sequences is 2^{25}.

ii. These sequences can either end when the number of Compacts reaches 13 or the number of non-Compacts does. Suppose the number of Compacts reaches 13 first and this happens at the $x+1^{st}$ car for $12 \le x \le 24$. Among the first x cars, there must be 12 Compacts and $x-12$ non-Compacts. Summing over all possible x, and double the result to properly account for the sequences where the number of non-Compacts reaches 13 first. The total number of distinct sequences is

$$2\sum_{x=12}^{24}\binom{x}{12} = 10400600$$

Alternatively, the problem can be approached in a different way. Again, assume that the number of Compacts reaches 13 first. Now choose 13 cars out of the 25 cars to be Compacts. The length of the sequence (as described as $(x+1)$ above) is actually determined by how we pick the 13 cars. For example, if we pick the first 13 cars to be all Compacts, x will be automatically set as 12. Therefore, the total number of ways to arrange the 13 Compacts is $\binom{25}{13}$, i.e. total number of distinct sequences is

$$2\binom{25}{13} = 10400600$$

3. We are given a biased coin with $P(heads) = p$ and $P(tails) = 1-p$.

We define Events A,B as the following:

A: Exactly k heads in N tosses.
B: At least one head was tossed.

From the definition of conditional probability, $P(A|B) = \frac{P(A\cap B)}{P(B)}$.

Note that $P(B) = 1 - P(B') = 1 - P(N\ straight\ tails) = 1-(1-p)^N$. We know that the probability of getting one result of k heads and $N-k$ tails is $p^k(1-p)^{N-k}$. Furthermore, there are $\binom{N}{k}$ ways of getting those k heads in N tosses. If $k=0$, then $A\cap B = \phi$, and $P(A\cap B) = 0$. For $k\ge 1$, $P(A\cap B) = P(A)$ because $A\cap B = A$. Thus our answer is:

$$P(exactly\ k\ heads\ in\ N\ tosses \mid at\ least\ one\ head) = \frac{\binom{N}{k}p^k(1-p)^{N-k}}{1-(1-p)^N}, for\ k = 1,2,\ldots,N$$

$$0,\ otherwise.$$

(b) We are asked to find the probability of the sequence $HTHTHTHTHT$, given that we tossed exactly 5 heads in 10 tosses. We note that each individual combination of 5 heads, 5 tails is equally likely. Thus, we need only find the total number of combinations of 5 heads and 5 tails, which is $\binom{10}{5}$. Thus,

$$P(HTHTHTHTHT \mid 5H,\ 5T) = \frac{1}{\binom{10}{5}}.$$

4. (a)

$$p_K(k) = \begin{cases} 0.3^{k-1}0.7 & k = 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases}$$

(b) The PMF for X is given by,

$$p_X(x) = \begin{cases} 0.7 & x = 1 \\ 0.3 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = (1)(0.7) + (0)(0.3) = 0.7$$

$$\begin{aligned} var(X) &= E[X^2] - E[X]^2 \\ &= 0.7 \cdot 1^2 + 0.3 \cdot 0^2 - 0.7^2 \\ &= 0.21 \end{aligned}$$

(c) M is a binomial random variable.

$$p_M(m) = \begin{cases} \binom{n}{m}0.7^m 0.3^{n-m} & m = 0, 1, 2, ..., n \\ 0 & \text{otherwise} \end{cases}$$

(d) Let A denote the event of exactly 3 heads on the first 6 tosses.
Let B denote the event of exactly 6 heads on the first 10 tosses.
We desire

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Now both A and B are binomial random variables. Therefore:

$$P(B) = \binom{10}{6} 0.7^6 0.3^4,$$

and

$$P(A) = \binom{6}{3} 0.7^3 0.3^3.$$

Now $P(B|A) =$ P(exactly three heads on the last 4 tosses) $= \binom{4}{3}0.7^3 0.3$.
Therefore

$$P(A|B) = \frac{\binom{4}{3}0.7^3 0.3 \binom{6}{3}0.7^3 0.3^3}{\binom{10}{6}0.7^6 0.3^4} = \frac{8}{21}$$

5. Suppose the president decides to investigate A first. Then her expected costs will be:

$$\mathbf{E}[\text{costs}] = D_A + pR_A + (1-p) \cdot D_B + (1-p) \cdot R_B$$

where as if she investigates B first, then

$$\mathbf{E}[\text{costs}] = p(D_A + R_A) + D_B + (1-p) \cdot R_B$$

In order that the first be smaller than the second, we need:

$$pD_B > (1-p)D_A$$

6. Let X be the number of royal flushes that we get in n hands. We can treat X as a binomial random variable with parameters n and $p = 1/649740$.

Let A be the event of getting at least one royal flush in n hands. Then, A^C is the event of getting no royal flush with a probability

$$P(A^C) = p_X(0) = \binom{n}{0} p^0 (1-p)^{n-0}.$$

Thus, we get $P(A) = 1 - P(A^C) = 1 - (1-p)^n$.

Solving the inequality $1 - (1-p)^n \geq 1 - 1/e$, we get $n \geq 649744$.

G1†. Let X be the total number of tosses.

(a) If $x \geq 2$, there are only two possible sequences of outcomes that lead to the event $\{X = x\}$, the sequence HH and TT. Therefore,

$$P(X = x) = 2 \cdot (\frac{1}{2})^x = (\frac{1}{2})^{x-1}.$$

It follows that

$$p_X(x) = \begin{cases} (\frac{1}{2})^{x-1} & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[X] = \sum_{x=2}^{\infty} x \cdot (\frac{1}{2})^{x-1} = 2 \cdot (-\frac{1}{2} + \sum_{x=1}^{\infty} x \cdot (\frac{1}{2})^x) = 2 \cdot (-\frac{1}{2} + \frac{1}{1/2}) = \boxed{3}.$$

To find the variance of X, we first compute $E[X^2]$. We have

$$E[X^2] = \sum_{x=2}^{\infty} x^2 \cdot (\frac{1}{2})^{x-1} = 2 \cdot (-\frac{1}{2} + \sum_{x=1}^{\infty} x^2 \cdot (\frac{1}{2})^x) = -1 + 2 \cdot 6 = 11.$$

Thus

$$var(X) = 11 - 3^2 = \boxed{2}.$$

(b) If $x > 2$, there are $x - 1$ sequences of outcomes that lead to the event $\{X = x\}$, the first contains only heads in the first $x - 2$ trials and the others are determined by the index of last occurrence of tail in the first $x - 2$ trials. For the case when $x = 2$, there is only one (hence $x - 1$) possible sequences of outcomes that lead to the event $\{X = x\}$. Therefore, for any $x \geq 2$,

$$P(X = x) = (x - 1)(\frac{1}{2})^x.$$

It follows that

$$p_X(x) = \begin{cases} (x-1)(\frac{1}{2})^x & \text{if } x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[X] = \sum_{x=2}^{\infty} x(x-1)(\frac{1}{2})^{x-1} = \sum_{x=1}^{\infty} x^2(\frac{1}{2})^x - \sum_{x=1}^{\infty} x(\frac{1}{2})^x = \boxed{4}.$$

†Required for 6.431; optional for 6.041