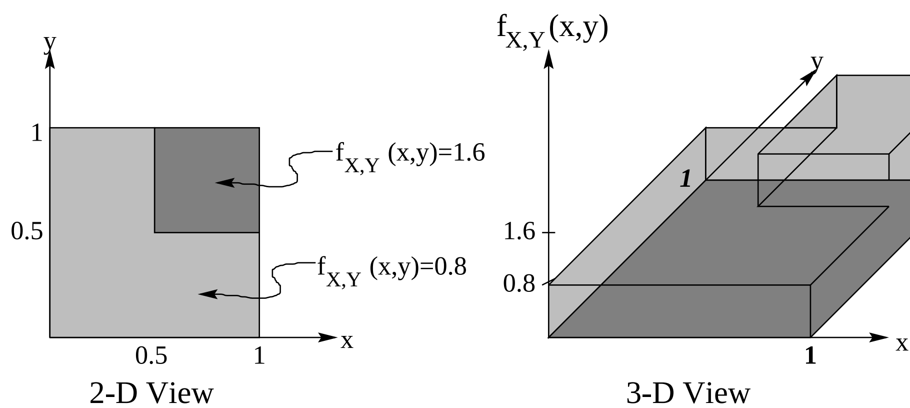
Problem Set 5
Due: March 18, 2009

- **Text Sections: 3.1-3.5**
- **Continuous random variables and PDFs, CDFs, Normal random variables, Joint PDFs, Conditioning.**

1. Continuous random variables X and Y each take on experimental values between zero and one, with the joint pdf indicated below (the cutoff between probability density 0.8 and 1.6 occurs at $x = 0.5$ and $y = 0.5$):



   (a) Are X and Y independent? Present a convincing argument for your answer.

   (b) Prepare neat, fully labelled plots for $f_X(x)$ and $f_{Y|X}(y \mid 0.75)$.

   (c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.

   (d) Let $W = \min\{X, Y\}$ and determine the cumulative distribution function (CDF) of W. You should be able to reason out this part without doing any formal integrals.

2. **Signal Classification:** A wire connecting two locations serves as the transmission medium for ternary-valued messages; in other words, any transmitted message between locations is known to be one of three possible symbols, each occurring with equal probability. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additive noise, assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. Assume that X is independent of N.

   (a) Suppose the transmitter encodes the three types of messages with the values -1, 0 and 1. At the other end, the received message is decoded according to the following rules:

   - If $Y > \frac{1}{2}$, then conclude the value 1 was sent.
   - If $Y < -\frac{1}{2}$. then conclude the value -1 was sent.
   - If $-\frac{1}{2} \leq Y \leq \frac{1}{2}$, then conclude the value 0 was sent.

   Determine the probability of error for this encoding/decoding scheme. Try to obtain your answer in terms of the standard normal CDF Φ, then look up the corresponding numerical value from the standard normal table (page 155 in your textbook).

(b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the three types of messages with the values -2, 0 and 2 while the receiver's decoding rules are:

- If $Y > 1$, then conclude the value 2 was sent.
- If $Y < -1$. then conclude the value -2 was sent.
- If $-1 \leq Y \leq 1$, then conclude the value 0 was sent.

Repeat part (a) for this modified encoding/decoding scheme, i.e. determine the probability of error achieved. Does it improve compared to the previous scheme?

(c) Motivated by your answer, can you suggest an approach to make the probability of error as small as desired?

3. Consider the following problem and a purported solution. Explain the flaw, and correct the solution.

Question: Let X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} 1, & x \in [0,1] \quad \text{and} \quad y \in [x, x+1]; \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_X(x)$, $f_Y(y)$, and $f_{Y|X}(y|x)$. Are X and Y independent?

Solution:

$$f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_x^{x+1} 1 \cdot dy = 1.$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx = \int_0^1 1 \cdot dx = 1.$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{1} = 1.$$

Since $f_{Y|X}(y|x)$ does not depend on x, we have that X and Y are independent. Alternatively, X and Y are independent because $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

4. Suppose n contestants on a TV game show spin the same, infinitely finely calibrated, but not necessarily fair, wheel of fortune. Suppose further that each spin is independent, and has a CDF $F(s)$, where F is a continuous function, $s \in [0, 2\pi], F(0) = 0$ and $F(2\pi) = 1$. After all n contestants have spun the wheel, label the value of their spins S_1 to S_n in increasing order. Now suppose that the host spins the wheel. Let S denote the value of his spin. Compute the following probabilities:

a) $\mathbf{P}(S = S_n)$

b) $\mathbf{P}(S > S_n)$

c) $\mathbf{P}(S > S_1)$

d) $\mathbf{P}(S_j < S < S_k)$ for $1 \leq j < k \leq n$.

Hint: For $n = 1$, $\mathbf{P}(S > S_1) = \mathbf{P}(S < S_1)$ by symmetry.

5. Beginning at time $t = 0$, we start using three light bulbs to illuminate a house. Light bulb A has a life exponentially distributed with mean 1 month, bulb B has a life exponentially distributed with mean 2 months, and bulb C has a life exponentially distributed with mean 3 months. The lifetimes of all three bulbs are independent of each other.

   (a) Let T be the time until the first failure of a bulb. Find the PDF of T. What kind of distribution does T have?

   (b) What is the probability that the first bulb to fail is bulb C?

G1[†]. Let X be a random variable that takes on values between 0 and c; that is, $\mathbf{P}(0 \leq X \leq c) = 1$.

   (a) Show that

$$\mathrm{var}(X) \leq \frac{c^2}{4}.$$

   (b) Show that there exists a random variable $\tilde{X}$ with $\mathbf{P}(0 \leq \tilde{X} \leq c) = 1$, such that

$$\mathrm{var}(\tilde{X}) = \frac{c^2}{4}.$$

   Give the PMF or PDF of $\tilde{X}$.

[†]Required for 6.431; optional for 6.041