Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2011)

Problem Set 9
Due: April 27, 2011

1. The outcomes of successive flips of a particular coin are dependent and are found to be described fully by the conditional probabilities

$$P(H_{n+1}|H_n) = 3/4 \quad P(T_{n+1}|T_n) = 2/3$$

where we have used the notation: Event H_k: Heads on kth toss; Event T_k: Tails on kth toss. We know that the first toss came up heads.

   (a) Determine the probability that the *first* tail will occur on the kth toss ($k = 2, 3, 4, \ldots$).

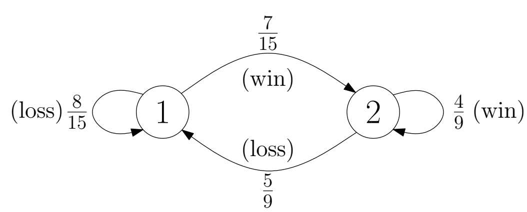
   (b) What is the approximate probability that flip 5000 will come up heads?

   (c) What is the approximate probability that flip 5000 will come up heads and flip 5002 will also come up heads?

   (d) Given that flips $5001, 5002, \ldots, 5000 + m$ all have the same result, what is the approximate probability that all of these m outcomes are heads? Simplify your answer as much as possible, and interpret your result for large values of m.

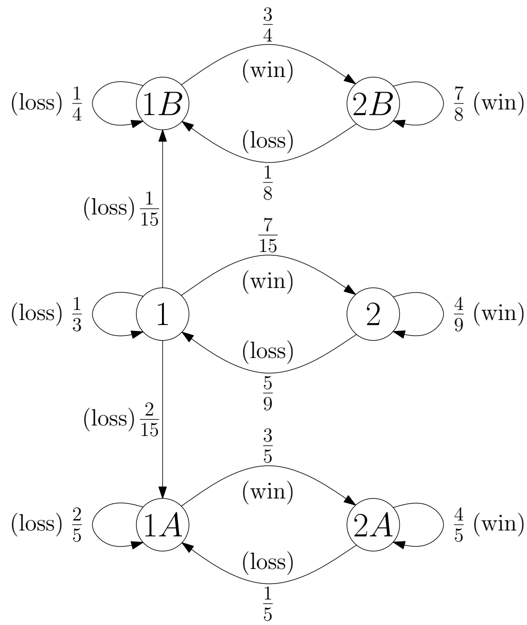
   (e) We are told that the 375th head just occurred on the 500th toss. Determine the expected value of the number of additional tosses required until we observe the 379th head.

2. Jack is a gambler who pays for his MIT tuition by spending weekends in Las Vegas. Lately he's been playing 21 at a table that returns cards to the deck and reshuffles them all before each hand. As he has a fixed policy in how he plays, his probability of winning a particular hand remains constant, and is independent of all other hands. There is a wrinkle, however; the dealer switches between two decks (deck #2 is more unfair to Jack than deck #1), depending on whether or not Jack wins. Jack's wins and losses can be modeled via the transitions of the following Markov chain, whose states correspond to the particular deck being used.



   (a) What is Jack's long term probability of winning?

Given that Jack loses and the dealer is not occupied with switching decks, with probability $\frac{2}{8}$ the dealer looks away for one second and with probability $\frac{1}{8}$ the dealer looks away for two seconds, independently of everything else. When this happens, Jack secretly inserts additional cards into both of the dealer's decks, transforming the decks into types $1A$ & $2A$ (when he has 1 second) or $1B$ & $2B$ (when he has 2 seconds). Jack slips cards into the decks at most once. The process can be described by the modified Markov chain in the picture. Assume in all future problems that play begins with the dealer using deck #1.



(b) What is the probability of Jack eventually playing with decks $1A$ and $2A$?

(c) What is Jack's long-term probability of winning?

(d) What is the expected time (as in number of hands) until Jack slips additional cards into the deck?

(e) What is the distribution of the number of times that the dealer switches from deck 2 to deck 1?

(f) What is the distribution of the number of wins that Jack has before he slips extra cards into the deck? *Hint*: Note that after some conditioning, we have a geometric number of geometric random variables, all of which are independent.

(g) What is the average net losses (number of losses minus the number of wins, sometimes negative) prior to Jack slipping additional cards into the deck?

(h) Given that after a very long period of time Jack is playing a hand with deck $1A$, what is the approximate probability that his previous hand was played with deck $2A$?

3. **Signal-to-Noise Ratio:** If random variable X has mean $\mu \neq 0$ and standard deviation $\sigma > 0$, the ratio $r = |\mu|/\sigma$ is called the measurement *signal-to-noise ratio*, or *SNR*, of X. The idea is that X can be expressed as $X = \mu + (X - \mu)$, with μ representing a deterministic, constant-valued "signal" and $(X - \mu)$ the random, zero-mean "noise." If we define $|(X - \mu)/\mu| = D$ as the relative deviation of X from its mean μ, show that for $\alpha > 0$,

$$\mathbf{P}(D \leq \alpha) \geq 1 - \frac{1}{r^2\alpha^2} \quad .$$

4. Demonstrate that the Chebyshev inequality is tight, that is, for every μ, $\sigma > 0$, and $c \geq \sigma$, construct a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}$$

Hint: You should be able to do this with a discrete random variable that takes on only 3 distinct values with nonzero probability.

G1†. let Y be the sum of n independent, identically distributed random variables,

$$Y = X_1 + X_2 + ... + X_n,$$

where each X_i is normal with mean m, and unit variance

$$f_X(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-m)^2}, \quad -\infty \leq x \leq \infty.$$

(a) Write down an integral expression for the exact value of $P(Y \geq \alpha)$. Do NOT try to evaluate the integral.

(b) Obtain an upper bound on $P(Y \geq \alpha)$ for our sum-of-normal random variable Y using the Chernoff bound. Choose a value for s which yields the best (i.e., smallest) upper bound. What is this smallest upper bound? For what values of α is your best bound valid?

Notice that the bound you obtain is much simpler to use than your integral expression in a.

†Required for 6.431 students and optional (not graded) for 6.041.