MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 1
Due: September 16, 2009

1. We are given that $\mathbf{P}(A^c) = 0.6$, $\mathbf{P}(B) = 0.3$, and $\mathbf{P}(A \cap B) = 0.2$. Determine $\mathbf{P}(A \cup B)$.

2. Consider two events, X_1 and X_2. Prove the following identities:
   (a) $\mathbf{P}(X_1 \cap X_2) \leq \mathbf{P}(X_1)$
   (b) $\mathbf{P}(X_1) \leq \mathbf{P}(X_1 \cup X_2)$
   (c) $\mathbf{P}(X_1 \cup X_2) \leq \mathbf{P}(X_1) + \mathbf{P}(X_2)$

3. You flip a fair coin 3 times. Since the coin is fair, you can assume that all sequences are equally likely. Determine the probability of the events below.
   (a) Three heads: HHH.
   (b) The sequence tail, head, tail: THT.
   (c) Any sequence in which the first flip and the last flip resulted in the same outcome.
   (d) Any sequence where the number of heads is greater than or equal to the number of tails.

4. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the sum of the results of each die. All outcomes that result in a particular sum are equally likely.
   (a) What is the probability of the sum being even?
   (b) What is the probability of Bob rolling a 4 and a 1?

5. Each of two people choose at random a number x and y, respectively, between zero and one. We take "at random" to mean "according to the uniform probability law" introduced in lecture. Consider the following events:

$$\begin{aligned} A &= \{\text{The magnitude of the difference of the two numbers is at most } 1/3\} \\ B &= \{\text{None of the numbers exceeds } 2/3\} \\ C &= \{\text{The two numbers are equal}\} \end{aligned}$$

   Find the following probabilities: $\mathbf{P}(A)$, $\mathbf{P}(B)$, $\mathbf{P}(A \cap B)$, $\mathbf{P}(C)$.

6. Mike and John are playing a friendly game of darts where the dart board is a disk with radius of 10in.

   Whenever a dart falls within 1in of the center, 50 points are scored. If the point of impact is between 1 and 3in from the center, 30 points are scored, if it is at a distance of 3 to 5in 20 points are scored and if it is further that 5in, 10 points are scored.

   Assume that both players are skilled enough to be able to throw the dart within the boundaries of the board.

   Mike can place the dart uniformly on the board (i.e., the probability of the dart falling in a given region is proportional to its area).

   (a) What is the probability that Mike scores 50 points on one throw?

(b) What is the probability of him scoring 30 points on one throw?

(c) John is right handed and is twice more likely to throw in the right half of the board than in the left half. Across each half, the dart falls uniformly in that region. Answer the previous questions for John's throw.

G1†. Consider an experiment whose sample space is the real line.

(a) Let $\{a_n\}$ be an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence that converges to b. Show that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

Here, the notation $[a, b]$ stands for the closed interval $\{x \mid a \leq x \leq b\}$. *Note:* This result seems intuitively obvious. The issue is to derive it using the axioms of probability theory.

(b) Let $\{a_n\}$ be a decreasing sequence that converges to a and $\{b_n\}$ an increasing sequence that converges to b. Is it true that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b])?$$

Note: You may use freely the results from the problems in the text in your proofs.

†Required for 6.431; optional for 6.041