Recitation 23 Solutions

1. (a) Here the prior PDF is

$$f_\Theta(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the conditional PDF of the observation is

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} 1/\theta, & \text{if } 0 \le x \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

Using Bayes' rule, and taking into account that $f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)$ is nonzero only if $0 \le x \le \theta \le 1$, we find that for any $x \in [0, 1]$, the posterior PDF is

$$f_{\Theta|X}(\theta \mid x) = \frac{f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)}{\displaystyle\int_0^1 f_\Theta(\theta') f_{X|\Theta}(x \mid \theta') d\theta'} = \frac{1/\theta}{\displaystyle\int_x^1 \frac{1}{\theta'} d\theta'} = \frac{1}{\theta \cdot |\log x|}, \quad \text{if } x \le \theta \le 1,$$

and $f_{\Theta|X}(\theta \mid x) = 0$ if $\theta < x$ or $\theta > 1$.

(b) Similar to the case where $n = 1$, we have

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} 1/\theta^n, & \text{if } \bar{x} \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\bar{x} = \max\{x_1, \ldots, x_n\}$.
The posterior PDF is

$$f_{\Theta|X}(\theta \mid x) = \begin{cases} \frac{c(\bar{x})}{\theta^n}, & \text{if } \bar{x} \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $c(\bar{x})$ is a normalizing constant that depends only on $\bar{x}$:

$$c(\bar{x}) = \frac{1}{\displaystyle\int_{\bar{x}}^1 \frac{1}{(\theta')^n} d\theta'}.$$

(c) In part a, we saw that for $x \in [0, 1]$, the posterior PDF is

$$f_{\Theta|X}(\theta \mid x) = \begin{cases} \frac{1}{\theta \cdot |\log\, x|}, & \text{if } x \le \theta \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a given x, $f_{\Theta|X}(\theta \mid x)$ is decreasing in θ, over the range $[x, 1]$ of possible values of Θ. Thus, the MAP estimate is equal to x. Note that this is an "optimistic" estimate. If Juliet is late by a small amount on the first date ($x \approx 0$), the estimate of future lateness is also small.

(d) The conditional expectation estimate turns out to be less "optimistic." In particular, we have

$$\mathbf{E}[\Theta \mid X = x] = \int_x^1 \theta \frac{1}{\theta \cdot |\log\, x|} d\theta = \frac{1 - x}{|\log\, x|}.$$

(e) Given $X = x$, the for any estimate $\hat{\theta}$, we have

$$\begin{aligned}
\mathbf{E}[(\hat{\theta} - \Theta)^2 \mid X = x] &= \int_x^1 (\hat{\theta} - \theta)^2 \cdot \frac{1}{|\log x|} d\theta \\
&= \int_x^1 (\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2) \cdot \frac{1}{\theta |\log x|} d\theta \\
&= \hat{\theta}^2 - \hat{\theta} \frac{2(1-x)}{|\log x|} + \frac{1-x^2}{2|\log x|}.
\end{aligned}$$

For the MAP estimate, $\hat{\theta} = x$, the conditional mean squared error is

$$\mathbf{E}[(\hat{\theta} - \Theta)^2 \mid X = x] = x^2 + \frac{3x^2 - 4x + 1}{2|\log x|}.$$

For the LMS estimate, $\hat{\theta} = (1-x)/|\log x|$, the conditional mean squared error is

$$\mathbf{E}[(\hat{\theta} - \Theta)^2 \mid X = x] = \frac{1-x^2}{2|\log x|} - \left(\frac{1-x}{\log x}\right)^2.$$

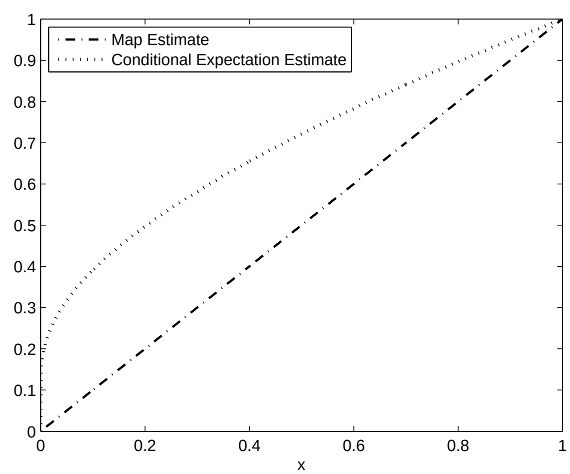

Figure 1: MAP and conditional expectation estimates as functions of the observation x.

2. (a) We will decide the alternative hypothesis is true if

$$\begin{aligned}
f_{X|\Theta}(x \mid 1) p_\Theta(1) &\geq f_{X|\Theta}(x \mid 0) p_\Theta(0) \\
2x \cdot p &\geq 1 \cdot (1-p) \\
x &\geq \boxed{\tfrac{1-p}{2p}, \quad \text{for } x \in [0,1]}.
\end{aligned}$$

If $p = 2/3$, the rule above corresponds to $x \geq 1/4$.
If $p = 1/2$, the rule above corresponds to $x \geq 1/2$.

If $p = 1/3$, the rule above corresponds to $x \geq 1$. In other words, we will always decide that the null hypothesis is true. Since the threshold is a monotonically decreasing function of p, this is true for any $p \leq 1/3$.

(b) If the null hypothesis is true, the error occurs when we decide the alternative hypothesis was true. For $p = 2/3$, this corresponds to the event $\{X \geq 1/4\}$. Therefore,

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}(\mathrm{X} \geq 1/4 \mid \Theta = 0) = \int_{1/4}^{1} \mathrm{f}_{\mathrm{X}|\Theta}(\mathrm{x} \mid 0)\, \mathrm{dx} = \int_{1/4}^{1} 1\, \mathrm{dx} = \boxed{\tfrac{3}{4}}.$$

(c) Similar to the computation above, we find for $p > 1/3$

$$\mathbf{P}(\text{error} \mid \Theta = 0) = \mathbf{P}\left(\mathrm{X} \geq \frac{1-\mathrm{p}}{2\mathrm{p}} \mid \Theta = 0\right) = \int_{\frac{1-\mathrm{p}}{2\mathrm{p}}}^{1} \mathrm{f}_{\mathrm{X}|\Theta}(\mathrm{x} \mid 0)\, \mathrm{dx} = \int_{\frac{1-\mathrm{p}}{2\mathrm{p}}}^{1} 1\, \mathrm{dx} = 1 - \frac{1-\mathrm{p}}{2\mathrm{p}} = \frac{3\mathrm{p}-1}{2\mathrm{p}}.$$

$$\mathbf{P}(\text{error} \mid \Theta = 1) = \mathbf{P}\left(\mathrm{X} < \frac{1-\mathrm{p}}{2\mathrm{p}} \mid \Theta = 1\right) = \int_{0}^{\frac{1-\mathrm{p}}{2\mathrm{p}}} \mathrm{f}_{\mathrm{X}|\Theta}(\mathrm{x} \mid 1)\, \mathrm{dx} = \int_{0}^{\frac{1-\mathrm{p}}{2\mathrm{p}}} 2\mathrm{x}\, \mathrm{dx} = \left(\frac{1-\mathrm{p}}{2\mathrm{p}}\right)^2.$$

Now using the total probability law, we find

$$\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid \Theta = 0)\mathrm{p}_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)\mathrm{p}_\Theta(1) \\
&= \frac{(3p-1)(1-p)}{2p} + \frac{(1-p)^2}{4p} = \boxed{\tfrac{(1-p)(5p-1)}{4p}}, \text{ for } p \geq 1/3.
\end{aligned}$$

For $p \leq 1/3$, we will always decide on the null hypothesis, and the resulting probability of error is

$$\mathbf{P}(\text{error}) = \mathbf{P}(\text{error} \mid \Theta = 0)\mathrm{p}_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1)\mathrm{p}_\Theta(1) = 0 \cdot (1-\mathrm{p}) + 1 \cdot \mathrm{p} = \boxed{p}.$$

For the boundary value of $p = 1/3$, both formulas yield $\mathbf{P}(\text{error}) = 1/3$.