Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem Set 5: Solutions

1. (a)

$$\begin{aligned} 1 &= \int_{x=1}^{2} \left(\int_{y=x}^{2} ay \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_{x=1}^{2} a \left(2 - \frac{x^2}{2} \right) \mathrm{d}x \\ &= a \left(2 - \frac{8}{6} + \frac{1}{6} \right) \\ &= \frac{5}{6} a. \end{aligned}$$

Therefore, $a = \frac{6}{5}$.

(b) For $1 \le x \le 2$,

$$\begin{aligned} f_X(x) &= \int_{x}^{2} \frac{6}{5} y \, \mathrm{d}y \\ &= \frac{6}{5} \left(2 - \frac{x^2}{2} \right). \end{aligned}$$

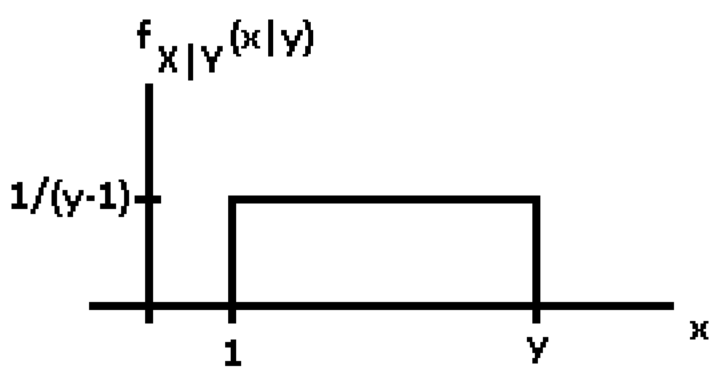
For all other values of x, $f_X(x) = 0$.

(c) For $1 \le y \le 2$,

$$\begin{aligned} f_Y(y) &= \int_{1}^{y} \frac{6}{5} y \, \mathrm{d}y \\ &= \frac{6}{5} \left(y^2 - y \right). \end{aligned}$$

For all other values of y, $f_Y(y) = 0$.

(d) For any given value of $Y = y$, $f_{X,Y}(x, y)$ is a constant function of x over the range $1 \le x \le y$ and zero otherwise. Hence, we have:



It follows that $\mathbf{E}[X|Y = y] = \frac{y+1}{2}$.

(e) Since X must be between 1 and 2, W must be between 1 and 4. We then know that the CDF of W satisfies

$$F_W(w) = 0, \quad w < 1,$$
$$F_W(w) = 1, \quad w > 4,$$

so that all we need to do is to find its value for $1 \leq w \leq 4$. In this range,

$$\begin{aligned} F_W(w) &= \mathbf{P}(W \leq w) \\ &= \mathbf{P}(X \leq \sqrt{w}) \\ &= \int_1^{\sqrt{w}} \frac{6}{5}\left(2 - \frac{x^2}{2}\right) \mathrm{d}x \\ &= \frac{6}{5}\left(2w^{\frac{1}{2}} - \frac{w^{\frac{3}{2}}}{6} - \frac{11}{6}\right). \end{aligned}$$

Taking derivatives, we have that

$$f_W(w) = \begin{cases} \frac{6}{5}\left(\frac{1}{\sqrt{w}} - \frac{1}{4}\sqrt{w}\right), & \text{if } 1 \leq w \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

2. Let X be a mixed random variable where the value of X is obtained according to the probability law of Y with probability p, and according to the probability law of Z with the complementary probability $1 - p$. The CDF of a mixed random variable is given, using the total probability theorem, by

$$\begin{aligned} F_X(x) &= \mathbf{P}(X \leq x) = p\mathbf{P}(Y \leq x) + (1-p)\mathbf{P}(Z \leq x) \\ &= pF_Y(x) + (1-p)F_Z(x). \end{aligned}$$

By differentiating, we obtain

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

Using the pdf, we can find the mean and the variance:

$$\begin{aligned} \mathbf{E}[X] &= p\int xf_Y(x)dx + (1-p)\int xf_Z(x)dx \\ &= p\mathbf{E}[Y] + (1-p)\mathbf{E}[Z]. \end{aligned}$$

It follows that the 2nd moment is

$$\mathbf{E}[X^2] = p\mathbf{E}[Y^2] + (1-p)\mathbf{E}[Z^2]$$

and so the variance is

$$\mathrm{var}(X) = p\mathbf{E}[Y^2] + (1-p)\mathbf{E}[Z^2] - (p\mathbf{E}[Y] + (1-p)\mathbf{E}[Z])^2$$

Using the information given in our problem, Let Y describe the lifetime of the energy saving bulb, and be an exponential r.v. with parameter $\lambda_y = 1/2100$; and let Z describe the lifetime of the incandescent bulb and be an exponential r.v. with parameter $\lambda_z = 1/700$. Aisha will choose an energy saving bulb with probability $p = 3/10$, and an incandescent bulb with probability $(1-p) = 7/10$. Let X be the distribution of the time until the randomly chosen bulb burns out.

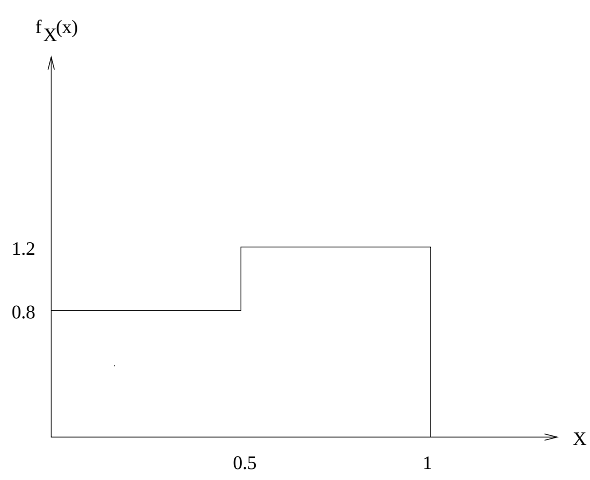
$$
\begin{aligned}
F_X(x) &= \begin{cases} 0, & x < 0 \\ \frac{3}{10}\left(1 - e^{-\frac{x}{2100}}\right) + \frac{7}{10}\left(1 - e^{-\frac{x}{700}}\right), & x \geq 0 \end{cases} \\
f_X(x) &= \begin{cases} \frac{1}{7000}e^{-\frac{x}{2100}} + \frac{1}{1000}e^{-\frac{x}{700}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
\mathbf{E}[X] &= \frac{3}{10} \cdot 2100 + \frac{7}{10} \cdot 700 = 1120 \text{ hours.} \\
\text{var}(X) &= \frac{3}{10} \cdot \frac{2}{(1/2100)^2} + \frac{7}{10} \cdot \frac{2}{(1/700)^2} - 1120^2 = 2077600.
\end{aligned}
$$

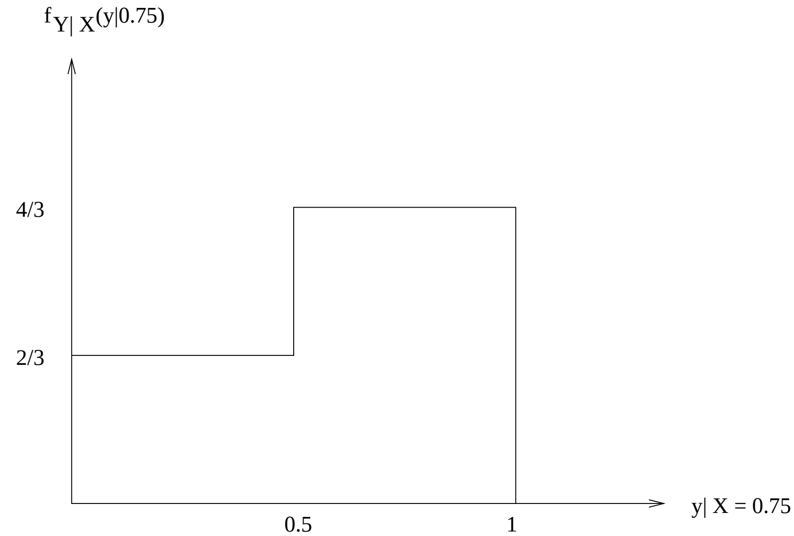
The probability the bulb lasts for longer than 1400 hours is

$$
\begin{aligned}
\mathbf{P}(X > 1400) &= 1 - \mathbf{P}(X \leq 1400) \\
&= 1 - F_X(1400) \\
&= 1 - \frac{3}{10}\left(1 - e^{-\frac{1400}{2100}}\right) - \frac{7}{10}\left(1 - e^{-\frac{1400}{700}}\right) \\
&= 0.2488.
\end{aligned}
$$

3. (a) X and Y are not independent because there exist x and y such that $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$. For instance, $f_{X,Y}(\frac{2}{3}, \frac{1}{3}) = 0.8$, $f_X(\frac{2}{3}) = \int_0^1 f_{X,Y}(\frac{2}{3}, y)dy = 1.2$, $f_Y(\frac{1}{3}) = \int_0^1 f_{X,Y}(x, \frac{1}{3})dx = 0.8$, but $f_{X,Y}(\frac{2}{3}, \frac{1}{3}) \neq f_X(\frac{2}{3})f_Y(\frac{1}{3})$.
We can see this intuitively in the graph: For example, if X is larger than 0.5, then y is more likely to be large.

(b) The plots are shown below.





$$f_X(x) = \begin{cases} 0.8, & 0 < x \leq 1/2 \\ 1.2, & 1/2 < x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad f_{Y|X}(y \mid 0.75) = \frac{f_{X,Y}(0.75,y)}{f_X(0.75)} = \begin{cases} 2/3, & 0 < y \leq 1/2 \\ 4/3, & 1/2 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Conditioned on event A, X and Y are independent. Thus

$$\mathbf{E}[R \mid A] = \mathbf{E}[XY \mid A] = \mathbf{E}[X \mid A]\mathbf{E}[Y \mid A] = (1/4)(1/2) = 1/8.$$

(d) It is easiest to see the CDF of W in this case as the integral of the PDF over an L-shaped area. For $0 < w \leq 1/2$ the CDF would be the integral over the PDF of the L-shaped area given by $((1)(w) + (w)(1-w))(0.8)$. Similarly, for $1/2 < w \leq 1$ the CDF would take on the values $(0.8)(3/4) + ((w-0.5)(0.5) + (1-w)(w-0.5))(1.6)$. Thus the entire CDF is given by

$$F_W(w) = \begin{cases} 0, & w \leq 0 \\ (2w - w^2)(0.8), & 0 < w \leq 1/2 \\ 1 - (1-w)^2(1.6), & 1/2 < w \leq 1 \\ 1, & w > 1 \end{cases}$$

4. (a) Let G represent the event that the weather is good. We are given $\mathbf{P}(G) = \frac{2}{3}$.
To find the PDF of X, we first find the PDF of W, since $X = s + W = 2 + W$. We know that given good weather, $W \sim N(0,1)$. We also know that given bad weather, $W \sim N(0,9)$. To find the unconditional PDF of W, we use the density version of the total probability theorem.

$$\begin{aligned} f_W(w) &= \mathbf{P}(G) \cdot f_{W|G}(w) + \mathbf{P}(G^c) \cdot f_{W|G^c}(w) \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{w^2}{2(9)}} \end{aligned}$$

We now perform a change of variables using $X = 2 + W$ to find the PDF of X:

$$f_X(x) = f_W(x-2) = \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-2)^2}{18}}.$$

(b) In principle, one can use the PDF determined in part (a) to compute the desired probability as

$$\int_1^3 f_X(x)\,dx.$$

It is much easier, however, to translate the event $\{1 \le X \le 3\}$ to a statement about W and then to apply the total probability theorem.

$$\mathbf{P}(1 \le X \le 3) = \mathbf{P}(1 \le 2 + W \le 3) = \mathbf{P}(-1 \le W \le 1)$$

We now use the total probability theorem.

$$\mathbf{P}(-1 \le W \le 1) \quad = \quad \mathbf{P}(G)\underbrace{\mathbf{P}(-1 \le W \le 1 \mid G)}_{a} + \mathbf{P}(G^c)\underbrace{\mathbf{P}(-1 \le W \le 1 \mid G^c)}_{b}$$

Since conditional on either G or G^c the random variable W is Gaussian, the conditional probabilities a and b can be expressed using Φ. Conditional on G, we have $W \sim N(0,1)$ so

$$a = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1.$$

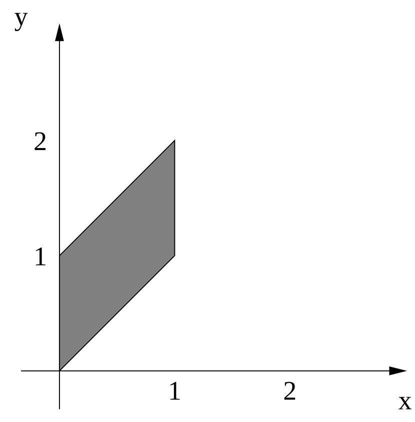
Conditional on G^c, we have $W \sim N(0,9)$ so

$$b = \Phi\left(\frac{1}{3}\right) - \Phi\left(-\frac{1}{3}\right) = 2\Phi\left(\frac{1}{3}\right) - 1.$$

The final answer is thus

$$\mathbf{P}(1 \le X \le 3) = \frac{2}{3}\left(2\Phi\left(1\right) - 1\right) + \frac{1}{3}\left(2\Phi\left(\frac{1}{3}\right) - 1\right).$$

5. (a) The shaded region represents nonzero probability:



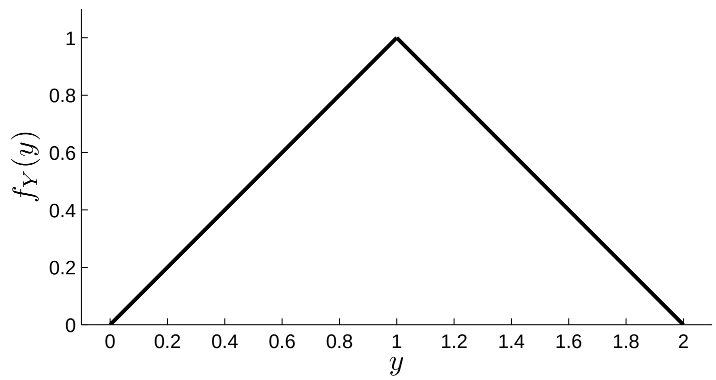
(b) Applying the definition of a marginal PDF,

for $0 \le y \le 1$,

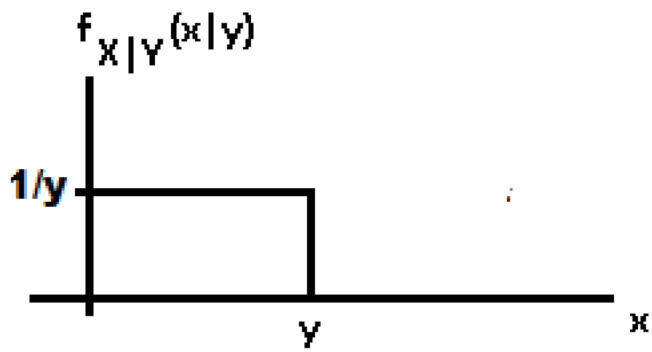
$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\;dx \\ &= \int_0^y 1\;dx \\ &= y; \end{aligned}$$

and for $1 \le y \le 2$,

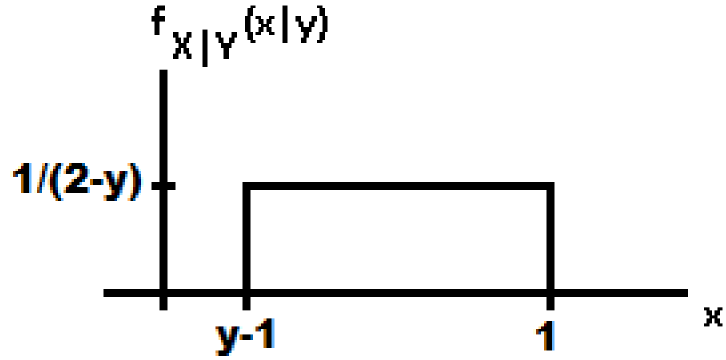
$$\begin{aligned} f_Y(y) &= \int_x f_{X,Y}(x,y)\ dx \\ &= \int_{y-1}^{1} 1\ dx \\ &= 2-y. \end{aligned}$$



(c) For $0 \le y \le 1$, we have



from which we conclude $\mathbf{E}[X|Y=y] = \frac{y}{2}$. Similarly, for $1 \le y \le 2$, we have



from which we conclude $\mathbf{E}[X|Y=y] = \frac{y}{2}$.

(d) By linearity of expectation, the expected value of a sum is the sum of the expected values. By inspection, $\mathbf{E}[X] = 1/2$ and $\mathbf{E}[Y] = 1$.
Thus, $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] = 3/2$.

6. (a) Let A be the event that the first coin toss resulted in heads. To calculate the probability $\mathbf{P}(A)$, we use the continuous version of the total probability theorem:

$$\mathbf{P}(A) = \int_0^1 \mathbf{P}(A \mid P = p) f_P(p)\ dp = \int_0^1 2p(1-p)\ dp = \frac{1}{3}$$

(b) Using Bayes rule,

$$\begin{aligned} f_{P|A}(p) &= \frac{\mathbf{P}(A \mid P = p) f_P(p)}{\mathbf{P}(A)} \\ &= \begin{cases} 6p(1-p), & \text{if } 0 \le p \le 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Let B be the event that the second toss resulted in heads. We have

$$\begin{aligned} \mathbf{P}(B \mid A) &= \int_0^1 \mathbf{P}(B \mid P = p, A) f_{P|A}(p)\ dp \\ &= \int_0^1 \mathbf{P}(B \mid P = p) f_{P|A}(p)\ dp \\ &= \int_0^1 6p^2(1-p)\ dp \\ &= \frac{1}{2} \end{aligned}$$

G1†. a) $X_k = \begin{cases} 2 & \textit{with probability } 1/2 \\ 1/4 & \textit{with probability } 1/2 \end{cases}$

b) $E[W_n] = E[X_1 \cdot X_2 \cdots X_k \cdots X_n] = E\left[\prod_{k=1}^n X_k\right]$

†Required for 6.431; optional for 6.041

Since $X_1, X_2, ..., X_n$ are independent random variables, $E\left[\prod_{k=1}^{n} X_k\right] = \prod_{k=1}^{n} E[X_k]$.

So $E[W_n] = E[X_1] \cdot E[X_2] \cdots E[X_n] = (E[X_1])^n = (2 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2})^n = (9/8)^n$

If n = 3, E$[W_n]$ = 1.424

If n= 6, E$[W_n]$= 2.027

If n =12, E$[W_n]$= 4.110

As n increases, your expected wealth increases exponentially.

c) $P[W_n \geq 1] = P[X_1 \cdot X_2 \cdots X_n \geq 1] = P[2^i \cdot (\frac{1}{4})^{n-i} \geq 1]$ where i is the number of heads obtained in n tosses, and n-i is the number of tails.

So for the wealth at n to be greater than 1, we need the number of heads obtained in n tosses to be at least double the number of tails obtained. This is equivalent to saying that at least two thirds of our tosses result in heads.

$P[W_n \geq 1] = P\left[at\ least\ \lceil \frac{2n}{3} \rceil\ heads\ in\ n\ tosses\right]$ The number of heads obtained in n tosses has a binomial distribution with parameter p=1/2.

Therefore, $P[W_n \geq 1] = P\left[at\ least\ \lceil \frac{2n}{3} \rceil\ heads\ in\ n\ tosses\right] = \sum_{k=\lceil \frac{2n}{3} \rceil}^{n} \binom{n}{k} \cdot (0.5)^k \cdot (0.5)^{n-k} = (0.5)^n \sum_{k=\lceil \frac{2n}{3} \rceil}^{n} \binom{n}{k}$

For n=3, this probability is 1/2. For n=6, it is 11/32, and for n=12 it is 397/2048.

As n increases, the probability of winning (i.e., the probability that your wealth is greater than or equal to your starting wealth) decreases.

d) At time n, an outcome is the sequence of heads and tails that we have so far. As n increases, the set of sequences that lead to Wn greater than or equal to 1 has decreasing probability. However, the gain associated with some of the sequences in the set increases exponentially. For example, the wealth after n heads is 2^n. The gain associated with this sequence is $2^n - 1$. This compensates for the decreasing probability of the sequence while calculating the expected value. Note that the set of sequences leading to $W_n < 1$ has increasing probability with n.

e) Let us first calculate the variance of the wealth at n. $Var[W_n] = E[W_n^2] - E[W_n]^2 = E[W_n^2] - \left(\frac{9}{8}\right)^{2n}$

To get the expected value of the wealth square, define a random variable

$$Y_k = X_k^2 = \begin{cases} 4 & with\ probability\ 1/2 \\ 1/16 & with\ probability\ 1/2 \end{cases}$$

$Y_1, Y_2, ...Y_n$ are independent identically distributed.

$E[Y_n] = 4 \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{2} = \frac{65}{32}$

$W_n^2 = (X_1 X_2 ... X_n)^2 = (X_1 X_2 \cdots X_n)(X_1 X_2 \cdots X_n) = Y_1 \cdot Y_2 \cdots Y_n$

So $E[W_n^2] = E[Y_1] \cdot E[Y_2] \cdots E[Y_n] = E[Y_1]^n = \left(\frac{65}{32}\right)^n$

So $Var[W_n] = \left(\frac{65}{32}\right)^n - \left(\frac{9}{8}\right)^{2n}$

The standard deviation of the wealth $= \sqrt{Var[W_n]} = \sqrt{\left(\frac{65}{32}\right)^n - \left(\frac{9}{8}\right)^{2n}} = \sqrt{\left(\frac{130}{64}\right)^n - \left(\frac{81}{64}\right)^n} \approx \left(\frac{130}{64}\right)^{n/2} = \left(\frac{65}{32}\right)^{n/2}$ for large n.