Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Problem Set 4

Due October 5, 2011

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x, y) = \begin{cases} c(x+y)^2, & \text{if } x \in \{1, 2, 4\} \text{ and } y \in \{1, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?

   (b) What is $\mathbf{P}(Y < X)$?

   (c) What is $\mathbf{P}(Y > X)$?

   (d) What is $\mathbf{P}(Y = X)$?

   (e) What is $\mathbf{P}(Y = 3)$?

   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

   (g) Find the expectations $\mathbf{E}[X]$, $\mathbf{E}[Y]$ and $\mathbf{E}[XY]$.

   (h) Find the variances $\text{var}(X)$, $\text{var}(Y)$ and $\text{var}(X + Y)$.

   (i) Let A denote the event $X \geq Y$. Find $\mathbf{E}[X \mid A]$ and $\text{var}(X \mid A)$.

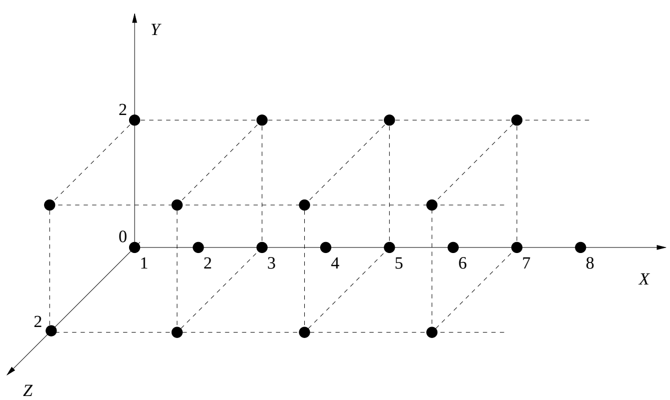
2. A simple example of a random variable is the *indicator* of an event A, which is denoted by I_A:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A and I_B are independent.

   (b) Show that if $X = I_A$, then $\mathbf{E}[X] = \mathbf{P}(A)$.

3. Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter p. If X is even, then Y and Z are equal to zero. If X is odd, (Y, Z) is uniformly distributed on the set $S = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$. The figure below shows all the possible values for the triple (X, Y, Z) that have $X \leq 8$. (Note that the X axis starts at 1 and that a complete figure would extend indefinitely to the right.)



   (a) Find the joint PMF $p_{X,Y,Z}(x, y, z)$.

(b) Answer with "yes" or "no" and one sentence of explanation:
   (i) Are Y and Z independent?
   (ii) Given that $Z = 2$, are X and Y independent?
   (iii) Given that $Z = 0$, are X and Y independent?
   (iv) Given that $Z = 2$, are X and Z independent?

(c) Give a formula that expresses $\mathbf{E}[X \mid Y = 2]$ as a function of p. (If the formula is in the form of a series, you do not need to evaluate the series.)

(d) Find $\text{var}(Y + Z \mid X = 5)$.

4. Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y the number of weeks that he won.

   (a) What is the probability that he played the lottery any particular week, given that he did not win anything that week?
   (b) Find the conditional PMF $p_{Y|X}(y \mid x)$.
   (c) Find the joint PMF $p_{X,Y}(x, y)$.
   (d) Find the marginal PMF $p_Y(y)$. *Hint:* One possibility is to start with the answer to part (c), but the algebra can be messy. But if you think intuitively about the procedure that generates Y, you may be able to guess the answer.
   (e) Find the conditional PMF $p_{X|Y}(x \mid y)$. Do this algebraically using previous answers.
   (f) Re-derive the answer to part (e) by thinking as follows: For each one of the $n - Y$ weeks that he did not win, the answer to part (a) should tell you something. Note that the answer you derive in this part may look different from your earlier answer. There is no need to show they are algebraically equivalent.

   In all parts of this problem, make sure to indicate the range of values for which your PMF formula applies.

5. (a) Let X be a random variable that takes nonnegative integer values. Show that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k).$$

   Hint: Express the right-hand side of the above formula as a double summation then interchange the order of the summations.

   (b) Use the formula in the previous part to find the expectation of a random variable Y whose PMF is defined as follows:

$$p_Y(y) = \frac{1}{b - a + 1}, \qquad y = a, a + 1, \ldots, b$$

   where a and b are nonnegative integers with $b > a$. Note that for $y = a, a + 1, \ldots, b$, $p_Y(y)$ does not depend explicitly on y since it is a uniform PMF.

G1†. Probabilities can have properties that are surprising and counter to one's basic intuition. As one application, probability can be helpful in picking the position in a game from which you are most likely to win. But the **nontransitive dice** below show an unexpected difficulty that can arise. Consider four dice, labeled A,B, C, and D. These are normal dice, in that each has six sides and the probability any given die lands on any particular side is 1/6 (regardless of the outcome of other tosses for that die or any of the tosses for all the other dice). They are unusual only in that the numbers on their six sides are different from standard dice. The numbers on the four dice are:

$$A : 0, 0, 4, 4, 4, 4 \qquad B : 3, 3, 3, 3, 3, 3 \qquad C : 2, 2, 2, 2, 6, 6 \qquad D : 1, 1, 1, 5, 5, 5$$

In a single game, I choose a die and toss it, and you choose a different die and toss it. The highest number wins. (For these dice, a tie is not possible.)

(a) Find the probabilities of the following events:

i. Die A beats die B
ii. Die B beats die C
iii. Die C beats die D
iv. Die D beats die A

(b) I pick my die first, and then you get to choose any one of the remaining three. Given my choice, how would you choose your die so that you will beat me in the toss with probability 2/3, *regardless of which die I chose*?

G2†. Expectations can also have properties that are surprising and counter to one's basic intuition. In many cases, the expectation of a random variable gives a "typical" value, and one normally expects the actual sample values observed to lie near the expectation. (In a few weeks we will study theorems in Chapter 5 of the text that guarantee and quantify this phenomenon.) But consider the simple **double-or-nothing game**, based on a sequence of independent fair coin tosses. At each toss that comes up heads you receive back twice your bet, and at each toss that comes up tails you lose your entire bet. Suppose you enter this game with a total wealth of $1.00 and bet your total wealth (if any remains) on each successive toss. Let the random variable W_n be your wealth after n tosses.

(a) Find the possible values of W_n, and find the probability of each value (i.e., find the pmf of W_n).

(b) Find your expected wealth after n tosses, $\mathbf{E}[W_n]$.

(c) Find the probability that $W_n = 0$ as $n \to \infty$. This is the probability that you will eventually lose all of your money. (Note that $\mathbf{E}[W_n] \neq 0$.)

(d) Give a simple explanation of the behavior in this case.

(e) Find the standard deviation of W_n.

†Required for 6.431; optional for 6.041