Problem Set 8
Due November 18, 2009

1. The first-order interarrival times for cars passing a checkpoint are independent random variables with PDF

$$f_T(t) = \begin{cases} 2e^{-2t}, & \text{for } t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

where the interarrival times are measured in minutes. The successive experimental values of the durations of these first-order interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

   (a) Determine the mean and the third moment of the first-order interarrival times.

   (b) Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.

   (c) Determine the PDF and the expected value for the total time required to use up the first dozen computer cards.

   (d) Consider the following two experiments:

      i. Pick a card at random from a group of completed cards and note the total time, Y, the card was in service. Find $\mathbf{E}[Y]$ and $\text{var}(Y)$.

      ii. Come to the corner at a random time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time W. Determine $\mathbf{E}[W]$ and $\text{var}(W)$.

2. Let X, Y, and Z be independent exponential random variables with parameters λ, μ, and ν, respectively. Find $\mathbf{P}(X < Y < Z)$.

3. For a series of dependent trials, the probability of success on any given trial is given by $(k+1)/(k+3)$, where k is the number of successes in the previous three trials. Define a state description and a set of transition probabilities which allow this process to be described as a Markov chain. Draw the state transition diagram. Try to use the smallest possible number of states.

4. For each of the following definitions of state X_k at time k ($k = 1, 2, \ldots$), determine whether the Markov property is satisfied and, when it is, specify the transition probabilities p_{ij}:

   (a) A six-sided die is rolled repeatedly.

      i. Let X_k denote the largest number rolled in the first k rolls.

      ii. Let X_k denote the number of sixes in the first k rolls.

      iii. At time k, let X_k be the number of rolls since the most recent six.

   (b) Let Y_k be the state of some discrete-time Markov process at time k (i.e., it is known Y_k satisfies the Markov property) with known transition probabilities q_{ij}.

      i. For a fixed integer $r > 0$, let $X_k = Y_{r+k}$.

      ii. Let $X_k = Y_{2k}$.

iii. Let $X_k = (Y_k, Y_{k+1})$; that is, the state X_k is defined by the sequence of state *pairs* in a given Markov process.

5. (a) Identify the transient, recurrent, and periodic states of the Markov chain described by

$$[p_{ij}] = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0.3 & 0.4 & 0 & 0 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0.3 & 0.4 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

(b) How many classes are formed by the recurrent states of this process?

6. Consider the following model for spread of a disease amongst a population of n individuals, who at any time may be susceptible, infected, or recovered:

- Between time t and $t+1$, each pair of individuals (i, j) come into contact with each other independently with probability p.
- Any susceptible individual at time t who comes into contact between times t and $t+1$ with a total of exactly one infected individual becomes recovered at time $t+1$ (as if he/she was exposed to low enough levels of the disease so as to produce immunity).
- Any susceptible individual at time t who comes into contact between times t and $t+1$ with more than one infected individual becomes infected at time $t+1$.
- Any infected individual at time t independently becomes recovered with probability q.
- Otherwise, individuals maintain their state (susceptible, infected, recovered).

(a) Given that the population consists of exactly k infected individuals and s susceptible individuals at time t, determine the PMF of the number of newly infected individuals at time $t+1$.

(b) What is the minimum number of states necessary to represent this Markov chain?

(c) What are the recurrent states?

(d) For $n = 3$, draw the Markov chain and label all transition probabilities.

†Required for 6.431; optional challenge problem for 6.041