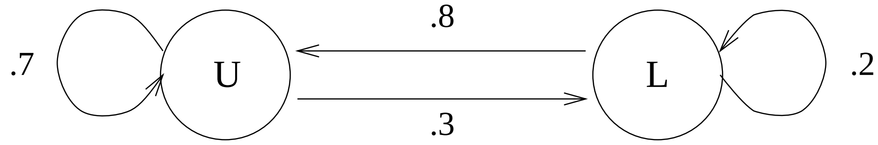
Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2010)

Problem Set 9: Solutions
Due: April 28, 2010

1. The state-transition diagram is the following:



(a) We are interested in finding the steady-state probabilites of the states in this Markov chain. Since this is a birth-death process, we use the local balance equations based on the frequency of transitions between two successive states and the normalization equation to solve for π_U and π_L.

$$\pi_L = \frac{\pi_U \cdot 3/10}{8/10} = \frac{3}{8}\pi_U$$

$$1 = \pi_L + \pi_U.$$

Solving this system of equations, we get,

$$\pi_U = \frac{8}{11} \qquad \pi_L = \frac{3}{11}.$$

Thus,

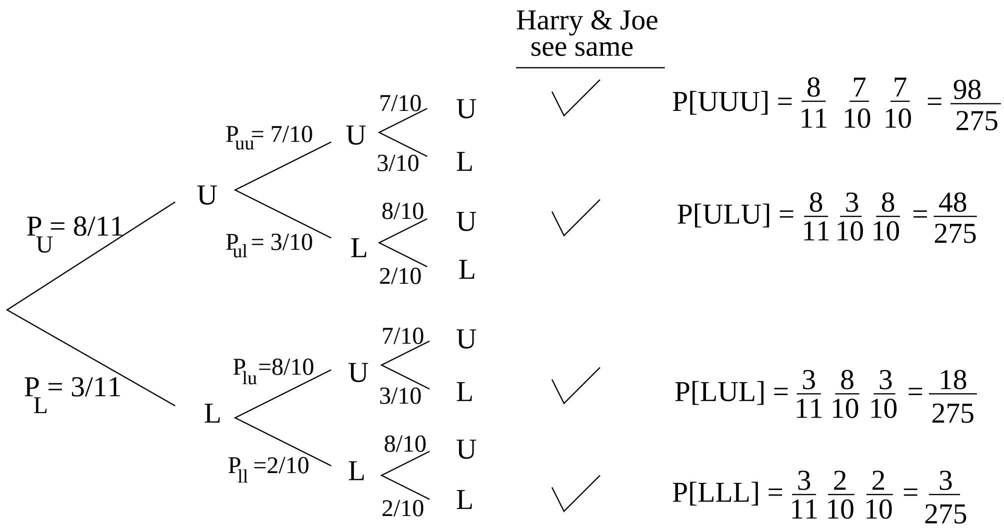
$$\mathbf{P}(\text{he unlocks the door}) = \pi_L \cdot p_{LU} = \frac{3}{11} \cdot \frac{8}{10} = \frac{12}{55}$$

and

$$\mathbf{P}(\text{he locks the door}) = \pi_U \cdot p_{UL} = \frac{8}{11} \cdot \frac{3}{10} = \frac{12}{55}.$$

So, the two events are equally likely.

(b) We can draw a tree of the possible outcomes of Mean Variance's two visits between Joe's arrival and Harry's.



$$\mathbf{P}(\text{both Joe and Harry see the same condition}) = \frac{98}{275} + \frac{48}{275} + \frac{18}{275} + \frac{3}{275} = \boxed{\frac{167}{275}}.$$

(c) Define

X = number of visits from hiring to locking
Y = number of visits from locking to unlocking.
W = number of visits from hiring to unlocking (this is the random variable of interest)

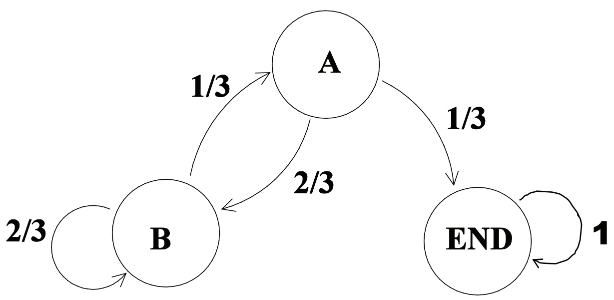
Note that $W = X + Y$. X is a geometric random variable with success probability equal to 0.3 and Y is a geometric random variable with success probability equal to 0.8:

$$\begin{aligned} p_X(x) &= \frac{3}{10}\left(\frac{7}{10}\right)^{x-1}, x = 1, 2, 3, \ldots \\ p_Y(y) &= \frac{8}{10}\left(\frac{2}{10}\right)^{y-1}, y = 1, 2, 3, \ldots \end{aligned}$$

Using the linearity property of expectation and the expected value of a geometric random variable we obtain,

$$\begin{aligned} E[W] &= E[X] + E[Y] \\ &= \frac{10}{3} + \frac{10}{8} \\ &\approx 4.583 \end{aligned}$$

2. We set up the Markov chain shown below.



States A and B indicate the type of the most recent event, except if a second A in a row occurs, in which case we move to state END. The problem is to find the expected time until we enter state END, starting from state B. Note that the times between transitions are i.i.d. exponential, with mean 1/3. This comes from the fact that the arrivals of type A and B can be seen as arrivals of a single merged Poisson process with rate 3. Also, notice that even though the last two arrivals before absorption into END are going to be A, the expected length of each of the last two interarrival times is still going to be $\frac{1}{3}$, and not 1 as one might mistakenly assume. In general, in a merged Poisson process, given that the next arrival is going to be of a particular type (say type A), the expected time until the next arrival is still just the expected interarrival time of the merged process(not the expected interarrival time of arrivals of type A). Thus, the desired expected time is 1/3 times the expected number of transitions. (We are using here the formula for the expectation of a sum of a random number of i.i.d. random variables.) Let t_i be the expected number of transitions until the end starting from state i. We have

$$t_B = 1 + \frac{2}{3}t_B + \frac{1}{3}t_A,$$

$$t_A = 1 + \frac{2}{3}t_B.$$

We solve and find $t_B = 12$, $t_A = 9$. Thus, the desired expected time is $12/3 = 4$.

3. (a) The process is in state 3 immediately before the first transition. After leaving state 3 for the first time, the process cannot go back to state 3 again. Hence J, which represents the number of transitions up to and including the transition on which the process leaves state 3 for the last time is a geometric random variable with success probability equal to 0.6. The variance for J is given by:

$$\sigma_J^2 = \frac{1-p}{p^2} = \frac{10}{9}$$

(b) There is a positive probability that we never enter state 4; i.e., $P(K < \infty) < 1$. Hence the expected value of K is ∞.

(c) The Markov chain has 3 different recurrent classes. The first recurrent class consists of states $\{1, 2\}$, the second recurrent class consists of state $\{7\}$ and the third recurrent class consists of states $\{4, 5, 6\}$. The probability of getting absorbed into the first recurrent class starting from the transient state 3 is,

$$\frac{1/10}{1/10 + 2/10 + 3/10} = \frac{1}{6}$$

which is the probability of transition to the first recurrent class given there is a change of state. Similarly, probability of absorption into second and third recurrent classes are $\frac{3}{6}$ and $\frac{2}{6}$ respectively.

Now, we solve the balance equations within each recurrent class, which give us the probabilities conditioned on getting absorbed from state 3 to that recurrent class. The unconditional steady-state probabilities are found by weighing the conditional steady-state probabilities by the probability of absorption to the recurrent classes.

The first recurrent class is a birth-death process. We write the following equations and solve for the conditional probabilities, denoted by p_1 and p_2.

$$p_1 = \frac{p_2}{2}$$

$$p_1 + p_2 = 1$$

Solving these equations, we get $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$. For the second recurrent class, $p_7 = 1$. The third recurrent class is also a birth-death process, we can find the conditional steady-state probabilities as follows,

$$p_4 = 2p_5$$

$$p_5 = 2p_6$$

$$p_4 + p_5 + p_6 = 1$$

and thus, $p_4 = \frac{4}{7},\ p_5 = \frac{2}{7},\ p_6 = \frac{1}{7}$.

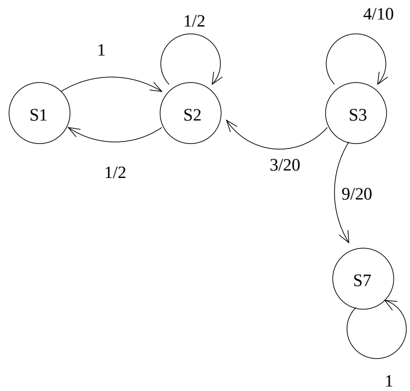
Using these data, the unconditional steady-state probabilities for all the states are found as follows:

$$\pi_1 = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$$
$$\pi_2 = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$$
$$\pi_3 = 0 \text{ (transient state)}$$
$$\pi_7 = 1 \cdot \frac{3}{6} = \frac{1}{2}$$
$$\pi_4 = \frac{4}{7} \cdot \frac{2}{6} = \frac{4}{21}$$
$$\pi_5 = \frac{2}{7} \cdot \frac{2}{6} = \frac{2}{21}$$
$$\pi_6 = \frac{1}{7} \cdot \frac{2}{6} = \frac{1}{21}$$

(d) The given conditional event, that the process never enters state 4, changes the absorption probabilities to the recurrent classes. The probability of getting absorbed to the first recurrent class is $\frac{1}{4}$, to the second recurrent class is $\frac{3}{4}$, and to the third recurrent class is 0. Hence, the steady state probabilities are given by,

$$\pi_1 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$
$$\pi_2 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$$
$$\pi_3 = \pi_4 = \pi_5 = \pi_6 = 0$$
$$\pi_7 = 1 \cdot \frac{3}{4} = \frac{3}{4}$$

For pedagogical purposes, let us actually draw what the new Markov chain would look like, given the event that the process never enters state 4. The resulting chain is shown below. Let us see how we came up with these transition probabilities.



We need to be careful when rescaling the new transition probabilities. First of all, it is clear that the probabilities within the recurrent classes $\{S1, S2\}$ and $\{S7\}$ don't get affected. We also note that the self loop transition probability of the transient state $S3$ doesn't get changed either.(this would be true for any other transient state)
To see that the self loop probability $p_{3,3}$ doesn't get changed, we condition on the event that we eventually enter S2 or S7. Let's call the new self loop probability, $q_{3,3}$.
Then,

$$q_{3,3} = P(X_1 = S3 |\ absorbed\ into\ 2\ or\ 7, X_0 = S3) = \frac{p_{3,3} * P(absorbed\ into\ 2\ or\ 7\ |X_1{=}S3,\ X_0{=}S3)}{P(absorbed\ into\ 2\ or\ 7\ |\ X_0{=}S3)}$$

$$= \frac{p_{3,3} * (a_{3,2} + a_{3,7})}{(a_{3,2} + a_{3,7})} = p_{3,3} = \frac{4}{10}$$

Now, we calculate $q_{3,7}$ and $q_{3,2}$.

$$q_{3,7} = P(X_1 = S7 |\ absorbed\ into\ 2\ or\ 7, X_0 = S3) = \frac{p_{3,7} * P(absorbed\ into\ 2\ or\ 7\ |X_1{=}S7,\ X_0{=}S3)}{P(absorbed\ into\ 2\ or\ 7\ |\ X_0{=}S3)}$$

$$= \frac{p_{3,7} * 1}{(a_{3,2} + a_{3,7})} = \frac{\frac{3}{10}}{\frac{1}{6} + \frac{1}{2}} = \frac{9}{20}$$

$$q_{3,2} = P(X_1 = S2 |\ absorbed\ into\ 2\ or\ 7, X_0 = S3) = \frac{p_{3,2} * P(absorbed\ into\ 2\ or\ 7\ |X_1{=}S2,\ X_0{=}S3)}{P(absorbed\ into\ 2\ or\ 7\ |\ X_0{=}S3)}$$

$$= \frac{p_{3,2} * 1}{(a_{3,2} + a_{3,7})} = \frac{\frac{1}{10}}{\frac{1}{6} + \frac{1}{2}} = \frac{3}{20}$$

Now, we can calculate the absorption probabilities of this new Markov chain.

The probability of getting absorbed into the recurrent class $\{1, 2\}$, starting from $S3$, is $\frac{\frac{3}{20}}{\frac{3}{20} + \frac{9}{20}} = \frac{1}{4}$. The probability of getting absorbed into the recurrent class $\{7\}$, starting from $S3$, is $\frac{\frac{9}{20}}{\frac{3}{20} + \frac{9}{20}} = \frac{3}{4}$. Thus, our calculated absorption probabilities match the probabilities we intuited earlier. The important thing to take away from this example is that, when doing problems of this sort, (i.e given we do/don't enter a particular set of recurrent classes), it is neccessary to rescale the transition probabilities of the new chain, coming out of ALL the transient states. In other words, to find each of the new transition probabilities, we condition on the given event, that we do or do not enter particular recurrent classes.

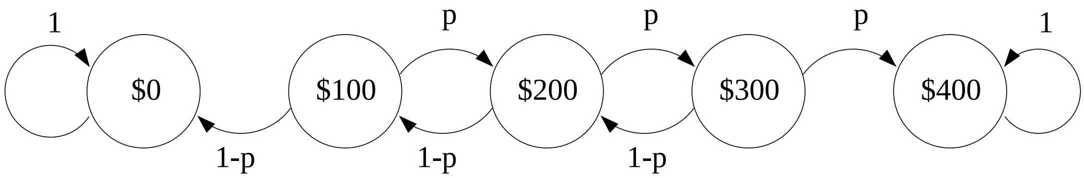
4. a) We calculate the simple cases first. The easiest decision is when Mary has \$100. She must bet \$100 because she can't bet \$200.

   When Mary has \$300, she should bet \$100. Whether she bets \$100 or \$200, she will meet her goal if she wins the next game. If she bets \$100 and loses, she will then have \$200. If she bets \$200 and loses, she will then have \$100. Everything else being equal, it's more advantageous to have \$200 than to have \$100.

   The more difficult decision is how much to bet when Mary has \$200. We'll investigate both possible strategies and decide which is preferable.

First, Mary can bet \$200. This case is easy to analyze. She will either win the desired amount or go "bust" on the next game. The probability that she will win is p.

Second, Mary can bet \$100. In this case, we have the following state transition diagram.



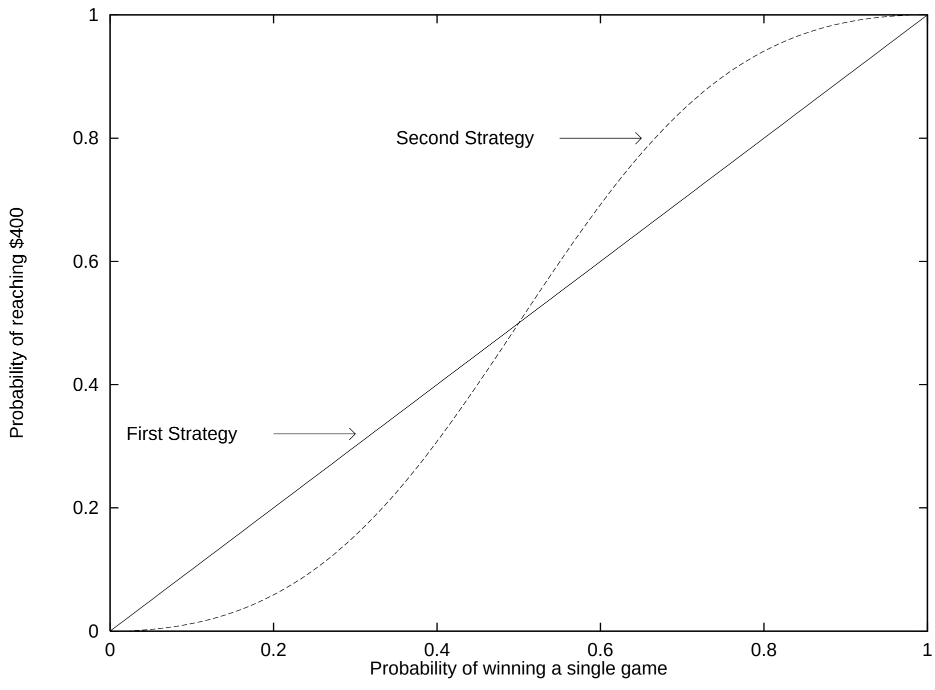
Winning and going "bust" are the two absorption states, and we want to find the probability of eventually winning, given that she starts with \$200. We will denote the probability that Mary wins given that she starts with j hundred dollars by a_j.

$$\begin{aligned} a_1 &= a_2 p \\ a_2 &= a_1(1-p) + a_3 p \\ a_3 &= a_2(1-p) + p \end{aligned}$$

A few simple substitutions yield the following.

$$a_2 = \frac{p^2}{1-2p+2p^2}$$

We need to compare p and a_2 to determine which strategy is optimal. Solving for the condition such that $p > a_2 = \frac{p^2}{1-2p+2p^2}$ yields that betting \$200 is advantageous when $p < 1/2$ and betting \$100 is advantageous when $p > 1/2$. When $p = 1/2$ neither strategy is better than the other. The figure below shows the graph of p against the absorption probability to state 400 for each strategy.



b) With $p = .75$, the optimal strategy is for Mary to bet \$100 when she has \$200. We need to find the expected time until absorption, given that Mary started with \$200.

Let μ_i = E(number of transitions to absorption starting from i(\$100)). We know that $\mu_0 = 0$ and $\mu_4 = 0$ because these are absorption states. We have the following relationship to determine the other μ_i's.

$$\mu_i = 1 + \sum_{j=1}^{3} p_{ij}\mu_j$$

So, we get the following three equations.

$$\begin{aligned}
\mu_1 &= 1 + p\mu_2 \\
\mu_2 &= 1 + (1-p)\mu_1 + p\mu_3 \\
\mu_3 &= 1 + (1-p)\mu_2
\end{aligned}$$

We need to solve for μ_2. Inserting $p = .75$, we get the following values for μ_i.

$$\begin{aligned}
\mu_1 &= 3.4 \\
\mu_2 &= 3.2 \\
\mu_3 &= 1.8
\end{aligned}$$

Therefore, the answer is $\boxed{3.2}$.