MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2011)

Tutorial 3 Solutions

September 29/30, 2011

1. *Consider a random variable X such that*

$$p_X(x) = \frac{x^2}{a} \text{ for } x \in \{-3,-2,-1,1,2,3\}, \quad \mathbf{P}(X = x) = 0 \text{ for } x \notin \{-3,-2,-1,1,2,3\},$$

where $a > 0$ is a real parameter.

(a) *Find a.*

Solution. The sum of the values of the PMF of a random variable over all values that it takes with positive probability must be equal to 1. Hence, we have

$$\begin{aligned} 1 &= \sum_{x=-3}^{3} p_X(x) \\ &= \frac{9}{a} + \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} \\ &= \frac{28}{a}, \end{aligned}$$

which implies that $a = 28$.

(b) *What is the PMF of the random variable $Z = X^2$?*

Solution. The following table shows the value of Z for a given value of X and the probability of that event.

x	-3	-2	-1	1	2	3
$p_X(x)$	9/28	1/7	1/28	1/28	1/7	9/28
$Z \mid X = x$	9	4	1	1	4	9

We see that Z can take only three possible values with non-zero probability, namely 1,4, and 9. In addition, for each value, there correspond two values of X. So we have, for example, $p_Z(9) = \mathbf{P}(Z = 9) = \mathbf{P}(X = -3) + \mathbf{P}(X = 3) = p_X(-3) + p_X(3)$. Hence the PMF of Z is given by

$$p_Z(z) = \begin{cases} 1/14 & \text{if } z = 1, \\ 2/7 & \text{if } z = 4, \\ 9/14 & \text{if } z = 9. \end{cases}$$

2. Check online solutions for problem 2.40.

3. Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the ith roll.

(a) What is the probability that exactly three of the rolls have result equal to 3? Each roll X_i can either be a 3 with probability 1/4 or not a 3 with probability 3/4. There are $\binom{6}{3}$ ways of placing the 3's in the sequence of six rolls. After we require that a 3 go in each of these spots, which has probability $(1/4)^3$, our only remaining condition is that either a 1 or a 2 go in the other three spots, which has probability $(3/4)^3$. So the probability of exactly three rolls of 3 in a sequence of six independent rolls is $\boxed{\binom{6}{3}(\frac{1}{4})^3(\frac{3}{4})^3}$.

(b) What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1? The probability of obtaining a 1 on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent result. We know that there are $\binom{6}{2}$ ways of rolling exactly two 1's. Of these $\binom{6}{2}$ ways, exactly $\binom{5}{1} = 5$ ways result in a 1 in the first roll, since we can place the remaining 1 in any of the five remaining rolls. The rest of the rolls must be either 2 or 3. Thus, the probability that the first roll is a 1 given exactly two rolls had an outcome of 1 is $\boxed{\frac{5}{\binom{6}{2}}}$.

(c) We are now told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. What is the probability of the sequence 121212? We want to find

$$\mathbf{P}(121212 \,|\, \text{exactly three 1's and three 2's}) = \frac{\mathbf{P}(121212)}{\mathbf{P}(\text{exactly 3 ones and 3 twos})}.$$

Any particular sequence of three 1's and three 2's will have the same probability: $(1/2)^3(1/4)^3$. There are $\binom{6}{3}$ possible rolls with exactly three 1's and three 2's. Therefore,

$$\mathbf{P}(121212 \,|\, \text{exactly three 1's and three 2's}) = \boxed{\frac{1}{\binom{6}{3}}}.$$

(d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a 3. Then

$$\mathbf{P}(A) = 1 - \mathbf{P}(\text{no rolls resulted in 3}) = 1 - \left(\frac{3}{4}\right)^6.$$

Now let K be the random variable representing the number of 3's in the 6 rolls. The (unconditional) PMF $p_K(k)$ for K is given by

$$p_K(k) = \binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k}.$$

We find the conditional PMF $p_{k|A}(k \mid A)$ using the definition of conditional probability:

$$p_{K|A}(k \mid A) = \frac{\mathbf{P}(\{K = k\} \cap A)}{\mathbf{P}(A)}.$$

Thus we obtain

$$p_{K|A}(k \mid A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}(\frac{1}{4})^k(\frac{3}{4})^{6-k} & \text{if } k = 1, 2, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $p_{K|A}(0 \mid A) = 0$ because the event $\{K = 0\}$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.