6.041/6.431 Fall 2011 Quiz 1
Wednesday, October 12, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
0		0
1.1		4
1.2		4
1.3i		3
1.3ii		3
1.4		6
1.5		5
1.6		6
1.7		6
1.8		7
1.9		6
Your Grade		50

- This quiz has 1 problem, worth a total of 50 points.
- You may tear apart pages 3 and 4, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- **Unless otherwise specified, you may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of** <u>numbers</u> **that could be evaluated using a calculator.** Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/13.

Problem 0: (1 point penalty) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Times
Patrick Jaillet	Aliaa Atwi	10 & 11 AM
Alan Willsky	Jagdish Ramakrishnan	1 & 2 PM
John Wyatt	Jimmy Li	2 PM & 3 PM

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem 1: (50 points)

Note: All parts can be done independently. Even so, if worried about carrying over a possible mistake in a previous part, you can use a symbol for the expression you found there, and use that symbol in the formulas for subsequent parts.

Note: **Algebraic expressions do not need to be simplified in your answers, unless otherwise noted.**

We start with a collection of n (distinguishable) individuals.

1. (**4 points**) Let k be a positive integer, less than n. In how many ways can these individuals be split into a blue team with k members and a red team with $n-k$ members? *(Individuals within each team are not ordered.)*

2. (**4 points**) In how many ways can these individuals be split into a blue, red, and white team? (The team sizes are not fixed; they can be any numbers in the range from 0 to n, but must add up to n. *Individuals within each team are not ordered.)*

For the remaining questions, assume that each individual is randomly assigned to exactly one team: a person is assigned to the blue team with probability p_B, the red team with probability p_R, and the white team with probability p_W (where $p_B, p_R, p_W > 0$ and $p_B + p_R + p_W = 1$); this is done independently for each person. Let N_B, N_R, N_W be the numbers of people in the respective teams.

3. (**6 points**) Suppose that $n = 5$. We are told that persons 1 and 2 belong to the blue team.
   (i) (**3 points**) Find the conditional PMF of N_B, conditioned on this information.
   (ii) (**3 points**) Are N_B and N_R independent, conditioned on this information? *(Give a 1-2 line justification.)*

4. (**5 points**) Write down the PMF and mean of $N_B + N_R$.

5. (**6 points**) Find $\mathbf{E}[N_B^2]$. (Your formula should be a simple closed form expression, as opposed to something of the form $\sum \cdots$).

6. (**6 points**) Let k be a given positive integer, less than n. Find the conditional PMF of N_B given that $N_B + N_R = k$. *Hint:* No algebra; think.

7. (**6 points**) Each person in the red team receives a random bonus of either \$1 or \$2. Either bonus value is equally likely, and the bonuses of different persons are independent. Let Z be the total amount of bonuses given to the red team. Write down the joint PMF of N_R and Z. (Make sure to specify the range of values where your formula applies.)

8. (**7 points**) Person i (for $i = 2, \ldots, n-1$) is happy if and only if persons $i-1$, i, and $i+1$ belong to the same team. Find the expected number of happy people. *(Persons 1 and n are never happy.)*

9. (**6 points**) Find the conditional probability that person 3 is happy, given that person 2 is happy.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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