# 6.041/6.431 Spring 2007 Quiz 2
# Wednesday, April 16, 7:30 - 9:30 PM

# DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

**Name:** ______________________________

**Recitation Instructor:** ______________________________

**TA:** ______________________________

| **Question** | | Score | Out of |
|---|---|---|---|
| **0** | | | 3 |
| **1a** | (i) | | 5 |
| | (ii) | | 5 |
| | (iii) | | 6 |
| | (iv) | | 6 |
| **1b** | | | 9 |
| **1c** | | | 9 |
| **1d** | | | 9 |
| **1e** | | | 9 |
| **2a** | (i) | | 6 |
| | (ii) | | 6 |
| | (iii) | | 6 |
| **2b** | (i) | | 7 |
| | (ii) | | 7 |
| **2c** | | | 7 |
| **Your Grade** | | | 100 |

# Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
**6.041/6.431: Probabilistic Systems Analysis**
(Spring 2007)

General Instructions:

- This quiz has 3 problems, worth a total of 100 points.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded.
- You are allowed two double sided, handwritten, 8.5 by 11 formula sheet plus a calculator.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A standard normal table is attached at the last page of this quiz.
- You have 2 hours to complete the quiz.
- Be neat! You will not get credit if we can't read it.

**Problem 0:** (3 points) Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

| Recitation Instructor | TA | Recitation Time |
|---|---|---|
| Shawn Staker | Said Francis | 10 AM |
| Shivani Agarwal | Ahmad Zamanian | 10 & 11 AM |
| Premal Shah | Sherman Jia | 11 & 12 PM |
| Peter Hagelstein | James Sun & Mabel Feng | 1 & 2 PM |
| Arthur Baggeroer (6.431) | Faisal Kashif | 10 & 11 AM |

**Problem 1:** Xavier and Wasima are participating in the 6.041 MIT marathon, where race times are defined by random variables[1]. Let $X$ and $W$ denote the race time of Xavier and Wasima respectively. All race times are in hours. Assume the race times for Xavier and Wasima are independent (i.e. $X$ and $W$ are independent). Xavier's race time, $X$, is defined by the following density

$$f_X(x) = \begin{cases} 2c, & \text{if } 2 \le x < 3, \\ c, & \text{if } 3 \le x \le 4, \\ 0, & \text{otherwise,} \end{cases}$$

where $c$ is an unknown constant. Wasima's race time, $W$, is uniformly distributed between 2 and 4 hours. The density of $W$ is then

$$f_W(w) = \begin{cases} \frac{1}{2}, & \text{if } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (i) (5 pts) Find the constant $c$

(ii) (5 pts) Compute $\mathbf{E}[X]$

(iii) (6 pts) Compute $\mathbf{E}[X^2]$

(iv) (6 pts) Provide a fully labeled sketch of the PDF of $2X + 1$

(b) (9 pts) Compute $\mathbf{P}(X \le W)$.

(c) (9 pts) Wasima is using a stopwatch to time herself. However, the stopwatch is faulty; it overestimates her race time by an amount that is uniformly distributed between 0 and $\frac{1}{10}$ hours, which is independent of the actual race time. Thus, if $T$ is the time measured by the stopwatch, then we have

$$f_{T|W}(t|w) = \begin{cases} 10, & \text{if } w \le t \le w + \frac{1}{10} \text{ and } 2 \le w \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{W|T}(w|t)$, when $t = 3$.

(d) (9 pts) Wasima realizes her stopwatch is faulty and buys a new stopwatch. Unfortunately, the new stopwatch is also faulty; this time, the watch adds random noise $N$ that is normally distributed with mean $\mu = \frac{1}{60}$ hours and variance $\sigma^2 = \frac{4}{3600}$. Find the probability that the watch overestimates the actual race time by more than 5 minutes, $\mathbf{P}(N > \frac{5}{60})$. For full credit express your final answer as a number.

(e) (9 pts) Wasima has a sponsor for the marathon! If Wasima finishes the marathon in $w$ hours, the sponsor pays her $\frac{24}{w}$ thousand dollars. Define

$$S = \frac{24}{W}$$

Find the PDF of $S$.

[1] A runner's race time is defined as the time required for a given runner to complete the marathon.

**Problem 2.** Consider the following family of **independent** random variables $N, A_1, B_1, A_2, B_2, \ldots$, where $N$ is a nonnegative discrete random variable and each $A_i$ or $B_i$ is normal with mean 1 and variance 1. Let $A = \sum_{i=1}^{N} A_i$ and $B = \sum_{i=1}^{N} B_i$. Recall that the sum of a fixed number of independent normal random variables is normal.

(a) Assume $N$ is geometrically distributed with a mean of $1/p$.

(i) (6 pts) Find the mean, $\mu_a$, and the variance, $\sigma_a^2$, of $A$.

(ii) (6 pts) Find $c_{ab}$, defined by $c_{ab} = \mathbf{E}[AB]$.

(iii) (6 pts) We observe $B$ (but not $N$) and wish to estimate $A$, using a linear estimator of the form $c + dB$, where $c$ and $d$ are constants. Find values for $c$ and $d$ that result in the smallest possible mean squared error. Express your answer in terms of constants such as $\mu_a$, $c_{ab}$, etc., without plugging in the values found in parts (i) and (ii).

(b) Now assume that $N$ can take only the values 1 (with probability 1/3) and 2 (with probability 2/3).

(i) (7 pts) Give a formula for the PDF of $A$.

(ii) (7 pts) Find the conditional probability $\mathbf{P}(N = 1 \mid A = a)$.

(c) (7 pts) Is it true that $\mathbf{E}[A \mid N] = \mathbf{E}[A \mid B, N]$ ? Either provide a proof, or an explanation why the equality does not hold.