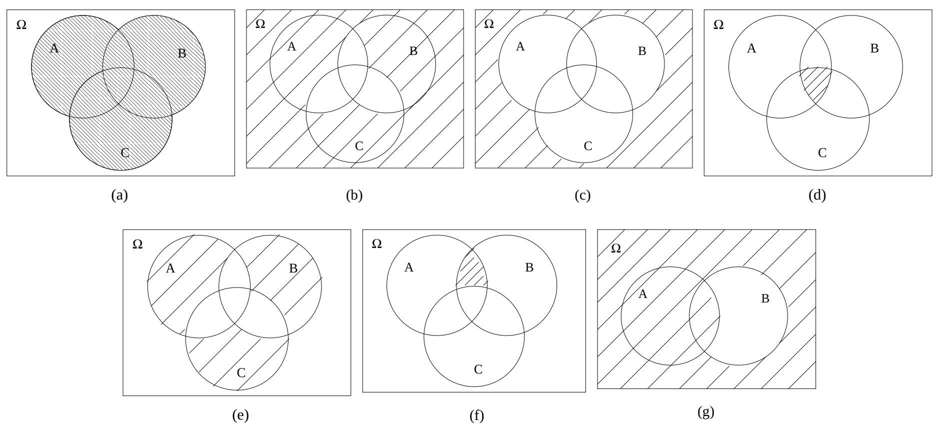
Problem Set 1: Solutions
Due: February 10, 2010

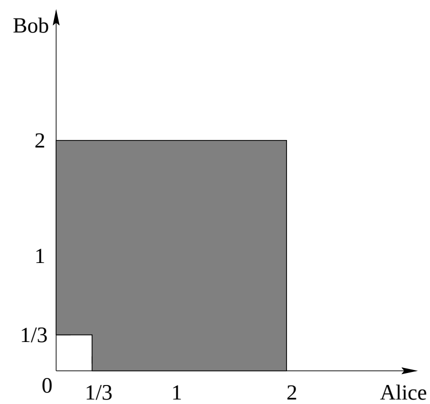
1. (a) $A \cup B \cup C$
   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
   (d) $A \cap B \cap C$
   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$
   (f) $A \cap B \cap C^c$
   (g) $A \cup (A^c \cap B^c)$



2. (a) Since the cars are all distinct, there are 20! ways to line them all up.
   (b) To find the probability that the cars will be alternating: US-made, foreign-made, . . . , we will count the number of "favorable" outcomes and divide by the total number of outcomes which we found in part (a) above. We count in the following manner: first lay the US-made cars down. We can do this in 10! ways, since the cars are distinct. Now lay the foreign-made cars in between the US-made cars. Again we can do this in 10! ways. Finally, we need to multiply by 2, since the sequence could begin either with a US-made car or with a foreign-made car. Thus we have a total of $2 \cdot 10! \cdot 10!$, and the final answer is
   $$\frac{2 \cdot 10! \cdot 10!}{20!}.$$
   Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose that the foreign-made cars are indistinguishable, and also suppose that the US-made cars are indistinguishable. Again we count the number of "favorable" outcomes in the same way: lay the US-made cars down in one way. Then there are two ways to lay the foreign-made cars down since the sequence can begin with either a US or a foreign car. Thus there are two favorable outcomes, out of a possible $\frac{20!}{10! \cdot 10!}$, and the two methods yield the same answer.

3. **P(B)**

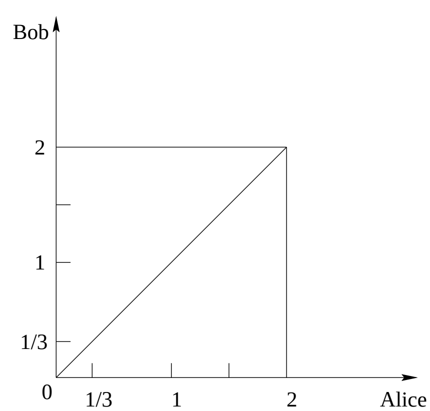
The shaded area in the following figure is the union of Alice's pick being greater than 1/3 and Bob's pick being greater than 1/3.



$$\begin{aligned}\mathbf{P}(B) &= 1 - \mathbf{P}(\text{both numbers are smaller than } 1/3) \\ &= 1 - \frac{\text{area of small square}}{\text{total sample area}} \\ &= 1 - \frac{(1/3)(1/3)}{4} = 1 - \frac{1}{36} = \boxed{35/36}\end{aligned}$$

P(C)

In the following figure, the diagonal line represents the set of points where the two selected numbers are equal.

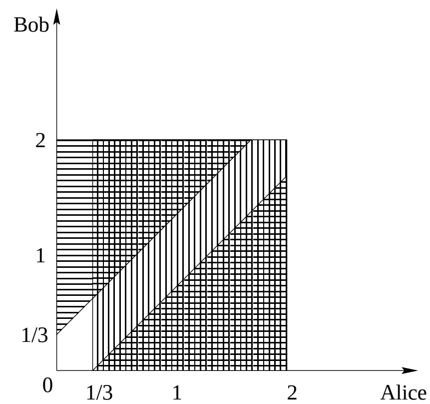


The line has an area of 0. Thus,

$$\mathbf{P}(C) = \frac{\text{area of line}}{\text{total sample area}} = \frac{0}{4} = \boxed{0}$$

$\mathbf{P}(A \cap D)$

Overlapping the diagrams we would get for $\mathbf{P}(A)$ and $\mathbf{P}(D)$,

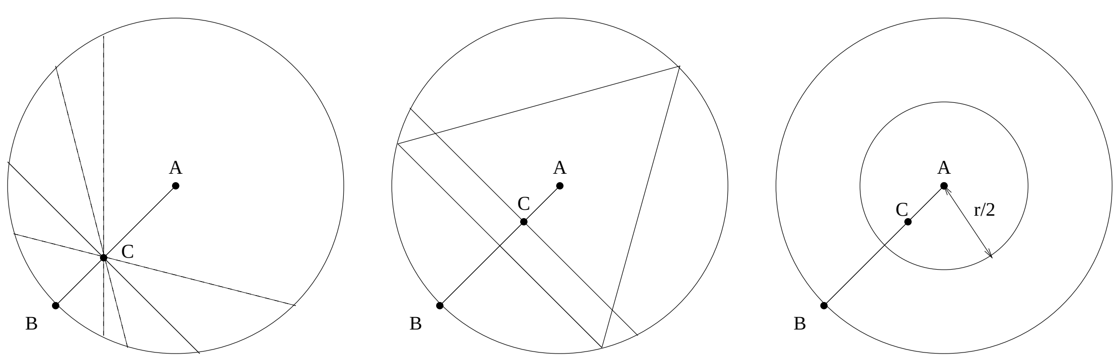


$$\begin{aligned}\mathbf{P}(A \cap D) &= \frac{\text{double shaded area}}{\text{total sample area}} \\ &= \frac{(5/3)(5/3)(1/2) + (4/3)(4/3)(1/2)}{4} = \frac{25/18 + 16/18}{4} = \boxed{41/72}\end{aligned}$$

4. See online solutions for Problem 1.50.

5. (a) Consider a random point C inside the circle. Of the many chords that pass through C, the one that has C as its midpoint is exactly the chord that is orthogonal to the radius AB that contains C (Figure (a)). This chord is longer than the side of the triangle if and only if C is closer to A than B (Figure (b)); this latter condition holds exactly when C is inside the circle that is also centered at A and has radius $\frac{r}{2}$, where r is the radius of the original circle (Figure (c)). The probability that this is true is

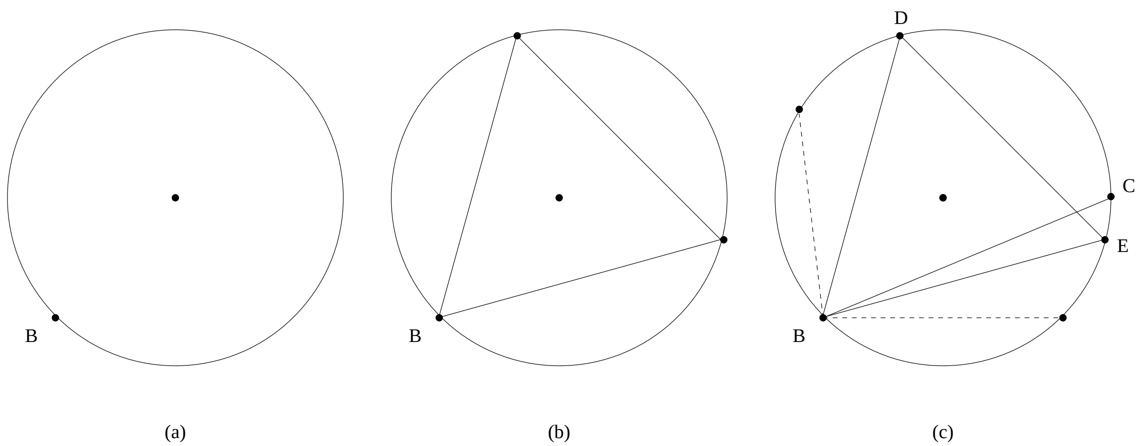
$$\frac{\text{area of circle of radius } \frac{r}{2}}{\text{area of circle of radius } r} = \frac{\pi(\frac{r}{2})^2}{\pi r^2} = \frac{1}{4}.$$



(a) (b) (c)

(b) Let B be the fixed point on the circumference (Figure (a)). Draw a copy of the triangle so that one of its vertices coincides with B (Figure (b)). Now pick a random point C on the circumference and consider the chord BC. From elementary geometry, BC is longer than the side of the triangle exactly when C lies between D and E, the other two vertices of the triangle besides B (Figure (c)). This happens with probability

$$\frac{\text{length of arc } DE}{\text{length of entire circumference}} = \frac{\frac{1}{3}2\pi r}{2\pi r} = \frac{1}{3}.$$



G1†. (a) Define event $E = A \cup B$. Then $E \cap C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Therefore $\mathbf{P}(E \cap C) = \mathbf{P}(A \cap C) + \mathbf{P}(B \cap C) - \mathbf{P}(A \cap B \cap C)$.

$$\begin{aligned}
\mathbf{P}(A \cup B \cup C) &= \mathbf{P}(E \cup C) \\
&= \mathbf{P}(E) + \mathbf{P}(C) - \mathbf{P}(E \cap C) \\
&= \mathbf{P}(A \cup B) + \mathbf{P}(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) + \mathbf{P}(A \cap B \cap C) \\
&= \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) - \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B \cap C)
\end{aligned}$$

(b) **Method 1:** An intuitive justification.
View $\cup_{k=1}^{n} A_k$ from a Venn Diagram perspective. This expression defines all area (or outcomes) in any set A_k. The first expression on the right hand side of the equation defines all outcomes in A_1, the second all outcomes in A_2 and not in A_1, the third all outcomes in A_3 and not in A_1 nor A_2, and so forth. Hence each set described by each expression is disjoint. Since the probability of the union of disjoint sets is the sum of the probabilities of each set, we arrive at our intended expression.

Method 2: A more rigorous inductive argument.
Base Case:

$\mathbf{P}(A_1) = \mathbf{P}(A_1)$

Inductive Step:

Assume

$$\mathbf{P}(\cup_{k=1}^{n-1} A_k) = \mathbf{P}(A_1) + \mathbf{P}(A_1^c \cap A_2) + \mathbf{P}(A_1^c \cap A_2^c \cap A_3) + \cdots + \mathbf{P}(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}).$$

Then

$$
\begin{aligned}
\mathbf{P}(\cup_{k=1}^{n} A_k) &= \mathbf{P}(\cup_{k=1}^{n-1} A_k \cup A_n) \\
&= \mathbf{P}(A_1) + \mathbf{P}(A_1^c \cap A_2) + \mathbf{P}(A_1^c \cap A_2^c \cap A_3) + \cdots + \mathbf{P}(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\
&\quad +(\mathbf{P}(A_n) - \mathbf{P}(\cup_{k=1}^{n-1} A_k \cap A_n)) \qquad (1) \\
&= \mathbf{P}(A_1) + \mathbf{P}(A_1^c \cap A_2) + \mathbf{P}(A_1^c \cap A_2^c \cap A_3) + \cdots + \mathbf{P}(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\
&\quad +\mathbf{P}((\cup_{k=1}^{n-1} A_k)^c \cap A_n) \qquad (2) \\
&= \mathbf{P}(A_1) + \mathbf{P}(A_1^c \cap A_2) + \mathbf{P}(A_1^c \cap A_2^c \cap A_3) + \cdots + \mathbf{P}(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n). \qquad (3)
\end{aligned}
$$

We get equation (1) by applying

$$\mathbf{P}(X \cup Y) = \mathbf{P}(X) + \mathbf{P}(Y) - \mathbf{P}(X \cap Y),$$

where $X = \cup_{k=1}^{n} A_k$ and $Y = A_n$. The last component of equation (2) is a direct application of

$$\mathbf{P}(Y) - \mathbf{P}(X \cap Y) = \mathbf{P}(X^c \cap Y),$$

where X and Y are defined as before. The last component of equation (3) results as a direct application of the identity

$$(\cup_{k=1}^{n-1} A_k)^c \equiv \cap_{k=1}^{n-1} A_k^c \qquad \text{for any sets } A_1, A_2, \ldots, A_k.$$