Recitation 16 Solutions
November 3, 2011

1. (a) R is a binomial random variable with parameters p and n. Hence,

$$\begin{aligned} p_R(r) &= \binom{n}{r}(1-p)^{n-r}p^r, \qquad \text{for } r = 0,\, 1,\, 2,\, \ldots,\, n, \\ \mathbf{E}[R] &= np, \\ \mathrm{var}(R) &= np(1-p). \end{aligned}$$

(b) Denote the event of interest by A. Note that $\mathbf{P}(A)$ depends on the number of packages n:

- If there is only one package, then that package will definitely be the only one in its truck, and so $\mathbf{P}(A) = 1$.
- If there are two or more packages, then A is the union of the following two disjoint events:
  - The first item is placed in the red truck and the remaining $n-1$ are placed in the green truck.
  - The first item is placed in the green truck and the remaining $n-1$ are placed in the red truck.

  Thus the probability of A is the sum of the probabilities of the two events above:

$$\mathbf{P}(A) = p(1-p)^{n-1} + (1-p)p^{n-1}.$$

Combining these results, we have

$$\mathbf{P}(A) = \begin{cases} 1, & \text{if } n = 1, \\ p(1-p)^{n-1} + (1-p)p^{n-1}, & \text{if } n = 2, 3, 4, \ldots. \end{cases}$$

(c) Denote the event of interest by B. Similar to part (b), note that $\mathbf{P}(B)$ depends on the number of packages n:

- If there is only one package, then the truck that gets that package will definitely contain exactly one package, and so $\mathbf{P}(B) = 1$.
- If there are two packages, then event B occurs only if each truck gets one package, which occurs with probability $\mathbf{P}(B) = p(1-p) + (1-p)p$.
- If there are three or more packages, then B is the union of the following two disjoint events:
  - Any one of the n packages is placed in the red truck and the remaining $n-1$ packages are placed in the green truck.
  - Any one of the n packages is placed in the green truck and the remaining $n-1$ packages are placed in the red truck.

  Thus the probability of B is the sum of the probabilities of the two events above. Note that these events are almost the same as the events in part (b), except that now *any* one package can be the one by itself, not just the first package. Therefore, we multiply each of the probabilities in part (b) by $\binom{n}{1}$ and obtain

$$\begin{aligned} \mathbf{P}(B) &= \binom{n}{1}p(1-p)^{n-1} + \binom{n}{1}(1-p)p^{n-1} \\ &= np(1-p)^{n-1} + n(1-p)p^{n-1}. \end{aligned}$$

Combining these three results, we have

$$\mathbf{P}(B) = \begin{cases} 1, & \text{if } n = 1, \\ p(1-p) + (1-p)p, & \text{if } n = 2, \\ np(1-p)^{n-1} + n(1-p)p^{n-1}, & \text{if } n = 3, 4, 5, \ldots. \end{cases}$$

(d) Random variables R and G are dependent: $G = n - R$. To find the expectation of D, we can either use the linearity of expectations or write G in terms of R:

$$\begin{aligned} \mathbf{E}[D] &= \mathbf{E}[R-G] = \mathbf{E}[R] - \mathbf{E}[G] = np - n(1-p) = 2np - n \\ &= \mathbf{E}[R-G] = \mathbf{E}[R-(n-R)] = \mathbf{E}[2R-n] = 2\mathbf{E}[R] - n = 2np - n \end{aligned}$$

Since R and G are dependent, we cannot split the variance of their difference into the sum of their respective variances, and so instead we write G in terms of R:

$$\text{var}(D) = \text{var}(R-G) = \text{var}(2R-n) = 4\text{var}(R) = 4np(1-p).$$

(e) Let C be the event that each of the first 2 packages is loaded onto the red truck. Given that C occurred, the random variable R becomes

$$2 + X_3 + X_4 + \cdots + X_n.$$

Hence,

$$\mathbf{E}[R \mid C] = \mathbf{E}[2 + X_3 + X_4 + \cdots + X_n] = 2 + (n-2)\mathbf{E}[X_i] = 2 + (n-2)p.$$

Similarly, the conditional variance of R is

$$\text{var}(R \mid C) = \text{var}(2 + X_3 + X_4 + \cdots + X_n) = (n-2)\text{var}(X_i) = (n-2)p(1-p).$$

Finally, given that the first two packages are loaded onto the red truck, the probability that a total of r packages are loaded onto the red truck is equal to the probability that $r-2$ of the remaining $n-2$ packages go to the red truck:

$$p_{R|C}(r) = \binom{n-2}{r-2}(1-p)^{n-r}p^{r-2}, \quad \text{for } r = 2, \ldots, n.$$

2. Problem 6.6, page 328 in text. See text for solutions.

3. Problem 6.4, page 327 in text. See text for solutions.