Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2010)

Problem Set 10
Due: May 5, 2010

1. In your summer internship, you are working for the world's largest producer of lightbulbs. Your manager asks you to estimate the quality of the production, that is, to estimate the probability p that a bulb produced by the factory is defectless. You are told to assume that all lightbulbs have the same probability of having a defect, and that defects in different lightbulbs are independent.

   (a) Supposing you test n randomly picked bulbs, what is a good estimate Z_n for p, such that Z_n converges to p in probability?

   (b) The management asks that the estimate is located in the range $p \pm 0.1$ with probability 0.95. Are 27 randomly picked bulbs enough for this specification? Give the reason. Solve this problem using Chebyshev's inequality, and then using the central limit theorem.

2. (a) Given the information $\mathbf{E}[X] = 7$ and $\text{var}(X) = 9$, use the Chebyshev inequality to find a lower bound for $\mathbf{P}(4 \leq X \leq 10)$.

   (b) Find the smallest and largest possible values of $\mathbf{P}(4 < X < 10)$, given the mean and variance information from part (a).

3. Many casino games are only slightly biased in favor of the casino, so that the casino makes a profit while customers maintain interest. Imagine such a game, where the probability of the casino winning is 0.51. Suppose you play 400 independent games, and let L denote the number of times you lose. Use whichever approximations to the binomial you feel are appropriate to calculate the following:

   (a) $\mathbf{P}(190 \leq L \leq 210)$

   (b) $\mathbf{P}(210 \leq L \leq 230)$

4. Let $X_1, X_2, \ldots$ be independent, identically distributed, continuous random variables with $\mathbf{E}[X] = 2$ and $\text{var}(X) = 9$. Define $Y_i = (0.5)^i X_i$, $i = 1, 2, \ldots$. Also define T_n and A_n to be the sum and the average, respectively, of the terms $Y_1, Y_2, \ldots, Y_n$.

   (a) Is Y_n convergent in probability to a real value y? If so, what is y? Explain.

   (b) Is T_n convergent in probability to a real value t? If so, what is t? Explain.

   (c) Is A_n convergent in probability to a real value a? If so, what is a? Explain.

5. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with (unknown but finite) mean μ and variance σ^2 where $\sigma^2 > 0$. For $i = 1, 2, \ldots$, let

$$Y_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}.$$

   (a) Are the random variables Y_i independent?

   (b) Are they identically distributed?

   (c) Let

$$M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

   Is M_n convergent in probability to μ? Prove your answer.

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G1†. (a) If U and V are random variables, and if ϵ is a scalar, explain why

$$\mathbf{P}(|U+V| \geq \epsilon) \leq \mathbf{P}(|U| \geq \epsilon/2) + \mathbf{P}(|V| \geq \epsilon/2).$$

(b) Let U_n and V_n be two sequences of random variables that converge (in probability) to a and b, respectively. Show that $U_n + V_n$ converges to $a + b$. *Hint:* The inequality from part (a) may come handy.