Problem Set 6
Due: March 28, 2011

1. Imagine a TV game show where each contestant i spins an infinitely calibrated "fair" wheel of fortune, which assigns him/her some real number x_i (not necessarily integer) between 1 and 100.

   a) Find $\mathrm{P}(x_1 < x_2)$. Explain your answer.

   b) Find $\mathrm{P}(x_1 < x_2,\ x_1 < x_3)$, i.e., find the probability that the first contestant will have the smallest value of the first three contestants.

   c) Consider a new random variable n, which is integer valued. n is the index of the first contestant who is assigned a smaller number than contestant 1. As an illustration, if contestant 1 has a smaller value than contestants 2, 3, and 4, but contestant 5 has a smaller value than contestant 1 ($x_5 < x_1$), then $n = 5$. Find $\mathrm{P}(n > n_0)$ as a function of n_0.

   d) Find $E[n]$.

   e) Now suppose n is the index of the first contestant who is assigned a <u>larger</u> number than contestant 1. Argue that the answers to (c) and (d) do not change.

2. Random variable X has the following PDF:

$$f_X(x) = \begin{cases} cx^{-2}, & 1 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the numerical value of c.
   (b) Let A be the event $X > 2$. Determine $P(A)$ and the conditional PDF of X given that A has occurred, i.e., $f_{X|A}(x)$.
   (c) Let $Y = X^2$. Compute the conditional expectation and the conditional variance of Y given A.

3. Let continuous random variables X, Y and Z be independent and identically distributed according to the uniform distribution in the unit interval $[0, 1]$.

   (a) Consider two new random variables defined by $V = XY$ and $W = Z^2$. Derive the joint PDF $f_{V,W}(v, w)$.
   (b) Show that $\mathbf{P}(XY < Z^2) = \frac{5}{9}$.

4. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

   (a) Show that X and $E[X \mid Y]$ are positively correlated.
   (b) Show that the correlation coefficient of Y and $E[X \mid Y]$ has the same sign as the correlation coefficient of X and Y.

5. Provide a new derivation of the formula

$$\mathrm{var}(X) = \mathrm{var}(\hat{X}) + \mathrm{var}(\tilde{X})$$

   using the law of total variance. Here, as in the text, $\hat{X} = E[X \mid Y]$, and $\tilde{X} = X - \hat{X}$.

G1†. A coin shows heads with probability p. Let X_n be the number of flips till a sequence of n consecutive heads is obtained. Find $E[X_n]$.