

Recitation 10 Solutions
October 13, 2011

1. (a) We know that the PDF must integrate to 1. Therefore we have

$$\int_{-\infty}^{\infty} f_Z(z) dz = \int_{-2}^1 \gamma(1+z^2) = \gamma \left(z + \frac{1}{3}z^3 \right) \Big|_{-2}^1 = 6\gamma.$$

From this we conclude $\gamma = 1/6$.

- (b) To find the CDF, we integrate:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^z f_Z(t) dt = \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6} \left(t + \frac{1}{3}t^3 \right) \Big|_{-2}^z, & \text{if } -2 \leq z \leq 1, \\ 1, & \text{if } z > 1 \end{cases} \\ &= \begin{cases} 0, & \text{if } z < -2, \\ \frac{1}{6} \left(z + \frac{1}{3}z^3 + \frac{14}{3} \right), & \text{if } -2 \leq z \leq 1, \\ 1, & \text{if } z > 1. \end{cases} \end{aligned}$$

2. See textbook, Problem 3.9, page 187.

3. (a) For $x \geq 0$,

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_0^x \lambda e^{-\lambda t} dt = \left[-e^{-\lambda t} \right]_0^x = 1 - e^{-\lambda x}.$$

For $x < 0$, we have $F_X(x) = \int_{-\infty}^x f_X(t) dt = 0$. Thus we conclude

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

- (b) The key step in the following computation uses integration by parts, whereby

$$\int_0^{\infty} u dv = uv \Big|_0^{\infty} - \int_0^{\infty} v du$$

is applied with $u = x$ and $v = -e^{-\lambda x}$:

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \left[-x e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = \frac{1}{\lambda}.$$

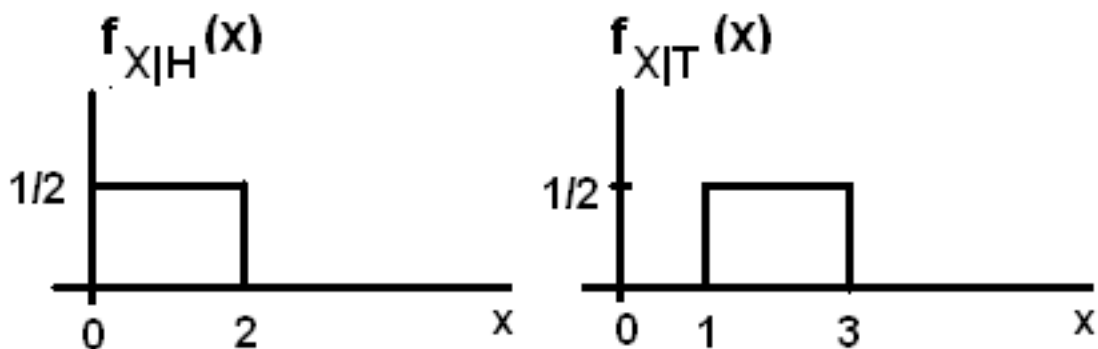
- (c) Integrating by parts with $u = x^2$ and $v = -e^{-\lambda x}$ in the second line below gives

$$\begin{aligned} \mathbf{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx \\ &= \left[-x^2 e^{-\lambda x} \right]_0^{\infty} + 2 \int_0^{\infty} x e^{-\lambda x} dx = \frac{2}{\lambda} \mathbf{E}[X] = \frac{2}{\lambda^2}. \end{aligned}$$

Combining with the previous computation, we obtain

$$\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}.$$

4. Let H be the event that the flip results in heads and T be the event that the flip results in tails. Then, $\mathbf{P}(H) = \frac{3}{4}$ and $\mathbf{P}(T) = \frac{1}{4}$. We are told $f_{X|H}(x)$ and $f_{X|T}(x)$ looks like:



From the total probability theorem, we have

$$f_X(x) = \mathbf{P}(H)f_{X|H}(x) + \mathbf{P}(T)f_{X|T}(x),$$

from which we conclude $f_X(x)$ looks like:

