6.041/6.431 Fall 2011 Final Exam
Tuesday, December 20, 9:00 A.M. - 12:00 NOON

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

- This exam has 5 problems, worth a total of 100 points.
- You may tear apart pages 3-8, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- For full credit, answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.
- You are allowed three two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- You have 180 minutes to complete the quiz.

Question	Score	Out of
1a		5
1b		5
2a		4
2b		4
2c		5
2d		6
2e		6
3a		4
3b		4
3c		5
3d		5
3e		6
3f		5
3g		5
4a		5
4b		5
4c		8
5a		6
5b		7
Your Grade		100

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem 0: Write your name, your assigned recitation instructor's name, and assigned TA's name on the cover of the quiz booklet. The Instructor/TA pairing is listed below.

Recitation Instructor	TA	Recitation Times
Patrick Jaillet	Aliaa Atwi	10 AM & 11 AM
Alan Willsky	Jagdish Ramakrishnan	1 PM & 2 PM
John Wyatt	Jimmy Li	2 PM & 3 PM

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Poisson with parameter $\lambda\tau$: (Describes the number N_τ arrivals in a Poisson process with rate λ, over an interval of length τ.)

$$p_{N_\tau}(k) = P(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, \ldots,$$

$$\mathbf{E}[N_\tau] = \lambda\tau, \qquad \text{var}(N_\tau) = \lambda\tau.$$

Exponential Random Variable with parameter λ : (Describes the time T until the first arrival.)

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[T] = \frac{1}{\lambda}, \qquad \text{var}(T) = \frac{1}{\lambda^2}.$$

Erlang PDF of order k : (Describes the kth arrival time.)

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0.$$

$$\mathbf{E}[Y_k] = \frac{k}{\lambda}, \qquad \text{var}(Y_k) = \frac{k}{\lambda^2}.$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

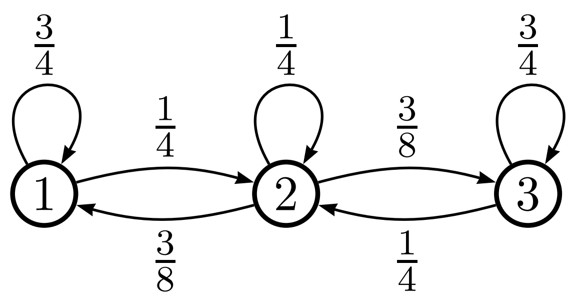
Problem 1: (10 points) Let $X_1, X_2, \ldots$ be independent, exponentially distributed random variables with parameter $\lambda = 8$, that is,

$$f_{X_i}(x) = \begin{cases} 8e^{-8x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

(a) **(5 points)** Find the PDF of X_1^4.

(b) **(5 points)** Write down the PDF of $X_1 + X_2 + X_3$.

Problem 2: (25 points) Consider a Markov chain $X_0, X_1, X_2, \ldots$ described by the transition diagram below. The chain starts at state 1, that is, $X_0 = 1$.



(a) **(4 points)** Find the probability that $X_2 = 3$.

(b) **(4 points)** Find the probability that the process is in state 3 immediately after the second change of state. (A "change of state" is a transition which is not a self-transition.)

(c) **(5 points)** Find (approximately) $\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001})$.

(d) **(6 points)** Let T be the first time that the state is equal to 3. Write down a system of equations that can be used to find $\mathbf{E}[T]$. (You don't have to solve the equations.)

(e) **(6 points)** Suppose now that the process starts at state 2, i.e., $X_0 = 2$. Let S be the first time by which both states 1 and 3 have been visited. Find $\mathbf{E}[S]$. *Note:* You can express your answer in terms of $\mathbf{E}[T]$ in part (d).

Problem 3: (34 points) A dedicated professor has been holding infinitely long office hours. Undergraduate students arrive according to a Poisson process, at a rate of $\lambda_U = 3$ per hour, while graduate students arrive according to an independent Poisson process, at a rate of $\lambda_G = 5$ per hour. An arriving student receives immediate attention (the previous student's stay is immediately terminated), and stays with the professor until the next student arrives. (So, the professor is always busy, talking with the most recently arrived student.)

(a) **(4 points)** What is the probability that exactly three undergraduates arrive between 10:00 pm and 10:30 pm?

(b) **(4 points)** What is the expected length of time that the kth arriving student will stay with the professor?

(c) **(5 points)** Given that the professor is currently talking with an undergraduate, what is the expected number of new arrivals until (and including) the time a graduate student arrives?

(d) **(5 points)** Given that the professor is currently talking with an undergraduate, what is the probability that 5 of the next 7 students will be undergraduates?

(e) **(6 points)** Evaluate (approximately) the probability that over the course of 24 hours the professor sees more than 205 students. You can leave your answer in the form $\Phi(z)$, where Φ is the CDF of the standard normal, and where z is a number for which you can give an algebraic expression, such as $(4-\sqrt{2})/2 \times \sqrt{13}$.

As rumors spread around campus, a worried department head drops in at midnight and begins observing the professor.

(f) **(5 points)** When the department head arrives, the professor is talking to an undergraduate. What is the expected length of time until the next student arrives, conditioned on the event that the next student will be an undergraduate?

(g) **(5 points)** What is the expected time that the department head will have to wait until the students he/she has seen (including the one seen when he/she arrived) include both an undergraduate and a graduate student?

Problem 4: (18 points) Let θ be an unknown constant. Let $W_1, \ldots, W_n$ be independent exponential random variables with parameter 1. Let $X_i = \theta + W_i$.

(a) **(5 points)** What is the maximum likelihood estimator of θ based on the single observation X_1?

(b) **(5 points)** What is the maximum likelihood estimator of θ based on the sequence of observations $X_1, \ldots, X_n$?

(c) **(8 points)** For some obscure reason, the boss wants you to construct a confidence interval of the particular form

$$[\hat{\Theta} - c,\ \hat{\Theta}],$$

where

$$\hat{\Theta} = \min_i \{X_i\},$$

and c is a constant that we need to choose. For $n = 1000$, how should the constant c be chosen so that we have a 95% confidence interval? (You do not need to evaluate c numerically. An answer such as, for example, $34 + \log(0.45)$ is fine.)

Problem 5: (13 points) Let Θ be an unknown random variable that we wish to estimate. It has a prior distribution with mean and variance equal to $\mu_\Theta = 1$ and $\text{var}(\Theta) = 2$, respectively. Let W be a noise term, another unknown random variable, independent of Θ, with mean and variance equal to $\mu_W = 3$ and $\text{var}(W) = 5$, respectively.

We have two different instruments that we can use to measure Θ. The first instrument yields a measurement of the form $X = \Theta + W$, the second a measurement of the form $X = 2\Theta + 3W$. We pick an instrument at random, with each instrument having probability 1/2 of being chosen, and we then record the value of X, without knowing which instrument was used.

(a) **(6 points)** Find $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$.

(b) **(7 points)** Give a formula for the optimal linear estimator of Θ given X. (Specify numerical values for any constants involved in the formula.)

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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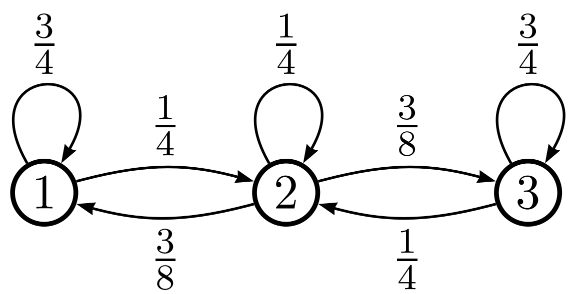

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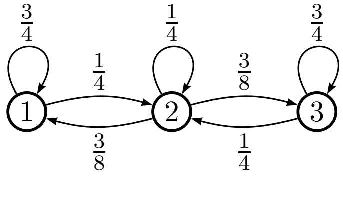

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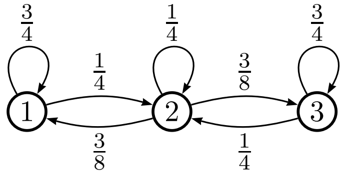

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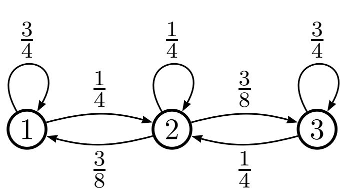

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