Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2009)

Problem Set 2
Due: February 18, 2009

1. The weather on any given day can be one of four different possibilites: sunny, cloudy, rainy, or snowy. Assume that a snowy day can happen only during the winter, and that each season has 90 days. Assume that: 1) within each season, cloudy days and rainy days are equally likely, 2) in the spring and the fall, sunny days and rainy days are equally likely, 3) in the summer, sunny days are twice as likely as rainy days, and 4) in the winter, sunny and snowy days are each one half as likely as rainy days. What is the probability that a day chosen at random from these 360 days is sunny?

2. **The Chess Problem.** This year's Belmont chess champion is to be selected by the following procedure. Bo and Ci, the leading challengers, first play a two-game match. If one of them wins both games, he gets to play a two-game *second round* with Al, the current champion. Al retains his championship unless a second round is required and the challenger beats Al in both games. If Al wins the initial game of the second round, no more games are played.

   Furthermore, we know the following:

   - The probability that Bo will beat Ci in any particular game is 0.6.
   - The probability that Al will beat Bo in any particular game is 0.5.
   - The probability that Al will beat Ci in any particular game is 0.7.

   Assume no tie games are possible and all games are independent.

   (a) Determine the apriori probabilities that
      i. the second round will be required.
      ii. Bo will win the first round.
      iii. Al will retain his championship this year.

   (b) Given that the second round is required, determine the conditional probabilities that
      i. Bo is the surviving challenger.
      ii. Al retains his championship.

   (c) Given that the second round was required and that it comprised only one game, what is the conditional probability that it was Bo who won the first round?

3. Imno Kolmogorov, an MIT Freshman, makes one to five new friends every week, with equal probability. The number of friends she makes during each week is independent from all other weeks. We are concerned with two consecutive weeks.

   Let event A be "Imno made a total of 10 friends during the two weeks". Let event B be "Imno made more than 5 friends during the two weeks."

   (a) Are events A and B independent?

   (b) Let C be the event "Imno made exactly 5 friends during the first week". Are A and B independent, conditioned on C?

(c) Is A independent of C? Is B independent of C?

(d) Given that Imno made a total of 6 friends in two weeks, what is the probability that she made exactly 2 friends in the first week? How about 3 friends in the first week?

(e) Given that Imno made a total of 6 friends in two weeks, what is the probability that she made exactly 2 friends in at least one of the weeks? How about exactly 3 friends in at least one of the weeks?

4. **Knives and Forks:** In the kitchen in your apartment, you put all your 10 forks in the left drawer and all 10 knives in the right drawer. Your roommate, who does not agree with your organizational approach, comes in, takes two forks from the left drawer and tosses them into the right drawer. She then takes at random an item (knife or fork) from the right drawer and tosses it in the left drawer.

   After this exchange, you come in and randomly pick up an item from a randomly chosen drawer. Given you have picked up a knife, what is the probability that you have opened the left drawer?

5. **Ternary Channel:** A communication system transmits one of three signals, s_1, s_2 and s_3, with equal probabilities. The transmission is corrupted by noise, causing the received signal to be changed according to the following table of conditional probabilities:

		Receive, j		
	$\mathrm{P}(s_j \mid s_i)$	s_1	s_2	s_3
	s_1	0.25	0.5	0.25
Send, i	s_2	0.04	0.9	0.06
	s_3	0.8	0.15	0.05

   For example, if s_1 is sent, the probability of receiving s_3 is 0.25. The entries of the table list the probability of s_j received, given that s_i is sent, i.e., $P(s_j \text{ received} \mid s_i \text{ sent})$.

   (a) Compute the (unconditional) probability that s_j is received for $j = 1, 2, 3$.

   (b) Compute the probability $P(s_i \text{ sent} \mid s_j \text{received})$ for $i, j = 1, 2, 3$.

G1†. Suppose that A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

G2†. Consider an experiment whose sample space is the real line where all intervals (open, closed, semi-open), as well as their complements, are events. Let $\{a_n\}$ be an increasing sequence of numbers that converges to a and let $\{b_n\}$ be a decreasing sequence that converges to b. Show that

$$\lim_{n\to\infty} P((a_n, b_n)) = P((a, b)).$$

Here, the notation (a, b) stands for the open interval $\{x \mid a < x < b\}$. *Note:* This result seems intuitively obvious. The issue is to derive it using the axioms of probability theory.

†Required for 6.431; optional for 6.041