

**Quiz 2 Solutions:**  
**November 3, 2009**

**Problem 2.** (49 points)

(a) (7 points)

We start by recognizing that  $f_X(x) = e^{-x}$  for  $x \geq 0$  and  $f_{Y|X}(y | x) = xe^{-xy}$  for  $y \geq 0$ . Furthermore,  $f_{X,Y}(x, y) = f_X(x) \cdot f_{Y|X}(y | x)$ . Substituting for  $f_X(x)$  and  $f_{Y|X}(y | x)$  yields,

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(1+y)x}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) (7 points)

The marginal PDF of  $Y$  can be found by integrating the joint PDF of  $X$  and  $Y$ .

$$\begin{aligned} f_Y(y) &= \int_X f_{X,Y}(x, y) dx \\ &= \int_0^\infty xe^{-(1+y)x} dx \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{(1+y)^2}, & y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(c) (7 points)

We are asked to compute the PDF of the random variable  $X$  while conditioning on another random variable  $Y$ . The conditional PDF of  $X$  given that  $Y = 2$  is

$$f_{X|Y}(x | 2) = \frac{f_{X,Y}(x, 2)}{f_Y(2)} = \frac{xe^{-3x}}{\frac{1}{3^2}}$$

$$f_{X|Y}(x | 2) = \begin{cases} 9xe^{-3x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(d) (7 points)

$$\begin{aligned} \mathbf{E}[X | Y = 2] &= \int_X x \cdot f_{X|Y}(x | 2) dx \\ &= 9 \int_0^\infty x^2 e^{-3x} dx \\ &= 9 \cdot \frac{2}{3^3} \\ &= \frac{2}{3}. \end{aligned}$$

(e) (7 points)

In the new universe in which  $X = 2$ , we are asked to compute the conditional PDF of  $Y$  given the event  $Y \geq 3$ .

$$f_{Y|X,Y \geq 3}(y | 2) = \frac{f_{Y|X}(y | 2)}{\mathbf{P}(Y \geq 3 | X = 2)}.$$

We first calculate the  $\mathbf{P}(Y \geq 3 | X = 2)$ .

$$\begin{aligned}\mathbf{P}(Y \geq 3 | X = 2) &= \int_3^\infty f_{Y|X}(y | 2) dy \\ &= \int_3^\infty 2e^{-2y} dy \\ &= 1 - F_{Y|X}(3 | 2) \\ &= 1 - (1 - e^{-2 \cdot 3}) \\ &= e^{-6},\end{aligned}$$

where  $F_{Y|X}(3|2)$  is the CDF of an exponential random variable with  $\lambda = 2$  evaluated at  $y = 3$ . Substituting the values of  $f_{Y|X}(y | 2)$  and  $\mathbf{P}(Y \geq 3 | X = 2)$  yields

$$f_{Y|X,Y \geq 3}(y | 2) = \begin{cases} 2e^6 e^{-2y}, & y \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively,  $f_{Y|X}(y | 2)$  is an exponential random variable with  $\lambda = 2$ . To compute the conditional PMF  $f_{Y|X,Y \geq 3}(y | 2)$ , we can apply the memorylessness property of an exponential variable. Therefore, this conditional PMF is also an exponential random variable with  $\lambda = 2$ , but it is shifted by 3.

(f) (7 points)

Let's define  $Z = e^{2X}$ . Since  $X$  is an exponential random variable that takes on non-negative values ( $X \geq 0$ ),  $Z \geq 1$ . We find the PDF of  $Z$  by first computing its CDF.

$$\begin{aligned}F_Z(z) &= \mathbf{P}(Z \leq z) \\ &= \mathbf{P}(e^{2X} \leq z) \\ &= \mathbf{P}(2X \leq \ln z) \\ &= \mathbf{P}(X \leq \frac{\ln z}{2}) \\ &= 1 - e^{-\frac{\ln z}{2}} \\ &= 1 - e^{\ln z^{-\frac{1}{2}}}\end{aligned}$$

The CDF of  $Z$  is:

$$F_Z(z) = \begin{cases} 1 - z^{-\frac{1}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Differentiating the CDF of  $Z$  yields the PDF

$$f_Z(z) = \begin{cases} \frac{1}{2}z^{-\frac{3}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Alternatively, you can apply the PDF formula for a strictly monotonic function of a continuous random variable. Recall if  $z = g(x)$  and  $x = h(z)$ , then

$$f_Z(z) = f_X(h(z)) \left| \frac{dh}{dz}(z) \right|.$$

In this problem,  $z = e^{2x}$  and  $x = \frac{1}{2}\ln z$ . Note that  $f_Z(z)$  is nonzero for  $z > 1$ . Since  $X$  is an exponential random variable with  $\lambda = 1$ ,  $f_X(x) = e^{-x}$ . Thus,

$$\begin{aligned} f_Z(z) &= e^{-\frac{1}{2}\ln z} \left| \frac{1}{2z} \right| \\ &= e^{\ln z^{-\frac{1}{2}}} \frac{1}{2z} \\ &= \frac{1}{2}z^{-\frac{3}{2}} \quad z \geq 1, \end{aligned}$$

where the second equality holds since the expression inside the absolute value is always positive for  $z \geq 1$ .

**Problem 3.** (10 points)

(a) (5 points) The quantity  $\mathbf{E}[X \mid Y]$  is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

If  $X$  and  $Y$  are not independent, then  $\mathbf{E}[X \mid Y]$  is a function of  $Y$  and is therefore a continuous random variable. However if  $X$  and  $Y$  are independent, then  $\mathbf{E}[X \mid Y] = \mathbf{E}[X]$  which is a number.

(b) (5 points) The quantity  $\mathbf{E}[\mathbf{E}[X \mid Y, N] \mid N]$  is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

If  $X$ ,  $Y$  and  $N$  are not independent, then the inner expectation  $G(Y, N) = \mathbf{E}[X \mid Y, N]$  is a function of  $Y$  and  $N$ . Furthermore  $\mathbf{E}[G(Y, N) \mid N]$  is a function of  $N$ , a discrete random variable. If  $X$ ,  $Y$  and  $N$  are independent, then the inner expectation  $\mathbf{E}[X \mid Y, N] = \mathbf{E}[X]$ , which is a number. The expectation of a number given  $N$  is still a number, which is a special case of a discrete random variable.

**Problem 4.** (25 points)

(a) (i) (5 points)

Using the Law of Iterated Expectations, we have

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Q]] = \mathbf{E}[Q] = \frac{1}{2}.$$

(ii) (5 points)

$X$  is a Bernoulli random variable with a mean  $p = \frac{1}{2}$  and its variance is  $\text{var}(X) = p(1-p) = 1/4$ .

(b) (7 points)

We know that  $\text{cov}(X, Q) = \mathbf{E}[XQ] - \mathbf{E}[X]\mathbf{E}[Q]$ , so first let's calculate  $\mathbf{E}[XQ]$ :

$$\mathbf{E}[XQ] = \mathbf{E}[\mathbf{E}[XQ \mid Q]] = \mathbf{E}[Q\mathbf{E}[X \mid Q]] = \mathbf{E}[Q^2] = \frac{1}{3}.$$

Therefore, we have

$$\text{cov}(X, Q) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

(c) (8 points)

Using Bayes' Rule, we have

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{p_X(1)} = \frac{f_Q(q)\mathbf{P}(X=1 \mid Q=q)}{\mathbf{P}(X=1)}, \quad 0 \leq q \leq 1.$$

Additionally, we know that

$$\mathbf{P}(X=1 \mid Q=q) = q,$$

and that for Bernoulli random variables

$$\mathbf{P}(X=1) = \mathbf{E}[X] = \frac{1}{2}.$$

Thus, the conditional PDF of  $Q$  given  $X=1$  is

$$\begin{aligned} f_{Q|X}(q \mid 1) &= \frac{1 \cdot q}{1/2} \\ &= \begin{cases} 2q, & 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

**Problem 5.** (21 points)

(a) (7 points)

$$\begin{aligned} \mathbf{P}(S \geq 1) &= \mathbf{P}(\min\{X, Y\} \geq 1) = \mathbf{P}(X \geq 1 \text{ and } Y \geq 1) = \mathbf{P}(X \geq 1)\mathbf{P}(Y \geq 1) \\ &= (1 - F_X(1))(1 - F_Y(1)) = (1 - \Phi(1))^2 \approx (1 - 0.8413)^2 \approx 0.0252. \end{aligned}$$

(b) (7 points)

Recalling Problem 2 of Problem Set 6, we have

$$\begin{aligned}\mathbf{P}(s \leq S \text{ and } L \leq \ell) &= \mathbf{P}(s \leq \min\{X, Y\} \text{ and } \max\{X, Y\} \leq \ell) \\ &= \mathbf{P}(s \leq X \text{ and } s \leq Y \text{ and } X \leq \ell \text{ and } Y \leq \ell) \\ &= \mathbf{P}(s \leq X \leq \ell) \mathbf{P}(s \leq Y \leq \ell) \\ &= (F_X(\ell) - F_X(s))(F_Y(\ell) - F_Y(s)).\end{aligned}$$

(c) (7 points)

Given that  $s \leq s + \delta \leq \ell$ , the event  $\{s \leq S \leq s + \delta, \ell \leq L \leq \ell + \delta\}$  is made up of the union of two disjoint possible events:

$$\{s \leq X \leq s + \delta, \ell \leq Y \leq \ell + \delta\} \cup \{s \leq Y \leq s + \delta, \ell \leq X \leq \ell + \delta\}.$$

In other words, either  $S = X$  and  $L = Y$ , or  $S = Y$  and  $L = X$ . Because the two events are disjoint, the probability of their union is equal to the sum of their individual probabilities.

Using also the independence of  $X$  and  $Y$ , we have

$$\begin{aligned}\mathbf{P}(s \leq S \leq s + \delta, \ell \leq L \leq \ell + \delta) &= \mathbf{P}(s \leq X \leq s + \delta, \ell \leq Y \leq \ell + \delta) \\ &\quad + \mathbf{P}(s \leq Y \leq s + \delta, \ell \leq X \leq \ell + \delta) \\ &= \mathbf{P}(s \leq X \leq s + \delta) \mathbf{P}(\ell \leq Y \leq \ell + \delta) \\ &\quad + \mathbf{P}(s \leq Y \leq s + \delta) \mathbf{P}(\ell \leq X \leq \ell + \delta) \\ &= \int_s^{s+\delta} f_X(x) dx \int_\ell^{\ell+\delta} f_Y(y) dy \\ &\quad + \int_s^{s+\delta} f_Y(y) dy \int_\ell^{\ell+\delta} f_X(x) dx\end{aligned}$$