LECTURE 18

Markov Processes – III

Readings: Section 7.4

Lecture outline

- Review of steady-state behavior
- Probability of blocked phone calls
- Calculating absorption probabilities
- Calculating expected time to absorption

Review

- Assume a single class of recurrent states, aperiodic;
  plus transient states. Then,

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j$$

where π_j does not depend on the initial conditions:

$$\lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = i) = \pi_j$$

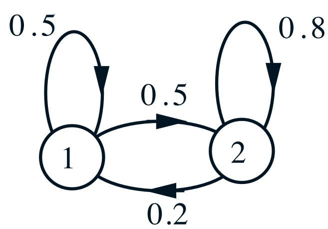
- $\pi_1, \ldots, \pi_m$ can be found as the unique solution to the balance equations

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$

together with

$$\sum_j \pi_j = 1$$

Example

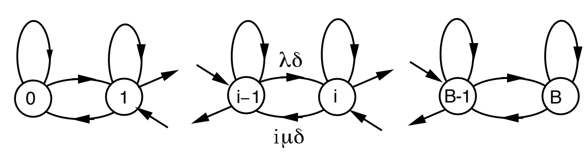


$\pi_1 = 2/7$, $\pi_2 = 5/7$

- Assume process starts at state 1.
- $\mathbf{P}(X_1 = 1,$ and $X_{100} = 1) =$
- $\mathbf{P}(X_{100} = 1$ and $X_{101} = 2)$

The phone company problem

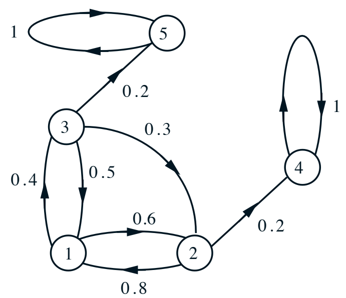
- Calls originate as a Poisson process, rate λ
  - Each call duration is exponentially distributed (parameter μ)
  - B lines available
- Discrete time intervals of (small) length δ



- Balance equations: $\lambda \pi_{i-1} = i\mu\pi_i$
- $\pi_i = \pi_0 \dfrac{\lambda^i}{\mu^i i!} \qquad \pi_0 = 1 / \sum_{i=0}^{B} \dfrac{\lambda^i}{\mu^i i!}$

Calculating absorption probabilities

- What is the probability a_i that: process eventually settles in state 4, given that the initial state is i?

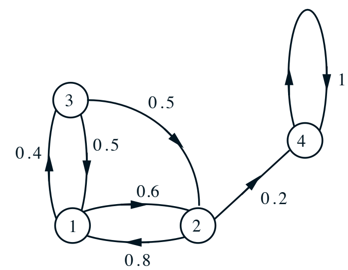


For $i = 4$, $a_i =$
For $i = 5$, $a_i =$

$$a_i = \sum_j p_{ij} a_j, \quad \text{for all other } i$$

– unique solution

Expected time to absorption



- Find expected number of transitions μ_i, until reaching the absorbing state, given that the initial state is i?

$\mu_i = 0$ for $i =$

For all other i: $\mu_i = 1 + \sum_j p_{ij} \mu_j$

– unique solution

Mean first passage and recurrence times

- Chain with one recurrent class; fix s recurrent
- **Mean first passage time from i to s:**

  $t_i \;=\; \mathbf{E}[\min\{n \geq 0 \text{ such that } X_n = s\} \mid X_0 = i]$

- $t_1, t_2, \ldots, t_m$ are the unique solution to

$$\begin{aligned} t_s &= 0, \\ t_i &= 1 + \sum_j p_{ij} t_j, \qquad \text{for all } i \neq s \end{aligned}$$

- **Mean recurrence time of s:**

  $t_s^* \;=\; \mathbf{E}[\min\{n \geq 1 \text{ such that } X_n = s\} \mid X_0 = s]$

- $t_s^* \;=\; 1 + \sum_j p_{sj} t_j$