Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2009)

Problem Set 6

Due: April 1st, 2009

- **Text Sections: 4.1, 4.2, 4.4, 8.3, 8.4**

1. Let X be a standard normal random variable, i.e. X has mean 0 and variance 1 with normal distribution. Then, find the density function of Y where :

   (a) $Y = \sqrt{|X|}$

   (b) $Y = -\ln |X|$

2. An ambulance travels back and forth, at a known constant speed v, along a road of length ℓ. In other words, at any moment in time, consider the location of the ambulance to be uniformly distributed over the interval $(0, \ell)$, and it is equally likely to be traveling in either direction. Also at some moment in time, an accident (not involving the ambulance itself) occurs at a point uniformly distributed on the road; that is, the accident's distance from the starting end of the road is also uniformly distributed over the interval $(0, \ell)$. Assume the location of the accident and the location of the ambulance are independent.

   Supposing the ambulance is capable of immediate U-turns, compute the CDF and PDF of the ambulance's travel time T to the location of the accident.

3. Suppose X is a unit normal random variable. Define a new random variable Y such that:

   $$Y = a + bX + cX^2.$$

   Find the correlation coefficient ρ for X, Y. Find the best linear estimator for Y based on observation of X.

4. Suppose $X \sim N(0, 1)$ and $Z = 0$ or 1 with equal probability. Now consider a random variable Y such that:

   $$Y = \begin{cases} X & \text{if } Z = 1 \\ -X & \text{if } Z = 0 \end{cases}$$

   (a) Are X, Y independent?

   (b) Are Y, Z independent?

   (c) Show that $Y \sim N(0, 1)$.

   (d) Show that $\text{cov}(X, Y) = 0$.

5. Let $M(s) = \frac{c}{\sqrt{s+1}}$ be the transform of a real valued random variable X for $s \in (-0.5, 0.5)$.

   (a) Compute c.

   (b) Compute the mean and variance of X using its transform $M(s)$.

6. X and Y are continuous, independent random variables. The transform of X is given by $M_X(s) = \frac{1}{s}(e^{4s} - e^{3s})$, and the distribution of Y is given by

   $$f_Y(y) = \begin{cases} 3c, & \text{for } -2 \le y \le -1, \\ c, & \text{for } 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the numerical value of the constant c.

(b) Compute the transform $M_Y(s)$.

(c) Find the mean and variance of Y.

(d) Find the transform $M_W(s)$, where $W = \alpha X + \beta Y + \gamma$.

(e) Determine the PDF of W for the case where $\alpha = 1$, $\beta = 1$, and $\gamma = 0$.

G1†. (a) Consider a deterministic random variable $X = a$, for some constant a. Find its transform $M_X(s)$ for $s \in \mathbb{R}$.

(b) Suppose Y is a random variable that takes values in $[0, L]$ for some finite number $L > 0$. Argue that for any $s \in (-\delta, \delta)$, $M_Y(s)$ is well defined for some $\delta > 0$.

(c) Let

$$S_n = \frac{1}{n}(Y_1 + \cdots + Y_n),$$

where $Y_1, \cdots, Y_n$ are independent and identically distributed random variables with Y_1 having the same distribution as Y above.

Show that

$$M_{S_n}(s) \to e^{s\mathbf{E}[Y]}$$

as $n \to \infty$ for a given $s \in (-\delta, \delta)$, where $\delta > 0$ is as defined in (b). What does this suggest?

†Required for 6.431; optional for 6.041