LECTURE 7

- **Readings:** Finish Chapter 2

Lecture outline

- Review
- Multiple random variables
  - Joint PMF
  - Conditioning
  - Independence
- Binomial distribution revisited
- A hat problem

Review

- Conditioning on A with $\mathbf{P}(A) > 0$ gives a probability law
  - Conditional PMF:

$$p_{X|A}(x) = \mathbf{P}(X = x \mid A)$$

  - Conditional expectation:

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x)$$

- Pair of discrete random variables have a joint PMF:

$$p_{X,Y}(x, y) = \mathbf{P}(\{X = x\} \cap \{Y = y\})$$

Marginalization

$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

Expected value rule

- Given a function g and random variables X and Y, $g(X, Y)$ is a random variable
- $\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$
- $\mathbf{E}[aX + bY + c] =$

Conditional PMF

- If $\mathbf{P}(\{Y = y\}) > 0$, we can condition on event $\{Y = y\}$:

$$p_{X|\{Y=y\}}(x) = \mathbf{P}(\{X = x\} \mid \{Y = y\}) = \frac{\mathbf{P}(\{X = x\} \cap \{Y = y\})}{\mathbf{P}(\{Y = y\})}$$

More compactly:

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} \qquad \text{(defined where } p_Y(y) > 0\text{)}$$

- Multiplication rule for random variables:

$$p_{X,Y}(x, y) = p_Y(y)\, p_{X|Y}(x \mid y) \qquad \text{and}$$
$$p_{X,Y}(x, y) = p_X(x)\, p_{Y|X}(y \mid x)$$

(where the conditional PMF is defined)

Bayes' rule

- Multiplication rule for random variables:

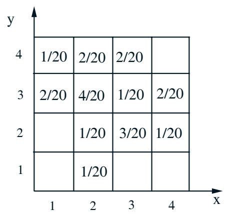
$$p_{X,Y}(x, y) = p_Y(y)\, p_{X|Y}(x \mid y) \qquad \text{and}$$
$$p_{X,Y}(x, y) = p_X(x)\, p_{Y|X}(y \mid x)$$

- Bayes' rule for random variables:

$$p_{X|Y}(x \mid y) = \frac{p_X(x)\, p_{Y|X}(y \mid x)}{p_Y(y)}$$

(where the conditional PMF is defined)

Joint, marginal, and conditional PMFs



Independence of random variable and event

- Recall: Events A and $\{X = x\}$ are independent when

$$\mathbf{P}(A \cap \{X = x\}) \;=\; \mathbf{P}(A)\mathbf{P}(\{X = x\}) \quad \Big[\; = \; \mathbf{P}(A) p_X(x) \;\Big]$$

– When $\mathbf{P}(A) > 0$, this is

$$p_{X|A}(x) \;=\; p_X(x)$$

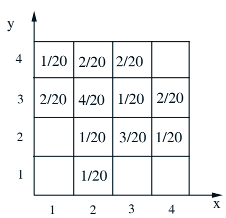
- Define: Event A and r.v. X are called *independent* when

$$\mathbf{P}(A \cap \{X = x\}) \;=\; \mathbf{P}(A) p_X(x) \qquad \text{for all } x$$

– When $\mathbf{P}(A) > 0$, this is

$$p_{X|A}(x) \;=\; p_X(x) \qquad \text{for all } x$$

Independence of random variable and event: Example



Independence of random variables

- Recall: Events $\{X = x\}$ and $\{Y = y\}$ are independent when

$$\mathbf{P}(\{X = x\} \cap \{Y = y\}) \;=\; \mathbf{P}(\{X = x\})\mathbf{P}(\{Y = y\})$$

or

$$p_{X,Y}(x, y) \;=\; p_X(x) p_Y(y)$$

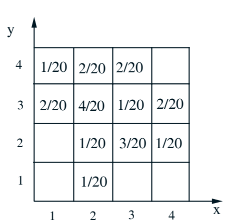
- Define: Random variables X and Y called *independent* when

$$p_{X,Y}(x, y) \;=\; p_X(x)\, p_Y(y) \qquad \text{for all } x \text{ and } y$$

- Random variables X, Y, Z are called *independent* when

$$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z) \qquad \text{for all } x, y, z$$

Independence of random variables: Example



Expectations

- When X, Y are independent,

$$\mathbf{E}[g(X)h(Y)] \;=\; \mathbf{E}[g(X)]\,\mathbf{E}[h(Y)]$$

$$\text{Var}(X + Y) \;=\; \text{Var}(X) + \text{Var}(Y)$$

Binomial mean and variance

- $X = \#$ of successes in n independent trials
  - probability of success p

$$\mathbf{E}[X] \;=\; \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k}$$

- $X_i = \begin{cases} 1, & \text{if success in trial } i; \\ 0, & \text{otherwise} \end{cases}$

- $\mathbf{E}[X_i] =$
- $\text{Var}(X_i) =$

- $\mathbf{E}[X] =$
- $\text{Var}(X) =$

A hat problem

- n people throw their hats in a box and pick one at random
  - X: number of people who get their own hat
  - Find $\mathbf{E}[X]$
- Daunting approach: Find $p_X(x)$ then compute $\mathbf{E}[X]$

A hat problem

- Define indicator variables:

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat;} \\ 0, & \text{otherwise.} \end{cases}$$

- $X = X_1 + X_2 + \cdots + X_n$
- $\mathbf{P}(X_i = 1) =$
- $\mathbf{E}[X_i] =$
- Are the X_is independent?
- $\mathbf{E}[X] =$

Variance in the hat problem

- $\text{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

$$X^2 = \sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j$$

- $\mathbf{E}[X_i^2] \;=\; \mathbf{P}(X_i = 0) \;=$
- For $i \neq j$,

$$\begin{aligned} \mathbf{E}[X_i X_j] &= \mathbf{P}(X_i = 1, X_j = 1) \\ &= \mathbf{P}(X_i = 1)\,\mathbf{P}(X_j = 1 \mid X_1 = 1) \\ &= \end{aligned}$$

- $\text{Var}(X) =$