Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2009)

Problem Set 7
Due November 9, 2009

1. You are visiting the rainforest, but unfortunately your insect repellent has run out. As a result, at each second, a mosquito lands on your neck with probability 0.5. If one lands, with probability 0.2 it bites you, and with probability 0.8 it never bothers you, independently of other mosquitoes. What is the expected time between successive bites? What is the variance of the time between successive bites?

2. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

   (a) Determine the probability that Fred gives away his first sample on his third call.

   (b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

   (c) Determine the probability that he gives away his second sample on his fifth call.

   (d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

   (e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

   (f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

3. All ships travel at the same speed through a wide canal. Eastbound ship arrivals at the canal are a Poisson process with an average arrival rate λ_E ships per day. Westbound ships arrive as an independent Poisson process with average arrival rate λ_W per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass it. Each ship takes t days to traverse the length of the canal.

   (a) Given that the pointer is pointing west:

      i. What is the probability that the next ship to pass it will be westbound?

      ii. What is the PDF for the remaining time until the pointer changes direction?

   (b) What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

   (c) We begin observing at an arbitrary time. Let V be the time we have to continue observing until we see the seventh eastbound ship. Determine the PDF for V.

4. Consider two independent Poisson processes, with arrival rates α and β, respectively. Determine:

   (a) The probability q that the next three arrivals come from the same process.

(b) The PMF of N, the number of arrivals from the first process that occur before the fourth arrival from the second process.

5. (a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)
   (b) Now suppose that the shuttles are no longer operating on a deterministic schedule, but rather their interdeparture times are independent and exponentially distributed with rate μ per hour. Find the PMF for the number of shuttles departing in one hour.
   (c) Let us define an "event" to be either the arrival of a passenger, or the departure of a shuttle (or both simultaneously). With the same assumptions as in (b) above, find the expected number of "events" that occur in one hour.
   (d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.
   (e) Find the PMF for the number of people on a shuttle.

6. A single dot is placed on a very long length of yarn at the textile mill. The yarn is then cut into lengths requested by different customers. The lengths are independent of each other, but all distributed according to the PDF $f_L(l)$. Let R be be the length of yarn purchased by that customer whose purchase included the dot. Determine the expected value of R in the following cases:
   (a) $f_L(l) = \lambda e^{-\lambda l}, \quad l \geq 0$
   (b) $f_L(l) = \frac{\lambda^3 l^2 e^{-\lambda l}}{2}, \quad l \geq 0$

7. Consider a Poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t)$ and M be the number of arrivals in $(0, t + s)$, where $t > 0$ and $s > 0$.
   (a) Find the joint PMF of N and M, $p_{N,M}(n, m)$.
   (b) Find $\mathbf{E}[NM]$.