Recitation 9 Solutions
October 6, 2011

1. a. To find the marginal distribution of X, you can easily sum over Y from the joint, $p_X(x) = \sum_{y=1}^{3} p_{X,Y}(x,y)$. Given the above table, we simply sum the columns to find $p_X(x)$. This leads to

$$p_X(x) = \begin{cases} 1/3, & \text{for } x = 1 \\ 1/9, & \text{for } x = 2 \\ 1/3, & \text{for } x = 3 \\ 2/9, & \text{for } x = 4 \\ 0, & \text{otherwise.} \end{cases}$$

b. Converting the Boolean expressions we find

$$\begin{aligned} A \cap B^c &= \{(2,1),(2,2),(3,1),(3,2),(4,1),(4,2)\} \\ (A \cap B^c) \cup C &= \{(2,1),(2,2),(3,1),(3,2),(4,1),(4,2),(4,3)\} \\ ((A \cap B^c) \cup C)^c &= \{(1,1),(1,2),(1,3),(2,3),(3,3)\} \end{aligned}$$

Summing these we find $\mathbf{P}(D) = 4/9$.

c. First we find the marginal PMF of Y,

$$p_Y(y) = \begin{cases} 7/18, & \text{y=1} \\ 2/9, & \text{y=2} \\ 7/18, & \text{y=3} \\ 0, & \text{otherwise,} \end{cases}$$

and the conditional PMF of Y,

$$p_{Y|A}(y) = \begin{cases} 7/11, & \text{y=1} \\ 4/11, & \text{y=2} \\ 0, & \text{otherwise.} \end{cases}$$

Now we have

$$\mathbf{E}[Y \mid A] = 2 \cdot p_{Y|A}(2) + 1 \cdot p_{Y|A}(1) = 8/11 + 7/11 = 15/11.$$

d. Yes they are independent. We write the joint conditional distribution

$y = 3$	3/10	1/5
$y = 2$	0	0
$y = 1$	3/10	1/5
	$x = 1$	$x = 4$

$p_{X,Y|E}(x,y)$, the conditional joint PMF of X and Y.

and the marginal conditional distribution of X and Y given E,

$$p_{X|E}(x) = \begin{cases} 3/5, & \text{x=1} \\ 2/5, & \text{x=4} \\ 0, & \text{otherwise,} \end{cases}$$

$$p_{Y|E}(y) = \begin{cases} 1/2, & \text{y=1} \\ 0, & \text{y=2} \\ 1/2, & \text{y=3} \\ 0, & \text{otherwise.} \end{cases}$$

Now we obtain that $p_{X,Y|E}(x,y) = p_{X|E}(x)p_{Y|E}(y)$, which proves the conditional independence of X and Y given E.

2. a. The number of apples received is a binomial random variable, so the probability is

$$\mathbf{P}(3 \text{ apples received}) = \binom{10}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^7.$$

b. X is a geometric random variable, thus $\mathbf{E}[X] = \frac{1}{p} = 3$.

c. The probability of any particular ordering is the same, similar to coin tossing problems. Thus we can use counting to find the desired probability. For the denominator count the number of ways you can choose 3 apples from 10 pieces of fruit $= \binom{10}{3}$. For the numerator count the total number of ways you can choose the 2 apples in the first 5, and then one apple in the last $5 = \binom{5}{2}\binom{5}{1}$. Thus,

$$\frac{\binom{5}{2}\binom{5}{1}}{\binom{10}{3}} = \frac{5}{12}.$$

Note you could also count oranges instead of apples with the same result.

d. Let W_i be the cost of the ith piece of fruit. Note $T = \sum_{i=1}^{10} W_i$. Using the linearity of expectation, and the identical distribution of W_i $\forall i$ we find

$$\mathbf{E}[T] = \sum_{i=1}^{10} \mathbf{E}[W_i] = 10\mathbf{E}[W] = 10\left(3\frac{2}{3} + 15\frac{1}{3}\right) = 10(2+5) = 70.$$

e. We now include the independence of the W_i's to the previous statements to claim the variance of a sum of W_i's is equal to the sum of variances.

$$\text{var}(T) = \text{var}(\sum_{i=1}^{10} W_i) = \sum_{i=1}^{10} \text{var}(W_i) = 10\text{var}(W).$$

To find the variance of W, we exploit $\text{var}(W) = \mathbf{E}[W^2] - \mathbf{E}[W]^2$. We have $\mathbf{E}[W] = 7$ from the previous part, thus we only require $\mathbf{E}[W^2]$ which is easily computed as

$$\begin{aligned} \mathbf{E}[W^2] &= 3^2\left(\frac{2}{3}\right) + 15^2\left(\frac{1}{3}\right) \\ &= 6 + 75 = 81. \end{aligned}$$

Putting it all together, we find

$$\begin{aligned} \text{var}(T) &= 10\text{var}(W) \\ &= 10(81 - 49) = 320. \end{aligned}$$

f. One approach is to use counting. The number of ways that you and Mais choose adjacent seats is $2(k-1)$, and the number of all ways of seating is $2\binom{k}{2}$. Therefore

$$\mathbf{P}(\text{You and Mais choose adjacent seats}) = \frac{2(k-1)}{2\binom{k}{2}} = \frac{2}{k}.$$

Another approach is to use sequential probability. Denote the seat you choose as A and the seat Mais choose as B. There are three ways for Mais and you to sit together: you choose the 1st seat and Mais chooses the 2nd seat; you choose the last seat and Mais chooses the second last seat; you choose a seat which is not at either end and Mais chooses the seat at the left or right to you. We have

$$\begin{aligned}
\mathbf{P}(A, B \text{ are adjacent seats}) &= \sum_{i=2}^{k-1} \mathbf{P}(A=i)(\mathbf{P}(B=i+1)+\mathbf{P}(B=i-1)) \\
&\quad + \mathbf{P}(A=1)\mathbf{P}(B=2) + \mathbf{P}(A=k)\mathbf{P}(B=k-1) \\
&= (k-2)\left(\frac{1}{k}\cdot\frac{1}{k-1}\right) + \frac{1}{k}\cdot\frac{1}{k-1} + \frac{1}{k}\cdot\frac{1}{k-1} \\
&= 2/k.
\end{aligned}$$