

## LECTURE 12

- **Readings:** Section 4.3; parts of Section 4.5 (mean and variance only; no transforms)

### Lecture outline

- Conditional expectation
  - Law of iterated expectations
  - Law of total variance
- Sum of a random number of independent r.v.'s
  - mean, variance

## Conditional expectations

- Given the value  $y$  of a r.v.  $Y$ :

$$\mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

(integral in continuous case)

- Stick example: stick of length  $\ell$   
break at uniformly chosen point  $Y$   
break again at uniformly chosen point  $X$
- $\mathbf{E}[X \mid Y = y] = \frac{y}{2}$  (number)

$$\mathbf{E}[X \mid Y] = \frac{Y}{2} \text{ (r.v.)}$$

- **Law of iterated expectations:**

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \sum_y \mathbf{E}[X \mid Y = y] p_Y(y) = \mathbf{E}[X]$$

- In stick example:  
 $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]] = \mathbf{E}[Y/2] = \ell/4$

### $\text{var}(X \mid Y)$ and its expectation

- $\text{var}(X \mid Y = y) = \mathbf{E}[(X - \mathbf{E}[X \mid Y = y])^2 \mid Y = y]$
- $\text{var}(X \mid Y)$ : a r.v. with value  $\text{var}(X \mid Y = y)$  when  $Y = y$
- **Law of total variance:**  
 $\text{var}(X) = \mathbf{E}[\text{var}(X \mid Y)] + \text{var}(\mathbf{E}[X \mid Y])$

### Proof:

- (a) Recall:  $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$
- (b)  $\text{var}(X \mid Y) = \mathbf{E}[X^2 \mid Y] - (\mathbf{E}[X \mid Y])^2$
- (c)  $\mathbf{E}[\text{var}(X \mid Y)] = \mathbf{E}[X^2] - \mathbf{E}[(\mathbf{E}[X \mid Y])^2]$
- (d)  $\text{var}(\mathbf{E}[X \mid Y]) = \mathbf{E}[(\mathbf{E}[X \mid Y])^2] - (\mathbf{E}[X])^2$

Sum of right-hand sides of (c), (d):  
 $\mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \text{var}(X)$

## Section means and variances

Two sections:

$y = 1$  (10 students);  $y = 2$  (20 students)

$$y = 1: \frac{1}{10} \sum_{i=1}^{10} x_i = 90 \quad y = 2: \frac{1}{20} \sum_{i=11}^{30} x_i = 60$$

$$\mathbf{E}[X] = \frac{1}{30} \sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$$

$$\mathbf{E}[X \mid Y = 1] = 90, \quad \mathbf{E}[X \mid Y = 2] = 60$$

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbf{E}[\mathbf{E}[X \mid Y]] = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 60 = 70 = \mathbf{E}[X]$$

$$\begin{aligned} \text{var}(\mathbf{E}[X \mid Y]) &= \frac{1}{3}(90 - 70)^2 + \frac{2}{3}(60 - 70)^2 \\ &= \frac{600}{3} = 200 \end{aligned}$$

### Section means and variances (ctd.)

$$\frac{1}{10} \sum_{i=1}^{10} (x_i - 90)^2 = 10 \quad \frac{1}{20} \sum_{i=11}^{30} (x_i - 60)^2 = 20$$

$$\text{var}(X | Y = 1) = 10 \quad \text{var}(X | Y = 2) = 20$$

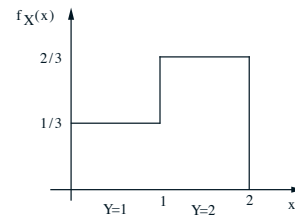
$$\text{var}(X | Y) = \begin{cases} 10, & \text{w.p. } 1/3 \\ 20, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbb{E}[\text{var}(X | Y)] = \frac{1}{3} \cdot 10 + \frac{2}{3} \cdot 20 = \frac{50}{3}$$

$$\begin{aligned} \text{var}(X) &= \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y]) \\ &= \frac{50}{3} + 200 \\ &= (\text{average variability } \mathbf{within} \text{ sections}) \\ &\quad + (\text{variability } \mathbf{between} \text{ sections}) \end{aligned}$$

### Example

$$\text{var}(X) = \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y])$$



$$\mathbb{E}[X | Y = 1] = \quad \mathbb{E}[X | Y = 2] =$$

$$\text{var}(X | Y = 1) = \quad \text{var}(X | Y = 2) =$$

$$\mathbb{E}[X] =$$

$$\text{var}(\mathbb{E}[X | Y]) =$$

### Sum of a random number of independent r.v.'s

- $N$ : number of stores visited  
( $N$  is a nonnegative integer r.v.)
- $X_i$ : money spent in store  $i$ 
  - $X_i$  assumed i.i.d.
  - independent of  $N$
- Let  $Y = X_1 + \dots + X_N$ 

$$\begin{aligned} \mathbb{E}[Y | N = n] &= \mathbb{E}[X_1 + X_2 + \dots + X_n | N = n] \\ &= \mathbb{E}[X_1 + X_2 + \dots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n] \\ &= n \mathbb{E}[X] \end{aligned}$$
- $\mathbb{E}[Y | N] = N \mathbb{E}[X]$

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y | N]] \\ &= \mathbb{E}[N \mathbb{E}[X]] \\ &= \mathbb{E}[N] \mathbb{E}[X] \end{aligned}$$

### Variance of sum of a random number of independent r.v.'s

- $\text{var}(Y) = \mathbb{E}[\text{var}(Y | N)] + \text{var}(\mathbb{E}[Y | N])$
  - $\mathbb{E}[Y | N] = N \mathbb{E}[X]$   
 $\text{var}(\mathbb{E}[Y | N]) = (\mathbb{E}[X])^2 \text{var}(N)$
  - $\text{var}(Y | N = n) = n \text{var}(X)$   
 $\text{var}(Y | N) = N \text{var}(X)$   
 $\mathbb{E}[\text{var}(Y | N)] = \mathbb{E}[N] \text{var}(X)$
- $$\begin{aligned} \text{var}(Y) &= \mathbb{E}[\text{var}(Y | N)] + \text{var}(\mathbb{E}[Y | N]) \\ &= \mathbb{E}[N] \text{var}(X) + (\mathbb{E}[X])^2 \text{var}(N) \end{aligned}$$