Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Problem Set 2
Due: September 22, 2011

1. Most mornings, Paul checks the weather report before deciding whether to carry an umbrella. If the forecast is "rain," the probability of actually having rain that day is 75%. On the other hand, if the forecast is "no rain," the probability of it actually raining is equal to 20%. During fall and winter the forecast is "rain" 70% of the time and during summer and spring it is 20%.

   (a) One day, Paul missed the forecast and it rained. What is the probability that the forecast was "rain" if it was during the winter? What is the probability that the forecast was "rain" if it was during the summer?

   (b) The probability of Paul missing the morning forecast is equal to 0.25 on any day in the year. If he misses the forecast, Paul will flip a fair coin to decide whether to carry an umbrella. On any day of a given season he sees the forecast, if it says "rain" he will always carry an umbrella, and if it says "no rain," he will not carry an umbrella. Are the events "Paul is carrying an umbrella," and "The forecast is no rain" independent? Does your answer depend on the season?

   (c) Paul is carrying an umbrella and it is not raining. What is the probability that he saw the forecast? Does it depend on the season?

2. You have a fair five-sided die. The sides of the die are numbered from 1 to 5. Each die roll is independent of all others, and all faces are equally likely to come out on top when the die is rolled. Suppose you roll the die twice.

   (a) Let event A be "the total of two rolls is 10", event B be "at least one roll resulted in 5", and event C be "at least one roll resulted in 1".

      i. Is event A independent of event B?

      ii. Is event A independent of event C?

   (b) Let event D be "the total of two rolls is 7", event E be "the difference between the two roll outcomes is exactly 1", and event F be "the second roll resulted in a higher number than the first roll".

      i. Are events E and F independent?

      ii. Are events E and F independent given event D?

3. As a marketing gimmick, the local produce market decided to give away 400 red apples and 600 green apples to unfortunate school children, because some of them are rotten. In fact, 10% of red apples and 30% of green apples are rotten. An unsuspecting child heads to the local market to collect his share, and decides to flip a fair coin to choose between red and green apples. After the child picks a color, he receives 3 apples of that color selected at random.

   (a) What is the probability that all 3 apples are rotten?

   (b) Given that the 3 apples are rotten, what is the probability that they are red?

4. Oscar has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6).

   On any given day, if the dog is in A and Oscar spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Oscar spends a day looking for it there, the conditional probability that he will find the dog that day is 0.15.

   The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only at night.

   (a) In which forest should Oscar look to maximize the probability he finds his dog on the first day of the search?

   (b) Given that Oscar looked in A on the first day but didn't find his dog, what is the probability that the dog is in A?

   (c) If Oscar flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

   (d) If the dog is alive and not found by the Nth day of the search, it will die that evening with probability $\frac{N}{N+2}$. Oscar has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?

5. For each of the following statements provide either a proof or a counterexample.

   (a) Suppose that events A, B, and C are independent. Then, $A \cap B^c$ is independent of C.

   (b) Suppose that events A, B, and C are independent. Then, A and B are conditionally independent given C.

   (c) Suppose that events A, B, and C are pairwise independent. Then, A and B are conditionally independent given C.

   (d) Suppose that events A and B are conditionally independent, given a third event C. Then, A and B are conditionally independent, given the event C^c.

6. Three people each roll a fair n-sided die once. Let A_{ij} be the event that person i and person j roll the same face. Show that the events A_{12}, A_{13}, and A_{23} are pairwise independent but are not independent. Assume $n > 1$!

G1†. Prove or disprove the following statement:

If A_1, A_2, and A_3 are events such that

$$\mathbf{P}(A_1 \mid A_2) > \mathbf{P}(A_1) \quad \text{and} \quad \mathbf{P}(A_2 \mid A_3) > \mathbf{P}(A_2),$$

then

$$\mathbf{P}(A_1 \mid A_3) > \mathbf{P}(A_1).$$

(To "prove," give a clear and convincing argument. To "disprove," provide a counterexample and an explanation of why it is a counterexample.)

G2†. In solving this problem, feel free to browse problems 43-45 in Chapter 1 of the text for ideas. If you need to, you may quote the results of these problems.

(a) Suppose that A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

(b) Prove that if $A_1, \ldots, A_n$ are independent events, then

$$\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \prod_{i=1}^{n}(1 - \mathbf{P}(A_i)).$$

†Required for 6.431; optional for 6.041