Problem Set 7: Solutions
Due November 9, 2009

1. Let X = time between successive bites.
   Let G = time until the next bite.
   We have $X = G - 1$.

   The mosquito bites occur according to a Bernoulli process with parameter $p = 0.1$. G is a geometric random variable, so we have

$$\mathbf{E}[G] = \frac{1}{p} = \frac{1}{0.1} = 10$$

$$\mathbf{E}[X] = \mathbf{E}[G-1] = 9$$

$$\text{var}(X) = \text{var}(G-1) = \text{var}(G) = \frac{1-p}{p^2} = \frac{1-0.1}{0.1^2} = 90.$$

2. A successful call occurs with probability $p = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

   (a) Fred will give away his first sample on the third call if the first two calls are failures and the third is a success. Since the trials are independent, the probability of this sequence of events is simply

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (b) The event of interest requires failures on the ninth and tenth trials and a success on the eleventh trial. For a Bernoulli process, the outcomes of these three trials are independent of the results of any other trials and again our answer is

$$(1-p)(1-p)p = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

   (c) We desire the probability that L_2, the second-order interarrival time is equal to five trials. We know that $p_{L_2}(l)$ is a Pascal PMF, and we have

$$p_{L_2}(5) = \binom{5-1}{2-1} p^2 (1-p)^{5-2} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}$$

   (d) Here we require the conditional probability that the experimental value of L_2 is equal to 5, given that it is greater than 2.

$$\begin{aligned} p_{L_2|L_2>2}(5|L_2>2) &= \frac{p_{L_2}(5)}{\mathbf{P}(L_2>2)} = \frac{p_{L_2}(5)}{1-p_{L_2}(2)} \\ &= \frac{\binom{5-1}{2-1}p^2(1-p)^{5-2}}{1-\binom{2-1}{2-1}p^2(1-p)^0} = \frac{4\cdot\left(\frac{1}{2}\right)^5}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{6} \end{aligned}$$

(e) The probability that Fred will complete at least five calls before he needs a new supply is equal to the probability that the experimental value of L_2 is greater than or equal to 5.

$$\begin{aligned}\mathbf{P}(L_2 \geq 5) = 1 - \mathbf{P}(L_2 \leq 4) &= 1 - \sum_{l=2}^{4} \binom{l-1}{2-1} p^2 (1-p)^{l-2} \\ &= 1 - \left(\frac{1}{2}\right)^2 - \binom{2}{1}\left(\frac{1}{2}\right)^3 - \binom{3}{1}\left(\frac{1}{2}\right)^4 = \frac{5}{16}\end{aligned}$$

(f) Let discrete random variable F represent the number of failures before Fred runs out of samples on his mth successful call. Since L_m is the number of trials up to and including the mth success, we have $F = L_m - m$. Given that Fred makes L_m calls before he needs a new supply, we can regard each of the F unsuccessful calls as trials in another Bernoulli process with parameter r, where r is the probability of a success (a disappointed dog) obtained by

$$\begin{aligned} r &= \mathbf{P}(\text{dog lives there} \mid \text{Fred did not leave a sample}) \\ &= \frac{\mathbf{P}(\text{dog lives there AND door not answered})}{1 - \mathbf{P}(\text{giving away a sample})} = \frac{\frac{1}{4}\cdot\frac{2}{3}}{1-\frac{1}{2}} = \frac{1}{3}\end{aligned}$$

We define X to be a Bernoulli random variable with parameter r. Then, the number of dogs passed up before Fred runs out, D_m, is equal to the sum of F Bernoulli random variables each with parameter $r = \frac{1}{3}$, where F is a random variable. In other words,

$$D_m = X_i + X_2 + X_3 + \cdots + X_F.$$

Note that D_m is a sum of a random number of independent random variables. Further, F is independent of the X_i's since the X_i's are defined in the conditional universe where the door is not answered, in which case, whether there is a dog or not does not affect the probability of that trial being a failed trial or not. From our results in class, we can calculate its expectation and variance by

$$\begin{aligned}\mathbf{E}[D_m] &= \mathbf{E}[F]\mathbf{E}[X], \\ \text{var}(D_m) &= \mathbf{E}[F]\text{var}(X) + (\mathbf{E}[X])^2 \text{var}(F),\end{aligned}$$

where we make the following substitutions:

$$\begin{aligned}\mathbf{E}[F] &= \mathbf{E}[L_m - m] = \frac{m}{p} - m = m, \\ \text{var}(F) &= \text{var}(L_m - m) = \text{var}(L_m) = \frac{m(1-p)}{p^2} = 2m, \\ \mathbf{E}[X] &= r = \frac{1}{3}, \\ \text{var}(X) &= r(1-r) = \frac{2}{9}.\end{aligned}$$

Finally, substituting these values, we have

$$\begin{aligned}\mathbf{E}[D_m] &= m \cdot \frac{1}{3} = \frac{m}{3} \\ \text{var}(D_m) &= m \cdot \frac{2}{9} + \left(\frac{1}{3}\right)^2 \cdot 2m = \frac{4m}{9}\end{aligned}$$

3. (a) We are given that the previous ship to pass the pointer was traveling westward.

   i. The direction of the next ship is independent of those of any previous ships. Therefore, we are simply looking for the probability that a westbound arrival occurs before an eastbound arrival, or

   $$\mathbf{P}(\text{next} = \text{westbound}) = \frac{\lambda_W}{\lambda_E + \lambda_W}$$

   ii. The pointer will change directions on the next arrival of an east-bound ship. By definition of the Poisson process, the remaining time until this arrival, denote it by X, is exponential with parameter λ_E, or

   $$f_X(x) = \lambda_E e^{-\lambda_E x}\ , x \geq 0 \quad .$$

   (b) For this to happen, no westbound ship can enter the channel from the moment the eastbound ship enters until the moment it exits, which consumes an amount of time t after the eastward ship enters the channel. In addition, no westbound ships may already be in the channel prior to the eastward ship entering the channel, which requires that no westbound ships enter for an amount of time t *before* the eastbound ship enters. Together, we require no westbound ships to arrive during an interval of time $2t$, which occurs with probability

   $$\frac{(\lambda_W 2t)^0 e^{-\lambda_W 2t}}{0!} = e^{-\lambda_W 2t} \quad .$$

   (c) Letting X be the first-order interarrival time for eastward ships, we can express the quantity $V = X_1 + X_2 + \ldots + X_7$, and thus the PDF for V is equivalent to the 7th order Erlang distribution

   $$f_V(v) = \frac{\lambda_E^7 v^6 e^{-\lambda_E v}}{6!}\ , v \geq 0 \quad .$$

4. (a) Whenever an arrival occurs, it comes from the first Poisson process with probability $\alpha/(\alpha+\beta)$, and this is independent from one arrival to the next. Therefore, the probability that all three arrivals come form the same process is equal to

   $$\left(\tfrac{\alpha}{\alpha+\beta}\right)^3 + \left(\tfrac{\beta}{\alpha+\beta}\right)^3.$$

   (b) The probability that $N = n$ is the probability that out of the first $n + 3$ arrivals, there were exactly n coming from the first process and that the $(n + 4)$th arrival was from process 2. Thus,

   $$p_N(n) = \tbinom{n+3}{3}\left(\tfrac{\beta}{\alpha+\beta}\right)^4\left(\tfrac{\alpha}{\alpha+\beta}\right)^n,\ n = 0, 1, \ldots$$

5. (a) Since the shuttles depart exactly every hour on the hour, the number of passengers that arrive in a one hour interval is the number of passengers on a shuttle. So, the arrivals are described by a Poisson process, and the expected number of arrivals (and therefore the expected number of passengers on a shuttle) is the mean of a Poisson random variable, or λ.

   $$\mathbf{E}[\text{number of passengers on a shuttle}] = \lambda$$

(b) Recall that in continuous time, each inter-arrival time in a Poisson process is described by the exponential distribution. Here, we consider the times inbetween shuttle arrivals with an exponential distribution, rate μ per hour. Then each shuttle arrival is a Poisson process. Let A be the number of shuttles arriving in one hour with parameter μ and the following distribution,

$$p_A(a) = \begin{cases} \frac{e^{-\mu}\mu^a}{a!}, & a = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(c) In the terminal, there is a Poisson process describing the arrival of passengers and another Poisson process describes the departures of shuttles. The "event" described includes either process (or both), so the event is a merged process (still Poisson). The two processes are independent from one another, so the merged process has an arrival rate of: $\lambda + \mu$.

$$\mathbf{E}[\text{number of events per hour}] = \lambda + \mu$$

(d) The wait time until the next shuttle is the inter-arrival time of the shuttles, which is exponential, with parameter μ. Recall that the exponential distribution is memoryless, so seeing 2λ people waiting around does not affect the expected wait time for a shuttle. So from the time the passenger arrives at the gate, the wait time is exponential with parameter μ. The expected value of an exponential is 1/rate, or $1/\mu$.

$$\mathbf{E}[\text{wait time} \mid 2\lambda \text{ people waiting}] = \frac{1}{\mu}$$

(e) To find the PMF for the number of passengers in a shuttle, we go back to part c, where we determined that the event of either a passenger arrival or shuttle departure is a merged Poisson process, with paramter $\lambda + \mu$. In the merged Poisson, the probability that the arrival was a passenger arrival is $\frac{\lambda}{\lambda+\mu}$, and the probability that the "arrival" was a shuttle departure is $\frac{\mu}{\lambda+\mu}$.

Let N be the number of people on a shuttle. There must be n successive passenger arrivals before a shuttle departure. Therefore, the PMF for N is:

$$p_N(n) = \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

One can also think of the PMF Of N as number of "failures" (passenger arrivals) until the first "success" (shuttle departure), but shifted to start at 0 rather than 1 in a standard geometric distribution.

6. The dot location of the yarn, as related to the size of the pieces of the yarn cut for any particular customer, can be viewed in light of the random incident paradox.

   (a) Here, the length of each piece of yarn is exponentially distributed. As explained on page 298 of the text, due to the memorylessness of the exponential, the distribution of the length of the piece of yarn containing the red dot is a second order Erlang. Thus, the $\mathbf{E}[R] = 2\mathbf{E}[L] = \frac{2}{\lambda}$.

(b) Think of exponentially-spaced marks being made on the yarn, so the length requested by the customers each involve *three* such sections of exponentially distributed lengths (since the PDF of L is third-order Erlang). The piece of yarn with the dot will have the dot in any one of these three sections, and the expected length of that section, by (a), will be $2/\lambda$, while the expected lengths of the other two sections will be $1/\lambda$. Thus, the total expected length containing the dot is $4/\lambda$.

7. (a) The event $\{M = m \cap N = n\}$ occurs when $\{N = n\}$ and $\{M - N = m - n\}$. That is, from $(0, t]$ there have to be n arrivals, and after t but prior to $t + s$ there have to be $m - n$ arrivals. Since the increment $(0, t]$ is disjoint from the increment $(t, t + s]$, the number of arrivals in each are independent and have a poisson distribution with rate λ. Symbolically,

$$\begin{aligned} p_{N,M}(n,m) &= p_N(n)p_{M|N}(m|n) \\ &= \left[\frac{(\lambda t)^n e^{-\lambda t}}{n!}\right]\left[\frac{(\lambda s)^{m-n}e^{-\lambda s}}{(m-n)!}\right]. \end{aligned}$$

(b) A similar principle is helpful here as well. We can rewrite $\mathbf{E}[NM]$ as

$$\begin{aligned} \mathbf{E}[NM] &= \mathbf{E}[N(M-N)+N^2] \\ &= \mathbf{E}[N]\mathbf{E}[M-N]+\mathbf{E}[N^2] \\ &= (\lambda t)(\lambda s)+\left[\mathrm{var}(N)+\mathbf{E}[N]^2\right] \\ &= (\lambda t)(\lambda s)+\lambda t+(\lambda t)^2, \end{aligned}$$

where the second equality is obtained via the independent increment property of the Poisson process.