Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2009)

Problem Set 1: Solutions
Due: February 11, 2009

1. Since all outcomes are equally likely we apply the discrete uniform probability law to solve the problem. To solve for any event we simply count the number of elements in the event and divide by the total number of elements in the sample space.

   There are 2 possible outcomes for each flip, and 3 flips. Thus there are $2^3 = 8$ elements (or sequences) in the sample space.

   (a) Any sequence has probability of 1/8. Therefore $\mathbf{P}(\{H, H, H\}) = \boxed{1/8}$.

   (b) This is still a single sequence, thus $\mathbf{P}(\{H, T, H\}) = \boxed{1/8}$.

   (c) The event of interest has 3 unique sequences, thus $\mathbf{P}(\{HHT, HTH, THH\}) = \boxed{3/8}$.

   (d) The sequences where there are more heads than tails are $A : \{HHH, HHT, HTH, THH\}$. 4 unique sequences gives us $\mathbf{P}(A) = \boxed{1/2}$.

2. (a) $A \cup B \cup C$

   (b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$

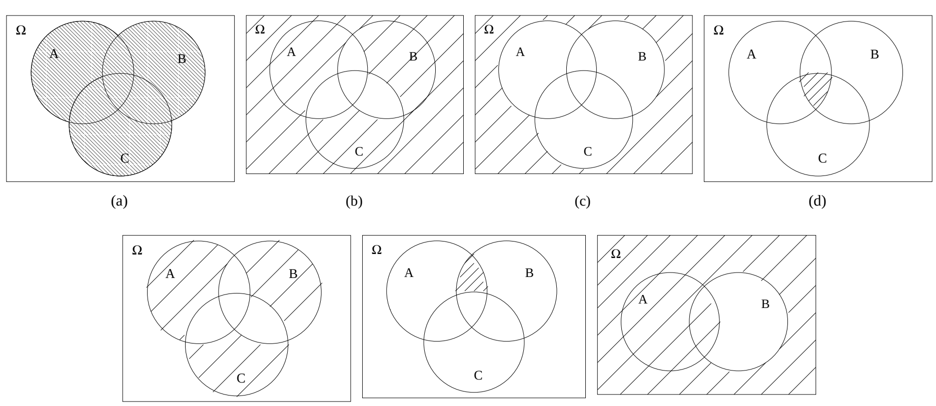
   (c) $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$

   (d) $A \cap B \cap C$

   (e) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

   (f) $A \cap B \cap C^c$

   (g) $A \cup (A^c \cap B^c)$



(a) (b) (c) (d) (e) (f) (g)

3. Using only probability axioms (page 9) and set algebra show the following:

   (a) If event B is a subset of event A, then $P(B) \leq P(A)$.
   Answer Using set notation[1] we can write the following for events B and A:

$$A = B \cup (A \cap B^c).$$

[1] Use Venn diagrams to convince yourself if needed.

Since the right hand side (RHS) is a union of 2 disjoint sets, the additivity axiom gives us

$$P(A) = P(B) + P(A \cap B^c).$$

Then, using the nonnegativity axiom

$$P(A \cap B^c) \geq 0$$

we find that

$$P(A) \geq P(B).$$

(b) For any two events A and B, $P(A \cup B) \geq P(A) + P(B) - 1$.
Answer
Use set notation to write

$$A \cup B = A \cup (B \cap A^c)$$

The RHS is a union of 2 disjoint sets, thus the additivity axiom gives us

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B \cap A^c)$$

By the first axiom, we have $\mathbf{P}(B \cap A^c) \geq 0$, hence

$$\mathbf{P}(A \cup B) \geq \mathbf{P}(A) \tag{1}$$

Finally, by the first axiom we have $\mathbf{P}(B^c) \geq 0$, by additivity we have $P(B) + P(B^c) = 1$, hence $\mathbf{P}(B) \leq 1$ and

$$0 \geq \mathbf{P}(B) - 1 \tag{2}$$

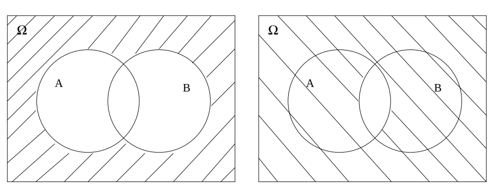
Combining equations 1 and 2 gives the required result,

$$\mathbf{P}(A \cup B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$$

4. We will appeal to Venn diagrams to overcome the various attempts at confusing us that are evident in this problem, especially in parts **(b)** and **(c)**.

(a) Define the following events, expressing each associated set in terms of the sets A and B:

E student has taken neither FYS nor calculus $\Rightarrow E = A^c \cap B^c$
F student has missed at least one of FYS or calculus $\Rightarrow F = A^c \cup B^c$



Event E   Event F

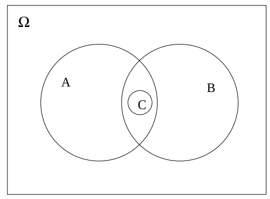
We are given $P(E) = 0.3$ and $P(F) = 0.8$, and we can express the event of interest (namely a student having *exactly* one of FYS or calculus) by $E^c \cap F$. The identity $P(E^c \cup F) = P(E^c) + P(F) - P(E^c \cap F)$ implies

$$P(E^c \cap F) = P(E^c) + P(F) - P(E^c \cup F).$$

Finally, noticing that $E^c \cup F = (A^c \cap B^c)^c \cup (A^c \cup B^c) = (A \cup B) \cup (A^c \cup B^c) = \Omega$, and therefore $P(E^c \cup F) = P(\Omega) = 1$, we obtain

$$P(E^c \cap F) = (1 - 0.3) + 0.8 - 1 = \boxed{0.5}.$$

(b) Students who have taken *at most one* of FYS or calculus is expressed by the set $A^c \cup B^c$. The new piece of data in part **(b)** is that $C \cap (A^c \cup B^c) = \emptyset$. Noticing that $C \cap (A^c \cup B^c) = (C \cap A^c) \cup (C \cap B^c)$, this further implies $C \cap A^c = C \cap B^c = \emptyset$ because if the union of two sets is empty, so must be each individual set. These properties between events A, B and C imply the following Venn diagram:



i. We can partition the set $E = A^c \cap B^c$, as defined in part (a) with given probability P(E) = 0.3, into the (disjoint) sets, $E \cap C$ and $E \cap C^c$; therefore, $P(E) = P(E \cap C) + P(E \cap C^c)$. However, from the previous paragraph we infer $P(E \cap C) = P(A^c \cap B^c \cap C) = 0$ because any subset of the empty set can only remain empty (i.e., $A^c \cap B^c \cap C \subseteq B^c \cap C = \emptyset$). Finally, putting it all together,

$$P(E \cap C^c) = P(A^c \cap B^c \cap C^c) = P(E) - P(E \cap C) = 0.3 - 0 = \boxed{0.3}.$$

ii. Define G as the event of interest, which can be expressed as $G = C^c \cap (E^c \cap F)$. From the Venn diagrams, we deduce

$$C^c \cap (E^c \cap F) = (E^c \cap F) \quad \Rightarrow \quad P(G) = P(E^c \cap F) = \boxed{0.5}.$$

(c) We already showed in part (b) that $P(A^c \cap B^c \cap C) = 0$. Applying De Morgan's laws, we have $P(A \cup B \cup C^c) = 1 - P(A^c \cap B^c \cap C) = 1$. There is insufficient information available, however, to compute the probabilities $P(A \cap B \cap C)$, $P(C)$, $P(A^c \cap B \cap C^c)$ or $P(A)$.

G1†. (a) Define event E = $A \cup B$. Then $E \cap C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Therefore $P(E \cap C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$$\begin{aligned} P(A \cup B \cup C) &= P(E \cup C) \\ &= P(E) + P(C) - P(E \cap C) \\ &= P(A \cup B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) - P(A \cap B) + P(A \cap B \cap C) \end{aligned}$$

(b) **Method 1:** An intuitive justification.
View $\cup_{k=1}^{n} A_k$ from a Venn Diagram perspective. This expression defines all area (or events) in any set A_k. The first expression on the right hand side of the equation defines all events in A_1, the second all events in A_2 & not in A_1, the third all events in A_3 & not in A_1 nor A_2, and so forth. Hence each set described by each expression is disjoint. Since the probability of disjoint sets is the sum of the probability of each set, we arrive at our intended expression.

Method 2: A more rigorous inductive argument.
Base Case:

$P(A_1) = P(A_1)$

Inductive Step:

Assume $P(\cup_{k=1}^{n-1} A_k) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1})$.

$$\begin{aligned}
P(\cup_{k=1}^{n} A_k) &= P(\cup_{k=1}^{n-1} A_k \cup A_n) && (3)\\
&= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\
&\quad +(P(A_n) - P(\cup_{k=1}^{n-1} A_k \cap A_n)) && (4)\\
&= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\
&\quad +P((\cup_{k=1}^{n-1} A_k)^c \cap A_n) && (5)\\
&= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n) && (6)
\end{aligned}$$

We get equation 4 by applying the following equation $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$, where $X = \cup_{k=1}^{n} A_k$ and $Y = A_n$. The last component of equation 5 is a direct application of the following equation $P(Y) - P(X \cap Y) = P(X^c \cap Y)$, where X and Y are defined as before. The last component of equation 6 results as a direct application of the following identity $(\cup_{k=1}^{n-1} A_k)^c \equiv \cap_{k=1}^{n-1} A_k^c$ for any sets $A_1, \cdots, A_k$.