Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2011)

Recitation 22: November 29, 2011

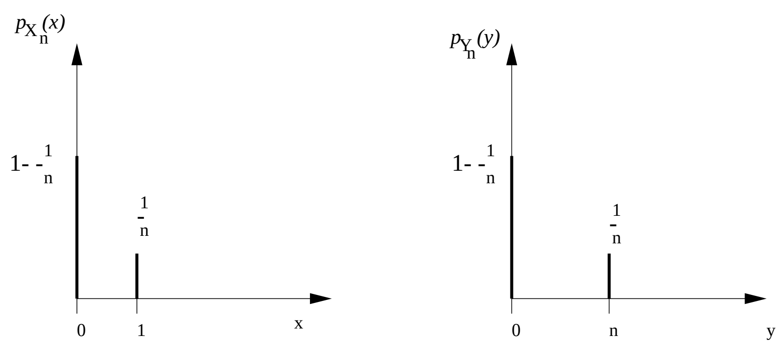
1. In your summer internship, you are working for the world's largest producer of lightbulbs. Your manager asks you to estimate the quality of production, that is, to estimate the probability p that a bulb produced by the factory is defectless. You are told to assume that all lightbulbs have the same probability of having a defect, and that defects in different lightbulbs are independent. Use Chebyshev inequality for the following questions.

   (a) Suppose that you test n randomly picked bulbs, what is a good estimate Z_n for p, such that Z_n converges to p in probability?

   (b) If you test 50 light bulbs, what is the probability that your estimate is in the range $p \pm 0.1$?

   (c) The management asks that your estimate falls in the range $p \pm 0.1$ with probability 0.95. How many light bulbs do you need to test to meet this specification?

2.



   Let X_n and Y_n have the distributions shown above.

   (a) Find the expected value and variance of X_n and Y_n.

   (b) What does the Chebyshev inequality tell us about the convergence of X_n? Y_n?

   (c) Is Y_n convergent in probability? If so, to what value?

   (d) If a sequence of random variables converges in probability to a, does the corresponding sequence of expected values converge to a? Prove or give a counter example.

   A sequence of random variables is said to converge to a number c in the **mean square**, if

$$\lim_{n\to\infty} \mathbf{E}\left[(X_n - c)^2\right] = 0.$$

   (e) Use Markov's inequality to show that convergence in the mean square implies convergence in probability.

   (f) Give an example that shows that convergence in probability does not imply convergence in the mean square.

3. Let $X_1, \ldots, X_{10}$ be independent random variables, uniformly distributed over the unit interval [0,1].

   (a) Estimate $\mathbf{P}(X_1 + \cdots + X_{10} \geq 7)$ using the Markov inequality.

   (b) Repeat part (a) using the Chebyshev inequality.

   (c) Repeat part (a) using the central limit theorem.

4. **Problem 10 in the textbook (page 290)**
   A factory produces X_n gadgets on day n, where the X_n are independent and identically distributed random variables, with mean 5 and variance 9.

   (a) Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.

   (b) Find (approximately) the largest value of n such that

   $$\mathbf{P}\left(X_1 + \cdots + X_n \geq 200 + 5n\right) \leq 0.05.$$

   (c) Let N be the first day on which the total number of gadgets produced exceeds 1000. Calculate an approximation to the probability that $N \geq 220$.