Lagrange Multipliers

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Setup: one constraint

Problem

Find the maximum and minimum values of $f(x, y)$ subject to the constraint

$$g(x, y) = k,$$

assuming the extreme values exist and $\nabla g \neq \mathbf{0}$ on the constraint curve.

- The constraint $g(x, y) = k$ restricts (x, y) to a curve C in the plane.
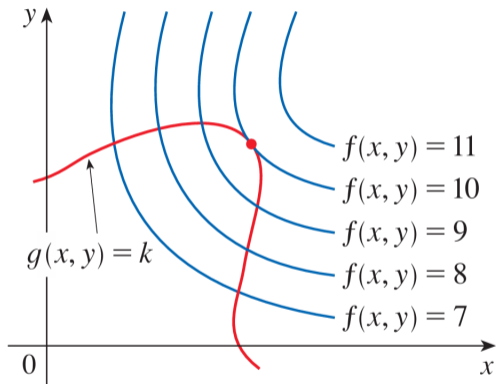- We look for points on C where f is as large or as small as possible.

Geometric idea

- The level curves of f are $f(x, y) = c$.
- Moving along the constraint curve $C : g(x, y) = k$, we want the largest (or smallest) level curve of f that still touches C.
- At such a contact point, the level curve of f and the constraint curve C have a common tangent line.
- Therefore their normals are parallel; equivalently, their gradients are parallel.
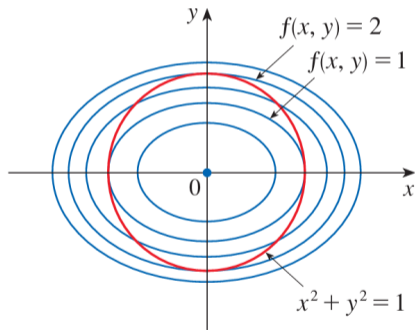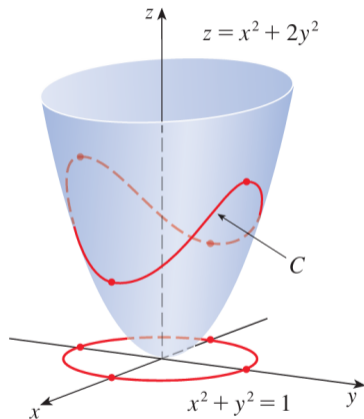
Key condition

At an extreme point on $g(x, y) = k$ (with $\nabla g \neq 0$),

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \quad \textit{forsomescalar} \lambda.$$

Figures: "tangent contact"

$f(x, y) = 11$
$f(x, y) = 10$
$f(x, y) = 9$
$f(x, y) = 8$
$f(x, y) = 7$
$g(x, y) = k$

Figures: "tangent contact"

Derivation via a curve on the constraint

Assume f has an extreme value on the curve C. Pick a smooth parametrization

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle \quad \text{with} \quad \mathbf{r}(t_0) = (x_0, y_0) \in C.$$

Define $h(t) = f(x(t), y(t)) = f(\mathbf{r}(t))$. At an extreme point along C, we have $h'(t_0) = 0$.

By the Chain Rule,

$$h'(t) = f_x(x(t), y(t)) \, x'(t) + f_y(x(t), y(t)) \, y'(t) = \nabla f(x(t), y(t)) \cdot \mathbf{r}'(t).$$

So $h'(t_0) = 0$ means $\nabla f(x_0, y_0) \perp \mathbf{r}'(t_0)$ (orthogonal to the tangent).

Since $g(\mathbf{r}(t)) = k$ is constant on C, similarly

$$0 = \frac{d}{dt} g(\mathbf{r}(t)) \Big|_{t=t_0} = \nabla g(x_0, y_0) \cdot \mathbf{r}'(t_0),$$

so $\nabla g(x_0, y_0) \perp \mathbf{r}'(t_0)$ as well.

Conclusion

$$\nabla f(x_0, y_0) = \lambda \, \nabla g(x_0, y_0).$$

Method: Lagrange multipliers (one constraint)

Steps

To find the maximum and minimum values of $f(x, y)$ subject to $g(x, y) = k$:

1. Find all points (x, y) and a number λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad and \quad g(x, y) = k.$$

2. Evaluate f at all points found in Step 1. The largest value is the maximum value of f; the smallest is the minimum value.

Writing $\nabla f = \lambda \nabla g$ in components gives

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad g(x, y) = k.$$

EVT / Continuity

True/False

Determine whether the statement is true or false:

$$f(x, y) = e^{1/x} + \sin(y^x) - 1$$

attains an **absolute maximum** on

$$R = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 \leq 1\}.$$

EVT / Continuity (Solution)

Idea

f is unbounded above on R.

1. Note $(0, 0) \in R$ because $(0 - 1)^2 + 0^2 = 1$.
2. But $e^{1/x}$ is not defined at $x = 0$, so f is not continuous on R (indeed, not even defined at $(0, 0)$).
3. Take the path $y = 0$ and $x \to 0^+$ along the line segment $\{(x, 0) : 0 < x \le 2\} \subset R$.
4. Then $\sin(0^x) = 0$ (for $x > 0$) so

$$f(x, 0) = e^{1/x} - 1 \xrightarrow[x \to 0^+]{} +\infty.$$

5. Therefore f is **unbounded above** on R, hence it cannot attain an absolute maximum.
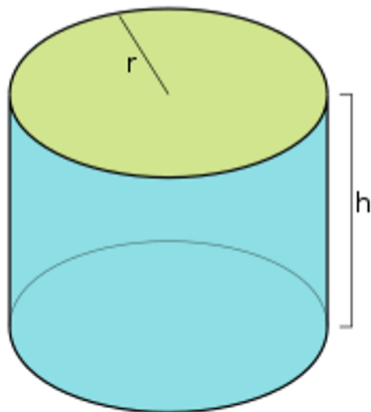
Conclusion

The statement is **False**.

Lagrange multipliers   (Cylinder optimization)

Problem

Use Lagrange multipliers to find the dimensions (r, h) of the right circular cylinder with **minimum surface area** (including the ends) and volume 32π.

Lagrange multipliers   (Cylinder solution)

1. **Volume constraint:** $V = \pi r^2 h = 32\pi \Rightarrow r^2 h = 32.$
2. Surface area (including ends):

$$S(r, h) = 2\pi r^2 + 2\pi r h.$$

Minimize S subject to $g(r, h) = r^2 h - 32 = 0.$
3. Gradients:

$$\nabla S = \langle 4\pi r + 2\pi h, \ 2\pi r \rangle, \qquad \nabla g = \langle 2rh, \ r^2 \rangle.$$

4. Lagrange equations $\nabla S = \lambda \nabla g$:

$$4\pi r + 2\pi h = 2\lambda rh, \qquad 2\pi r = \lambda r^2, \qquad r^2 h = 32.$$

5. From $2\pi r = \lambda r^2$ (with $r > 0$): $\lambda = \dfrac{2\pi}{r}$. Plug into the first:

$$4\pi r + 2\pi h = 2\left(\frac{2\pi}{r}\right) rh = 4\pi h \Rightarrow 4r + 2h = 4h \Rightarrow h = 2r.$$

6. Use $r^2 h = 32$:

$$r^2(2r) = 32 \Rightarrow r^3 = 16 \Rightarrow r = \sqrt[3]{16} = 2\sqrt[3]{2}, \quad h = 2r = 4\sqrt[3]{2}.$$

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Directional derivative

Problem

Compute the directional derivative of

$$f(x, y) = x^3 + xy - 4$$

in the direction $\mathbf{u} = \langle 1, 0 \rangle$ at the point $(1, 1)$.

Directional derivative   (Solution)

1. $\nabla f(x, y) = \langle 3x^2 + y, \ x \rangle$.
2. $\nabla f(1, 1) = \langle 3(1)^2 + 1, \ 1 \rangle = \langle 4, 1 \rangle$.
3. $u = \langle 1, 0 \rangle$ is already a unit vector.
4. $D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot u = \langle 4, 1 \rangle \cdot \langle 1, 0 \rangle = 4$.

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Directional derivative

Problem

Compute the directional derivative of

$$f(x, y) = y^2 - x^3 y$$

in the direction $\mathbf{u} = \langle 2, 1 \rangle$ at the point $(-2, 3)$.

Directional derivative   (Solution)

1. $\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -3x^2 y, \ 2y - x^3 \rangle.$

2. At $(-2, 3)$: $-3x^2 y = -3 \cdot 4 \cdot 3 = -36$ and $2y - x^3 = 6 - (-8) = 14$. So $\nabla f(-2, 3) = \langle -36, 14 \rangle.$

3. Unit direction: $u = \dfrac{1}{\sqrt{5}} \langle 2, 1 \rangle.$

4. Directional derivative:

$$D_{\mathbf{u}} f(-2, 3) = \nabla f(-2, 3) \cdot u = \frac{1}{\sqrt{5}} \left((-36) \cdot 2 + 14 \cdot 1 \right) = \frac{-58}{\sqrt{5}}.$$

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Max rate of change

Problem

Find the **maximum rate of change** of

$$f(x, y) = e^x + xy^2$$

at the point $(0, 3)$.

Max rate of change   (Solution)

1. $\nabla f(x, y) = \langle e^x + y^2, \ 2xy \rangle$.
2. $\nabla f(0, 3) = \langle 1 + 9, \ 2 \cdot 0 \cdot 3 \rangle = \langle 10, 0 \rangle$.
3. The maximum rate of change is $\|\nabla f(0, 3)\| = \sqrt{10^2 + 0^2} = 10$.

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Direction of max increase

Problem

Determine the direction of the **maximum rate of change** of

$$f(x, y) = \sin(xy)$$

at the point $(\pi, 2)$.

Direction of max increase   (Solution)

1. $\nabla f(x, y) = \langle y\cos(xy), \ x\cos(xy)\rangle$.
2. At $(\pi, 2)$ we have $xy = 2\pi$ so $\cos(xy) = \cos(2\pi) = 1$.
3. $\nabla f(\pi, 2) = \langle 2, \pi\rangle$.
4. Direction of maximum increase is the unit vector:

$$\frac{\nabla f(\pi, 2)}{\|\nabla f(\pi, 2)\|} = \frac{\langle 2, \pi\rangle}{\sqrt{4 + \pi^2}}.$$

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Counting critical points

Problem

Let

$$f(x, y) = x^2 + 2y^2 + xy^2.$$

How many critical points does f have?

Counting critical points   (Solution)

1. $\nabla f = \langle f_x, f_y \rangle = \langle 2x + y^2,\ 4y + 2xy \rangle$.
2. Solve $\nabla f = \mathbf{0}$:

$$2x + y^2 = 0, \qquad 4y + 2xy = 2y(2 + x) = 0.$$

3. From $2y(2 + x) = 0$: either $y = 0$ or $x = -2$.
4. If $y = 0$, then $2x + 0 = 0 \Rightarrow x = 0$. So $(0, 0)$.
5. If $x = -2$, then $2(-2) + y^2 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$. So $(-2, 2)$ and $(-2, -2)$.

Answer

There are **3** critical points: $(0, 0)$, $(-2, 2)$, $(-2, -2)$.

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Local extrema (Second Derivatives Test)

Problem

Find and classify the local extrema of

$$f(x, y) = x^2 y - x - y^2.$$

Local extrema (Solution)

1. $f_x = 2xy - 1, \quad f_y = x^2 - 2y$.
2. Solve $f_x = f_y = 0$:

$$x^2 - 2y = 0 \Rightarrow y = \frac{x^2}{2}, \qquad 2x\left(\frac{x^2}{2}\right) - 1 = 0 \Rightarrow x^3 = 1.$$

So $x = 1$ and $y = \frac{1}{2}$. Critical point: $\left(1, \frac{1}{2}\right)$.

3. Second derivatives: $f_{xx} = 2y, \ f_{yy} = -2, \ f_{xy} = 2x$.
4. Discriminant:

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2y)(-2) - (2x)^2 = -4y - 4x^2.$$

At $\left(1, \frac{1}{2}\right)$:

$$D = -4 \cdot \frac{1}{2} - 4 \cdot 1^2 = -2 - 4 = -6 < 0.$$

Conclusion

$\left(1, \frac{1}{2}\right)$ is a **saddle point**. No local max/min.

Local extrema   (Solution)

1. $f_x = 2xy - 1, \quad f_y = x^2 - 2y.$
2. Solve $f_x = f_y = 0$:

$$x^2 - 2y = 0 \Rightarrow y = \frac{x^2}{2}, \qquad 2x\left(\frac{x^2}{2}\right) - 1 = 0 \Rightarrow x^3 = 1.$$

   So $x = 1$ and $y = \frac{1}{2}$. Critical point: $\left(1, \frac{1}{2}\right)$.

3. Second derivatives: $f_{xx} = 2y, \ f_{yy} = -2, \ f_{xy} = 2x.$
4. Discriminant:

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2y)(-2) - (2x)^2 = -4y - 4x^2.$$

   At $\left(1, \frac{1}{2}\right)$:

$$D = -4 \cdot \frac{1}{2} - 4 \cdot 1^2 = -2 - 4 = -6 < 0.$$

Conclusion

$\left(1, \frac{1}{2}\right)$ is a **saddle point**. No local max/min.

Local extrema   (Solution)

1. $f_x = 2xy - 1, \quad f_y = x^2 - 2y$.
2. Solve $f_x = f_y = 0$:

$$x^2 - 2y = 0 \Rightarrow y = \frac{x^2}{2}, \qquad 2x\left(\frac{x^2}{2}\right) - 1 = 0 \Rightarrow x^3 = 1.$$

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Conclusion

$\left(1, \frac{1}{2}\right)$ is a **saddle point**. No local max/min.

Local extrema (Second Derivatives Test)

Problem

Find and classify the local extrema of

$$f(x, y) = x^4 + xy^2 - xy.$$

Local extrema   (Solution)

1. $f_x = 4x^3 + y^2 - y, \quad f_y = 2xy - x = x(2y - 1).$

2. Solve $f_y = 0$: either $x = 0$ or $y = \frac{1}{2}$.

3. If $x = 0$, then $f_x = y^2 - y = y(y - 1) = 0 \Rightarrow y = 0$ or 1. So $(0, 0)$ and $(0, 1)$.

4. If $y = \frac{1}{2}$, then $f_x = 4x^3 + \frac{1}{4} - \frac{1}{2} = 4x^3 - \frac{1}{4} = 0$, so $x^3 = \frac{1}{16}$ and

$$x = \frac{1}{\sqrt[3]{16}}.$$

   So $\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$.

5. Second derivatives: $f_{xx} = 12x^2, \quad f_{yy} = 2x, \quad f_{xy} = 2y - 1$. At $(0, 0)$:
   $D = 0 \cdot 0 - (-1)^2 < 0 \quad \Rightarrow$ saddle.
   At $(0, 1)$: $D = 0 \cdot 0 - (1)^2 < 0 \quad \Rightarrow$ saddle.
   At $\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$: $f_{xy} = 0, f_{xx} > 0, f_{yy} > 0 \Rightarrow D > 0$ and $f_{xx} > 0$.

Conclusion

$(0, 0)$ and $(0, 1)$ are **saddle points**.

$\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$ is a **local minimum**.

Local extrema   (Solution)

1. $f_x = 4x^3 + y^2 - y, \quad f_y = 2xy - x = x(2y - 1).$
2. Solve $f_y = 0$: either $x = 0$ or $y = \frac{1}{2}$.
3. If $x = 0$, then $f_x = y^2 - y = y(y - 1) = 0 \Rightarrow y = 0$ or 1. So $(0, 0)$ and $(0, 1)$.
4. If $y = \frac{1}{2}$, then $f_x = 4x^3 + \frac{1}{4} - \frac{1}{2} = 4x^3 - \frac{1}{4} = 0$, so $x^3 = \frac{1}{16}$ and

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   $D = 0 \cdot 0 - (-1)^2 < 0 \quad \Rightarrow$ saddle.
   At $(0, 1)$: $D = 0 \cdot 0 - (1)^2 < 0 \quad \Rightarrow$ saddle.
   At $\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$: $f_{xy} = 0, f_{xx} > 0, f_{yy} > 0 \Rightarrow D > 0$ and $f_{xx} > 0$.

Conclusion

$(0, 0)$ and $(0, 1)$ are **saddle points**.
$\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$ is a **local minimum**.

Local extrema   (Solution)

1. $f_x = 4x^3 + y^2 - y$, $f_y = 2xy - x = x(2y - 1)$.
2. Solve $f_y = 0$: either $x = 0$ or $y = \frac{1}{2}$.
3. If $x = 0$, then $f_x = y^2 - y = y(y - 1) = 0 \Rightarrow y = 0$ or 1. So $(0,0)$ and $(0,1)$.
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Conclusion

$(0,0)$ and $(0,1)$ are **saddle points**.
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Conclusion

$(0, 0)$ and $(0, 1)$ are **saddle points**.
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1. $f_x = 4x^3 + y^2 - y, \quad f_y = 2xy - x = x(2y - 1).$
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Conclusion

$(0, 0)$ and $(0, 1)$ are **saddle points**.

$\left(\frac{1}{\sqrt[3]{16}}, \frac{1}{2} \right)$ is a **local minimum**.

Absolute extrema on a disk

Problem

Find and classify the absolute extrema of

$$f(x, y) = x^2 + 2y^2 - 3$$

on the disk $D = \{(x, y) : x^2 + y^2 \leq 3\}$.

Absolute extrema on a disk (Solution)

1. Interior critical points: $\nabla f = \langle 2x, 4y \rangle = \mathbf{0}$ gives $(0,0)$.

$$f(0,0) = -3.$$

2. On the boundary $x^2 + y^2 = 3$, maximize/minimize $x^2 + 2y^2 - 3$. Since $2y^2$ has the larger coefficient, the maximum happens when y^2 is as large as possible on the circle.

3. On $x^2 + y^2 = 3$, the largest possible y^2 is 3 (at $x = 0$), giving

$$f(0, \pm\sqrt{3}) = 0 + 2 \cdot 3 - 3 = 3.$$

Answer

Absolute minimum: -3 at $(0,0)$.
Absolute maximum: 3 at $(0, \pm\sqrt{3})$.

Absolute extrema on a disk   (Solution)

1. Interior critical points: $\nabla f = \langle 2x, 4y \rangle = \mathbf{0}$ gives $(0,0)$.

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Answer

Absolute minimum: -3 at $(0,0)$.
Absolute maximum: 3 at $(0, \pm\sqrt{3})$.

Critical points + global extrema on regions

Problem

Given

$$f(x, y) = x^2 + y^2 - 2x + 2y - xy + 2,$$

(a) Find the critical points and classify them.

(b) Find the absolute max/min on $R_1 = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

(c) Find the absolute max/min on $R_2 = \{(x, y) : x^2 + y^2 \leq 4\}$.

Problem (a)   (Solution: critical point + classification)

1. $\nabla f = \langle 2x - 2 - y,\ 2y + 2 - x \rangle$.

2. Solve $\nabla f = \mathbf{0}$:

$$2x - y - 2 = 0, \qquad -x + 2y + 2 = 0.$$

From the first: $y = 2x - 2$. Substitute into second:

$$-x + 2(2x - 2) + 2 = 0 \Rightarrow 3x - 2 = 0 \Rightarrow x = \frac{2}{3}, \quad y = -\frac{2}{3}.$$

3. Hessian:

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -1.$$

4. Discriminant $D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$ and $f_{xx} > 0$.

Conclusion

Critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ is a **local minimum**.
(Value: $f\left(\frac{2}{3}, -\frac{2}{3}\right) = \frac{2}{3}$.)

Problem (a)   (Solution: critical point + classification)

1. $\nabla f = \langle 2x - 2 - y, \ 2y + 2 - x \rangle$.

2. Solve $\nabla f = \mathbf{0}$:
$$2x - y - 2 = 0, \qquad -x + 2y + 2 = 0.$$

   From the first: $y = 2x - 2$. Substitute into second:
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Conclusion

Critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ is a **local minimum**.
(Value: $f\left(\frac{2}{3}, -\frac{2}{3}\right) = \frac{2}{3}$.)

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4. Discriminant $D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$ and $f_{xx} > 0$.

Conclusion

Critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ is a **local minimum**.
(Value: $f\left(\frac{2}{3}, -\frac{2}{3}\right) = \frac{2}{3}$.)

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1. $\nabla f = \langle 2x - 2 - y,\ 2y + 2 - x \rangle$.

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3. Hessian:

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -1.$$

4. Discriminant $D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$ and $f_{xx} > 0$.

Conclusion

Critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ is a **local minimum**.
(Value: $f\left(\frac{2}{3}, -\frac{2}{3}\right) = \frac{2}{3}$.)

Problem (b)   (Solution: absolute extrema on the square)

Plan

No interior critical points in R_1 (since $y = -2/3$ is outside). So check the **boundary**.

1. Edge $x = 0$: $f(0, y) = y^2 + 2y + 2$ on $0 \leq y \leq 1$. Min at $y = 0$ gives 2, max at $y = 1$ gives 5.

2. Edge $x = 1$: $f(1, y) = y^2 + y + 1$ on $0 \leq y \leq 1$. Min at $y = 0$ gives 1, max at $y = 1$ gives 3.

3. Edge $y = 0$: $f(x, 0) = x^2 - 2x + 2$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 1, max at $x = 0$ gives 2.

4. Edge $y = 1$: $f(x, 1) = x^2 - 3x + 5$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 3, max at $x = 0$ gives 5.

Answer on R_1

Absolute **minimum** is 1 at $(1, 0)$.
Absolute **maximum** is 5 at $(0, 1)$.

Problem (b)   (Solution: absolute extrema on the square)

Plan

No interior critical points in R_1 (since $y = -2/3$ is outside). So check the **boundary**.

1. Edge $x = 0$: $f(0, y) = y^2 + 2y + 2$ on $0 \leq y \leq 1$. Min at $y = 0$ gives 2, max at $y = 1$ gives 5.
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3. Edge $y = 0$: $f(x, 0) = x^2 - 2x + 2$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 1, max at $x = 0$ gives 2.
4. Edge $y = 1$: $f(x, 1) = x^2 - 3x + 5$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 3, max at $x = 0$ gives 5.

Answer on R_1

Absolute **minimum** is 1 at $(1, 0)$.
Absolute **maximum** is 5 at $(0, 1)$.

Problem (b)   (Solution: absolute extrema on the square)

Plan

No interior critical points in R_1 (since $y = -2/3$ is outside). So check the **boundary**.

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3. Edge $y = 0$: $f(x, 0) = x^2 - 2x + 2$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 1, max at $x = 0$ gives 2.
4. Edge $y = 1$: $f(x, 1) = x^2 - 3x + 5$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 3, max at $x = 0$ gives 5.

Answer on R_1

Absolute **minimum** is 1 at $(1, 0)$.
Absolute **maximum** is 5 at $(0, 1)$.

Problem (b)   (Solution: absolute extrema on the square)

Plan

No interior critical points in R_1 (since $y = -2/3$ is outside). So check the **boundary**.

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3. Edge $y = 0$: $f(x, 0) = x^2 - 2x + 2$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 1, max at $x = 0$ gives 2.
4. Edge $y = 1$: $f(x, 1) = x^2 - 3x + 5$ on $0 \leq x \leq 1$. Min at $x = 1$ gives 3, max at $x = 0$ gives 5.

Answer on R_1

Absolute **minimum** is 1 at $(1, 0)$.
Absolute **maximum** is 5 at $(0, 1)$.

Problem (c)   (Solution: absolute extrema on the disk)

1. The interior critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ lies in $x^2 + y^2 \leq 4$, so it is a candidate:

$$f\left(\tfrac{2}{3}, -\tfrac{2}{3}\right) = \tfrac{2}{3}.$$

2. For the boundary $x^2 + y^2 = 4$, use Lagrange multipliers:

$$\nabla f = \lambda \nabla(x^2 + y^2).$$

That is, $2x - 2 - y = 2\lambda x, \qquad 2y + 2 - x = 2\lambda y, \qquad x^2 + y^2 = 4.$

3. Solving gives boundary candidates: $(2, 0), \ (0, -2), \ (\sqrt{2}, -\sqrt{2}), \ (-\sqrt{2}, \sqrt{2})$.

4. Evaluate:

$$f(2, 0) = 2, \quad f(0, -2) = 2, \quad f(\sqrt{2}, -\sqrt{2}) = 8 - 4\sqrt{2}, \quad f(-\sqrt{2}, \sqrt{2}) = 8 + 4\sqrt{2}.$$

Answer on R_2

Absolute **minimum**: $\dfrac{2}{3}$ at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$.

Absolute **maximum**: $8 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$.

Problem (c)   (Solution: absolute extrema on the disk)

1. The interior critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ lies in $x^2 + y^2 \leq 4$, so it is a candidate:
$$f\left(\tfrac{2}{3}, -\tfrac{2}{3}\right) = \tfrac{2}{3}.$$

2. For the boundary $x^2 + y^2 = 4$, use Lagrange multipliers:
$$\nabla f = \lambda \nabla(x^2 + y^2).$$

That is, $2x - 2 - y = 2\lambda x$, $\qquad 2y + 2 - x = 2\lambda y$, $\qquad x^2 + y^2 = 4$.

3. Solving gives boundary candidates: $(2, 0)$, $(0, -2)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

4. Evaluate:
$$f(2, 0) = 2, \quad f(0, -2) = 2, \quad f(\sqrt{2}, -\sqrt{2}) = 8 - 4\sqrt{2}, \quad f(-\sqrt{2}, \sqrt{2}) = 8 + 4\sqrt{2}.$$

Answer on R_2

Absolute **minimum**: $\dfrac{2}{3}$ at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$.

Absolute **maximum**: $8 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$.

Problem (c)   (Solution: absolute extrema on the disk)

1. The interior critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ lies in $x^2 + y^2 \le 4$, so it is a candidate:
$$f\left(\tfrac{2}{3}, -\tfrac{2}{3}\right) = \tfrac{2}{3}.$$

2. For the boundary $x^2 + y^2 = 4$, use Lagrange multipliers:
$$\nabla f = \lambda \nabla(x^2 + y^2).$$

   That is, $2x - 2 - y = 2\lambda x, \qquad 2y + 2 - x = 2\lambda y, \qquad x^2 + y^2 = 4.$

3. Solving gives boundary candidates: $(2, 0)$, $(0, -2)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

4. Evaluate:
$$f(2,0) = 2, \quad f(0,-2) = 2, \quad f(\sqrt{2}, -\sqrt{2}) = 8 - 4\sqrt{2}, \quad f(-\sqrt{2}, \sqrt{2}) = 8 + 4\sqrt{2}.$$

Answer on R_2

Absolute **minimum**: $\dfrac{2}{3}$ at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$.

Absolute **maximum**: $8 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$.

Problem (c)   (Solution: absolute extrema on the disk)

1. The interior critical point $\left(\frac{2}{3}, -\frac{2}{3}\right)$ lies in $x^2 + y^2 \leq 4$, so it is a candidate:

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2. For the boundary $x^2 + y^2 = 4$, use Lagrange multipliers:

$$\nabla f = \lambda \nabla(x^2 + y^2).$$

   That is, $2x - 2 - y = 2\lambda x$, $2y + 2 - x = 2\lambda y$, $x^2 + y^2 = 4.$
3. Solving gives boundary candidates: $(2, 0)$, $(0, -2)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.
4. Evaluate:

$$f(2, 0) = 2, \quad f(0, -2) = 2, \quad f(\sqrt{2}, -\sqrt{2}) = 8 - 4\sqrt{2}, \quad f(-\sqrt{2}, \sqrt{2}) = 8 + 4\sqrt{2}.$$

Answer on R_2

Absolute **minimum**: $\dfrac{2}{3}$ at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$.

Absolute **maximum**: $8 + 4\sqrt{2}$ at $(-\sqrt{2}, \sqrt{2})$.

Lagrange multipliers

Problem

Find the maximum and minimum values of

$$f(x, y) = xy^2$$

subject to the constraint $x^2 + y^2 = 3$.

Lagrange multipliers (Solution)

1. Constraint: $g(x, y) = x^2 + y^2 = 3$ so $\nabla g = \langle 2x, 2y \rangle$.

2. $\nabla f = \langle y^2, 2xy \rangle$. Solve $\nabla f = \lambda \nabla g$:

$$y^2 = 2\lambda x, \qquad 2xy = 2\lambda y, \qquad x^2 + y^2 = 3.$$

3. From $2xy = 2\lambda y$: either $y = 0$ or $x = \lambda$.

4. If $y = 0$, then $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$, and $f = xy^2 = 0$.

5. If $x = \lambda$ and $y \neq 0$, then $y^2 = 2\lambda x = 2x^2$. Combine with $x^2 + y^2 = 3$:

$$x^2 + 2x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

   Then $y^2 = 2$, so $y = \pm\sqrt{2}$.

6. Evaluate $f = xy^2 = x \cdot 2$:

$$f(1, \pm\sqrt{2}) = 2, \qquad f(-1, \pm\sqrt{2}) = -2.$$

Answer

Maximum value 2 at $(1, \pm\sqrt{2})$.

Minimum value -2 at $(-1, \pm\sqrt{2})$.

Lagrange multipliers (Solution)

1. Constraint: $g(x, y) = x^2 + y^2 = 3$ so $\nabla g = \langle 2x, 2y \rangle$.
2. $\nabla f = \langle y^2, 2xy \rangle$. Solve $\nabla f = \lambda \nabla g$:

$$y^2 = 2\lambda x, \qquad 2xy = 2\lambda y, \qquad x^2 + y^2 = 3.$$

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Answer

Maximum value 2 at $(1, \pm\sqrt{2})$.

Minimum value -2 at $(-1, \pm\sqrt{2})$.

Lagrange multipliers

Problem

Find the maximum and minimum values of

$$f(x, y, z) = xyz$$

subject to $x^2 + y^2 + z^2 = 1$.

Lagrange multipliers   (Solution)

1. Constraint $g = x^2 + y^2 + z^2 = 1$, so $\nabla g = \langle 2x, 2y, 2z \rangle$.

2. $\nabla f = \langle yz, xz, xy \rangle$. Solve $\nabla f = \lambda \nabla g$:

$$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z.$$

3. If $xyz \neq 0$, divide equations pairwise:

$$\frac{yz}{xz} = \frac{x}{y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2,$$

   and similarly $x^2 = z^2$, so $|x| = |y| = |z|$.

4. With $x^2 + y^2 + z^2 = 1$, we get $3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$, similarly for y, z.

5. Then

$$|xyz| = \left(\frac{1}{\sqrt{3}}\right)^3 = \frac{1}{3\sqrt{3}}.$$

Answer

Maximum value $\dfrac{1}{3\sqrt{3}}$ (when $xyz > 0$ and $|x| = |y| = |z| = \frac{1}{\sqrt{3}}$).

Minimum value $\dfrac{1}{}$ (when $xyz < 0$ with the same magnitudes).

Lagrange multipliers   (Solution)

1. Constraint $g = x^2 + y^2 + z^2 = 1$, so $\nabla g = \langle 2x, 2y, 2z \rangle$.
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Lagrange multipliers (Solution)

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Minimum value $\dfrac{1}{}$ (when $xyz < 0$ with the same magnitudes).

Lagrange multipliers (distance)

Problem

On the circle $x^2 + y^2 = 4$, find the point **closest** to $(3, 3)$ and the point **farthest** from $(3, 3)$.

Lagrange multipliers   (Solution)

1. Min/max distance $\Leftrightarrow$ min/max of squared distance:

$$F(x, y) = (x - 3)^2 + (y - 3)^2$$

subject to $g(x, y) = x^2 + y^2 = 4$.

2. $\nabla F = \langle 2(x - 3), 2(y - 3) \rangle, \quad \nabla g = \langle 2x, 2y \rangle$. Set $\nabla F = \lambda \nabla g$:

$$2(x - 3) = 2\lambda x, \qquad 2(y - 3) = 2\lambda y, \qquad x^2 + y^2 = 4.$$

3. If $\lambda \neq 1$, subtracting gives $x = y$. Then $2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ with the same sign.

4. Candidates: $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

5. Compare distances to $(3, 3)$: $(\sqrt{2}, \sqrt{2})$ is closer, $(-\sqrt{2}, -\sqrt{2})$ is farther.

Answer

Closest: $(\sqrt{2}, \sqrt{2})$.   Farthest: $(-\sqrt{2}, -\sqrt{2})$.

Lagrange multipliers   (Solution)

1. Min/max distance $\Leftrightarrow$ min/max of squared distance:

$$F(x, y) = (x - 3)^2 + (y - 3)^2$$

subject to $g(x, y) = x^2 + y^2 = 4$.

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Answer

Closest: $(\sqrt{2}, \sqrt{2})$.   Farthest: $(-\sqrt{2}, -\sqrt{2})$.

Lagrange multipliers

Problem

Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = x^2 + 2y^2 + xy^2$$

subject to

$$g(x, y) = 2x + y^2 = 1.$$

Lagrange multipliers   (Solution)

1. $\nabla f = \langle 2x + y^2,\ 4y + 2xy \rangle, \quad \nabla g = \langle 2,\ 2y \rangle.$

2. Solve $\nabla f = \lambda \nabla g$:

$$2x + y^2 = 2\lambda, \qquad 4y + 2xy = 2\lambda y, \qquad 2x + y^2 = 1.$$

3. From the last equation and the first: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$.

4. Plug into the second:

$$4y + 2xy = (2\lambda)y = 1 \cdot y \Rightarrow 3y + 2xy = 0 \Rightarrow y(3 + 2x) = 0.$$

5. Case 1: $y = 0$. Then $2x = 1 \Rightarrow x = \frac{1}{2}$ and

$$f\left(\tfrac{1}{2}, 0\right) = \tfrac{1}{4}.$$

6. Case 2: $3 + 2x = 0 \Rightarrow x = -\frac{3}{2}$. Then

$$2\left(-\tfrac{3}{2}\right) + y^2 = 1 \Rightarrow -3 + y^2 = 1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2,$$

and

$$f\left(-\tfrac{3}{2}, \pm 2\right) = \frac{17}{4}.$$

7. Check existence of a minimum: along the constraint $x = \frac{1 - y^2}{2}$. As $|y| \to \infty$, the

Lagrange multipliers   (Solution)

1. $\nabla f = \langle 2x + y^2,\ 4y + 2xy \rangle, \quad \nabla g = \langle 2,\ 2y \rangle.$

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6. Case 2: $3 + 2x = 0 \Rightarrow x = -\frac{3}{2}$. Then

$$2\left(-\tfrac{3}{2}\right) + y^2 = 1 \Rightarrow -3 + y^2 = 1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2,$$

and

$$f\left(-\tfrac{3}{2}, \pm 2\right) = \frac{17}{4}.$$

Check existence of a minimum: along the constraint $x = \frac{1 - y^2}{2}$. As $|y| \to \infty$, the

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$$\nabla f = \langle 1, 1, 1 \rangle, \quad \nabla g = \langle 1, -1, 1 \rangle, \quad \nabla h = \langle 2x, 2y, 0 \rangle.$$

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Constraint $x^2 + y^2 = 1$ gives $y = \pm 1$, so $\mu = \pm 1$ accordingly.

From $x - y + z = 1$, with $x = 0$: $z = 1 + y$. So $z = 2$ when $y = 1$, and $z = 0$ when $y = -1$.

Evaluate $f = x + y + z$:

$$f(0, 1, 2) = 3, \qquad f(0, -1, 0) = -1.$$

Answer

Maximum $= 3$ at $(0, 1, 2)$, minimum $= -1$ at $(0, -1, 0)$.

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Exercise

Problem

$f(x, y) = x^2 + y^2, \qquad xy = 1$

(Solution)

Let $g(x, y) = xy$. Then $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \langle y, x \rangle$. Lagrange system:

$$\langle 2x, 2y \rangle = \lambda \langle y, x \rangle, \qquad xy = 1,$$

$$so \qquad 2x = \lambda y, \qquad 2y = \lambda x, \qquad xy = 1.$$

Multiply the first two equations:

$$(2x)(2y) = (\lambda y)(\lambda x) \; \Rightarrow \; 4xy = \lambda^2 xy.$$

Since $xy = 1$, we get $\lambda^2 = 4$, so $\lambda = \pm 2$. If $\lambda = 2$, then $2x = 2y \Rightarrow x = y$. With $xy = 1$, this gives $(x, y) = (1, 1)$ or $(-1, -1)$. If $\lambda = -2$, then $2x = -2y \Rightarrow x = -y$, which would give $xy = -x^2 \neq 1$, impossible. Evaluate:

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Also on $xy = 1$ we have $y = 1/x$, so $f(x, 1/x) = x^2 + x^{-2} \to \infty$ as $x \to 0$ or $|x| \to \infty$.

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Minimum value is 2 (at $(1, 1)$ and $(-1, -1)$). There is no maximum value.

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