Double Integral

Ali Saraeb

Trick 1: Distance $\Rightarrow$ squared distance

Key idea

If you want the **closest/farthest** point, you are optimizing a *distance*

$$d(x,y) = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Since $\sqrt{}$ is **increasing**, minimizing/maximizing d is the same as minimizing/maximizing

$$D(x,y) = d(x,y)^2 = (x-x_0)^2 + (y-y_0)^2.$$

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Mini-example

Closest point on $x^2 + y^2 = 4$ to $(3,3)$: optimize $D = (x-3)^2 + (y-3)^2$ on the circle, then take $\sqrt{D}$ at the end.

Trick 2: Log trick for products/ratios (avoid product/quotient rule)

Key idea (must have $F > 0$)

If $F(x, y, \dots) > 0$, then

$$\text{maximize/minimize } F \quad \Longleftrightarrow \quad \text{maximize/minimize } \ln F,$$

because $\ln(\cdot)$ is **increasing**.

- $\ln(\text{product}) = \text{sum of logs}$.

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Example (ratio $\rightarrow$ reduce scale, then log)

Maximize $F(x, y) = \dfrac{x^2 y e^x}{(x^2+1)^3}$ subject to $x + y = 1$.

Maximize $2\ln x + \ln y + x - 3ln(x^2+1)$ subject to $x + y = 1$.

Trick 3: Lagrange gives candidates, but max/min may not exist

Important warning

Lagrange multipliers find **critical candidates** *assuming an absolute max/min exists*. If the constraint set is **unbounded** (not closed/bounded), or f is **not continuous** on it, an absolute maximum/minimum may **not exist**.

- (You *can* still have a minimum: here the minimum is 2 at $(1, 1)$ and $(-1, -1)$.)

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Example: no maximum

Let $f(x,y) = x^2 + y^2$ on the constraint $xy = 1$.
Write $y = \frac{1}{x}$:

$$f(x, 1/x) = x^2 + \frac{1}{x^2} \xrightarrow[|x|\to\infty]{} \infty \quad \text{and} \quad f(x, 1/x) \xrightarrow[x\to 0]{} \infty.$$

So f has **no maximum** on $xy = 1$ (it is unbounded above).

- (You *can* still have a minimum: here the minimum is 2 at $(1,1)$ and $(-1,-1)$.)

Double integrals: what are we computing?

Big picture

A **double integral** adds up values of $f(x,y)$ over a region $R \subset \mathbb{R}^2$:

$$\iint_R f(x,y)\,dA.$$

- If $f(x,y) \geq 0$, then $\displaystyle\iint_R f\,dA$ is the **volume** under $z = f(x,y)$ above the region R.

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- Think: "area" in 2D becomes "volume" in 3D.

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- Think: "area" in 2D becomes "volume" in 3D.
- The tiny area element is

$$dA = dx\, dy \quad \text{or} \quad dA = dy\, dx$$

(same dA, just a different order of integration later).

Definition via a Double Riemann Sum (rectangle $R = [a, b] \times [c, d]$)

Definition

For $R = [a, b] \times [c, d]$, partition $[a, b]$ into m pieces and $[c, d]$ into n pieces:

$$\Delta x = \frac{b-a}{m}, \qquad \Delta y = \frac{d-c}{n}, \qquad \Delta A = \Delta x \, \Delta y.$$

Pick a sample point (x_i^*, y_j^*) in each sub-rectangle. Then

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A.$$

- Each term $f(x_i^*, y_j^*)\Delta A$ is like "height $\times$ base area" of a tiny box.

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- Each term $f(x_i^*, y_j^*)\Delta A$ is like "height $\times$ base area" of a tiny box.
- Add all boxes $\Rightarrow$ total volume.

Iterated integrals (Fubini's Theorem for rectangles)

Fubini (rectangle case)

If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

- **Inner bounds** can depend on the **outer variable**.

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How to read the order

$$\int_a^b \int_c^d f(x, y)\,\underline{dy}\,dx \quad \text{means: integrate in } y \text{ first (inner), then in } x \text{ (outer).}$$

- **Inner bounds** can depend on the **outer variable**.
- For rectangles, bounds are constants either way.

Double integrals over a general region D

Two common descriptions of D

Type I (vertical slices):

$$D = \{(x, y) \mid a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x)\}.$$

Type II (horizontal slices):

$$D = \{(x, y) \mid c \leq y \leq d, \quad h_1(y) \leq x \leq h_2(y)\}.$$

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Iterated integral setups

If D is Type I: $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

If D is Type II: $$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Trick

To switch the order in $\int\int f(x,y)\,dy\,dx$:

1. Sketch the region from the bounds.
2. Outer boundaries must be constants. Inner boundaries can only depend on the outer variable. i.e must be of the form $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ or $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.
3. Outer variable is fixed. Draw ray along the axis of the inner variable.

Problem

Select the iterated integral equivalent to $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$.

Choices:

(A) $\displaystyle\int_0^2 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$ (B) $\displaystyle\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

(C) $\displaystyle\int_{y/2}^{\sqrt{y}} \int_0^2 f(x,y)\,dx\,dy$ (D) $\displaystyle\int_0^4 \int_{\sqrt{y}}^{y/2} f(x,y)\,dx\,dy$ (E) $\displaystyle\int_0^2 \int_{\sqrt{y}}^{y/2} f(x,y)\,dx\,dy$

Solution (region $\to$ new bounds)

1. The region is

$$0 \leq x \leq 2, \qquad x^2 \leq y \leq 2x.$$

   So y is between the parabola $y = x^2$ (bottom) and the line $y = 2x$ (top).

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So y is between the parabola $y = x^2$ (bottom) and the line $y = 2x$ (top).
2. Find where they meet:
$$x^2 = 2x \implies x(x-2) = 0 \implies x = 0 \text{ or } 2.$$
At those points, $y = 0$ and $y = 4$. So the overall y-range is $\boxed{0 \le y \le 4}$.

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3. For a fixed $y \in [0, 4]$, solve each boundary for x:
$$y = x^2 \implies x = \sqrt{y} \quad (\text{since } x \ge 0), \qquad y = 2x \implies x = \frac{y}{2}.$$

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$$y = x^2 \implies x = \sqrt{y} \quad (\text{since } x \ge 0), \qquad y = 2x \implies x = \frac{y}{2}.$$
4. Left/right bounds: $\frac{y}{2} \le x \le \sqrt{y}$.

Answer

Equivalent integral (order $dx\,dy$)

$$\iint_R f(x,y)\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

Multiple choice

$\boxed{\text{Correct choice: (B)}}$

Trick

- Draw the reigon.
- Ask yourself if I send a ray parrallel to x-axis do the starting and ending curves change as we vary the ray? Ask same question for y-axis.
- If when going parrallel to y-axis, for example, you always encounter the same starting and ending curve, then that's the right inner variable i.e. $\int_{.}^{.} \int_{.}^{.} \dots dydx$

Problem

Consider the integral $\iint_D 5x \, dA$, where D is the region in the **first quadrant** bounded by the y-axis and the curves $y + x = 2, \qquad y - x^2 = 0.$

(i) Sketch and shade the region D. Label any intersection points.

(ii) What does $\iint_D 5x \, dA$ represent geometrically?

(iii) Evaluate the integral.

Solution (i): sketch + bounds

Step 1: rewrite the boundary curves

$y + x = 2 \;\Rightarrow\; y = 2 - x$ (a line), $\qquad y - x^2 = 0 \;\Rightarrow\; y = x^2$ (a parabola).

Step 2: intersection points

Solve $x^2 = 2 - x$:

$$x^2 + x - 2 = 0 \;\Rightarrow\; (x+2)(x-1) = 0 \;\Rightarrow\; x = 1 \text{ (in Q1)},$$

so the curves meet at $\boxed{(1,1)}$. On the y-axis ($x = 0$): the parabola gives $(0,0)$ and the line gives $(0,2)$.

Region description

$$\boxed{0 \le x \le 1, \qquad x^2 \le y \le 2 - x.}$$

Solution (ii): geometric meaning

Recall

$\iint_D f(x, y)\, dA$ is the **volume** under $z = f(x, y)$ above the region D (when $f \geq 0$).

Here

Since the integrand is $5x$, the surface is

$$z = 5x \qquad \text{(a plane)}.$$

So $\iint_D 5x\, dA$ represents the **volume of the solid**:

$$\boxed{\text{above } D \text{ in the } xy\text{-plane, below } z = 5x, \text{ and above } z = 0.}$$

Solution (iii): evaluate $\iint_D 5x \, dA$

Trick

Sending a ray along y-axis we always encounter $y = -x + 2$ then $y = x^2$, so integrate in y first.

$$\begin{aligned}
\iint_D 5x \, dA &= \int_0^1 \int_{x^2}^{2-x} 5x \, dy \, dx \\
&= \int_0^1 5x\left[(2-x) - x^2\right] dx \\
&= 5\int_0^1 \left(2x - x^2 - x^3\right) dx \\
&= 5\left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 5\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 5 \cdot \frac{5}{12} = \boxed{\frac{25}{12}}.
\end{aligned}$$

Trick

If the inner integral looks impossible (here: $\int \sin(x^2)\, dx$), try to **switch the order**.

1. Draw the reigon D.
2. Interchange order of integration and update boundaries.

Problem

Consider the integral

$$\int_0^3 \int_{y^2}^9 y \sin(x^2)\, dx\, dy.$$

(a) Sketch the region of integration.

(b) Evaluate the integral.

Solution (a): describe the region D

Read bounds as inequalities

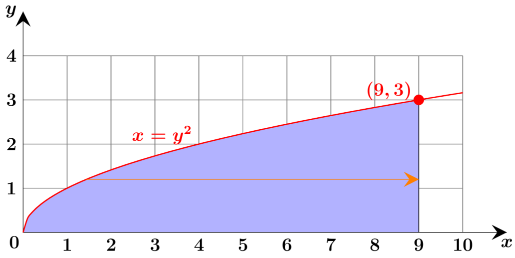
$$0 \le y \le 3, \qquad y^2 \le x \le 9.$$

So D is in the first quadrant, between the curve $x = y^2$ and the vertical line $x = 9$, from $y = 0$ up to $y = 3$.

Equivalent description (solve for y)

From $x \ge y^2$ with $y \ge 0$, we get $0 \le y \le \sqrt{x}$. Also x runs from 0 to 9. Hence

$$\boxed{D = \{(x, y) \mid 0 \le x \le 9, \;\; 0 \le y \le \sqrt{x}\}.}$$



y
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
x
$(9, 3)$
$x = y^2$

Solution (b): switch order, then integrate

Switch the order

Using $D = \{0 \le x \le 9,\ 0 \le y \le \sqrt{x}\}$,

$$\int_0^3 \int_{y^2}^9 y \sin(x^2)\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y \sin(x^2)\, dy\, dx.$$

$$\begin{aligned}\int_0^9 \int_0^{\sqrt{x}} y \sin(x^2)\, dy\, dx &= \int_0^9 \sin(x^2) \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx \\ &= \int_0^9 \frac{x}{2} \sin(x^2)\, dx.\end{aligned}$$

Last step: substitution

Let $u = x^2$, so $du = 2x\, dx$, hence $\frac{x}{2}\, dx = \frac{1}{4}\, du$.

Finish

$$\int_0^9 \frac{x}{2}\sin(x^2)\,dx = \frac{1}{4}\int_0^{81} \sin u\,du$$
$$= \frac{1}{4}\Big[-\cos u\Big]_0^{81} = \frac{1}{4}\big(1-\cos 81\big).$$

Answer

$$\boxed{\int_0^3 \int_{y^2}^9 y\sin(x^2)\,dx\,dy = \frac{1}{4}\big(1-\cos 81\big).}$$

Trick

Remember to draw a plane (e.g $z = y$)

- you can pick 3 points on the plane and trace the triangle or
- pick a point the plane and go in direction of the normal
- set one of the vairbale to 0, draw the line, and then extend it to 3D.

Problem

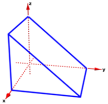
Consider the integral

$$\int_0^1 \int_0^2 y \, dy \, dx.$$

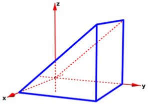
(a) Evaluate the integral. *(No partial credit.)*

(b) Interpret it as a volume. Which picture best represents the solid?

(1)



(2)



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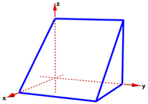

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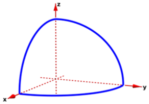

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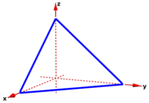


Solution (a): evaluate

$$\int_0^1 \int_0^2 y \, dy \, dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^2 dx = \int_0^1 2 \, dx = 2.$$

Answer

$$\boxed{2}$$

Solution (b): what solid is it?

Geometric meaning (straight to the point)

$$\iint_R y \, dA$$

is the volume under the surface $\boxed{z = y}$ above the rectangle

$$\boxed{0 \le x \le 1, \quad 0 \le y \le 2}$$

and above $z = 0$.

Key shape clue

Since $z = y$ does **not** depend on x, the solid is a **prism extruded in the x-direction**. For each fixed x, the cross-section in the yz-plane is the triangle $0 \le y \le 2, \qquad 0 \le z \le y$. So it's a **triangular prism (a wedge)**.

Trick

If the **inner integral has no nice antiderivative** (here: $\int e^{y^4}\, dy$), try reversing the order so the inner integral.

Problem

Compute the iterated integral

$$\int_0^1 \int_{x^{1/3}}^1 e^{y^4}\, dy\, dx.$$

Solution (step 1): describe the region and swap order

Read the bounds as a region D

$$0 \le x \le 1, \qquad x^{1/3} \le y \le 1.$$

So D is above $y = x^{1/3}$ and below $y = 1$, for $0 \le x \le 1$.

Rewrite D the other way

From $x^{1/3} \le y$ we get $x \le y^3$. Also y runs from 0 to 1. Hence

$$\boxed{0 \le y \le 1, \qquad 0 \le x \le y^3.}$$

So the reversed-order integral is

$$\int_0^1 \int_0^{y^3} e^{y^4}\, dx\, dy.$$

Solution (step 2): evaluate

$$\int_0^1 \int_0^{y^3} e^{y^4}\,dx\,dy = \int_0^1 \left[x\,e^{y^4}\right]_0^{y^3} dy$$
$$= \int_0^1 y^3 e^{y^4}\,dy.$$

Substitution

Let $u = y^4$. Then $du = 4y^3\,dy$, so $y^3\,dy = \frac{1}{4}\,du$. When $y = 0$, $u = 0$. When $y = 1$, $u = 1$.

$$\int_0^1 y^3 e^{y^4}\,dy = \frac{1}{4}\int_0^1 e^u\,du = \frac{1}{4}\left[e^u\right]_0^1 = \boxed{\frac{e-1}{4}}.$$

Problem

Let R be the region in the first quadrant bounded by

$$x = 0, \qquad y = x^2, \qquad y = 8 - x^2.$$

Evaluate

$$\iint_R (x + y)\, dA.$$

Solution (step 1): describe the region and bounds

Intersect the two curves to find the x-range

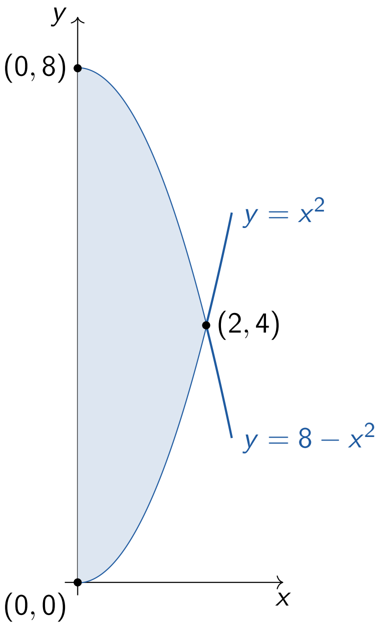
$$x^2 = 8 - x^2 \implies 2x^2 = 8 \implies x^2 = 4 \implies x = 2 \quad \text{(first quadrant)}.$$

So $\boxed{0 \le x \le 2}$.

For each fixed $x \in [0, 2]$

Lower curve: $\boxed{y = x^2}$, Upper curve: $\boxed{y = 8 - x^2}$. So

$$\boxed{x^2 \le y \le 8 - x^2.}$$



y
$(0, 8)$
$y = x^2$
$(2, 4)$
$y = 8 - x^2$
$(0, 0)$
x

Solution (step 2): set up and compute

$$\iint_R (x+y)\,dA = \int_0^2 \int_{x^2}^{8-x^2} (x+y)\,dy\,dx.$$

Inner integral

$$\int (x+y)\,dy = xy + \frac{y^2}{2}.$$

So

$$\int_{x^2}^{8-x^2} (x+y)\,dy = \left(xy + \tfrac{y^2}{2}\right)\Big|_{y=x^2}^{y=8-x^2}.$$

Finish (clean algebra)

Compute the difference:

$$\left(xy + \tfrac{y^2}{2}\right)\Big|_{x^2}^{8-x^2} = \left(x(8-x^2) + \frac{(8-x^2)^2}{2}\right) - \left(x(x^2) + \frac{(x^2)^2}{2}\right)$$

$$= \left(8x - x^3 + \frac{64 - 16x^2 + x^4}{2}\right) - \left(x^3 + \frac{x^4}{2}\right)$$

$$= 8x - x^3 + 32 - 8x^2 + \frac{x^4}{2} - x^3 - \frac{x^4}{2}$$

$$= 32 + 8x - 8x^2 - 2x^3.$$

Now integrate in x:

$$\iint_R (x+y)\,dA = \int_0^2 \left(32 + 8x - 8x^2 - 2x^3\right) dx$$

$$= \left[32x + 4x^2 - \frac{8}{3}x^3 - \frac{1}{2}x^4\right]_0^2$$

$$= 64 + 16 - \frac{64}{\ } - 8 = 72 - \frac{64}{\ } = \boxed{\frac{152}{\ }}$$

Trick

Recall the fundemental theorem of calculus

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Problem

Suppose the second partial derivatives of f are continuous on

$$R = \{(x, y) : 0 \leq x \leq a,\ 0 \leq y \leq b\}.$$

Show that

$$\iint_R \frac{\partial^2 f}{\partial x\, \partial y}(x, y)\, dA = f(a, b) - f(a, 0) - f(0, b) + f(0, 0).$$

Solution

Solution

$$
\begin{aligned}
\iint_R f_{xy}(x,y)\,dA &= \int_0^b \int_0^a f_{xy}(x,y)\,dx\,dy \\
&= \int_0^b \Big[f_y(x,y)\Big]_{x=0}^{x=a} dy = \int_0^b \big(f_y(a,y) - f_y(0,y)\big)\,dy \\
&= \Big[f(a,y) - f(0,y)\Big]_{y=0}^{y=b} \\
&= f(a,b) - f(a,0) - f(0,b) + f(0,0).
\end{aligned}
$$

Trick (seperable integrand)

If the region is a rectangle $R = [a, b] \times [c, d]$ and the integrand factors:

$$f(x, y) = g(x)\, h(y),$$

then you can separate:

$$\iint_R g(x)h(y)\, dA = \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right).$$

Problem

Evaluate

$$\iint_{[0,2]\times[1,3]} x^2\, e^y\, dA.$$

Solution

Since $x^2 e^y = (x^2)(e^y)$ and $R = [0, 2] \times [1, 3]$,

$$\begin{aligned}\iint_R x^2 e^y \, dA &= \left(\int_0^2 x^2 \, dx \right) \left(\int_1^3 e^y \, dy \right) \\ &= \left[\frac{x^3}{3} \right]_0^2 \cdot [e^y]_1^3 = \frac{8}{3} (e^3 - e).\end{aligned}$$

Problem

Find the volume

Find the volume of the solid bounded by

$$y^2 + z^2 = 9, \qquad z = 0, \qquad y = 0, \qquad x = 0, \qquad 2x + y = 2.$$

Set up as a *double* integral

Because $y^2 + z^2 = 9$ and $z \geq 0$, the top surface is

$$z = \sqrt{9 - y^2}.$$

So the volume is the double integral

$$V = \iint_D \sqrt{9 - y^2}\, dA.$$

The region D is in the xy-plane and is bounded by

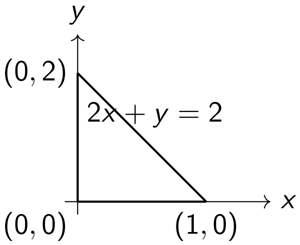
$$x = 0, \quad y = 0, \quad 2x + y = 2 \;\; (\text{i.e. } \; x = 1 - \tfrac{y}{2}).$$

Thus

$$D = \{(x, y) : 0 \leq y \leq 2, \;\; 0 \leq x \leq 1 - \tfrac{y}{2}\}.$$

$$\boxed{V = \int_0^2 \int_0^{1-\frac{y}{2}} \sqrt{9 - y^2}\, dx\, dy}$$

Compute the double integral



Integrate in x first (since the integrand has no x):

$$V=\int_0^2\Big[\,x\sqrt{9-y^2}\,\Big]_0^{1-\frac{y}{2}}dy=\int_0^2\Big(1-\tfrac{y}{2}\Big)\sqrt{9-y^2}\,dy.$$

Split:

$$V=\int_0^2\sqrt{9-y^2}\,dy-\frac{1}{2}\int_0^2 y\sqrt{9-y^2}\,dy.$$

(1)

$$\int \sqrt{9-y^2}\,dy \quad \text{let } y = 3\sin\theta, \quad dy = 3\cos\theta\,d\theta$$

$$\sqrt{9-y^2} = 3\cos\theta \quad \Rightarrow \quad \int \sqrt{9-y^2}\,dy = 9\int \cos^2\theta\,d\theta = 9\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\Big(\theta + \tfrac{1}{2}\sin 2\theta\Big) + C$$

$$\theta = \arcsin\Big(\frac{y}{3}\Big), \qquad \sin 2\theta = 2\sin\theta\cos\theta = 2\Big(\frac{y}{3}\Big)\Big(\frac{\sqrt{9-y^2}}{3}\Big) = \frac{2y\sqrt{9-y^2}}{9}$$

$$\boxed{\int \sqrt{9-y^2}\,dy = \frac{y}{2}\sqrt{9-y^2} + \frac{9}{2}\arcsin\Big(\frac{y}{3}\Big) + C.}$$

So

$$\int_0^2 \sqrt{9-y^2}\,dy = \frac{1}{2}\Big(2\sqrt{5} + 9\arcsin(2/3)\Big).$$

(2) Let $u = 9-y^2$, so $du = -2y\,dy$:

$$\int_0^2 y\sqrt{9-y^2}\,dy = -\frac{1}{2}\int_9^5 u^{1/2}\,du = \frac{1}{2}\int_5^9 u^{1/2}\,du = \frac{1}{3}\big(27-5\sqrt{5}\big).$$