Midterm 1 Review

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Practice 1: Critical Points & Second Derivative Test

Given

$$g(x, y) = x^2 + 4xy + 5y^2 - 2x + 6y,$$

1. Find all critical points of g.
2. Use the Second Derivative Test to classify each critical point.

Solution 1

$$g_x = 2x + 4y - 2, \qquad g_y = 4x + 10y + 6.$$

Solve

$$2x + 4y = 2 \Rightarrow x + 2y = 1, \qquad 4x + 10y = -6.$$

From $x = 1 - 2y$:

$$4(1 - 2y) + 10y = -6 \Rightarrow 4 - 8y + 10y = -6 \Rightarrow 2y = -10 \Rightarrow y = -5,$$

$$x = 1 - 2(-5) = 11.$$

So the only critical point is $(11, -5)$.
Hessian:

$$g_{xx} = 2, \quad g_{yy} = 10, \quad g_{xy} = 4, \quad D = g_{xx}g_{yy} - g_{xy}^2 = 2 \cdot 10 - 16 = 4 > 0.$$

Since $g_{xx} = 2 > 0$, $\Rightarrow$ **local minimum at** $(11, -5)$.

Practice 2: Lagrange Multipliers (Max/Min)

Use Lagrange multipliers to find the maximum and minimum values of

$$F(x, y, z) = x^2 yz \quad \text{subject to} \quad x^2 + y^2 + z^2 = 9.$$

Solution 2

Let $h(x, y, z) = x^2 + y^2 + z^2$. Then

$$\nabla F = (2xyz,\ x^2 z,\ x^2 y), \qquad \nabla h = (2x, 2y, 2z),$$

and $\nabla F = \lambda \nabla h$ gives

$$2xyz = 2\lambda x, \quad x^2 z = 2\lambda y, \quad x^2 y = 2\lambda z.$$

Assume $xyz \neq 0$. From the last two equations:

$$\frac{x^2 z}{x^2 y} = \frac{2\lambda y}{2\lambda z} \Rightarrow \frac{z}{y} = \frac{y}{z} \Rightarrow z^2 = y^2.$$

So $z = \pm y$. Also, using $2xyz = 2\lambda x$ (with $x \neq 0$) gives $\lambda = yz = \pm y^2$. Then $x^2 z = 2\lambda y$ implies $x^2 = 2y^2$ (in either sign case).

Use the constraint:

$$x^2 + y^2 + z^2 = 2y^2 + y^2 + y^2 = 4y^2 = 9 \Rightarrow y^2 = \frac{9}{4},$$

$$x^2 = \frac{9}{2}, \qquad z^2 = \frac{9}{4}.$$

Now $F = x^2yz = x^2(\pm y^2)$, so

$$F = \frac{9}{2}\left(\pm\frac{9}{4}\right) = \pm\frac{81}{8}.$$

$$\boxed{F_{\text{max}} = \frac{81}{8}, \qquad F_{\text{min}} = -\frac{81}{8}.}$$

Practice 3: Double Integral Over a Region

Evaluate

$$\iint_R (x^2 + y)\, dA,$$

where R is the region bounded by

$$y = |x| \quad \text{and} \quad y = 2 - |x|.$$

Solution 3

The curves intersect when $|x| = 2 - |x| \Rightarrow |x| = 1$, so $-1 \le x \le 1$. For each x, the bounds are $|x| \le y \le 2 - |x|$. Thus

$$\iint_R (x^2 + y)\, dA = \int_{-1}^{1} \int_{|x|}^{2-|x|} (x^2 + y)\, dy\, dx.$$

By symmetry (integrand and region even in x):

$$= 2 \int_0^1 \int_x^{2-x} (x^2 + y)\, dy\, dx.$$

Compute inner integral:

$$\int_x^{2-x} (x^2 + y)\, dy = \left[x^2 y + \frac{y^2}{2} \right]_x^{2-x} = 2 - 2x + 2x^2 - 2x^3.$$

So

$$\iint_R (x^2 + y)\, dA = 2 \int_0^1 (2 - 2x + 2x^2 - 2x^3)\, dx = 4 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right) = \boxed{\frac{7}{3}}.$$

Practice 4: Write as Iterated Integrals in Two Orders

Let R be the triangle with vertices $(0,0)$, $(5,0)$, and $(2,4)$. Write $\iint_R f(x,y)\,dA$ as an iterated integral in:

1. the order $dy\,dx$,
2. the order $dx\,dy$.

Solution 4

Lines:

$$\overline{(0,0)(2,4)}: \; y = 2x, \qquad \overline{(5,0)(2,4)}: \; y = -\frac{4}{3}x + \frac{20}{3}.$$

Order $dy\,dx$ (split at $x = 2$):

$$\boxed{\iint_R f\,dA = \int_0^2 \int_0^{2x} f(x,y)\,dy\,dx + \int_2^5 \int_0^{-\frac{4}{3}x+\frac{20}{3}} f(x,y)\,dy\,dx.}$$

Order $dx\,dy$ (single integral): for $0 \le y \le 4$,

$$x \text{ runs from } x = \frac{y}{2} \text{ to } x = 5 - \frac{3}{4}y.$$

$$\boxed{\iint_R f\,dA = \int_0^4 \int_{\frac{y}{2}}^{5-\frac{3}{4}y} f(x,y)\,dx\,dy.}$$

Practice 5: Volume Between a Paraboloid and a Plane

Find the volume of the solid bounded by

$$z = 10 - x^2 - y^2 \quad \text{and} \quad z = 2.$$

Solution 5

Intersection: $10 - x^2 - y^2 = 2 \Rightarrow x^2 + y^2 = 8$ (disk $0 \le r \le 2\sqrt{2}$).
Volume:

$$V = \iint_{r \le 2\sqrt{2}} \left[(10 - r^2) - 2\right] dA = \iint_{r \le 2\sqrt{2}} (8 - r^2)\, dA.$$

Polar coordinates ($dA = r\,dr\,d\theta$):

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (8 - r^2)\, r\,dr\,d\theta.$$

$$\int_0^{2\sqrt{2}} (8r - r^3)\, dr = \left[4r^2 - \frac{r^4}{4}\right]_0^{2\sqrt{2}} = 32 - 16 = 16.$$

Hence

$$V = \int_0^{2\pi} 16\, d\theta = 32\pi, \qquad \boxed{V = 32\pi.}$$

1.

Suppose that

$$f(x,y) = x^3 + 6x^2y + a\,xy^2 + b\,y^3$$

for some constants a and b. Find a and b such that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{for every } (x,y).$$

(Equivalently: $f_{xx} + f_{yy} = 0$ for all (x,y).)

Solution to 1

Compute first derivatives:

$$f_x = 3x^2 + 12xy + ay^2, \qquad f_y = 6x^2 + 2axy + 3by^2.$$

Then

$$f_{xx} = 6x + 12y, \qquad f_{yy} = 2ax + 6by.$$

So

$$f_{xx} + f_{yy} = (6x + 12y) + (2ax + 6by) = 2(3 + a)x + 6(2 + b)y.$$

For this to be 0 for all x, y, the coefficients must vanish:

$$2(3 + a) = 0 \Rightarrow a = -3, \qquad 6(2 + b) = 0 \Rightarrow b = -2.$$

$$\boxed{a = -3, \quad b = -2.}$$

2.

Consider the hyperbolic paraboloid

$$z = 2x^2 - 3y^2.$$

(a) In what (unit) direction does z have its maximum rate of change at the point $(2, 1)$?

(b) What is the maximum rate of change in the direction in (a)?

Solution to 2

Let $f(x, y) = 2x^2 - 3y^2$. Then

$$\nabla f(x, y) = (f_x, f_y) = (4x, -6y).$$

At $(2, 1)$:

$$\nabla f(2, 1) = (8, -6), \qquad \|\nabla f(2, 1)\| = \sqrt{8^2 + (-6)^2} = \sqrt{100} = 10.$$

(a) Maximum increase direction is the *unit* gradient:

$$\boxed{\frac{\nabla f(2, 1)}{\|\nabla f(2, 1)\|} = \left(\frac{8}{10}, \frac{-6}{10}\right) = \left(\frac{4}{5}, -\frac{3}{5}\right)}.$$

(b) The maximum rate of change equals $\|\nabla f(2, 1)\|$:

$$\boxed{10}.$$

3.

Find and classify the critical points (local maxima, local minima, or saddle points) of

$$f(x, y) = x^3 + y^3 - 3xy.$$

Solution to 3

$$f_x = 3x^2 - 3y, \qquad f_y = 3y^2 - 3x.$$

Set $f_x = f_y = 0$:

$$3x^2 - 3y = 0 \Rightarrow y = x^2, \qquad 3y^2 - 3x = 0 \Rightarrow x = y^2.$$

Substitute $y = x^2$ into $x = y^2$:

$$x = (x^2)^2 = x^4 \Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

Then $y = x^2$ gives critical points $(0,0)$ and $(1,1)$.
Second derivatives:

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3,$$
$$D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.$$

At $(0,0)$: $D = -9 < 0 \Rightarrow$ saddle point. At $(1,1)$: $D = 27 > 0$ and $f_{xx}(1,1) = 6 > 0 \Rightarrow$ local minimum.

$$\boxed{(0,0) \text{ is a saddle, } (1,1) \text{ is a local minimum}}$$

6.

Let D be the region in the xy-plane bounded by the x-axis, the vertical line $x = 1$, and the line $y = 2x$.

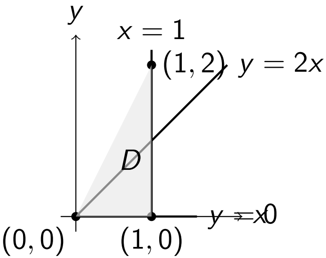
(a) Sketch the region D.

(b) Compute $\displaystyle\iint_D \sqrt{1 - x^2}\, dA$.

Solution to 6(a): Sketch and bounds

The boundaries are $y = 0$, $x = 1$, and $y = 2x$. So D is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.
A convenient description:

$$0 \le x \le 1, \qquad 0 \le y \le 2x.$$



Solution to 6(b): The integral

Using $0 \leq x \leq 1$, $0 \leq y \leq 2x$,

$$\iint_D \sqrt{1-x^2}\,dA = \int_0^1 \int_0^{2x} \sqrt{1-x^2}\,dy\,dx.$$

Integrate in y:

$$= \int_0^1 \sqrt{1-x^2}\,(2x)\,dx.$$

Let $u = 1 - x^2$, so $du = -2x\,dx$. Then

$$\int_0^1 2x\sqrt{1-x^2}\,dx = -\int_{u=1}^{u=0} \sqrt{u}\,du = \int_0^1 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{2}{3}.$$

$$\boxed{\iint_D \sqrt{1-x^2}\,dA = \frac{2}{3}.}$$

1. Directional/Partial Derivatives at the Origin

Consider the function

$$f(x, y) = |xy|^{1/2} + x.$$

(a) By using the definition of partial derivatives (difference quotients), show that $f_x(0, 0)$ and $f_y(0, 0)$ exist.

Solution to 1(a)

By definition,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}.$$

Compute:

$$f(h,0) = |h \cdot 0|^{1/2} + h = h, \qquad f(0,0) = 0.$$

So

$$f_x(0,0) = \lim_{h \to 0} \frac{h}{h} = 1.$$

Similarly,

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}.$$

But $f(0,h) = |0 \cdot h|^{1/2} + 0 = 0$, hence

$$f_y(0,0) = \lim_{h \to 0} \frac{0}{h} = 0.$$

$$\boxed{f_x(0,0) = 1, \qquad f_y(0,0) = 0.}$$

2. Optimization: Minimum-Cost Closed Box

You want to design a **closed** rectangular box at minimum cost with volume 60 cubic inches. The price per square inch is:

- 3 cents for the **top and bottom**,
- 2 cents for the **front and back**,
- 1 cent for the **sides**.

(a) Does the Extreme Value Theorem guarantee a minimizing box? Why/why not?
(b) Find the dimensions that minimize the cost.

Solution to 2(a)

Let the dimensions be $x, y, z > 0$ with constraint $xyz = 60$. The feasible set $\{(x, y, z) : x, y, z > 0,\ xyz = 60\}$ is **not closed and bounded** in $\mathbb{R}^3$, so it is not compact.
Therefore the Extreme Value Theorem **does not apply directly**.
(Nonetheless, a minimum can still exist; EVT just does not guarantee it on this non-compact set.)

Solution to 2(b)

Let $x, y, z > 0$ and $xyz = 60$. Areas and costs:

$$\text{top/bottom: } 2(xy) \cdot 3 = 6xy, \quad \text{front/back: } 2(xz) \cdot 2 = 4xz, \quad \text{sides: } 2(yz) \cdot 1 = 2yz.$$

So the cost (in cents) is

$$C(x, y, z) = 6xy + 4xz + 2yz, \qquad \text{with } xyz = 60.$$

Eliminate $z = \dfrac{60}{xy}$:

$$C(x, y) = 6xy + 4x\frac{60}{xy} + 2y\frac{60}{xy} = 6xy + \frac{240}{y} + \frac{120}{x}.$$

Set partial derivatives to zero:

$$C_x = 6y - \frac{120}{x^2} = 0, \qquad C_y = 6x - \frac{240}{y^2} = 0.$$

From $C_x = 0$: $y = \dfrac{20}{x^2}$. Plug into $C_y = 0$:

$$6x = \frac{240}{(20/x^2)^2} = \frac{240}{400/x^4} = \frac{240x^4}{400} = \frac{3}{5}x^4 \;\Rightarrow\; x^3 = 10 \;\Rightarrow\; x = \sqrt[3]{10}.$$

Then

$$y = \frac{20}{x^2} = \frac{20}{10^{2/3}} = 2 \cdot 10^{1/3} = 2\sqrt[3]{10}, \qquad z = \frac{60}{xy} = \frac{60}{(\sqrt[3]{10})(2\sqrt[3]{10})} = 3\sqrt[3]{10}.$$

$$\boxed{(x, y, z) = \left(\sqrt[3]{10},\ 2\sqrt[3]{10},\ 3\sqrt[3]{10}\right).}$$

5. Steepest Ascent on a Surface

You are standing on a mountain whose shape is

$$z = -2x^2 - 3y^2 + 1500,$$

at the point $(5, 10, 1150)$.
(a) If you walk up the mountain along the **steepest** path, what is the slope of that path at $(5, 10, 1150)$?

Solution to 5(a)

Let $z = f(x,y) = -2x^2 - 3y^2 + 1500$. Then

$$\nabla f(x,y) = \left(f_x, f_y\right) = (-4x, -6y).$$

At $(5,10)$:

$$\nabla f(5,10) = (-20,-60), \qquad \|\nabla f(5,10)\| = \sqrt{(-20)^2 + (-60)^2} = \sqrt{4000} = 20\sqrt{10}.$$

The maximum (steepest) uphill slope equals $\|\nabla f\|$. Hence

$$\boxed{\text{slope of steepest ascent at } (5,10,1150) \text{ is } 20\sqrt{10}\ (= \sqrt{4000}).}$$

6(a). Volume Setup (Do NOT Evaluate)

Set up (but do not evaluate) an integral that gives the volume of the solid bounded **below** by the sphere

$$x^2 + y^2 + (z - 4)^2 = 16$$

and **above** by the cone

$$z = 8 - \sqrt{3(x^2 + y^2)}.$$

Give explicit bounds/domain.

6(a). Volume Setup (Double Integral)

Set up (do not evaluate) a *double* integral for the volume of the solid bounded **below** by the sphere

$$x^2 + y^2 + (z - 4)^2 = 16$$

and **above** by the cone

$$z = 8 - \sqrt{3(x^2 + y^2)}.$$

Solution 6(a): Write $V = \iint_D (z_{\text{top}} - z_{\text{bot}})\, dA$

Let $r = \sqrt{x^2 + y^2}$.

Top surface (cone):

$$z_{\text{top}}(x, y) = 8 - \sqrt{3}\, r.$$

Bottom surface (sphere):

$$x^2 + y^2 + (z-4)^2 = 16 \;\Rightarrow\; (z-4)^2 = 16 - r^2 \;\Rightarrow\; z = 4 \pm \sqrt{16 - r^2}.$$

Since the sphere is the *lower* boundary, take

$$z_{\text{bot}}(x, y) = 4 - \sqrt{16 - r^2}.$$

Hence

$$V = \iint_D \left[\left(8 - \sqrt{3}\, r\right) - \left(4 - \sqrt{16 - r^2}\right) \right] dA = \iint_D \left(4 - \sqrt{3}\, r + \sqrt{16 - r^2}\right) dA.$$

Solution 6(a): Find the projection D in the xy-plane

The boundary of D comes from where the two surfaces meet:

$$8 - \sqrt{3}\, r \;=\; 4 - \sqrt{16 - r^2}.$$

Rearrange:

$$4 + \sqrt{16 - r^2} = \sqrt{3}\, r.$$

At intersection we have $z \geq 0$, so the relevant solution gives

$$r = 2\sqrt{3}.$$

Therefore the projection is the disk

$$D = \{(x, y) : x^2 + y^2 \leq (2\sqrt{3})^2 = 12\}.$$

Double-integral form in polar coordinates $(dA = r\, dr\, d\theta)$:

$$\boxed{V = \int_0^{2\pi} \int_0^{2\sqrt{3}} \left(4 - \sqrt{3}\, r + \sqrt{16 - r^2}\right) r\, dr\, d\theta.}$$

(Do not evaluate.)

Solution 6(a): Same setup as a double integral in x, y

Using $D = \{(x, y) : x^2 + y^2 \leq 12\}$ and $r = \sqrt{x^2 + y^2}$,

$$\boxed{V = \iint_{x^2+y^2\leq 12} \left[\left(8 - \sqrt{3(x^2 + y^2)}\right) - \left(4 - \sqrt{16 - (x^2 + y^2)}\right)\right] dA.}$$

(Do not evaluate.)

Solution to 6(a): Cylindrical setup

Use cylindrical coordinates $x = r\cos\theta,\ \ y = r\sin\theta$.
Sphere:

$$r^2 + (z-4)^2 = 16 \ \Rightarrow\ z = 4 \pm \sqrt{16 - r^2}.$$

Since the sphere is the *lower* boundary, use

$$z_{\text{low}}(r) = 4 - \sqrt{16 - r^2}.$$

Cone:

$$z_{\text{up}}(r) = 8 - \sqrt{3}\, r.$$

Find intersection: set $4 - \sqrt{16 - r^2} = 8 - \sqrt{3}\, r$. This occurs at $r = 2\sqrt{3}$ (and also $r = 0$, the common tip at $z = 8$). Thus $0 \le r \le 2\sqrt{3}$, $0 \le \theta \le 2\pi$, and

$$z_{\text{low}}(r) \le z \le z_{\text{up}}(r).$$

So a correct volume integral is

$$\boxed{V = \int_0^{2\pi} \int_0^{2\sqrt{3}} \int_{4-\sqrt{16-r^2}}^{8-\sqrt{3}\, r} r\, dz\, dr\, d\theta.}$$

6(b). Region + Evaluate a Double Integral

Consider the integral

$$\int_0^1 \int_0^{\frac{\sqrt{1-x}}{\sqrt{3}}} e^{-y^3+y}\, dy\, dx.$$

(i) Draw and label the region of integration.

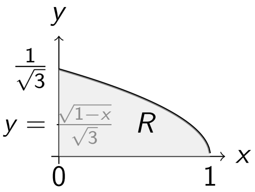
(ii) Evaluate the integral.

Solution to 6(b)(i): Region

The region is

$$0 \le x \le 1, \qquad 0 \le y \le \frac{\sqrt{1-x}}{\sqrt{3}}.$$

Equivalently, it is the set under the curve $y = \frac{\sqrt{1-x}}{\sqrt{3}}$ above $y = 0$.



Solution to 6(b)(ii): Evaluate by reversing order

From $y \le \dfrac{\sqrt{1-x}}{\sqrt{3}}$ we get

$$\sqrt{3}\,y \le \sqrt{1-x} \;\Rightarrow\; 3y^2 \le 1-x \;\Rightarrow\; x \le 1-3y^2.$$

Also $y \ge 0$ and when $x=0$, $y \le \frac{1}{\sqrt{3}}$. So the region can be written as

$$0 \le y \le \frac{1}{\sqrt{3}}, \qquad 0 \le x \le 1-3y^2.$$

Reverse the order:

$$\int_0^1 \int_0^{\frac{\sqrt{1-x}}{\sqrt{3}}} e^{-y^3+y}\,dy\,dx = \int_0^{1/\sqrt{3}} \int_0^{1-3y^2} e^{-y^3+y}\,dx\,dy.$$

Integrate in x:

$$= \int_0^{1/\sqrt{3}} (1-3y^2)\,e^{-y^3+y}\,dy.$$

Let $u=-y^3+y$. Then $du=(1-3y^2)\,dy$.

Hence

$$\int_0^{1/\sqrt{3}} (1-3y^2)e^{-y^3+y}\,dy = [e^u]_{y=0}^{y=1/\sqrt{3}} = e^{-y^3+y}\Big|_0^{1/\sqrt{3}}.$$

Compute at $y = \frac{1}{\sqrt{3}}$:

$$-y^3 + y = -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}.$$

Therefore the value is

$$\boxed{e^{\frac{2}{3\sqrt{3}}} - 1.}$$