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Definition 6.2

A polygonal curve in the plane is the union of finitely many line segments such that each segment starts at the end of the previous one and no point lies in more than one segment, except the end of one segment and the beginning of the next one coincide.
A simple open polygonal curve is one homeomorphic to a closed interval.
A simple closed polygonal curve is one homeomorphic to a unit circle.

Definition 6.3

- A drawing of a graph G is a function that maps each vertex $v \in V(G)$ to a point $f(v)$ in the plane, and each uv-edge to a simple polygonal $f(u)f(v)$-curve in the plane.

Plane Graphs

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Note 6.4

A plane embedding corresponds to an embedding of the graph in the sphere through a stereographic projection.

Faces

Theorem 6.5 (Jordan Curve Theorem)

If C is a simple closed polygonal curve in the plane, then the complement of C in the plane consists of two connected components each with C as the boundary.

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  - the vertices of G^* are the faces of G;

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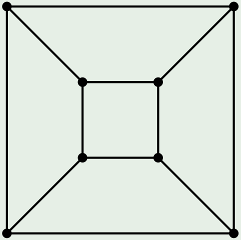
- The length of a face is the number of edges in the boundary of the face, with cut-edges counted twice.
- The dual graph G^* of a non-empty plane graph G is the graph such that
  - the vertices of G^* are the faces of G;
  - the edges of G^* are the edges of G;
  - a vertex and an edge of G^* are incident if and only if the edge is the boundary of the corresponding face of G.

Example of a Dual Graph

Example 6.7



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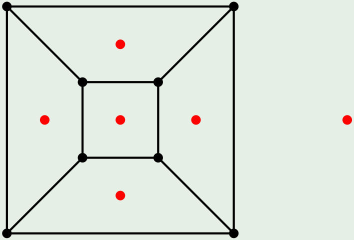

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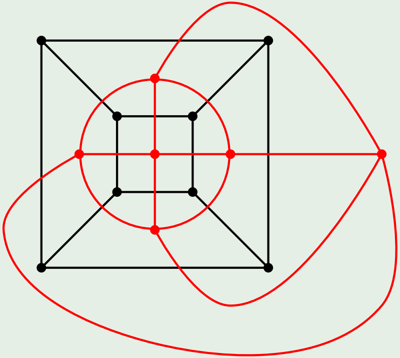

Properties of Dual Graphs

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Theorem 6.9

Edges in a plane graph form a cycle if and only if the edges in the dual graph form a bond.

Properties of Dual Graphs, Continued

Theorem 6.10

If a plane graph G is connected, then the Tutte connectivity of G is the same as the Tutte connectivity of G^*.

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The following are equivalent for a plane graph G:

(A) *G is bipartite;*

(B) *every face of G has even length;*

(C) *G^* is Eulerian.*

Note 6.12

- *Deleting an edge or a vertex from a plane graph results in a plane graph.*

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- *Contracting an edge in a plane graph can be visualized as sliding the two endvertices towards each other until they meet, pulling all incident edges along.*
- *Thus the class of planar graphs is minor-closed, that is, all minors of planar graphs are also planar.*

Euler's Formula

Theorem 6.13 (Euler's Formula)

If a connected non-empty plane graph has v vertices, e edges, and f faces, then $v - e + f = 2$.

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Proof.

We proceed by induction on v.

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Suppose $v > 1$.

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Suppose $v > 1$. Since G is connected, it has a non-loop edge.

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Suppose $v > 1$. Since G is connected, it has a non-loop edge. Contract such an edge to obtain a plane graph with $v' = v - 1$ vertices, $e' = e - 1$ edges, and $f' = f$ faces.

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Corollaries of Euler's Formula

Note 6.14

- *Euler's Formula implies that all plane embeddings of connected graphs have the same number of faces.*

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Corollary 6.15

If G is a planar graph whose order v is at least 3, whose size is e, and whose girth g is at least 3 but finite, then

$$e \leq \frac{(v-2)g}{g-2}.$$

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If G is simple, then $e \leq 3v - 6$.

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Example 6.16

Is K_5 planar?
No, since $e = 10 > 3v - 6 = 9$.

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Is $K_{3,3}$ planar?
No, since

$$e = 9 > \frac{(v-2)g}{g-2} = 8.$$

Statement of the Kuratowski Theorem

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Lemma 6.19

If F is the edge-set of the boundary of a face of a plane graph G, then G has an plane embedding in which F is the boundary of the infinite face.

Minimally No-Planar Graphs

Definition 6.20

A graph is minimally non-planar if it is non-planar, but every proper subgraph of it is planar.

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Every minimal non-planar graph is 2-connected.

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Proof.

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If G is disconnected, we can embed one component of G inside one face of the rest of G. Similarly, if G has a cut-vertex v, let G_1, G_2, ..., G_k be the subgraphs of G induced by v together with the components of $G - v$.

□

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Kuratowski Graphs and Topological Minors

Lemma 6.22

Suppose $G = H_1 \oplus_2 H_2$ is non-planar.

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Definition 6.23

- A Kuratowski subgraph is a subgraph isomorphic to a subdivision of K_5 or of $K_{3,3}$.
- A vertex of a graph G is a branch vertex of a Kuratowski subgraph H of G, if its degree in H exceeds two.

Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G.

Tutte's Version of Kuratowski's Theorem

Definition 6.25

A plane embedding is convex if every face except the infinite one is a convex polygon.

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A plane embedding is convex if every face except the infinite one is a convex polygon.

Theorem 6.26 (Tutte 1960–63)

If G is a simple 3-connected graph with neither K_5 nor $K_{3,3}$ as the topological minor, then G has a convex embedding in the plane with no three vertices on a line.