Vertex Covers

Definition 3.12

- A vertex cover of G is a set S of vertices such that every edge of G is incident with at least one element of S.
- The vertices in S cover the edges of G.

Theorem 3.13 (König-Egerváry 1931)

If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover in G.

Easy Direction.

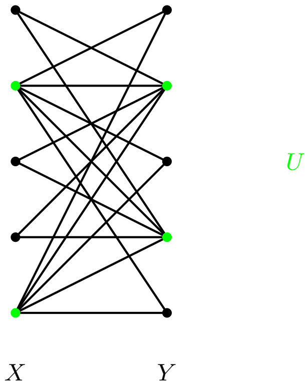
Since distinct vertices must be used to cover the edges of a matching, we have $|U| \geq |M|$ whenever U is a vertex cover and M is a matching. $\square$

Proof of König-Egerváry Theorem, Continued

Given a minimum vertex cover U, we construct a matching of size $|U|$.

Proof of König-Egerváry Theorem, Continued

Given a minimum vertex cover U, we construct a matching of size $|U|$. Suppose G has bipartition $\{X, Y\}$.



Proof of König-Egerváry Theorem, Continued

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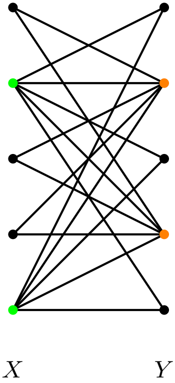

Proof of König-Egerváry Theorem, Continued

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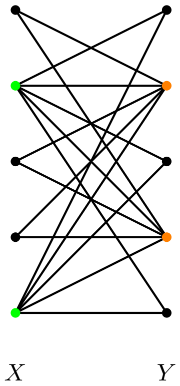

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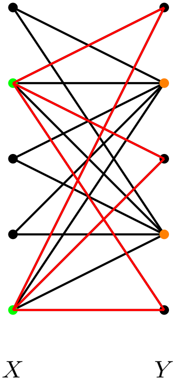

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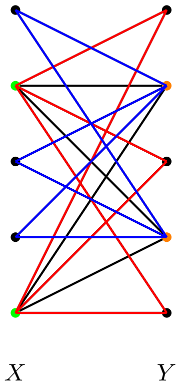

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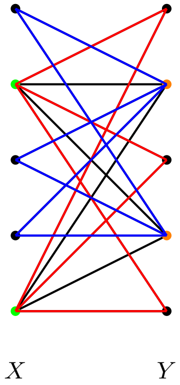

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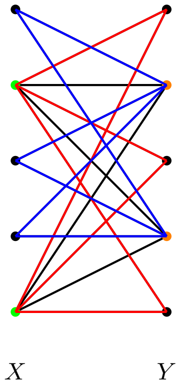

Proof of König-Egerváry Theorem, Continued

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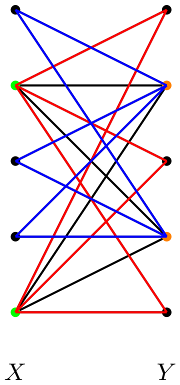

Proof of König-Egerváry Theorem, Continued

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