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- A graph is Hamiltonian if it has a Hamilton cycle.

Sufficient Condition for Hamiltonicity

Theorem 8.2 (Dirac 1952)

Every graph of order $n \geq 3$ *and* $\delta \geq n/2$ *is Hamiltonian.*

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Let $P = x_0 x_1 \ldots x_k$ be a longest path in G. By the maximality of P, all neighbors of x_0 and all neighbors of x_k lie on P. Hence at least $n/2$ of the vertices x_0, x_1, ..., x_{k-1} are adjacent to x_k,

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Another Sufficient Condition

Theorem 8.4

Every graph G with $|G| \geq 3$ and $\kappa(G) \geq \alpha(G)$ is Hamiltonian.

A Necessary Condition

Theorem 8.5

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Proof.

When leaving a component of $G - S$, a Hamilton cycle can go only to S and the arrivals in S must occur at different vertices of S. Hence S must have at least as many vertices as $G - S$ has components. $\square$

Grinberg's Theorem

Theorem 8.6 (Grinberg 1968)

If G is a loopless plane graph with a Hamilton cycle C, and G has f_i' faces of length i inside C and f_i'' faces of length i outside C, then $\sum_i (i-2)(f_i' - f_i'') = 0$.

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Corollary 8.7

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Theorem 8.8 (Tutte 1956)

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Every 4-connected planar graph is Hamiltonian.

Theorem 8.9 (Thomas, Yu 1994)

Every 4-connected projective graph is Hamiltonian.

Theorem 8.10 (Thomas, Yu 1997)

Every 5-connected toroidal graph is Hamiltonian.