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- The distance between sets U and W of vertices of G, written $d(U, W)$, is the length of a shortest uw-path where $u \in U$ and $w \in W$, or infinity if no such path exists.

Induction for Trees

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Then T' is acyclic.
Suppose u and w are vertices of T'. Then, in T there is a uw-path P.
But P cannot contain v as $d_T(v) = 1$, and so it also lies in T'. $\square$

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(D) *For every two vertices u and v, the graph G contains exactly one uv-path.*

Edge Exchange

Theorem 2.4

If T and T' are two spanning trees of a connected graph G and $e \in E(T) \setminus E(T')$, then there is an edge $e' \in E(T') \setminus E(T)$ such that $T \setminus e \cup e'$ is a spanning tree of G.

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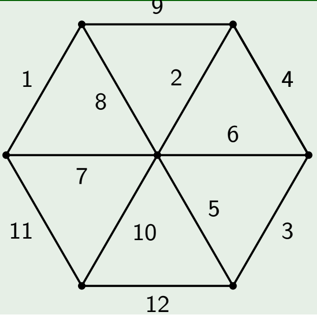

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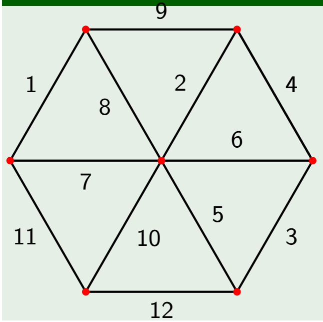

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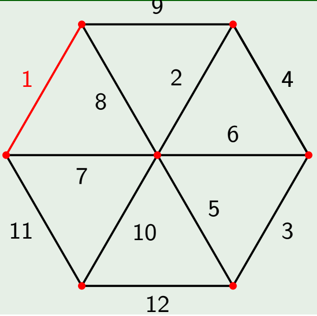

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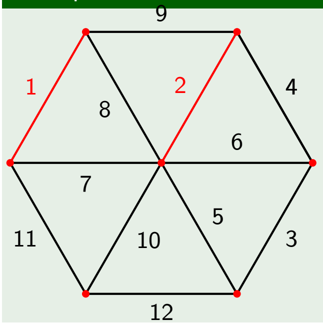

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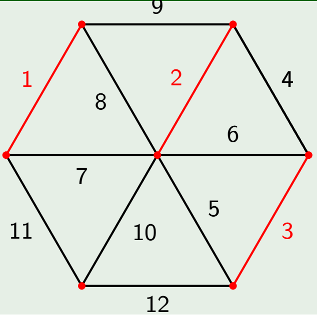

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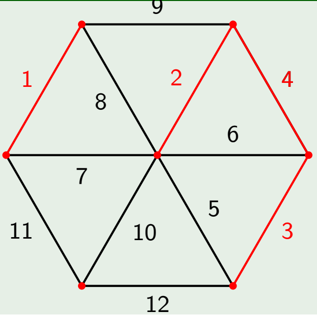

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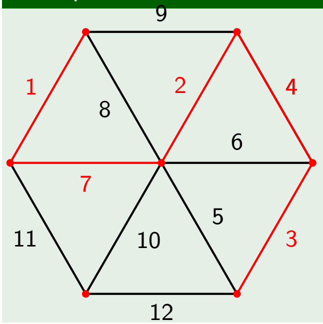

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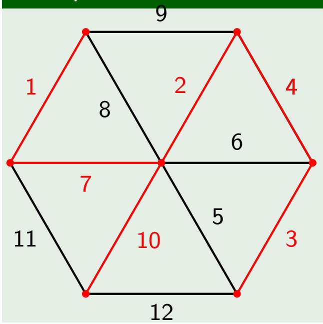

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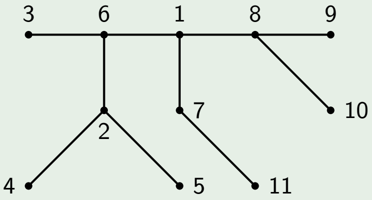

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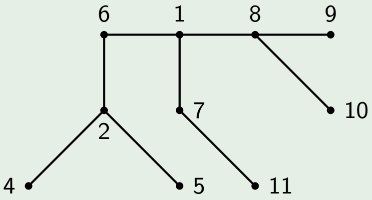

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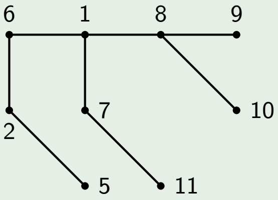

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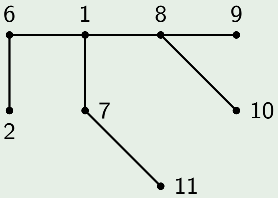

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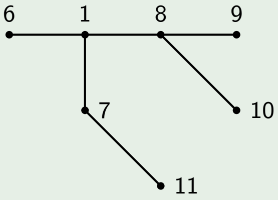

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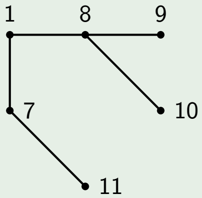

Prüfer sequence: 6 , 2 , 2 , 6 , 1

Prüfer Sequences

To find a Prüfer sequence $f(T)$ of a labeled tree T,

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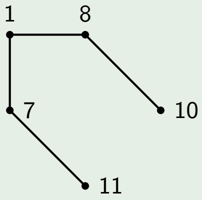

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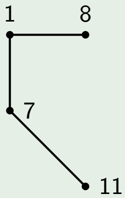

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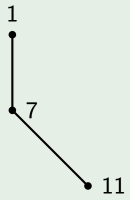

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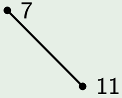

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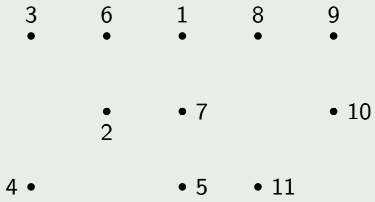

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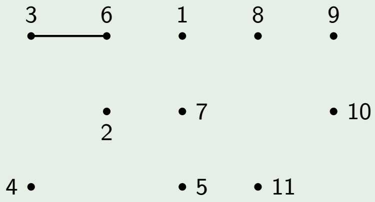

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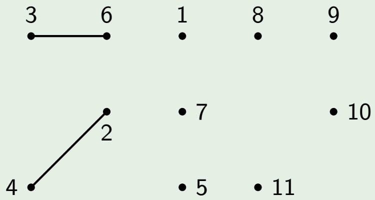

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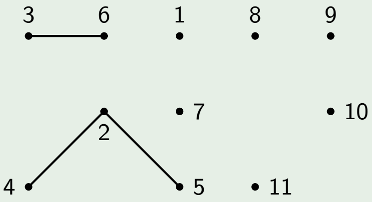

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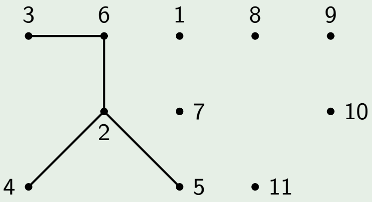

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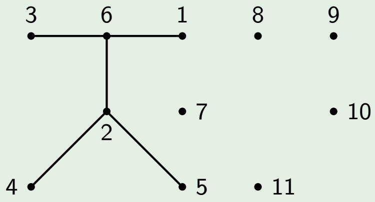

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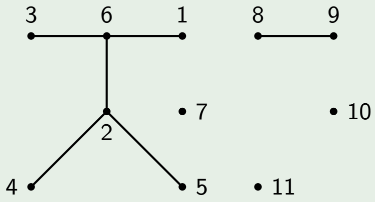

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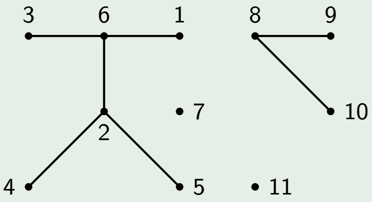

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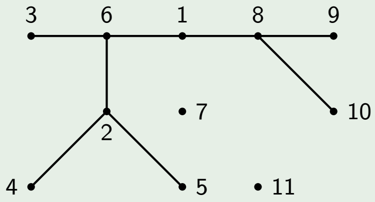

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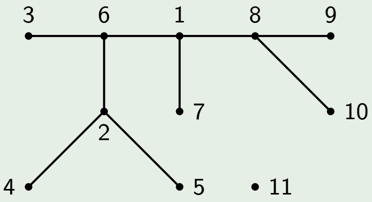

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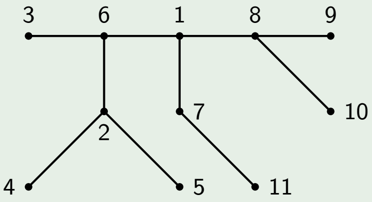

Counting Trees with Prescribed Degrees

Corollary 2.11

The number of trees with vertex set $\{1, 2, \ldots, n\}$ *in which vertices* 1, 2, $\ldots$, n *have respective degrees* d_1, d_2, $\ldots$, d_n *is*

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Minors

Definition 2.12

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Note 2.13

The order of operations of deleting and contracting to get a minor of a graph is irrelevant.

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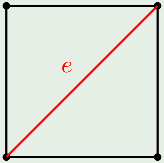

G

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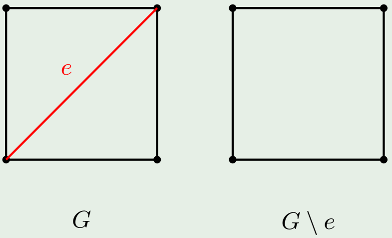

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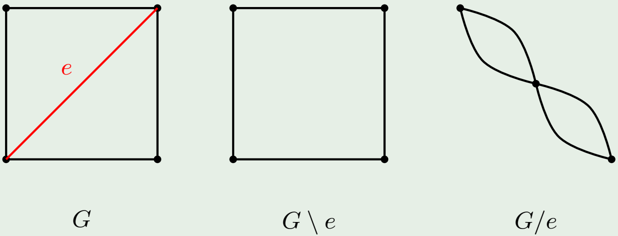

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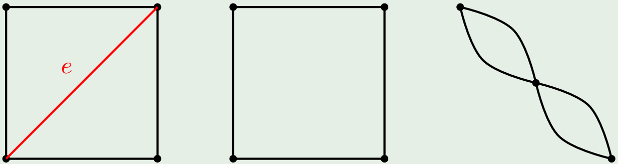

G $G \setminus e$ G/e

$$\tau(G) = \tau(G \setminus e) + \tau(G/e) = 4 + 4 = 8$$

Proof of Spanning Tree Formula

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- The spanning trees of $G \setminus e$ are precisely the spanning trees of G that avoid e.
- The spanning trees of G/e correspond to the spanning trees of G using e. (If T is a spanning tree of G/e, then $E(T) \cup e$ form the edge-set of a spanning tree of G.)
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Using the deletion-contraction formula for calculating the number of spanning trees is inefficient. A much more efficient method is to construct a special matrix, called the Laplacian of the graph, and to compute its determinant.