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Proof.

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Color greedily: Order the vertices arbitrarily as $v_1, v_2, \ldots, v_n$. Starting with $k = 1$, color each vertex v_k with the smallest color not used among the vertices $v_1, v_2, \ldots, v_{k-1}$ that are neighbors of v_k. □

Brooks' Theorem

Theorem 7.2 (Brooks 1941)

If G is a connected simple graph other than a clique and an odd cycle, then $\chi(G) \leq \Delta(G)$.

5-Color Theorem

Theorem 7.3 (Heawood 1890)

Every loopless planar graph has a proper 5*-coloring.*

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Theorem 7.4 (4-Color Theorem, Appel and Haken 1977)

Every loopless planar graph has a proper 4-coloring.

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Theorem 7.6 (Four-Color Theorem, restated)

If G is a planar loopless graph, then $P_G(4) > 0$.

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Theorem 7.9 (Strong Graph Theorem (formerly Berge's Strong Graph Conjecture), Chudnovsky, Robertson, Seymour, Thomas 2002)

A graph is perfect if and only if it has no induced subgraph that is an odd cycle of length at least five or its complement.

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- A graph is k-edge-colorable if it has a proper k-edge-coloring.
- The chromatic index or edge chromatic number $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

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- A k-edge-coloring of a graph G is a labeling $f : E(G) \to \{1, 2, \dots, k\}$.
- The labels are colors and the edge-set with one color is a color class.
- A k-edge-coloring is proper if adjacent edges have different colors, or equivalently, if every color class is a matching.
- A graph is k-edge-colorable if it has a proper k-edge-coloring.
- The chromatic index or edge chromatic number $\chi'(G)$ of a loopless graph G is the least k such that G is k-edge-colorable.

Note 7.11

$\Delta(G) \leq \chi'(G)$.

Edge-Coloring of Bipartite Graphs

Theorem 7.12 (König 1916)

If G is bipartite, then $\chi'(G) = \Delta(G)$.

Vizing's Theorem

Theorem 7.13 (Vizing 1964–65, Gupta 1966)

If G *is simple, then* $\chi'(G) \leq \Delta(G) + 1$.

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But there are graphs for which $\mathrm{ch}(G) \neq \chi(G)$. Consider $K_{3,3}$ where each side of the bipartition has lists $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. The list-chromatic number of this graph is 3, while the chromatic number is 2.

Every Planar Graph Is 5-Choosable

Theorem 7.14 (Thomassen 1994)

Every planar graph is 5-choosable.