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Connectivity Examples

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- An n-wheel W_n is obtained from C_n by adding a new vertex and joining it to all vertices of C_n. If $n \geq 3$, then $\kappa(W_n) = 3$.

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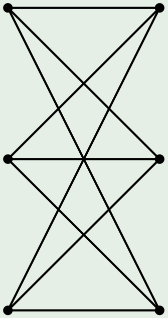

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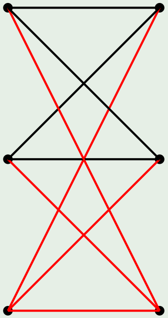

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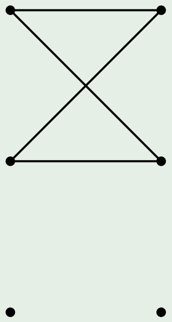

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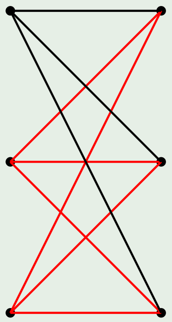

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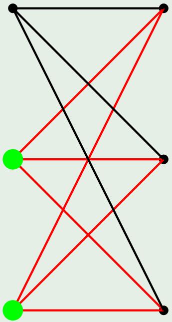

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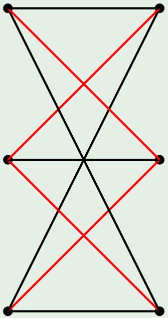

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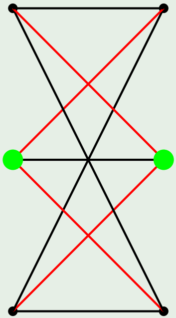

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Whitney's Theorem

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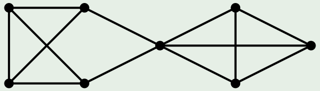

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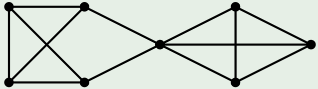

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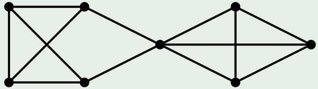

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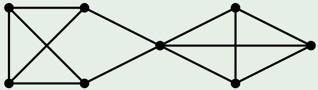

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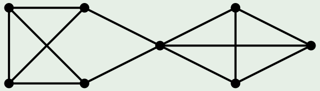

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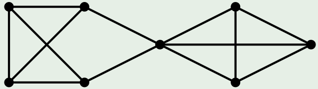

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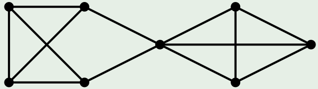

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- If G has a k-separation for some k, then Tutte connectivity of G is $\min\{j : G$ has a j separation$\}$, and ∞ if no k-separation exists.

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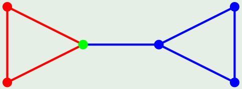
1-separations: 

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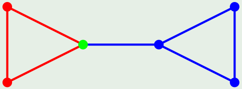
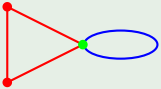
1-separations:  

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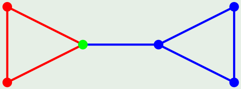

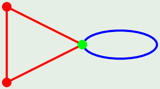

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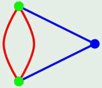

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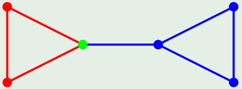

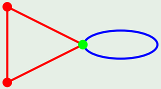

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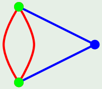

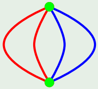

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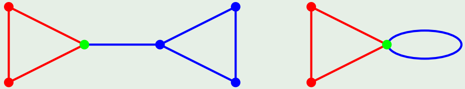

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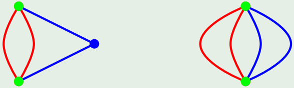

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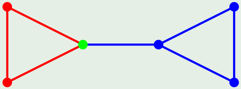
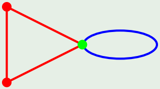
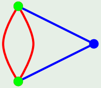
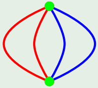
1-separations:  

2-separations:  

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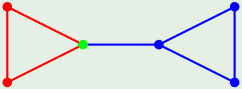
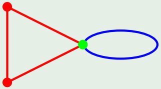
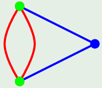
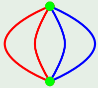
1-separations:  

2-separations:  

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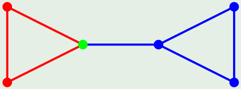
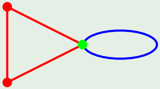
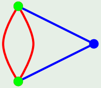
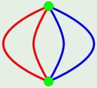
1-separations:  

2-separations:  

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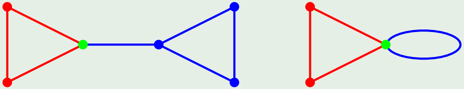

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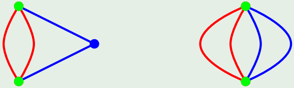

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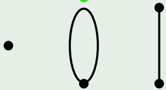

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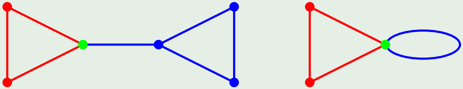

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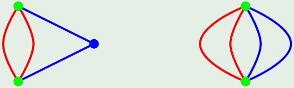

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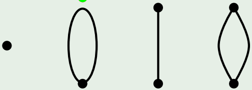

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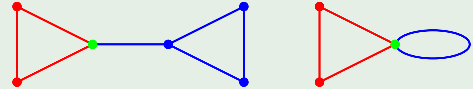

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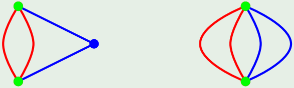

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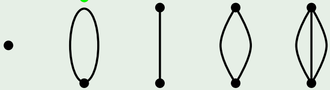

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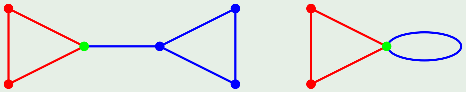

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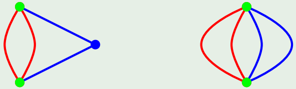

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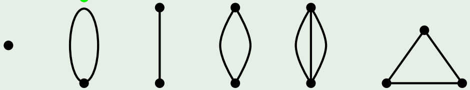

Tutte Connectivity vs. Vertex Connectivity

Theorem 5.12

If G is a graph on at least 3 vertices and $G \ncong K_3$, then the Tutte connectivity of G is $\min(\kappa(G), g(G))$, where $g(G)$ is the girth of G, that is, the length of a shortest cycle in G.

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Note 5.14

A block of a non-empty graph is an isolated vertex, a loop-graph, a graph on two vertices with a positive number of edges between those vertices, or is vertex-2-connected.

Block Tree

Note 5.15

Two distinct blocks in a graph share at most one vertex since otherwise their union would be Tutte-2-connected.

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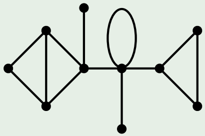

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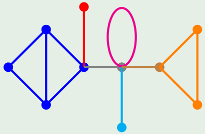

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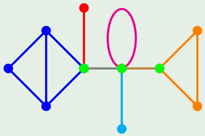

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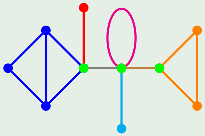

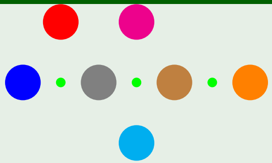

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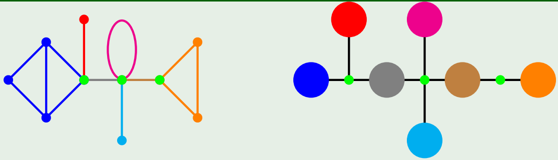

Whitney's Characterization of 2-Connected Graphs

Definition 5.18

Two paths are internally-disjoint if neither contains a non-endpoint of the other.

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Theorem 5.19 (Whitney)

A graph with at least three vertices is 2-connected if and only if each pair u and v of vertices is connected by a pair internally-disjoint uv-paths.

Expansion Lemma

Lemma 5.20 (Expansion Lemma)

If G is a k-connected graph and G' is obtained from G by adding a new vertex y adjacent to at least k vertices of G, then G' is also k-connected.

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Proof.

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$\square$

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Suppose S is a separating set of G'. If $y \in S$, then $S - y$ separates G, so $|S| \geq k + 1$. If $y \notin S$ and $N(y) \subseteq S$, then $|S| \geq k$.

$\square$

Expansion Lemma

Lemma 5.20 (Expansion Lemma)

If G is a k-connected graph and G' is obtained from G by adding a new vertex y adjacent to at least k vertices of G, then G' is also k-connected.

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Suppose S is a separating set of G'. If $y \in S$, then $S - y$ separates G, so $|S| \geq k + 1$. If $y \notin S$ and $N(y) \subseteq S$, then $|S| \geq k$. Otherwise, S must separate G, and again $|S| \geq k$. $\square$

Characterization of 2-Connected Graphs

Theorem 5.21

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(D) *$\delta \geq 1$ and every pair of edges of G lies on a common cycle.*

Subdivisions

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  - deleting an edge;
  - deleting a vertex; and
  - contracting an edge incident with a vertex of degree two (un-subdividing an edge).

Corollary 5.23

A subdivision of a 2-connected graph is also 2-connected.

Ears

Definition 5.24

- A path addition to G is the addition to G of a path of length $\ell \geq 1$ between two vertices of G, introducing $\ell - 1$ new vertices.

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- An ear decomposition is a partition of $E(G)$ into sets $R_0, R_1, \ldots, R_k$ so that $C = R_0$ is a cycle, and R_i, for $i > 0$, is a path addition to the graph $R_0 \cup R_1 \cup \ldots \cup R_{i-1}$.

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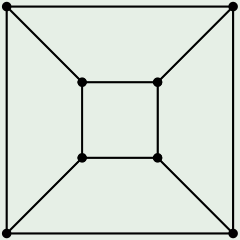
Example 5.25



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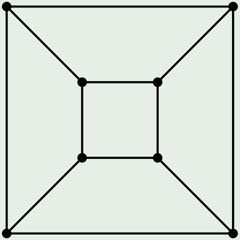

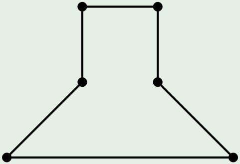

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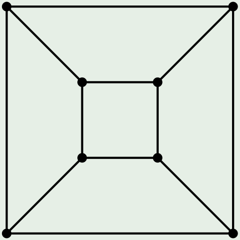

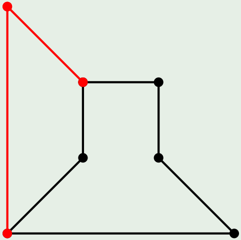

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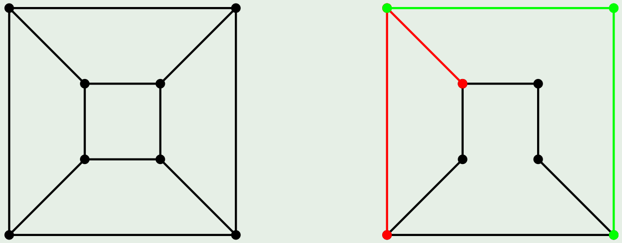

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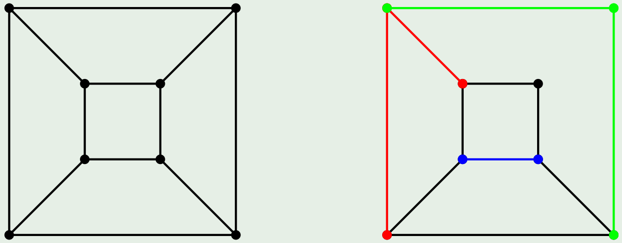

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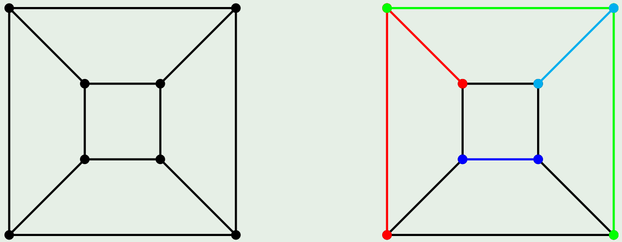

Whitney's Ear Decomposition

Theorem 5.26 (Whitney's Ear Decomposition)

A simple graph is 2*-connected if and only if it has an ear decomposition.*

Whitney's Ear Decomposition

Theorem 5.26 (Whitney's Ear Decomposition)

A simple graph is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle of some ear decomposition.

Closed-Ear Decomposition

Definition 5.27

A closed-ear decomposition of a graph G is a partition of $E(G)$ into sets R_0, $R_1, \ldots, R_k$ such that R_0 is a cycle and R_i for $i > 0$ is either a path addition

Closed-Ear Decomposition

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A closed-ear decomposition of a graph G is a partition of $E(G)$ into sets R_0, R_1, ..., R_k such that R_0 is a cycle and R_i for $i > 0$ is either a path addition or a cycle with exactly one vertex in $R_0 \cup R_1 \cup \ldots R_{i-1}$ (closed ear).

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Theorem 5.28

A simple graph is 2-edge-connected if and only if it has a closed-ear decomposition.

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Theorem 5.28

A simple graph is 2-edge-connected if and only if it has a closed-ear decomposition. Moreover, every cycle in a 2-edge-connected graph is the initial cycle in some closed-ear decomposition.

The Menger Theorem

Theorem 5.29 (Menger 1927)

If x and y are non-adjacent distinct vertices of a graph G, then the minimum size of a vertex-cut separating x from y equals the maximum number of pairwise internally-disjoint xy-paths.

The Edge Version of Menger's Theorem

Theorem 5.30 (Edge Version of Menger's Theorem)

If x and y are distinct vertices of a graph, then the minimum size $\kappa'(x, y)$ of the set of edges that separate x from y equals the maximum number $\lambda'(x, y)$ of pairwise edge-disjoint xy-paths.

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Definition 5.31

The line graph of a graph G, written $L(G)$, is a simple graph whose vertex set is $E(G)$ with two vertices adjacent if the corresponding edges are adjacent in G.

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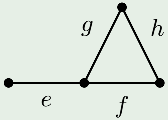

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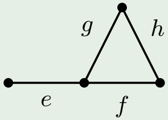

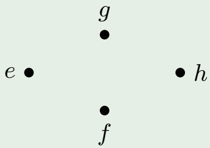

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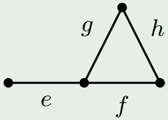

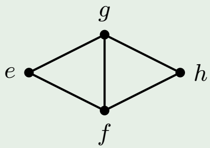

Tutte's Wheel Theorem

Theorem 5.34 (Tutte's Wheel Theorem)

If G is a Tutte-3-connected graph on at least four vertices that is not a wheel, then there is an edge e of G such that at least one of G/e and $G \setminus e$ is also Tutte-3-connected.

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Lemma 5.35 (Thomassen 1980)

Every 3-connected graph G on at least five vertices has an edge e such that G/e is 3-connected.