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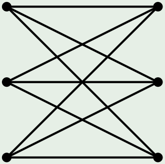

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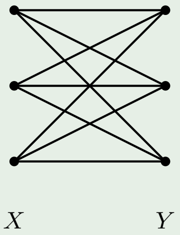

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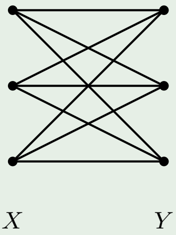


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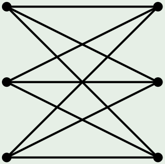

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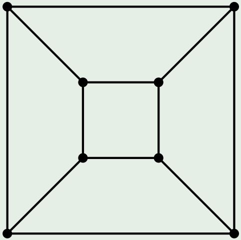

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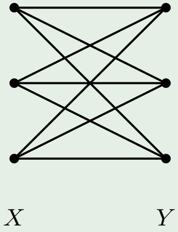


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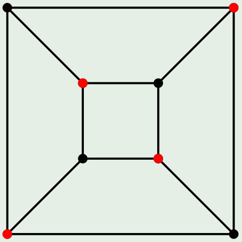

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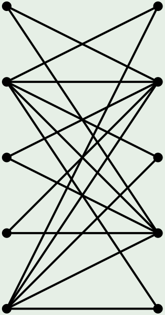

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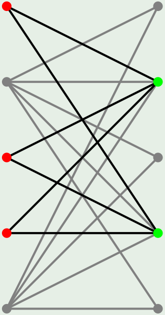

Does $G$ have a matching that saturates all vertices on the left side?
No! Look at $S$, which has $3$ elements, and $N(S)$, which has only $2$ elements.

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**Theorem 3.6 (Hall's Marriage Theorem, 1935)**

*Suppose $G$ is a bipartite graph with bipartition $\{X, Y\}$. The graph $G$ has a matching saturating $X$ if and only if $|N(S)| \geq |S|$ for every subset $S$ of $X$.*

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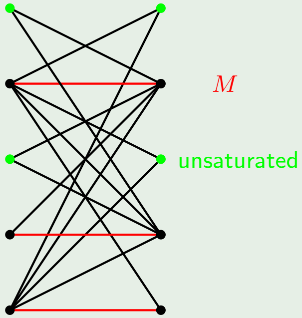

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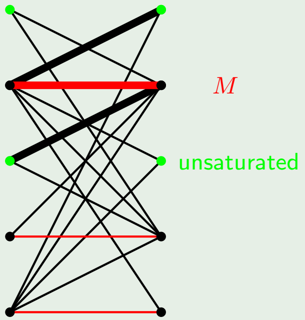

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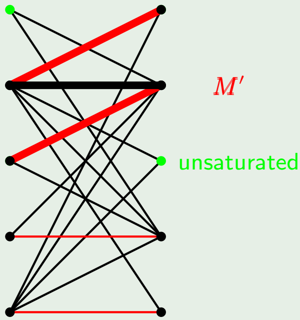

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Counting the edges by endpoints in $X$ and by endpoints in $Y$, we conclude that $k|X| = k|Y|$, and so $|X| = |Y|$, and so every matching saturating $X$ is perfect. Consider $S \subseteq X$, and suppose that there are $m$ edges between $S$ and $N(S)$. Since $G$ is $k$-regular, we have $m = k|S|$. Since these $m$ edges are incident to $N(S)$, we have $m \leq k|N(S)|$. Hence $k|S| \leq k|N(S)|$ and the Hall's condition holds. $\square$

# Vertex Covers

**Definition 3.12**

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**Theorem 3.13 (König-Egerváry 1931)**

*If $G$ is a bipartite graph, then the maximum size of a matching in $G$ equals the minimum size of a vertex cover in $G$.*

# Matchings in Non-Bipartite Graphs

**Note 3.14**

- *Hall's Marriage Theorem 3.6 does not make sense for non-bipartite graphs.*

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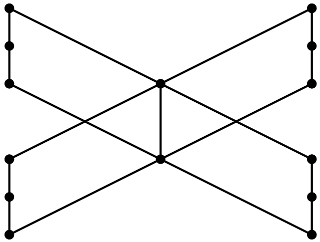
Does the graph below have a perfect matching?



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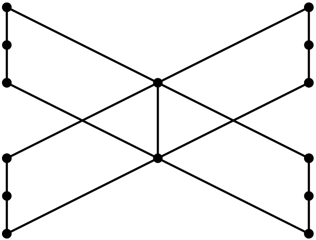

No, since removing the two vertices in the middle leaves more than two components of odd order.

# Tutte's 1-Factor Theorem

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**Theorem 3.16 (Tutte 1-Factor)**

*A graph $G$ has a perfect matching if and only if $q(G - S) \leq |S|$ for every $S \subseteq V(G)$.*

# Petersen's Theorem

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A an edge $e$ of $G$ is a cut-edge if $G \setminus e$ has more connected components than $G$.

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## Corollary 3.18 (Petersen 1891)

*Every simple $3$-regular graph with no cut-edge has a perfect matching.*