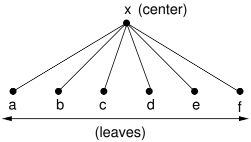
Centrality:

- Represents a “measure of importance”.
  - Usually for nodes.
  - Some measures can also be defined for edges (or subgraphs, in general).

Point Centrality – A Qualitative Measure

Example:



- The **center** node is "structurally more important" than the other nodes.

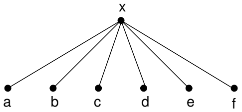
Reasons for the importance of the center node:

- The center node has the maximum possible degree.
- It lies on the shortest path ("geodesic") between any pair of other nodes (leaves).
- It is the closest node to each leaf.

Degree Centrality – A Quantitative Measure

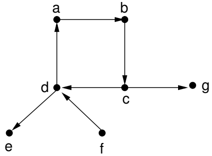
- For an **undirected** graph, the **degree** of a node is the number of edges incident on that node.
- For a **directed** graph, both **indegree** (i.e., the number of incoming edges) and **outdegree** (i.e., the number of outgoing edges) must be considered.

Example 1:



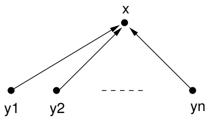
- Degree of $x = 6$.
- For all other nodes, degree $= 1$.

Example 2:



- Indegree of $b = 1$.
- Outdegree of $d = 2$.

When does a large indegree imply higher importance?

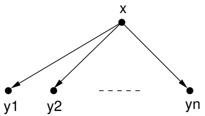


- Consider the Twitter network.
- Think of x as a **celebrity** and the other nodes as followers of x.

- For a different context, think of each node in the directed graph as a web page.
- Each of the nodes y_1, y_2, ..., y_n has a link to x.
- The larger the value of n, the higher is the "importance" of x (a crude definition of **page rank**).

Degree Centrality (continued)

When does a large outdegree imply higher importance?



- Consider the hierarchy in an organization.
- Think of x as the manager of y_1, y_2, ..., y_n.
- Large outdegree may mean more "power".

Undirected graphs:

- High degree nodes are called **hubs** (e.g. airlines).
- High degree may also also represent higher **risk**.

  Example: In disease propagation, a high degree node is more likely to get infected compared to a low degree node.

Normalized Degree

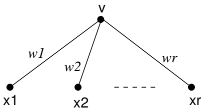
Definition: The **normalized degree** of a node x is given by

$$\text{Normalized Degree of } x = \frac{\text{Degree of } x}{\text{Maximum possible degree}}$$

- Useful in comparing degree centralities of nodes between two networks.

Example: A node with a degree of 5 in a network with 10 nodes may be relatively more important than a node with a degree of 5 in a network with a million nodes.

Weighted Degree Centrality (Strength):



- Weighted degree (or strength) of $v = w_1 + w_2 + \ldots + w_r$.

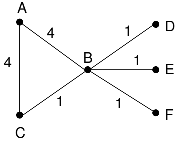
Degree Centrality (continued)

Assuming an **adjacency list** representation

- for an undirected graph $G(V, E)$, the degree (or weighted degree) of all nodes can be computed in **linear** time (i.e., in time $O(|V| + |E|)$) and
- for a directed graph $G(V, E)$, the indegree or outdegree (or their weighted versions) of all nodes can be computed in **linear** time.

Combining degree and strength:

Motivating Example:



- A and B have the same strength.
- However, B seems more central than A.

Combining Degree and Strength (continued)

- Let d and s be the degree and strength of a node v respectively.
- Let α be a parameter satisfying the condition $0 \leq \alpha \leq 1$.
- The combined measure for node $v = d^{\alpha} \times s^{1-\alpha}$.
- When $\alpha = 1$, the combined measure is the **degree**.
- When $\alpha = 0$, the combined measure is the **strength**.
- A suitable value of α must be chosen for each context.

Farness and Closeness Centralities

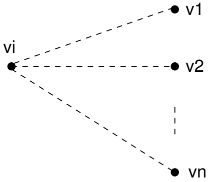
Assumptions:

- Undirected graphs. (Extension to directed graphs is straightforward.)
- Connected graphs.
- No edge weights. (Extension to weighted graphs is also straightforward.)

Notation:

- Nodes of the graph are denoted by v_1, v_2, ..., v_n. The set of all nodes is denoted by V.
- For any pair of nodes v_i and v_j, d_{ij} denotes the number of edges in a shortest path between v_i and v_j.

Farness and Closeness Centralities (continued)



- A schematic showing shortest paths between node v_i and the other nodes of an undirected graph.

Definition: The **farness centrality** f_i of node v_i is given by

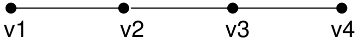
$$\begin{aligned} f_i &= \text{Sum of the distances between } v_i \text{ and the other nodes} \\ &= \sum_{v_j \in V - \{v_i\}} d_{ij} \end{aligned}$$

Definition: The **closeness centrality** (or **nearness centrality**) η_i of node v_i is given by $\eta_i = 1/f_i$.

Note: If a node x has a larger closeness centrality value compared to a node y, then x is more central than y.

Farness and Closeness Centralities (continued)

Example 1:

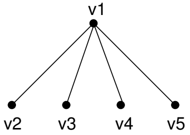


- $f_1 = 1 + 2 + 3 = 6$. So, $\eta_1 = 1/6$.
- $f_2 = 1 + 1 + 2 = 4$. So, $\eta_2 = 1/4$.
- $f_3 = 2 + 1 + 2 = 4$. So, $\eta_3 = 1/4$.
- $f_4 = 3 + 2 + 1 = 6$. So, $\eta_4 = 1/6$.

So, in the above example, nodes v_2 and v_3 are more central than nodes v_1 and v_4.

Farness and Closeness Centralities (continued)

Example 2:



- $f_1 = 4$. So, $\eta_1 = 1/4$.
- For every other node, the farness centrality value = 7; so the closeness centrality value = 1/7.
- Thus, v_1 is more central than the other nodes.

Remarks:

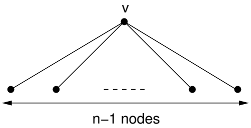
- For any graph with n nodes, the **farness centrality** of each node is **at least** $n - 1$.

  Reason: Each of the other $n - 1$ nodes must be at a distance of at least 1.

Farness and Closeness Centralities (continued)

Remarks (continued):

- Since the farness centrality of each node is at least $n-1$, the **closeness centrality** of any node must be **at most** $1/(n-1)$.



- For the star graph on the left, the closeness centrality of the center node v is exactly $1/(n-1)$.

- If G is an n-clique, then the closeness centrality of each node of G is $1/(n-1)$.

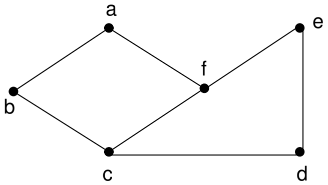
An Algorithm for Computing Farness and Closeness

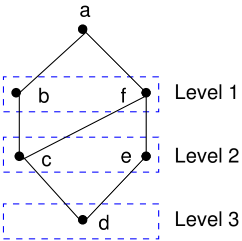
Assumptions: The given undirected graph is **connected** and does **not** have edge weights.

Computing Farness (or closeness) Centrality (Idea):

- A Breadth-First-Search (BFS) starting at a node v_i will find shortest paths to all the other nodes.

Example:





An Algorithm for Farness ... (continued)

Let $G(V, E)$ denote the given graph.

- Recall that the time for doing a BFS on $G = O(|V| + |E|)$.
- So, farness (or closeness) centrality for any node of G can be computed in $O(|V| + |E|)$ time.
- By carrying out a BFS from each node, the time to compute farness (or closeness) centrality for **all** nodes of G $= O(|V|(|V| + |E|))$.
- The time is $O(|V|^3)$ for **dense** graphs (where $|E| = \Omega(|V|^2)$) and $O(|V|^2)$ for **sparse** graphs (where $|E| = O(|V|)$).

Eccentricity Measure

- Recall that **farness centrality** of a node v_i is given by

$$f_i \;=\; \sum_{v_j \in V - \{v_i\}} d_{ij}$$

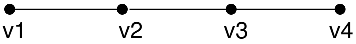
- The **eccentricity** μ_i of node v_i is defined by replacing the **summation** operator $\left(\sum\right)$ by the **maximization** operator; that is,

$$\mu_i \;=\; \max_{v_j \in V - \{v_i\}} \{d_{ij}\}$$

- This measure was studied by two graph theorists (Gert Sabidussi and Seifollah L. Hakimi).
- **Interpretation:** If μ_i denotes the eccentricity of node v_i, then every other node is within a distance of **at most** μ_i from v_i.
- If the eccentricity of node x is less than that of y, then x is more central than y.

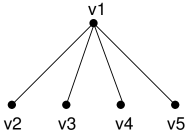
Examples: Eccentricity Computation

Example 1:



- $\mu_1 \;=\; \max\{1, 2, 3\} = 3.$
- $\mu_2 \;=\; \max\{1, 1, 2\} = 2.$
- $\mu_3 \;=\; \max\{2, 1, 1\} = 2.$
- $\mu_4 \;=\; \max\{3, 2, 1\} = 3.$

Example 2:

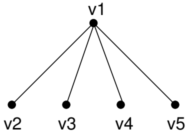


- $\mu_1 \;=\; 1.$
- For every other node, eccentricity $= 2$.

Eccentricity – Additional Definitions

Definition: A node v of a graph which has the smallest eccentricity among all the nodes is called a **center** of the graph.

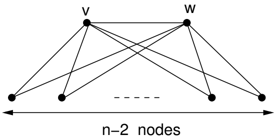
Example:



- The center of this graph is v_1.
  (The eccentricity of $v_1 = 1$.)

Note: A graph may have two or more centers.

Example:



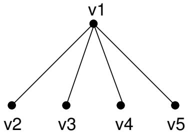
- Both v and w are centers of this graph.
  (Their eccentricities are $= 1$.)
- If G is clique on n nodes, then every node of G is a center.

Eccentricity – Additional Definitions (continued)

Definition: The smallest eccentricity value among all the nodes is called the **radius** of the graph.

Note: The value of the radius is the eccentricity of a center.

Example:



- The radius of this graph is 1 (since v_1 is the center of this graph and the eccentricity of $v_1 = 1$.)

Facts:

- The **largest eccentricity value** is the **diameter** of the graph.
- For any graph, the diameter is at most twice the radius. **(Students should try to prove this result.)**

An Algorithm for Computing Eccentricity

Let $G(V, E)$ denote the given graph.

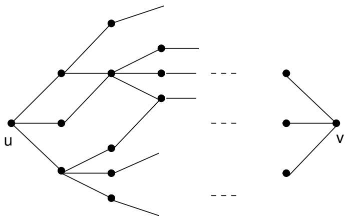
- **Recall:** By carrying out a BFS from node v_i, the shortest path distances between v_i and all the other nodes can be found in $O(|V| + |E|)$ time.
- So, the eccentricity of any node of G can be computed in $O(|V| + |E|)$ time.
- By repeating the BFS for each node, the time to compute eccentricity for **all** nodes of $G \; = \; O(|V|(|V| + |E|))$.
- So, the radius, diameter and all centers of G can be found in $O(|V|(|V| + |E|))$ time.

Random Walk Based Centrality (Brief Discussion)

Motivation:

- Definitions of centrality measures (such as **closeness** centrality) assume that "information" propagates along shortest paths.
- This may not be appropriate for certain other types of propagation. For example, propagation of diseases is a **probabilistic** phenomenon.

Idea of Random Walk Distance in a Graph:



Random Walk ... (Brief Discussion)

Random Walk Algorithm – Outline:

- Suppose we want to find the random walk distance from u to v.
- **Initialize:** Current Node $= u$ and No. of steps $= 0$.
- **Repeat**

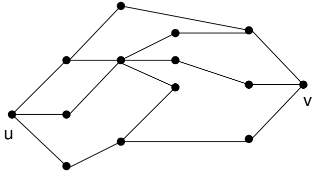
  1. Randomly choose a neighbor x of the Current Node.
  2. No. of steps $=$ No. of steps $+ 1$.
  3. Set Current Node $= x$.

  Until Current Node $= v$.

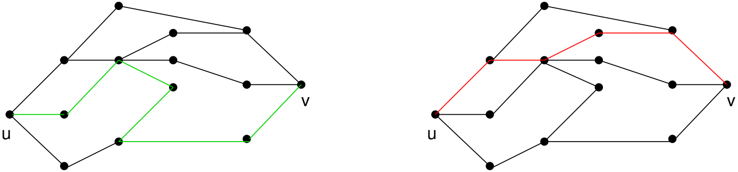
Note: In Step 1 of the loop, if the Current Node has degree d, probability of choosing any neighbor is $1/d$.

Examples of Random Walks

A graph for carrying out a random walk:



Examples of random walks on the above graph:



Random Walk ... (Brief Discussion)

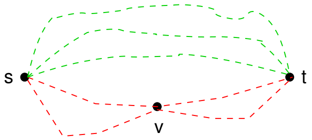
Definition: The **random walk distance** (or **hitting time**) from u to v is the expected number of steps used in a random walk that starts at u and ends at v.

- One can define farness/closeness centrality measures based on random walk distances.
- **Weakness:** Even for undirected graphs, the random walk distances are **not symmetric**; that is, the random walk distance from u to v may **not** be the same as the random walk distance from v to u.

Betweenness Centrality (for Nodes)

- Measures the importance of a node using the **number of shortest paths** in which the node appears.

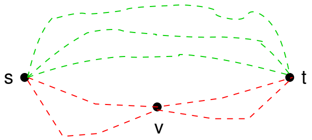
Consider a node v and two other nodes s and t.



- Each shortest path between s and t shown in **green doesn't** pass through node v.
- Each shortest path between s and t shown in **red** passes through node v.

Betweenness Centrality ... (continued)

Notation: Any shortest path between nodes s and t will be called an s-t **shortest path**.



- Let σ_{st} denote the number of all s-t shortest paths.
- Let $\sigma_{st}(v)$ denote the number of all s-t shortest paths that pass through node v.

Consider the ratio $\dfrac{\sigma_{st}(v)}{\sigma_{st}}$:

- This gives the fraction of s-t shortest paths passing through v.
- The larger the ratio, the more important v is with respect to the pair of nodes s and t.
- To properly measure the importance of a node v, we need to consider all pairs of nodes (not involving v).

Betweenness Centrality ... (continued)

Definition: The **betweenness centrality** of a node v, denoted by $\beta(v)$, is defined by

$$\beta(v) \;=\; \sum_{\substack{s\,,\,t \\ s\neq v\,,\,t\neq v}} \left[\frac{\sigma_{st}(v)}{\sigma_{st}}\right]$$

Interpreting the above formula: Suppose we want to compute $\beta(v)$ for some node v. The formula suggests the following steps.

- Set $\beta(v) = 0$.
- For each pair of nodes s and t such that $s \neq v$ and $t \neq v$,
  1. Compute σ_{st} and $\sigma_{st}(v)$.
  2. Set $\beta(v) \;=\; \beta(v) + \sigma_{st}(v)/\sigma_{st}$.
- Output $\beta(v)$.

Note: For two nodes x and y, if $\beta(x) > \beta(y)$, then x is more central than y.

Examples: Betweenness Computation

Example 1:

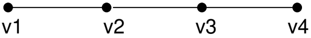
Note: Here, there is **only one** path between any pair of nodes. (So, that path is also the shortest path.)

Consider the computation of $\beta(v_2)$ first.

- The s-t pairs to be considered are: (v_1, v_3), (v_1, v_4) and (v_3, v_4).
- For the pair (v_1, v_3):
  - The number of shortest paths between v_1 and v_3 is 1; thus, $\sigma_{v_1,v_3} = 1$.
  - The (only) path between v_1 and v_3 passes through v_2; thus, $\sigma_{v_1,v_3}(v_2) = 1$.
  - So, the ratio $\sigma_{v_1,v_3}(v_2)/\sigma_{v_1,v_3} = 1$.
- In a similar manner, for the pair (v_1, v_4), the ratio $\sigma_{v_1,v_4}(v_2)/\sigma_{v_1,v_4} = 1$.

Examples: Betweenness Computation (continued)

Computation of $\beta(v_2)$ continued:



- For the pair (v_3, v_4):
  - The number of shortest paths between v_3 and v_4 is 1; thus, $\sigma_{v_3,v_4} = 1$.
  - The (only) path between v_3 and v_4 **does not** pass through v_2; thus, $\sigma_{v_3,v_4}(v_2) = 0$.
  - So, the ratio $\sigma_{v_3,v_4}(v_2)/\sigma_{v_3,v_4} = 0$.

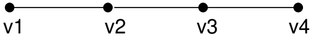
Therefore,

$$\begin{aligned} \beta(v_2) &= \quad 1 \quad \text{(for the pair } (v_1, v_3)\text{)} \\ &\quad + 1 \quad \text{(for the pair } (v_1, v_4)\text{)} \\ &\quad + 0 \quad \text{(for the pair } (v_3, v_4)\text{)} \\ &= 2. \end{aligned}$$

Note: In a similar manner, $\beta(v_3) = 2$.

Examples: Betweenness Computation

Example 1: (continued)



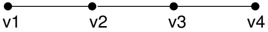
Now, consider the computation of $\beta(v_1)$.

- The s-t pairs to be considered are: (v_2, v_3), (v_2, v_4) and (v_3, v_4).
- For each of these pairs, the number of shortest paths is 1.
- v_1 **doesn't** lie on any of these shortest paths.
- Thus, for each pair, the fraction of shortest paths that pass through $v_1 = 0$.
- Therefore, $\beta(v_1) = 0$.

Note: In a similar manner, $\beta(v_4) = 0$.

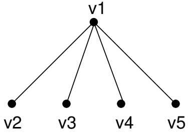
Examples: Betweenness Computation (continued)

Summary for Example 1:



- $\beta(v_1) = \beta(v_4) = 0.$
- $\beta(v_2) = \beta(v_3) = 2.$

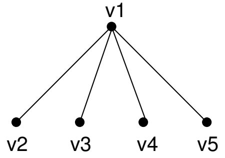
Example 2:



- Here also, there is **only one** path between any pair of nodes.
- Consider the computation of $\beta(v_1)$ first.

Examples: Betweenness Computation (continued)

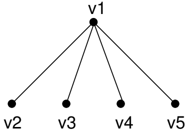
Computation of $\beta(v_1)$ (continued):



- We must consider all pairs of nodes from $\{v_2, v_3, v_4, v_5\}$.
- The number of such pairs = 6. (They are: (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_5), (v_4, v_5).)
- For each pair, there is only one path between them and the path passes through v_1.
- Therefore, the ratio contributed by each pair is 1.
- Since there are 6 pairs, $\beta(v_1) = 6$.

Examples: Betweenness Computation (continued)

Computation of $\beta(v_2)$:



- We must consider all pairs of nodes from $\{v_1, v_3, v_4, v_5\}$.
- The number of such pairs = 6.

- For each pair, there is only one path between them and the path **doesn't** pass through v_2.
- Therefore, $\beta(v_2) = 0$.

Notes:

- In a similar manner, $\beta(v_3) = \beta(v_4) = \beta(v_5) = 0$.
- Summary for Example 2:
  - $\beta(v_1) = 6$ and
  - $\beta(v_i) = 0$, for $i = 2, 3, 4, 5$.

Computing Betweenness: Major Steps

Requirement: Given graph $G(V, E)$, compute $\beta(v)$ for each node $v \in V$.

Note: A straightforward algorithm and its running time will be discussed.

Major steps: Consider one node (say, v) at a time.

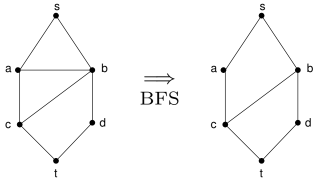
- For a given pair of nodes s and t, where $s \neq v$ and $t \neq v$, compute the following values:
  1. The no. of s-t shortest paths (i.e., the value of σ_{st}).
  2. The no. of s-t shortest paths passing through v (i.e., the value of $\sigma_{st}(v)$).

Major Step 1: Computing the **number** of shortest paths between a pair of nodes s and t.

Method: Breadth-First-Search (BFS) from node s followed by a **top down** computation procedure.

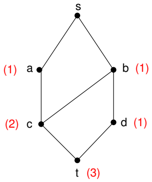
Example for Major Step 1

(a) Carrying out a BFS:



Note: The edge $\{a, b\}$ does not play any role in the computation of σ_{st}.

(b) Computing the value of σ_{st}:



- For each node, the value shown in **red** gives the number of shortest paths from s to that node.
- These numbers are computed through a top-down computation (to be explained in class).
- In this example, $\sigma_{st} = 3$.

Running Time of Major Step 1

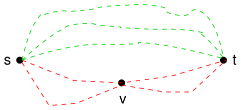
Assume that $G(V, E)$ is the given graph.

- For each node s, the time for BFS starting at s is $O(|V| + |E|)$.
- For the chosen s, computing the σ_{st} value for for all other nodes t can also be done in $O(|V| + |E|)$ time.
- So, the computation time for each node s is $O(|V| + |E|)$.
- Since there are $|V|$ nodes, the time for Major Step 1 is $O(|V|(|V| + |E|)$.
- The running time is $O(|V|^3)$ for **dense** graphs and $O(|V|^2)$ for **sparse** graphs.

Idea for Major Step 2

Goal of Major Step 2: Given an (s, t) pair and a node v (which is neither s nor t), compute $\sigma_{st}(v)$, the number of s-t shortest paths **passing through** v.

Idea:



- Compute the the number of of s-t shortest paths that **don't** pass through v (i.e., the number of **green** paths). Let $\gamma_{st}(v)$ denote this value.

- Then, $\sigma_{st}(v) \;=\; \sigma_{st} - \gamma_{st}(v)$.

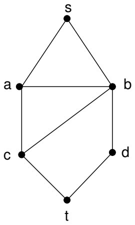
How can we compute $\gamma_{st}(v)$?

- If we delete node v from the graph, all the **green** paths remain in the graph.

- So, $\gamma_{st}(v)$ can be computed by considering the graph G_v obtained by deleting v and all the edges incident on v.

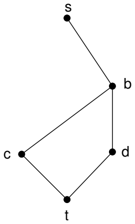
Example for Major Step 2

Graph $G(V, E)$:



Goal: Compute the number of s-t shortest paths that **don't** pass through a.

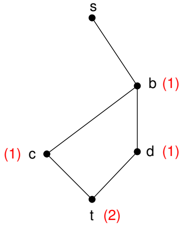
Graph G_a :



- The number of s-t shortest paths in G that **don't** pass through a is the number of s-t shortest paths in G_a.
- The required computation is exactly that of Major Step 1, except that it must be done for graph G_v.

Example for Major Step 2 (continued)

Graph G_a :



- For each node, the number in **red** gives the number of shortest paths between s and the node in G_a.
- From the figure, $\gamma_{st}(a) = 2$.
- Since $\sigma_{st} = 3$, $\sigma_{st}(a) = 3 - 2 = 1$.

Running Time of Major Step 2

As before, assume that $G(V, E)$ is the given graph.

- For each node $v \in V$, the following steps are carried out.
  - Construct graph G_v. (This can be done in $O(|V| + E|)$ time.)
  - For each node s of G_v, computing the number of s-t shortest paths for all other nodes can be done in $O(|V| + |E|)$ time.
  - Since there are $|V| - 1$ nodes G_v, the time for Major Step 2 for each node v is $O(|V|(|V| + |E|)$.
- So, over all the nodes $v \in V$, the running time for Major Step 2 is $O(|V|^2(|V| + |E|))$.
- The running time is $O(|V|^4)$ for **dense** graphs and $O(|V|^3)$ for **sparse** graphs.

A Review of Concepts Related to Matrices

Review of Concepts Related to Matrices

Example:

$$\begin{bmatrix} 7 & -8 & -14 \\ 2 & 4 & -3 \end{bmatrix}$$

- A matrix with 2 rows and 3 columns.
- Also referred to as a 2×3 matrix.
- This matrix is **rectangular**.
- In a **square** matrix, the number of rows equals the number of columns.

Notation: For an $m \times n$ matrix A, a_{ij} denotes the entry in row i and column j of A, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Matrix addition or subtraction:

- Two matrices can be added (or subtracted) only if they have the same number of rows and columns.
- The result is obtained by adding (or subtracting) the **corresponding** entries.

Example:

$$\begin{bmatrix} 7 & -8 & -14 \\ 2 & 4 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -15 \\ 3 & 6 & -7 \end{bmatrix}$$

Review of Matrices (continued)

Matrix multiplication:

- Given matrices P and Q, the product PQ is defined only when **the number of columns of** P = **the number of rows of** Q.
- If P is an $m \times n$ matrix and Q is an $n \times r$ matrix, the product PQ is an $m \times r$ matrix.

Example: (The procedure will be explained in class.)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 7 & 6 \end{bmatrix}$$

Main diagonal of a square matrix:

$$\begin{bmatrix} \mathbf{3} & 4 & 5 & 0 \\ 2 & \mathbf{4} & 3 & 7 \\ 3 & 1 & \mathbf{9} & 4 \\ 7 & 9 & 2 & \mathbf{8} \end{bmatrix}$$

- A 4×4 (square) matrix.
- The **main diagonal** entries are in **blue**.

Review of Matrices (continued)

Identity Matrix: For any positive integer n, the $n \times n$ **identity** matrix, denoted by I_n, has 1's along the main diagonal and 0's in every other position.

Example: Identity matrix I_4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Property: For any $n \times n$ matrix A, $I_n A = A I_n = A$.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 3 & 2 & 2 & 3 \\ 1 & 2 & 3 & -2 \\ 2 & 1 & 1 & 4 \\ 7 & 5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 & 3 \\ 1 & 2 & 3 & -2 \\ 2 & 1 & 1 & 4 \\ 7 & 5 & 4 & 1 \end{bmatrix}$$

Review of Matrices (continued)

Definition: An $n \times n$ matrix A is **symmetric** if $a_{ij} = a_{ji}$ for all i and j, $1 \le i, j \le n$.

Example:

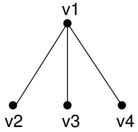
$$\begin{matrix} 2 & 3 & 7 \\ 3 & 4 & 9 \\ 7 & 9 & 6 \end{matrix}$$

- A 3×3 symmetric matrix.
- Observe the symmetry around the main diagonal.

Notes:

- For any n, the identity matrix I_n is symmetric.
- For any undirected graph G, its adjacency matrix is symmetric.

Example: An undirected graph and its adjacency matrix.



$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Review of Matrices (continued)

Definition: Let P be an $m \times n$ matrix, where p_{ij} is the entry in row i and column j, $1 \leq i \leq m$ and $1 \leq j \leq n$. The **transpose** of P, denoted by P^T, is an $n \times m$ matrix obtained by making each row of P into a column of P^T.

Examples:

$$P = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}_{1\times 4} \qquad P^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}_{4\times 1}$$

$$Q = \begin{bmatrix} 7 & -8 & -14 \\ 2 & 4 & -3 \end{bmatrix}_{2\times 3} \qquad Q^T = \begin{bmatrix} 7 & 2 \\ -8 & 4 \\ -14 & -3 \end{bmatrix}_{3\times 2}$$

Note: If a matrix A is symmetric, then $A^T = A$.

Review of Matrices (continued)

Example – Multiplying a matrix by a number (scalar):

$$3 \times \begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -15 & 12 \end{bmatrix}.$$

Determinant of a square matrix:

- For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value of the **determinant** is given by

  $$\mathrm{Det}(A) = ad - bc.$$

Example: Suppose $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. Then

$$\mathrm{Det}(A) = (-2 \times 2) - (3 \times -1) = -1.$$

Review of Matrices (continued)

Example: Computing the determinant of a 3×3 matrix.

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$\mathrm{Det}(B)$ can be computed as follows.

$$\begin{aligned} \mathrm{Det}(B) &= 3 \times \mathrm{Det} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} - 1 \times \mathrm{Det} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \\ &\quad + 0 \times \mathrm{Det} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \\ &= 3(-5) - 1(2) + 0 \\ &= -17. \end{aligned}$$

Note: In the expression for $\mathrm{Det}(B)$, the signs of the successive terms on the right side **alternate**.

Eigenvalues of a square matrix: If A is an $n \times n$ matrix, the **eigenvalues** of A are the solutions to the **characteristic** equation

$$\mathrm{Det}(A - \lambda I_n) = 0$$

where λ is a variable.

Example: Suppose

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}.$$

Note that

$$\lambda I_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

So,

$$A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}.$$

Hence,

$$\mathrm{Det}(A - \lambda I_2) = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4.$$

Example (continued):

So, the characteristic equation for A is given by

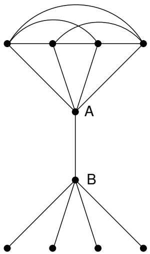
$$\lambda^2 - 3\lambda - 4 \ = \ 0$$

- The solutions to this equation are: $\lambda \ = \ 4$ and $\lambda \ = -1$.
- These are the **eigenvalues** of the matrix A.
- The largest eigenvalue (in this case, $\lambda \ = \ 4$) is called the **principal** eigenvalue.
- For each eigenvalue λ of A, there is a 2×1 matrix (vector) $\mathbf{x}$ such that $A\mathbf{x} \ = \ \lambda\mathbf{x}$. Such a vector is called an **eigenvector** of the eigenvalue λ. (This vector can be computed efficiently.)
- For the above matrix A, for the principal eigenvalue $\lambda = 4$, an eigenvector $\mathbf{x}$ is given by

$$\mathbf{x} \ = \ \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Eigenvector Centrality

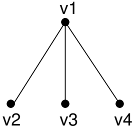
Degree centrality vs Eigenvector centrality:



- Nodes A and B both have degree 5.
- The four nodes (other than A) to which B is adjacent may be "unimportant" (since they don't have any interactions among themselves).
- So, A seems more central than B.
- Eigenvector centrality was proposed to capture this.

Eigenvector Centrality (continued)

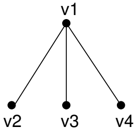
Example: Consider the following undirected graph and its adjacency matrix. (The matrix is **symmetric**.)



$$\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}$$

- We want the centrality of each node to be a **function** of the centrality values of its neighbors.
- The simplest function is the **sum** of the centrality values.
- A scaling factor λ is used to allow for more general solutions.

Eigenvector Centrality (continued)



- **Notation:** Let x_i denote the centrality of node v_i, $1 \leq i \leq 4$.

The equations to be satisfied by the unknowns x_1, x_2, x_3 and x_4 are:

$$\begin{aligned} x_1 &= \frac{1}{\lambda}(x_2 + x_3 + x_4) \\ x_2 &= \frac{1}{\lambda}(x_1) \\ x_3 &= \frac{1}{\lambda}(x_1) \\ x_4 &= \frac{1}{\lambda}(x_1) \end{aligned}$$

- Must avoid the **trivial** solution $x_1 = x_2 = x_3 = x_4 = 0$.
- So, additional constraint: $x_i > 0$, for at least one $i \in \{1, 2, 3, 4\}$.

Eigenvector Centrality (continued)

Rewriting the equations, we get:

$$
\begin{aligned}
\lambda\, x_1 &= x_2 + x_3 + x_4 \\
\lambda\, x_2 &= x_1 \\
\lambda\, x_3 &= x_1 \\
\lambda\, x_4 &= x_1
\end{aligned}
$$

Matrix version:

$$
\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

Note: The matrix on the right side of the above equation is the **adjacency matrix** of the graph.

Eigenvector Centrality (continued)

Using $\mathbf{x}$ for the vector $[x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T$, and A for the adjacency matrix of the graph, the equation becomes:

$$\lambda \mathbf{x} \; = \; A \mathbf{x}$$

Observation: λ is an **eigenvalue** of matrix A and $\mathbf{x}$ is the corresponding **eigenvector**.

Goal: To use the numbers in an eigenvector as the centrality values for nodes.

Theorem: **[Perron-Frobenius Theorem]**
If a matrix A has **non-negative entries** and is **symmetric**, then all the values in the the eigenvector corresponding to the **principal eigenvalue** of A are **positive**.

Eigenvector Centrality (continued)

Algorithm for Eigenvector Centrality:

Input: The adjacency matrix A of an undirected graph $G(V, E)$.

Output: The eigenvector centrality of each node of G.

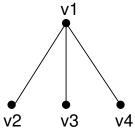
Steps of the algorithm:

1. Compute the principal eigenvalue λ^* of A.
2. Compute the eigenvector $\mathbf{x}$ corresponding to the eigenvalue λ^*.
3. Each component of $\mathbf{x}$ gives the eigenvector centrality of the corresponding node of G.

Running time: $O(|V|^3)$.

Eigenvector Centrality (continued)

Example: Consider the following graph and its adjacency matrix A.



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- The characteristic equation for matrix A is $\lambda^4 - 3\lambda^2 = 0$.
- The eigenvalues are: $-\sqrt{3}$, 0, 0 and $\sqrt{3}$.
- The principal eigenvalue λ^* of $A = \sqrt{3}$.
- The corresponding eigenvector $= \begin{bmatrix} 0.707 \\ 0.408 \\ 0.408 \\ 0.408 \end{bmatrix}$.
- Note that the center node v_1 has a larger eigenvector centrality value than the other nodes.

Centralization Index for a Graph

- A measure of the extent to which the centrality value of a most central node differs from the centrality of the other nodes.
- Value depends on which centrality measure is used.

Definition of Centralization Index:

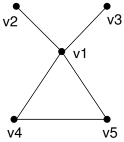
- Let C be any centrality measure and let $G(V, E)$ be a graph with n nodes.
- **Notation:** For any node $v \in V$, $C(v)$ denotes the centrality value of v.
- Let v^* be a node of maximum centrality in G with respect to C.
- Define $Q_G \; = \; \sum_{v \in V} [\, C(v^*) - C(v) \,]$.

Centralization Index ... (continued)

Definition of Centralization Index (continued):

- Let Q^* be the maximum value of Q_G over all graphs with n nodes.
- The **centralization index** C_G of G is the ratio Q_G/Q^*.
- C_G provides an indication of how close G is to the graph with the maximum value Q^*.

Example: We will use the following graph G and **degree centrality**.



- Node with highest degree centrality $= v_1$.
- $Q_G = \sum_{i=2}^{5} [\mathrm{degree}(v_1) - \mathrm{degree}(v_i)] = 10.$