Stochastic Electrodynamics

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Fundamental Concepts and Historical Culture of Physics

1 Introduction

Stochastic Electrodynamics (SED) is a compelling theoretical framework attempting to regenerate the qualities of quantum mechanics with a classical perspective. SED mainly tries to explain quantum behavior through a reformulation of classical electrodynamics that incorporates a universally present random electromagnetic field. This approach stands out because of its physical clarity and the solvability of its equations, allowing for rigorous comparisons with experimental results and other theories, notably quantum mechanics.

The origin of SED theory is rooted in the broader epistemological challenge faced by contemporary physics theories: the uneasy coexistence and lack of a seamless transition between classical and quantum theories. Historically, physics has aimed for the unification of disparate and distinct theories, or at least the establishment of a coherent correspondence between their concepts and laws. Despite significant efforts, a satisfactory unification has not yet been made, igniting a sense of crisis within the scientific community. This condition highlights the main goal of scientific journey: to build a comprehensive and unified understanding of nature.

Before quantum theory physicists had the idea that they have solved basically every problem and have a global understanding of nature. However, over the next fifty years, quantum theory came to dominate, reshaping all the traditional ideas and establishing our today's understanding of nature. This paradigm shift made classical theories being overshadowed by quantum concepts, which began to redefine fundamental notions such as entropy and macroscopic quantum behavior. Despite the eventual rise of quantum mechanics, classical theories have made a turn back by their explanatory power, particularly in describing dynamical macroscopic systems in physics, chemistry, and biology. This revival has again attracted interest in exploring whether a classical theory, like SED, could provide a coherent understanding of a quantum phenomena. Specifically, SED investigates whether the introduction of a universally present random electromagnetic field can illuminate the mysterious aspects of quantum mechanics. For a good discussion abot this matter I encourage you to read preface of the book "Dynamical Systems and Microphysics" by Biaquiere, Fer and Marzollo [1].

2 Ground State of a Harmonic Oscillator [2]

The basic idea of SED is to find a random background electromagnetic field that mimic the qualities of quantum systems for a classical system. As usual we choose the Harmonic Oscillator for the first step of calculations. Here we know that a charged particle having charge e and mass m radiates while undergoing a non-uniform, so we find the background field such that it keep the oscillator in motion and recover the lost energy from radiation. The backward radiation force experienced by the particle is

$$\overrightarrow{F}_{rad} = \frac{2e^2}{3c^3}.(\ddot{\overrightarrow{v}}) \tag{1}$$

And the differential equation for the harmonic oscillator then is:

$$\ddot{x} + \omega^2 x = \frac{2e^2}{3c^3}\dddot{x} \tag{2}$$

Adding the external electromagnetic force:

$$\ddot{x} + \omega^2 x = \frac{eE_{(t)}}{m} + \frac{2e^2}{3c^3}\dddot{x} \tag{3}$$

We may also introduce $\varepsilon \equiv \frac{e^2\omega^2}{3c^3} \ll 1$ since the radiation term is small compared to the natural frequency of the harmonic oscillator. And we take the field at the same order of ε as it's going to cancel out radiation effects. Now by purturbing x and solving the differential equation to the first order of ε we have:

$$\ddot{x}^{(0)} + \omega^2 x^{(0)} = 0 \tag{4}$$

$$\ddot{x}^{(1)} + \omega^2 x^{(1)} = \frac{eE_{(t)}}{m} + \frac{2\varepsilon}{\omega^2}\dddot{x}^{(0)} \tag{5}$$

$$\Rightarrow \ddot{x}^{(1)} + 2\varepsilon\dot{x}^{(0)} + \omega^2 x^{(1)} = \frac{eE_{(t)}}{m} \tag{6}$$

So by keeping in mind that we should keep x to the first order of ε, this is the equation we have to solve:

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = \frac{eE_{(t)}}{m} \tag{7}$$

Now it's time to take the randomness of the field in account. As $E_{(t)}$ is a random electromagnetic field, we can take the Fourier transform as

$$E_{(t)} = \sum_{-\infty}^{+\infty} E_n e^{in\frac{2\pi}{T}t} \tag{8}$$

Where the orthogonality of E_n have been took into account.

Separation of the solutions give us the solution to homogeneous equation (2) x_c for $x_{(0)} = x_0$ and $\dot{x}_{(0)} = \dot{x}_0$ and the special solution x_R for Eq. (7) for $x_{(0)} = 0$ and $\dot{x}_{(0)} = 0$. So we can calculate $x = x_C + x_R$. We leave the explicit expression for x_C for a reason that would be shown in a moment. To calculate it, we take the Fourier transform of x_R:

$$x_R \equiv \frac{e}{m}\sum_{-\infty}^{+\infty} E_n u_{n(t)} \tag{9}$$

$$\Rightarrow p_R = m\dot{x} \equiv \frac{e}{m}\sum_{-\infty}^{+\infty} E_n \dot{u}_{n(t)} \tag{10}$$

Then from equation 7 and 8:

$$\sum_{-\infty}^{+\infty}\left(\ddot{u}_{n(t)} + 2\varepsilon\dot{u}_{n(t)} + \omega^2 u_{n(t)}\right) = \sum_{-\infty}^{+\infty} e^{in\frac{2\pi}{T}t} \tag{11}$$

$$\Rightarrow \ddot{u}_{n(t)} + 2\varepsilon\dot{u}_{n(t)} + \omega^2 u_{n(t)} = e^{in\frac{2\pi}{T}t} \tag{12}$$

Which is an equation we can finally solve. The answer for $u_{n(t)}$ to the first order of ε (and also large t) is:

$$u_{n(t)} = \frac{e^{in\frac{2\pi}{T}t} - cos(\omega t)e^{-\varepsilon t} - \frac{\varepsilon}{\omega}sin(\omega t)e^{-\varepsilon t} + i\frac{2\pi n}{\omega T}sin(\omega t)e^{-\varepsilon t}}{\omega^2 - \frac{4\pi^2 n^2}{T^2} + i\frac{4\pi n\varepsilon}{T}} \tag{13}$$

Now we have the enough equipment to calculate the average values $\langle x_R^n \rangle$ and $\langle p_R^n \rangle$. $\langle x_R \rangle$ would be zero since it's a sum on random components of E_n which would be averaged to zero. To calculate $\left\langle x_R^2 \right\rangle$ we will write:

$$x_R^2 = \frac{e^2}{m^2}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} E_n E_m u_{n(t)} u_{m(t)} \tag{14}$$

Then:

$$\left\langle x_R^2 \right\rangle = \frac{e^2}{m^2}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} \langle E_n E_m \rangle u_{n(t)} u_{m(t)} \tag{15}$$

Since u_m and u_n are orthogonal, the only m that would not vanish is $m = -n$:

$$\Rightarrow \left\langle x_R^2 \right\rangle = \frac{e^2}{m^2}\sum_{-\infty}^{+\infty} \langle E_n E_{-n} \rangle u_{n(t)} u_{-n(t)} \tag{16}$$

And by going to the limit of large T:

$$\left\langle x_R^2 \right\rangle = \frac{e^2}{m^2}\int_0^{+\infty} 2\langle E_n E_{-n} \rangle u_{n(t)} u_{-n(t)} dn \tag{17}$$

But we can derive the summation term on E_n from another quantity. Suppose we are calculating v, the electromagnetic energy density. Since we don't have an explicit magnetic field, the term for v would be:

$$v = \frac{1}{4\pi T}\int_0^T \left|E_{(t)}^2\right| dt = \frac{1}{4\pi T}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}\int_0^T E_m E_n e^{i(m+n)\frac{2\pi}{T}t} dt \tag{18}$$

$$\Rightarrow v = \frac{1}{4\pi}\sum_{-\infty}^{+\infty} E_n E_{-n} \tag{19}$$

$$= \frac{1}{2\pi}\sum_{0}^{+\infty} E_n E_{-n} \tag{20}$$

$$= \frac{1}{2\pi}\int_0^{+\infty} E_n E_{-n} dn \tag{21}$$

In the last equation we assumed large T which enable us to do an integral over infinite count of n. We also can express energy density as an integral over all the frequencies:

$$\langle v \rangle = \int_0^{\infty} I_{(\nu)} d\nu \tag{22}$$

Where we can say $\nu \equiv \frac{n}{T}$ and integrate over n. Then the last two equations will give us $I_{(\nu)} = \frac{T}{2\pi}\langle E_n E_{-n} \rangle$

Putting the last expression in equation 17 will give us:

$$\left\langle x_R^2 \right\rangle = \frac{4\pi e^2}{m^2} \int_0^\infty I_{(\nu)} u_{(\nu,t)} u_{(-\nu,t)} d\nu \tag{23}$$

Where we again used the notation $\nu \equiv \frac{n}{T}$. The integrand is small duo to the exponential terms in $u_{(\nu,t)}$ except for the resonance frequency, where the denominator goes to zero:

$$\left|\omega^2 - \frac{4\pi^2 n_{res}^2}{T^2} + \frac{4i\pi n_{res}}{T}\varepsilon\right|^2 = \left|\omega^2 - 4\pi^2\nu_{res}^2 + 4i\pi\nu_{res}\varepsilon\right|^2 = 0 \tag{24}$$

$$\Rightarrow \left(\omega^2 - (2\pi\nu_{res})^2\right)^2 + (4\pi\nu_{res})\varepsilon^2 = 0 \tag{25}$$

$$\Rightarrow \nu_{res} \simeq \omega/2\pi \tag{26}$$

So we can take out $I_{(\nu_r es)}$

$$\left\langle x_R^2 \right\rangle = \frac{4\pi e^2 I_{(\nu_{res})}}{m^2} \int_0^\infty u_{(\nu,t)} u_{(-\nu,t)} d\nu \tag{27}$$

Calculating this integral is not that hard but it's rather long, so I skip the calculations and write the answer

$$\left\langle x_R^2 \right\rangle = \frac{\pi e^2 I_{(\nu_{res})}}{2m^2\varepsilon\omega^2} \left(1 - e^{-2\varepsilon t} - \frac{2\varepsilon}{\omega} sin(\omega t) cos(\omega t) e^{-2\varepsilon t}\right) \tag{28}$$

By the same procedure we can calculate the mean value $\left\langle p_R^2 \right\rangle$:

$$\left\langle p_R^2 \right\rangle = \frac{\pi e^2 I_{(\nu_{res})}}{2\varepsilon} \left(1 - e^{-2\varepsilon t} + \frac{2\varepsilon}{\omega} sin(\omega t) cos(\omega t)\right) \tag{29}$$

And the general form of:

$$\left\langle x_R^{2m} p_R^{2n} \right\rangle = \frac{(2m)!(2n)!}{m!n!2^{(m+n)}} \left\langle x_R^2 \right\rangle^m \left\langle p_R^2 \right\rangle^n \tag{30}$$

Note that only the mean values of even powers of x and p would not vanish.

These mean values that we found are rather common ones. They are obtained from the normal distribution for x_R and p_R. So we have found the phase space distribution of x_R and p_R.

$$\rho_{R(x_R,p_R,t)} = \frac{1}{2\pi\sqrt{\left\langle x_R^2 \right\rangle \left\langle p_R^2 \right\rangle}} exp\left(-\frac{x_R^2}{2\left\langle x_R^2 \right\rangle} - \frac{p_R^2}{2\left\langle p_R^2 \right\rangle}\right) \tag{31}$$

And by putting $t \to \infty$ in equations 28 and 29, we have the distribution in large times:

$$\left\langle x_R^2 \right\rangle = \frac{\pi e^2 I_{(\nu_{res})}}{2m^2\varepsilon\omega^2}, \qquad \left\langle p_R^2 \right\rangle = \frac{\pi e^2 I_{(\nu_{res})}}{2\varepsilon} \tag{32}$$

$$\Rightarrow \rho_{R(x_R,p_R,t)} = \frac{\varepsilon^2 m\omega}{\pi^2 e^2 I_{(\nu_{res})}} exp\left(-\frac{\varepsilon m^2\omega^2}{\pi e^2 I_{(\nu_{res})}} x_R^2 - \frac{\varepsilon}{\pi e^2 I_{(\nu_{res})}} p_R^2\right) \tag{33}$$

But our aim is to find the whole distribution function $\rho_{(x,p,t)}$ for any initial conditions. A system at (x,p) and time t can be an evolution of many different (x_0,p_0) systems in time, so we should take all the origin points into account:

$$\rho_{(x,p,t)} = \int \rho_{(x_0,p_0)} P_{(x,p,t|x_0,p_0)} dx_0 dp_0 \tag{34}$$

Where $P_{(x,p,t|x_0,p_0)}$ is the possibility of a system at point (x_0, p_0) to evolve to (x, p) after some time t. Let's look at this equation in a more intuitive way. $\rho_{R(x,p,t)}$ is the distribution function generated around initial conditions $x_{(0)} = 0$ and $p_{(0)} = 0$. These initial conditions results to $x_{C(t)} = 0$ and $p_{C(t)} = 0$. But if the initial conditions weren't fixed at zero, then the $(x_{C(t)}, p_{C(t)})$ would evolve and change coordinates in phase space, which we can see as a motion of distribution function source (figure 1(a)). Thus by assuming enough time has passed from each configuration of (x_0, p_0), we can write the whole distribution function as

$$\rho_{(x,p,t)} = \int \rho_{(x_0,p_0)} \rho_{R(x_R,p_R,\infty)} dx_0 dp_0 \tag{35}$$

$$= \int \rho_{(x_0,p_0)} \frac{1}{\pi\alpha} exp\left(-\frac{m\omega}{\alpha} x_R^2 - \frac{p_R^2}{m\omega\alpha}\right) dx_0 dp_0 \tag{36}$$

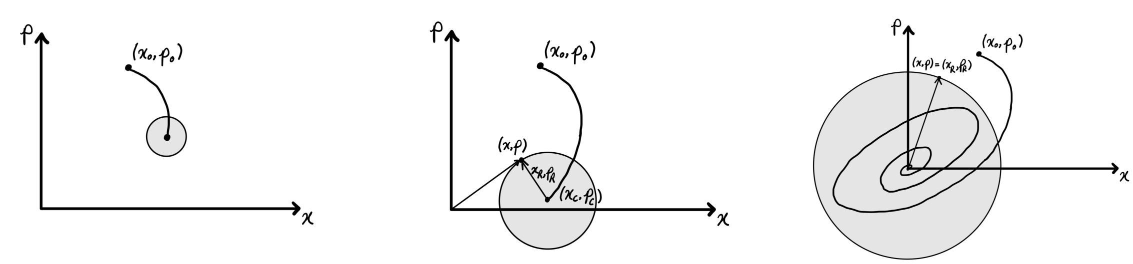
$$\simeq \frac{1}{\pi\alpha} exp\left(-\frac{m\omega}{\alpha} x^2 - \frac{p^2}{m\omega\alpha}\right) \int \rho_{(x_0,p_0)} dx_0 dp_0 \tag{37}$$

$$\alpha \equiv \frac{\pi e^2 I_{(\nu_{res})}}{m\omega\varepsilon} \tag{38}$$

We used the fact that in this limit $x_R \simeq x$ since x_C is a damping term and vanishes in large t (figure 1(c)). Now we know that $\int \rho_{(x_0,p_0)} dx_0 dp_0 = 1$ as $\rho_{(x_0,p_0)}$ is a normalized distribution function. So the final result for $\rho_{(x,p,t)}$ is:

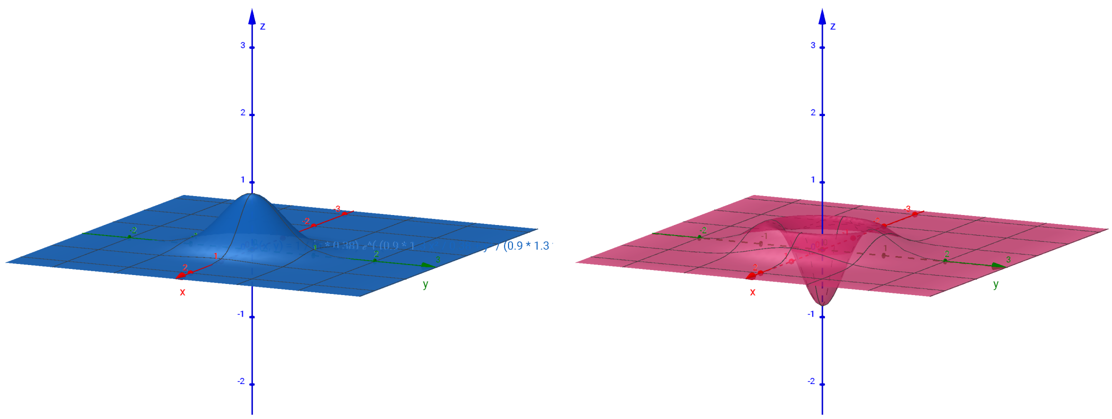
$$\rho_{(x,p,t)} = \frac{1}{\pi\alpha} exp\left(-\frac{m\omega}{\alpha} x^2 - \frac{p^2}{m\omega\alpha}\right) \tag{39}$$

The interesting point here is that the distribution is not time dependent and is stationary (figure 2(a)), meaning that we have found a random field which successfully canceled out the radiation effects. And more fascinating it is identical to the quantum mechanical ground state of harmonic oscillator, if we choose $\alpha = \hbar$. This is the significant result of the Stochastic Electrodynamics.



(a) (b) (c)

Fig. 1: Phase space intuition of equation 35



(a) ground state (b) first excited state

Fig. 2: Phase space distribution function for ground state and first excited state

3 Excited States and Mixture Ensembles

After calculating the ground state distribution function of the phase space of a harmonic oscillator under the influence of a random electromagnetic field, now we wish to calculate the same quantity for an ensemble of many non-interactive harmonic oscillators at temperature T. To do so we use micro-canonical ensemble and define the distribution functions ρ_n regarding to a harmonic oscillator at energy $E_n = n\hbar\omega$:

$$\rho_{(T)} = \frac{1}{Z}\sum_{n=0}^{\infty}\rho_n e^{-E_n/kT} \tag{40}$$

Z will be given as $\left(1 - e^{(-\hbar/kT)}\right)$ from the normalization. By defining $\eta \equiv e^{-\hbar/kT}$ we have:

$$\rho_{(T)} = \left(1+\eta\right)\sum_{n=0}^{\infty}\rho_n \eta^n \tag{41}$$

At $T = 0$ the distribution function is that we derived in the last section. This ρ_0 is the ground state distribution function of this system and the corresponding energy density $I_0 \equiv I_{(\nu_{res})}$ would then interpret as background -Black Body- radiation energy of the system. The radiation of the system at temperature T then, from Planck distribution, would be:

$$I_{T(\nu)} = \frac{1+\eta}{1-\eta} I_{0(\nu)} \tag{42}$$

And by putting this energy density in the former distribution function ρ_0 we get

$$\rho_{(T)} = \frac{1}{\pi\hbar}\frac{1-\eta}{1+\eta}exp\left(-\frac{1-\eta}{1+\eta}\left(\frac{m\omega}{\hbar}x_R^2 + \frac{p_R^2}{m\omega\hbar}\right)\right) \tag{43}$$

By comparing this result to the equation 41 derived from the micro-canonical ensemble, we have: So we can find the distribution function of each state by expanding the right hand side of this

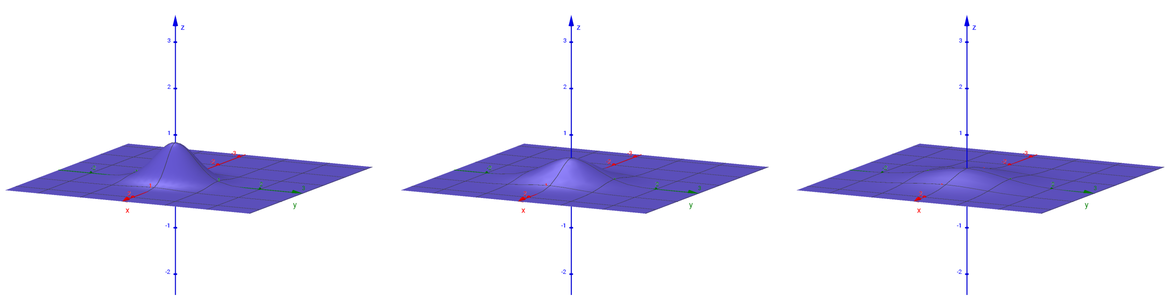
$$(1+\eta)\sum_{n=0}^{\infty}\rho_n\eta^n = \frac{1}{\pi\hbar}\frac{1-\eta}{1+\eta}exp\left(-\frac{1-\eta}{1+\eta}\left(\frac{m\omega}{\hbar}x_R^2 + \frac{p_R^2}{m\omega\hbar}\right)\right) \tag{44}$$

So we can find the distribution function of each state by expanding the right hand side of this equation. As an example this procedure gives the distribution function of the first excited state ρ_1 as below:

$$\rho_1 = \frac{1}{\pi\hbar}\left(\frac{2m\omega}{\hbar}x_R^2 + \frac{2}{m\omega\hbar}p_R^2 - 1\right)exp\left(-\frac{m\omega}{\hbar}x_R^2 - \frac{p_R^2}{m\omega\hbar}\right) \tag{45}$$

This distribution function also gives all the mean values of position and momentum exactly equal to those of quantum theory. It happens for ρ_n for all other n meaning for all other excited states too. By this conclusion then, the mission of Stochastic Electrodynamics is completed as to justify quantum harmonic oscillator problem results by a classical theory.

There is just one problem bothering the mind. Harmonic oscillators can't be solely in a specific excited state, as their distribution function get negative values for some x and p (figure 2(b)). For a more detailed discussion view section 5 of the [2]. There Marshall states that it's not a major problem, since it means they're not physical stationary states (in classical manner), which is not a rather strange thing to say in the quantum view too as the excited states are always on the verge of a downfall to ground state.



(a) for $\eta = 0$ (b) $\eta = 0.25$ (c) $\eta = 0.5$

Fig. 3: Phase space distribution function of an ensemble in temperature T. As you can see the function is the same at $\eta = 0$ as ground state in figure 2(a)

4 Discussion [3, 4]

Stochastic Electrodynamics have been tested by other problems in modern physics too. For instance the hydrogen atom, also known as the Kepler problem, is one of the fundamental systems where SED has been applied to. Solving this problem using SED involves incorporating the random electromagnetic fields to explain the stability and energy levels of the hydrogen atom [5]. While the results are promising, showing similar ground state energy levels as those predicted by quantum mechanics, discrepancies arise when dealing with higher energy states and transitions. These discrepancies highlight the limitations of SED in fully replicating the quantum mechanical model for atomic systems. SED also addresses foundational issues such as Bell's inequality and the EPR paradox [6], which are central to understanding quantum entanglement and non-locality. SED's approach, based on classical fields with added randomness, attempts to explain these phenomena without invoking inherently quantum concepts. However, while SED can reproduce some statistical correlations predicted by quantum mechanics, it struggles to account for the full spectrum of observed quantum non-locality, indicating gaps in its explanatory power.

The Compton effect, which involves the scattering of X-rays by electrons, presents another critical test for SED. The theory's predictions for the shift in wavelength due to the scattering process can be compared to quantum mechanical results [7]. While SED can approximate the overall behavior observed in experiments, it does not match the precision of quantum electrodynamics (QED), particularly in the finer details of the scattering process. Lastly the Casimir effect, which arises from quantum fluctuations in the vacuum, provides a unique scenario where SED's inclusion of a random electromagnetic field can be directly tested. SED's predictions for the Casimir force between two uncharged, parallel plates align closely with experimental results, showcasing its potential in explaining certain quantum phenomena from a classical perspective. However, discrepancies in quantitative predictions persist, suggesting that while SED captures some aspects of the underlying physics, it may lack the completeness of a fully quantum theoretical treatment.

SED has demonstrated significant strengths, particularly in its ability to provide classical analogs for quantum phenomena and its solvability, which allows for straightforward mathematical treatment. For instance, the ground state solutions for the harmonic oscillator within SED closely match those of quantum mechanics in terms of average energy and probability distributions. These successes underline SED's potential as a pedagogical tool, offering insights into the transition from classical to quantum descriptions. However, SED faces notable challenges. The disagreements between SED and quantum theory become particularly visible in systems with non-linear dynamics or multiple frequencies. For example, SED struggles to account for sharp energy levels and precise spectral lines, which are well-established in quantum mechanics or its inability to fully justify the anharmonic oscillator [8]. These issues highlight the difficulties in extending SED beyond simple systems and suggest inherent limitations in its classical framework.

Experimental validation also remains a crucial area for assessing SED. While certain predictions, such as those related to the Casimir effect, align well with observations, others fall short. Future experiments, particularly those designed to probe the nuances of random electromagnetic fields and their interactions with matter, will be vital in determining the viability of SED as a comprehensive theory.

5 Conclusion

SED represents an intriguing attempt to bridge classical and quantum physics, offering a unique perspective on quantum phenomena through classical means. Its ability to reproduce certain quantum results without referring to quantum mechanics' full formalism suggests that it can provide valuable insights into the foundational aspects of physics. However, its limitations in explaining complex, multi-frequency systems and the full spectrum of quantum behavior shows that it may not serve as a standalone theory of nature.

Ultimately, SED should be viewed as a complementary framework that enhances our understanding of the classical-quantum transition rather than a replacement for quantum mechanics. Its value lies in deepening our conceptual comprehension of physical phenomena and inspiring new approaches to unresolved questions in modern physics. As research progresses, SED may continue to evolve, potentially offering more robust solutions or paving the way for entirely new theoretical developments.

References

[1] A. Blaquiere, F. Fer, and A. Marzollo, “Dynamical systems and microphysics,” *Springer*, 1980.

[2] T. Marshall, “Random electrodynamics,” *RSPA*, vol. 276, no. 1367, 1963.

[3] S. Diner, “Stochastic electrodynamics general considerations,” *In: Blaquiere, Fer, Marzollo (eds) Dynamical Systems and Microphysics*, pp. 98–110, 1980.

[4] P. Claverie, “Stochastic electrodynamics: Methods and results,” *In: Blaquiere, Fer, Marzollo (eds) Dynamical Systems and Microphysics*, pp. 110–135, 1980.

[5] T. Marshall, “The kepler problem in stochastic electrodynamics,” *In: Blaquiere, Fer, Marzollo (eds) Dynamical Systems and Microphysics*, pp. 135–144, 1980.

[6] E. Santos, “Stochastic electrodynamics and the bell inequalities,” *Open Questions in Quantum Physics*, pp. 283–269, 1985.

[7] A. Barranco and H. Franca, “Stochastic electrodynamics and the compton effect,” *IFUSP-P-799*, 1989.

[8] L. Pesquera, “The anharmonic oscillator in stochastical electrodynamics (s.e.d.) : The problem of ”radiation balance” at each frequency.,” *In: Blaquiere, Fer, Marzollo (eds) Dynamical Systems and Microphysics*, pp. 145–151, 1980.