MATH402 - Homework #1 Sample Solutions

Q1 The company owns 800 hectares (ha) of land. The table for the houses is given below.

Type	Profit	Cost	Water Usage	Required Area
I	200	145	20 ℓ per day	1 ha
II	240	165	27 ℓ per day	1,5 ha
III	300	215	32 ℓ per day	2 ha

Also, there are some rules that the company has to obey:

1- At least half of the houses have to be Type-I.

- Let's say that x, y, z are the numbers of the houses for Type I, II, and III, respectively. Then, we can formulate this condition as

$$x \geqslant \frac{x+y+z}{2} \Rightarrow 2x \geqslant x+y+z \Rightarrow \boxed{x \geqslant y+z}$$

2- Recreational areas require 15% of the total area.

- Let's say that t is the number of the recreational areas. Then, we can obtain

$$\text{Recreational Areas} = 800 \times \frac{15}{100} = 120 \text{ ha.}$$

We know that each recreational area requires 0,5 ha.

$$\Rightarrow \boxed{t = \frac{120}{0{,}5} = 240} \quad \Rightarrow \boxed{x + 1{,}5y + 2z + 120 \leqslant 800}$$

3- For every 15 houses, there has to be at least one recreational areas, which require 0,5 ha of area. In addition, building cost and water usage per day are 125 and 25 ℓ, respectively.

- We can add an item to the table such as

Type	Profit	Cost	Water Usage	Required Area
Recreational Area	0	125	25 ℓ per day	0,5 ha

$$\frac{x+y+z}{15} \leq t \Rightarrow 15t \geq x+y+z \Rightarrow \boxed{3600 \geq x+y+z}$$

4- Water usage cannot exceed 8500 ℓ per day.

$$20x+27y+32z+25t \leq 8500$$

$t=240$

$$\Rightarrow \boxed{2500 \geq 20x+27y+32z}$$

After we obtain the constraints, we can write the optimization problem.

$$\max f(x,y,z) = 200x+240y+300z \quad \text{subject to} \quad \begin{array}{l} -x+y+z \leq 0, \\ x+1{,}5y+2z \leq 680, \\ x+y+z \leq 3600, \\ 20x+27y+32z \leq 2500, \\ x \geq 0,\ y \geq 0,\ z \geq 0. \end{array}$$

Q2) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.

a) If the function f is convex, then by definition

$$f(\lambda y+(1-\lambda)x) \leq \lambda f(y)+(1-\lambda)f(x), \quad \forall \lambda \in [0,1],$$

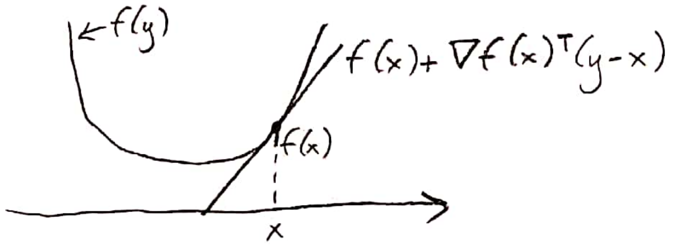
$$f(x+\lambda(y-x)) \leq f(x)+\lambda(f(y)-f(x)),$$

$$\Rightarrow f(y)-f(x) \geq \frac{f(x+\lambda(y-x))-f(x)}{\lambda}, \quad \forall \lambda \in (0,1]$$

As $\lambda \to 0$, we get

$$f(y)-f(x) \geq \nabla f(x)^T(y-x)$$

$$\Rightarrow \boxed{f(y) \geq f(x)+\nabla f(x)^T(y-x)} \quad \blacksquare$$



$\Rightarrow$ The first order Taylor expansion at any point is less than or equal to the function.

b) Let's assume that $\nabla^2 f(x) \geq 0, \ \forall x \in \mathbb{R}^n$. By using Taylor's Theorem we have

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x), \quad \text{for } z \in [x,y]$$

We know that

$$\frac{1}{2}(y-x)^T \nabla^2 f(z)(y-x) \geq 0 \quad \text{for } z \in [x,y] \text{ from assumption.}$$

Then, we have

$$f(y) \geq f(x) + \nabla f(x)^T (y-x).$$

Let's define a variable such that

$$k = \lambda x + (1-\lambda) y$$

We have

$$f(x) \geq f(k) + \nabla f(k)^T (x-k) \qquad (1)$$

$$f(y) \geq f(k) + \nabla f(k)^T (y-k) \qquad (2)$$

Multiplying (1) by λ, (2) by $(1-\lambda)$ and adding, we get

$$\lambda f(x) + (1-\lambda) f(y) \geq f(k) + \nabla f(k)^T (\lambda x - \lambda k + y - \lambda y - k + \lambda k)$$

$$= f(k) + \nabla f(k)^T (\underbrace{\lambda x + (1-\lambda) y}_{=k} - k)$$

$$= f(k) = f(\lambda x + (1-\lambda) y) \Rightarrow f \text{ is a convex function.} \ \blacksquare$$

c) The set $C_\alpha = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is convex.

Proof: Let's choose two points $x, y \in C_\alpha$, $\lambda \in [0,1]$

$$x \in C_\alpha \Rightarrow f(x) \leq \alpha, \qquad y \in C_\alpha \Rightarrow f(y) \leq \alpha$$

We know that f is convex. Then,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda \alpha + (1-\lambda)\alpha = \alpha$$

$$\Rightarrow \lambda x + (1-\lambda) y \in C_\alpha \ \blacksquare$$

d) f is a convex function. Let $k = \dfrac{Ax+b}{c^T x + d} = \lambda y + (1-\lambda) z$

Then,

$$f(\lambda y + (1-\lambda) z) \leq \lambda f(y) + (1-\lambda) f(z)$$

Multiplying both sides with a positive constant c^Tx+d satisfies

$$\underbrace{(c^Tx+d)f(\lambda y+(1-\lambda)z)}_{g(k)} \leq [c^Tx+d](\lambda f(y)+(1-\lambda)f(z)).$$

$$g(\lambda y+(1-\lambda)z) \leq \lambda g(y)+(1-\lambda)g(z), \quad \lambda \in [0,1]$$

Therefore, $g(x)$ is also a convex function. ∎

Q3) $f(x)=c^Tx+\frac{1}{2}x^THx$, where $c=[1\ 1\ 1\ 1]^T$, $H=\begin{bmatrix} 4 & 4 & 4 & 3 \\ 4 & 7 & 3 & 3 \\ 4 & 3 & 5 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

a) We should check $\nabla f(x)$ for the stationary point x^*.

$$\nabla f(x)=c+Hx=0 \Rightarrow x=-H^{-1}c$$

For existence of a stationary point, H should be non-singular matrix.

$$H^{-1}=\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 1/2 & 1/2 & 0 \\ -2 & 1/2 & 3/2 & 0 \\ -1 & 0 & 0 & 4/3 \end{bmatrix}$$ $\Rightarrow$ H is nonsingular. Therefore, there is a stationary point x^*.

b) $$x^*=-H^{-1}c=-\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 1/2 & 1/2 & 0 \\ -2 & 1/2 & 3/2 & 0 \\ -1 & 0 & 0 & 4/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/3 \end{bmatrix}$$

First-order necessary condition is satisfied $\Rightarrow \nabla f(x^*)=0$

Let's check second-order necessary condition.

$\nabla^2 f(x)=H \Rightarrow H$ should be positive semi-definite or positive definite (2nd order sufficient condition). Let's check

$$H=\begin{bmatrix} 4 & 4 & 4 & 3 \\ 4 & 7 & 3 & 3 \\ 4 & 3 & 5 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

<u>Theorem:</u> Let A be an $n\times n$ symmetric matrix, and let A_k be the submatrix of A obtained by taking the upper left-hand corner $k\times k$ submatrix of A. Furthermore, let $\Delta_k = \det(A_k)$, the k^{th} principal minor of A. Then,

1- A is positive definite iff $\Delta_k > 0$ for $k=1,2,\dots,n$;
2- A is negative definite iff $(-1)^k\Delta_k > 0$ for $k=1,2,\dots,n$;
3- A is positive semi-definite if $\Delta_k \geq 0$ for $k=1,2,\dots,n-1$ and $\Delta_n = 0$;
4- A is negative semi-definite if $(-1)^k\Delta_k \geq 0$ for $k=1,2,\dots,n-1$ and $\Delta_n = 0$.

Therefore, we should check the determinant of every leading principal sub-matrix of H.

$$H_{11} = 4 > 0$$

$$H_{2\times 2} = \begin{vmatrix} 4 & 4 \\ 4 & 7 \end{vmatrix} = 4\times 7 - 4\times 4 = 12 > 0$$

$$H_{3\times 3} = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 7 & 3 \\ 4 & 3 & 5 \end{vmatrix} = 8 > 0$$

$$\det(H) = 6 > 0$$

H is positive definite. Then, second-order sufficient condition is satisfied.
$\Rightarrow x^*$ is the strict local minimum of f.

c) The point x^* is the strictly global minimizer due to

$$f(x^*) = -\frac{1}{6} < f(x) \quad \forall x.$$

It is a unique minimizer, because $\nabla f(x^*) = 0$ is satisfied only with $x^* = [0 \;\; 0 \;\; 0 \;\; -1/3]^T$.

<u>Q4</u> $x^2+y^2-2=0$
$x-y=0$

a) $\left.\begin{array}{l} x^2+y^2=2 \\ x=y \end{array}\right\} \; 2x^2 = 2 \Rightarrow x^2 = 1 \rightarrow \boxed{x=y=-1}, \; \boxed{x=y=1}$

b) $f_1(x,y) = x^2 + y^2 - 2$

$f_2(x,y) = x - y$

Newton's method can be shown as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{f(x_n, y_n)}{f'(x_n, y_n)}$$

For the first iteration, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{f(x_0, y_0)}{f'(x_0, y_0)}$$

$$f(x_0, y_0) = \begin{bmatrix} x_0^2 + y_0^2 - 2 \\ x_0 - y_0 \end{bmatrix} \qquad f'(x_0, y_0) = \begin{bmatrix} 2x_0 & 2y_0 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left(\begin{bmatrix} 2x_0 & 2y_0 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_0^2 + y_0^2 - 2 \\ x_0 - y_0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_0 & 2y_0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{x_0 + y_0} \begin{bmatrix} 1/2 & y_0 \\ 1/2 & -x_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{x_0 + y_0} \begin{bmatrix} 1/2 & y_0 \\ 1/2 & -x_0 \end{bmatrix} \begin{bmatrix} x_0^2 + y_0^2 - 2 \\ x_0 - y_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{x_0 + y_0} \begin{bmatrix} \frac{x_0^2 + y_0^2 - 2}{2} + x_0 y_0 - y_0^2 \\ \frac{x_0^2 + y_0^2 - 2}{2} - x_0^2 + x_0 y_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \frac{1}{2(x_0 + y_0)} \begin{bmatrix} x_0^2 + y_0^2 - 2 + 2x_0 y_0 - 2y_0^2 \\ x_0^2 + y_0^2 - 2 - 2x_0^2 + 2x_0 y_0 \end{bmatrix}$$

$$= \frac{1}{2(x_0 + y_0)} \begin{bmatrix} 2x_0^2 + 2x_0 y_0 - x_0^2 - y_0^2 + 2 - 2x_0 y_0 + 2y_0^2 \\ 2x_0 y_0 + 2y_0^2 - x_0^2 - y_0^2 + 2 + 2x_0^2 - 2x_0 y_0 \end{bmatrix}$$

$$= \frac{1}{2(x_0 + y_0)} \begin{bmatrix} x_0^2 + y_0^2 + 2 \\ x_0^2 + y_0^2 + 2 \end{bmatrix} \xrightarrow{\text{Generalize}} \boxed{\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{x_n^2 + y_n^2 + 2}{2(x_n + y_n)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

c) If this system converges to a value. Then, for a large value of n, we have

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \frac{x_n^2+y_n^2+2}{2(x_n+y_n)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_n = y_n = \frac{x_n^2+y_n^2+2}{2(x_n+y_n)} \Rightarrow x_n = y_n = \frac{2x_n^2+2}{4x_n} \Rightarrow 4x_n^2 = 2x_n^2+2$$

$$2x_n^2 = 2$$

$$\boxed{x_n^2 = 1} \quad x_n = -1 \text{ or } x_n = 1$$

We know that $x_n = y_n$. Therefore,

If $x_n + y_n > 0$, then $(x^*, y^*)^T = (1,1)^T$,

If $x_n + y_n < 0$, then $(x^*, y^*)^T = (-1,-1)^T$.

d) We have the general formula such that

$$x_{n+1} = \frac{x_n^2+1}{2x_n}$$

The error value can be defined as

$$e_{n+1} = x_{n+1} - x^* = \frac{x_n^2+1}{2x_n} - x^*$$

For $x^* = 1$:

$$e_{n+1} = \frac{x_n^2+1}{2x_n} - 1 = \frac{x_n^2-2x_n+1}{2x_n} = \frac{(x_n-1)^2}{2x_n} = \frac{e_n^2}{2x_n}$$

For $x^* = -1$:

$$e_{n+1} = \frac{x_n^2+1}{2x_n} + 1 = \frac{x_n^2+2x_n+1}{2x_n} = \frac{(x_n+1)^2}{2x_n} = \frac{e_n^2}{2x_n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|^2} = \frac{1}{2|x^*|} = \frac{1}{2}$$

Hence this sequence converges <u>quadratically</u> with rate constant $\frac{1}{2}$.

Q5 The objective function is

$$f(x_1,x_2)=\frac{x_1^2}{2}+x_1\cos x_2$$

a) $$\nabla f(x_1,x_2)=\begin{bmatrix} x_1+\cos x_2 \\ -x_1\sin x_2 \end{bmatrix}$$

$$\nabla^2 f(x_1,x_2)=\begin{bmatrix} 1 & -\sin x_2 \\ -\sin x_2 & -x_1\cos x_2 \end{bmatrix}$$

b) $\nabla f(x_1,x_2)=0$ for minima of f.

$$\Rightarrow \begin{matrix} x_1+\cos x_2=0 \\ -x_1\sin x_2=0 \end{matrix} \Rightarrow \begin{matrix} x_1=-\cos x_2 \\ \cos x_2\sin x_2=0 \end{matrix} \Rightarrow x_2^*=\frac{k\pi}{2}, \text{ for } k=0,1,2,\ldots$$

Then, $x_1^*=-\cos x_2^*=-\cos\left(\frac{k\pi}{2}\right)$, for $k=0,1,2,3,\ldots$

Hence, the stationary points can be shown as

$$x^*=\begin{bmatrix} -\cos\left(\frac{k\pi}{2}\right) \\ \frac{k\pi}{2} \end{bmatrix}, \quad \text{for } k=0,1,2,3,\ldots$$

For $k=0,4,8,\ldots \Rightarrow x^{*1}=\begin{bmatrix} -1 \\ 2t\pi \end{bmatrix}$ for $t=0,1,2,3,\ldots$

For $k=1,5,9,\ldots \Rightarrow x^{*2}=\begin{bmatrix} 0 \\ k\pi/2 \end{bmatrix}$

For $k=2,6,10,\ldots \Rightarrow x^{*3}=\begin{bmatrix} 1 \\ k\pi/2 \end{bmatrix}$

For $k=3,7,11,\ldots \Rightarrow x^{*4}=\begin{bmatrix} 0 \\ k\pi/2 \end{bmatrix}$.

Let's check Hessian matrix for these stationary points.

$$\nabla^2 f(x^{*1}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |\nabla^2 f(x^{*1})| = 1 > 0$$

$\nabla^2 f(x^{*1})$ is positive definite.

Therefore, $x^{*1} = [-1,\ 2t\pi]^T$ is a local minimizer for $t = 0, 1, 2, 3, \ldots$

$$\nabla^2 f(x^{*2}) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \Rightarrow |\nabla^2 f(x^{*2})| = -1 < 0$$

$\nabla^2 f(x^{*2})$ is indefinite.

Therefore, $x^{*2} = [0,\ k\pi/2]^T$ is neither a minimizer nor a maximizer of f. It is a saddle point. $(k = 1, 5, 9, \ldots)$

$$\nabla^2 f(x^{*3}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |\nabla^2 f(x^{*3})| = 1 > 0$$

$\nabla^2 f(x^{*3})$ is positive definite.

Therefore, $x^{*3} = [1,\ (2t+1)\pi]^T$ is a local minimizer for $t = 0, 1, 2, 3, \ldots$

$$\nabla^2 f(x^{*4}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |\nabla^2 f(x^{*4})| = -1 < 0$$

$\nabla^2 f(x^{*4})$ is indefinite.

Therefore, $x^{*4} = [0,\ k\pi/2]^T$ is neither a minimizer nor a maximizer of f. It is a saddle point. $(k = 3, 7, 11, \ldots)$

$\Rightarrow$ As a result $x^* = [-1,\ 2t\pi]^T$ and $x^* = [1,\ (2t+1)\pi]^T$ are local minimizers for $t = 0, 1, 2, 3, \ldots$

c) $p_0 = (-1, 0)$ $\quad x_0 = (0, \pi/4)$

$$f(x_0) = 0 + 0 \cdot \cos\left(\frac{\pi}{4}\right) = 0 \qquad \nabla f(x_0) = \begin{bmatrix} 0 + \cos(\pi/4) \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \end{bmatrix}$$

For descent direction: $\nabla f(x_0)^T p_0 < 0$

$$\nabla f(x_0)^T p_0 = [\sqrt{2}/2 \quad 0] \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\sqrt{2}/2 < 0 \Rightarrow \text{This is a descent direction.}$$

d) $x_0 + \alpha p_0 = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \pi/4 \end{bmatrix}$

$f(x_0) = 0$

$$f(x_0 + \alpha p_0) = f\left(\begin{bmatrix} -\alpha \\ \pi/4 \end{bmatrix}\right) = \frac{\alpha^2}{2} - \alpha \cos\left(\frac{\pi}{4}\right) = \alpha\left(\frac{\alpha}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= \frac{\alpha}{2}(\alpha - \sqrt{2})$$

$$\nabla f(x_0) = \begin{bmatrix} \sqrt{2}/2 \\ 0 \end{bmatrix}$$

$$\nabla f(x_0 + \alpha p_0) = \begin{bmatrix} -\alpha + \cos(\pi/4) \\ \alpha \sin(\pi/4) \end{bmatrix} = \begin{bmatrix} -\alpha + \sqrt{2}/2 \\ \sqrt{2}\alpha/2 \end{bmatrix}$$

Let's check the Wolfe conditions for $c_1 = 0.1$ and $c_2 = 0.8$

$$f(x_0 + \alpha p_0) \leq f(x_0) + 0.1\alpha \nabla f(x_0)^T p_0 \quad (1)$$

$$\nabla f(x_0 + \alpha p_0)^T p_0 \geq 0.8 \nabla f(x_0)^T p_0 \quad (2)$$

$$\frac{\alpha}{2}(\alpha - \sqrt{2}) \leq 0 - 0.1\alpha \frac{\sqrt{2}}{2} \quad (1a)$$

$$\alpha - \frac{\sqrt{2}}{2} \geq -0.8 \frac{\sqrt{2}}{2} \quad (2a)$$

$\Rightarrow$ From (2a), we have

$$\alpha \geq \frac{\sqrt{2}}{2}(1 - 0.8) = \frac{\sqrt{2}}{10}$$

$\Rightarrow$ From (1a), we have

$$\frac{1}{2}(\alpha - \sqrt{2}) \leq -0.1 \frac{\sqrt{2}}{2} \Rightarrow \alpha - \sqrt{2} \leq -0.1\sqrt{2}$$

$$\Rightarrow \alpha \leq 0.9\sqrt{2}$$

$\Rightarrow$ As a result,

$\boxed{\frac{\sqrt{2}}{10} \leq \alpha \leq \frac{9\sqrt{2}}{10}}$ for Wolfe conditions.

e) $x_0 = (0, \pi/4)^T \quad p_0 = (-1, 0)^T$

For exact line search, we should find α value such as

$$\underset{\alpha}{\operatorname{argmin}}\, f(x_k + \alpha p_k) \Rightarrow \frac{\partial f(x_k + \alpha p_k)}{\partial \alpha} = 0$$

$$f(x_k + \alpha p_k) = \frac{(x_1 + \alpha p_1)^2}{2} + (x_1 + \alpha p_1)\cos(x_2 + \alpha p_2)$$

$$= \frac{1}{2}(x_1^2 + 2\alpha p_1 x_1 + \alpha^2 p_1^2) + x_1 \cos(x_2 + \alpha p_2) + \alpha p_1 \cos(x_2 + \alpha p_2)$$

$$\frac{\partial f(x_k + \alpha p_k)}{\partial \alpha} = p_1 x_1 + \alpha p_1^2 - x_1 \sin(x_2 + \alpha p_2) p_2 + p_1 \cos(x_2 + \alpha p_2) - \alpha p_1 \sin(x_2 + \alpha p_2) p_2$$

$$= 0$$

By using x_0 and p_0, we have

$$\alpha - \cos\left(\frac{\pi}{4}\right) = 0 \Rightarrow \alpha = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f(x_0) = 0 \qquad \nabla f(x_0) = [\sqrt{2}/2 \quad 0]^T$$

$$x_1 = x_0 + \alpha p_0 = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix} + \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \pi/4 \end{bmatrix}$$

$$f(x_1) = \frac{(-\sqrt{2}/2)^2}{2} - \frac{\sqrt{2}}{2}\cos\left(\frac{\pi}{4}\right) = \frac{2}{8} - \frac{2}{4} = -\frac{1}{4}$$