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Assignment 3

Q# 1

(a) Prove that $||u+v||^2 = ||u||^2 + 2\langle u,v\rangle + ||v||^2$

Taking L.H.S:

$= \langle u+v \rangle . \langle u+v \rangle$

$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

$= ||u||^2 + 2\langle u,v\rangle + ||v||^2$

Hence, proved L.H.S = R.H.S

(b) Prove that $\langle u+v, u-v\rangle = ||u||^2 - ||v||^2$

Taking L.H.S:-

$= \langle u, u-v\rangle + \langle v, u-v\rangle$

$= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle$

$= ||u||^2 - ||v||^2$

LHS = RHS proved

(c) If $||u|| = 4$ $v = (2,-1,-2)$ and u v are orthogonal find $||u+v||^2$

$||u+v||^2 = ||u||^2 + 2\langle u,v\rangle + ||v||^2$

$(4)^2 + 2\langle 0,\rangle + \langle v,v\rangle$

$||u+v||^2 = 25$

$||u+v|| = 5$

Q#2

$V_1 = U_1 = 1$

$$\langle U_2, V_1 \rangle = \int_{-1}^{1} x \, dx = 0$$

$$V_2 = U_2 - \frac{\langle U_2 V_1 \rangle V_1}{\|V_1\|} = U_2 = x$$

$$\langle U_3, V_1 \rangle = \int_{-1}^{1} x^2 \, dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

$$\langle U_3, V_2 \rangle = \int_{-1}^{1} x^3 \, dx = \left.\frac{x^4}{4}\right|_{-1}^{1} = 0$$

$$\|V_1\|^2 = \langle V_1 \cdot V_1 \rangle = \int_{-1}^{1} 1 \, dx = x\Big|_{-1}^{1} = 2$$

$$V_3 = U_3 - \frac{\langle U_3, V_1 \rangle}{\|V_1\|} V_1 - \frac{\langle U_3, V_2 \rangle}{\|V_2\|} V_2$$

$$= x^2 - \frac{1}{3}$$

The orthogonal basis are $W = \{1, x, x^2 - 1/3\}$
for $W = \{V_1, V_2, V_3\}$

Q#3

$||x||_1$, $||x||_2$, $||x||_\infty$

i) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$||x||_1 = |2| + |3| = 5$

$||x||_2 = \sqrt{(2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{13}$

$||x||_\infty = \max(|2|, |3|) = |3|$

ii) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$||x_1|| = |0| + |2| = 2$

$||x_2|| = \sqrt{(0)^2 + (2)^2} = \sqrt{4} = 2$

$||x||_\infty = \max(|0|, |2|) = |2|$

iii) $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$

$||x||_1 = |-4| + |-1| = 5$

$||x||_2 = \sqrt{(-4) + (-1)^2} = \sqrt{17}$

$||x||_\infty = \max(|-4|, |-1|) = 4$

Q#4

QR decomposition.

$A = QR$

$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -2 \end{bmatrix}$

$Det(A) = -5$

QR decomposition possible

Using Gram-Schmidt Process to find Q

$V_1 = U_1 = (2, 1, 0)$

$V_2 = U_2 - \frac{\langle U_2, V_1 \rangle}{||V||^2} V_1$

$N_2 = (-1, 2, -1) - \frac{0}{\sqrt{5}}(2, 1, 0) = (-1, 2, -1)$

$V_3 = U_3 - \frac{\langle U_3, V_1 \rangle}{||V_1||^2} V_1 - \frac{\langle U_3, V_2 \rangle V_2}{||V_2||^2}$

$$V_3 = (1,-2,-2) - \frac{0}{\sqrt{5}}(2,1,0) - \left(\frac{-7}{\sqrt{6}}(-1,2,1)\right)$$

$$V_3 = (1,-2,-2) - \left(\frac{7}{6}, -\frac{7}{3}, \frac{-7}{6}\right) = \left(\frac{-1}{6}, \frac{1}{3}, \frac{-5}{6}\right)$$

$$W = \left\{ V_1 = (2,1,0),\ V_2 = (-1,2,1),\ V_3 = \left(\frac{-1}{6}, 1/3, -5/6\right) \right\}$$

Converting the above orthogonal basis to orthonormal basis

$$\hat{V}_1 = \frac{V_1}{\|V_1\|} = (2/\sqrt{5}, 1/\sqrt{5}, 0)$$

$$\hat{V}_2 = \frac{V_2}{\|V_2\|} = (-1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6})$$

$$\hat{V}_3 = \frac{V_3}{\|V_3\|} = \left(-\frac{\sqrt{30}}{30}, \frac{\sqrt{30}}{15}, -\frac{\sqrt{30}}{6}\right)$$

$$Q = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{6} & -\sqrt{30}/30 \\ 1/\sqrt{5} & 2/\sqrt{6} & \sqrt{30}/15 \\ 0 & 1/\sqrt{6} & -\sqrt{30}/6 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -\sqrt{30}/30 & \sqrt{30}/15 & -\sqrt{30}/6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{6} & -7\sqrt{6}/\sqrt{6} \\ 0 & 0 & \sqrt{30}/\sqrt{6} \end{bmatrix}$$

QR decomposition of A is

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{6} & -\sqrt{30}/\sqrt{30} \\ 1/\sqrt{5} & 2/\sqrt{6} & \sqrt{30}/15 \\ 0 & 1/\sqrt{6} & -\sqrt{30}/6 \end{bmatrix} = Q$$

$$R = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{6} & -7\sqrt{6}/6 \\ 0 & 0 & \sqrt{30}/6 \end{bmatrix}$$

Q#5

$$A = \begin{bmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -7-\lambda & 24 & 0 & 0 \\ 24 & 7-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 24 \\ 0 & 0 & 24 & 7-\lambda \end{bmatrix}$$

$$-7-\lambda \begin{bmatrix} 7-\lambda & 0 & 0 \\ 0 & -7-\lambda & 24 \\ 0 & 24 & 7-\lambda \end{bmatrix} - 24 \begin{bmatrix} 24 & 0 & 0 \\ 0 & -7-\lambda & 24 \\ 0 & 24 & 7-\lambda \end{bmatrix}$$

$$(-7-\lambda)(7-\lambda)\{(\cancel{60}-7-\lambda)(7-\lambda) - 576\} - 24(24)\{(-7-\lambda)$$
$$(7-\lambda) - 576\} = 0$$

$$(-49 + \lambda^2)(\lambda^2 - 625) - 576(\lambda^2 - 625) = 0$$

$$(\lambda^2 - 625)(-49 + \lambda^2 - 576) = 0$$

$$(\lambda^2 - 625)^2 = 0$$

$$\lambda_1 = -25, \quad \lambda_2 = 25 \quad \lambda_3 = -25, \quad \lambda_4 = 25$$

for $\lambda = 25$

$$\begin{bmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{bmatrix}$$

$$R_1/-32 \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{bmatrix} = R_2 - 24R_1 \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{bmatrix}$$

$$R_2/-32 \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_3 - 24R_2 \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 24 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 3/4s$

$x_2 = s$

$x_3 = 3/4t$

$x_4 = t$

$$s \begin{bmatrix} 3/4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3/4 \\ 0 \\ 1 \end{bmatrix}$$

eigen vectors are $\left\{ \begin{bmatrix} 3/4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3/4 \\ 0 \\ 1 \end{bmatrix} \right\}$

for $\lambda = -25$

$$= R_1/18 \begin{bmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{bmatrix}$$

$$= R_2 - 24R_1 \begin{bmatrix} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R_2/18 \begin{bmatrix} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 24 & 32 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= R_3 - 24R_1 \begin{bmatrix} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -4/3\,s$

$x_2 = s$

$x_3 = -4/3\,t$

$x_4 = t$

Eigen vectors
$$\left\{ \underset{P_3}{\begin{bmatrix} 0 \\ 0 \\ -4/3 \\ 1 \end{bmatrix}}, \underset{P_4}{\begin{bmatrix} -4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \right\}$$

Applying Gram-Schmidt proceu to bo the the basis $\{P_1, P_2\}$ and $\{P_3, P_4\}$ to normalize

$$P = \begin{bmatrix} 3/4 & 0 & 0 & -4/3 \\ 0 & 3/4 & 0 & 1 \\ 1 & 0 & -4/3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$P_2 \begin{bmatrix} 3/5 & 0 & 0 & -4/5 \\ 0 & 3/5 & 0 & 3/5 \\ 4/5 & 0 & -4/5 & 0 \\ 0 & 4/5 & 3/5 & 0 \end{bmatrix}$$

$$P^T A P_2 \begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 25 \end{bmatrix}$$