* Notation O *

Prouver que :

- $n = O(n)$ $\Leftrightarrow$ $\begin{cases} f(n) = O(g(n)) \\ f(n) = n \in \mathbb{N} \\ g(n) = n \in \mathbb{N}. \end{cases}$

$n = O(n) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0 :$

on divise des deux côtés par n, car $n \geq 1$.

$$n \leq c \cdot n$$
$$\Rightarrow 1 \leq c$$

Comme $1 \leq c$ nous assure que $n = O(n)$, il suffit de prendre $n_0 = 1$ pour que la relation reste vraie.

- $2n = O(3n) \Leftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) = 2n \in \mathbb{N} \\ g(n) = 3n \in \mathbb{N} \end{cases}$

$2n = O(3n) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0 :$

$$2n \leq c \times (3n)$$

on divise par n, car $n \geq 1$

$$\Rightarrow 2 \leq 3c$$
$$\Rightarrow (2/3) \leq c$$

Comme $(2/3) \leq c$ nous assure que $2n = O(3n)$ il suffit de prendre $n_0 = 1$.

• $n+2 = O(n) \Leftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) := n+2 \in \mathbb{N} \\ g(n) = n \in \mathbb{N} \end{cases}$

$n+2 = O(n) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0$

$$n+2 \leq c\, n$$

$C = 2$ et $n_0 = 1$ marche, car:

$$n+2 \leq 2n.$$

• $\sqrt{n} = O(n) \Leftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) = \sqrt{n} \in \mathbb{R}^+ \\ g(n) = n \in \mathbb{N}. \end{cases}$

$\sqrt{n} = O(n) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0:$

$$\sqrt{n} \leq c\, n$$

$$\Rightarrow 1 / \sqrt{n} \leq c$$

Pour $n \geq 1$: $n^2 \geq n$

$$\Rightarrow \sqrt{n^2} \geq \sqrt{n}$$

$$\Rightarrow n \geq \sqrt{n}$$

2/ $C = 1$ et $n_0 = 1$.

• $\log_2(n) = O(n)$: $\begin{cases} f(n) = O(g(n)) \\ f(n) = \log_2(n) \\ g(n) = n \end{cases}$

$\log(n) = O(n) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0 :$

$$\log(n) \leq c . n$$

$$\Rightarrow \frac{\log(n)}{n} \leq c$$

Par ailleurs, $\forall n \geq 1$: $\log(n) < n$

il suffit de prendre $c = 1$ et $n_0 = 1$.

• $n = O(n^2)$: $\begin{cases} f(n) = O(g(n)) \\ f(n) = n \\ g(n) = n^2 . \end{cases}$

Ceci est trivialement vrai, démo :

$n = O(n^2) \Leftrightarrow \exists n_0 \in \mathbb{N}, \exists c \geq 0, \forall n \geq n_0 :$

$$n \leq c . n^2$$

$$\Rightarrow 1/n \leq c$$

or : $\forall n \geq 1$: $n^2 \geq n$

il suffit de prendre $c = n_0 = 1$.

* Notation o *

- $\sqrt{n} = o(n) \iff \begin{cases} f(n) = o(g(n)) \\ f(n) = \sqrt{n} \in \mathbb{R}^+ \\ g(n) = n \in \mathbb{N} \end{cases}$

il suffit de remarquer que :

$\forall n \geq \underbrace{1}_{= n_0}$: $\dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}} \xrightarrow[n \to +\infty]{} 0$

i.e : $\lim\limits_{n \to +\infty} \dfrac{1}{\sqrt{n}} = 0.$

Donc, $\sqrt{n} = o(n)$.

- $\log(n) = o(n) \iff \begin{cases} f(n) = o(g(n)) \\ f(n) = \log(n) \\ g(n) = n \end{cases}$

il suffit de remarquer que :

$\forall n \geq \underbrace{1}_{= n_0}$: $\dfrac{\log(n)}{n} = \dfrac{\left(\dfrac{\ln(n)}{\ln(10)}\right)}{n}$

$= \dfrac{1}{\ln(10)} \cdot \left(\dfrac{\ln(n)}{n}\right)$

Ainsi : $\lim\limits_{n \to +\infty} \dfrac{\log(n)}{n} = \lim\limits_{n \to +\infty} \dfrac{1}{\ln(10)} \left(\dfrac{\ln(n)}{n}\right)$

situation $\dfrac{\infty}{\infty}$
$\Rightarrow$ Règle d'hôpital

$\overset{?}{=} \dfrac{1}{\ln(10)} \cdot \lim\limits_{n \to +\infty} \left(\dfrac{\ln(n)}{n}\right)$

$= (\ln(10))^{-1} \cdot \lim\limits_{n \to +\infty} \left(\dfrac{\ln'(n)}{n'}\right)$

$= (\ln(10))^{-1} \lim\limits_{n \to +\infty} \left(\dfrac{1/n}{1}\right)$

$= 0.$

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• $n = o(n^2) \Leftrightarrow \begin{cases} f(n) = o(g(n)) \\ f(n) = n \\ g(n) = n^2 \end{cases}$

$\forall n \geq 1.$

$$\frac{n}{n^2} = \frac{1}{n} \xrightarrow[+\infty]{} 0.$$

• $\log(n) = o(\sqrt{n}) \Leftrightarrow \begin{cases} \log(n) = o(\sqrt{n}) \\ f(n) \qquad g(n) \end{cases}$

$\forall n \geq 1.$

$$\frac{\log(n)}{\sqrt{n}} = \frac{\frac{\ln(n)}{\ln(10)}}{\sqrt{n}}$$

$$= \left(\frac{1}{\ln(10)}\right) \frac{\ln(n)}{\sqrt{n}}$$

$$= \left(\frac{1}{\ln(10)}\right) \cdot 2 \cdot \left(\frac{\frac{1}{2}\ln(n)}{\sqrt{n}}\right)$$

$$= \left(\frac{2}{\ln(10)}\right) \cdot \frac{\ln(n^{1/2})}{n^{1/2}}$$

$$\xrightarrow[n \to +\infty]{} 0$$

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