FEM 45418

File 13

HRZ Lumping Scheme

why HRZ? ? ? ?

procedure:

- Use ? ? ? of the consistant mass matrix
- scale them such that ? ? ? ?
- Compute the total man of the element m
- Compute the "S" number

$$S = \sum m_{ii}$$ → associated with ? ? D.O.F. that are mutually ? ? ? and in the same direction.

- scale all ? ? ? by multiplying them by the ratio m/s

Example:

Bar element
$$[m] = \frac{m}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$S = \Sigma m_{ii} = 4\left(\frac{m}{6}\right)$$

$$S = \Sigma m_{ii} = 4\left(\frac{m}{6}\right)$$

total mass $\equiv m$

$$\frac{m}{S} = \frac{m}{4(m/6)} = \frac{3}{2} \longrightarrow \text{for } u \text{ or } x\text{-dir.}$$

coefficient

$$\frac{m}{S} = \frac{m}{4(m/6)} = \frac{3}{2} \longrightarrow \text{for } v \text{ or } y\text{-dir}$$

$$[m] = \begin{bmatrix} \frac{m}{3}\times\frac{3}{2} & & & 0 \\ & \frac{m}{3}\times\frac{3}{2} & & \\ & & \frac{m}{3}\times\frac{3}{2} & \\ 0 & & & \frac{m}{3}\times\frac{3}{2} \end{bmatrix} = \frac{m}{2}\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

HRZ lump mass matrix for bar element

HRZ Lumping (Beam element)

$$[m] = \frac{m}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$ ← consistant mass matrix

total $m = m$

$$S = \sum m_{ii} = \frac{2 \times 156 \times m}{420}$$

$$\frac{m}{S} = \frac{420}{2 \times 156}$$

$$[m] = \begin{bmatrix} \frac{m}{2} & & & \\ & \frac{mL^2}{78} & & 0 \\ & 0 & \frac{m}{2} & \\ & & & \frac{mL^2}{78} \end{bmatrix}$$ ⟶ we had before $$[m] = \begin{bmatrix} m/2 & & & \\ & \frac{m}{2} \times \frac{\alpha L^2}{210} & & 0 \\ & 0 & m/2 & \\ & & & \frac{m}{2} \times \frac{\alpha L^2}{210} \end{bmatrix}$$

$\alpha = 17.5$ or $\frac{m}{2} \times \frac{17.5 L^2}{210}$

$= \frac{mL^2}{24}$

$\alpha = 0 \rightarrow$ No rotary term

Using ~~WRM~~ method as explained before

just the terms associated with ???

$$\bar{G} = \frac{m}{420}\left(156 + 54 + 156 + 54 + \cancel{22L} + \cancel{13L} - \cancel{22L} - \cancel{13L}\right)$$

$$\bar{G} = m$$

Just the terms associated with ???

$$\sum g_{ii} = 2 \times 156 \times \frac{m}{420}$$

$$\Rightarrow G_{11} = \frac{g_{ii}\,\bar{G}}{\sum g_{ii}} = \frac{156 \times m}{420 \times 2 \times 156 \times m/420} = \frac{m}{2}$$

$$G_{22} = \frac{m\,4L^2/420 \times m}{2 \times 156 \times \frac{m}{420}} = \frac{mL^2}{78}$$

Natural Frequency and mode shapes

Assumptions:

+ no damper
+ no externally applied load (to unrestrained nodes)
+ under initial conditions, the structure undergoes harmonic motion

$\Rightarrow \quad \{D\} = \{\bar{D}\}\sin\omega t$ or $\{\ddot{D}\} = -\omega^2\{\bar{D}\}\sin\omega t$

amplitude

circular frequency (rad/s)

$$[m]\{\ddot{D}\} + \cancel{[C]\{\dot{D}\}}^{\,0} + [K]\{D\} = \cancel{\{R_{ext}\}}^{\,0}$$

$$\big[[K] - \omega^2[m]\big]\{\bar{D}\} = \{0\}$$

if $\big[[K] - \omega^2[m]\big] \rightarrow$ singular $\Rightarrow$ Trivial solution;

$\{\bar{D}\} = \{0\}$

For nontrivial solution: $\det\big[[K]-\omega^2[m]\big]=0$

+ For each eigenvalue (ω) $\Rightarrow$ There exist an eigenvector $\{\bar{D}\}_i$

+ If [K] and [m] are both positive definite : all $\omega > 0$

This happens when: [K] has all rigid-body modes constrained

and

[m] is either consistant or lumped

with strictly ? ? ?

+ one or more zero eigenvalue(s) (i.e. $\omega = 0$)

(for semidefinite [K] and [m])

for a - - ? - - ? - ? - structure

which has positive semidefinite [K] (each $\omega = 0$ is

associated - - ? - - ? - ? ?

+ with optimal lumping sometimes we encounter

some nonrotational coefficients to be zero or negative

Typically:

each $m_{ii} = 0$ => will result in an infinite eigenvalue

$\omega \simeq \infty$

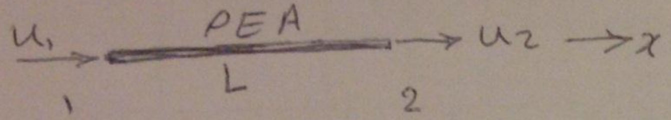
each $m_{ii} < 0$ => " " " a negative eigenvalue

$\omega < 0$

$\Downarrow$

gives rise to an imaginary frequency

Example: a 1-D unsupported Bar



$$\left([k]-\omega^2[m]\right)\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

For consistant mass matrix:

$$\left(\frac{AE}{L}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}-\omega^2\frac{\rho AL}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\omega^2\left(\omega^2\rho AL-12\frac{AE}{L}\right)=0$$

$$\Longrightarrow \quad \omega_1=0$$

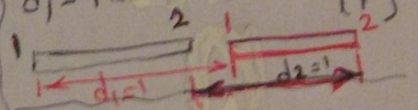
$$\omega_2=\frac{2}{L}\sqrt{3\,E/\rho}=\frac{3.464}{L}\sqrt{E/\rho}$$

$\omega_1=0$

$$\left(\frac{AE}{L}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}-0\right)\begin{Bmatrix}d_1\\d_2\end{Bmatrix}=0$$

Assume:

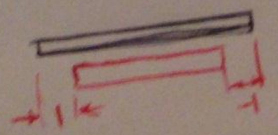
$d_1=1 \Rightarrow d_2=1$ $\begin{Bmatrix}1\\1\end{Bmatrix}$



$$\omega_2=\frac{3.464}{L}\sqrt{E/\rho}$$

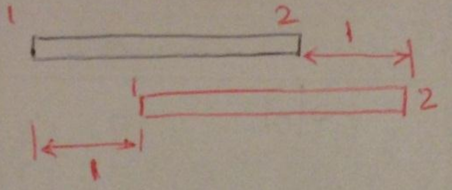
Assume:

$d_1=1 \rightarrow d_2=-1$ $\begin{Bmatrix}1\\-1\end{Bmatrix}$



eigenvector corresponding to $\omega_1 = 0$

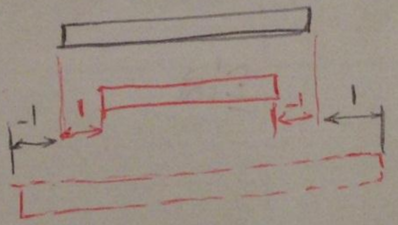
$$d_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad d_2 =$$



rigid body motion in x_ dir.

eigenvector corresponding to $\omega_2 = \frac{3.464}{L}\sqrt{E/\rho}$

$$d_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \quad d_2 =$$



axial straining mode

__Exact solution:__ $\omega_2 = \frac{\pi}{L}\sqrt{E/\rho}$

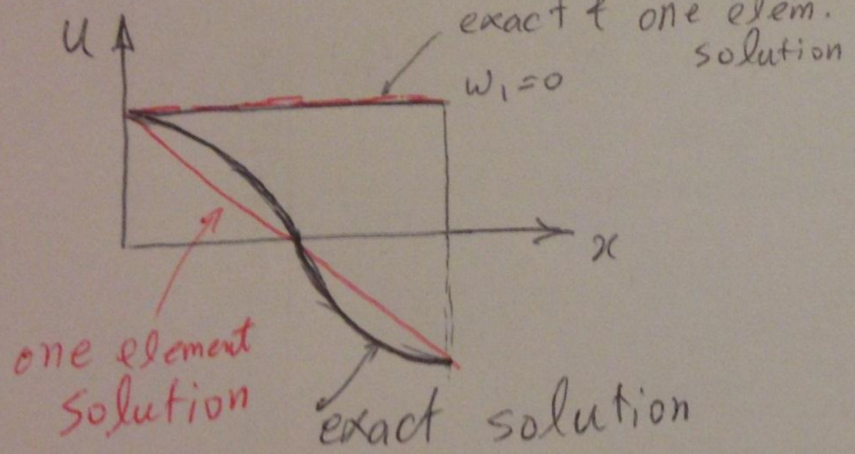
for a bar without support with Length L

$\Rightarrow$ Consistant mass matrix over predicts the natural frequency by % for This example.

For lump mass matrix:

$$\left(\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \omega^2\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$\omega_1 = 0$ & $\omega_2 = \frac{2}{L}\sqrt{E/\rho}$ exact: $\omega_2 = \frac{\pi}{L}\sqrt{E/\rho}$



Consistant mass matrix

over stimates ω_2

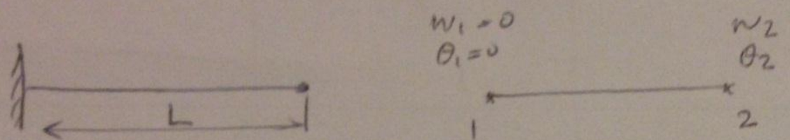
$\omega_2 = \frac{3.464}{L}\sqrt{E/\rho}$

by approximately %10 here

Lump mass matrix:

understimates by approximately %30 here

Example 2:

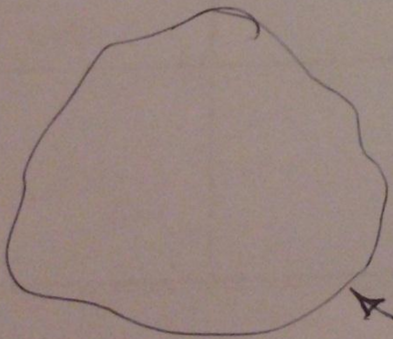


Consistant mass matrix:

$$\left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho A L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{Symm.} & & & 4L^2 \end{bmatrix}\right)\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho A L}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

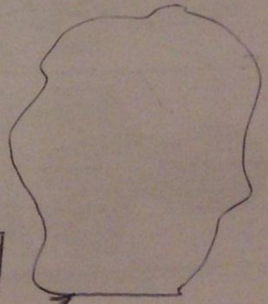
$$\begin{vmatrix} 12 - 156a & -6L + 22La \\ -6L + 22La & 4L^2 - 4L^2 a \end{vmatrix} = 0 \quad \leftarrow a = \frac{\omega^2 \rho A L^4}{420 EI}$$



$$\omega_1 = 3.533\left(\frac{EI}{\rho A L^4}\right)^{1/2} \qquad \omega_2 = 34.81\left(\frac{EI}{\rho A L^4}\right)^{1/2}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

First mode $\begin{bmatrix} w_2 = 1 \\ \theta_2 = \frac{1.38}{L} \end{bmatrix}$ Second mode $\begin{bmatrix} w_2 = 1 \\ \theta_2 = \frac{7.62}{L} \end{bmatrix}$



Lump mass matrix:

$$\left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & \text{symm.} & 12 & -6L \\ & & & 4L^2 \end{bmatrix} - \omega^2 \frac{\rho A L}{2}\begin{bmatrix} 1 & & & \\ & \frac{\alpha L^2}{210} & & 0 \\ & & 1 & \\ & 0 & & \frac{\alpha L^2}{210} \end{bmatrix}\right) = \begin{bmatrix} 0 \end{bmatrix}$$

without the rotary inertia $\Rightarrow \alpha = 0$

$$\left.\begin{aligned} \omega_1 &= 2.449\left(\frac{EI}{\rho A L^4}\right)^{1/2} \\ \omega_2 &= \text{not obtainable} \end{aligned}\right\} \quad \begin{aligned} W_1 &= 1 \\ \theta_2 &= 1.5/L \end{aligned} \rightarrow \begin{Bmatrix} 1 \\ 1.5/L \end{Bmatrix}$$

mode shape

Exact solution:

$$\omega_1 = 3.516\left(\frac{EI}{\rho A L^4}\right)^{1/2}$$

$$\omega_2 = 22.03\left(\frac{EI}{\rho A L^4}\right)^{1/2}$$

$\Rightarrow$

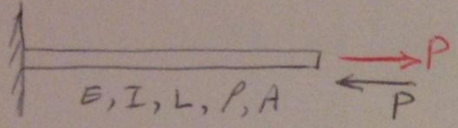
+ Consistant mass matrix overstimates

+ lump mass matrix understimates

mesh refinment

???

Effect of Tension-compression on Vibrations



$$\left[K_{ij} + K^{\sigma}_{ij} - \omega^2 M_{ij}\right][d] = [F]$$

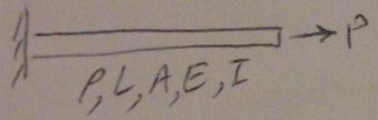
Buckling check:

$$\left([K] + \lambda[K_\sigma]\right)[d] = \{0\}$$

Vibrations check:

$$\left([K] - \omega^2[M]\right)[d] = \{0\}$$

Example:



$$\left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix} + \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ & 4 & -3L & -1 \\ & & 36 & -3L \\ \text{symm.} & & & 4 \end{bmatrix} - \frac{\omega^2 m}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm.} & & & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \{0\}$$

$$\beta = \frac{P}{30\frac{EI}{L^2}}$$

$$\begin{vmatrix} 12 + 36\beta - \frac{156\omega^2}{420A} & -6L - 3L\beta + \frac{\omega^2}{A}\frac{22L}{420} \\ \text{symm.} & 4L^2 + 4L^2\beta - \frac{\omega^2}{A}\frac{4L^2}{420} \end{vmatrix} = 0$$

$$A = \frac{EI}{L^3 m}$$

If $P = 0 \rightarrow \beta = 0 \rightarrow \omega_1 = 3.534\sqrt{A}$

If $P = -2.986 \frac{EI}{L^2} = P_{cr} \rightarrow \beta = -0.08287$

$$\rightarrow \left(9.0167 - \frac{\omega^2}{A}\frac{156}{420}\right)4\left(0.91713 - \frac{\omega^2}{420A}\right) - \left(5.7514 - \frac{\omega^2}{A}\frac{22}{420}\right)^2 = 0$$

$$\Longrightarrow \omega = ?$$

? ? ?

IF $P = -\frac{1}{2} P_{cr} \longrightarrow \beta = -0.04144$

$$\longrightarrow \omega_1 = 2.493\sqrt{A}$$

IF $P = +\frac{1}{2} P_{cr} \rightarrow \beta = +0.04144$

$$\longrightarrow \omega_1^2 = 18.735 A$$

$$-\frac{P}{P_{cr}} + \left(\frac{\omega}{\omega_0}\right)^2 = 1 \rightarrow \left(\frac{\omega}{\omega_0}\right)^2 = 1 + \frac{P}{P_{cr}} \rightarrow \omega = \omega_0\sqrt{1 + \frac{P}{P_{cr}}}$$

no axial force

if $P < 0 \rightarrow \omega < \omega_0$

if $P > 0 \rightarrow \omega > \omega_0$