ГЛАВА 1

ОБЩИЕ СВЕДЕНИЯ

Обозначения, используемые в этой главе:

- Величины без риски: реальные траектории систем
- Величины с рисками: предсказанные траектории
- L-стоимость перехода
- $(\cdot;t)$ - значение, предсказанное в момент t
- T - горизонт предсказания
- Оптимальное значение функции $J^*(x(t)) = J(x(t), \bar{u}^*(t))$

1.1 Терминальная задача MPC

Приведем математическую формулировку для задачи с нулевым терминальным множеством:

Системная динамика: $\dot{x} = f(x,u),\ x(0) = x_0,\ x,u \in \mathbb{R}^n$

Ограничения: $x(t) \in X,\ u \in U,\ \forall t \geq 0$

Предположения:

- $f(0,0) \Rightarrow x_1 = 0$– точка равновесия для $u_1 = 0$
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ – дважды непрерывно дифференцируема
- U компактное множество (ограниченное и замкнутое)
- X связанное и закрытое множество
- $(0,0) \in int(X \times U)$

Задача MPC:

В момент t, дано начальное состояние $x(t)$

$$\min_{\bar{u}(\cdot,t)} J(x(t), \bar{u}(\cdot;t))$$

with $J(x(t), \bar{u}(\cdot;t)) = \int_t^{t+T} L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau$

такое, что

$$\dot{\bar{x}} = f(x,u), \bar{x}(t;t) = x(t)$$

$$\bar{u}(\tau;t) \in U, \bar{x}(\tau;t) \in X,\ \forall \tau \in [t, t+T]$$

$$\bar{x}(t+T;t) = 0$$

Оптимальное управление открытой системы:

$$\bar{u}^*(\cdot;t) = arg\ min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot;t))$$

Отметим, что настоящая траектория системы может отличаться от предсказанной

Предположения:

- $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ непрерывная и

$$\begin{cases} L(0,0) = 0 \\ L(x,u) > 0 \\ \forall (x,u) \neq (0,0) \end{cases} \tag{1.1}$$

- $J^*(x)$ непрерывна в точке $x = 0$

Алгоритм MPC

1. В момент времени t, вычисляем $x(t)$ и решаем оптимизационную задачу MPC
2. Применяем $u_{MPC}(\tau) = \bar{u}^*(\tau, t) \forall t \in [t, t+\delta)$ на временном промежутке δ
3. Устанавливаем $t := t + \delta$ и переходим к шагу 1

Достижимость: Задача MPC достижима в момент времени t если существует хотя бы одно управление $\bar{u}(\cdot;t)$, удовлетворяющее ограничениям.

Theorem 1.1.1 Предположим, что

1. предположения выполняются
2. задача с нулевым терминальным множеством достижима в момент времени $t = 0$

Тогда верно следующее:

- задача MPC рекуррентно достижима
- получаемая в результате замкнутая система является асимптотически стабильной

Пусть $D \subset \mathbb{R}^n$ является множеством всех точек, где выполняется (??). Тогда D называется областью притяжения замкнутой системы.

Приведем идею доказательства, представленной выше теоремы, так как такой это универсальный подход и он еще будет использоваться в дальнейшем для наших целей.

Доказательство.

1. рекуррентная достижимость доказывается по индукции
2. - достижима в $t = 0$ по предположению индукции
   - допустим, что достижима в момент t. Рассмотрим следующее управление:

$$\bar{u}(\tau; t+\delta) = \begin{cases} \bar{u}^*(\tau; t) \; \tau \in [t+\delta, t+T] \\ 0 \; \tau \in [t+T, t+\delta+T] \end{cases}$$

3. асимптотическая стабильность

   Идея в использовании функции $J^*(x(t))$ в качестве функции Ляпунова.

   Рассмотрим:

$$J(x(t+\delta), \bar{u}(\cdot; t+\delta)) = \int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), \bar{u}(\tau; t+\delta))d\tau =$$

$$= \int_{t+\delta}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau + \int_{t+T}^{t+\delta+T} L(0,0)d\tau(=0) =$$

$$= J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

   из оптимальности

$$J^*(x(t+\delta)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) \leq J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

   по индукции

$$J^*(x(\infty))(\geq 0) \leq J^*(x(0))(finite) - \int_{0}^{\infty} L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$$

Lemma 1 (Barbalat's) *ϕ uniformly continuous $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$*

$$\lim_{t\to\infty} \int_0^t \phi(\tau) d\tau < \infty \Rightarrow \phi(t) \to 0, t \to \infty$$

Из леммы Barbalat's $L \to 0$ при $t \to \infty \Rightarrow$. Следовательно мы получаем, что $\|x_{MPC}(t)\| \to 0$ при $t \to \infty \Rightarrow$, что и означает сходимость.

1.2 Robust MPC

Рассмотрим линейную (дискретную) систему: $x(t+1) = Ax(t) + Bu(t) + w(t)$ in short $x^+ = Ax + Bu + w$

Constraints: $x(t) \in X, u(t) \in U, \forall t = 0, 1...$

Bound on w: W is a compact,convex set which contains 0. $w(t) \in W \ \forall t = 0, 1, ...$

Main idea: Use additional error feeedback s.t. real systems state contained in a "tube"around some nominal system state.

Repetition of QI-MPC in discrete time:

Nominal system:

$$z^+ = Az + Bv$$

At time t, given $z(t)$, solve

$$\min_{v(\cdot|t)} \hat{J}(z(t), v(\cdot|t)) = \sum_{i=t}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

s.t.

$$z(i+1|t) = Az(i|t) + Bv(i|t), z(t|t) = z(t)$$

$$z(i|t) \in Z, v(i|t) \in V, t \leq i \leq t+N-1$$

$$z(t+N|t) \in Z^f \subseteq Z$$

$\Rightarrow$ optimizer $V^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Assumption 1:

- Cost is quadratic $L(z, v) = z^T Qz + v^t Rv, Q, R > 0$
- There exists a local auxiliary controller $k^{loc} = Kx$ s.t.
    1. Z^f is invariant with $Z^+ = (A+BK)z$, $A_k = A+BK$, i.e. $A_k Z^f \subseteq Z^f$
    2. $Kz \in V \forall z \in Z^f$

3. $F(A_k z) - F(z) \leq -L(z, Kz) \forall z \in Z^f$

From Assumption 1 it follows (as in continuous time) that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \leq -L(z(t), v_{MPC}(t))$$

Since L is quadratic, there exists constants $c_2 > c_1 > 0$ s.t. $\forall z \in Z_N$ - feasible set

1. $c_1 |z|^2 \leq \hat{J}^*(z)$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1 |z|^2$
3. $\hat{J}^*(z) \leq c_2 |z|^2$

Why is (3) true?
From Assumption 1.3 $\forall z \in Z^f$

$$\hat{J}^*(z) \leq \hat{J}(z, Kz(\cdot)) = \sum_{i=1}^{N-1} L(z(i), Kz(i)) + F(z(N)) \leq$$

N times apply Assumption 1.3

$$\leq F(z) = z^T P z \leq \lambda_{max}(P) |z|^2$$

Influence of disturbance:

Определение 1.1 Minkowski set addition:

$$A, B \subseteq \mathbb{R}^n A \oplus B = \{a + b | a \in A, b \in B\}$$

Pontryagin set difference:

$$A, B \subseteq \mathbb{R}^n A \ominus B = \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$$

$$(A \ominus B) \oplus B \subseteq A$$

$$A \subseteq (A \oplus B) \ominus B$$

Определение 1.2 Robust positively invariant set (RPI set):
S is RPI set for $x^+ = Ax + w$ if $AS \oplus W \subseteq S$ (or equivalently $Ax + w \in S \forall x \in S, \forall w \in W$)

Пример 1.1 $x^+ = 0.5x + w$. $w \in [-5, 5]$. So RPI set: $S = [-20, 20]$, minimal RPI set: $S = [-10, 10]$

Minimal RPI set:

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

(Minkowski set addition), min. RPI set exists and is bounded if A is Schour table.

Why?

Current state at time t is x,

possible states at time $t+1$: $Ax \oplus W$

$t+2$: $A(Ax \oplus W) \oplus W = A^2 x \oplus AW \oplus W$

....

$t+j$: $A^j x \oplus \sum_{k=0}^{j-1} A^k w$

$\Rightarrow$ by choosing j large enough we can reach any state in S_∞

$\Rightarrow$ any RPI set must satisfy $S_\infty \subseteq S$

Remains to show: S_∞ is an RPI set

$$AS_\infty \oplus W = A\sum_{i=0}^{\infty} A^i w \oplus W = \sum_{i=1}^{\infty} A^i w \oplus W = S_\infty$$

S_∞ in general is difficult to compute

$\Rightarrow$ can compute invariant outer approximations of S_∞ (with bounded complexity)

Пример 1.2 Calculate RPI

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

For the system given and bounded disturbances

$$x^+ = \frac{1}{2}x + w,\ \ w \in [-5, 5]$$

$$S_\infty = \sum_{i=0}^{\infty} (\frac{1}{2})^i [-5, 5] = [-10, 10]$$

Central idea in tube-based MPC

Use additional error feedback around some nominal input:

$$u_{MPC} = v_{MPC}(x) + K(x - z)$$

Proposition 1

Let $x^+ = Ax+Bu+w$ and $z^+ = Az+Bv$. If $x \in Z \oplus S$ and $u = v+K(x-z)$, then $X^+ \in Z^+ \oplus S$ (RPI set for $x^+ = (A+BK)x+w$)

image to be inserted

Доказательство.

Let $e(t) := x(t) - z(t) \rightarrow$

$$e^+ = x^+ - z^+ = Ax + B(v + K(x-z)) + w - Az - Bv = (A+BK)e + w$$

As S is RPI for $e^+ = A_k e + w$, we obtain $e \in S \Rightarrow e \in S\ \forall w \in W$

Hence $x \in Z \oplus S \Rightarrow x^+ \in Z^+ \oplus S\ \forall w \in W$

Robust MPC scheme

MPC problem for robust tube-based MPC: At time t, given $x(t)$, solve

$$\min_{z(t|t), v(\cdot|t)} J(x(t), v(\cdot|t)) = \sum_{i=1}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

$$s.t. z(i+1|t) = Az(i|t) + Bv(i|t)$$

$$z(i|t) \in Z = X \ominus S$$

$$v(i|t)\ \in V = U \ominus KS$$

$$t \leq i \leq t+N-1$$

$$z(t+N|t) \in Z_N \subseteq Z$$

Initial condition $x(t) \in z(t|t) \oplus S$

$\rightarrow$ optimizer: $z^*(t|t), v^*(\cdot|t) \rightarrow$ optimal value function $J^*(x(t))$

$\rightarrow$ applied input: $u(t) = v^*(t|t) + K(x(t) - z^*(t|t))$

Important: Tightened input/state constraints for the nominal predictions ensure fulfilment of original input/state constraints for real (disturbed) closed-loop system.

Properties of robust MPC scheme (in the following $z^*(x(t)) := z^*(t|t)$)

- a feasible set $X_N = Z_N \oplus S \subseteq X$
- $J^*(x) = \hat{J}^*(z^*(x))$ by definition of J^* and $\hat{J}^*$
- $J^*(x) = 0\ \forall x \in S$

Why?

If $x \in S$, then $z(x) = 0$ and $v(\cdot|t) = 0$ is a feasible solution. Hence $J^*(x) \leq \hat{J}(0,0) = 0$

$\Rightarrow J^*(x) = 0$ and $z^*(X) = 0$

"S serves an origin for the disturbed system"

Theorem 1.2.1 Suppose that Assumption 1 holds and the robust MPC problem is feasible at $t = 0$.

Then:

1. robust MPC problem is recursively feasible
2. closed-loop system robustly exponentially converges to S
3. closed-loop system satisfies input/state constraints, i.e. $x(t) \in X$, $u(t) \in U$ $\forall t = 0, 1...$

Доказательство.

i) Consider candidate solution at time $t + 1$

$$\tilde{V}(i|t+1) = \begin{cases} v^*(i|t) \; t+1 \leq i \leq t+N-1 \\ k^{loc}(z^*(t+N|t)) \; i = t+N \end{cases}$$

$$\tilde{z}(t+1|t+1) = z^*(t+1|t)$$

it is feasible because $x(t+1) \in z^*(t+1|t) \oplus S$ by proposition 1

image to be inserted

iii) follows from Proposition 1 + definition of tightened constraints

ii) from (1-3) inequalities described below

1. $\hat{J}^*(z) \geq C_1|z|^2$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1|z|^2$
3. $\hat{J}^*(z) \leq c_2|z|^2$

$J^*(x) = \hat{J}^*(z^*(x))$

we obtain the following $\forall x \in X_N$

4. $J^*(x) = \hat{J}^*(z^*(x)) \geq^{(1)} c_1|z^*(x)|^2$
5. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(3)} c_2|z^*(x)|^2$

So now we will show convergence to 0

$$J^*(x(t+1)) - J^*(x(t)) = \hat{J}^*(z^*(x(t+1))) - \hat{J}^*(z^*(x(t))) \leq$$

$$\leq \hat{J}^*(z^*(x(t+1|t))) - \hat{J}^*(z^*(x(t))) \leq^{(2)}$$

$$-c_1|z^*(x(t))|^2 \leq -\frac{c_1}{c_2}J^*(x(t))$$

$\Rightarrow$

$$J^*(x(t+1)) \leq (1-\frac{c_1}{c_2})J^*(x(t))$$

where $\gamma := 1-\frac{c_1}{c_2},\ \gamma \in (0,1)$

$$J^*(x(i)) = \gamma^i J^*(x(0)) \leq^{(5)} c_2\gamma^i|z^*(x(0))|^2$$

$\Rightarrow^{(4)}$

$$|z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$\Rightarrow z^*(x(t))$ exponentially converges to 0.

Recall: $x(i) \in z^*(x(i)) \oplus S \Rightarrow$

$$|x(i)|_S \leq |z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$|x(i)|_S$ - point-to-set distance

Extensions:

- Nonlinear systems: difficult to compute RPI sets
  - approaches based on input-to-state stability(ISS)
  - approaches which apply MPC two times:
    * first for nominal input
    * to determine local error feedback(Rawlings and Mayne chapter 3-6)
- Linear systems with parametric uncertainties

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

$$(A(t), B(t)) \in \rho : con(A_j, B_j), j = 1, ..., J\ \forall \geq 0$$

Note. *co*- convex

Define: $\bar{A} := \frac{1}{J}\sum_{i=0}^{J} A_i,\ \bar{B} := \frac{1}{J}\sum_{i=0}^{J} B_i$

$$x(t+1) = \bar{A}x(t) + \bar{B}u(t) + w(t)$$

$$w(t) \in W := (A-\bar{A})x + (B-\bar{B})u|(A,B) \in \rho, x \in X, u \in U$$

W is compact if X,U are compact

$\rightarrow$ can apply tube MPC as before but: can slow down more!

If ρ is "small enough closed-loop asymptotically to zero

Intuition: x converges to the RPI set $S \rightarrow W$ gets smaller

$\rightarrow x$ converges to RPI set

Invariant approximations of the minimal RPI set S_∞ is difficult to compute

$$S_\infty := \sum_{i=0}^{\infty} A^i w$$

Define $S_k := \sum_{i=0}^{k-1} A^i w \ k \geq 1$

In general, S_k for a finite k are not RPI sets (this is the case if only if A is nilpotent)

Theorem 1.2.2 If $0 \in int(W)$ and A is Schur, then there exists an integer $k > 0$ and $\alpha \in [0, 1)$ s.t.

$$A^k W \subseteq \alpha W \tag{1.2}$$

If (1.2) holds, then

$$S(\alpha, k) := (1-\alpha)^{-1} S_k$$

is an RPI set for the system $x^+ = Ax + w$

Доказательство.

i)(1.2) is a direct consequence of our assumptions ii) want to show that $AS(\alpha, k) \oplus W \subseteq S(\alpha, k)$

$$AS(\alpha, k) \oplus W = (1-\alpha)^{-1} \sum_{i=1}^{k} A^i W \oplus W =$$

$$= (1-\alpha)^{-1} A^k W \oplus \sum_{i=1}^{k-1} A^i W (1-\alpha)^{-1} \oplus W$$

As far as $A^k W \subseteq \alpha W$ by (1.2)

$$(1-\alpha)^{-1} \alpha W \oplus W \oplus \sum_{i=1}^{k-1} A^i W (1-\alpha)^{-1}$$

As $(1-\alpha)^{-1} \alpha W \oplus W = [(1-\alpha)^{-1}\alpha + 1] W = (1-\alpha)^{-1} W$

Then we get

$$= (1-\alpha)^{-1} \sum_{i=0}^{k-1} A^i W = S(\alpha, k)$$

Remark:

- for a given k s.t. (1.2) can be satisfied, we want to find the smallest possible α ("small scaling factor")

- for a given α, one wants to find the smallest possible k s.t. (1.2) holds ("low complexity"of RPI set)

  $\Rightarrow$ tradeoff between small α and small k needs to be found

- one can determine "how good"$S(\alpha, k)$ is compared to S_∞

  $\Rightarrow$ can specify suboptimality degree of approximation a priori

  Possible algorithm to determine RPI set

  1. fix $\alpha \in (0, 1)$ and $k > 0$ (integer)
  2. check whether (1.2) holds:
     - if yes: $S(\alpha, k)$ is a RPI set
     - if not: set $k := k + 1$ and go to (2)