Differential equations

Lemma 1 (Cromwall)**.** Suppose that $0 \le \phi(t) \le c + L\int_0^t \phi(\tau)d\tau$, $c, L > 0$, ϕ – continuous. Then $\phi(t) \le ce^{Lt}$.

Definition. Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \le \epsilon,\ \forall t \ge 0$.

Definition. Equilibrium point $x = 0$ is asymptotically stable if it is stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t\to\infty} x(t) \to 0$.

Nonlinear systems

Definition. Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \ge 0,\ \exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

Definition. Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0\ \ \exists \delta > 0$, s.t $\forall t_0 \ge 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

Definition. Point $x^* = 0$ asymptotically stable if it is stable and $\forall t_0 \ge 0\ \ \exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

Definition. Point $x^* = 0$ uniformly asymptotically stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

Definition. Convergence: $\forall \eta > 0\ \ \forall t_0 \ge 0$, $\exists T > 0$ such that $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

Definition. Uniform convergence: $\forall \eta > 0\ \ \exists T > 0$ such that $\forall t_0 \ge 0$ and $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

Definition. Point $x^* = 0$ is globally uniformly asymptotically stable if it is uniformly stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta\ \ \ \exists T > 0$ such that $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\|x(t)\| < \eta$, $\forall t \ge t_0 + T$.

Theorem 0.1 (Lyapunov's direct method)**.** Let $f : [0, \infty) \times D \to R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \to R$ with:

- $W_1(x) \le V(t, x) \le W_2(x)$, $\forall t \ge 0,\ x \in D$
- $\dot V(t, x) \le 0$, $\forall t \ge 0,\ x \in D$

where $W_1, W_2 : D \to R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.
If further $\dot V(t, x) \le -W_3(x)$, $\forall t \ge 0,\ x \in D$ with $W_3 : D \to R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.
If $D = R^n$ and W_1 is radialy unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

Definition. A function $\alpha : [0, \delta) \to [0, \infty)$ is (of) "class K" if it is continuous, strictly increasing, and $\alpha(0) = 0$.

Definition. A function $\alpha : [0, \delta) \to [0, \infty)$ is "class K_∞" if $\alpha \in K$ and $\lim_{r\to\infty} \to \infty$.

Definition. A function $\beta : [0, \delta) \times [0, \delta) \to [0, \infty)$ is "class KL if it is continuous , $\beta(\cdot, s) \in K$ for all fixed s, and for each fixed r, $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s\to\infty} \beta(r, s) = 0$

Lemma 2. The equilibrium $x^* = 0$ of $\dot x(t) = f(t, x(t))$ is uniformly stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \alpha(||x(t_0)||)$.

Lemma 3. The equilibrium $x^* = 0$ of $\dot x(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \beta(||x(t_0)||, t - t_0)$.

System with inputs

Definition. System (**??**) is input-to-state stable (ISS) if $\exists \beta \in KL$, $\gamma \in K$ s.t. $\forall x_0 \in R^n$, $\forall t \ge 0$ follows $||x(t)|| \le \beta(||x_0||, t) + \gamma(\sup_{\tau\in[0,t]} ||u(\tau)||)$.

Theorem 0.2. Suppose that there exists a continuously differentiable function $V : R^n \to R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x, u) \le -\alpha_3(||x||)$, $\forall x : ||x|| \ge \rho(||u||)$. Then (**??**) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Theorem 0.3. Assume that:

- f is globally Lipschitz;
- $x = 0$ is a globally exponentially stable EP for $\dot x = f(x, 0)$

Then the system (**??**) is ISS.

Theorem 0.4 (Artstein)**.** There exists $k : R^n \to R^m$ (state feedback) which is continuous on $R^n \backslash \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot x = f(x) + G(x)k(x)$ iff there exists a CLF.

Sontag's formula"
Fix $c \ge 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$-cb(x) - \frac{a(x)+\sqrt{a(x)^2+(b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T$$
$$0,\ \ b(x) = 0$$

Backstepping

Integrator backstepping

$$\dot x_1 = f_1(x_1) + g_1(x_1)x_2 \tag{1}$$
$$\dot x_2 = u$$

$$u = (-\frac{\partial V_1}{\partial e_1} g_1(e_1) + \dot{\alpha_1}) - k_2 e_2,\ k_2 > 0 \tag{2}$$

$$\dot{x_1} = f_1(x_1) + g_1(x_1)x_2$$
$$\dot{x_2} = f_2(x_1, x_2) + g_2(x_1, x_2)u$$

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)}(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha_1}$$
$$-k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2))$$

$$\alpha_i(x_1, \dots x_i) = \frac{1}{g_i}(\dot\alpha_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}} g_{i-1}$$
$$-k_i(x_i - \alpha_{i-1}) - f_i)$$

Systems with inputs and outputs

Two-step approach:

1. Bring $x(t)$ to $S := \{x \in \mathbb{R}^n | S(x) = 0\}$ in finite time
2. Have $x(t)$ going to zero asymptotically (on S)
   - switching between nodes 1 and 2
   - mode 2 is "sliding mode"

$$V(X) = \frac{1}{2}s(x)^2$$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat u sgn(s(x))),\ \hat u > 0$$

$$\dot x = f(x) + g(x)\sigma(x) + g(x)u$$

If $|\sigma(x)| \le \beta(x)$

$$u = -\frac{L_f s(x)}{L_g s(x)} -$$
$$-\frac{1}{L_g s(x)}(\hat u + \beta(x))|L_g s(x)| sgn(s(x))$$

Definition (dissipativity)**.**

$$S(x(t)) \le S(x_0) + \int_0^t s(u(\tau), y(\tau))d\tau \tag{3}$$

Introduce "available storage" $S_a(x)$

$$sup_{u:[0,T]\to\mathbb{R}^m, T\ge0, x(0)=0}(-\int_0^T s(u(\tau), y(\tau)))d\tau$$

Theorem 0.5. System is dissipative w.r.t. the supply rate s iff $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$ Moreover, if $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$, then S_a is a storage function and $S(x) \ge S_a(x)\ \forall x \in \mathbb{R}^n$ for all storage functions S.

If system is dissipative then $x = 0$ is asymptotically stable.

$$\dot x = f(x, u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$
$$y = h(x),\ y \in \mathbb{R}^m \tag{4}$$

Definition. System is passive if it is dissipative w.r.t. supply rate $s(u, y) = u^T y$

Definition. System is zero-state observable (ZSO) if (for $u(t) = 0$) $y(t) = 0$ for all $t \ge 0 \Rightarrow x(t) = 0$ for all $t \ge 0$

Theorem 0.6. Let system (4) be i) passive in differentiable storage set ii)ZSO. Then the feedback $u = -Py,\ P > 0$ renders the origin asymptotically stable.

Theorem 0.7. Feedback interconnection with $u \equiv 0$. H_1 and H_2 are ZSO and dissipative with $S_1,\ S_2$ w.r.t.

$$s_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i,\ i = 1, 2,$$
$$\rho, \nu \in \mathbb{R}$$

The origin $(x_1, x_2) = (0, 0)$ for interconnection is asymptotically stable if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$.

If is sabisfied with $v_i = 0$: "output - feedback passive". If (**??**) satisfied with $p_i = 0$: "input - feadforward passive".