Nonlinear Control

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Intro

Goals of Course

- overview over moder nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control method learn the mathmatic basis

Differential equations $\dot{x} = f(x)$

Nonlinear differential eqution $\dot{x} = f(t, x)$

System with input $\dot{x} = f(x, u)$

System with input and output $\dot{x} = f(x, u), \quad y = g(x, u)$

Input-output methods

Scope

[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997

1   Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)),\ x(0) = x_0 \tag{1}$$

Where $f : D \to R^n,\ D \subset R^n$ is open, [here we should explain, what means open set].

Solution to 1 $x : I_{x_0} \to D,\ t \to x(t)$ is differentiable

Interval existence solution

Questions:

\# existence of solution

\# "how large" is I_{x_0}

\# uniquence of solution

Usaly we will add some restrictions on f functions, like continuous.

1.1 Existence of solutions

Definition. Function $f : D \to R^n$ is contiuous at $x' \in D$ if for $\forall \epsilon > 0 \; \exists \delta > 0$ such that for $\forall x \in D$, $\|x - x'\| < \delta => \|f(x) - f(x')\| < \epsilon$

Function $f : D \to R^n$ is continuous on D if it's continuous at $\forall x' \in D$

Theorem 1.1 (Piano). If $f : D \to R^n$ continuous, then for each $x_0 \in D \; \exists x : (-\epsilon, \epsilon) \to D, \; \epsilon > 0$ satisfying (1).

Further, given a compact sed $U \subset D$, then $\exists \alpha > 0$ s.t. $\forall x_0 \in U \; \exists x : (-\epsilon, \epsilon) \to D$ satisfying (1).

Example. Consider equation $\dot{x}(t) = x(t)^2, \;\; x(0) = x_0 = 0$. Solution $x(t) = -\frac{1}{t-c}, \;\; c = \frac{1}{x_0}$. In this example solution exist in interval $(-c, c)$.

But, what about the number of solutions? Which conditions we should add to garanty uniquence of solution?

1.2 Uniquence of solutions

Definition. Function $f : D \to R^n$ is locally Lipshitz (continuous???) on D if $\forall x \in D$ there is a neighborhood $N(x) \subset D$ and $\exists L > 0$ s.t.

$$\|f(x_1) - f(x_2)\| \le L\|x_1 - x_2\| \tag{2}$$

For all $x_1, x_2 \in N$.

- Lipschiz on $W \in D$ if (2) holds $\forall x_1, x_2 \in W$ (with same L)

- globally Lipschitz if (2) holds $\forall x_1, x_2 \in R^n$ (with same L)

We have

\# localy Lipschitz functions are continuous

\# differenciable functions are locally Lipschitz

\# locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

Lemma 1 (Cromwall). Suppose that $0 \leq \phi(t) \leq c + L \int_0^t \phi(\tau)d\tau$, $c, L > 0$, ϕ − continuous. Then $\phi(t) \leq ce^{Lt}$.

Proof. $c + L \int_0^t \phi(\tau)d\tau := \psi(t)$, $\dot{\psi}(t) = L\phi(t) \leq L\psi(t)$.

Consider $\frac{d}{dt}\left(\psi(t)e^{-LT}\right) = e^{-Lt}\dot{\psi}(t) - L\psi(t) () \leq 0$, thus $\psi(t)e^{-LT}$ is decreased, and as a result we have $\phi(t)e^{-Lt} \leq \psi(t)e^{-Lt} \leq \psi(0) = c$

$\square$

Theorem 1.2 (Picard Lindelof). If function $f : D \to R^n$ is localy Lipschitz then for $\forall x_0 \in D$ $\exists! x : (-\epsilon, \epsilon) \to D$, $\epsilon > 0$ satisfying (1).

Proof. * existence from Piano theorem

Proof of uniqueness

Consider two solutions $x_1(.)$ and $x_2(.)$ to (1). $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$, $x_1(0) = x_2(0)$. Then we can integrate equality: $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau))d\tau$. $|x_1(t) - x_2(t)| \leq \int_0^t |f(x_1(\tau)) - f(x_2(\tau))|d\tau \leq L \int_0^t |f(x_1(\tau)) - f(x_2(\tau))|d\tau$. Now we can apply Cromwall's lemma with $c = 0$ and $\phi(t) = |x_1(t) - x_2(t)|$, then $\phi(t) \leq 0$, then $x_1(t) = x_2(t)$, $\forall t \in (-\epsilon, \epsilon)$ $\square$

Example.

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$Solutions\ x(t) = \begin{cases} \frac{1}{4}(t - c)^2, & \text{if } t \geq c \geq 0 \\ 0, & \text{else} \end{cases}$$

Global existance & uniqueness

- sufficient condition: f globally Lipschitz

- another sufficient condition: solution entirely lies in a coplex set

- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff $\exists$ solution $V : R^n \to R \geq 0$ s.t. $\frac{\partial V}{\partial x}f(x) \leq V(x)$, $\forall x \in R^n$

1.3 Lyapunov stability

If functions $\dot{V}(x) < 0$, $\forall x \in D$
$\{0\}$, then x^* is asymptotically stable.

Definition. Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \leq \epsilon, \; \forall t \geq 0$.

Definition. Equilibrium point $x = 0$ is asymptotikaly stable if it stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t \to \infty} x(t) \to 0$.

Theorem 1.3 (Lyapunov's direct method)**.** Let $x^* = 0 \in D$ be an equilibrium point of (1), i.e., $f(0) = 0$. Let $f : D \to R^n$ is continious. If there exist a differetiable $V : D \to R$ s.t.

1. $V(x^*) = 0, \; V(x) > 0, \; \forall x \in D$
   $\{0\}$

2. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0, \; \forall x \in D$

then $x^* = 0$ is stable.

Proof. Fix compact $U = \{x : V(x) \leq c\}$ s.t. $U \in D$. By Piano: exist $\alpha > 0$ s.t. any solution x with $x_0 \in U$ exists at least on the interval $[0, \alpha)$.

TODO proof is not full $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Lyapunovs direct method gives us:

- stability

- convergence (if $V < 0$)

- subset of the region of attraction (all compact $U = \{x : V(x) \leq c\} \in D$)

- existance of solution for all times

2 Nonlinear systems

In this section we consider function $f : R \times D \to R^n$, where $D \subseteq R^n$, and D is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \tag{3}$$

The origin $x^* \in D$ is an equilinrium point for (3), if $f(t, 0) = 0, \quad \forall t \geq 0$.

Remark: EP (equilibrium point) $x^* = 0$ can be translation of a nonzero solution.

Suppose $\overline{y}$ is a solution of $\dot{y} = g(t, y)$.

Change of coordinates: $x(t) = y(t) - \overline{y}(t)$, then $\dot{x}(t) = \dot{y}(t) - \dot{\overline{y}}(t) = g(t, x(t) + \overline{y}(t)) - \dot{\overline{y}}(t) := f(t, x(t))$. Since $\dot{\overline{y}}(t) = g * t, \overline{y}(t))$, then $f(t, 0) = 0, \; \forall t \geq 0$.

Existance and uniquence of solution to (3):

- if f continuous, then exist local colution

- if f continuous and locally Lipschitz in x^*, then exist local uniq solution

Now we need new stability definitions.

Definition. Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \geq 0$, $\exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

Definition. Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0$ $\exists \delta > 0$, s.t $\forall t_0 \geq 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

Definition. Point $x^* = 0$ asymptoticaly stable if it is stable and $\forall t_0 \geq 0$ $\exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t \to \infty} ||x(t)|| \to 0$.

Definition. Point $x^* = 0$ uniformly asymptoticaly stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $\lim_{t \to \infty} ||x(t)|| \to 0$.

Definition. Convergence: $\forall \eta > 0$ $\forall t_0 \geq 0$, $\exists T > 0$ such that $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

Definition. Uniform convergence: $\forall \eta > 0$ $\exists T > 0$ such that $\forall t_0 \geq 0$ and $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

Example. Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0$$

Solution $x(t) = x(t_0)\frac{1+t_0}{1+t}$. It is uniformly stable, because we can choose $\delta = \epsilon$. But does $x(t)$ convergence uniformly? Answer is no.

Definition. Point $x^* = 0$ is globaly uniformly asymptotically stable if it is uniformly stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta$ $\exists T > 0$ such that $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $||x(t)|| < \eta$, $\forall t \geq t_0 + T$.

2.1 Lyapunov's direct method

Consider some function $V : [0, \infty) \times D \to R$, $(t, x) \to V(t, x)$ such that $\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x)$.

Theorem 2.1 (Lyapunov's direct method). Let $f : [0, \infty) \times D \to R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \to R$ with:

- $W_1(x) \leq V(t, x) \leq W_2(x)$, $\forall t \geq 0$, $x \in D$

- $\dot{V}(t, x) \leq 0$, $\forall t \geq 0$, $x \in D$

where $W_1, W_2 : D \to R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.

If further $\dot{V}(t,x) \leq -W_3(x)$, $\forall t \geq 0$, $x \in D$ with $W_3 : D \to R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.

If $D = R^n$ and W_1 is radialy unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Check function $V(t,x) = \frac{1}{2}x^2$ as candidat for Lyapunov's function. Then $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and $\dot{V}(t,x) = -(1+t)x^2 \leq -x^2(t) =: W_3(x)$. Then from theorem we have, that $X^* = 0$ is globally uniformly asymptotically stable.

2.2 Exponential stability

Definition. Point $X^* = 0$ is an exponentially stable EP of (3) if $\exists \lambda, c, k > 0$ s.t. $t \geq t_0 \geq 0$ and all $\|x_0\| < c$ follows $\|x(t)\| \leq K\|x(t_0)\|e^{\lambda(t-t_0)}$.

Remark: from exponential stability follows uniformly asymptoticaly stability.

Lemma 2 (Auxilarity result). Let $\dot{x}(t) = f(t, x(t))$, f scalar and $\dot{\xi}(t) \leq f(t, \xi(t))$ with $\xi(t_0) \leq x(t_0)$. Then $\xi(t) \leq x(t)$ $\forall t \geq t_0$.

Theorem 2.2. Let $f : [0, \infty) \times D \to R^n$ be continuous and $x^* = 0 \in D$ be an EP.

If there is a differentiable function $V : [0, \infty) \times D \to R$ and constants $k_1, k_2, k_3, a > 0$ s.t.

1. $k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$, $\forall t \geq 0$, $x \in D$

2. $\dot{V}(t,x) \leq -k_3\|x\|^a$

then $x^* = 0$ is exponentially stable.

If $D = R^n$, then X^* is globally exponential stable.

Proof. For $c > 0$ small enough, trajectories initialized in $\{x : k_2\|x\|^a < c\}$ remain bounded and in D. From 1) and 2) we can conclude $\dot{V} \leq -\frac{k_3}{k_2}V$. Then from previous Lemma $V(t, x(t)) \leq V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}$. Then $\|x(t)\| \leq [from 1)] \leq \left(\frac{V(t,x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0,x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{k_2\|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}}$

$\left(\frac{k_2}{k_1}\right)^{\frac{1}{a}}\|x(t_0)\|e^{-\frac{k_3}{k_2 a}(t-t_0)}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Here $V(t,x) = \frac{1}{2}x^2$ then X^* is exponentially stable.

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2.3 Comparsion function

Definition. A function $\alpha : [0, \delta) \to [0, \infty)$ is (of) "klass K" if it is continous, strictly increasing, and $\alpha(0) = 0$.

Definition. A function $\alpha : [0, \delta) \to [0, \infty)$ is "class K_∞ if $\alpha in K$ and $\lim_{r \to \infty} \to \infty$.

Example. Function $\alpha(r) = \tan^{-1}(r) -$ class K

Function $\alpha(r) = r^k -$ class K_∞

Definition. A function $\beta : [0, \delta) \times [0, \delta) \to [0, \infty)$ is "class KL if it is continuous , $\beta(\cdot, s) \in K$ for all fixed s, and for each fixed r, $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s \to \infty} \beta(r, s) = 0$

Example. Function $\beta(x, s) = max(r, r^2)e^s$ belong class KL.

Properties of compasion functions:

- If $\alpha \in K$ on $[0, \delta)$, then α^{-1} is defined on $[0, \alpha(\delta))$ and $\alpha^{-}1 \in K$.

- If $\alpha \in K_\infty$, then $\alpha^{-1} \in K_\infty$

- If $\alpha_1, \alpha_2 \in K$, then $\alpha_1 \circ \alpha_2 \in K$ (same for K_∞

- If $\alpha_1, \alpha_2 \in K$, $\beta \in KL$ then $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we conseider comparsion funcctions and stability definitions.

Lemma 3. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \leq \alpha(||x(t_0)||)$.

(only sufficiency). Given $\epsilon > 0$ choose $\delta < \min(c, \alpha^{-1}(\epsilon))$. Then from $||x(t_0)|| < \delta$ follows $||x(t)|| \leq \alpha(||x(t_0)||) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$. □

Lemma 4. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

(only sufficiency). Let $||x(t_0)|| < c$. Then $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(||x_{t_0}||, 0)$. This gives us uniform stability. □

Lemma 5. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

(only sufficiency). Let $||x(t_0)|| < c$. Then $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(||x_{t_0}||, 0)$. This gives us uniform stability. □

Lemma 6. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable iff previous lemma holds for all $x_0 \in R^n$.

Now consider comparsion functios and Lyapunov functions

If $W : R^n \to R$ is continuous and positive definite, then $\forall r > 0$ $\exists \alpha_1, \alpha_2 \in K$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in B_r(0) = \{x|||x|| \leq r\}$.

If W is radially unbounded, then $\exists \alpha_1, \alpha_2 \in K_\infty$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in R^n$.

Lemma 7 (Auxility). Consider $\dot{y} = \alpha(y)$, $y(t_0) = y_0 > 0$, $\alpha \in K$. Then $\exists \beta \in KL$ s.t. $y(t) = \beta(y_0, t - t_0)$.

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t, x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where W_1, W_2, W_3 – continuous and positive defined.

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in K$ such that $\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)$ and $\dot{V}(t, x) \leq -\alpha_3(||x||)$.

Proof uniform stability:

$$||x(t)|| \leq \alpha_1^{-1}(V(t, x(t)) \leq [\alpha_1 in K] \leq \alpha_1^{-1}(V(t_0, x(t_0)) \leq \alpha_1^{-1}(\alpha_2(||x(t_0)||)).$$

Proof uniform convergence

$\dot{V} \leq -\alpha_3(||x||) \leq -\alpha_3(\alpha_2^{-1}(V))$. We know, that $\alpha_3 \circ \alpha_2^{-1} \in K$. By comparsion lemma, $V(t, x(t)) \leq W(t)$, where W solves $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$ with $W(t_0) = V(t_0, x(t_0))$. By auxility lemma $\exists \beta \in KL$ s.t. $V(t, x(t)) \leq \beta(V(t_0, x(t_0)), t-t_0)$, then $||x(t)|| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\beta(V(t_0, x(t_0)), t-t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(||x(t_0)||), t - t_0)) =: \bar{\beta}(||x(t_0)||, t - t_0)$. From this follows uniform asymptotic stability since $\bar{\beta} \in KL$.

2.4   Converse theorems

Theorem 2.3. Let $X^* = 0$ be an EP of $\dot{x}(t) = f(t, x(t))$ with $f : [0, \infty) \times R^n \to R^n$ continously differentiable and $\frac{\partial f}{\partial x}$ bounded in R^n, uniformly in t ($||\frac{\partial f}{\partial x}(t, x)|| \leq L$ for all $x \in R^n$, $t \geq 0$, $L > 0$.

If $x^* = 0$ is globally exponentially stale, then exists differentiable $V : [0, \infty) \times R^n \to R$ and $c_1, c_2, c_3, c_4 > 0$ s.t. $c_1||x||^2 \leq V(t, x) \leq c_2||x||^2$, $\dot{V}(t, x) \leq -c_3||x||^2$ and $||\frac{\partial V}{\partial x}|| \leq c_4||x||$.

Proof. Let $\Phi(\tau; t, x)$ – solution to $\dot{x}(t) = f(t, x(t))$ which static at (t, x).

$V(t, x) = \int_t^{t+\delta} \Phi^T(\tau; t, x)\Phi(\tau; t, x)d\tau$, $\delta > 0$. Upper bound: $V(t, x) = \int_t^{t+\delta} ||\Phi(\tau; t, x)||_2^2 d\tau \leq$ [*exponential stability*] $\leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)}d\tau||x||_2^2 = \frac{k^2}{2\lambda}(1 - e^{-2\lambda\delta})||x||_2^2$.

Lower bound: since $||\frac{\partial V}{\partial x}|| \leq L$, then $||f(t, x)||_2 \leq L||x||_2$. Thus by comparation lemma $||\Phi(\tau; t, x)||_2^2 \geq ||x||_2^2 e^{-2L(\tau-t)}$. Set it in $V(t, x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)}d\tau||x||_2^2 = \frac{1}{2L}(1 - e^{-2L\delta})||x||_2^2$.

Decrease conditions: $\dot{V}(t, x) = \cdots \leq -(1 - k^2 e^{-2\lambda\delta})||x||_2^2.$ $\qquad\qquad$ □

3   System with inputs

Consider equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \qquad\qquad (4)$$

where $f : R^n \to R^n$.

Assumption: f in localy Lipschitz.

Exageneous signa $u : R \to R^n$.

Input can be "bad" (disturbance) or "good" (control).

3.1   Input-to-state stability

Motivation: LTI system $\dot{x} = Ax + Bu, \quad x(0) = x_0$.

Solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$. If A is Hurwitz, then $||e^{At}|| \leq ce^{-\lambda t}$ for some $c, \lambda > 0$.

How large can x grow for some bounded u? $||x(t)|| \leq ||e^{At}||||x_0|| + \int_0^t ||e^{A(t-\tau)}||||B||||u(\tau)||d\tau \leq e^{-\lambda t}c||x_0|| + \int_0^t e^{-\lambda(t-\tau)}c||B||||u(\tau)||d\tau = ce^{-\lambda t}||x_0|| + (1 - e^{-\lambda t}) \frac{c}{\lambda}||B|| \sup_{\tau \in [0,t]} ||u(\tau)||.$

- $ce^{-\lambda t}||x_0||$ class KL in $(||x_0||, t)$
- $(1 - e^{-\lambda t})$ less than 1
- $\frac{c}{\lambda}||B|| \sup ||u(\tau)||$ class K

If $\sup_{\tau \in [0,t]} ||u(\tau)||$ is bounded than $\dot{x}$ remains bounded. Even more: the smaller $\sup_{\tau \in [0,t]} ||u(\tau)||$, the smaller $||x(t)||$.

Definition. System (4) is input-to-state stable (ISS) if $\exists \beta \in KL, \gamma \in K$ s.t. $\forall x_0 \in R^n, \forall t \geq 0$ follows $||x(t)|| \leq \beta(||x_0||, t) + \gamma(\sup_{\tau \in [0,t]} ||u(\tau)||).$

Remarks:

- From ISS follows O-GAS (global assymptotical stability of $x = 0$ for $\dot{x} = f(x, 0)$
- γ can be interpreted as "gain" w.r.t. u

- if $\lim_{t \to \infty} u(t) = 0$ then $\lim_{t \to \infty} x(t) = 0$

Example. Consider equation $\dot{x} = -x + xu$. System is O-GASS, not ISS (for example $u \equiv \alpha \Rightarrow \dot{x} = x(\alpha - 1)$ all solution diverge).

Example. Consider equation $\dot{x} = -3x + (1 + 2x^2)u$. System is O-GASS, not ISS (for example $u \equiv 1$, $x_0 = 2$, $x(t) = \frac{3 - e^t}{3 - 2e^t}$ has a finite escape time.

Theorem 3.1. Suppose that there exists a continuously differentiable function $V : R^n \to R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(||x||)$, $\forall x : ||x|| \geq \rho(||u||)$. Then (4) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof. Idea: same as Lyapunovs direct method when x is "outside" of ball $\{x | ||x|| \leq \rho(||u||)\}$

TODO Picture $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example. Consider equality $\dot{x} = -x^3 + u$. Let $V(x) = \frac{1}{2}x^2$, then $\dot{V} = -x^4 + xu = [0 < \Theta < 1] = -(1 - \Theta)x^4 - \Theta x^4 + xu \leq -(1 - \Theta)x^4$ for all $x : ||x|| \geq \left(\frac{||u||}{\Theta}\right)^{\frac{1}{3}}$. Thus, system is ISS with $\gamma(v) = \rho(v) = \left(\frac{v}{\Theta}\right)^{\frac{1}{3}}$.

4   Backstepping

These remarks from the last lecture, so should be added to the last chapter

$$\forall x \neq 0 : \inf_u \frac{\partial V}{\partial x} f(x, u) < 0 \; \dot{x} = f(x) + G(x)u$$

So this leads to

$$\forall x \neq 0 L_G V(x) = 0 \Rightarrow L_f V(x) \neq 0$$

Remark: The last formula is "optimal" if minimize:

$$\int_0^\infty \frac{1}{2} p(x)b(x)^T b(x) + \frac{1}{2p(x)} u^T u dt$$

$b(x) := (L_G V(x))^T$

where $c > 0$

$$p(x) = \begin{cases} c + \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} & b(x) \neq 0 \\ c & b(x) = 0 \end{cases}$$

It still works if $u = \lambda h(x)$ with $\lambda \in [\frac{1}{2}; \infty)$ is applied (large "gain margin")

This is Backstepping

Integrator backstepping

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \qquad (5)$$
$$\dot{x}_2 = u$$

where $x_1 \in \mathbb{R}^m$, x_2, $u \in \mathbb{R}$ (single input)

image to be inserted

Assumption: we know (smooth) "feedback" $\alpha_1 : \mathbb{R}^n \to \mathbb{R}$, and positive definite, differentiable $v_1 : \mathbb{R}^m \to \mathbb{R}$

s.t. $L_{f_1+g_1\alpha_1}V_1(x)$ is negative definite $\Rightarrow$ origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1)$ is asymptotically stable

Goal: Compute feedback $u = k(x)$ which stabilises (5). Backstepping constructs $u = \alpha_2(x_1, x_2)$ s.t. $(e_1, e_2) = (x_1 - 0, x_2 - \alpha_1(x_1)) = 0$ error coordinates

Rewrite (5) :

$$\dot{x}_1 = f_1(x_1) + g_1\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1))$$
$$\dot{x}_2 = u$$

image to be inserted

In error coordinates

$$\dot{e}_1 = f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \qquad (6)$$
$$\dot{e}_2 = u - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial e_1}\dot{e}_1 = u - \frac{\partial \alpha_1}{\partial e_1}$$

"backstepping" α_1 through the integrator

Define $V_2(e_2) := \frac{1}{2}e_2^2$, and

$$V(e_1, e_2) = V_1(e_1) + V_2(e_2)$$

$$\dot{V}(e_1, e_2) = \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)\alpha_1(e_1)) + \frac{\partial V_1}{\partial e_1}g_1(e_1)e_2 + \frac{\partial V_2}{\partial e_2}(u - \dot{\alpha}_1)$$

as far as $L_{f_1+g_1\alpha_1}V_1$ -negative definite and $\frac{\partial V_2}{\partial e_2} \to e_2$

Choose

$$u = (-\frac{\partial V_1}{\partial e_1}g_1(e_1) + \dot{\alpha}_1)("canaling\ terms") - k_2e_2("stabilizing\ term")k_2 > 0 \qquad (7)$$

$\Rightarrow$ Then $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0$, $\forall(e_1, e_2) \neq 0$

$\Rightarrow$ Then $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2 e_2^2 < 0 \ \forall (e_1, e_2) \neq 0$

$\Rightarrow (e_1, e_2) = (0,0)$ is an asymptotically stable EP for (6) with u as in (7)

Remark: $(e_1, e_2) \rightarrow (0,0)$ doesnot necessarily imply that $(x_1, x_2) \rightarrow 0$ for $u = \alpha_2(x_1, x_2) = -\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1))$

where $u \leftarrow$ (7) the original coordinates and $\dot{\alpha}_1 \leftarrow \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + g_1(x_1)x_2)$

But $(x_1, x_2) = (0,0)$ is asymptotically stable if $\alpha_1(0) = 0$ why? $(e_1, e_2) \rightarrow 0 \Rightarrow x_1 \rightarrow 0 \ x_2 \rightarrow \alpha_1(0) = 0$

Example.
$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = u$$

Choose $\alpha_1(x_1) = -k \ (k > 0) \rightarrow \dot{x}_1 = -kx_1 \Rightarrow V_1(x_1) = \frac{1}{2}x_1^2$

Then:
$$e_1 = x_1 - 0 \ \dot{e}_1 = e_1(e_2 - k)$$
$$e_2 = x_2 + k \ \dot{e}_2 = u$$

Backstepping yields: $u = -e_1^2 - k_2 e_2 \ k_2 > 0 \Rightarrow (e_1, e_2) = (0,0)$ is asymptotically stabilized

$(x_1, x_2) = (0, -k)$ is asymptotically stabilized

Can we choose different α_1 s.t. $(x_1, x_2) = (0,0)$ is stabilized?

Yes, e.g.
$$\alpha_1(x_1) = -x_1^2 \Rightarrow \dot{x}_1 = -x_1^3 \ V_1(x_1) = \frac{1}{2}x_1^2$$

So we have equations

$$e_1 = x_1 - 0 \ \dot{e}_1 = e_1(e_2 - e_1^2)$$
$$e_2 = x_2 + x_1^2 \ \dot{e}_2 = u + 2e_1^2(e_2 - e_1^2)$$

Backstepping results in

$$u = -e_1^2 - 2e_1^2(e_2 - e_1^2) - k_2 e_2, \ k_2 > 0 \Rightarrow (e_1, e_2) \rightarrow (0,0), \ (x_1, x_2) \rightarrow (0,0)$$

Generalization-1

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)u$$

Assumption: $g_2(x_1, x_2) \neq 0 \forall x_1, x_2 \Rightarrow$ Input transformation: $u = \frac{1}{g_2(x_1, x_2)}(V - f_2(x_1, x_2)) \Rightarrow \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$ $\dot{x}_2 = V \Rightarrow$ can apply integrator backstepping to determine V results in

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)}(-\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2))$$

5 Practice Exercise 3

Motivation: Lyapunov Theory

$$\dot{x} = f(x, u)$$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

Definition. (CLF) A function $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF if it is continuous differentiable, positive definite, radially unbounded and $\forall x \neq 0$ $\inf_u < \nabla V(x), f(x, u) >< 0$

In order to find CLFs, we restrict our analysis to input -affine systems

$$\dot{x} = f(x) + G(x)u$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$

Proposition: A continuous, differentiable, positive definite and radially unbounded. $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF iff
$$\forall x \neq 0 \ L_G V(x) = 0 \Rightarrow L_f V(x) < 0$$

Image to be inserted

Problem 1

Consider $\dot{x} = cos(x) + (1 + e^x)u$ where $f(x) = cos(x)$- drift and $g(x) = 1 + e^x$

Let $V : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{2}x^2$. Clearly, continuous differentiable, positive definite and radially unbounded. Moreover, for any nonzero x, we have $L_G V(x) \neq 0$.

Thus, for any $x \neq 0$, there exists a control that readers $< \nabla V(x), f(x) + g(x)u >$ negative. Givn this CLF, there exists a state feedback $u = u(x)$, e.g.

$$u(x) = -\frac{kx + cos(x)}{1 + e^x}, \ k > 0$$

Problem

Consider

$$\dot{x_1} = -x_1^3 + x_2 e^{x_1} cos(x_2)$$
$$\dot{x_2} = x_1^5 sin(x_2) + u$$

Take $V : \mathbb{R}^2 \to \mathbb{R}, \ (x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2)$

For any $x \neq 0$, we have

$$\inf_{u \in \mathbb{R}} (L_f V(x) + L_G V(x)u) = \begin{cases} L_f V(x), & if \ L_G V(x) = 0 \\ -\infty & else \end{cases}$$

In particular,
$$L_f V(x) = \cdots = x_1(-x_1^3 + x_2 e_1^x cos(x_2)) + x_2 x_1^5 sin(x_2)$$
$$L_G V(x) = \cdots = x_2$$

However,
$$L_f V(x)|_{x_2=0} = -x_1^4 < 0 \ \forall x_1 \neq 0$$

Image to be inserted

Concluding that V is a CLF.

Problem 2:

$\dot{x} = Ax + Bu$, input defined system where (A, B) is stabilizable, there exists $K \in \mathbb{R}^{m \times n}$ s.t. $A + BK$ is Hurwitz (cf. KRT).The latter is equivalent to the existance $P = P^T > 0$ s.t. $P(A+BK) + (A+BK)^T P < 0$ (cf. Khalil theorem 4,6)

Let $V : \mathbb{R}^n \to \mathbb{R}, x \mapsto < x, Px >$. Moreover, $\forall x \neq 0 \exists u = Kx$ s.t. $< \nabla V(x), Ax + Bu > < 0$, since

$$< \nabla V(x), Ax + Bu > =^{u=Kx} < x, (P(A+BK) + (A+BK)^T P)x > < 0$$

In addition,
$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\|K\|} > 0 \ \forall x \neq 0, \ \|x\| < \delta \ \exists u = Kx \ \|u\| < \epsilon$$

s.t. $L_f V(x) + L_G V(x)u < 0$ since $\|u\| = \|Kx\| \leq \|K\|\|x\| < \|K\|\delta = \epsilon$

Problem 3

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be continuous, differentiable consider

$$m\dot{v} = -g\nabla P(q) + F, \ m, g > 0$$

a) Hamiltonian form. Let $x := (q, v)$. Then $\dot{x} = (-\frac{g}{m}\nabla P(q) + \frac{1}{m}F) = \begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} \frac{g}{m}\nabla P(q) \\ V \end{bmatrix} +$

$\begin{bmatrix} \frac{1}{m}I \end{bmatrix} F = \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) + G(x)$ given $H(x) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q)$

b) "CLF". Take H as a CLF candidate. Then, for any x

$$< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x)+G(x)F >=< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) > + < \nabla H(x), G(x)F >= [< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla$$

Strictly speaking, H is no CLF, but it reveals how to choose F s.t. the origin is GAS.

For any point x for which there exists no control F s.t. $< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) + G(x)F >< 0$

Choose $F = 0$. Why? Using the Krasovsky-Lasallle inv. principle, we conclude that the origin is GAS, since any solution in $\{x | \dot{H}(x) = 0\}$ verifies $v(t) \equiv 0$, implying $\dot{v}(t) \equiv 0$ s.t.

$$0 = -\frac{g}{m} \nabla P(q(t)) + \frac{1}{m} P(t)$$

The last part equals 0. Since $F = 0$ (by choice) and $\nabla P(q) = 0$ iff $q = 0$ we conclude that $\dot{H}(x) = 0$ can only be "maintained" at the origin.

Problem 4

Consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ux_2 + u^3$$

show that $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$ is CLF and let $V : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\ddot{x} + u\dot{x} - u^3 = 0$$

For any x and u, we have $< \nabla V(x), f(x, u) >= \cdots = x_1(2x_2 - ux_2 + u^3) + x_2(x_2 - ux_2 + u^3) = x_1 h_1 + x_2 h_2$

Image to be inserted

Hence if $u < 0$ and $-u$ "large", then we can render $< \nabla V(x), f(x, u) >< 0$.