1 MPC

Formulation of control problem

$$\begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0 \\ y = h(x) \end{cases}$$

Objective of MPC: Find stabilizing control strategy that:

- minimize objective function: $J = \int_t^\infty F(x(\tau), u(\tau))d\tau$
- satisfies constraints: $u(\tau) \in U,\ x(\tau) \in X$

Closed-loop optimal control vs Open-loop optimal control

Closed-loop: Feedback $u = k(x)$

- + Feedback present
- + suit for uncertainty disturbances
- - Finding closed solution hardly possible

Open-loop optimal control: Input trajectory $u = u(t, x_0)$

- + Computation often feasible
- - No feedback
- - Don't know much about system

MPC - repeated open-loop optimal control in feedback fashion.

2 Zero-terminal constraint MPC

Mathematical formulation of NMPC problem:

System dynamics: $\dot{x} = f(x,u),\ x(0) = x_0,\ x, u \in \mathbb{R}^n$

Constraints: $x(t) \in X,\ u \in U,\ \forall t \geq 0$

Assumptions:

- $f(0,0) \Rightarrow x_1 = 0$- equilibrium point for $u_1 = 0$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ - twice continuously differentiable
- U is a compact set (closed and bounded)
- X is a connected and closed set
- $(0,0) \in int(X \times U)$

MPC optimization problem:

At time t, given initial state $x(t)$

$$\min_{\bar{u}(\cdot,t)} J(x(t), \bar{u}(\cdot;t))$$

with $J(x(t), \bar{u}(\cdot;t)) = \int_t^{t+T} L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau$

s.t.

$$\begin{gathered}\dot{\bar{x}} = f(x,u), \bar{x}(t;t) = x(t) \\ \bar{u}(\tau;t) \in U, \bar{x}(\tau;t) \in X,\ \forall\tau \in [t, t+T] \\ \bar{x}(t+T;t) = 0\end{gathered}$$

Optimal open-loop solution:

$$\bar{u}^*(\cdot;t) = arg\ min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot;t))$$

Notation used throughout the chapter:

- Quantities without bar: real system trajectories
- Quantities with bar: predicted trajectories
- L-stage cost
- $(\cdot;t)$ - predicted at time t
- T - prediction horizon
- Optimal value function $J^*(x(t)) = J(x(t), \bar{u}^*(t))$

Real trajectories deviate from predicted one!

Assumptions:

- $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and

$$\begin{cases} L(0,0) = 0 \\ L(x,u) > 0 \\ \forall(x,u) \neq (0,0) \end{cases} \tag{1}$$

- $J^*(x)$ is continuous at $x = 0$

MPC algorithm:

1. At sampling time t, measure $x(t)$ and solve MPC optimization problem
2. Apply $u_{MPC}(\tau) = \bar{u}^*(\tau, t) \forall t \in [t, t+\delta)$ with sampling time δ
3. Set $t := t + \delta$ and go to step 1

Feasibility: The MPC problem is feasible at time t if there exists at least one $\bar{u}(\cdot; t)$ s.t. constraints satisfied.

Theorem 2.1. Suppose that

(i) assumptions are satisfied

(ii) and that zero-terminal constraint MPC problem is feasible at $t = 0$

Then:

- MPC problem is recursively feasible
- resulting closed-loop system is asymptotically stable

Let $D \subset \mathbb{R}^n$ be the set of all points for which (ii) holds. The D is a region of attraction for the closed loop.

Proof. 1. recursive feasibility: by induction

2. - feasible at $t = 0$ by assumption
   - assume: feasibility at t. Consider the candidate solution:

$$\bar{u}(\tau; t+\delta) = \begin{cases} \bar{u}^*(\tau; t) \ \tau \in [t+\delta, t+T] \\ 0 \ \tau \in [t+T, t+\delta+T] \end{cases}$$

3. asymptotic stability

   Idea: use $J^*(x(t))$ as "Lyapunov function"

   Consider:

$$J(x(t+\delta), \bar{u}(\cdot; t+\delta)) = \int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), \bar{u}(\tau; t+\delta))d\tau =$$

$$= \int_{t+\delta}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau + \int_{t+T}^{t+\delta+T} L(0, 0)d\tau (= 0) =$$

$$= J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

by optimality

$$J^*(x(t+\delta)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) \leq J^*(x(t)) - \int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau$$

by induction

$$J^*(x(\infty))(\geq 0) \leq J^*(x(0))(finite) - \int_0^{\infty} L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$$

□

Lemma 1 (Barbalat's lemma)**.** ϕ uniformly continuous $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$$\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau < \infty \Rightarrow \phi(t) \to 0, t \to \infty$$

From Barbalat's lemma $L \to 0$ when $t \to \infty \Rightarrow$ L pos.def. $\|x_{MPC}(t)\| \to 0$ when $t \to \infty \Rightarrow$ convergence

Lyapunov stability: using standard arguments (J^* is continuous at $x = 0$)

Lessons learned:

- feasibility $\Rightarrow$ stability
- value function is Lyapunov function
- have to prove recursive feasibility
- suboptimal solution is sufficient for stability

3 Quasi - infinite horizon MPC

Goal: Relax (restrictive) zero-terminal zero-terminal constraint

Idea: terminal region + local CLF(controller Lyapunov functiom)

MPC optimization problem: At time t

$$\min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot;t)) = \int_t^{t+T} L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau + F(\bar{x}(t+T;t))$$

$F(\bar{x}(t+T;t))$ - terminal cost

s.t.

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \bar{x}(t;t) = x(t)$$
$$\bar{x}(t;t) \in X \;\; \bar{u}(t;t) \in U \;\; \forall \tau \in [t, t+T]$$
$$\bar{x}(t+T;t) \in X^f$$

X^f - terminal region

Optimal solution: $\bar{u}^*(\cdot, t), \;\; J^*(x(t))$

Assumption 1: Terminal region + terminal controller

There exists an auxillary local controller $u = k^{loc}(x)$ s.t.

1. X^f is positively invariant $\dot{x} = f(x, k^{loc}(x))$
2. $k^{loc}(x) \in U \;\; \forall x \in X^f$
3. $\dot{F}(x) + L(x, k^{loc}(x)) \leq 0 \;\; \forall x \in X^f$

$\Rightarrow$ F is a local control-Lyapunov function.

Theorem 3.1. Suppose Assumption 1 holds and MPC problem is feasible at $t = 0$. Then:

- recursive feasibility
- closed-loop is asymptotically stable

Proof. 1. Recursive feasibility by induction

- feasible at $t = 0$ by assumption
- assume feasibility at t

candidate

$$\bar{u}(\tau; t+\delta) = \begin{cases} \bar{u}^*(\tau; t) \;\; \tau \in [t, t+T] \\ k^{loc}(\bar{x}(\tau; t+\delta)) \;\; \tau \in [t+T, t+\delta+T] \end{cases}$$

$\Rightarrow$ this is a feasible solution at $t + \delta$

2. asymptotic stability

$$J^*(x(t+\delta)) - J^*(x(t)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) - J^*(x(t)) =$$
$$\int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), \bar{u}(\tau; t+\delta))d\tau + F(\bar{x}(t+\delta+T; t+\delta)) -$$
$$-\int_{t}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau - F(\bar{x}^*(t+T; t)) =$$
$$= \int_{t+T}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), k^{loc}(\bar{x}(\tau; t+\delta)))d\tau + F(\bar{x}(t+\delta+T; t+\delta)) -$$

$$-\int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau - F(\bar{x}^*(t+T;t)) \leq$$

As far as from Assumption 1.3 we have the sum of three terms is ≤ 0

$$-\int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau$$

$\Rightarrow J^*(x(\infty)) \leq J^*(x(0)) - \int_0^\infty L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$ From here: some steps as in zero-terminal constraint rose.

$\square$

How can Assumption 1 be satisfied?

Assume:

- quadratic state cost $L(x,u) = x^TQx + u^TRu,\ Q, R > 0$
- linearization at the origin is stabilizable $\dot{x} = Ax + Bu$ $A = \frac{\partial F}{\partial x}(0,0)$ $B = \frac{\partial F}{\partial u}(0,0)$

Approach:

- Linear auxiliary controller $k^{loc}(x) = Kx$
- Quadratic terminal cost function $F(x) = x^TPx,\ P > 0$
- Terminal region $X_\alpha^f = \{x \in \mathbb{R}^n | x^TPx \leq \alpha\}$ for some $\alpha > 0$
- Determine P, K, α s.t. Assumption 1.1-1.3 hold:

For (Assumption 1.3):

$$\frac{d}{dt}x(t)^TPx(t) \leq -x(t)^t(Q + K^TRK)x(t) = -x(t)^TQ^*x(t)$$

$[x^TQx + u^TRu = [u = Kx] = x^T(Q + K^TRK)x)]$

$$\frac{d}{dt}x(t)^TPx(t) = f(x, Kx)^TPx + x^TPf(x, Kx)$$

$[f(x, Kx) = (A + BK)x + \phi(x), A + BK = A_K,\ K$ is chosen s.t. A_BK is Hurwitz$]$

Upper bound for $x^TP\phi(x)$: $L_\phi := sup\{\frac{|\phi(x)|}{|x|}, x \in X_\alpha^f, x \neq 0\}$

$$x^TP\phi(x) \leq |x^TP||\phi(x)| \leq \|P\|L_\phi|x|^2 \leq \frac{\|P\|L_\phi}{\lambda_{min}(P)}x^TPx \tag{2}$$

We choose α small enough s.t.

$$L_\phi \leq \frac{k\lambda_{min}(P)}{\|P\|} \tag{3}$$

for some $k > 0$. Plug this into (2): $x^T P\phi(x) \leq kx^T Px$. Insert this into $\frac{d}{dt}x^T Px \leq x^T(A_K P + PA_K)x + 2kx^T Px$

$$= x^T((A_K + kI)^T P + P(A_K + kI))x$$

ensure that it $\leq -x^T Q^* x$

$\Rightarrow$ Lyapunov equation which can be solved if any only if $A_K + kI$ is Hurwitz

$$\Leftrightarrow k < -max\ Re\{\lambda(A_K)\} \tag{4}$$

$$\Rightarrow \ (A_K + kI)^T P + P(A_K + kI) = -Q^* \tag{5}$$

Design procedure

1. Compute K s.t. $(A + BK)$ is Hurwitz
2. Choose $k > 0$ s.t. (4) and solve (5)
3. Find largest possible α_1 s.t. $Kx \in U,\ \forall x \in X_\alpha^f$
4. Find the largest $\alpha \in (0, \alpha_1]$ s.t. (3) holds

Alternative to the (4) step

Solve optimization problem

$$\max_x x^T P\phi(x) - kx^T Px s.t.\ x^T Px \leq \alpha \tag{6}$$

Iterate this by reducing α from α_1 until optimal value of (4) is nonpositive

Degrees of freedom in design

- calculation of K
- choice of k - tradeoff between "large" terminal region and "large" P

4 Unconstrained MPC

Goal:Guarantee stability + degree of suboptimality without stabilizing terminal constraint + cost

Setup:

- $\dot{x} = f(x, u), x(0) = x_0$

- input constraints $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m \forall t \geq 0$

Infinite-horizon cost function: $J_\infty(x_0, \bar{u}(\cdot; 0)) = \int_0^\infty L(\bar{x}(\tau; 0), \bar{u}(\tau; 0)) d\tau \Rightarrow$ optimal value function $J^*_\infty(x_0)$

Assumption: $J^*_\infty(x_0) < \infty, \forall x_0 \Rightarrow$ system is asymptotically stabilizable

Finite-horizon cost function: $J_\infty(x(t), \bar{u}(\cdot; t)) = \int_0^\infty L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau$

Infinite-horizon performance resulting from application of MPC controller: $J^{MPC}_\infty(x_0) = \int_0^\infty L(\bar{x}_{MPC}(\tau), \bar{u}_{MPC}(\tau))$

Definition. Suboptimality index α: $\alpha J^{MPC}_\infty(x_0) \leq J^*_\infty(x_0) \forall x_0$

- $\alpha \leq 1$ by optimality of J^*_∞
- $\alpha > 0$ implies closed-loop stability (Barb.lemma)

Proposition 1: Relaxed dynamic programming

Assume $\exists \alpha \in (0, 1] s.t. \forall x \in \mathbb{R}^n$

$J^*_T(x(t+\delta)) \leq J^*_T(x(t)) - \alpha \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau (*)$

Then the estimate

$$\alpha J^*_\infty(x(t)) \leq \alpha J^{MPC}_\infty(x(t)) \leq J^*_T(x(t)) \leq J^*_\infty(x(t)) \tag{7}$$

holds for all $x \in \mathbb{R}^n$

Proof.

- 1 and 3 inequalities follow from optimality (by definition)
- 2 inequality follows from summing up (*) over all sampling instances

$$J^*_T(x(N\delta)) \leq J^*_T(x_0) - \alpha \int_0^{N\delta} L(x_{MPC}(t), u_{MPC}(t)) dt \tag{8}$$

$$N \to \infty : J^*_T(x_0) \geq \alpha J^{MPC}_\infty(x_0) \tag{9}$$

□

Central idea (image to be inserted)

$L^*(t; t) = L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))$

$$(c) : J^*_T(x(t+\delta)) \leq \frac{1}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau : (b) \tag{10}$$

$$(b) : \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau \leq \gamma \int_t^{t+\delta} L^*(\tau; t) d\tau : (a) \tag{11}$$

Theorem 4.1. Assume $\exists c \in (0;1]$ and $\gamma > 0$ s.t. 10 - 11 holds. Then (*) holds with $\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon}$

Proof.

$$J_T^*(x(t+\delta)) - J_T^*(x(t)) = J_T^*(x(t+\delta)) - \int_t^{t+T} L^*(\tau;t)d\tau \leq^{(1)}$$

$$\leq \frac{1-\epsilon}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau - \int_t^{t+\delta} L^*(\tau;t)d\tau \leq^{(2)}$$

$$\leq (\gamma \frac{1-\epsilon}{\epsilon} - 1) \int_t^{t+\delta} L^*(\tau;t)d\tau$$

$-\alpha := \gamma \frac{1-\epsilon}{\epsilon} - 1$ □

Assumption 1: Asymptotic Controlability

For all x, $\exists$ some input trajectory $\hat{u}_x(\cdot)$ with $\hat{u}_x(t) \in \mathbb{U}, \forall t \geq 0$ s.t.

$$L(\hat{x}(t), \hat{u}(t)) \leq \beta(t) min_u L(x,u), \forall t > 0$$

with $\beta : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ - continuous, positive, strictly decreasing with $\lim_{t\rightarrow 0} \beta(t) = 0 \Rightarrow \int_0^\infty \beta(\tau)d\tau < \infty$
$B(t) = \int_0^t \beta(\tau)d\tau$

Typical example: (image to be inserted)

How to compute ϵ and γ:

Lemma 2. Let Assumption 1 hold. Then the inequality

$$J_T^*(x(t+\delta)) \leq \int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t) \tag{12}$$

holds for all $t' \in [\delta, T]$

(image to be inserted)

Proof. Consider

$$\bar{u}(\tau;t+\delta) = \begin{cases} \bar{u}^*(\tau;t), \tau \in [t+\delta, t+t'] \\ \hat{u}_{x'}(\tau - t - t'), \tau \in [t+t', t+\delta+T] \end{cases}$$

$$J_T^*(x(t+\delta)) \leq J_T(x(t+\delta), \bar{u}(\cdot; t+\delta)) =$$

$$= \int_{t+\delta}^{t+t'} L^*(\delta;t)d\delta + \int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau-t-t'), \hat{(}\tau-t-t'))d\tau \leq$$

by Assumption 1

$$\int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau-t-t'), \hat{(}\tau-t-t'))d\tau \leq L^*(t+t';t) \int_0^{T+\delta-t'} \beta(\tau)d\tau$$

as far as $B(t) = \int_0^t \beta(\tau) d\tau$

$$\leq \int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t)$$

□

Calculation of ϵ from (12):

$$J_T^*(x(t+\delta)) \leq \min_{t' \in [\delta,T]} (\int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t)) \leq$$

$$\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau + B(T) \min_{t' \in [\delta,T]} L^*(t+t';t)$$

as far as $\min_{t' \in [\delta,T]} L^*(t+t';t) \leq \frac{1}{T-\delta} \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau$ minimum is less or equal that the average

$$= (1 + \frac{B(T)}{T-\delta}) \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau$$

$(1 + \frac{B(T)}{T-\delta}) = \frac{1}{\epsilon}$

Lemma 3.

$$\int_{t+t'}^{t+T} L^*(\tau;t)d\tau \leq B(T-t')L^*(t+t';t) \forall t' \in [0;T]$$

Proof. Analogues to lemma 1.

Calculation of γ:

$$\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau \leq \int_{t+\hat{t}}^{t+T} L^*(\tau;t)d\tau (\forall \hat{t} \in [0,\delta]) \leq$$

$$\leq \min_{\hat{t} \in [0,\delta]} (B(T-\hat{t})L^*(t+\hat{t};t)) \leq$$

$$\leq B(T) \min_{\hat{t} \in [0,\delta]} L^*(t+\hat{t};t) \leq \frac{B(T)}{\delta} \int_t^{t+\delta} L^*(\tau;t)d\tau$$

□

Denote $\gamma = \frac{B(T)}{\delta}$

$\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon} = 1 - \frac{B(T)}{\delta} (\frac{B(T)}{T-\delta})$

Alternative computation of ϵ (less conservative):

We want to compute ϵ s.t.

$$\epsilon \leq \frac{\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau}{J_T^*(x(t+\delta))}$$

Idea: Minimize

$$\epsilon = \min_{L_t, J_T^*} \frac{\int_\delta^T L_t(\tau; t) d\tau}{J_T^*(x(t+\delta))} \tag{13}$$

$J_T^* = 1$ - without loss of generality s.t. $0 \leq L_t \forall \tau \in [\delta, T]$

$$J_T^*(x(t+\delta)) \leq \int_\delta^{t'} L_t(\tau) d\tau + B(T+\delta-t') L_t(t') \forall t' \in [\delta, T]$$

Due to linearity in L_t, without loss of generality we can set $J_T^* = 1$.

$\Rightarrow$ infinite dimensional linear problem

Idea for solution: second constraint has to be active for all times

Differentiate (12) with relation to t'

$$0 = L_t(t') + \frac{dB(T+\delta-t')}{dt'} L_t(t') + B(\tau+\delta-t')\dot{L}_t(t')$$

as far as $\frac{dB(T+\delta-t')}{dt'} = \beta(T+\delta-t')$

$$\begin{cases} \dot{L}_t(t') = \frac{\beta(T+\delta-t')-1}{B(T+\delta-t')} L_t(t') \\ \text{initial condition } L_t(t) = \frac{1}{B(\tau)} \end{cases}$$

Solution:

$$\bar{L}_t(t') = \frac{1}{B(T+\delta-t')} e^{-\int_\delta^{t'} \frac{1}{B(T+\delta-\tau)} d\tau}$$

Have to show: $\bar{L}_t$ is a minimizer of (13)

$$\int_\delta^T \bar{L}_t(\tau) d\tau \leq \int_{delta}^T L_t(\tau) d\tau$$

for all feasible L_t

Proof. Assume $\exists L_t$ s.t.

$$\int_\delta^T L_t(\tau) d\tau < \int_\delta^T \bar{L}_t(\tau) d\tau$$

Then $\exists \hat{t} \in [\delta, T]$ s.t. $\int_\delta^{\hat{t}} L_t(\tau) d\tau \leq \int_\delta^{\hat{t}} \bar{L}_t(\tau) d\tau$ and $\bar{L}_t(\hat{t}) > L_t(\hat{t})$

But then

$$1 = \int_\delta^{\hat{t}} \bar{L}_t(\tau) d\tau + B(T+\delta-\hat{t})\bar{L}_t(\hat{t}) > \int_\delta^{\hat{t}} L_t(\tau) d\tau + B(T+\delta-\hat{t}) L_t(\hat{t}) \tag{14}$$

the sign equality from (13) with equality. $\square$

Contradiction:

L_t cannot be a feasible solution of (13) $\Rightarrow$

$$\epsilon = \int_{\delta}^{T} \bar{L}_t(\tau)d\tau = 1 - e^{-\int_0^{T-\delta} \frac{1}{B(T-\tau)}d\tau}$$

Similarly, better estimate for γ can be obtained

$\alpha = 1 - \gamma\frac{1-\epsilon}{\epsilon}$ For $T \to \infty$: both estimates for $\epsilon \to 1 \Rightarrow \alpha \to 1$ as $T \to \infty \Rightarrow$ closed-loop stability for T large enough

5 Robust MPC

Consider linear (discrete-time) sytem: $x(t+1) = Ax(t) + Bu(t) + w(t)$ in short $x^+ = Ax + Bu + w$

Constraints: $x(t) \in X, u(t) \in U, \forall t = 0, 1...$

Bound on w: W is a compact,convex set which contains 0. $w(t) \in W \forall t = 0, 1, ...$

Main idea: Use additional error feeedback s.t. real systems state contained in a "tube" around some nominal system state.

Repetition of QI-MPC in discrete time: Nominal system:

$$z^+ = Az + Bv$$

At time t, given $z(t)$, solve

$$\min_{v(\cdot|t)} \hat{J}(z(t), v(\cdot|t)) = \sum_{i=t}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

s.t.

$$z(i+1|t) = Az(i|t) + Bv(i|t), z(t|t) = z(t)$$
$$z(i|t) \in Z, v(i|t) \in V, t \leq i \leq t+N-1$$
$$z(t+N|t) \in Z^f \subseteq Z$$

$\Rightarrow$ optimizer $V^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Assumption 1:

- Cost is quadratic $L(z, v) = z^T Qz + v^t Rv, Q, R > 0$
- There exists a local auxiliary controller $k^{loc} = Kx$ s.t.

1. Z^f is invariant with $Z^+ = (A + BK)z$, $A_k = A + BK$, i.e. $A_k Z^f \subseteq Z^f$
2. $Kz \in V \forall z \in Z^f$
3. $F(A_k z) - F(z) \leq -L(z, Kz) \forall z \; in Z^f$

From Assumption 1 it follows (as in continuous time) that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \leq -L(z(t), v_{MPC}(t))$$

Since L is quadratic, there exists constants $c_2 > c_1 > 0$ s.t. $\forall z \in Z_N$ - feasible set

1. $c_1 |z|^2 \leq \hat{J}^*(z)$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1 |z|^2$
3. $\hat{J}^*(z) \leq c_2 |z|^2$

Why is (3) true?

From Assumption 1.3 $\forall z \in Z^f$

$$\hat{J}^*(z) \leq \hat{J}(z, Kz(\cdot)) = \sum_{i=1}^{N-1} L(z(i), Kz(i)) + F(z(N)) \leq$$

N times apply Assumption 1.3

$$\leq F(z) = z^T P z \leq \lambda_{max}(P) |z|^2$$

Influence of disturbance:

Definition. Mainkowski set addition:

$$A, B \subseteq \mathbb{R}^n A \oplus B = \{a + b | a \in A, b \in B\}$$

Pontryagin set difference:

$$A, B \subseteq \mathbb{R}^n A \ominus B = \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$$

$$(A \ominus B) \oplus B \subseteq A$$

$$A \subseteq (A \oplus B) \ominus B$$

Definition. Robust positively invariant set (RPI set):

S is RPI set for $x^+ = Ax + w$ if $AS \oplus W \subseteq S$ (or equivalently $Ax + w \in S \forall x \in S, \forall w \in W$)

Example. $x^+ = 0.5x + w$. $w \in [-5, 5]$. So RPI set: $S = [-20, 20]$, minimal RPI set: $S = [-10, 10]$

Minimal RPI set:

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

(Minkowski set addition), min. RPI set exists and is bounded if A is Schour table.

Why?

Current state at time t is x,

possible states at time $t+1$: $Ax \oplus W$

$t+2$: $A(Ax \oplus W) \oplus W = A^2 x \oplus AW + w$

....

$t+j$: $A^j x \oplus \sum_{k=0}^{j-1} A^k w$

$\Rightarrow$ by choosing j large enough we can reach any state in S_∞

$\Rightarrow$ any RPI set must satisfy $S_\infty \subseteq S$

Remains to show: S_∞ is an RPI set

$$AS_\infty \oplus W = A \sum_{i=0}^{\infty} A^i w \oplus W = \sum_{i=1}^{\infty} A^i w \oplus W = S_\infty$$

S_∞ in general is difficult to compute

$\Rightarrow$ can compute invariant outer approximations of S_∞ (with bounded complexity)

Example. Calculate RPI

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

For the system given and bounded disturbances

$$x^+ = \frac{1}{2} x + w, \ w \in [-5, 5]$$

$$S_\infty = \sum_{i=0}^{\infty} (\frac{1}{2})^i [-5, 5] = [-10, 10]$$

Central idea in tube-based MPC

Use additional error feedback around some nominal input:

$$u_{MPC} = v_{MPC}(x) + K(x - z)$$

Proposition 1

Let $x^+ = Ax + Bu + w$ and $z^+ = Az + Bv$. If $x \in Z \oplus S$ and $u = v + K(x - z)$, then $X^+ \in Z^+ \oplus S$ (RPI set for $x^+ = (A + BK)x + w$)

image to be inserted

Proof. Let $e(t) := x(t) - x(t) \rightarrow$

$$e^+ = x^+ - z^+ = Ax + B(v + K(x - z)) + w - Az - Bv = (A + BK)e + w$$

As S is RPI for $e^+ = A_k e + w$, we obtain $e \in S \Rightarrow e\ inS \forall w \in W$

Hence $x \in Z \oplus S \Rightarrow x^+ \in Z^+ \oplus S \forall w \in W$

$\square$

Robust MPC scheme

MPC problem for robust tube-based MPC: At time t, given $x(t)$, solve

$$\begin{gathered}
\min_{z(t|t), v(\cdot|t)} J(x(t), v(\cdot|t)) = \sum_{i=1}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t)) \\
s.t. z(i+1|t) = Az(i|t) + Bv(i|t) \\
z(i|t) \in Z = X \ominus S \\
v(i|t)\ inV = U \ominus KS \\
t \leq i \leq t + N - 1 \\
z(t+N|t) \in Z^+ \subseteq Z
\end{gathered}$$

Initial condition $x(t) \in z(t|t) \oplus S$

$\rightarrow$ optimizer: $z^*(t|t), v^*(\cdot|t) \rightarrow$ optimal value function $J^*(x(t))$

$\rightarrow$ applied input: $u(t) = v^*(t|t) + K(x(t) - z^*(t|t))$

Important: Tightened input/state constraints for the nominal predictions ensure fulfilment of original input/state constraints for real (disturbed) closed-loop system.

Properties of robust MPC scheme (in the following $z^*(x(t)) := z^*(t|t)$)

- a feasible set $X_N = Z_N \oplus S \subseteq X$
- $J^*(x) = \hat{J}^*(z^*(x))$ by definition of J^* and $\hat{J}^*$
- $J^*(x) = 0\ \forall x \in S$

Why?

If $x \in S$, then $z(x) = 0$ and $v(\cdot|t) = 0$ is a feasible solution. Hence $J^*(x) \leq \hat{J}(0,0) = 0$

$\Rightarrow J^*(x) = 0$ and $z^*(X) = 0$

"S serves an origin for the disturbed system"

Theorem 5.1. Suppose that Assumption 1 holds and the robust MPC problem is feasible at $t = 0$.

Then:

(i) robust MPC problem is recursively feasible

(ii) closed-loop system robustly exponentially converges to S

(iii) closed-loop system satisfies input/state constraints, i.e. $x(t) \in X$, $u(t) \in U$ $\forall t = 0, 1...$

Proof. i) Consider candidate solution at time $t+1$

$$\tilde{V}(i|t+1) = \begin{cases} v^*(i|t) \; t+1 \leq i \leq t+N-1 \\ k^{loc}(z^*(t+N|t)) \; i = t+N \end{cases}$$

$$\tilde{z}(t+1|t+1) = z^*(t+1|t)$$

it is feasible because $x(t+1) \in z^*(t+1|t) \oplus S$ by proposition 1

image to be inserted

iii) follows from Proposition 1 + definition of tightened constraints

ii) from (1-3) inequalities described below

1. $\hat{J}^*(z) \geq C_1|z|^2$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1|z|^2$
3. $\hat{J}^*(z) \leq c_2|z|^2$

$J^*(x) = \hat{J}^*(z^*(x))$

we obtain the following $\forall x \in X_N$

4. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(1)} |z^*(x)|^2$
5. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(3)} c_2|z^*(x)|^2$

So now we will show convergence to 0

$$J^*(x(t+1)) - J^*(x(t)) = \hat{J}^*(z^*(x(t+1))) - \hat{J}^*(z^*(x(t))) \leq$$

$$\leq \hat{J}^*(z^*(x(t+1|t))) - \hat{J}^*(z^*(x(t))) \leq^{(2)}$$

$$-c_1|z^*(x(t))|^2 \leq -\frac{c_1}{c_2}J^*(x(t))$$

$\Rightarrow$

$$J^*(x(t+1)) \leq (1-\frac{c_1}{c_2})J^*(x(t))$$

where $\gamma := 1 - \frac{c_1}{c_2},\ \gamma \in (0,1)$

$$J^*(x(i)) = \gamma^i J^*(x(0)) \leq^{(5)} c_2\gamma^i|z^*(x(0))|^2$$

$\Rightarrow^{(4)}$

$$|z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$\Rightarrow z^*(x(t))$ exponentially converges to 0.

Recall: $x(i) \in z^*(x(i)) \oplus S \Rightarrow$

$$|x(i)|_S \leq |z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$\square$

$|x(i)|_S$ - point-to-set distance

Extensions:

- Nonlinear systems: difficult to compute RPI sets
  - approaches based on input-to-state stability(ISS)
  - approaches which apply MPC two times:
    * first for nominal input
    * to determine local error feedback(Rawlings and Mayne chapter 3-6)
- Linear systems with parametric uncertainties

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

  $(A(t), B(t)) \in \rho : con(A_j, B_j), j = 1, ..., J\ \forall \geq 0$

Note. *co*- convex

Define: $\bar{A} := \frac{1}{J}\sum_{i=0}^{J} A_i,\ \bar{B} := \frac{1}{J}\sum_{i=0}^{J} B_i$

$$x(t+1) = \bar{A}x(t) + \bar{B}(t) + w(t)$$

$$w(t) \in W := (A - \bar{A})x + (B - \bar{B})u|(A, B) \in \rho, x \in X, u \in U$$

W is compact if X,U are compact

$\rightarrow$ can apply tube MPC as before but: can slow down more!

If ρ is "small enough", closed-loop asymptotically to zero

Intuition: x converges to the RPI set $S \rightarrow W$ gets smaller

$\rightarrow x$ converges to RPI set

Invariant approximations of the minimal RPI set S_∞ is difficult to compute

$$S_\infty := \sum_{i=0}^{\infty} A^i w$$

Define $S_k := \sum_{i=0}^{k-1} A^i w \; k \geq 1$

In general, S_k for a finite k are not RPI sets (this is the case if only if A is nilpotent)

Theorem 5.2. If $0 \in int(W)$ and A is Schur, then there exists an integer $k > 0$ and $\alpha \in [0, 1)$ s.t.

$$A^k W \subseteq \alpha W \tag{15}$$

If (15) holds, then

$$S(\alpha, k) := (1 - \alpha)^{-1} S_k$$

is an RPI set for the system $x^+ = Ax + w$

Proof. i)(15) is a direct consequence of our assumptions ii) want to show that $AS(\alpha, k) \oplus W \subseteq S(\alpha, k)$

$$AS(\alpha, k) \oplus W = (1 - \alpha)^(-1) \sum_{i=1}^{k} A^i W \oplus W =$$

$$= (1 - \alpha)^{-1} A^k W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1} \oplus W$$

As far as $A^k W \subseteq \alpha W$ by (15)

$$(1 - \alpha)^{-1} \alpha W \oplus W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1}$$

As $(1 - \alpha)^{-1} \alpha W \oplus W = [(1 - \alpha)^{-1} \alpha + 1] W = (1 - \alpha)^{-1} W$

Then we get

$$= (1-\alpha)^{-1} \sum_{i=0}^{k-1} A^i W = S(\alpha, k)$$

$\square$

Remark:

- for a given k s.t. (15) can be satisfied, we want to find the smallest possible α ("small scaling factor")
- for a given α, one wants to find the smallest possible k s.t. (15) holds ("low complexity" of RPI set)

  $\Rightarrow$ tradeoff between small α and small k needs to be found
- one can determine "how good" $S(\alpha, k)$ is compared to S_∞

  $\Rightarrow$ can specify suboptimality degree of approximation a priori

  Possible algorithm to determine RPI set

  1. fix $\alpha \in (0, 1)$ and $k > 0$ (integer)
  2. check whether (15) holds:
     - if yes: $S(\alpha, k)$ is a RPI set
     - if not: set $k := k + 1$ and go to (2)