1 MPC

Formulation of control problem

$$\begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0 \\ y = h(x) \end{cases}$$

Objective of MPC: Find stabilizing control strategy that:

- minimize objective function: $J = \int_t^\infty F(x(\tau), u(\tau))d\tau$
- satisfies constraints: $u(\tau) \in U,\ x(\tau) \in X$

Closed-loop optimal control vs Open-loop optimal control

Closed-loop: Feedback $u = k(x)$

- + Feedback present
- + suit for uncertainty disturbances
- - Finding closed solution hardly possible

Open-loop optimal control: Input trajectory $u = u(t, x_0)$

- + Computation often feasible
- - No feedback
- - Don't know much about system

MPC - repeated open-loop optimal control in feedback fashion.

2 Zero-terminal constraint MPC

Mathematical formulation of NMPC problem:

System dynamics: $\dot{x} = f(x,u),\ x(0) = x_0,\ x, u \in \mathbb{R}^n$

Constraints: $x(t) \in X,\ u \in U,\ \forall t \geq 0$

Assumptions:

- $f(0,0) \Rightarrow x_1 = 0$- equilibrium point for $u_1 = 0$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ - twice continuously differentiable
- U is a compact set (closed and bounded)
- X is a connected and closed set
- $(0, 0) \in int(X \times U)$

MPC optimization problem:

At time t, given initial state $x(t)$

$$\min_{\bar{u}(\cdot,t)} J(x(t), \bar{u}(\cdot; t))$$

with $J(x(t), \bar{u}(\cdot; t)) = \int_t^{t+T} L(\bar{x}(\tau; t), \bar{u}(\tau; t))d\tau$

s.t.

$$\begin{gathered}
\dot{\bar{x}} = f(x, u), \bar{x}(t; t) = x(t) \\
\bar{u}(\tau; t) \in U, \bar{x}(\tau; t) \in X,\ \forall \tau \in [t, t + T] \\
\bar{x}(t + T; t) = 0
\end{gathered}$$

Optimal open-loop solution:

$$\bar{u}^*(\cdot; t) = arg\ min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot; t))$$

Notation used throughout the chapter:

- Quantities without bar: real system trajectories
- Quantities with bar: predicted trajectories
- L-stage cost
- $(\cdot; t)$ - predicted at time t
- T - prediction horizon
- Optimal value function $J^*(x(t)) = J(x(t), \bar{u}^*(t))$

Real trajectories deviate from predicted one!

Assumptions:

- $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and

$$\begin{cases} L(0, 0) = 0 \\ L(x, u) > 0 \\ \forall (x, u) \neq (0, 0) \end{cases} \tag{1}$$

- $J^*(x)$ is continuous at $x = 0$

MPC algorithm:

1. At sampling time t, measure $x(t)$ and solve MPC optimization problem
2. Apply $u_{MPC}(\tau) = \bar{u}^*(\tau, t) \forall t \in [t, t+\delta)$ with sampling time δ
3. Set $t := t + \delta$ and go to step 1

Feasibility: The MPC problem is feasible at time t if there exists at least one $\bar{u}(\cdot; t)$ s.t. constraints satisfied.

Theorem 2.1. Suppose that

(i) assumptions are satisfied

(ii) and that zero-terminal constraint MPC problem is feasible at $t = 0$

Then:

- MPC problem is recursively feasible
- resulting closed-loop system is asymptotically stable

Let $D \subset \mathbb{R}^n$ be the set of all points for which (ii) holds. The D is a region of attraction for the closed loop.

Proof. 1. recursive feasibility: by induction

2. - feasible at $t = 0$ by assumption
   - assume: feasibility at t. Consider the candidate solution:

$$\bar{u}(\tau; t+\delta) = \begin{cases} \bar{u}^*(\tau; t) \ \tau \in [t+\delta, t+T] \\ 0 \ \tau \in [t+T, t+\delta+T] \end{cases}$$

3. asymptotic stability

   Idea: use $J^*(x(t))$ as "Lyapunov function"

   Consider:

$$J(x(t+\delta), \bar{u}(\cdot; t+\delta)) = \int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), \bar{u}(\tau; t+\delta)) d\tau =$$

$$= \int_{t+\delta}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau + \int_{t+T}^{t+\delta+T} L(0, 0) d\tau (= 0) =$$

$$= J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau$$

by optimality

$$J^*(x(t+\delta)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) \leq J^*(x(t)) - \int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau$$

by induction

$$J^*(x(\infty))(\geq 0) \leq J^*(x(0))(finite) - \int_0^{\infty} L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$$

$\square$

Lemma 1 (Barbalat's lemma)**.** ϕ uniformly continuous $\phi : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$

$$\lim_{t\rightarrow\infty} \int_0^t \phi(\tau)d\tau < \infty \Rightarrow \phi(t) \rightarrow 0, t \rightarrow \infty$$

From Barbalat's lemma $L \rightarrow 0$ when $t \rightarrow \infty \Rightarrow$ L pos.def. $\|x_{MPC}(t)\| \rightarrow 0$ when $t \rightarrow \infty \Rightarrow$ convergence

Lyapunov stability: using standard arguments (J^* is continuous at $x = 0$)

Lessons learned:

- feasibility $\Rightarrow$ stability
- value function is Lyapunov function
- have to prove recursive feasibility
- suboptimal solution is sufficient for stability

3 Quasi - infinite horizon MPC

Goal: Relax (restrictive) zero-terminal zero-terminal constraint

Idea: terminal region + local CLF(controller Lyapunov functiom)

MPC optimization problem: At time t

$$\min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot;t)) = \int_t^{t+T} L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau + F(\bar{x}(t+T;t))$$

$F(\bar{x}(t+T;t))$ - terminal cost

s.t.

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \bar{x}(t;t) = x(t)$$
$$\bar{x}(t;t) \in X \;\; \bar{u}(t;t) \in U \;\; \forall \tau \in [t, t+T]$$
$$\bar{x}(t+T;t) \in X^f$$

X^f - terminal region

Optimal solution: $\bar{u}^*(\cdot, t)$, $J^*(x(t))$

Assumption 1: Terminal region + terminal controller

There exists an auxillary local controller $u = k^{loc}(x)$ s.t.

1. X^f is positively invariant $\dot{x} = f(x, k^{loc}(x))$
2. $k^{loc}(x) \in U \;\; \forall x \in X^f$
3. $\dot{F}(x) + L(x, k^{loc}(x)) \leq 0 \;\; \forall x \in X^f$

$\Rightarrow$ F is a local control-Lyapunov function.

Theorem 3.1. Suppose Assumption 1 holds and MPC problem is feasible at $t = 0$. Then:

- recursive feasibility
- closed-loop is asymptotically stable

Proof. 1. Recursive feasibility by induction

- feasible at $t = 0$ by assumption
- assume feasibility at t

candidate

$$\bar{u}(\tau; t+\delta) = \begin{cases} \bar{u}^*(\tau; t) \;\; \tau \in [t, t+T] \\ k^{loc}(\bar{x}(\tau; t+\delta)) \;\; \tau \in [t+T, t+\delta+T] \end{cases}$$

$\Rightarrow$ this is a feasible solution at $t + \delta$

2. asymptotic stability

$$J^*(x(t+\delta)) - J^*(x(t)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) - J^*(x(t)) =$$

$$\int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), \bar{u}(\tau; t+\delta))d\tau + F(\bar{x}(t+\delta+T; t+\delta)) -$$

$$- \int_{t}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau - F(\bar{x}^*(t+T; t)) =$$

$$= \int_{t+T}^{t+\delta+T} L(\bar{x}(\tau; t+\delta), k^{loc}(\bar{x}(\tau; t+\delta)))d\tau + F(\bar{x}(t+\delta+T; t+\delta)) -$$

$$-\int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau - F(\bar{x}^*(t+T;t)) \leq$$

As far as from Assumption 1.3 we have the sum of three terms is ≤ 0

$$-\int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau$$

$\Rightarrow J^*(x(\infty)) \leq J^*(x(0)) - \int_0^\infty L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$ From here: some steps as in zero-terminal constraint rose.

$\square$

How can Assumption 1 be satisfied?

Assume:

- quadratic state cost $L(x,u) = x^T Qx + u^T Ru,\ Q, R > 0$
- linearization at the origin is stabilizable $\dot{x} = Ax + Bu$ $A = \frac{\partial F}{\partial x}(0,0)$ $B = \frac{\partial F}{\partial u}(0,0)$

Approach:

- Linear auxiliary controller $k^{loc}(x) = Kx$
- Quadratic terminal cost function $F(x) = x^T Px,\ P > 0$
- Terminal region $X_\alpha^f = \{x \in \mathbb{R}^n | x^T Px \leq \alpha\}$ for some $\alpha > 0$
- Determine P, K, α s.t. Assumption 1.1-1.3 hold:

For (Assumption 1.3):

$$\frac{d}{dt} x(t)^T Px(t) \leq -x(t)^t (Q + K^T RK)x(t) = -x(t)^T Q^* x(t)$$

$[x^T Qx + u^T Ru = [u = Kx] = x^T (Q + K^T RK)x)]$

$$\frac{d}{dt} x(t)^T Px(t) = f(x, Kx)^T Px + x^T Pf(x, Kx)$$

$[f(x, Kx) = (A + BK)x + \phi(x), A + BK = A_K,\ K$ is chosen s.t. $A + BK$ is Hurwitz$]$

Upper bound for $x^T P\phi(x)$: $L_\phi := sup\{\frac{|\phi(x)|}{|x|}, x \in X_\alpha^f, x \neq 0\}$

$$x^T P\phi(x) \leq |x^T P||\phi(x)| \leq \|P\| L_\phi |x|^2 \leq \frac{\|P\| L_\phi}{\lambda_{min}(P)} x^T Px \tag{2}$$

We choose α small enough s.t.

$$L_\phi \leq \frac{k\lambda_{min}(P)}{\|P\|} \tag{3}$$

for some $k > 0$. Plug this into (2): $x^T P\phi(x) \leq kx^T Px$. Insert this into $\frac{d}{dt}x^T Px \leq x^T(A_K P + PA_K)x + 2kx^T Px$

$$= x^T((A_K + kI)^T P + P(A_K + kI))x$$

ensure that it $\leq -x^T Q^* x$

$\Rightarrow$ Lyapunov equation which can be solved if any only if $A_K + kI$ is Hurwitz

$$\Leftrightarrow k < -max\ Re\{\lambda(A_K)\} \tag{4}$$

$$\Rightarrow \ (A_K + kI)^T P + P(A_K + kI) = -Q^* \tag{5}$$

Design procedure

1. Compute K s.t. $(A + BK)$ is Hurwitz
2. Choose $k > 0$ s.t. (4) and solve (5)
3. Find largest possible α_1 s.t. $Kx \in U,\ \forall x \in X^f_{\alpha_1}$
4. Find the largest $\alpha \in (0, \alpha_1]$ s.t. (3) holds

Alternative to the (4) step

Solve optimization problem

$$\max_x x^T P\phi(x) - kx^T Px\ s.t.\ x^T Px \leq \alpha \tag{6}$$

Iterate this by reducing α from α_1 until optimal value of (4) is nonpositive

Degrees of freedom in design

- calculation of K
- choice of k - tradeoff between "large" terminal region and "large" P

4 Unconstrained MPC

Goal:Guarantee stability + degree of suboptimality without stabilizing terminal constraint + cost

Setup:

- $\dot{x} = f(x, u), x(0) = x_0$

- input constraints $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m \forall t \geq 0$

Infinite-horizon cost function: $J_\infty(x_0, \bar{u}(\cdot;0)) = \int_0^\infty L(\bar{x}(\tau;0), \bar{u}(\tau;0))d\tau \Rightarrow$ optimal value function $J_\infty^*(x_0)$

Assumption: $J_\infty^*(x_0) < \infty, \forall x_0 \Rightarrow$ system is asymptotically stabilizable

Finite-horizon cost function: $J_T(x(t), \bar{u}(\cdot;t)) = \int_0^T L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau$

Infinite-horizon performance resulting from application of MPC controller:

$J_\infty^{MPC}(x_0) = \int_0^\infty L(\bar{x}_{MPC}(\tau), \bar{u}_{MPC}(\tau))d\tau$

Definition. Suboptimality index α: $\alpha J_\infty^{MPC}(x_0) \leq J_\infty^*(x_0) \forall x_0$

- $\alpha \leq 1$ by optimality of J_∞^*
- $\alpha > 0$ implies closed-loop stability (Barb.lemma)

Proposition 1: Relaxed dynamic programming

Assume $\exists \alpha \in (0,1] s.t. \forall x \in \mathbb{R}^n$

$J_T^*(x(t+\delta)) \leq J_T^*(x(t)) - \alpha \int_t^{t+\delta} L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))d\tau \quad (*)$

Then the estimate

$$\alpha J_\infty^*(x(t)) \leq \alpha J_\infty^{MPC}(x(t)) \leq J_T^*(x(t)) \leq J_\infty^*(x(t)) \tag{7}$$

holds for all $x \in \mathbb{R}^n$

Proof.

- 1 and 3 inequalities follow from optimality (by definition)
- 2 inequality follows from summing up (*) over all sampling instances

$$J_T^*(x(N\delta)) \leq J_T^*(x_0) - \alpha \int_0^{N\delta} L(\bar{x}_{MPC}(t), \bar{u}_{MPC}(t))dt \tag{8}$$

$$N \to \infty : J_T^*(x_0) \geq \alpha J_\infty^{MPC}(x_0) \tag{9}$$

□

Central idea (image to be inserted)

$L^*(t;t) = L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))$

$$(c) : J_T^*(x(t+\delta)) \leq \frac{1}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau : (b) \tag{10}$$

$$(b): \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau \leq \gamma \int_{t}^{t+\delta} L^*(\tau;t)d\tau : (a) \tag{11}$$

Theorem 4.1. Assume $\exists \epsilon \in (0;1]$ and $\gamma > 0$ s.t. 10 - 11 holds. Then (*) holds with $\alpha = 1 - \gamma\frac{1-\epsilon}{\epsilon}$

Proof.

$$J_T^*(x(t+\delta)) - J_T^*(x(t)) = J_T^*(x(t+\delta)) - \int_t^{t+T} L^*(\tau;t)d\tau \leq^{(10)}$$

$$\leq \frac{1-\epsilon}{\epsilon}\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau - \int_t^{t+\delta} L^*(\tau;t)d\tau \leq^{(11)}$$

$$\leq (\gamma\frac{1-\epsilon}{\epsilon} - 1)\int_t^{t+\delta} L^*(\tau;t)d\tau$$

$-\alpha := \gamma\frac{1-\epsilon}{\epsilon} - 1$ □

Assumption 1: Asymptotic Controlability

For all x, $\exists$ some input trajectory $\hat{u}_x(\cdot)$ with $\hat{u}_x(t) \in \mathbb{U}, \forall t \geq 0$ s.t.

$$L(\hat{x}(t), \hat{u}(t)) \leq \beta(t)\min_u L(x,u), \forall t > 0$$

with $\beta : \mathbb{R} \to \mathbb{R}_{\geq 0}$ - continuous, positive, strictly decreasing with $\lim_{t\to 0}\beta(t) = 0 \Rightarrow \int_0^\infty \beta(\tau)d\tau < \infty$
$B(t) = \int_0^t \beta(\tau)d\tau$

Typical example: (image to be inserted)

How to compute ϵ and γ:

Lemma 2. Let Assumption 1 hold. Then the inequality

$$J_T^*(x(t+\delta)) \leq \int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t) \tag{12}$$

holds for all $t' \in [\delta, T]$

(image to be inserted)

Proof. Consider

$$\bar{u}(\tau;t+\delta) = \begin{cases} \bar{u}^*(\tau;t), \tau \in [t+\delta, t+t'] \\ \hat{u}_{x'}(\tau - t - t'), \tau \in [t+t', t+\delta+T] \end{cases}$$

$$J_T^*(x(t+\delta)) \leq J_T(x(t+\delta), \bar{u}(\cdot;t+\delta)) =$$

$$= \int_{t+\delta}^{t+t'} L^*(\delta;t)d\delta + \int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau-t-t'), \hat{u}(\tau-t-t'))d\tau \leq$$

by Assumption 1

$$\int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau-t-t'), \hat{u}(\tau-t-t'))d\tau \leq L^*(t+t';t)\int_0^{T+\delta-t'} \beta(\tau)d\tau$$

as far as $B(t) = \int_0^t \beta(\tau)d\tau$

$$\leq \int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t)$$

□

Calculation of ϵ from (12):

$$J_T^*(x(t+\delta)) \leq \min_{t'\in[\delta,T]}(\int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t)) \leq$$

$$\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau + B(T)\min_{t'\in[\delta,T]} L^*(t+t';t)$$

as far as $\min_{t'\in[\delta,T]} L^*(t+t';t) \leq \frac{1}{T-\delta}\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau$ minimum is less or equal that the average

$$= (1+\frac{B(T)}{T-\delta})\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau$$

$(1+\frac{B(T)}{T-\delta}) = \frac{1}{\epsilon}$

Lemma 3.

$$\int_{t+t'}^{t+T} L^*(\tau;t)d\tau \leq B(T-t')L^*(t+t';t)\forall t' \in [0;T]$$

Proof. Analogues to lemma 2.

Calculation of γ:

$$\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau \leq \int_{t+\hat{t}}^{t+T} L^*(\tau;t)d\tau(\forall\hat{t}\in[0,\delta]) \leq$$

$$\leq \min_{\hat{t}\in[0,\delta]}(B(T-\hat{t})L^*(t+\hat{t};t)) \leq$$

$$\leq B(T)\min_{\hat{t}\in[0,\delta]} L^*(t+\hat{t};t) \leq \frac{B(T)}{\delta}\int_t^{t+\delta} L^*(\tau;t)d\tau$$

□

Denote $\gamma = \frac{B(T)}{\delta}$

$\alpha = 1-\gamma\frac{1-\epsilon}{\epsilon} = 1-\frac{B(T)}{\delta}(\frac{B(T)}{T-\delta})$

Alternative computation of ϵ (less conservative):

We want to compute ϵ s.t.

$$\epsilon \leq \frac{\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau}{J_T^*(x(t+\delta))}$$

Idea: Minimize

$$\epsilon = \min_{L_t, J_T^*} \frac{\int_\delta^T L_t(\tau; t) d\tau}{J_T^*(x(t+\delta))} \tag{13}$$

$J_T^* = 1$ - without loss of generality s.t. $0 \leq L_t \forall \tau \in [\delta, T]$

$$J_T^*(x(t+\delta)) \leq \int_\delta^{t'} L_t(\tau) d\tau + B(T+\delta-t')L_t(t') \forall t' \in [\delta, T]$$

Due to linearity in L_t, without loss of generality we can set $J_T^* = 1$.

$\Rightarrow$ infinite dimensional linear problem

Idea for solution: second constraint has to be active for all times

Differentiate (12) with relation to t'

$$0 = L_t(t') + \frac{dB(T+\delta-t')}{dt'} L_t(t') + B(\tau+\delta-t')\dot{L}_t(t')$$

as far as $\frac{dB(T+\delta-t')}{dt'} = \beta(T+\delta-t')$

$$\begin{cases} \dot{L}_t(t') = \frac{\beta(T+\delta-t')-1}{B(T+\delta-t')} L_t(t') \\ \text{initial condition } L_t(t) = \frac{1}{B(t)} \end{cases}$$

Solution:

$$\bar{L}_t(t') = \frac{1}{B(T+\delta-t')} e^{-\int_\delta^{t'} \frac{1}{\beta(T+\delta-\tau)} d\tau}$$

Have to show: $\bar{L}_t$ is a minimizer of (13)

$$\int_\delta^T \bar{L}_t(\tau) d\tau \leq \int_\delta^T L_t(\tau) d\tau$$

for all feasible L_t

Proof. Assume $\exists L_t$ s.t.

$$\int_\delta^T L_t(\tau) d\tau < \int_\delta^T \bar{L}_t(\tau) d\tau$$

Then $\exists \hat{t} \in [\delta, T]$ s.t. $\int_\delta^{\hat{t}} L_t(\tau) d\tau \leq \int_\delta^{\hat{t}} \bar{L}_t(\tau) d\tau$ and $\bar{L}_t(\hat{t}) > L_t(\hat{t})$

But then

$$1 = \int_\delta^{\hat{t}} \bar{L}_t(\tau) d\tau + B(T+\delta-\hat{t})\bar{L}_t(\hat{t}) > \int_\delta^{\hat{t}} L_t(\tau) d\tau + B(T+\delta-\hat{t})L_t(\hat{t}) \tag{14}$$

the sign equality from (13) with equality.

Contradiction:

L_t cannot be a feasible solution of (13) $\Rightarrow$

$$\epsilon = \int_\delta^T \bar{L}_t(\tau)d\tau = 1 - e^{-\int_0^{T-\delta} \frac{1}{B(T-\tau)} d\tau}$$

$\square$

Similarly, better estimate for γ can be obtained

$\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon}$ For $T \to \infty$: both estimates for $\epsilon \to 1 \Rightarrow \alpha \to 1$ as $T \to \infty \Rightarrow$ closed-loop stability for T large enough

5 Robust MPC

Consider linear (discrete-time) sytem: $x(t+1) = Ax(t) + Bu(t) + w(t)$ in short $x^+ = Ax + Bu + w$

Constraints: $x(t) \in X, u(t) \in U, \forall t = 0, 1...$

Bound on w: W is a compact,convex set which contains 0. $w(t) \in W \ \forall t = 0, 1, ...$

Main idea: Use additional error feeedback s.t. real systems state contained in a "tube" around some nominal system state.

Repetition of QI-MPC in discrete time:

Nominal system:

$$z^+ = Az + Bv$$

At time t, given $z(t)$, solve

$$\min_{v(\cdot|t)} \hat{J}(z(t), v(\cdot|t)) = \sum_{i=t}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

s.t.

$$z(i+1|t) = Az(i|t) + Bv(i|t), z(t|t) = z(t)$$
$$z(i|t) \in Z, v(i|t) \in V, t \leq i \leq t+N-1$$
$$z(t+N|t) \in Z^f \subseteq Z$$

$\Rightarrow$ optimizer $V^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Assumption 1:

- Cost is quadratic $L(z, v) = z^T Qz + v^t Rv, Q, R > 0$

- There exists a local auxiliary controller $k^{loc} = Kx$ s.t.
    1. Z^f is invariant with $Z^+ = (A + BK)z$, $A_k = A + BK$, i.e. $A_k Z^f \subseteq Z^f$
    2. $Kz \in V \forall z \in Z^f$
    3. $F(A_k z) - F(z) \leq -L(z, Kz) \forall z \in Z^f$

From Assumption 1 it follows (as in continuous time) that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \leq -L(z(t), v_{MPC}(t))$$

Since L is quadratic, there exists constants $c_2 > c_1 > 0$ s.t. $\forall z \in Z_N$ - feasible set

1. $c_1 |z|^2 \leq \hat{J}^*(z)$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1 |z|^2$
3. $\hat{J}^*(z) \leq c_2 |z|^2$

Why is (3) true?

From Assumption 1.3 $\forall z \in Z^f$

$$\hat{J}^*(z) \leq \hat{J}(z, Kz(\cdot)) = \sum_{i=1}^{N-1} L(z(i), Kz(i)) + F(z(N)) \leq$$

N times apply Assumption 1.3

$$\leq F(z) = z^T P z \leq \lambda_{max}(P) |z|^2$$

Influence of disturbance:

Definition. Minkowski set addition:

$$A, B \subseteq \mathbb{R}^n A \oplus B = \{a + b | a \in A, b \in B\}$$

Pontryagin set difference:

$$A, B \subseteq \mathbb{R}^n A \ominus B = \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$$

$$(A \ominus B) \oplus B \subseteq A$$

$$A \subseteq (A \oplus B) \ominus B$$

Definition. Robust positively invariant set (RPI set):

S is RPI set for $x^+ = Ax + w$ if $AS \oplus W \subseteq S$ (or equivalently $Ax + w \in S \forall x \in S, \forall w \in W$)

Example. $x^+ = 0.5x + w$. $w \in [-5, 5]$. So RPI set: $S = [-20, 20]$, minimal RPI set: $S = [-10, 10]$

Minimal RPI set:

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

(Minkowski set addition), min. RPI set exists and is bounded if A is Schour table.

Why?

Current state at time t is x,

possible states at time $t+1$: $Ax \oplus W$

$t+2$: $A(Ax \oplus W) \oplus W = A^2 x \oplus AW \oplus W$

....

$t+j$: $A^j x \oplus \sum_{k=0}^{j-1} A^k w$

$\Rightarrow$ by choosing j large enough we can reach any state in S_∞

$\Rightarrow$ any RPI set must satisfy $S_\infty \subseteq S$

Remains to show: S_∞ is an RPI set

$$AS_\infty \oplus W = A \sum_{i=0}^{\infty} A^i w \oplus W = \sum_{i=1}^{\infty} A^i w \oplus W = S_\infty$$

S_∞ in general is difficult to compute

$\Rightarrow$ can compute invariant outer approximations of S_∞ (with bounded complexity)

Example. Calculate RPI

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

For the system given and bounded disturbances

$$x^+ = \frac{1}{2}x + w,\ w \in [-5, 5]$$

$$S_\infty = \sum_{i=0}^{\infty} (\frac{1}{2})^i [-5, 5] = [-10, 10]$$

Central idea in tube-based MPC

Use additional error feedback around some nominal input:

$$u_{MPC} = v_{MPC}(x) + K(x - z)$$

Proposition 1

Let $x^+ = Ax + Bu + w$ and $z^+ = Az + Bv$. If $x \in Z \oplus S$ and $u = v + K(x - z)$, then $X^+ \in Z^+ \oplus S$ (RPI set for $x^+ = (A + BK)x + w$)

image to be inserted

Proof. Let $e(t) := x(t) - z(t) \rightarrow$

$$e^+ = x^+ - z^+ = Ax + B(v + K(x - z)) + w - Az - Bv = (A + BK)e + w$$

As S is RPI for $e^+ = A_k e + w$, we obtain $e \in S \Rightarrow e \in S \ \forall w \in W$

Hence $x \in Z \oplus S \Rightarrow x^+ \in Z^+ \oplus S \ \forall w \in W$

□

Robust MPC scheme

MPC problem for robust tube-based MPC: At time t, given $x(t)$, solve

$$\min_{z(t|t), v(\cdot|t)} J(x(t), v(\cdot|t)) = \sum_{i=1}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$
$$s.t. z(i+1|t) = Az(i|t) + Bv(i|t)$$
$$z(i|t) \in Z = X \ominus S$$
$$v(i|t) \ \in V = U \ominus KS$$
$$t \leq i \leq t + N - 1$$
$$z(t+N|t) \in Z_N \subseteq Z$$

Initial condition $x(t) \in z(t|t) \oplus S$

$\rightarrow$ optimizer: $z^*(t|t), v^*(\cdot|t) \rightarrow$ optimal value function $J^*(x(t))$

$\rightarrow$ applied input: $u(t) = v^*(t|t) + K(x(t) - z^*(t|t))$

Important: Tightened input/state constraints for the nominal predictions ensure fulfilment of original input/state constraints for real (disturbed) closed-loop system.

Properties of robust MPC scheme (in the following $z^*(x(t)) := z^*(t|t)$)

- a feasible set $X_N = Z_N \oplus S \subseteq X$
- $J^*(x) = \hat{J}^*(z^*(x))$ by definition of J^* and $\hat{J}^*$
- $J^*(x) = 0 \ \forall x \in S$

Why?

If $x \in S$, then $z(x) = 0$ and $v(\cdot|t) = 0$ is a feasible solution. Hence $J^*(x) \leq \hat{J}(0,0) = 0$

$\Rightarrow J^*(x) = 0$ and $z^*(X) = 0$

"S serves an origin for the disturbed system"

Theorem 5.1. Suppose that Assumption 1 holds and the robust MPC problem is feasible at $t = 0$.

Then:

(i) robust MPC problem is recursively feasible

(ii) closed-loop system robustly exponentially converges to S

(iii) closed-loop system satisfies input/state constraints, i.e. $x(t) \in X$, $u(t) \in U$ $\forall t = 0, 1...$

Proof. i) Consider candidate solution at time $t+1$

$$\tilde{V}(i|t+1) = \begin{cases} v^*(i|t) \ t+1 \leq i \leq t+N-1 \\ k^{loc}(z^*(t+N|t)) \ i = t+N \end{cases}$$

$$\tilde{z}(t+1|t+1) = z^*(t+1|t)$$

it is feasible because $x(t+1) \in z^*(t+1|t) \oplus S$ by proposition 1

image to be inserted

iii) follows from Proposition 1 + definition of tightened constraints

ii) from (1-3) inequalities described below

1. $\hat{J}^*(z) \geq C_1|z|^2$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1|z|^2$
3. $\hat{J}^*(z) \leq c_2|z|^2$

$J^*(x) = \hat{J}^*(z^*(x))$

we obtain the following $\forall x \in X_N$

4. $J^*(x) = \hat{J}^*(z^*(x)) \geq^{(1)} c_1|z^*(x)|^2$
5. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(3)} c_2|z^*(x)|^2$

So now we will show convergence to 0

$$J^*(x(t+1)) - J^*(x(t)) = \hat{J}^*(z^*(x(t+1))) - \hat{J}^*(z^*(x(t))) \leq$$

$$\leq \hat{J}^*(z^*(x(t+1|t))) - \hat{J}^*(z^*(x(t))) \leq^{(2)}$$

$$-c_1|z^*(x(t))|^2 \leq -\frac{c_1}{c_2}J^*(x(t))$$

$\Rightarrow$

$$J^*(x(t+1)) \leq (1 - \frac{c_1}{c_2})J^*(x(t))$$

where $\gamma := 1 - \frac{c_1}{c_2},\ \gamma \in (0,1)$

$$J^*(x(i)) = \gamma^i J^*(x(0)) \leq^{(5)} c_2\gamma^i|z^*(x(0))|^2$$

$\Rightarrow^{(4)}$

$$|z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$\Rightarrow z^*(x(t))$ exponentially converges to 0.

Recall: $x(i) \in z^*(x(i)) \oplus S \Rightarrow$

$$|x(i)|_S \leq |z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}}\sqrt{\gamma}^i|z^*(x(0))|$$

$\square$

$|x(i)|_S$ - point-to-set distance

Extensions:

- Nonlinear systems: difficult to compute RPI sets
  - approaches based on input-to-state stability(ISS)
  - approaches which apply MPC two times:
    - first for nominal input
    - to determine local error feedback(Rawlings and Mayne chapter 3-6)
- Linear systems with parametric uncertainties

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

  $(A(t), B(t)) \in \rho : con(A_j, B_j), j = 1, ..., J\ \forall \geq 0$

Note. *co*- convex

Define: $\bar{A} := \frac{1}{J}\sum_{i=0}^{J} A_i,\ \bar{B} := \frac{1}{J}\sum_{i=0}^{J} B_i$

$$x(t+1) = \bar{A}x(t) + \bar{B}u(t) + w(t)$$

$$w(t) \in W := (A - \bar{A})x + (B - \bar{B})u|(A, B) \in \rho, x \in X, u \in U$$

W is compact if X,U are compact

$\rightarrow$ can apply tube MPC as before but: can slow down more!

If ρ is "small enough", closed-loop asymptotically to zero

Intuition: x converges to the RPI set $S \rightarrow W$ gets smaller

$\rightarrow x$ converges to RPI set

Invariant approximations of the minimal RPI set S_∞ is difficult to compute

$$S_\infty := \sum_{i=0}^{\infty} A^i w$$

Define $S_k := \sum_{i=0}^{k-1} A^i w \ k \geq 1$

In general, S_k for a finite k are not RPI sets (this is the case if only if A is nilpotent)

Theorem 5.2. If $0 \in int(W)$ and A is Schur, then there exists an integer $k > 0$ and $\alpha \in [0, 1)$ s.t.

$$A^k W \subseteq \alpha W \tag{15}$$

If (15) holds, then

$$S(\alpha, k) := (1 - \alpha)^{-1} S_k$$

is an RPI set for the system $x^+ = Ax + w$

Proof. i)(15) is a direct consequence of our assumptions ii) want to show that $AS(\alpha, k) \oplus W \subseteq S(\alpha, k)$

$$AS(\alpha, k) \oplus W = (1 - \alpha)^{-1} \sum_{i=1}^{k} A^i W \oplus W =$$

$$= (1 - \alpha)^{-1} A^k W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1} \oplus W$$

As far as $A^k W \subseteq \alpha W$ by (15)

$$(1 - \alpha)^{-1} \alpha W \oplus W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1}$$

As $(1 - \alpha)^{-1} \alpha W \oplus W = [(1 - \alpha)^{-1} \alpha + 1] W = (1 - \alpha)^{-1} W$

Then we get

$$= (1-\alpha)^{-1} \sum_{i=0}^{k-1} A^i W = S(\alpha, k)$$

$\square$

Remark:

- for a given k s.t. (15) can be satisfied, we want to find the smallest possible α ("small scaling factor")
- for a given α, one wants to find the smallest possible k s.t. (15) holds ("low complexity" of RPI set)

  $\Rightarrow$ tradeoff between small α and small k needs to be found
- one can determine "how good" $S(\alpha, k)$ is compared to S_∞

  $\Rightarrow$ can specify suboptimality degree of approximation a priori

  Possible algorithm to determine RPI set

  1. fix $\alpha \in (0, 1)$ and $k > 0$ (integer)
  2. check whether (15) holds:
     - if yes: $S(\alpha, k)$ is a RPI set
     - if not: set $k := k + 1$ and go to (2)

6 Explicit MPC

Idea: Compute an "explicit MPC control law" by solving the optimization problem for all x.

Linear discrete-time systems $x^+ = Ax + Bu$

Polytopic input+state constraints $C_x x \leq d_x$, $C_u u \leq d_u$

MPC problem. At time t, given $x(t)$, solve

$$\min_{u(\cdot|t)} = \sum_{k=t}^{t+N-1} L(x(k|t), u(k|t)) + F(x(t+N|t))$$

s.t.

$$x(k+1|t) = Ax(k|t) + Bu(k|t)$$

$$x(t|t) = x(t)$$

$$C_x x(k|t) \leq d_x$$

$$C_u u(k|t) \leq d_u$$

for $t \leq k \leq t+N-1$ and with terminal constraint

$$C^f x(t+N|t) \leq d^f$$

with $L(x,u) = x^T Q x + u^T R u,\ Q, R > 0,\ F(x) = x^T P x$

$\Rightarrow$ optimizer: $u^*(\cdot|t)$

Define:

$$X := [x^T(t+1|t), \ldots, x^T(t+N|t)]^T \tag{16}$$

$$U := [u^T(t+1|t), \ldots, u^T(t+N-1|t)]^T \tag{17}$$

- rewrite cost function

$$F(x(t), U) = x^T(t) Q x(t) + X^T \tilde{Q} X + U^T \tilde{R} U \tag{18}$$

with $\tilde{Q} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P \end{bmatrix}$, $\tilde{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$

- rewrite predicted states: $x(t+k|t) = A^k x(t) + \sum_{j=0}^{k-1} A^j B u(t+k-j-1|t),\ k = 1, .., N$

$$\Rightarrow X = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x(t) + \begin{bmatrix} B & 0 & \ldots & & 0 \\ AB & B & 0 & \ldots & 0 \\ \ldots & & & & \\ A^{N-1}B & A^{N-2}B & \ldots & \ldots & B \end{bmatrix} U \tag{19}$$

where $\Omega = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}$ and $\Gamma = \begin{bmatrix} B & 0 & \ldots & & 0 \\ AB & B & 0 & \ldots & 0 \\ \ldots & & & & \\ A^{N-1}B & A^{N-2}B & \ldots & \ldots & B \end{bmatrix}$

Plugging (19) in (18) : $J(x(t), U) = \frac{1}{2} x^T(t) Y x(t) + \frac{1}{2} U^T H U + x^T(t) F U$ with

$$Y = 2(Q + \Omega^T \tilde{Q} \Omega)$$

$$H = 2(\Gamma^T \tilde{Q} \Gamma + \tilde{R})$$

$$F = 2\Omega^T \tilde{Q} \Gamma$$

- rewrite constraints: using (19)

$$\begin{bmatrix} C_x & & \\ & \ddots & \\ & & C_x \end{bmatrix} (\Omega x(t) + \Gamma U) \leq \begin{bmatrix} d_x \\ \vdots \\ d_x \end{bmatrix}$$

$\Rightarrow$ constraints in total

$$GU \leq W + Ex(t)$$

$$\min_U \frac{1}{2}U^T HU + x^T FU + \frac{1}{2}x^T Yx \tag{20}$$

s.t. $GU \leq W + Ex$

multiparametric quadratic program (MPQP)

Last step: define $z := U + H^{-1}F^T x$, H^{-1} - pos.definite

$$\min_z \frac{1}{2}z^T Hz + \frac{1}{2}x^T \tilde{Y}x \tag{21}$$

s.t. $Gz \leq W + Sx$

$$\tilde{Y} := Y - FH^{-1}F^T$$
$$S := E + GH^{-1}F^T$$

This is a (strictly) convex optimization problem with (by assumption) feasible set with non-empty interior $\rightsquigarrow$ Slater's condition satisfied

$\Rightarrow$ Optimal solution (unique!) is characterized by following KKT conditions

$$Hz + G^T\lambda = 0,\ \lambda \in \mathbb{R}^q \tag{22}$$

∇(cost + λ^Tconstraints) = 0, complementary slackness condition below

$$\lambda^i(G^i z - W^i - S^i x) = 0\ i = 1, ...q \tag{23}$$

i - th component of vector λ or G

$$\lambda \geq 0 \tag{24}$$
$$Gz \leq W + Sx \tag{25}$$

image to be inserted

Definition. optimal active set $z^*(x)\ldots$ optimal solution to (21) for a given x

$$A(x) := \{i \in \{1, \ldots q\} | G^i z^*(x) = W^i + S^i x\}$$

$\rightsquigarrow G^A,\ W^A,\ S^A \ldots$ matricies containing rows of $G,\ U,\ S$ associated to indices in A

Assumption: Linear independence constraint qualification (LICQ) G^A has full row rank ("gradient of active constraints are linearly independent")

From (22)

$$\rightsquigarrow z^*(x) = -H^{-1}G^T\lambda = -H^{-1}(G^A)^T\lambda^A \tag{26}$$

For all active constraints, we have $G^A z(x) = W^A + S^A x \rightarrow^{(26)} -G^A H^{-1}(G^A)^T \lambda^A = W^A + S^A x$

$$\lambda^A = -(G^A H^{-1}(G^A)^T)^{-1}(W^A + S^A x) \tag{27}$$

Plug (27) into (26):

$$z^*(x) = H^{-1}(G^A)^T(G^A H^{-1}(G^A)^T)^{-1}(W^A + S^A x) \tag{28}$$

For a given active constraint set, optimal solution is an affine function of x.

Characterization of the set where A is an optimal active set (critical region CR^A)

Use (24), (25): Plug (28) into (25)

$$GH^{-1}(G^A)^T(G^A H^{-1}(G^A)^T)^{-1}(W^A + S^A x) \leq W + Sx \tag{29}$$

Plug (27) into (24)

$$-(G^A H^{-1}(G^A)^T)^{-1}(W^A + S^A x) \geq 0 \tag{30}$$

(29) + (30) describe critical region CR^A, on which the optimal solution is given by (27) -(28)

image to be inserted

Algorithm:

- Take some $x_0 \in X$ (e.g., $x_0 = 0$)
- Solve (21) with $x = x_0 \rightsquigarrow z * (x_0)$
- Identify active constraints $\rightsquigarrow G^A, W^A, S^A$
- Calculate the corresponding critical region CR^A via (29) and (30) $\Rightarrow$ inside this critical region, solution to (21) is given by (28)
- Step over all facets of this first critical region, take new x_0 go to (19)

What happens if more than two critical regions are adjacent to a critical region?

Procedure needs to be suitably adapted in case of degenerate problems (LICQ assumption cost satisfied)

Can merge critical regions if union is convex and the resulting control is the same

Main bottleneck: number of regions can grow very quickly

- online point-location problem becomes complex

Theorem 6.1. For linear MPC (linear system, linear constraints, quadratic cost), the resulting MPC positive definite controller $u_{MPC}(x)$ is continuous and piecewise affine over polyhedral regions. The optimal value function to (20), $F^*(x)$, is continuous, convex and piecewise quadratic.

Proof.
- $u_{MPC}(x)$ is affine by (28)
- since solution is unique and a point on the boundary of two regions belongs to both regions, the two control laws must be equal on the boundary $\Rightarrow u_{MPC}$ - continuous
- optimal value function of (21)

$$J_z^*(x) = \frac{1}{2} z^*(x)^T H z^*(x)$$

continuous and piecewise quadratic $\rightsquigarrow$ can show that it is convex (by standard arguments) $\rightsquigarrow u^*(x) = z^*(x) - H^{-1}F^T x$

$$J^*(x) = \frac{1}{2} U^*(x)^T H U^*(x) + x^T F U^*(x) + \frac{1}{2} x^T Y x = \cdots = J_z^*(x) + \frac{1}{2} x^T (Y - FH^{-1}F^T) x$$

First term is convex, continuous piecewise quadratic, the second term is continuous quadratic + convex.

Why convex? $\begin{bmatrix} Y & F^T \\ F & H \end{bmatrix} \geq 0$ as $J(x, u(\cdot)) \geq 0 \rightsquigarrow$ Schur complement implies $Y - FH^{-1}F^T \geq 0$

□

7 Economic MPC

Motivation: (setpoin) stabilization is often not primary control objective $\Rightarrow$ "general" cost function L.

Assumption: $L : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (need not be positive defined) X, U is compact.

Definition. Optimal steady-state $(x_s, u_s) = \arg\min_{x^+ = f(x,u);\ x \in X;\ u \in U} L(x, u)$

Example.

$$\begin{aligned} x^+ &= xu \\ X = U &= [-\delta, \delta] \\ L(x, u) &= g(x) + (u + 1)^2 \end{aligned}$$

Optimal steady state is $(x_s, u_s) = (0, -1)$

Optimal operating condition:

$$\begin{aligned} x &= (-1; +1; -1; \dots) \\ u &= (-1; +1; -1; \dots) \end{aligned}$$

Economic MPC problem

At each time t, given $x(t)$, we should minimize $J(x(t), u(\cdot|t)$ s.t.

$$J(x(t), u(\cdot|t) \sum_{k=t}^{t+N-1} L(x(k|t), u(k|t))$$

With conditions:

$$\begin{aligned}
&x(k+1|t) = f(x(k|t), u(k|t)) \\
&x(t|t) = x(t) \\
&x(k|t) \in X \\
&u(k|t) \in U \\
&x(t+N|t) = x_s
\end{aligned}$$

Remark: this can be extended to terminal region/cost framework.

Closed-loop average perfomance

Theorem 7.1. The closed-loop asymptatic average perfomance is at least as good as the optimal steady-state cost i.e.,

$$\lim_{T\to\infty} \sup \frac{\sum_{k=0}^{T-1} (x(k), u(k)}{T} \leq L(x_s, u_s)$$

Proof. Lets present Imagine a completely new way of proving that we have not used before in this course.

Joke. We again consider difference $J^*(x(t+1)) - J^*(x(t))$.

$$J^*(x(t+1)) - J^*(x(t)) \leq -L(x(t), u(t)) + L(x_s, u_s)$$

From this equation we can get:

$$\frac{\sum_{k=0}^{T-1} \left(J^*(x(k+1)) - J^*(x(t))\right)}{T} \leq \frac{\sum_{k=0}^{T-1} \left(L(x_s, u_s) + -L(x(t), u(t))\right)}{T}$$

$$\lim_{T\to\infty}\inf LHS \leq \lim_{T\to\infty}\inf RHS =$$

$$\begin{aligned} &= L(x_s,u_s) + \lim T\to\infty \inf \frac{\sum_{k=0}^{T-1} -L(x(k),u(k))}{T} = \\ &= L(x_s,u_s) - \lim T\to\infty \sup \frac{\sum_{k=0}^{T-1} L(x(k),u(k))}{T} \end{aligned} \tag{31}$$

Also, we can rewrite

$$\begin{aligned} \frac{\sum_{k=0}^{T-1}(J^*(x(k+1)) - J^*(x(t)))}{T} &= \lim T\to\infty \inf \frac{J^*(x(T)) - J^*(x(0))}{T} \geq \\ &\geq \lim T\to\infty \inf \frac{-J^*(x(0))}{T} \geq 0 \end{aligned} \tag{32}$$

Now combine (31) and (32) and we get

$$\lim T\to\infty \sup \frac{\sum_{k=0}^{T-1} L(x(k),u(k))}{T} \leq L(x_s,u_s)$$

□

Classify optimal operating condition

Definition. A system is optimaly operated at steady-state if

$$\lim T\to\infty \inf \frac{\sum_{k=0}^{T-1} L(x(k),u(k))}{T} \geq L(x_s,u_s)$$

for all faesible sequences $(x(\cdot),u(\cdot))$.

Definition. A system is istrictly dissipative w.r.t. supply rate $s(x,u)$ if there exists a storage function $\lambda : X \to \mathbb{R}_{\geq 0}$ s.t.

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u)$$

For strictlly dissipativity RHS=$s(x,u) - \rho((x-x_0))$, where ρ positive defined.

Theorem 7.2. A system is optimally operated at steady state if it is dissipative w.r.t. $s(x,u) = L(x,u) - L(x_s,u_s)$.

Proof.

$$\begin{aligned} \lim_{T\to\infty}\inf \frac{\sum_{t=0}^{T-1}\lambda(x(t+1)) - \lambda(x(t))}{T} &\leq \lim_{T\to\infty}\inf \frac{\sum_{t=0}^{T-1} s(x(t),u(t))}{T} = \\ &= \lim_{T\to\infty}\inf \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} - L(x_s,u_s) \end{aligned} \tag{33}$$

Note, that $\lim_{T\to\infty} \inf \frac{\sum_{t=0}^{T-1} \lambda(x(t+1))-\lambda(x(t))}{T} \geq 0$ $\square$

Example. Consider system

$$\begin{aligned} &x(t+1) = x(t)u(t) \\ &X = U = [-5, 5] \\ &L(x,u) = (x-1)^2 + \delta(u-2)^2, \quad 0 < \delta < 1 \end{aligned} \tag{34}$$

Set of all feasible steady-states $(x,u)|x = 0 \ \ or \ \ u = 1$. Let $(x_s, u_s) = (1,1)$, then $L(x_s, u_s) = \delta$.

$s(x,u) = L(x,u) - L(x_s,u_s) = (x-1)^2 + \delta(u-2)^2 - \delta$, we will search $\lambda(x)$ in form

$$\lambda(x) = ax + c$$

From dissipativity enequation we shooud have $\lambda(f(x,u))-\lambda(x) \leq s(x,u)$, then compare $\lambda(f(x,u)) - \lambda(x) = axu - ax$ with $s(x,u) = (x-1)^2 + \delta(u-2)^2 - \delta$.

Consider $s(x,u) - (\lambda(f(x,u)) - \lambda(x)) = g(x,u)$ as function, and find parameters a such that $g(x,u) \geq 0$, $\forall x \in X, \quad u \in U$. And minimum in point $(x_s, u_s) = (1,1)$

$$\triangledown g(x,u) = \begin{bmatrix} 2(x+1) - au + a \\ 2\delta(u-2) - ax \end{bmatrix}$$

Function g is convex.

$$\begin{bmatrix} 2(x+1) - au + a \\ 2\delta(u-2) - ax \end{bmatrix}_{(1,1)} = 0$$

Thus $a = -2\delta$, and from requrements, that $\lambda(x) = ax + c \geq 0 \quad \forall x \in X = [-5,5]$ we have $c \geq -ax = -2\delta x \geq [x = 5] \geq -10\delta$.

And now we can use previous theorem to prove, that system is optimaly operated at steady state $(x_s, u_s) = (1,1)$.

Consider case, when system is not dissipative. Change previous constraint for u, let $u \leq 0$: then new optimal steady-state $(x_s, u_s) = (0,0)$ and $L(x_s, u_s) = 4\delta + 1$.

Consider 2-period orbits:

$$\begin{aligned} x(\cdot) &= (1, -\frac{1}{3}, 1, -\frac{1}{3}, \dots) \\ u(\cdot) &= (-\frac{1}{3}, -3, -\frac{1}{3}, -3, \dots) \end{aligned} \tag{35}$$

With this orbits

$$\lim_{T\to\infty} \inf \frac{\sum_{t=0}^{T-1} L(x(t),u(t))}{T} = \frac{L(1,-\frac{1}{3})+L(-\frac{1}{3},-3)}{2} == \frac{\frac{4}{3}^2}{2} + \delta\frac{\frac{-7}{3}+(-3)^2}{2}$$

And for small enough δ this less then $L(0,0)$. That mean, that system is not optimal operated at steady state and not dissipative w.r.t. chosen $s(x,u)$.

Closed-loop convergence to the optimal steady state

Theorem 7.3. Suppose the system is strictly dissipative w.r.t. the supply rate $s(x,u) = L(x,u) - L(x_s,u_s)$. Then the closed-loop system asymptotically converge to the optimal steady state x_s.

Remark: x_s is asymptotically stable if $J^*(x)$ and $\lambda(x)$ are continuous at x_s.

Proof. Define "rotated" cost function

$$\tilde{L}(x,u) = L(x,u) - L(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

Auxiliary optimization problem $\tilde{P}$

$$\min_{u(\cdot|t)} \tilde{J}(x(t),u(t)) = \sum_{k=t}^{t+N-1} \tilde{L}(x(k|t),u(k|t))$$

s.t. some constraints as in original MPC problem

Claim: P and $\tilde{P}$ have the same minimizer. Proof of claim: feasible sets coincide

$$\tilde{J}(x(t),u(t)) = \sum_{k=t}^{t+N-1} [L(x(k|t),u(k|t))+\lambda(x(k|t))-\lambda(f(x(k|t),u(k|t)))-L(x_s,u_s)] = \lambda(x(t|t))-\lambda(x(t+N|t))-NL(x$$

$\Rightarrow$ $\tilde{J}$ and J only differ by a constant $\Rightarrow$ P and $\tilde{P}$ have the same minimizer.

$\Rightarrow$ we can use $\tilde{P}$ to analyze stability of the closed loop. $\tilde{L}(x,u) \geq \rho(|x-x_s|)$, $\tilde{L}$ is positive definite (from strictly dissipative) $\Rightarrow$ can apply standard MPC stability results. □

Average constraints:

Definition. Asymptotic average constraints $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ - (auxiliary) output specifying average constraints.

Assumption: h is continuous

- $Av[h(x,u)] \in Y$
- $Av[h(x,u)] := \bar{h} : \exists t_k \to \infty \ s.t. \ \lim_{n\to\infty} \frac{\sum_{t=0}^{t_n-1} h(x(t),u(t))}{t_n} = \bar{h}$
  $Av[h(x,u)]$ "set of asymptotic average values of h".

How to modify the MPC problem accordingly?

Adaptation of optimal steady state computation

$$(x_s, u_s) = argmin_{x=f(x,u), x\in X, u\in U, h(x,u)\in Y} L(x,u)$$

Additional constraint:

$$\sum_{k=t}^{t+N-1} h(x(k|t), u(k|t)) \in Y_t$$

with $Y_{t+1} := Y_t \oplus Y \oplus -h(x(t), u(t))$, $Y_0 = NY \oplus Y_\infty$, Y_∞ - arbitrary compact set.

1. How to guarantee recursive feasibility?

   Candidate solution at time $t+1$

   $$\tilde{u}(k|t+1) = \begin{cases} u^*(k|t) & t+1 \leq k \leq t+N+1 \\ u_s & k = t+N \end{cases}$$

   $$\sum_{k=t+1}^{t+N} h(\tilde{x}(k|t+1), \tilde{u}(k|t+1)) = \sum_{k=t}^{t+k-1} h(x^*(k|t), u^*(k|t))[\in Y_t] - h(x(t), u(t)) + h(x_s, u_s)[\in Y]$$

   $\in Y_t \oplus Y \oplus -h(x(t), u(t)) = Y_{t+1}$

2. Satisfaction of average constraint for the resulting closed loop:

   Assumption: Y is convex. Solve the recursion for Y_t to obtain

   $$Y_t = Y_\infty \oplus (t+N)Y \oplus -\sum_{k=0}^{t-1} h(x(k), u(k))$$

   the last item is closed loop sequence.

   From the additional constraint in optimal problem:

   $$\sum_{k=0}^{t-1} h(x(k), u(k)) + \sum_{k=t}^{t+N-1} h(x(k|t), u(k|t)) \in Y_\infty \oplus (t+N)Y$$

   For each sequence t_n s.t. $\lim_{n\to\infty} \sum_{k=0}^{t_n-1} \frac{h(x(k),u(k))}{t_n}$ exists, we obtain

   $$\lim_{n\to\infty} \frac{\sum_{k=0}^{t_n-1} h(x(k), u(k))}{t_n} \in \lim_{n\to\infty} \frac{Y_\infty \oplus (t_n+N)Y}{t_n} = Y$$

   $\Rightarrow$ closed-loop system satisfies average constraints.

Transient average constraint:

$$\sum_{k=t}^{t+T-1} \frac{h(x(k), u(k))}{T} \in Y, \ \forall t \geq 0$$

for $T = 1$, standard pointwise-in-time

Constraints are recovered

Assumption: $Y = \mathbb{R}^p_{\leq 0}$

$T + N - 1$ constraints in total,described below

$$\begin{cases} \sum_{t-T+\tau}^{t-1} h(x(k), u(k)) + \sum_{i=t}^{t+\tau-1} h(x(i|t), u(i|t)) \geq 0 \ \tau = 1, ..., T \\ \sum_{i=t+j+1}^{t+T+j} h(x(i|t), u(i|t)) \leq 0, \ j = 0, ..., N-2 \\ u(i|t) = u_s \ i = t+N, ..., t+N-2+T \end{cases}$$

- recursive feasibility can be shown using standard candidate sequence
- closed-loop transient average constraint satisfaction directly follows from first additional constraint