# LaTeX $2_\varepsilon$ Cheat Sheet

## Differential equations

**Lemma 1** (Cromwall)**.** Suppose that $0 \le \phi(t) \le c + L\int_0^t \phi(\tau)d\tau$, $c, L > 0$, $\phi$ – continuous. Then $\phi(t) \le ce^{Lt}$.

**Definition.** Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \le \epsilon,\ \forall t \ge 0$.

**Definition.** Equilibrium point $x = 0$ is asymptotically stable if it is stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t\to\infty} x(t) \to 0$.

## Nonlinear systems

**Definition.** Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \ge 0,\ \exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

**Definition.** Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0\ \ \exists \delta > 0$, s.t $\forall t_0 \ge 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

**Definition.** Point $x^* = 0$ asymptotically stable if it is stable and $\forall t_0 \ge 0\ \ \exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

**Definition.** Point $x^* = 0$ uniformly asymptotically stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

**Definition.** Convergence: $\forall \eta > 0\ \ \forall t_0 \ge 0$, $\exists T > 0$ such that $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

**Definition.** Uniform convergence: $\forall \eta > 0\ \ \exists T > 0$ such that $\forall t_0 \ge 0$ and $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

**Definition.** Point $x^* = 0$ is globally uniformly asymptotically stable if it is uniformly stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta\ \ \ \exists T > 0$ such that $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\|x(t)\| < \eta$, $\forall t \ge t_0 + T$.

**Theorem 0.1** (Lyapunov's direct method)**.** Let $f : [0,\infty) \times D \to R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0,\infty) \times D \to R$ with:

- $W_1(x) \le V(t,x) \le W_2(x)$, $\forall t \ge 0,\ x \in D$
- $\dot{V}(t,x) \le 0$, $\forall t \ge 0,\ x \in D$

where $W_1, W_2 : D \to R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.
If further $\dot{V}(t,x) \le -W_3(x)$, $\forall t \ge 0,\ x \in D$ with $W_3 : D \to R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.
If $D = R^n$ and $W_1$ is radialy unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

**Definition.** A function $\alpha : [0,\delta) \to [0,\infty)$ is (of) "class $K$" if it is continuous, strictly increasing, and $\alpha(0) = 0$.

**Definition.** A function $\alpha : [0,\delta) \to [0,\infty)$ is "class $K_\infty$" if $\alpha \in K$ and $\lim_{r\to\infty} \to \infty$.

**Definition.** A function $\beta : [0,\delta) \times [0,\delta) \to [0,\infty)$ is "class $KL$ if it is continuous , $\beta(\cdot, s) \in K$ for all fixed $s$, and for each fixed $r$, $\beta(r,\cdot)$ is strictly decreasing: $\lim_{s\to\infty} \beta(r,s) = 0$

**Lemma 2.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \alpha(||x(t_0)||)$.

**Lemma 3.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \beta(||x(t_0)||, t - t_0)$.

## System with inputs

**Definition.** System (**??**) is input-to-state stable (ISS) if $\exists \beta \in KL$, $\gamma \in K$ s.t. $\forall x_0 \in R^n$, $\forall t \ge 0$ follows $||x(t)|| \le \beta(||x_0||, t) + \gamma(\sup_{\tau\in[0,t]} ||u(\tau)||)$.

**Theorem 0.2.** Suppose that there exists a continuously differentiable function $V : R^n \to R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x,u) \le -\alpha_3(||x||)$, $\forall x : ||x|| \ge \rho(||u||)$. Then (**??**) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

**Theorem 0.3.** Assume that:

- $f$ is globally Lipschitz;
- $x = 0$ is a globally exponentially stable EP for $\dot{x} = f(x, 0)$

Then the system (**??**) is ISS.

**Theorem 0.4** (Artstein)**.** There exists $k : R^n \to R^m$ (state feedback) which is continuous on $R^n \backslash \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff there exists a CLF.

Sontag's formula"
Fix $c \ge 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$k(x) = \begin{cases} -cb(x) - \frac{a(x)+\sqrt{a(x)^2+(b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T, & b(x) \ne 0 \\ 0, \quad b(x) = 0 \end{cases}$$

## Backstepping

Integrator backstepping

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \qquad (1)$$
$$\dot{x}_2 = u$$

$$u = (-\frac{\partial V_1}{\partial e_1} g_1(e_1) + \dot{\alpha_1}) - k_2 e_2,\ k_2 > 0 \qquad (2)$$

$$\dot{x_1} = f_1(x_1) + g_1(x_1)x_2$$
$$\dot{x_2} = f_2(x_1, x_2) + g_2(x_1, x_2)u$$

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)}(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha_1} - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2))$$

$$\alpha_i(x_1, \dots x_i) = \frac{1}{g_i}(\dot{\alpha}_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}} g_{i-1} - k_i(x_i - \alpha_{i-1}) - f_i)$$

## Systems with inputs and outputs

Two-step approach:

1. Bring $x(t)$ to $S := \{x \in \mathbb{R}^n | S(x) = 0\}$ in finite time
2. Have $x(t)$ going to zero asymptotically (on $S$)
   - switching between nodes 1 and 2
   - mode 2 is "sliding mode"

$$V(X) = \frac{1}{2} s(x)^2$$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} sgn(s(x))),\ \hat{u} > 0$$

$$\dot{x} = f(x) + g(x)\sigma(x) + g(x)u$$

If $|\sigma(x)| \le \beta(x)$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|) sgn(s(x)))$$

**Definition** (dissipativity)**.**

$$S(x(t)) \le S(x_0) + \int_0^t s(u(\tau), y(\tau)) d\tau \qquad (3)$$

Introduce "available storage"

$$S_a(x) := \sup_{u:[0,T]\to\mathbb{R}^m, T\ge 0, x(0)=0} (-\int_0^T s(u(\tau), y(\tau)))$$

**Theorem 0.5.** System is dissipative w.r.t. the supply rate $s$ iff $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$

Moreover, if $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$, then $S_a$ is a storage function and $S(x) \ge S_a(x)\ \forall x \in \mathbb{R}^n$ for all storage functions $S$.

If system is dissipative then $x = 0$ is asymptotically stable.

$$\dot{x} = f(x, u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \qquad (4)$$
$$y = h(x),\ y \in \mathbb{R}^m$$

**Definition.** System is passive if it is dissipative w.r.t. supply rate $s(u, y) = u^T y$

**Definition.** System is zero-state observable (ZSO) if (for $u(t) = 0$) $y(t) = 0$ for all $t \ge 0 \Rightarrow x(t) = 0$ for all $t \ge 0$

**Theorem 0.6.** Let system (4) be i) passive in differentiable storage set ii)ZSO. Then the feedback $u = -Py$, $P > 0$ renders the origin asymptotically stable.

**Theorem 0.7.** Feedback interconnection with $u \equiv 0$. $H_1$ and $H_2$ are ZSO and dissipative with $S_1,\ S_2$ w.r.t.

$$s_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i,\ i = 1, 2,\ \rho, \nu \in \mathbb{R} \qquad (5)$$

The origin $(x_1, x_2) = (0, 0)$ for interconnection is asymptotically stable if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$.

If is sabisfied with $v_i = 0$: "output - feedback passive". If (5) satisfied with $p_i = 0$: "input - feadforward passive".