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ГЛАВА 1

MPC

Formulation of control problem

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \\ y = h(x) \end{cases} \quad (1.1)$$

Objective of MPC:

Find stabilizing control strategy that:

- minimize objective function: $J = \int_t^\infty F(x(\tau), u(\tau)) d\tau$
- satisfies constraints: $u(\tau) \in U \quad x(\tau) \in X$

Closed-loop optimal control vs Open-loop optimal control

Closed-loop:

Feedback $u = k(x)$

- + Feedback present
- + suit for uncertainty disturbances
- - Finding closed solution hardly possible

Open-loop optimal control:

Input trajectory $u = u(t, x_0)$

- + Computation often feasible
- - No feedback
- - Don't know much about system

MPC - repeated open-loop optima; control in feedback fashion.

ГЛАВА 2

ZERO-TERMINAL CONSTRAINT MPC

Mathematical formulation of NMPC problem:

System dynamics: $\dot{x} = f(x, u)$ $x(0) = x_0$ $x, u \in \mathbb{R}^n$

Constraints: $x(t) \in X$ $u \in U \forall t \geq 0$

Assumptions:

- $f(0, 0) \Rightarrow x_1 = 0$ - equilibrium point for $u_1 = 0$
- $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ - twice continuously differentiable
- U is a compact set (closed and bounded)
- X is a connected and closed set
- $(0, 0) \in \text{int}(X \times U)$

MPC optimization problem:

At time t , given initial state $x(t)$

$$\min_{\bar{u}(\cdot, t)} J(x(t), \bar{u}(\cdot; t))$$

with $J(x(t), \bar{u}(\cdot; t)) = \int_t^{t+T} L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau$
s.t.

$$\dot{\bar{x}} = f(x, u), \bar{x}(t; t) = x(t)$$

$$\bar{u}(\tau; t) \in U, \bar{x}(\tau; t) \in X, \forall \tau \in [t, t+T]$$

$$\bar{x}(t+T; t) = 0$$

Optimal open-loop solution:

$$\bar{u}^*(\cdot; t) = \text{argmin}_{\bar{u}(\cdot; t)} J(x(t), \bar{u}(\cdot; t))$$

Notation:

- Quantities without bar: real system trajectories
- Quantities with bar: predicted trajectories

- L -stage cost
- $(\cdot; t)$ - predicted at time t
- T - prediction horizon
- Optimal value function $J^*(x(t)) = J(x(t), \bar{u}^*(t))$

Real trajectories deviate from predicted one!

Assumptions:

- $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous and

$$\begin{cases} L(0, 0) = 0 \\ L(x, u) > 0 \\ \forall (x, u) \neq (0, 0) \end{cases} \quad (2.1)$$

- $J^*(x)$ is continuous at $x = 0$

MPC algorithm:

1. At sampling time t , measure $x(t)$ and solve MPC optimization problem
2. Apply $u_{MPC}(\tau) = \bar{u}^*(\tau, t) \forall \tau \in [t, t + \delta)$ with sampling time δ
3. Set $t := t + \delta$ and go to step 1

Feasibility: The MPC problem is feasible at time t if there exists at least one $\bar{u}(\cdot; t)$ s.t. constraints satisfied.

Theorem:

Suppose that

- (i) assumptions are satisfied
- (ii) and that zero-terminal constraint MPC problem is feasible at $t = 0$

Then:

- MPC problem is recursively feasible
- resulting closed-loop system is asymptotically stable

Let $D \subset \mathbb{R}^n$ be the set of all points for which (ii) holds. The D is a region of attraction for the closed loop.

Proof.

1. recursive feasibility: by induction

2. • feasible at $t = 0$ by assumption

• assume: feasibility at t . Consider the candidate solution:

$$\bar{u}(\tau; t + \delta) = \begin{cases} \bar{u}^*(\tau; t) & \tau \in [t + \delta, t + T] \\ 0 & \tau \in [t + T, t + \delta + T] \end{cases}$$

3. asymptotic stability

Idea: use $J^*(x(t))$ as "Lyapunov function"

Consider:

$$\begin{aligned} J(x(t + \delta), \bar{u}(\cdot; t + \delta)) &= \int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), \bar{u}(\tau; t + \delta)) d\tau = \\ &= \int_{t+\delta}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau + \int_{t+T}^{t+\delta+T} L(0, 0) d\tau (= 0) = \\ &= J^*(x(t)) - \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau \end{aligned}$$

by optimality

$$J^*(x(t+\delta)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) \leq J^*(x(t)) - \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau$$

by induction

$$J^*(x(\infty)) (\geq 0) \leq J^*(x(0)) (finite) - \int_0^\infty L(x_{MPC}(\tau), u_{MPC}(\tau)) d\tau$$

Barbalat's lemma:

ϕ uniformly continuous $\phi : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$

$$\lim_{t \rightarrow \infty} \int_0^t \phi(\tau) d\tau < \infty \Rightarrow \phi(t) \rightarrow 0, t \rightarrow \infty$$

From Barbalat's lemma $L \rightarrow 0$ when $t \rightarrow \infty \Rightarrow$ L pos.def. $\|x_{MPC}(t)\| \rightarrow 0$ when $t \rightarrow \infty \Rightarrow$ convergence

Lyapunov stability: using standard arguments (J^* is continuous at $x = 0$)

Lessons learned:

• feasibility \Rightarrow stability

- value function is Lyapunov function
- have to prove recursive feasibility
- suboptimal solution is sufficient for stability

ГЛАВА 3

QUASI - INFINITE HORIZON MPC

Goal: Relax (restrictive) zero-terminal zero-terminal constraint

Idea: terminal region + local CLF(controller Lyapunov function)

MPC optimization problem: At time t

$$\min_{\bar{u}(\cdot; t)} J(x(t), \bar{u}(\cdot; t)) = \int_t^{t+T} L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau + F(\bar{x}(t+T; t))$$

$F(\bar{x}(t+T; t))$ - terminal cost

s.t.

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \bar{x}(t; t) = x(t)$$

$$\bar{x}(t; t) \in X \quad \bar{u}(t; t) \in U \quad \forall \tau \in [t, t+T]$$

$$\bar{x}(t+T; t) \in X^f$$

X^f - terminal region

Optimal solution: $\bar{u}^*(\cdot, t)$, $J^*(x(t))$

Assumption 1: Terminal region + terminal controller

There exists an auxiliary local controller $u = k^{loc}(x)$ s.t.

1. X^f is positively invariant $\dot{x} = f(x, k^{loc}(x))$

2. $k^{loc}(x) \in U \quad \forall x \in X^f$

3. $\dot{F}(x) + L(x, k^{loc}(x)) \leq 0 \quad \forall x \in X^f$

$\Rightarrow F$ is local control-Lyapunov function.

Theorem.

Suppose Assumption 1 holds and MPC problem is feasible at $t = 0$. Then:

- recursive feasibility
- closed-loop is asymptotically stable

Proof.

1. Recursive feasibility by induction

- feasible at $t = 0$ by assumption

- assume feasibility at t

candidate

$$\bar{u}(\tau; t + \delta) = \begin{cases} \bar{u}^*(\tau; t) & \tau \in [t, t + T] \\ k^{loc}(\bar{x}(\tau; t + \delta)) & \tau \in [t + T, t + \delta + T] \end{cases}$$

\Rightarrow this is a feasible solution at $t + \delta$

2. asymptotic stability

$$\begin{aligned} J^*(x(t + \delta)) - J^*(x(t)) &\leq J(x(t + \delta), \bar{u}(\cdot; t + \delta)) - J^*(x(t)) = \\ &\int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), \bar{u}(\tau; t + \delta)) d\tau + F(\bar{x}(t + \delta + T; t + \delta)) - \\ &\quad - \int_t^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau - F(\bar{x}^*(t + T; t)) = \\ &= \int_{t+T}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), k^{loc}(\bar{x}(\tau; t + \delta))) d\tau + F(\bar{x}(t + \delta + T; t + \delta)) - \\ &\quad - \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau - F(\bar{x}^*(t + T; t)) \leq \end{aligned}$$

As far as from Assumption 1.3 we have the sum of three terms is ≤ 0

$$- \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau$$

$\Rightarrow J^*(x(\infty)) \leq J^*(x(0)) - \int_0^\infty L(x_{MPC}(\tau), u_{MPC}(\tau)) d\tau$ From here: some steps as in zero-terminal constraint rose.

How can Assumption 1 be satisfied?

Assume:

- quadratic state cost $L(x, u) = x^T Q x + u^T R u$, $Q, R > 0$
- linearization at the origin is stabilizable $\dot{x} = Ax + Bu$ $A = \frac{\partial F}{\partial x}(0, 0)$ $B = \frac{\partial F}{\partial u}(0, 0)$

Approach:

- Linear auxiliary controller $k^{loc}(x) = Kx$
- Quadratic terminal cost function $F(x) = x^T P x$, $P > 0$

- Terminal region $X_\alpha^f = \{x \in \mathbb{R}^n | x^T P x \leq \alpha\}$ for some $\alpha > 0$
- Determine P, K, α s.t. Assumption 1.1-1.3 hold:

For (Assumption 1.3):

$$\frac{d}{dt}x(t)^T P x(t) \leq -x(t)^T (Q + K^T R K) x(t) = -x(t)^T Q^* x(t)$$

$$[x^T Q x + u^T R u = [u = Kx] = x^T (Q + K^T R K) x]$$

$$\frac{d}{dt}x(t)^T P x(t) = f(x, Kx)^T P x + x^T P f(x, Kx)$$

$[f(x, Kx) = (A + BK)x + \phi(x), A + BK = A_K, K \text{ is chosen s.t. } A_K \text{ is Hurwitz}]$
Upper bound for $x^T P \phi(x)$: $L_\phi := \sup\{\frac{|\phi(x)|}{|x|}, x \in X_\alpha^f, x \neq 0\}$

$$x^T P \phi(x) \leq |x^T P| |\phi(x)| \leq \|P\| L_\phi |x|^2 \leq \frac{\|P\| L_\phi}{\lambda_{\min}(P)} x^T P x \quad (3.1)$$

We choose α small enough s.t.

$$L_\phi \leq \frac{k \lambda_{\min}(P)}{\|P\|} \quad (3.2)$$

for some $k > 0$. Plug this into (3.1): $x^T P \phi(x) \leq k x^T P x$. Insert this into $\frac{d}{dt}x^T P x \leq x^T (A_K P + P A_K) x + 2k x^T P x$

$$= x^T ((A_K + kI)^T P + P(A_K + kI)) x$$

ensure that it $\leq -x^T Q^* x$

\Rightarrow Lyapunov equation which can be solved if and only if $A_K + kI$ is Hurwitz

$$\Leftrightarrow k < -\max \operatorname{Re}\{\lambda(A_K)\} \quad (3.3)$$

$$\Rightarrow (A_K + kI)^T P + P(A_K + kI) = -Q^* \quad (3.4)$$

Design procedure

1. Compute K s.t. $(A + BK)$ is Hurwitz
2. Choose $k > 0$ s.t. (3.3) and solve (3.4)
3. Find largest possible α_1 s.t. $Kx \in U, \forall x \in X_\alpha^f$

4. Find the largest $\alpha \in (0, \alpha_1]$ s.t. (3.2) holds

Alternative to the (4) step
Solve optimization problem

$$\max_x x^T P \phi(x) - k x^T P x \text{ s.t. } x^T P x \leq \alpha \quad (3.5)$$

Iterate this by reducing α from α_1 until optimal value of (4) is nonpositive
Degrees of freedom in design

- calculation of K
- choice of k - tradeoff between "large" terminal region and "large" P

ГЛАВА 4

UNCONSTRAINED MPC

Goal: Guarantee stability + degree of suboptimality without stabilizing terminal constraint + cost

Setup:

- $\dot{x} = f(x, u), x(0) = x_0$
- input constraints $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m \forall t \geq 0$

Infinite-horizon cost function: $J_\infty(x_0, \bar{u}(\cdot; 0)) = \int_0^\infty L(\bar{x}(\tau; 0), \bar{u}(\tau; 0)) d\tau \Rightarrow$
optimal value function $J_\infty^*(x_0)$

Assumption: $J_\infty^*(x_0) < \infty, \forall x_0 \Rightarrow$ system is asymptotically stabilizable

Finite-horizon cost function: $J_\infty(x(t), \bar{u}(\cdot; t)) = \int_0^\infty L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau$

Infinite-horizon performance resulting from application of MPC controller:

$$J_\infty^{MPC}(x_0) = \int_0^\infty L(\bar{x}_{MPC}(\tau), \bar{u}_{MPC}(\tau)) d\tau$$

Definition. Suboptimality index α : $\alpha J_\infty^{MPC}(x_0) \leq J_\infty^*(x_0) \forall x_0$

- $\alpha \leq 1$ by optimality of J_∞^*
- $\alpha > 0$ implies closed-loop stability (Barb.lemma)

Proposition 1: Relaxed dynamic programming

Assume $\exists \alpha \in (0, 1] s.t. \forall x \in \mathbb{R}^n$

$$J_T^*(x(t + \delta)) \leq J_T^*(x(t)) - \alpha \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t)) d\tau (*)$$

Then the estimate

$$\alpha J_\infty^*(x(t)) \leq \alpha J_\infty^{MPC}(x(t)) \leq J_T^*(x(t)) \leq J_\infty^*(x(t)) \quad (4.1)$$

holds for all $x \in \mathbb{R}^n$

Proof

- 1 and 3 inequalities follow from optimality (by definition)
- 2 inequality follows from summing up (*) over all sampling instances

$$J_T^*(x(N\delta)) \leq J_T^*(x_0) - \alpha \int_0^{N\delta} L(x_{MPC}(t), u_{MPC}(t)) dt \quad (4.2)$$

$$N \rightarrow \infty : J_T^*(x_0) \geq \alpha J_\infty^{MPC}(x_0) \quad (4.3)$$

Central idea (image to be inserted)

$$L^*(t; t) = L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))$$

$$(c) : J_T^*(x(t + \delta)) \leq \frac{1}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau : (b) \quad (4.4)$$

$$(b) : \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau \leq \gamma \int_t^{t+\delta} L^*(\tau; t) d\tau : (a) \quad (4.5)$$

Theorem 1:

Assume $\exists c \in (0; 1]$ and $\gamma > 0$ s.t. 4.4 - 4.5 holds. Then (*) holds with $\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon}$

Proof.

$$\begin{aligned} J_T^*(x(t + \delta)) - J_T^*(x(t)) &= J_T^*(x(t + \delta)) - \int_t^{t+T} L^*(\tau; t) d\tau \leq^{(1)} \\ &\leq \frac{1-\epsilon}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau - \int_t^{t+\delta} L^*(\tau; t) d\tau \leq^{(2)} \\ &\leq (\gamma \frac{1-\epsilon}{\epsilon} - 1) \int_t^{t+\delta} L^*(\tau; t) d\tau \end{aligned}$$

$$-\alpha := \gamma \frac{1-\epsilon}{\epsilon} - 1$$

Assumption 1: Asymptotic Controllability

For all x , \exists some input trajectory $\hat{u}_x(\cdot)$ with $\hat{u}_x(t) \in \mathbb{U}, \forall t \geq 0$ s.t.

$$L(\hat{x}(t), \hat{u}(t)) \leq \beta(t) \min_u L(x, u), \forall t > 0$$

with $\beta : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ - continuous, positive, strictly decreasing with $\lim_{t \rightarrow 0} \beta(t) = 0$
 $\Rightarrow \int_0^\infty \beta(\tau) d\tau < \infty$ $B(t) = \int_0^t \beta(\tau) d\tau$

Typical example: (image to be inserted)

How to compute ϵ and γ : Lemma 1: Let Assumption 1 hold. Then the inequality

$$J_T^*(x(t + \delta)) \leq \int_{t+\delta}^{t+t'} L^*(\tau; t) d\tau + B(T + \delta - t') L^*(t + t'; t) \quad (4.6)$$

holds for all $t' \in [\delta, T]$

(image to be inserted)

Proof.

Consider

$$\bar{u}(\tau; t + \delta) = \left\{ \begin{array}{l} \bar{u}^*(\tau; t), \tau \in [t + \delta, t + t'] \\ \hat{u}_{x'}(\tau - t - t'), \tau \in [t + t', t + \delta + T] \end{array} \right\}$$

$$\begin{aligned} J_T^*(x(t + \delta)) &\leq J_T(x(t + \delta), \bar{u}(\cdot; t + \delta)) = \\ &= \int_{t+\delta}^{t+t'} L^*(\delta; t) d\delta + \int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau - t - t'), \hat{(\tau - t - t')}) d\tau \leq \end{aligned}$$

by Assumption 1

$$\int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau - t - t'), \hat{(\tau - t - t')}) d\tau \leq L^*(t + t'; t) \int_0^{T+\delta-t'} \beta(\tau) d\tau$$

as far as $B(t) = \int_0^t \beta(\tau) d\tau$

$$\leq \int_{t+\delta}^{t+t'} L^*(\tau; t) d\tau + B(T + \delta - t') L^*(t + t'; t)$$

Calculation of ϵ from (4.6):

$$J_T^*(x(t + \delta)) \leq \min_{t' \in [\delta, T]} \left(\int_{t+\delta}^{t+t'} L^*(\tau; t) d\tau + B(T + \delta - t') L^*(t + t'; t) \right) \leq$$

$$\int_{t+\delta}^{t+T} L^*(\tau; t) d\tau + B(T) \min_{t' \in [\delta, T]} L^*(t + t'; t)$$

as far as $\min_{t' \in [\delta, T]} L^*(t + t'; t) \leq \frac{1}{T-\delta} \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau$ minimum is less or equal that the average

$$= \left(1 + \frac{B(T)}{T - \delta}\right) \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau$$

$$\left(1 + \frac{B(T)}{T - \delta}\right) = \frac{1}{\epsilon}$$

Lemma 2:

$$\int_{t+t'}^{t+T} L^*(\tau; t) d\tau \leq B(T - t') L^*(t + t'; t) \forall t' \in [0; T]$$

Proof. Analogues to lemma 1.

Calculation of γ :

$$\int_{t+\delta}^{t+T} L^*(\tau; t) d\tau \leq \int_{t+\hat{t}}^{t+T} L^*(\tau; t) d\tau (\forall \hat{t} \in [0, \delta]) \leq$$

$$\begin{aligned} &\leq \min_{\hat{t} \in [0, \delta]} (B(T - \hat{t})L^*(t + \hat{t}; t)) \leq \\ &\leq B(T) \min_{\hat{t} \in [0, \delta]} L^*(t + \hat{t}; t) \leq \frac{B(T)}{\delta} \int_t^{t+\delta} L^*(\tau; t) d\tau \end{aligned}$$

Denote $\gamma = \frac{B(T)}{\delta}$

$$\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon} = 1 - \frac{B(T)}{\delta} \left(\frac{B(T)}{T-\delta} \right)$$

Alternative computation of ϵ (less conservative):

We want to compute ϵ s.t.

$$\epsilon \leq \frac{\int_{t+\delta}^{t+T} L^*(\tau; t) d\tau}{J_T^*(x(t + \delta))}$$

Idea: Minimize

$$\epsilon = \min_{L_t, J_T^*} \frac{\int_{\delta}^T L_t(\tau; t) d\tau}{J_T^*(x(t + \delta))} \quad (4.7)$$

$J_T^* = 1$ - without loss of generality s.t. $0 \leq L_t \forall \tau \in [\delta, T]$

$$J_T^*(x(t + \delta)) \leq \int_{\delta}^{t'} L_t(\tau) d\tau + B(T + \delta - t') L_t(t') \forall t' \in [\delta, T]$$

Due to linearity in L_t , without loss of generality we can set $J_T^* = 1$.

\Rightarrow infinite dimensional linear problem

Idea for solution: second constraint has to be active for all times

Differentiate (4.6) with relation to t'

$$0 = L_t(t') + \frac{dB(T + \delta - t')}{dt'} L_t(t') + B(T + \delta - t') \dot{L}_t(t')$$

as far as $\frac{dB(T + \delta - t')}{dt'} = \beta(T + \delta - t')$

$$\left\{ \begin{array}{l} \dot{L}_t(t') = \frac{\beta(T + \delta - t') - 1}{B(T + \delta - t')} L_t(t') \\ \text{initial condition } L_t(t) = \frac{1}{B(\tau)} \end{array} \right\}$$

Solution:

$$\bar{L}_t(t') = \frac{1}{B(T + \delta - t')} e^{-\int_{\delta}^{t'} \frac{1}{B(T + \delta - \tau)} d\tau}$$

Have to show: \bar{L}_t is a minimizer of (4.7)

$$\int_{\delta}^T \bar{L}_t(\tau) d\tau \leq \int_{\delta}^T L_t(\tau) d\tau$$

for all feasible L_t

Proof.

Assume $\exists L_t$ s.t.

$$\int_{\delta}^T L_t(\tau) d\tau < \int_{\delta}^T \bar{L}_t(\tau) d\tau$$

Then $\exists \hat{t} \in [\delta, T]$ s.t. $\int_{\delta}^{\hat{t}} L_t(\tau) d\tau \leq \int_{\delta}^{\hat{t}} \bar{L}_t(\tau) d\tau$ and $\bar{L}_t(\hat{t}) > L_t(\hat{t})$

But then

$$1 = \int_{\delta}^{\hat{t}} \bar{L}_t(\tau) d\tau + B(T + \delta - \hat{t}) \bar{L}_t(\hat{t}) > \int_{\delta}^{\hat{t}} L_t(\tau) d\tau + B(T + \delta - \hat{t}) L_t(\hat{t}) \quad (4.8)$$

the sign equality from (4.7) with equality.

Contradiction:

L_t cannot be a feasible solution of (4.7) \Rightarrow

$$\epsilon = \int_{\delta}^T \bar{L}_t(\tau) d\tau = 1 - e^{-\int_0^{T-\delta} \frac{1}{B(T-\tau)} d\tau}$$

Similarly, better estimate for γ can be obtained

$\alpha = 1 - \gamma \frac{1-\epsilon}{\epsilon}$ For $T \rightarrow \infty$: both estimates for $\epsilon \rightarrow 1 \Rightarrow \alpha \rightarrow 1$ as $T \rightarrow \infty \Rightarrow$ closed-loop stability for T large enough

ГЛАВА 5

ROBUST MPC

Consider linear (discrete-time) sytem: $x(t+1) = Ax(t) + Bu(t) + w(t)$ in short $x^+ = Ax + Bu + w$

Constraints: $x(t) \in X, u(t) \in U, \forall t = 0, 1, \dots$

Bound on w : W is a compact, convex set which contains 0. $w(t) \in W \forall t = 0, 1, \dots$

Main idea: Use additional error feeedback s.t. real systems state contained in a "tube" around some nominal system state.

Repetition of QI-MPC in discrete time: Nominal system:

$$z^+ = Az + Bv$$

At time t , given $z(t)$, solve

$$\min_{v(\cdot|t)} \hat{J}(z(t), v(\cdot|t)) = \sum_{i=t}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

s.t.

$$z(i+1|t) = Az(i|t) + Bv(i|t), z(t|t) = z(t)$$

$$z(i|t) \in Z, v(i|t) \in V, t \leq i \leq t+N-1$$

$$z(t+N|t) \in Z^f \subseteq Z$$

\Rightarrow optimizer $V^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Assumption 1:

- Cost is quadratic $L(z, v) = z^T Q z + v^T R v, Q, R > 0$
- There exists a local auxiliary controller $k^{loc} = Kx$ s.t.

1. Z^f is invariant with $Z^+ = (A+BK)z, A_k = A+BK$, i.e. $A_k Z^f \subseteq Z^f$
2. $Kz \in V \forall z \in Z^f$
3. $F(A_k z) - F(z) \leq -L(z, Kz) \forall z \text{ in } Z^f$

From Assumption 1 it follows (as in continuous time) that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \leq -L(z(t), v_{MPC}(t))$$

Since L is quadratic, there exists constants $c_2 > c_1 > 0$ s.t. $\forall z \in Z_N$ - feasible set

1. $c_1|z|^2 \leq \hat{J}^*(z)$
2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1|z|^2$
3. $\hat{J}^*(z) \leq c_2|z|^2$

Why is (3) true?

From Assumption 1.3 $\forall z \in Z^f$

$$\hat{J}^*(z) \leq \hat{J}(z, Kz(\cdot)) = \sum_{i=1}^{N-1} L(z(i), Kz(i)) + F(z(N)) \leq$$

N times apply Assumption 1.3

$$\leq F(z) = z^T P z \leq \lambda_{\max}(P)|z|^2$$

Influence of disturbance: Definition.

Mainkowski set addition:

$$A, B \subseteq \mathbb{R}^n \quad A \oplus B = \{a + b | a \in A, b \in B\}$$

Pontryagin set difference:

$$A, B \subseteq \mathbb{R}^n \quad A \ominus B = \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$$

$$(A \ominus B) \oplus B \subseteq A$$

$$A \subseteq (A \oplus B) \ominus B$$

Definition. Robust positively invariant set (RPI set):

S is RPI set for $x^+ = Ax + w$ if $AS \oplus W \subseteq S$ (or equivalently $Ax + w \in S \forall x \in S, \forall w \in W$)

Example:

$x^+ = 0.5x + w$. $w \in [-5, 5]$. So RPI set: $S = [-20, 20]$, minimal RPI set: $S = [-10, 10]$

Minimal RPI set:

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

(Minkowski set addition), min. RPI set exists and is bounded if A is Schour table.

Why?

Current state at time t is x ,

possible states at time $t + 1$: $Ax \oplus W$

$t + 2$: $A(Ax \oplus W) \oplus W = A^2x \oplus AW + w$

.....

$t + j$: $A^jx \oplus \sum_{k=0}^{j-1} A^k w$

\Rightarrow by choosing j large enough we can reach any state in S_∞

\Rightarrow any RPI set must satisfy $S_\infty \subseteq S$

Remains to show: S_∞ is an RPI set

$$AS_\infty \oplus W = A \sum_{i=0}^{\infty} A^i w \oplus W = \sum_{i=1}^{\infty} A^i w \oplus W = S_\infty$$

S_∞ in general is difficult to compute

\Rightarrow can compute invariant outer approximations of S_∞ (with bounded complexity)