# Nonlinear Control

Stuttgart university, May 2018

## Intro

Goals of Course

- overview over moder nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control method learn the mathmatic basis

Differential equations $\dot{x} = f(x)$

Nonlinear differential eqution $\dot{x} = f(t, x)$

System with input $\dot{x} = f(x, u)$

System with input and output $\dot{x} = f(x, u), \quad y = g(x, u)$

Input-output methods

Scope


[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997


## 1 Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)),\ x(0) = x_0 \tag{1}$$

Where $f : D \to R^n,\ D \subset R^n$ is open, [here we should explain, what means open set].

Solution to 1 $x : I_{x_0} \to D,\ t \to x(t)$ is differentiable

Interval existence solution

Questions:

# existence of solution

# "how large" is $I_{x_0}$

# uniqueness of solution

Usually we will add some restrictions on $f$ functions, like continuous.

### 1.1 Existence of solutions

**Definition.** Function $f : D \to R^n$ is continuous at $x' \in D$ if for $\forall \epsilon > 0$ $\exists \delta > 0$ such that for $\forall x \in D$, $\|x - x'\| < \delta => \|f(x) - f(x')\| < \epsilon$

Function $f : D \to R^n$ is continuous on D if it's continuous at $\forall x' \in D$

**Theorem 1.1** (Piano)**.** If $f : D \to R^n$ continuous, then for each $x_0 \in D$ $\exists x : (-\epsilon, \epsilon) \to D,\ \epsilon > 0$ satisfying (1).

Further, given a compact set $U \subset D$, then $\exists \alpha > 0$ s.t. $\forall x_0 \in U$ $\exists x : (-\epsilon, \epsilon) \to D$ satisfying (1).

**Example.** Consider equation $\dot{x}(t) = x(t)^2,\ \ x(0) = x_0 = 0$. Solution $x(t) = -\frac{1}{t-c},\ \ c = \frac{1}{x_0}$. In this example solution exist in interval $(-c, c)$.

But, what about the number of solutions? Which conditions we should add to garantie uniqueness of solution?

### 1.2 Uniquence of solutions

**Definition.** Function $f : D \to R^n$ is locally Lipshitz (continuous???) on D if $\forall x \in D$ there is a neighborhood $N(x) \subset D$ and $\exists L > 0$ s.t.

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2|| \tag{2}$$

For all $x_1, x_2 \in N$.

- Lipschitz on $W \in D$ if (2) holds $\forall x_1, x_2 \in W$ (with same L)
- globally Lipschitz if (2) holds $\forall x_1, x_2 \in R^n$ (with same L)

We have

# locally Lipschitz functions are continuous

# differentiable functions are locally Lipschitz

# locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

**Lemma 1** (Cromwall)**.** Suppose that $0 \le \phi(t) \le c + L\int_0^t \phi(\tau)d\tau$, $c, L > 0$, $\phi$ – continuous. Then $\phi(t) \le ce^{Lt}$.

*Proof.* $c + L\int_0^t \phi(\tau)d\tau := \psi(t)$, $\dot{\psi}(t) = L\phi(t) \le L\psi(t)$.

Consider $\frac{d}{dt}\left(\psi(t)e^{-LT}\right) = e^{-Lt}\left(\dot{\psi}(t) - L\psi(t)\right) \le 0$, thus $\psi(t)e^{-LT}$ is decreased, and as a result we have $\phi(t)e^{-Lt} \le \psi(t)e^{-Lt} \le \psi(0) = c$

□

**Theorem 1.2** (Picard Lindelof)**.** If function $f : D \to R^n$ is localy Lipschitz then for $\forall x_0 \in D$ $\exists! x : (-\epsilon, \epsilon) \to D,\ \epsilon > 0$ satisfying (1).

*Proof.* * existence from Piano theorem

Proof of uniqueness

Consider two solutions $x_1(.)$ and $x_2(.)$ to (1). $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$, $x_1(0) = x_2(0)$. Then we can integrate equality: $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau))d\tau$. $|x_1(t) - x_2(t)| \le \int_0^t |f(x_1(\tau)) - f(x_2(\tau))|d\tau \le L\int_0^t |x_1(\tau) - x_2(\tau)|d\tau$. Now we can apply Cromwall's lemma with $c = 0$ and $\phi(t) = |x_1(t) - x_2(t)|$, then $\phi(t) \le 0$, then $x_1(t) = x_2(t),\ \forall t \in (-\epsilon, \epsilon)$ □

**Example.**

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \ge 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$\textit{Solutions } x(t) = \begin{cases} \frac{1}{4}(t-c)^2, & \text{if } t \ge c \ge 0 \\ 0, & \text{else} \end{cases}$$

Global existance & uniqueness

- sufficient condition: $f$ globally Lipschitz
- another sufficient condition: solution entirely lies in a coplex set
- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff $\exists$ solution $V : R^n \to R \ge 0$ s.t. $\frac{\partial V}{\partial x} f(x) \le V(x)$, $\forall x \in R^n$

### 1.3 Lyapunov stability

If functions $\dot{V}(x) < 0$, $\forall x \in D \setminus \{0\}$, then $x^*$ is asymptotically stable.

**Definition.** Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \leq \epsilon,\ \forall t \geq 0$.

**Definition.** Equilibrium point $x = 0$ is asymptotically stable if it is stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t\to\infty} x(t) \to 0$.

**Theorem 1.3** (Lyapunov's direct method)**.** Let $x^* = 0 \in D$ be an equilibrium point of (1), i.e., $f(0) = 0$. Let $f : D \to R^n$ is continuous. If there exists a differentiable $V : D \to R$ s.t.

1. $V(x^*) = 0$, $V(x) > 0$, $\forall x \in D \setminus \{0\}$
2. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, $\forall x \in D$

then $x^* = 0$ is stable.

*Proof.* Fix compact $U = \{x : V(x) \leq c\}$ s.t. $U \in D$. By Piano: exist $\alpha > 0$ s.t. any solution $x$ with $x_0 \in U$ exists at least on the interval $[0, \alpha)$.

TODO proof is not full □

Lyapunovs direct method gives us:

- stability
- convergence (if $V < 0$)
- subset of the region of attraction (all compact $U = \{x : V(x) \leq c\} \in D$)
- existance of solution for all times

# 2 Nonlinear systems

In this section we consider function $f : R \times D \to R^n$, where $D \subseteq R^n$, and $D$ is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \tag{3}$$

The origin $x^* \in D$ is an equilibrium point for (3), if $f(t, 0) = 0, \quad \forall t \geq 0$.

Remark: EP (equilibrium point) $x^* = 0$ can be translation of a nonzero solution.

Suppose $\overline{y}$ is a solution of $\dot{y} = g(t, y)$.

Change of coordinates: $x(t) = y(t) - \overline{y}(t)$, then $\dot{x}(t) = \dot{y}(t) - \dot{\overline{y}}(t) = g(t, x(t) + \overline{y}(t)) - \dot{\overline{y}}(t) := f(t, x(t))$. Since $\dot{\overline{y}}(t) = g(t, \overline{y}(t))$, then $f(t, 0) = 0,\ \forall t \geq 0$.

Existence and uniqueness of solution to (3):

- if $f$ continuous, then exist local solution
- if $f$ continuous and locally Lipschitz in $x^*$, then exist local unique solution

Now we need new stability definitions.

**Definition.** Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \geq 0,\ \exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

**Definition.** Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0\ \ \exists \delta > 0$, s.t $\forall t_0 \geq 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

**Definition.** Point $x^* = 0$ asymptotically stable if it is stable and $\forall t_0 \geq 0\ \ \exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

**Definition.** Point $x^* = 0$ uniformly asymptotically stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

**Definition.** Convergence: $\forall \eta > 0\ \ \forall t_0 \geq 0,\ \exists T > 0$ such that $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

**Definition.** Uniform convergence: $\forall \eta > 0\ \ \exists T > 0$ such that $\forall t_0 \geq 0$ and $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

**Example.** Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0$$

Solution $x(t) = x(t_0)\frac{1+t_0}{1+t}$. It is uniformly stable, because we can choose $\delta = \epsilon$. But does $x(t)$ convergence uniformly? Answer is no.

**Definition.** Point $x^* = 0$ is globally uniformly asymptotically stable if it is uniformly stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta \quad \exists T > 0$ such that $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $\|x(t)\| < \eta$, $\forall t \geq t_0 + T$.

## 2.1 Lyapunov's direct method

Consider some function $V : [0, \infty) \times D \to R,\ (t, x) \to V(t, x)$ such that $\dot{V}(t, x) = \frac{\partial}{\partial t}V(t, x) + \frac{\partial}{\partial x}V(t, x)f(t, x)$.

**Theorem 2.1** (Lyapunov's direct method)**.** Let $f : [0, \infty) \times D \to R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \to R$ with:

- $W_1(x) \leq V(t, x) \leq W_2(x)$, $\forall t \geq 0,\ x \in D$

- $\dot{V}(t,x) \leq 0,\ \forall t \geq 0,\ \ x \in D$

where $W_1, W_2 : D \to R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.

If further $\dot{V}(t,x) \leq -W_3(x),\ \forall t \geq 0,\ \ x \in D$ with $W_3 : D \to R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.

If $D = R^n$ and $W_1$ is radialy unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

**Example.** Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Check function $V(t,x) = \frac{1}{2}x^2$ as candidate for Lyapunov's function. Then $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and $\dot{V}(t,x) = -(1+t)x^2 \leq -x^2(t) =: W_3(x)$. Then from theorem we have, that $X^* = 0$ is globally uniformly asymptotically stable.

## 2.2 Exponential stability

**Definition.** Point $X^* = 0$ is an exponentially stable EP of (3) if $\exists \lambda, c, k > 0$ s.t. $t \geq t_0 \geq 0$ and all $\|x_0\| < c$ follows $\|x(t)\| \leq K\|x(t_0)\|e^{\lambda(t-t_0)}$.

Remark: from exponential stability follows uniformly asymptotically stability.

**Lemma 2** (Auxiliary result)**.** Let $\dot{x}(t) = f(t, x(t))$, f scalar and $\dot{\xi}(t) \leq f(t, \xi(t))$ with $\xi(t_0) \leq x(t_0)$. Then $\xi(t) \leq x(t)\ \ \forall t \geq t_0$.

**Theorem 2.2.** Let $f : [0,\infty) \times D \to R^n$ be continuous and $x^* = 0 \in D$ be an EP.

If there is a differentiable function $V : [0,\infty) \times D \to R$ and constants $k_1, k_2, k_3, a > 0$ s.t.

1. $k_1||x||^a \leq V(t,x) \leq k_2\|x\|^a,\ \forall t \geq 0,\ \ x \in D$
2. $\dot{V}(t,x) \leq -k_3\|x\|^a$

then $x^* = 0$ is exponentially stable.

If $D = R^n$, then $X^*$ is globally exponential stable.

*Proof.* For $c > 0$ small enough, trajectories initialized in $\{x : k_2||x||^a < c\}$ remain bounded and in D. From 1) and 2) we can conclude $\dot{V} \leq -\frac{k_3}{k_2}V$. Then from previous Lemma $V(t,x(t)) \leq V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}$. Then $||x(t)|| \leq [from(1)] \leq \left(\frac{V(t,x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0,x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{k_2||x(t_0)||^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} ||x(t_0)||e^{-\frac{k_3}{k_2 a}(t-t_0)}$ □

**Example.** Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Here $V(t,x) = \frac{1}{2}x^2$ then $X^*$ is exponentially stable.

### 2.3 Comparsion function

**Definition.** A function $\alpha : [0, \delta) \to [0, \infty)$ is (of) "klass $K$" if it is continous, strictly increasing, and $\alpha(0) = 0$.

**Definition.** A function $\alpha : [0, \delta) \to [0, \infty)$ is "class $K_\infty$ if $\alpha in K$ and $\lim_{r\to\infty} \to \infty$.

**Example.** Function $\alpha(r) = \tan^{-1}(r)$ – class $K$

Function $\alpha(r) = r^k$ – class $K_\infty$

**Definition.** A function $\beta : [0, \delta) \times [0, \delta) \to [0, \infty)$ is "class $KL$ if it is continuous , $\beta(\cdot, s) \in K$ for all fixed $s$, and for each fixed $r$, $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s\to\infty} \beta(r, s) = 0$

**Example.** Function $\beta(x, s) = max(r, r^2)e^s$ belong class $KL$.

Properties of compasion functions:

- If $\alpha \in K$ on $[0, \delta)$, then $\alpha^{-1}$ is defined on $[0, \alpha(\delta))$ and $\alpha^-1 \in K$.
- If $\alpha \in K_\infty$, then $\alpha^{-1} \in K_\infty$
- If $\alpha_1, \alpha_2 \in K$, then $\alpha_1 \circ \alpha_2 \in K$ (same for $K_\infty$
- If $\alpha_1, \alpha_2 \in K$, $\beta \in KL$ then $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we conseider comparsion funcctions and stability definitions.

**Lemma 3.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists\alpha \in K$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall||x(t_0)|| < c$ and $||x(t)|| \leq \alpha(||x(t_0)||)$.

*(only sufficiency).* Given $\epsilon > 0$ choose $\delta < \min(c, \alpha^{-1}(\epsilon))$. Then from $||x(t_0)|| < \delta$ follows $||x(t)|| \leq \alpha(||x(t_0)||) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$. ☐

**Lemma 4.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists\beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

*(only sufficiency).* Let $\|x(t_0)\| < c$. Then $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x_{t_0}\|, 0)$. This gives us uniform stability. ☐

**Lemma 5.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists\beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

*(only sufficiency).* Let $||x(t_0)|| < c$. Then $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(||x_{t_0}||, 0)$. This gives us uniform stability. ☐

**Lemma 6.** The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable iff previous lemma holds for all $x_0 \in R^n$.

Now consider comparsion functios and Lyapunov functions

If $W : R^n \to R$ is continuous and positive definite, then $\forall r > 0 \quad \exists \alpha_1, \alpha_2 \in K$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in B_r(0) = \{x|||x|| \leq r\}$.

If $W$ is radially unbounded, then $\exists \alpha_1, \alpha_2 \in K_\infty$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in R^n$.

**Lemma 7** (Auxility)**.** Consider $\dot{y} = \alpha(y), \quad y(t_0) = y_0 > 0, \ \alpha \in K$. Then $\exists \beta \in KL$ s.t. $y(t) = \beta(y_0, t - t_0)$.

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t,x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where $W_1, W_2, W_3$ – continuous and positive defined.

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in K$ such that $\alpha_1(||x||) \leq V(t,x) \leq \alpha_2(||x||)$ and $\dot{V}(t,x) \leq -\alpha_3(||x||)$.

Proof uniform stability:

$||x(t)|| \leq \alpha_1^{-1}(V(t,x(t)) \leq [\alpha_1 in K] \leq \alpha_1^{-1}(V(t_0,x(t_0)) \leq \alpha_1^{-1}(\alpha_2(||x(t_0)||))$.

Proof uniform convergence

$\dot{V} \leq -\alpha_3(||x||) \leq -\alpha_3(\alpha_2^{-1}(V))$. We know, that $\alpha_3 \circ \alpha_2^{-1} \in K$. By comparsion lemma, $V(t,x(t)) \leq W(t)$, where $W$ solves $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$ with $W(t_0) = V(t_0, x(t_0))$. By auxility lemma $\exists \beta \in KL$ s.t. $V(t,x(t)) \leq \beta(V(t_0,x(t_0)), t-t_0)$, then $||x(t)|| \leq \alpha_1^{-1}(V(t,x(t))) \leq \alpha_1^{-1}(\beta(V(t_0,x(t_0)),t-t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(||x(t_0)||), t-t_0)) =: \bar{\beta}(||x(t_0)||, t-t_0)$. From this follows uniform asymptotic stability since $\bar{\beta} \in KL$.

## 2.4 Converse theorems

**Theorem 2.3.** Let $X^* = 0$ be an EP of $\dot{x}(t) = f(t,x(t))$ with $f : [0,\infty) \times R^n \to R^n$ continously differentiable and $\frac{\partial f}{\partial x}$ bounded in $R^n$, uniformly in t ($||\frac{\partial f}{\partial x}(t,x)|| \leq L$ for all $x \in R^n, \ t \geq 0, \ L > 0$.

If $x^* = 0$ is globally exponentially stale, then exists differentiable $V : [0,\infty) \times R^n \to R$ and $c_1, c_2, c_3, c_4 > 0$ s.t. $c_1||x||^2 \leq V(t,x) \leq c_2||x||^2$, $\dot{V}(t,x) \leq -c_3||x||^2$ and $\left\|\frac{\partial V}{\partial x}\right\| \leq c_4||x||$.

*Proof.* Let $\Phi(\tau;t,x)$ – solution to $\dot{x}(t) = f(t,x(t))$ which static at $(t,x)$.

$V(t,x) = \int_t^{t+\delta} \Phi^T(\tau;t,x)\Phi(\tau;t,x)d\tau, \quad \delta > 0$. Upper bound: $V(t,x) = \int_t^{t+\delta} ||\Phi(\tau;t,x)||_2^2 d\tau \leq$ [*exponential stability*] $\leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau ||x||_2^2 = \frac{k^2}{2\lambda}(1-e^{-2\lambda\delta})||x||_2^2$.

Lower bound: since $\left\|\frac{\partial V}{\partial x}\right\| \leq L$, then $||f(t,x)||_2 \leq L||x||_2$. Thus by comparation lemma $||\Phi(\tau;t,x)||_2^2 \geq ||x||_2^2 e^{-2L(\tau-t)}$. Set it in $V(t,x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau ||x||_2^2 = \frac{1}{2L}(1-e^{-2L\delta})||x||_2^2$.

Decrease conditions: $\dot{V}(t,x) = \cdots \leq -(1-k^2e^{-2\lambda\delta})||x||_2^2$. $\square$

# 3 System with inputs

Consider equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{4}$$

where $f : R^n \to R^n$.

Assumption: $f$ in localy Lipschitz.

Exageneous signa $u : R \to R^n$.

Input can be "bad" (disturbance) or "good" (control).

## 3.1 Input-to-state stability

Motivation: LTI system $\dot{x} = Ax + Bu, \quad x(0) = x_0$.

Solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$. If A is Hurwitz, then $||e^{At}|| \leq ce^{-\lambda t}$ for some $c, \lambda > 0$.

How large can x grow for some bounded $u$? $||x(t)|| \leq ||e^{At}||||x_0|| + \int_0^t ||e^{A(t-\tau)}||||B||||u(\tau)||d\tau \leq e^{-\lambda t}c||x_0|| + \int_0^t e^{-\lambda(t-\tau)}c||B||||u(\tau)||d\tau = ce^{-\lambda t}||x_0|| + (1-e^{-\lambda t})\ \frac{c}{\lambda}||B|| \sup_{\tau\in[0,t]} ||u(\tau)||$.

- $ce^{-\lambda t}||x_0||$ class $KL$ in $(||x_0||, t)$
- $(1-e^{-\lambda t})$ less than 1
- $\frac{c}{\lambda}||B|| \sup ||u(\tau)||$ class $K$

If $\sup_{\tau\in[0,t]} ||u(\tau)||$ is bounded than $\dot{x}$ remains bounded. Even more: the smaller $\sup_{\tau\in[0,t]} ||u(\tau)||$, the smaller $||x(t)||$.

**Definition.** System (4) is input-to-state stable (ISS) if $\exists\beta \in KL$, $\gamma \in K$ s.t. $\forall x_0 \in R^n$, $\forall t \geq 0$ follows $||x(t)|| \leq \beta(||x_0||, t) + \gamma(\sup_{\tau\in[0,t]} ||u(\tau)||)$.

Remarks:

- From ISS follows O-GAS (global assymptotical stability of $x = 0$ for $\dot{x} = f(x, 0)$
- $\gamma$ can be interpreted as "gain" w.r.t. $u$

- if $\lim_{t\to\infty} u(t) = 0$ then $\lim_{t\to\infty} x(t) = 0$

**Example.** Consider equation $\dot{x} = -x + xu$. System is O-GASS, not ISS (for example $u \equiv \alpha \Rightarrow \dot{x} = x(\alpha - 1)$ all solution diverge).

**Example.** Consider equation $\dot{x} = -3x + (1 + 2x^2)u$. System is O-GASS, not ISS (for example $u \equiv 1$, $x_0 = 2$, $x(t) = \frac{3-e^t}{3-2e^t}$ has a finite escape time.

**Theorem 3.1.** Suppose that there exists a continuously differentiable function $V : R^n \to R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x,u) \leq -\alpha_3(||x||)$, $\forall x : ||x|| \geq \rho(||u||)$. Then (4) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

*Proof.* Idea: same as Lyapunovs direct method when $x$ is "outside" of ball $\{x | ||x|| \leq \rho(||u||)\}$

TODO Picture □

**Example.** Consider equality $\dot{x} = -x^3 + u$. Let $V(x) = \frac{1}{2}x^2$, then $\dot{V} = -x^4 + xu = [0 < \Theta < 1] = -(1-\Theta)x^4 - \Theta x^4 + xu \leq -(1-\Theta)x^4$ for all $x : ||x|| \geq \left(\frac{||u||}{\Theta}\right)^{\frac{1}{3}}$. Thus, system is ISS with $\gamma(v) = \rho(v) = \left(\frac{v}{\Theta}\right)^{\frac{1}{3}}$.

Remarks:

- Existence of $V$ is both neccessary and sufficient for ISS;
- (**??**) is equivalent to $\frac{\partial V}{\partial x} f(x,u) \leq -\alpha_4(||x||) + \alpha_5(||u||), \forall x, u$ for some $\alpha_4, \alpha_5 \in K$;
- If $x_1 = 0$ is a globally asymptotically stable EP of $\Sigma_1$ and $\Sigma_2$ is ISS w.r.t. "input" $x_1$, then $(x_1, x_2) = (0,0)$ is a globally asymptotically stable EP for the cascaded system.

**Theorem 3.2.** Assume that:

- $f$ is globally Lipschitz;
- $x = 0$ is a globally exponentially stable EP for $\dot{x} = f(x, 0)$

Then the system (4) is ISS.

*Proof.* Sketch: $\exists$ continuous differentiable $V$:

$$c_1||x||^2 \leq V(x) \leq c_2||x||^2$$
$$\frac{\partial V}{\partial x} f(x,0) \leq -c_3||x||^2$$
$$||\frac{\partial V}{\partial x}|| \leq c_4||x||$$

Then:
$\frac{\partial V}{\partial x} f(x,u) = \frac{\partial V}{\partial x} f(x,0) + \frac{\partial V}{\partial x}(f(x,u) - f(x,0) \leq -c_3||x||^2 + c_4||x||L||u|| = -c_3(1-\theta)||x||^2 - \theta||x||^2 + c_4L||x||||u|| \leq -c_3(1-\theta)||x||^2$
if $||x|| \geq \frac{c_4 L}{\theta c_3}||u||$. □

### 3.2 Control Lyapunov functions

Motivation: Lyapunov theory for control systems.

(input affine systems)
$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u,$
$f : R^n \to R^n, g : R^n \to R^n, G : R^n \to R^{n\times m}$
$u : t \to u(t), R \to R^m$ is a control signal (decision variable).

**Definition.** A function $V : R^n \to R$ is a control Lyapunov function (CLF) if it's differentiable positive definite, radially unbounded and

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) < 0 \tag{5}$$

Remark:
Concept can be generalized to systems $\dot{x} = f(x, u)$. Then 5 becomes

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot f(x, u)) < 0$$

**Theorem 3.3** (Artstein)**.** There exists $k : R^n \to R^m$ (state feedback) which is continuous on $R^n \backslash \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff there exists a CLF.

How to find CLFs?

Proposition:
Condition (5) is equivalent to

$$\forall x \neq 0, \quad \frac{\partial V}{\partial x} G(x) = 0 \implies L_f V(x) < 0 \tag{6}$$

Remark:

$$\frac{\partial V}{\partial x} G(x) = (\nabla V(x) g_1(x), \ldots \nabla V(x) g_m(x)) =: L_G V(x)$$
$$(6) \iff \forall x \neq 0, \quad L_f V(x) \geq 0 \implies L_G V(x) \neq 0$$

*Proof.* $\Longleftarrow$:
Assume (6) holds. Then:

$$\inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) = \inf_u L_f V(x) + L_G V(x) u < 0$$

Why?

- If $L_G V(x) = 0$, then by (6) $L_f V(x) < 0$;
- If $L_G V(x) \neq 0$, then (at least) for one $i$ we have $\nabla V(x) \cdot g_i(x) \neq 0 \implies$ set $u_i = -c \nabla V(x) \cdot g_i(x)$.

$\Longrightarrow$:
If (5) holds for some $x$ with $L_G V(x) = 0$, then we must have $L_f V(x) < 0$. $\square$

Example (discontinuous control):

$$\dot{x} = \begin{cases} 1 - u, & u \geq 1 \\ -1 - u, & u \leq -1 \\ 0, & else \end{cases}$$

If you want to move the system you need to apply control $|u| \geq 1$.
Using

$$V(x) = \begin{cases} x + 1, & x > 0 \\ x - 1, & x \leq 0 \end{cases}$$

results in closed loop $\dot{x} = -x$ - asymptotically stable.
$V(x) = x^2$ is a CLF.

**Theorem 3.4.** There exists a continuous $k : R^n \to R^m$, smooth on $R^n \backslash \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff:

- there exists a (smooth)CLF V;
- $\forall \varepsilon > 0 \;\; \exists \delta > 0 : \;\; \forall x : 0 < ||x|| < \delta$
  $\exists u \in R^m : ||u|| < \varepsilon$ s.t. $L_f V(x) + L_G V(x) u < 0$

How to construct a globally stabilizing state feedback $k$ from knowledge of a CLF?

"Sontag's formula"
Fix $c \geq 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$k(x) = \begin{cases} -cb(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)}, & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases}$$

Proposition: Let $V : R^n \to R$ be a CLF and $k$ as above. Then $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$

*Proof.* $\dot{V} = L_f V(x) + L_G V(x) k(x) = a(x) - cb(x)^T b(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T b(x) = -cb(x)^T b(x) - \sqrt{a(x)^2 + (b(x)^T b(x))^2} < 0 \;\; \forall x \neq 0$ s.t. $L_G V(x) \neq 0$

$\dot{V} = L_f V(x) + L_G V(x) \cdot 0 < 0 \;\; \forall x \neq 0$ s.t. $L_G V(x) = 0$ (since $V$ is CLF)

$\Longrightarrow V$ - Lyapunov function $\Longrightarrow \ldots$ $\square$

Remarks:

- Sontag's formula is smooth on $R^n \backslash \{0\}$;
- Sontag's formula is continuous at $x = 0$ iff small control property holds.

$$\forall x \neq 0 : \inf_u \frac{\partial V}{\partial x} f(x, u) < 0 \ \dot{x} = f(x) + G(x)u$$

So this leads to

$$\forall x \neq 0 L_G V(x) = 0 \Rightarrow L_f V(x) \neq 0$$

Remark: The last formula is "optimal" if minimize:

$$\int_0^\infty \frac{1}{2} p(x) b(x)^T b(x) + \frac{1}{2p(x)} u^T u dt$$

$b(x) := (L_G V(x))^T$

where $c > 0$

$$p(x) = \begin{cases} c + \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} & b(x) \neq 0 \\ c & b(x) = 0 \end{cases}$$

It still works if $u = \lambda h(x)$ with $\lambda \in [\frac{1}{2}; \infty)$ is applied (large "gain margin")

# 4 Backstepping

Integrator backstepping

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1) x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{7}$$

where $x_1 \in \mathbb{R}^m,\ x_2,\ u \in \mathbb{R}$ (single input)

image to be inserted

Assumption: we know (smooth) "feedback" $\alpha_1 : \mathbb{R}^n \to \mathbb{R}$, and positive definite, differentiable $v_1 : \mathbb{R}^m \to \mathbb{R}$

s.t. $L_{f_1 + g_1 \alpha_1} V_1(x)$ is negative definite $\Rightarrow$ origin of $\dot{x_1} = f_1(x_1) + g_1(x_1)\alpha_1(x_1)$ is asymptotically stable

Goal: Compute feedback $u = k(x)$ which stabilises (7). Backstepping constructs $u = \alpha_2(x_1, x_2)$ s.t. $(e_1, e_2) = (x_1 - 0, x_2 - \alpha_1(x_1)) = 0$ error coordinates

Rewrite (7) :

$$\dot{x}_1 = f_1(x_1) + g_1 \alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1))$$

$$\dot{x}_2 = u$$

image to be inserted

In error coordinates

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 &= u - \dot{\alpha_1} = u - \frac{\partial \alpha_1}{\partial e_1}\dot{e}_1 = u - \frac{\partial \alpha_1}{\partial e_1}\end{aligned} \tag{8}$$

"backstepping" $\alpha_1$ through the integrator

Define $V_2(e_2) := \frac{1}{2}e_2^2$, and

$$V(e_1, e_2) = V_1(e_1) + V_2(e_2)$$

$$\dot{V}(e_1, e_2) = \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)\alpha_1(e_1)) + \frac{\partial V_1}{\partial e_1}g_1(e_1)e_2 + \frac{\partial V_2}{\partial e_2}(u - \dot{\alpha_1})$$

as far as $L_{f_1+g_1\alpha_1}V_1$ -negative definite and $\frac{\partial V_2}{\partial e_2} \to e_2$

Choose

$$u = (-\frac{\partial V_1}{\partial e_1}g_1(e_1) + \dot{\alpha_1})(\text{"}\textit{canaling terms}\text{"}) - k_2e_2(\text{"}\textit{stabilizing term}\text{"})k_2 > 0 \tag{9}$$

$\Rightarrow$ Then $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0, \ \forall(e_1, e_2) \neq 0$

$\Rightarrow$ Then $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0 \ \forall(e_1, e_2) \neq 0$

$\Rightarrow$ $(e_1, e_2) = (0, 0)$ is an asymptotically stable EP for (8) with $u$ as in (9)

Remark: $(e_1, e_2) \to (0, 0)$ doesnot necessarily imply that $(x_1, x_2) \to 0$ for $u = \alpha_2(x_1, x_2) = -\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha_1} - k_2(x_2 - \alpha_1(x_1))$

where $u \leftarrow$ (9) the original coordinates and $\dot{\alpha_1} \leftarrow \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + g_1(x_1)x_2)$

But $(x_1, x_2) = (0, 0)$ is asymptotically stable if $\alpha_1(0) = 0$ why? $(e_1, e_2) \to 0 \Rightarrow x_1 \to 0$ $x_2 \to \alpha_1(0) = 0$

**Example.**

$$\dot{x_1} = x_1x_2$$

$$\dot{x_2} = u$$

Choose $\alpha_1(x_1) = -k$ $(k > 0) \to \dot{x_1} = -kx_1 \Rightarrow V_1(x_1) = \frac{1}{2}x_1^2$

Then:

$$e_1 = x_1 - 0 \ \dot{e_1} = e_1(e_2 - k)$$

$$e_2 = x_2 + k \; e_2 = u$$

Backstepping yields: $u = -e_1^2 - k_2 e_2 \; k_2 > 0 \Rightarrow (e_1, e_2) = (0,0)$ is asymptotically stabilized

$(x_1, x_2) = (0, -k)$ is asymptotically stabilized

Can we choose different $\alpha_1$ s.t. $(x_1, x_2) = (0,0)$ is stabilized?

Yes, e.g.

$$\alpha_1(x_1) = -x_1^2 \Rightarrow \dot{x_1} = -x_1^3 \; V_1(x_1) = \frac{1}{2}x_1^2$$

So we have equations

$$e_1 = x_1 - 0 \; \dot{e_1} = e_1(e_2 - e_1^2)$$
$$e_2 = x_2 + x_1^2 \; \dot{e_2} = u + 2e_1^2(e_2 - e_1^2)$$

Backstepping results in

$$u = -e_1^2 - 2e_1^2(e_2 - e_1^2) - k_2 e_2, \; k_2 > 0 \Rightarrow (e_1, e_2) \to (0,0), \; (x_1, x_2) \to (0,0)$$

Generalization-1

$$\dot{x_1} = f_1(x_1) + g_1(x_1)x_2$$
$$\dot{x_2} = f_2(x_1, x_2) + g_2(x_1, x_2)u$$

Assumption: $g_2(x_1, x_2) \neq 0 \forall x_1, x_2 \Rightarrow$ Input transformation: $u = \frac{1}{g_2(x_1,x_2)}(V - f_2(x_1, x_2)) \Rightarrow \dot{x_1} = f_1(x_1) + g_1(x_1)x_2 \; \dot{x_2} = V \Rightarrow$ can apply integrator backstepping to determine $V$ results in

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)}(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha_1} - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2))$$

<u>Generalization 2:</u> (Backstepping through 2 integrators)

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1}$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3, \quad x_2, x_3 \in R$$
$$\dot{x}_3 = f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u, \quad u \in R$$

Assumption: $g_2, g_3$ nowhere zero.

Shown before: $\exists \alpha_2$: for $x_3 = \alpha_2(x_1, x_2)$ $(e_1, e_2) \to 0$
Thus $e_3 := x_3 - \alpha_2(x_1, x_2)$

Input transformation:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(V - f_3(x_1, x_2, x_3))$$

$\implies \dot{x}_1 = \ldots, \dot{x}_2 = \ldots, \dot{x}_3 = V \implies$ can apply backstepping once more.

In "error" coordinates:

$$\dot{e}_1 = f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1))$$

$$\dot{e}_2 = f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1$$

$$\dot{e}_3 = V - \dot{\alpha}_2$$

Define $V_3(e_3) = \frac{1}{2}e_3^2, V(e_1, e_2, e_3) = \sum\limits_{i=1}^{3} V_i(e_i)$
$\dot{V}(e_1, e_2, e_3) = \underline{\frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) + e_2(f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))}(e_3 + \underline{\alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1)} + e_3(V - \dot{\alpha}_2)$

All the underlined terms were designed (previously) to be $= L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 < 0$

So: $\dot{V}(e_1, e_2, e_3) = L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 + e_2g_2(e_1, e_2 + \alpha_1(e_1))e_3 + e_3(V - \dot{\alpha}_2)$

Structurally it is exactly the same as it was in backstepping through 1.

Choose:

$$V = \dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1)) - k_3e_3$$

$\dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1))$ - "cancelling terms".
$k_3e_3$ - "stabilizing term".

In original coordinates:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(\dot{\alpha}_2 - (x_2 - \alpha_1(x_1))g_2(x_1, x_2) - k_3(x_3 - \alpha_2(x_1, x_2)) - f_3(x_1, x_2, x_3))$$

We need $\alpha_1, \alpha_2$ to compute $u$.

General backstepping recursion:
Systems in "strict feedback form":
$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1}$
$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$
$\ldots$
$\dot{x}_k = f_k(x_1, \ldots x_k) + g_k(x_1, \ldots x_k)u, \quad x_2, \ldots x_k, u \in R$
$g_2, \ldots g_k$ nowhere zero, $f_i, g_i$ (sufficiantly) smooth, as it is needed in $\alpha_i$.

Backstepping recursion:

1. "Input data": a CLF $V_1$ for $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$ with a (smooth) feedback $u_1 = \alpha_1 x_1$ which as. stabilizes the origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$.

2. for $i = 2, \dots k$:
   construct a CLF $V_i(e_i) = \frac{1}{2}e_i^2, V = \sum_{j=1}^{i} V_j(e_j)$ and a feedback $\alpha_1$ which as. stabilizes origin of $(e_1, \dots e_i) = (x_1, x_2 - \alpha_1(x_1), \dots, x_i - \alpha_{i-1}(x_1, \dots x_{i-1}))$
   $$\alpha_i(x_1, \dots x_i) = \frac{1}{g_i}(\dot{\alpha}_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}} g_{i-1} - k_i(x_i - \alpha_{i-1} - f_i)$$

3. apply $u = \alpha_k(x_1, \dots x_k)$

Backstepping and CLFs:
Backstepping is sensitive to uncertainties in $f_i, g_i$ (due to cancelling terms)
$\implies$ Sontag's formula is more practical $\implies$ we can use it since $V$ is CLF.

Error system is input affine (using input transformation)
$\dot{e} = f(e) + g(e)V$

with $f(e) = \begin{pmatrix} f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dots \\ -\alpha_{k-1} \end{pmatrix}, g(e) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$

Claim:
$V(e) = \sum_{i=1}^{k} V_i(e_i)$ is a CLF.

*Proof.* For input affine systems we need to show $L_g V = 0 \implies L_f V < 0, \quad \forall e \neq 0$.
$\dot{V}(e) = L_{f_1 + g_1 \alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 + e_{k-1} g_{k-1}(\dots) e_k - e_k \dot{\alpha}_{k-1} + e_k u$.
Here $e_k u = L_g V$ and the rest is $L_f V$.
Assume $L_g V = 0 \iff e_k = 0$
$\implies L_f V = L_{f_1 + g_1 \alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 < 0 \ \ \forall e \neq 0$ with $e_k = 0$. $\square$

$\implies$ We can apply Sontag's formula to construct $V$.

This theory can be extended to systems with $x_2, \dots x_k, u \in R^m$ ("block backstepping").

# 5 Systems with inputs and outputs

Study/control systems $\dot{x} = f(x, u)$ with "output" $y(t) = W(x(t))$

## 5.1 Sliding mode control

Motivating example

$$\dot{x_1} = x_2$$

$$\dot{x_2} = u \Rightarrow \dot{y} = x_2 + u$$

$$y = x_1 + x_2$$

Choose:

$$u = \begin{cases} -x_2 - 1, & y > 0 \\ -x_2 + 1, & y < 0 \\ -x_2, & y = 0 \end{cases}$$

$$\Rightarrow \dot{y} = \begin{cases} -1, & y > 0 \\ +1, & y < 0 \\ 0, & y = 0 \end{cases}$$

Solutions(Laratheodory) are if $y(0) > 0$, then

$$y(t) = \begin{cases} y(0) - t, & t \leq y(0) \\ 0, & t > y(0) \end{cases}$$

If $y(0) < 0$, then

$$y(t) = \begin{cases} y(0) + t, & t \leq y(0) \\ 0, & t > -y(0) \end{cases}$$

Key property: choose $u$ s.t. $y(t)$ goes to zero in finite time $\Rightarrow x(t)$ tends $\{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = 0\}$ in finite time

Consider dynamics on $S$

$$\begin{cases} \dot{x_1} = x_2 (x_2 = -x_1 \text{ if } y = 1) = -x_1 \\ \dot{x_2} = u = -x_2 \end{cases}$$

globally as stable

Two "phases"

1. solutions converge to $S$ in finite time
2. solutions converge to zero ("on S") asymptotically

$\rightsquigarrow$ "sliding mode" control

Remark: in (1) "finite time convergence is crucial"

General procedure:

$$\dot{x} = f(x) + g(x)uy = h(x) = s(x)$$

$f : \mathbb{R}^n \to \mathbb{R}^n,\ y : \mathbb{R}^n \to \mathbb{R}^n,\ s : \mathbb{R}^n \to \mathbb{R}$

$u$ - scalar input, $s(x)$ - sliding

single input, single output

Assumptions: $y$ has relative degree 1, well - defined globally, i.e. $L_g s(x) \neq 0 \ \forall \in \mathbb{R}^n$

# 6 Exercises

## 6.1 Exercise 1

Problem 1:

*Proof.* For any $t \geq 0$, we have

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}(V \circ x)(t) = \langle \nabla V(x(t)), \frac{d}{dt}x(t) \rangle = \langle \nabla V(x(t)), f(x(t)) \rangle = L_f V(x(t))$$

□

Problem 2:

*Proof.*

**Lemma 8.** Given the assumptions in Problem 2, if there exists a solution $x : [0, +\infty] \to R^n, t \to x(t)$, of $\dot{x} = f(x)$ s.t. $x(t) \in K$ for any $t \geq 0$, where $k \subset R^n$ is a compact with $O \in K$ (O - origin), then $x(t) \xrightarrow{t \to +\infty} 0$.

Clearly, for any $c > 0, lev_{\leq c} V$ is positive invariant w.r.t $\dot{x} = f(x)$. Given $c > 0$, let $x_0 \in lev_{\leq c} V$, i.e., $V(x_0) \leq c$. Then, for any $t \geq 0$

$$V(x(t)) = V(x_0) + \int_0^t \frac{d}{dt} V(x(\tau)) d\tau < V(x_0) \leq c,$$

i.e. $x(t) \in lev_{\leq c} V$ for any $t \geq 0$.
Then, for any $x_0 \in lev_{\leq c} V$ there exists a solution $x : [0, +\infty] \to R^n$ of $\dot{x} = f(x)$ s.t. $x(t) \in lev_{\leq c} V$ for all $t \geq 0$. Clearly, $O \in lev_{\leq c} V$. We conclude by using the above Lemma ($K = lev_{\leq c} V$). □

Problem 3:

*Proof.* Let $r > 0$. By assumption, there exists $c > 0$ s.t. $\overline{B(0, r)} \subset lev_{\leq c} V$.
Since any bounded set $lev_{\leq c} V$ is a subset of the region of attraction, and since the sublevel sets are arbitrary large, $R^n$ is also the region of attraction.
A condition that ensures that for any $c > 0, lev_{\leq c} V$ is bounded is $V(x) \xrightarrow{||x|| \to +\infty} +\infty$. □

Problem 4:

*Proof.* Let $P : R^2 \to R^2$ be continuously differentiable. Consider

$$m\dot{v} = -g\nabla P(q).$$

Consider $x = (q, v), \dot{q} = v, \dot{v} = -\frac{g}{m}\nabla P(q)$. Let $H : R^2 \to R$ be defined by

$$H(q, v) = \frac{1}{2}||v||^2 + \frac{g}{m}P(q).$$

We have

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla H(q, v)$$

Since $P$ is positive definite, then $H$ is positive definite.
Then

$$L_{\begin{pmatrix} & I \\ -I & \end{pmatrix}\nabla H} H(q, v) = \langle \nabla H(q, v), \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla H(q, v)\rangle = 0 \quad \forall (q, v) \in R^2 \times R^2$$

$\implies$ the origin is stable. ☐

Problem 5:

*Proof.* For any $t \geq 0$, we have
$\frac{d}{dt}V(t, x(t)) = \frac{d}{dt}(V \circ (id_R, x))(t) = [id_R : R \to R, t- > t] = \langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \frac{d}{dt}(id_R(t), x(t))\rangle =$
$\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix}\rangle = \frac{\partial}{\partial t}V(t, x(t)) + \langle \frac{\partial}{\partial t}V(t, x(t)), f(t, x(t))\rangle = L_{\begin{pmatrix} 1 \\ f \end{pmatrix}} V(x(t)).$

$g(t, x(t)) := \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix}$ ☐

Problem 6:

*Proof.* Consider $\dot{x} = a\sin(\omega t), \quad x(0) = x_0 \in R \quad a, \omega > 0$.
This is solved by $x(t) = -\frac{a}{\omega}\cos(\omega t) + \frac{a}{\omega} + x_0$.
Clearly, $x$ is bounded on $[0, +\infty]$ since $x(t) \geq x_0$, and $x(t) \leq x_0 + 2\frac{a}{\omega}$ for any $t \geq 0$.
Choose $\varepsilon = \frac{a}{\omega}$ and $t_0 = 0$. Then $\forall \delta > 0 \ \exists x_0 \in B(0, \delta)$, namely $x_0$, s.t. $\exists t \geq t_0$, namely $t = \frac{\pi}{\omega}$, with $x(t) \notin B(0, \varepsilon) \ \ (x(\frac{\pi}{\omega}) = 2\frac{a}{\omega} > \varepsilon)$. ☐

Short notes:

Problem 7:
Take $V(t,x) = \frac{1}{2}x^2$.

Problem 8:
Take $V(t,x) = x_1^2 + (1+e^{-2t})x_2^2$.

## 6.2 Exercise 2

Problem 1:

*Proof.* a) Since $\alpha_1$ is continuous and strictly increasing:

$$\forall x, y \in [0,\delta), x < y \;\; \alpha_1(x) < \alpha_1(y)$$

$\implies \alpha_1$ is injective, i.e.

$$\forall x, y \in [0,\delta), x \neq y \implies \alpha_1(x) \neq \alpha_1(y).$$

Clearly, $\alpha_1 : [0,\delta) \to \alpha_1([0,\delta))$ is surjective, i.e.

$$\forall y \in \alpha_1([0,\delta)) \;\; \exists x \in [0,\delta) : \quad \alpha_1(x) = y$$

Thus $\alpha_1$ is bijective.
Define $\alpha_1^{-1} : [0, \alpha_1(\delta)) \to [0,\delta)$ by $\alpha_1^{-1}(\alpha_1(x)) = x$.

b) From a) we have $\alpha_3^{-1} \in K$. Since $\alpha_3 \in K_\infty, \alpha_3 {-1}$ is defined om $[0,+\infty)$ and $\alpha_3^{-1}(r) \xrightarrow{r\to\infty} \infty$

c) Let $\alpha = \alpha_1 \circ \alpha_2$. Then we have $\alpha(0) = \alpha_1(\alpha_2(0)) = 0$ and $\alpha(r) > 0$ whenever $r > 0$. Moreover, for any $x, y$:

$$x < y \implies \alpha_2(x) < \alpha_2(y) \implies \alpha(x) = \alpha_1(\alpha_2(x)) < \alpha_1(\alpha_2(y)) = \alpha(y)$$

It is continuous (as composition of continuous functions).

d) From c) we have $\alpha := \alpha_3 \circ \alpha_4 \in K, \alpha$ is defined on $[0,+\infty)$ since $\alpha_3, \alpha_4 \in K_\infty$ and

$$r \to +\infty \implies \alpha_4(r) \to +\infty \implies \alpha(r) \to +\infty$$

e) For each $s, r \mapsto \beta(\alpha_2(r), s)$ is of class $K$.
Thus $r \mapsto \alpha_1(\beta(\alpha_2(r), s)) \in K$.
For each $r, s \mapsto \beta(\alpha_2(r), s)$ decreases.
Hence, $s \mapsto \alpha_1(\beta(\alpha_2(r), s))$ decreases.
Moreover,

$$\alpha_1(\beta(\alpha_2(r), s)) \xrightarrow{s\to+\infty} 0$$

$\square$

Problem 3:

*Proof.* For $u = 0$ the origin is UGAS. Consider $V : [0, +\infty) \times R \to R, \quad (t, x) \mapsto \frac{1}{2}x^2$.
We have

$$\frac{\partial}{\partial t}V(t,x)+\frac{\partial}{\partial x}V(t,x)f(t,x,u) = (\sin(t)-2)x^2+xu \leq -x^2+|x||u| = -(1-\theta)x^2-\theta x^2+|x||u|, \;\; \theta \in (0,1)$$

Hence, whenever $|x| \geq \frac{|u|}{\theta}$, the system is ISS with $\gamma = \frac{r}{\theta}$. □

Problem 4:

*Proof.*

$$\dot{x} = -x + (x^2 + 1)d \tag{10}$$

$$\dot{x} = -2x - x^3 + (x^2 + 1)d \tag{11}$$

System (10): Clearly, the system is 0-GAS. However, for $d = 1$ and $x > 1$ we have $x^2 + 1 > x$.

$$f(x, 1) = -x + (x^2 + 1) > 0$$

and thus $\dot{x} > 0$. Hence, if $x(0) = x_0 > 1$, the solution diverges (in finite time).
$\implies$ System (10) isn't ISS.

System (11): It is 0-GAS. Moreover, for any finite $d$ there exists a "large" $x$ s.t.

$$2x + x^3 > (x^2 + 1)d$$

$$\implies f(x, d) = -2x - x^3 + (x^2 + 1)d < 0$$

and $\dot{x} < 0 \implies$ System 11 is ISS.
Consider $V : R \to R, x \mapsto \frac{1}{2}x^2$ s.t

$$V'(x)f(x, d) = -2x^2 - x^4 + x(x^2 + 1)d \leq -x^2 - x^2(x^2 + 1) + (x^2 + 1)|x||d|$$

Hence, whenever $|x| \geq |d|$,

$$V'(x)f(x, d) \leq -x^2$$

s.t. system (11) is ISS with $\gamma(r) = r$. □

Problem 5:

*Proof.*

$$\langle \nabla V(x), -\nabla V(x)+\delta u\rangle \leq -||\nabla V(x)||^2+|\langle \nabla V(x), \delta u\rangle| \leq [YI] \leq -||\nabla V(x)||^2+\frac{1}{2}||\nabla V(x)||^2+\frac{\delta^2}{2}||u||^2$$

Young's inequality:

$$\forall x, y : \quad |\langle x, y\rangle| \leq \varepsilon\frac{||x||^p}{p} + \frac{||y||^q}{\varepsilon q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \varepsilon > 0$$

Hence, whenever $||x|| > \frac{\delta}{\sqrt{c}}||u||, t \mapsto ||x(t)||$ is decreasing.
Moreover whenever $||x|| \geq \frac{\delta}{\sqrt{c\theta}}||u||, \theta \in (0, 1)$, we have $\langle \nabla V(x), -\nabla V(x)+\delta u\rangle \leq -\frac{c}{2}(1-\theta)||x||^2 \implies$ ISS. □

## 6.3 Exercise 3

Motivation: Lyapunov Theory

$$\dot{x} = f(x, u)$$

$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

**Definition.** (CLF) A function $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF if it is continuous differentiable, positive definite, radially unbounded and $\forall x \neq 0 \ \inf_u < \nabla V(x), f(x, u) >< 0$

In order to find CLFs, we restrict our analysis to input -affine systems

$$\dot{x} = f(x) + G(x)u$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, \ G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$

Proposition: A continuous, differentiable, positive definite and radially unbounded. $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF iff

$$\forall x \neq 0 \ L_G V(x) = 0 \Rightarrow L_f V(x) < 0$$

Image to be inserted

Problem 1

Consider $\dot{x} = cos(x) + (1 + e^x)u$ where $f(x) = cos(x)$- drift and $g(x) = 1 + e^x$

Let $V : \mathbb{R} \to \mathbb{R}, \ x \mapsto \frac{1}{2}x^2$. Clearly, continuous differentiable, positive definite and radially unbounded. Moreover, for any nonzero $x$, we have $L_G V(x) \neq 0$.

Thus, for any $x \neq 0$, there exists a control that readers $< \nabla V(x), f(x) + g(x)u >$ negative. Givn this CLF, there exists a state feedback $u = u(x)$, e.g.

$$u(x) = -\frac{kx + cos(x)}{1 + e^x}, \ k > 0$$

Problem

Consider

$$\dot{x_1} = -x_1^3 + x_2 e^{x_1} cos(x_2)$$
$$\dot{x_2} = x_1^5 sin(x_2) + u$$

Take $V : \mathbb{R}^2 \to \mathbb{R}, \ (x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2)$

For any $x \neq 0$, we have

$$\inf_{u \in \mathbb{R}} (L_f V(x) + L_G V(x)u) = \begin{cases} L_f V(x), & if \ L_G V(x) = 0 \\ -\infty & else \end{cases}$$

In particular,
$$L_f V(x) = \cdots = x_1(-x_1^3 + x_2 e_1^x cos(x_2)) + x_2 x_1^5 sin(x_2)$$
$$L_G V(x) = \cdots = x_2$$

However,
$$L_f V(x)|_{x_2=0} = -x_1^4 < 0 \ \forall x_1 \neq 0$$

Image to be inserted

Concluding that $V$ is a CLF.

Problem 2:

$\dot{x} = Ax + Bu$, input defined system where $(A, B)$ is stabilizable, there exists $K \in \mathbb{R}^{m\times n}$ s.t. $A+BK$ is Hurwitz (cf. KRT).The latter is equivalent to the existance $P = P^T > 0$ s.t. $P(A+BK)+(A+BK)^T P < 0$ (cf. Khalil theorem 4,6)

Let $V : \mathbb{R}^n \to \mathbb{R}, x \mapsto < x, Px >$. Moreover, $\forall x \neq 0 \exists u = Kx$ s.t. $< \nabla V(x), Ax + Bu >< 0$, since
$$< \nabla V(x), Ax + Bu > =^{u=Kx} < x, (P(A + BK) + (A + BK)^T P)x >< 0$$

In addition,
$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\|K\|} > 0 \ \forall x \neq 0, \ \|x\| < \delta \ \exists u = Kx \ \|u\| < \epsilon$$
s.t. $L_f V(x) + L_G V(x)u < 0$ since $\|u\| = \|Kx\| \leq \|K\|\|x\| < \|K\|\delta = \epsilon$

Problem 3

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be continuous, differentiable consider
$$m\dot{v} = -g\nabla P(q) + F, \ m, g > 0$$

a) Hamiltonian form. Let $x := (q, v)$. Then $\dot{x} = (-\frac{g}{m}\nabla P(q) + \frac{1}{m}F) = \begin{bmatrix} & I \\ -I & \end{bmatrix} \begin{bmatrix} \frac{g}{m}\nabla P(q) \\ V \end{bmatrix} + \begin{bmatrix} \\ \frac{1}{m}I \end{bmatrix} F = \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) + G(x)$ given $H(x) = \frac{1}{2}\|\nu\|^2 + \frac{g}{m}P(q)$

b) "CLF". Take $H$ as a CLF candidate. Then, for any $x$

$$< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) + G(x)F > = < \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) > + < \nabla H(x), G(x)F > = [< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla$$

Strictly speaking, $H$ is no CLF, but it reveals how to choose $F$ s.t. the origin is GAS.

For any point $x$ for which there exists no control $F$ s.t. $< \nabla H(x), \begin{bmatrix} & I \\ -I & \end{bmatrix} \nabla H(x) + G(x)F >< 0$

Choose $F = 0$. Why? Using the Krasovsky-Lasallle inv. principle, we conclude that the origin is GAS, since any solution in $\{x|\dot{H}(x) = 0\}$ verifies $v(t) \equiv 0$, implying $\dot{v}(t) \equiv 0$ s.t.

$$0 = -\frac{g}{m} \triangledown P(q(t)) + \frac{1}{m} P(t)$$

The last part equals 0. Since $F = 0$ (by choice) and $\triangledown P(q) = 0$ iff $q = 0$ we conclude that $\dot{H}(x) = 0$ can only be "maintained" at the origin.

Problem 4

Consider

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -ux_2 + u^3$$

show that $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$ is CLF and let $V : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\ddot{x} + u\dot{x} - u^3 = 0$$

For any $x$ and $u$, we have $< \triangledown V(x), f(x,u) >= \cdots = x_1(2x_2 - ux_2 + u^3) + x_2(x_2 - ux_2 + u^3) = x_1h_1 + x_2h_2$

Image to be inserted

Hence if $u < 0$ and $-u$ "large", then we can render $< \triangledown V(x), f(x,u) >< 0$.

## 6.4 Exercise 4

Consider

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1) + g_2(x_1, x_2)u \end{cases} \tag{12}$$

Using the "preliminary control"

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \check{u} \end{cases} \tag{13}$$

$$u = \frac{1}{g_2(x_1, x_2)}(\check{u} - f_2(x_1, x_2))$$

Idea: Look at the upper(-most) system only and consider $x_2$ as a "virtual control".

Assumptions: Suppose

- $\exists$ CLF $V_1$;
- $\exists$ (smooth) feedback $\alpha_1$ s.t. $L_{f_1+g_1\alpha_1}V_1 < 0$.

Now, add and subtract $g_1\alpha_1$ in 13 s.t.

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1)) \\ \dot{x}_2 = \check{u} \end{cases} \tag{14}$$

Next, introduce $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

$$\begin{cases} \dot{e}_1 = f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 = \check{u} - \dot{\alpha}_1(e_1) \end{cases} \tag{15}$$

Problem 1:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

*Proof.*

1. Choose "virtual control":

   $$x_2 = -(k+1)x_1 =: \alpha_1(x_1), \quad k > 0$$

   The origin of $\dot{x}_1 = -kx_1$ is GAS.
   (Take $V_1 : R \to R, \quad x_1 \mapsto \frac{1}{2}x_1^2$ s.t. $\dot{V}_1(x_1) = -kx_1^2 < 0$ for all $x_1 \neq 0$)

2. Error coordinates:
   Let $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

   $$\dot{e}_1 = -ke_1 + e_2$$

   $$\dot{e}_2 = u + (k+1)(-ke_1 + e_2)$$

3. "Composite CLF":
   Define $V : R \times R \to R, \quad (e_1, e_2) \mapsto V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

   $$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(u + (k+1)(-ke_1 + e_2) + e_1)$$

4. Choose control:
   Let $u = -e_1 - (k+1)(e_2 - ke_1) - ke_2$
   s.t. $\dot{V}(e_1, e_2) = -ke_1^2 - ke_2^2 < 0$ for all $(e_1, e_2) \neq (0, 0)$

Remark: The closed-loop system reads:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

□

Problem 2:

$$\dot{x}_1 = x_1(x_2 - k), \quad k > 0$$

$$\dot{x}_2 = u$$

*Proof.* 1. $x_2 = 0 =: \alpha_1(x_1)$
The origin of $\dot{x}_1 = -kx_1$ is GAS ($V_1(x_1) = \frac{1}{2}x_1^2$)

2. $(e_1, e_2) := (x_1, x_2)$ s.t.

$$\dot{e}_1 = e_1(e_2 - k)$$
$$\dot{e}_2 = u$$

3. $V(e_1, e_2) = V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(e_1^2 + u)$$

4. $u = -e_1^2 - ke_2$

□

Problem 3:

$$\dot{x}_1 = x_1(x_2 - k)$$
$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$
$$\dot{x}_3 = u$$

*Proof.* 1. From problem 2:
$\dot{x}_2 = x_2(x_3 - k) - x_1^2 = -x_1^2 - kx_2 = u$ in Problem 2.
The origin of

$$\dot{x}_1 = x_1(x_2 - k)$$
$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

is GAS.
And this is true for $x_3 = 0 =: \alpha_2(x_1, x_2)$.

2. $(e_1, e_2, e_3) := (x_1 - 0, x_2 - \alpha_1(x_1), x_3 - \alpha_2(x_1, x_2))$ s.t.

$$\dot{e}_1 = e_1(e_2 - k)$$
$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$
$$\dot{e}_3 = u$$

3. $V(e_1, e_2, e_3) = V_1(e_1) + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2$ s.t.

4. $u = -e_2^2 - ke_3$

□

Problem 4:

$$\dot{x}_1 = x_1(x_2 - k)$$
$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$
$$\dot{x}_3 = x_3(x_4 - k) - x_2^2$$
$$\dot{x}_4 = u$$

*Proof.* 1. Is GAS for

$$x_3(x_4 - k) - x_2^2 = -x_2^2 - kx_3$$

which is attained for $x_4 = 0 =: \alpha_3(x_1, x_2, x_3)$.

2.

$$\dot{e}_1 = e_1(e_2 - k)$$
$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$
$$\dot{e}_3 = e_3(e_4 - k) - e_2^2$$
$$\dot{e}_4 = u$$

...

3. $u = -e_3^2 - ke_4$

□

Problem 5:

$$\dot{x}_1 = x_1(x_2 - k)$$
$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$
$$\cdots$$
$$\dot{x}_i = x_i(x_{i+1} - k) - x_{i-1}^2$$
$$\cdots$$
$$\dot{x}_n = u$$

*Proof.* We will always have $u = e_{n-1}^2 - ke_n$.
Let $V : R \times \cdots \times R \to R, \quad (e_1, \dots e_n) \mapsto \sum_{i=1}^n V_i(e_i)$, where $V_i(e_i) = \frac{1}{2}e_i^2, \quad i = 2, \dots n.$
We have $\dot{V}(e_1, \dots e_n) = L_{f_1+g_1\alpha_1}V_1(e_1) - k\sum_{i=2}^{n-1} e_i^2 + e_n u + e_{n-1}g_{n-1}(x_1, \dots x_{n-1})e_n - e_n\dot{\alpha}_{n-1}(x_1, \dots x_{n-1})$.
We observe that for $\alpha_i$ being zero, the inequality

$$e_{n-1}g_{n-1}(x_1, \dots x_{n-1})e_n - e_n\dot{\alpha}_{n-1}(x_1, \dots x_{n-1}) + e_n u < 0$$

hence $e_{n-1}^2 e_n + e_n u < 0$ for non-zero $e$.
It is solved by $u = e_{n-1}^2 - ke_n, \quad k > 0.$ □

## 6.5 Exercise 5

Consider the SISO system

$$\dot{x} = f(x) + g(x)(u + \sigma(x))$$
$$y = s(x)$$

$f, g : R^n \to R^n, \quad \sigma : R^n \to R$ and bounded, $\quad s : R^n \to R$

Design steps for SMC:

1. If no output is provided, design a sliding surface $S := \{x \in R^n | s(x) = 0\}$ s.t.
   (a) the system has rel. degree one;
   (b) for $y(t) \equiv 0$, all solutions converge to the origin ("zero dynamics" have GAS origin)
2. Choose a control s.t. the sliding surface is reached (in finite time), e.g.

$$v(x) = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot sgn(s(x))), \quad \hat{u} > 0$$

Problem 1:

$$\dot{x}_1 = (x_2 - x_1)x_1^2$$
$$\dot{x}_2 = x_2 + u$$

Sliding surface $S$, $\quad s : R^2 \to R, \quad (x_1, x_2) \mapsto x_2$

*Proof.* (a) For the given S, we have $L_g s(x) = 1$ for any $x \in R^2$.
Moreover, from

$$\dot{s}(x) = L_f s(x) + L_g s(x) u$$

(we want $= 0$) we have that for

$$u = -\frac{L_f s(x)}{L_g s(x)} = -x_2$$

the "dynamics on $S$" (i.e. $x_2 = 0$) reduced to

$$\dot{\eta} = -\eta^3$$

whose origin is GAS.

(b) Consider

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot sgn(s(x))) = -x_2 - \hat{u} \cdot sgn(x_2), \quad \hat{u} > 0$$

such that $x(t)$ "tends to $S$" in finite time (phase 1). Moreover, "on S", $x(t)$ converges to the origin $t \to +\infty$ (phase 2).

□

Remark: Given a system in regular form

$$x = (\eta, \xi)^T$$
$$\dot{\eta} = f_1(\eta, \xi)$$
$$\dot{\xi} = f_2(\eta, \xi) + g_2(\eta, \xi) u$$

choose $s(x) = \xi - \Phi(\eta)$, s.t. $\Phi$ as. stabilizes $\dot{\eta} = f_1(\eta, \Phi(\eta))$.

Problem 2:

$$\dot{x}_1 = -x_1 \cos x_2 + x_1 x_2$$
$$\dot{x}_2 = x_1 \cos x_1 + \sigma(x) + u$$

*Proof.* (a) (For the design of sliding surface pretend that uncertainty $\sigma(x) = 0$)

Let $S := \{x \in R^2 | s(x) = 0\}$ be def. by $s : R^2 \to R, \quad (x_1, x_2) \mapsto x_2(-\Phi(x_1) = 0)$. We have $L_g s(x) = 1$ for all $x \in R^2$.

From

$$\dot{s}(x) = L_f s(x) + L_g s(x) u$$

(we want $= 0$) s.t. for $u = -\frac{L_f s(x)}{L_g s(x)} (= -x_1 \cos x_1)$ the "dynamics on $S$" (i.e. $x_2 = 0$) reads

$$\dot{\eta} = -\eta$$

whose origin is GAS.

(b) Take

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|) \cdot sgn(s(x)))(= -x_1 \cos x_1 - (\hat{u} + (x_1^2 + x_2^2)) \cdot sgn(x_2)), \quad \hat{u} > 0$$

Consider the Lyapunov(-like) function $V(x) = \frac{1}{2} s(x)^2$ s.t.

$$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x)))$$

Choosing $u$ as above

$\dot{V}(x) = s(x)(-(\hat{u} + \beta(x)|L_g s(x)|) \cdot sgn(s(x)) + \sigma(x) L_g s(x)) \leq -(\hat{u} + \beta(x)|L_g s(x)|)|s(x)| + |\sigma(x)||L_g s(x)||s(x)| \leq -\hat{u}|s(x)| < 0$ for $s(x) \neq 0$

$\square$

Problem 3:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^3 + \sigma(x) + u$$

$s(x) = x_2 + x_1, \quad u = -x_2 + x_1^3 - 2 \cdot sgn(s(x))$

*Proof.* (a) Given $S$, we have $L_g s(x) = 1$ for all $x \in R^2$. The "dynamics on $S$" (i.e. $x_1 + x_2 = 0$) reads

$$\dot{\eta}_1 = -\eta_1$$

$$\dot{\eta}_2 = -\eta_2$$

whose origin is GAS.

(b) Take $V(x) = \frac{1}{2} s(x)^2$ s.t.

$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x))) \leq -\hat{u}|L_g s(x)||s(x)| + |\sigma(x)||L_g s(x)||s(x)| \leq [|\sigma(x)| \leq c] \leq -(\hat{u} - c)|L_g s(x)||s(x)|$.

Hence, for $c < \hat{u} = 2$ there exists $\varepsilon > 0$ s.t. $\dot{V}(x) \leq -\varepsilon |s(x)| < 0$ for $s(x) \neq 0$

$\square$

### 6.6 Exercise 6

Problem 1:

$$\dot{x} = xu(x^2 + u)$$

$$\dot{y} = h(x)$$

$s : R \times R \to R, \quad (u, y) \mapsto uy^2 + u^2 y$
$S : R \to R, \quad x \mapsto \frac{x^2}{2}$

*Proof.* Clearly, S is non-negative. Moreover:
$\dot{S}(x) = x^2 u(x^2 + u) = x^4 u + x^2 u^2 = [h(x) = x^2] = s(u, x^2)$
for all $x, u \in R$ with $h : R \to R, x \mapsto x^2$. $\square$

Problem 2:

$$\dot{x} = u, \quad x(0) = x_0$$

$$y = x$$

$s : R^n \times R^n \to R, \quad (u, y) \mapsto < u, y >$

*Proof.* For any $x_0 \in R^n$, we have

$$S_a(x_0) = \sup_{u:[0,t]\to R^n,\ t\geq 0,\ x(0)=x_0} \left(-\int_0^t < u(\tau), y(\tau) > d\tau\right) =$$

$$= \sup_{-//-} \left(-\frac{1}{2}\int_0^t \frac{d}{d\tau}||x(\tau)||^2 d\tau\right) = \sup_{-//-} \left(-\frac{1}{2}||x(t)||^2 + \frac{1}{2}||x(0)||^2\right) \leq \frac{1}{2}||x_0||^2$$

$\implies$ av. storage is finite $\implies$ system is dissipative. Moreover, we have for any $x_0 \in R^n$,

$$S_r(x_0) = \inf_{u:[-t,0]\to R^n,\ t\geq 0,\ x(-t)=0,\ x(0)=x_0} \int_{-t}^0 < u(\tau), y(\tau) > d\tau = \inf_{-//-} \left(\frac{1}{2}||x_0||^2 - \frac{1}{2}||x(-t)||^2\right) = \frac{1}{2}||x_0||^2$$

($S_a = S_r \implies$ this is a unique stor. func.)
Hence the (lossless) system is reachable (from 0 to any $x_0$). $\square$

Problem 3:

*Proof.* Consider the Lyapunov func. cand. $V(x) = S_1(x_1) + S_2(x_2)$ s.t.
$\dot{V}(x) \leq S_1(u_1, y_1) + S_2(u_2, y_2) = S_1(u_1, y_1) + S_2(y_1, -u_1) = 0 \implies$ origin is stable. $\square$

Remark: the above problem captures many stability results (in the frequency domain). Particular choices of supply rates are:

- $s_i(u_i, y_i) = ||u_i||^2 - ||y_i||^2, i = 1, 2$ (small-gain theorem);

- $s_i(u_i, y_i) =< u_i, y_i >, i = 1, 2$ (positive operator theorem);
- $s_1(u_1, y_1) =< u_1 + ay_1, u_1 + by_1 >$
  $s_2(u_2, y_2) = -ab < u_2 - \frac{1}{a}y_2, u_2 - \frac{1}{b}y_2 >$ (conic operator theorem).

Problem 4:

$$\dot{x} = f(x) + G(x)u$$

$$y = h(x)$$

$s : R^m \times R^m \to R, \quad (u, y) \mapsto ||u||^2 - ||y||^2$

*Proof.* Take $V = S$ s.t.

$$\dot{V}(x) \leq ||u||^2 - ||h(x)||^2, \ \forall x \in R^n, \ \forall u \in R^m$$

Then the (continuous) state feedback $u = \gamma h(x)$ for some $|\gamma|^2 < 1$, s.t.

$$\dot{V}(x) \leq (|\gamma|^2 - 1)||h(x)||^2 < 0, \ \forall x \neq 0$$

□

Problem 5:

*Proof.* Take $S(x) =< x, P_x >$ s.t.

$$\dot{S}(x) =< x, (PA + A^T P)x > +2 < x, PBu >$$

Add and subtract $\gamma^2||u||^2$ and $\frac{1}{\gamma^2}||B^T Px||^2$.

$$\dot{S}(x) =< x, (PA + A^T P + \frac{1}{\gamma^2}PBB^T P)x > +\gamma^2||u||^2 - \gamma^2||u\frac{1}{\gamma^2}B^T Px||^2$$

Add and subtract $||y||^2$.

$$\dot{S}(x) =< x, (PA + A^T P + \frac{1}{\gamma^2}PBB^T P + C^T C)x > +\gamma^2||u||^2 - ||y||^2 - \gamma^2||u - \frac{1}{\gamma^2}B^T Px||^2$$

$$\dot{S}(x) \leq \gamma^2||u||^2 - ||y||^2$$

□

## 6.7 Exercise 7

**Definition.** A mapping $\Phi : R \to R, \ u \mapsto \Phi(u)$, belongs to the sector

- $[0, +\infty]$ if $u\Phi(u) \geq 0, \ \forall u \in R$;
- $[\alpha, +\infty]$ if $u(\Phi(u) - \alpha u) \geq 0, \ \forall u \in R$ and some $\alpha \in R$;

- $[0, \beta]$ if $\Phi(u)(\Phi(u) - \beta u) \leq 0, \ \forall u \in R$ and some $\beta \in R$;
- $[\alpha, \beta]$ if $(\Phi(u) - \alpha u)(\Phi(u) - \beta u) \leq 0, \ \forall u \in R$ and some $\alpha, \beta \in R$;

Notation: we write, e.g., $\Phi \in [0, +\infty]$.

Problem 1:

$$\dot{x} = x^3 - kx + u, \ k > 0$$

$$y = x$$

*Proof.* Take, e.g., $S : R \to R, x \mapsto \frac{x^2}{2}$ $(S \geq 0)$ s.t.

$$\dot{S}(x) = x^2(x^2 - k) + yu \leq yu$$

whenever $x \in [-\sqrt{k}, \sqrt{k}]$.

Let $\bar{x} \in R$ and take $u = -\bar{x}^3 + k\bar{x}$ with init. condition $x(0) = \bar{x}$, s.t. we have $x(t) = \bar{x}$ for all $t \geq 0$. If the system is passive, then along this (constant) solution we must have

$$S(x(t)) - S(\bar{x}) \leq \int_0^t u(\tau)y(\tau)d\tau, \ t \geq 0$$

This inequality, however, is violated for $\bar{x} \notin [-\sqrt{k}, \sqrt{k}]$ and hence $[-\sqrt{k}, \sqrt{k}]$ must be the largest interval. $\square$

Problem 2:

$$\dot{x} = -x + \frac{1}{\beta}h(x) + u, \ \beta > 0$$

$$y = h(x)$$

$S(x) = \int_0^x h(\sigma)d\sigma, \ h \in [0, \beta]$

*Proof.* Clearly, we have $S \geq 0$ since $h \in [0, \beta]$.
Moreover,

$$\dot{S}(x) = S'(x)\dot{x} = \dot{x}\frac{d}{dx}\int_0^x h(\sigma)d\sigma = h(x)\dot{x} = \frac{1}{\beta}h(x)(h(x) - \beta x) + yu \leq yu$$

since $h \in [0, \beta]$. $\square$

Problem 3:

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + kx_2 + u, \ k > 0 \\ y = x_2 \end{cases}$$

*Proof.* Take $S : R^2 \to R,\ (x_1, x_2) \mapsto \frac{x_1^2}{2} + \frac{x_2^2}{2}$ s.t. $\dot{S}(x) = uy + ky^2$.
Let $u = -\Phi(y),\ \Phi : R \to R$ satisfying $\Phi \in [l, +\infty]$ for some $l > k$ $(\nu_2 + \rho_1 > 0)$ s.t.

$$\dot{S}(x) = -y\Phi(y) + ky^2 \leq -(l-k)y^2$$

Since the system $H_1$ is ZSO the origin is GAS. □

Problem 4:

*Proof.* Take $S(x) = S_1(x_1) + S_2(x_2)$ s.t.

$$\dot{S}(x) \leq < u_1, y_1 > -\rho_1||y_1||^2 - \nu_1||u_1||^2 + < u_2, y_2 > -\rho_2||y_2||^2 - \nu_2||u_2||^2$$

Using that

$$< u_1, y_1 > + < u_2, y_2 > = < u - y_2, y_1 > + < v + y_1, y_2 > = < u, y_1 > + < v, y_2 >$$

and

$$||u_1||^2 = ||u||^2 - 2 < u, y_2 > + ||y_2||^2$$
$$||u_2||^2 = ||v||^2 + 2 < v, y_1 > + ||y_1||^2$$

we obtain
$\dot{S}(x) = - < \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} (\nu_2 + \rho_1)I_m & \\ & (\nu_1 + \rho_2)I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} > - < \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \nu_1 I_m & \\ & \nu_2 I_m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} > + < \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} I_m & 2\nu_1 I_m \\ -2\nu_2 I_m & I_m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} > \leq [Coshi - Schwarz] \leq -a||(y_1, y_2)||^2 + b||(u, v)||||(y_1, y_2)|| + c||(u, v)||^2$
with $a = \min\{\nu_2 + \rho_1, \nu_1 + \rho_2\} > 0,\ b = ||N|| \geq 0$ and $c = ||M|| \geq 0$.
Hence,
$\dot{S}(x) \leq -\frac{1}{2a}(b||(u, v)|| - a||(y_1, y_2)||)^2 + \frac{b^2}{2a}||(u, v)||^2 - \frac{a}{2}||(y_1, y_2)||^2 + c||(u, v)||^2 \leq \frac{b^2 + 2ac}{2a}||(u, v)||^2 - \frac{a}{2}||(y_1, y_2)||^2$ □

Problem 5:

*Proof.* Take $V(x) = < x, Px >$ s.t.

$$\dot{V}(x) = < x, (PA + A^T P)x > -2\Phi(y) < x, PB >$$

Add and subtract $2\Phi(y)^2$ and $2\Phi(y)BCx$ yields
$\dot{V}(x) = -\varepsilon < x, Px > - < x, L^T Lx > -2\Phi(y) < x, PB - BC^T > -2\Phi(y)^2 + 2\Phi(y)(\Phi(y) - By) = -\varepsilon < x, Px > -|Lx - \sqrt{2}\Phi(y)|^2 + 2\Phi(y)(\Phi(y) - By) \leq -\varepsilon < x, Px >$. □