# 1 Intro

Goals of Course

- overview over moder nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control method learn the mathmatic basis

Differential equations $\dot{x} = f(x)$

Nonlinear differential eqution $\dot{x} = f(t, x)$

System with input $\dot{x} = f(x, u)$

System with input and output

$$\dot{x} = f(x, u)$$

$$y = g(x, u) \tag{1}$$

Input-output methods

Scope


[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997

## 2 Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)),\ x(0) = x_0 \tag{2}$$

Where $f : D \to R^n,\ D \subset R^n$ is open, [here we should explain, what means open set].

Solution to 2 $x : I_{x_0} \to D,\ t \to x(t)$ is differentiable

Interval existence solution

Questions:

- # existence of solution
- # "how large"is $I_{x_0}$
- # uniquence of solution

Usaly we will add some restrictions on $f$ functions, like continuous.

### 2.1 Existence of solutions

Function $f : D \to R^n$ is contiuous at $x' \in D$ if for $\forall \epsilon > 0\ \exists \delta > 0$ such that for $\forall x \in D$, $\|x - x'\| < \delta => \|f(x) - f(x')\| < \epsilon$

Function $f : D \to R^n$ is continuous on D if it's continuous at $\forall x' \in D$

If $f : D \to R^n$ continuous, then for each $x_0 \in D\ \exists x : (-\epsilon, \epsilon) \to D,\ \epsilon > 0$ satisfying (2).

Further, given a compact sed $U \subset D$, then $\exists \alpha > 0$ s.t. $\forall x_0 \in U\ \exists x : (-\epsilon, \epsilon) \to D$ satisfying (2).

Consider equation $\dot{x}(t) = x(t)^2,\quad x(0) = x_0 = 0$. Solution $x(t) = -\frac{1}{t-c},\quad c = \frac{1}{x_0}$. In this example solution exist in interval $(-c, c)$.

But, what about the number of solutions? Which conditions we should add to garanty uniquence of solution?

### 2.2 Uniquence of solutions

Function $f : D \to R^n$ is locally Lipshitz (continuous???) on D if $\forall x \in D$ there is a neighborhood $N(x) \subset D$ and $\exists L > 0$ s.t.

$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2|| \tag{3}$$

For all $x_1, x_2 \in N$.

- Lipschiz on $W \in D$ if (3) holds $\forall x_1, x_2 \in W$ (with same L)
- globally Lipschitz if (3) holds $\forall x_1, x_2 \in R^n$ (with same L)

We have

# localy Lipschitz functions are continuous

# differenciable functions are locally Lipschitz

# locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

Suppose that $0 \leq \phi(t) \leq c + L\int_0^t \phi(\tau)d\tau,\ c, L > 0,\ \phi$ – continuous. Then $\phi(t) \leq c \exp Lt$. Proof. $c + L\int_0^t \phi(\tau)d\tau := \psi(t)$, $\dot{\psi}(t) = L\phi(t) \leq L\psi(t)$.

Consider $\frac{d}{dt}\left(\psi(t)\exp -LT\right) = \exp -Lt\dot{\psi}(t) - L\psi(t)\,() \leq 0$, thus $\psi(t)\exp -LT$ is decreased, and as a result we have $\phi(t)\exp -Lt \leq \psi(t)\exp -Lt \leq \psi(0) = c$

If function $f : D \to R^n$ is localy Lipschitz then for $\forall x_0 \in D\ \ \exists! x : (-\epsilon, \epsilon) \to D,\ \epsilon > 0$ satisfying (2).

Proof:

* existence from Piano theorem

Proof of uniqueness

Consider two solutions $x_1(.)$ and $x_2(.)$ to (2). $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$, $x_1(0) = x_2(0)$. Then we can integrate equality: $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau))d\tau$. $|x_1(t) - x_2(t)| \leq \int_0^t |f(x_1(\tau)) - f(x_2(\tau))|d\tau \leq L\int_0^t |f(x_1(\tau)) - f(x_2(\tau))|d\tau$. Now we can apply Cromwall's lemma with $c = 0$ and $\phi(t) = |x_1(t) - x_2(t)|$, then $\phi(t) \leq 0$, then $x_1(t) = x_2(t),\ \forall t \in (-\epsilon, \epsilon)$

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$Solutions\ x(t) = \begin{cases} \frac{1}{4}(t-c)^2, & \text{if } t \geq c \geq 0 \\ 0, & \text{else} \end{cases}$$

Global existance & uniqueness

- sufficient condition: $f$ globally Lipschitz
- another sufficient condition: solution entirely lies in a coplex set
- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff $\exists$ solution $V : R^n \to R \geq 0$ s.t. $\frac{\partial V}{\partial x} f(x) \leq V(x)$, $\forall x \in R^n$

### 2.3 Lyapunov stability

If functions $\dot{V}(x) < 0$, $\forall x \in D$
$\{0\}$, then $x^*$ is asymptotically stable.

Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \leq \epsilon, \; \forall t \geq 0$.

Equilibrium point $x = 0$ is asymptotikaly stable if it stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t \to \infty} x(t) \to 0$.

Let $x^* = 0 \in D$ be an equilibrium point of (2), i.e., $f(0) = 0$. Let $f : D \to R^n$ is continious. If there exist a differetiable $V : D \to R$ s.t.

1. $V(x^*) = 0$, $V(x) > 0$, $\forall x \in D$
   $\{0\}$
2. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$, $\forall x \in D$

then $x^* = 0$ is stable.

Proof. Fix compact $U = \{x : V(x) \leq c\}$ s.t. $U \in D$. By Piano: exist $\alpha > 0$ s.t. any solution $x$ with $x_0 \in U$ exists at least on the interval $[0, \alpha)$.

TODO proof is not full

Lyapunovs direct method gives us:

- stability
- convergence (if $V < 0$)
- subset of the region of attraction (all compact $U = \{x : V(x) \leq c\} \in D$)
- existance of solution for all times

# 3 Nonlinear systems

In this section we consider function $f : R \times D \to R^n$, where $D \subseteq R^n$, and $D$ is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \tag{4}$$

The origin $x^* \in D$ is an equilinrium point for (4), if $f(t, 0) = 0, \quad \forall t \geq 0$.

Remark: EP (equilibrium point) $x^* = 0$ can be translation of a nonzero solution.

Suppose $\overline{y}$ is a solution of $\dot{y} = g(t, y)$.

Change of coordinates: $x(t) = y(t) - \overline{y}(t)$, then $\dot{x}(t) = \dot{y}(t) - \dot{\overline{y}}(t) = g(t, x(t) + \overline{y}(t)) - \dot{\overline{y}}(t) := f(t, x(t))$. Since $\dot{\overline{y}}(t) = g * t, \overline{y}(t))$, then $f(t, 0) = 0, \ \forall t \geq 0$.

Existance and uniquence of solution to (4):

- if $f$ continuous, then exist local colution
- if $f$ continuous and locally Lipschitz in $x^*$, then exist local uniq solution

Now we need new stability definitions.

Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \geq 0, \ \exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0 \ \ \exists \delta > 0$, s.t $\forall t_0 \geq 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \geq t_0$.

Point $x^* = 0$ asymptoticaly stable if it is stable and $\forall t_0 \geq 0 \ \ \exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t \to \infty} ||x(t)|| \to 0$.

Point $x^* = 0$ uniformly asymptoticaly stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $\lim_{t \to \infty} ||x(t)|| \to 0$.

Convergence: $\forall \eta > 0 \ \ \forall t_0 \geq 0, \exists T > 0$ such that $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

Uniform convergence: $\forall \eta > 0 \ \ \exists T > 0$ such that $\forall t_0 \geq 0$ and $\forall t \geq t_0 + T$ follows $||x(t)|| < \eta$.

Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0 \tag{5}$$

Solution $x(t) = x(t_0)\frac{1+t_0}{1+t}$. It is uniformly stable, because we can choose $\delta = \epsilon$. But does $x(t)$ convergence uniformly? Answer is no.

Point $x^* = 0$ is globaly uniformly asymptotically stable if it is uniformly stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta \quad \exists T > 0$ such that $\forall t_0 \geq 0$ from $||x_0|| < c$ follows $\|x(t)\| < \eta$, $\forall t \geq t_0 + T$.

### 3.1 Lyapunov's direct method

Consider some function $V : [0, \infty) \times D \to R$, $(t, x) \to V(t, x)$ such that $\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x)$.

Let $f : [0, \infty) \times D \to R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \to R$ with:

- $W_1(x) \leq V(t, x) \leq W_2(x)$, $\forall t \geq 0,\ x \in D$
- $\dot{V}(t, x) \leq 0$, $\forall t \geq 0,\ x \in D$

where $W_1, W_2 : D \to R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.

If further $\dot{V}(t, x) \leq -W_3(x)$, $\forall t \geq 0,\ x \in D$ with $W_3 : D \to R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.

If $D = R^n$ and $W_1$ is radialy unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

Consider the equation $\dot{x}(t) = -(1 + t)x(t)$.

Check function $V(t, x) = \frac{1}{2}x^2$ as candidat for Lyapunov's function. Then $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and $\dot{V}(t, x) = -(1 + t)x^2 \leq -x^2(t) =: W_3(x)$. Then from theorem we have, that $X^* = 0$ is globally uniformly asymptotically stable.

### 3.2 Exponential stability

Point $X^* = 0$ is an exponentially stable EP of (4) if $\exists \lambda, c, k > 0$ s.t. $t \geq t_0 \geq 0$ and all $\|x_0\| < c$ follows $\|x(t)\| \leq K\|x(t_0)\|e^{\lambda(t-t_0)}$.

Remark: from exponential stability follows uniformly asymptoticaly stability.

Let $\dot{x}(t) = f(t, x(t))$, f scalar and $\dot{\xi}(t) \leq f(t, \xi(t)$ with $\xi(t_0) \leq x(t_0)$. Then $\xi(t) \leq x(t) \quad \forall t \geq t_0$.

Let $f : [0, \infty) \times D \to R^n$ be continuous and $x^* = 0 \in D$ be an EP.

If there is a differentiable function $V : [0, \infty) \times D \to R$ and constants $k_1, k_2, k_3, a > 0$ s.t.

1. $k_1||x||^a \leq V(t, x) \leq k_2\|x\|^a$, $\forall t \geq 0,\ x \in D$
2. $\dot{V}(t, x) \leq -k_3\|x\|^a$

then $x^* = 0$ is exponentially stable.

If $D = R^n$, then $X^*$ is globally exponential stable.

Proof. For $c > 0$ small enough, trajectories initialized in $\{x : k_2||x||^a < c\}$ remain bounded and in D. From 1) and 2) we can conclude $\dot{V} \leq -\frac{k_3}{k_2}V$. Then from previous Lemma $V(t, x(t)) \leq V(t_0, x(t_0)) \exp -\frac{k_3}{k_2}(t - t_0)$. Then $||x(t)|| \leq [from 1)] \leq \left(\frac{V(t,x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0,x(t_0)) \exp -\frac{k_3}{k_2}(t-t_0)}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{k_2||x(t_0)||^a \exp -\frac{k_3}{k_2}(t-t_0)}{k_1}\right)^{\frac{1}{a}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} ||x(t_0)|| \exp -\frac{k_3}{k_2 a}(t - t_0)$

Consider the equation $\dot{x}(t) = -(1 + t)x(t)$.

Here $V(t, x) = \frac{1}{2}x^2$ then $X^*$ is exponentially stable.

### 3.3 Comparsion function

A function $\alpha : [0, \delta) \to [0, \infty)$ is (of) "klass $K$"if it is continous, strictly increasing, and $\alpha(0) = 0$.

A function $\alpha : [0, \delta) \to [0, \infty)$ is "class $K_\infty$ if $\alpha in K$ and $\lim_{r\to\infty} \to \infty$.

Function $\alpha(r) = \tan^{-1}(r)$ – class $K$

Function $\alpha(r) = r^k$ – class $K_\infty$

A function $\beta : [0, \delta) \times [0, \delta) \to [0, \infty)$ is "class $KL$ if it is continuous , $\beta(\cdot, s) \in K$ for all fixed $s$, and for each fixed $r$, $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s\to\infty} \beta(r, s) = 0$

Function $\beta(x, s) = max(r, r^2) \exp s$ belong class $KL$.

Properties of compasion functions:

- If $\alpha \in K$ on $[0, \delta)$, then $\alpha^{-1}$ is defined on $[0, \alpha(\delta))$ and $\alpha^-1 \in K$.
- If $\alpha \in K_\infty$, then $\alpha^{-1} \in K_\infty$
- If $\alpha_1, \alpha_2 \in K$, then $\alpha_1 \circ \alpha_2 \in K$ (same for $K_\infty$
- If $\alpha_1, \alpha_2 \in K$, $\beta \in KL$ then $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we conseider comparsion funcctions and stability definitions.

The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists\alpha \in K$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall||x(t_0)|| < c$ and $||x(t)|| \leq \alpha(||x(t_0)||)$.

Proof. (only sufficiency) Given $\epsilon > 0$ choose $\delta < \min(c, \alpha^{-1}(\epsilon))$. Then from $||x(t_0)|| < \delta$ follows $||x(t)|| \leq \alpha(||x(t_0)||) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$.

The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists\beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

Proof. (only sufficiency) Let $\|x(t_0)\| < c$. Then $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x_{t_0}\|, 0)$. This gives us uniform stability.

The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$.

Proof. (only sufficiency) Let $||x(t_0)|| < c$. Then $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $||x(t)|| \leq \beta(||x(t_0)||, t - t_0) < \beta(||x_{t_0}||, 0)$. This gives us uniform stability.

The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable iff previous lemma holds for all $x_0 \in R^n$.

Now consider comparsion functios and Lyapunov functions

If $W : R^n \to R$ is continuous and positive definite, then $\forall r > 0 \;\; \exists \alpha_1, \alpha_2 \in K$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in B_r(0) = \{x|||x|| \leq r\}$.

If $W$ is radially unbounded, then $\exists \alpha_1, \alpha_2 \in K_\infty$ s.t. $\alpha_1(||x||) \leq W(x) \leq \alpha_2(|x||)$ for all $x \in R^n$.

Consider $\dot{y} = \alpha(y), \;\; y(t_0) = y_0 > 0, \; \alpha \in K$. Then $\exists \beta \in KL$ s.t. $y(t) = \beta(y_0, t - t_0)$.

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t, x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where $W_1, W_2, W_3$ – continuous and positive defined.

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in K$ such that $\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)$ and $\dot{V}(t, x) \leq -\alpha_3(||x||)$.

Proof uniform stability:

$||x(t)|| \leq \alpha_1^{-1}(V(t, x(t)) \leq [\alpha_1 in K] \leq \alpha_1^{-1}(V(t_0, x(t_0)) \leq \alpha_1^{-1}(\alpha_2(||x(t_0)||))$.

Proof uniform convergence

$\dot{V} \leq -\alpha_3(||x||) \leq -\alpha_3(\alpha_2^{-1}(V))$. We know, that $\alpha_3 \circ \alpha_2^{-1} \in K$. By comparsion lemma, $V(t, x(t)) \leq W(t)$, where $W$ solves $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$ with $W(t_0) = V(t_0, x(t_0))$. By auxility lemma $\exists \beta \in KL$ s.t. $V(t, x(t)) \leq \beta(V(t_0, x(t_0)), t - t_0)$, then $||x(t)|| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\beta(V(t_0, x(t_0)), t - t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(||x(t_0)||), t - t_0)) =: \bar{\beta}(||x(t_0)||, t - t_0)$. From this follows uniform asymptotic stability since $\bar{\beta} \in KL$.

### 3.4 Converse theorems

Let $X^* = 0$ be an EP of $\dot{x}(t) = f(t, x(t))$ with $f : [0, \infty) \times R^n \to R^n$ continously differentiable and $\frac{\partial f}{\partial x}$ bounded in $R^n$, uniformly in t ($||\frac{\partial f}{\partial x}(t, x)|| \leq L$ for all $x \in R^n, \; t \geq 0, \; L > 0$.

If $x^* = 0$ is globally exponentially stale, then exists differentiable $V : [0, \infty) \times R^n \to R$ and

$c_1, c_2, c_3, c_4 > 0$ s.t. $c_1||x||^2 \leq V(t,x) \leq c_2||x||^2$, $\dot{V}(t,x) \leq -c_3||x||^2$ and $\left\|\frac{\partial V}{\partial x}\right\| \leq c_4||x||$.

Proof. Let $\Phi(\tau; t, x)$ – solution to $\dot{x}(t) = f(t, x(t))$ which static at $(t, x)$.

$V(t,x) = \int_t^{t+\delta} \Phi^T(\tau; t, x)\Phi(\tau; t, x)d\tau, \quad \delta > 0$. Upper bound: $V(t,x) = \int_t^{t+\delta} ||\Phi(\tau; t, x)||_2^2 d\tau \leq$ $[\textit{exponential stability}] \leq \int_t^{t+\delta} k^2 \exp -2\lambda(\tau - t)d\tau||x||_2^2 = \frac{k^2}{2\lambda}(1 - \exp -2\lambda\delta)||x||_2^2$.

Lower bound: since $\left\|\frac{\partial V}{\partial x}\right\| \leq L$, then $||f(t,x)||_2 \leq L||x||_2$. Thus by comparation lemma $||\Phi(\tau; t, x)||_2^2 \geq ||x||_2^2 \exp -2L(\tau - t)$. Set it in $V(t,x) \geq \int_t^{t+\delta} \exp -2L(\tau - t)d\tau||x||_2^2 = \frac{1}{2L}(1 - \exp -2L\delta)||x||_2^2$.

Decrease conditions: $\dot{V}(t,x) = \cdots \leq -(1 - k^2 \exp -2\lambda\delta)||x||_2^2$.