

# Nonlinear Control

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## Intro

Goals of Course

- overview over modern nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control methods learn the mathematic basis

Differential equations  $\dot{x} = f(x)$

Nonlinear differential equation  $\dot{x} = f(t, x)$

System with input  $\dot{x} = f(x, u)$

System with input and output  $\dot{x} = f(x, u), \quad y = g(x, u)$

Input-output methods

Scope

[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997

## 1 Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \quad (1)$$

Where  $f : D \rightarrow R^n$ ,  $D \subset R^n$  is open, [here we should explain, what means open set].

Solution to 1  $x : I_{x_0} \rightarrow D$ ,  $t \rightarrow x(t)$  is differentiable

Interval existence solution

Questions:

- # existence of solution
- # "how large" is  $I_{x_0}$
- # uniqueness of solution

Usually we will add some restrictions on  $f$  functions, like continuous.

## 1.1 Existence of solutions

**Definition.** Function  $f : D \rightarrow R^n$  is continuous at  $x' \in D$  if for  $\forall \epsilon > 0 \exists \delta > 0$  such that for  $\forall x \in D, \|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < \epsilon$

Function  $f : D \rightarrow R^n$  is continuous on  $D$  if it's continuous at  $\forall x' \in D$

**Theorem 1.1** (Piano). If  $f : D \rightarrow R^n$  continuous, then for each  $x_0 \in D \exists x : (-\epsilon, \epsilon) \rightarrow D, \epsilon > 0$  satisfying (1).

Further, given a compact set  $U \subset D$ , then  $\exists \alpha > 0$  s.t.  $\forall x_0 \in U \exists x : (-\epsilon, \epsilon) \rightarrow D$  satisfying (1).

**Example.** Consider equation  $\dot{x}(t) = x(t)^2, x(0) = x_0 = 0$ . Solution  $x(t) = -\frac{1}{t-c}, c = \frac{1}{x_0}$ . In this example solution exist in interval  $(-c, c)$ .

But, what about the number of solutions? Which conditions we should add to guarantee uniqueness of solution?

## 1.2 Uniqueness of solutions

**Definition.** Function  $f : D \rightarrow R^n$  is locally Lipschitz (continuous???) on  $D$  if  $\forall x \in D$  there is a neighborhood  $N(x) \subset D$  and  $\exists L > 0$  s.t.

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \quad (2)$$

For all  $x_1, x_2 \in N$ .

- Lipschitz on  $W \in D$  if (2) holds  $\forall x_1, x_2 \in W$  (with same  $L$ )
- globally Lipschitz if (2) holds  $\forall x_1, x_2 \in R^n$  (with same  $L$ )

We have

- # locally Lipschitz functions are continuous

# differentiable functions are locally Lipschitz

# locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

**Lemma 1** (Cromwall). Suppose that  $0 \leq \phi(t) \leq c + L \int_0^t \phi(\tau) d\tau$ ,  $c, L > 0$ ,  $\phi$  - continuous. Then  $\phi(t) \leq ce^{Lt}$ .

*Proof.*  $c + L \int_0^t \phi(\tau) d\tau := \psi(t)$ ,  $\dot{\psi}(t) = L\phi(t) \leq L\psi(t)$ .

Consider  $\frac{d}{dt} (\psi(t)e^{-Lt}) = e^{-Lt} (\dot{\psi}(t) - L\psi(t)) \leq 0$ , thus  $\psi(t)e^{-Lt}$  is decreased, and as a result we have  $\phi(t)e^{-Lt} \leq \psi(t)e^{-Lt} \leq \psi(0) = c$

□

**Theorem 1.2** (Picard Lindelof). If function  $f : D \rightarrow R^n$  is locally Lipschitz then for  $\forall x_0 \in D$   $\exists ! x : (-\epsilon, \epsilon) \rightarrow D$ ,  $\epsilon > 0$  satisfying (1).

*Proof.* \* existence from Piano theorem

Proof of uniqueness

Consider two solutions  $x_1(\cdot)$  and  $x_2(\cdot)$  to (1).  $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$ ,  $x_1(0) = x_2(0)$ . Then we can integrate equality:  $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau)) d\tau$ .  $|x_1(t) - x_2(t)| \leq \int_0^t |f(x_1(\tau)) - f(x_2(\tau))| d\tau \leq L \int_0^t |x_1(\tau) - x_2(\tau)| d\tau$ . Now we can apply Cromwall's lemma with  $c = 0$  and  $\phi(t) = |x_1(t) - x_2(t)|$ , then  $\phi(t) \leq 0$ , then  $x_1(t) = x_2(t)$ ,  $\forall t \in (-\epsilon, \epsilon)$  □

**Example.**

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$\text{Solutions } x(t) = \begin{cases} \frac{1}{4}(t-c)^2, & \text{if } t \geq c \geq 0 \\ 0, & \text{else} \end{cases}$$

Global existance & uniqueness

- sufficient condition:  $f$  globally Lipschitz
- another sufficient condition: solution entirely lies in a coplex set
- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff  $\exists$  solution  $V : R^n \rightarrow R \geq 0$  s.t.  $\frac{\partial V}{\partial x} f(x) \leq -V(x)$ ,  $\forall x \in R^n$

### 1.3 Lyapunov stability

If functions  $\dot{V}(x) < 0, \forall x \in D \setminus \{0\}$ , then  $x^*$  is asymptotically stable.

**Definition.** Equilibrium point  $x = 0$  is stable if  $\forall \epsilon > 0 \exists \delta > 0$  s.t. from  $\|x_0\| < \delta$  follows  $\|x(t)\| \leq \epsilon, \forall t \geq 0$ .

**Definition.** Equilibrium point  $x = 0$  is asymptotically stable if it is stable and exist  $\delta > 0$  s.t. from  $\|x_0\| < \delta$  follows  $\lim_{t \rightarrow \infty} x(t) \rightarrow 0$ .

**Theorem 1.3** (Lyapunov's direct method). Let  $x^* = 0 \in D$  be an equilibrium point of (1), i.e.,  $f(0) = 0$ . Let  $f : D \rightarrow R^n$  is continuous. If there exists a differentiable  $V : D \rightarrow R$  s.t.

1.  $V(x^*) = 0, V(x) > 0, \forall x \in D \setminus \{0\}$
2.  $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0, \forall x \in D$

then  $x^* = 0$  is stable.

*Proof.* Fix compact  $U = \{x : V(x) \leq c\}$  s.t.  $U \in D$ . By Piano: exist  $\alpha > 0$  s.t. any solution  $x$  with  $x_0 \in U$  exists at least on the interval  $[0, \alpha)$ .

TODO proof is not full □

Lyapunovs direct method gives us:

- stability
- convergence (if  $V < 0$ )
- subset of the region of attraction (all compact  $U = \{x : V(x) \leq c\} \in D$ )
- existance of solution for all times

## 2 Nonlinear systems

In this section we consider function  $f : R \times D \rightarrow R^n$ , where  $D \subseteq R^n$ , and  $D$  is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \quad (3)$$

The origin  $x^* \in D$  is an equilibrium point for (3), if  $f(t, 0) = 0, \forall t \geq 0$ .

Remark: EP (equilibrium point)  $x^* = 0$  can be translation of a nonzero solution.

Suppose  $\bar{y}$  is a solution of  $\dot{y} = g(t, y)$ .

Change of coordinates:  $x(t) = y(t) - \bar{y}(t)$ , then  $\dot{x}(t) = \dot{y}(t) - \dot{\bar{y}}(t) = g(t, x(t) + \bar{y}(t)) - \dot{\bar{y}}(t) := f(t, x(t))$ . Since  $\dot{\bar{y}}(t) = g(t, \bar{y}(t))$ , then  $f(t, 0) = 0, \forall t \geq 0$ .

Existence and uniqueness of solution to (3):

- if  $f$  continuous, then exist local solution
- if  $f$  continuous and locally Lipschitz in  $x^*$ , then exist local unique solution

Now we need new stability definitions.

**Definition.** Point  $x^* = 0$  is stable if  $\forall \epsilon > 0$  and  $\forall t_0 \geq 0, \exists \delta > 0$  s.t. from  $\|x_0\| < \delta$  follows  $\|x(t)\| < \epsilon, \forall t \geq t_0$ .

**Definition.** Point  $x^* = 0$  is uniformly stable if  $\forall \epsilon > 0 \exists \delta > 0$ , s.t  $\forall t_0 \geq 0$ , from  $\|x_0\| < \delta$  follows  $\|x(t)\| < \epsilon, \forall t \geq t_0$ .

**Definition.** Point  $x^* = 0$  asymptotically stable if it is stable and  $\forall t_0 \geq 0 \exists c > 0$ , s.t from  $\|x_0\| < c$  follows  $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$ .

**Definition.** Point  $x^* = 0$  uniformly asymptotically stable if it is uniformly stable and  $\exists c > 0$ , s.t  $\forall t_0 \geq 0$  from  $\|x_0\| < c$  follows  $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$ .

**Definition.** Convergence:  $\forall \eta > 0 \forall t_0 \geq 0, \exists T > 0$  such that  $\forall t \geq t_0 + T$  follows  $\|x(t)\| < \eta$ .

**Definition.** Uniform convergence:  $\forall \eta > 0 \exists T > 0$  such that  $\forall t_0 \geq 0$  and  $\forall t \geq t_0 + T$  follows  $\|x(t)\| < \eta$ .

**Example.** Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0$$

Solution  $x(t) = x(t_0) \frac{1+t_0}{1+t}$ . It is uniformly stable, because we can choose  $\delta = \epsilon$ . But does  $x(t)$  convergence uniformly? Answer is no.

**Definition.** Point  $x^* = 0$  is globally uniformly asymptotically stable if it is uniformly stable with  $\delta \rightarrow \infty$  for  $\epsilon \rightarrow \infty$  and  $\forall c, \eta \exists T > 0$  such that  $\forall t_0 \geq 0$  from  $\|x_0\| < c$  follows  $\|x(t)\| < \eta, \forall t \geq t_0 + T$ .

## 2.1 Lyapunov's direct method

Consider some function  $V : [0, \infty) \times D \rightarrow R, (t, x) \rightarrow V(t, x)$  such that  $\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x)$ .

**Theorem 2.1** (Lyapunov's direct method). Let  $f : [0, \infty) \times D \rightarrow R^n$  is continuous and let  $x^* = 0$  be equilibrium point. If there is a differentiable function  $V : [0, \infty) \times D \rightarrow R$  with:

- $W_1(x) \leq V(t, x) \leq W_2(x), \forall t \geq 0, x \in D$

- $\dot{V}(t, x) \leq 0, \forall t \geq 0, x \in D$

where  $W_1, W_2 : D \rightarrow R$  continuous and positive definite, then  $x^* = 0$  is uniformly stable.

If further  $\dot{V}(t, x) \leq -W_3(x), \forall t \geq 0, x \in D$  with  $W_3 : D \rightarrow R$  continuous and positive definite, the  $x^* = 0$  is uniformly asymptotically stable.

If  $D = R^n$  and  $W_1$  is radially unbounded then  $X^* = 0$  is globally uniformly asymptotically stable.

**Example.** Consider the equation  $\dot{x}(t) = -(1+t)x(t)$ .

Check function  $V(t, x) = \frac{1}{2}x^2$  as candidate for Lyapunov's function. Then  $W_1(x) = W_2(x) = \frac{1}{2}x^2$  and  $\dot{V}(t, x) = -(1+t)x^2 \leq -x^2(t) =: W_3(x)$ . Then from theorem we have, that  $X^* = 0$  is globally uniformly asymptotically stable.

## 2.2 Exponential stability

**Definition.** Point  $X^* = 0$  is an exponentially stable EP of (3) if  $\exists \lambda, c, k > 0$  s.t.  $t \geq t_0 \geq 0$  and all  $\|x_0\| < c$  follows  $\|x(t)\| \leq K\|x(t_0)\|e^{\lambda(t-t_0)}$ .

Remark: from exponential stability follows uniformly asymptotically stability.

**Lemma 2** (Auxiliary result). Let  $\dot{x}(t) = f(t, x(t))$ ,  $f$  scalar and  $\dot{\xi}(t) \leq f(t, \xi(t))$  with  $\xi(t_0) \leq x(t_0)$ . Then  $\xi(t) \leq x(t) \quad \forall t \geq t_0$ .

**Theorem 2.2.** Let  $f : [0, \infty) \times D \rightarrow R^n$  be continuous and  $x^* = 0 \in D$  be an EP.

If there is a differentiable function  $V : [0, \infty) \times D \rightarrow R$  and constants  $k_1, k_2, k_3, a > 0$  s.t.

1.  $k_1\|x\|^a \leq V(t, x) \leq k_2\|x\|^a, \forall t \geq 0, x \in D$
2.  $\dot{V}(t, x) \leq -k_3\|x\|^a$

then  $x^* = 0$  is exponentially stable.

If  $D = R^n$ , then  $X^*$  is globally exponential stable.

*Proof.* For  $c > 0$  small enough, trajectories initialized in  $\{x : k_2\|x\|^a < c\}$  remain bounded and in  $D$ . From 1) and 2) we can conclude  $\dot{V} \leq -\frac{k_3}{k_2}V$ . Then from previous Lemma  $V(t, x(t)) \leq$

$$V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}. \quad \text{Then } \|x(t)\| \leq [from(1)] \leq \left(\frac{V(t, x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} \leq$$

$$\left(\frac{k_2\|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(t_0)\| e^{-\frac{k_3}{k_2 a}(t-t_0)} \quad \square$$

**Example.** Consider the equation  $\dot{x}(t) = -(1+t)x(t)$ .

Here  $V(t, x) = \frac{1}{2}x^2$  then  $X^*$  is exponentially stable.

## 2.3 Comparison function

**Definition.** A function  $\alpha : [0, \delta) \rightarrow [0, \infty)$  is (of) "class  $K$ " if it is continuous, strictly increasing, and  $\alpha(0) = 0$ .

**Definition.** A function  $\alpha : [0, \delta) \rightarrow [0, \infty)$  is "class  $K_\infty$ " if  $\alpha \in K$  and  $\lim_{r \rightarrow \infty} \alpha(r) = \infty$ .

**Example.** Function  $\alpha(r) = \tan^{-1}(r)$  – class  $K$

Function  $\alpha(r) = r^k$  – class  $K_\infty$

**Definition.** A function  $\beta : [0, \delta) \times [0, \delta) \rightarrow [0, \infty)$  is "class  $KL$ " if it is continuous,  $\beta(\cdot, s) \in K$  for all fixed  $s$ , and for each fixed  $r$ ,  $\beta(r, \cdot)$  is strictly decreasing:  $\lim_{s \rightarrow \infty} \beta(r, s) = 0$

**Example.** Function  $\beta(x, s) = \max(r, r^2)e^{-s}$  belongs to class  $KL$ .

Properties of comparison functions:

- If  $\alpha \in K$  on  $[0, \delta)$ , then  $\alpha^{-1}$  is defined on  $[0, \alpha(\delta))$  and  $\alpha^{-1} \in K$ .
- If  $\alpha \in K_\infty$ , then  $\alpha^{-1} \in K_\infty$
- If  $\alpha_1, \alpha_2 \in K$ , then  $\alpha_1 \circ \alpha_2 \in K$  (same for  $K_\infty$ )
- If  $\alpha_1, \alpha_2 \in K$ ,  $\beta \in KL$  then  $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we consider comparison functions and stability definitions.

**Lemma 3.** The equilibrium  $x^* = 0$  of  $\dot{x}(t) = f(t, x(t))$  is uniformly stable iff  $\exists \alpha \in K$  and  $c > 0$  s.t.  $\forall t \geq t_0, \forall \|x(t_0)\| < c$  and  $\|x(t)\| \leq \alpha(\|x(t_0)\|)$ .

(only sufficiency). Given  $\epsilon > 0$  choose  $\delta < \min(c, \alpha^{-1}(\epsilon))$ . Then from  $\|x(t_0)\| < \delta$  follows  $\|x(t)\| \leq \alpha(\|x(t_0)\|) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$ .  $\square$

**Lemma 4.** The equilibrium  $x^* = 0$  of  $\dot{x}(t) = f(t, x(t))$  is uniformly asymptotically stable iff  $\exists \beta \in KL$  and  $c > 0$  s.t.  $\forall t \geq t_0, \forall \|x(t_0)\| < c$  and  $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$ .

(only sufficiency). Let  $\|x(t_0)\| < c$ . Then  $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$ . This means uniform convergence.  $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x(t_0)\|, 0)$ . This gives us uniform stability.  $\square$

**Lemma 5.** The equilibrium  $x^* = 0$  of  $\dot{x}(t) = f(t, x(t))$  is globally uniformly asymptotically stable iff previous lemma holds for all  $x_0 \in \mathbb{R}^n$ .

Now consider comparison functions and Lyapunov functions

If  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and positive definite, then  $\forall r > 0 \exists \alpha_1, \alpha_2 \in K$  s.t.  $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$  for all  $x \in B_r(0) = \{x \mid \|x\| \leq r\}$ .

If  $W$  is radially unbounded, then  $\exists \alpha_1, \alpha_2 \in K_\infty$  s.t.  $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$  for all  $x \in \mathbb{R}^n$ .

**Lemma 6** (Auxiliary). Consider  $\dot{y} = \alpha(y)$ ,  $y(t_0) = y_0 > 0$ ,  $\alpha \in K$ . Then  $\exists \beta \in KL$  s.t.  $y(t) = \beta(y_0, t - t_0)$ .

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t, x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where  $W_1, W_2, W_3$  – continuous and positive defined.

Then  $\exists \alpha_1, \alpha_2, \alpha_3 \in K$  such that  $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$  and  $\dot{V}(t, x) \leq -\alpha_3(\|x\|)$ .

Proof uniform stability:

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq [\alpha_1 \in K] \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)).$$

Proof uniform convergence

$\dot{V} \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V))$ . We know, that  $\alpha_3 \circ \alpha_2^{-1} \in K$ . By comparison lemma,  $V(t, x(t)) \leq W(t)$ , where  $W$  solves  $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$  with  $W(t_0) = V(t_0, x(t_0))$ . By auxiliary lemma  $\exists \beta \in KL$  s.t.  $V(t, x(t)) \leq \beta(V(t_0, x(t_0)), t - t_0)$ , then  $\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\beta(V(t_0, x(t_0)), t - t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)) =: \bar{\beta}(\|x(t_0)\|, t - t_0)$ . From this follows uniform asymptotic stability since  $\bar{\beta} \in KL$ .

## 2.4 Converse theorems

**Theorem 2.3.** Let  $X^* = 0$  be an EP of  $\dot{x}(t) = f(t, x(t))$  with  $f : [0, \infty) \times R^n \rightarrow R^n$  continuously differentiable and  $\frac{\partial f}{\partial x}$  bounded in  $R^n$ , uniformly in  $t$  ( $\|\frac{\partial f}{\partial x}(t, x)\| \leq L$  for all  $x \in R^n$ ,  $t \geq 0$ ,  $L > 0$ ).

If  $x^* = 0$  is globally exponentially stable, then exists differentiable  $V : [0, \infty) \times R^n \rightarrow R$  and  $c_1, c_2, c_3, c_4 > 0$  s.t.  $c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2$ ,  $\dot{V}(t, x) \leq -c_3\|x\|^2$  and  $\|\frac{\partial V}{\partial x}\| \leq c_4\|x\|$ .

*Proof.* Let  $\Phi(\tau; t, x)$  – solution to  $\dot{x}(t) = f(t, x(t))$  which is static at  $(t, x)$ .

$V(t, x) = \int_t^{t+\delta} \Phi^T(\tau; t, x) \Phi(\tau; t, x) d\tau$ ,  $\delta > 0$ . Upper bound:  $V(t, x) = \int_t^{t+\delta} \|\Phi(\tau; t, x)\|_2^2 d\tau \leq [exponential\ stability] \leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau \|x\|_2^2 = \frac{k^2}{2\lambda} (1 - e^{-2\lambda\delta}) \|x\|_2^2$ .

Lower bound: since  $\|\frac{\partial V}{\partial x}\| \leq L$ , then  $\|f(t, x)\|_2 \leq L\|x\|_2$ . Thus by comparison lemma  $\|\Phi(\tau; t, x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$ . Set it in  $V(t, x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau \|x\|_2^2 = \frac{1}{2L} (1 - e^{-2L\delta}) \|x\|_2^2$ .

Decrease conditions:  $\dot{V}(t, x) = \dots \leq -(1 - k^2 e^{-2\lambda\delta}) \|x\|_2^2$ . □



### 3 System with inputs

Consider equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (4)$$

where  $f : R^n \rightarrow R^n$ .

Assumption:  $f$  is locally Lipschitz.

Exogenous signal  $u : R \rightarrow R^n$ .

Input can be "bad" (disturbance) or "good" (control).

#### 3.1 Input-to-state stability

Motivation: LTI system  $\dot{x} = Ax + Bu$ ,  $x(0) = x_0$ .

Solution:  $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ . If  $A$  is Hurwitz, then  $\|e^{At}\| \leq ce^{-\lambda t}$  for some  $c, \lambda > 0$ .

How large can  $x$  grow for some bounded  $u$ ?  $\|x(t)\| \leq \|e^{At}\| \|x_0\| + \int_0^t \|e^{A(t-\tau)}\| \|B\| \|u(\tau)\| d\tau \leq e^{-\lambda t} c \|x_0\| + \int_0^t e^{-\lambda(t-\tau)} c \|B\| \|u(\tau)\| d\tau = ce^{-\lambda t} \|x_0\| + (1 - e^{-\lambda t}) \frac{c}{\lambda} \|B\| \sup_{\tau \in [0, t]} \|u(\tau)\|$ .

- $ce^{-\lambda t} \|x_0\|$  class  $KL$  in  $(\|x_0\|, t)$
- $(1 - e^{-\lambda t})$  less than 1
- $\frac{c}{\lambda} \|B\| \sup \|u(\tau)\|$  class  $K$

If  $\sup_{\tau \in [0, t]} \|u(\tau)\|$  is bounded then  $\dot{x}$  remains bounded. Even more: the smaller  $\sup_{\tau \in [0, t]} \|u(\tau)\|$ , the smaller  $\|x(t)\|$ .

**Definition.** System (4) is input-to-state stable (ISS) if  $\exists \beta \in KL, \gamma \in K$  s.t.  $\forall x_0 \in R^n, \forall t \geq 0$  follows  $\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\sup_{\tau \in [0, t]} \|u(\tau)\|)$ .

Remarks:

- From ISS follows O-GAS (global asymptotical stability of  $x = 0$  for  $\dot{x} = f(x, 0)$ )
- $\gamma$  can be interpreted as "gain" w.r.t.  $u$
- if  $\lim_{t \rightarrow \infty} u(t) = 0$  then  $\lim_{t \rightarrow \infty} x(t) = 0$

**Example.** Consider equation  $\dot{x} = -x + xu$ . System is O-GASS, not ISS (for example  $u \equiv \alpha \Rightarrow \dot{x} = x(\alpha - 1)$  all solution diverge).

**Example.** Consider equation  $\dot{x} = -3x + (1 + 2x^2)u$ . System is O-GASS, not ISS (for example  $u \equiv 1$ ,  $x_0 = 2$ ,  $x(t) = \frac{3-e^t}{3-2e^t}$  has a finite escape time).

**Theorem 3.1.** Suppose that there exists a continuously differentiable function  $V : R^n \rightarrow R$  and  $\alpha_1, \alpha_2 \in K_\infty$  and  $\alpha_3, \rho \in K$  such that  $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ ,  $\forall x \in R^n$  and  $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x\|)$ ,  $\forall x : \|x\| \geq \rho(\|u\|)$ . Then (4) is ISS with  $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

*Proof.* Idea: same as Lyapunovs direct method when  $x$  is "outside" of ball  $\{x : \|x\| \leq \rho(\|u\|)\}$

TODO Picture □

**Example.** Consider equality  $\dot{x} = -x^3 + u$ . Let  $V(x) = \frac{1}{2}x^2$ , then  $\dot{V} = -x^4 + xu = [0 < \Theta < 1] = -(1 - \Theta)x^4 - \Theta x^4 + xu \leq -(1 - \Theta)x^4$  for all  $x : \|x\| \geq \left(\frac{\|u\|}{\Theta}\right)^{\frac{1}{3}}$ . Thus, system is ISS with  $\gamma(v) = \rho(v) = \left(\frac{v}{\Theta}\right)^{\frac{1}{3}}$ .

Remarks:

- Existence of  $V$  is both necessary and sufficient for ISS;
- (??) is equivalent to  $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_4(\|x\|) + \alpha_5(\|u\|)$ ,  $\forall x, u$  for some  $\alpha_4, \alpha_5 \in K$ ;
- If  $x_1 = 0$  is a globally asymptotically stable EP of  $\Sigma_1$  and  $\Sigma_2$  is ISS w.r.t. "input"  $x_1$ , then  $(x_1, x_2) = (0, 0)$  is a globally asymptotically stable EP for the cascaded system.

**Theorem 3.2.** Assume that:

- $f$  is globally Lipschitz;
- $x = 0$  is a globally exponentially stable EP for  $\dot{x} = f(x, 0)$

Then the system (4) is ISS.

*Proof.* Sketch:  $\exists$  continuous differentiable  $V$ :

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x, 0) \leq -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

Then:

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) &= \frac{\partial V}{\partial x} f(x, 0) + \frac{\partial V}{\partial x} (f(x, u) - f(x, 0)) \leq -c_3\|x\|^2 + c_4\|x\|L\|u\| = -c_3(1 - \theta)\|x\|^2 - \theta\|x\|^2 + \\ c_4L\|x\|\|u\| &\leq -c_3(1 - \theta)\|x\|^2 \\ \text{if } \|x\| &\geq \frac{c_4L}{\theta c_3}\|u\|. \end{aligned} \quad \square$$

### 3.2 Control Lyapunov functions

Motivation: Lyapunov theory for control systems.

(input affine systems)

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i = f(x) + G(x)u,$$

$$f : R^n \rightarrow R^n, g : R^n \rightarrow R^n, G : R^n \rightarrow R^{n \times m}$$

$u : t \rightarrow u(t), R \rightarrow R^m$  is a control signal (decision variable).

**Definition.** A function  $V : R^n \rightarrow R$  is a control Lyapunov function (CLF) if it's differentiable positive definite, radially unbounded and

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) < 0 \quad (5)$$

Remark:

Concept can be generalized to systems  $\dot{x} = f(x, u)$ . Then 5 becomes

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot f(x, u)) < 0$$

**Theorem 3.3** (Artstein). There exists  $k : R^n \rightarrow R^m$  (state feedback) which is continuous on  $R^n \setminus \{0\}$  s.t.  $x^* = 0$  is globally asymptotically stable EP for  $\dot{x} = f(x) + G(x)k(x)$  iff there exists a CLF.

How to find CLFs?

Proposition:

Condition (5) is equivalent to

$$\forall x \neq 0, \quad \frac{\partial V}{\partial x} G(x) = 0 \implies L_f V(x) < 0 \quad (6)$$

Remark:

$$\frac{\partial V}{\partial x} G(x) = (\nabla V(x)g_1(x), \dots, \nabla V(x)g_m(x)) =: L_G V(x)$$

$$(6) \iff \forall x \neq 0, \quad L_f V(x) \geq 0 \implies L_G V(x) \neq 0$$

*Proof.*  $\Leftarrow$ :

Assume (6) holds. Then:

$$\inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) = \inf_u L_f V(x) + L_G V(x)u < 0$$

Why?

- If  $L_G V(x) = 0$ , then by (6)  $L_f V(x) < 0$ ;
- If  $L_G V(x) \neq 0$ , then (at least) for one  $i$  we have  $\nabla V(x) \cdot g_i(x) \neq 0 \implies$  set  $u_i = -c \nabla V(x) \cdot g_i(x)$ .

$\implies$ :

If (5) holds for some  $x$  with  $L_G V(x) = 0$ , then we must have  $L_f V(x) < 0$ . □

Example (discontinuous control):

$$\dot{x} = \begin{cases} 1 - u, & u \geq 1 \\ -1 - u, & u \leq -1 \\ 0, & \text{else} \end{cases}$$

If you want to move the system you need to apply control  $|u| \geq 1$ .

Using

$$u(x) = \begin{cases} x + 1, & x > 0 \\ x - 1, & x \leq 0 \end{cases}$$

results in closed loop  $\dot{x} = -x$  - asymptotically stable.

$V(x) = x^2$  is a CLF.

**Theorem 3.4.** There exists a continuous  $k : R^n \rightarrow R^m$ , smooth on  $R^n \setminus \{0\}$  s.t.  $x^* = 0$  is globally asymptotically stable EP for  $\dot{x} = f(x) + G(x)k(x)$  iff:

- there exists a (smooth) CLF  $V$ ;
- $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall x : 0 < \|x\| < \delta$   
 $\exists u \in R^m : \|u\| < \varepsilon$  s.t.  $L_f V(x) + L_G V(x)u < 0$

How to construct a globally stabilizing state feedback  $k$  from knowledge of a CLF?

"Sontag's formula"

Fix  $c \geq 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$k(x) = \begin{cases} -cb(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T, & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases}$$

Proposition: Let  $V : R^n \rightarrow R$  be a CLF and  $k$  as above. Then  $x^* = 0$  is globally asymptotically stable EP for  $\dot{x} = f(x) + G(x)k(x)$

*Proof.*  $\dot{V} = L_f V(x) + L_G V(x)k(x) = a(x) - cb(x)^T b(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T b(x) = -cb(x)^T b(x) - \sqrt{a(x)^2 + (b(x)^T b(x))^2} < 0 \quad \forall x \neq 0$  s.t.  $L_G V(x) \neq 0$

$\dot{V} = L_f V(x) + L_G V(x) \cdot 0 < 0 \quad \forall x \neq 0$  s.t.  $L_G V(x) = 0$  (since  $V$  is CLF)

$\implies V$  - Lyapunov function  $\implies \dots$  □

Remarks:

- Sontag's formula is smooth on  $R^n \setminus \{0\}$ ;
- Sontag's formula is continuous at  $x = 0$  iff small control property holds.

$$\forall x \neq 0 : \inf_u \frac{\partial V}{\partial x} f(x, u) < 0 \quad \dot{x} = f(x) + G(x)u$$

So this leads to

$$\forall x \neq 0 L_G V(x) = 0 \Rightarrow L_f V(x) \neq 0$$

Remark: The last formula is "optimal" if minimize:

$$\int_0^\infty \frac{1}{2} p(x) b(x)^T b(x) + \frac{1}{2p(x)} u^T u dt$$

$$b(x) := (L_G V(x))^T$$

where  $c > 0$

$$p(x) = \begin{cases} c + \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} & b(x) \neq 0 \\ c & b(x) = 0 \end{cases}$$

It still works if  $u = \lambda h(x)$  with  $\lambda \in [\frac{1}{2}; \infty)$  is applied (large "gain margin")

## 4 Backstepping

Integrator backstepping

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{7}$$

where  $x_1 \in \mathbb{R}^m$ ,  $x_2, u \in \mathbb{R}$  (single input)

image to be inserted

Assumption: we know (smooth) "feedback"  $\alpha_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ , and positive definite, differentiable  $v_1 : \mathbb{R}^m \rightarrow \mathbb{R}$

s.t.  $L_{f_1+g_1\alpha_1} V_1(x)$  is negative definite  $\Rightarrow$  origin of  $\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1)$  is asymptotically stable

Goal: Compute feedback  $u = k(x)$  which stabilises (7). Backstepping constructs  $u = \alpha_2(x_1, x_2)$  s.t.  $(e_1, e_2) = (x_1 - 0, x_2 - \alpha_1(x_1)) = 0$  error coordinates

Rewrite (7) :

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1))$$

$$\dot{x}_2 = u$$

image to be inserted

In error coordinates

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 &= u - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial e_1} \dot{e}_1\end{aligned}\tag{8}$$

"backstepping"  $\alpha_1$  through the integrator

Define  $V_2(e_2) := \frac{1}{2}e_2^2$ , and

$$\begin{aligned}V(e_1, e_2) &= V_1(e_1) + V_2(e_2) \\ \dot{V}(e_1, e_2) &= \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)\alpha_1(e_1)) + \frac{\partial V_1}{\partial e_1}g_1(e_1)e_2 + \frac{\partial V_2}{\partial e_2}(u - \dot{\alpha}_1)\end{aligned}$$

as far as  $L_{f_1+g_1\alpha_1}V_1$  -negative definite and  $\frac{\partial V_2}{\partial e_2} \rightarrow e_2$

Choose

$$u = (-\frac{\partial V_1}{\partial e_1}g_1(e_1) + \dot{\alpha}_1)(\text{"canaling terms"}) - k_2e_2(\text{"stabilizing term"}), \quad k_2 > 0 \tag{9}$$

$\Rightarrow$  Then  $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0, \forall (e_1, e_2) \neq 0$

$\Rightarrow (e_1, e_2) = (0, 0)$  is an asymptotically stable EP for (8) with  $u$  as in (9)

Remark:  $(e_1, e_2) \rightarrow (0, 0)$  doesnot necessarily imply that  $(x_1, x_2) \rightarrow 0$  for  $u = \alpha_2(x_1, x_2) = -\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1))$

where  $u \leftarrow$  (9) the original coordinates and  $\dot{\alpha}_1 \leftarrow \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + g_1(x_1)x_2)$

But  $(x_1, x_2) = (0, 0)$  is asymptotically stable if  $\alpha_1(0) = 0$  why?  $(e_1, e_2) \rightarrow 0 \Rightarrow x_1 \rightarrow 0 \quad x_2 \rightarrow \alpha_1(0) = 0$

**Example.**

$$\begin{aligned}\dot{x}_1 &= x_1x_2 \\ \dot{x}_2 &= u\end{aligned}$$

Choose  $\alpha_1(x_1) = -k$  ( $k > 0$ )  $\rightarrow \dot{x}_1 = -kx_1 \Rightarrow V_1(x_1) = \frac{1}{2}x_1^2$

Then:

$$\begin{aligned}e_1 &= x_1 - 0, \quad \dot{e}_1 = e_1(e_2 - k) \\ e_2 &= x_2 + k, \quad \dot{e}_2 = u\end{aligned}$$

Backstepping yields:  $u = -e_1^2 - k_2 e_2$ ,  $k_2 > 0 \Rightarrow (e_1, e_2) = (0, 0)$  is asymptotically stabilized

$(x_1, x_2) = (0, -k)$  is asymptotically stabilized

Can we choose different  $\alpha_1$  s.t.  $(x_1, x_2) = (0, 0)$  is stabilized?

Yes, e.g.

$$\alpha_1(x_1) = -x_1^2 \Rightarrow \dot{x}_1 = -x_1^3, \quad V_1(x_1) = \frac{1}{2}x_1^2$$

So we have equations

$$\begin{aligned} e_1 &= x_1 - 0, \quad \dot{e}_1 = e_1(e_2 - e_1^2) \\ e_2 &= x_2 + x_1^2, \quad \dot{e}_2 = u + 2e_1^2(e_2 - e_1^2) \end{aligned}$$

Backstepping results in

$$u = -e_1^2 - 2e_1^2(e_2 - e_1^2) - k_2 e_2, \quad k_2 > 0 \Rightarrow (e_1, e_2) \rightarrow (0, 0), \quad (x_1, x_2) \rightarrow (0, 0)$$

Generalization-1

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

Assumption:  $g_2(x_1, x_2) \neq 0, \forall x_1, x_2 \Rightarrow$  Input transformation:  $u = \frac{1}{g_2(x_1, x_2)}(V - f_2(x_1, x_2)) \Rightarrow \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad \dot{x}_2 = V \Rightarrow$  can apply integrator backstepping to determine  $V$  results in

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)} \left( -\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2) \right)$$

Generalization 2: (Backstepping through 2 integrators)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1} \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3, \quad x_2, x_3 \in R \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u, \quad u \in R \end{aligned}$$

Assumption:  $g_2, g_3$  nowhere zero.

Shown before:  $\exists \alpha_2$ : for  $x_3 = \alpha_2(x_1, x_2) \quad (e_1, e_2) \rightarrow 0$

Thus  $e_3 := x_3 - \alpha_2(x_1, x_2)$

Input transformation:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(V - f_3(x_1, x_2, x_3))$$

$\Rightarrow \dot{x}_1 = \dots, \dot{x}_2 = \dots, \dot{x}_3 = V \Rightarrow$  can apply backstepping once more.

In "error" coordinates:

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dot{e}_2 &= f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1 \\ \dot{e}_3 &= V - \dot{\alpha}_2\end{aligned}$$

Define  $V_3(e_3) = \frac{1}{2}e_3^2, V(e_1, e_2, e_3) = \sum_{i=1}^3 V_i(e_i)$

$$\dot{V}(e_1, e_2, e_3) = \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) + e_2(f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1) + e_3(V - \dot{\alpha}_2))$$

All the underlined terms were designed (previously) to be  $= L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 < 0$

So:  $\dot{V}(e_1, e_2, e_3) = L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 + e_2g_2(e_1, e_2 + \alpha_1(e_1))e_3 + e_3(V - \dot{\alpha}_2)$

Structurally it is exactly the same as it was in backstepping through 1.

Choose:

$$V = \dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1)) - k_3e_3$$

$\dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1))$  - "cancelling terms".

$k_3e_3$  - "stabilizing term".

In original coordinates:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(\dot{\alpha}_2 - (x_2 - \alpha_1(x_1))g_2(x_1, x_2) - k_3(x_3 - \alpha_2(x_1, x_2)) - f_3(x_1, x_2, x_3))$$

We need  $\alpha_1, \alpha_2$  to compute  $u$ .

General backstepping recursion:

Systems in "strict feedback form":

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1}$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

$\dots$

$$\dot{x}_k = f_k(x_1, \dots, x_k) + g_k(x_1, \dots, x_k)u, \quad x_2, \dots, x_k, u \in R$$

$g_2, \dots, g_k$  nowhere zero,  $f_i, g_i$  (sufficiently) smooth, as it is needed in  $\alpha_i$ .

Backstepping recursion:

1. "Input data": a CLF  $V_1$  for  $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$  with a (smooth) feedback  $u_1 = \alpha_1x_1$  which as. stabilizes the origin of  $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$ .



2. for  $i = 2, \dots, k$ :

construct a CLF  $V_i(e_i) = \frac{1}{2}e_i^2$ ,  $V = \sum_{j=1}^i V_j(e_j)$  and a feedback  $\alpha_1$  which as. stabilizes origin of  $(e_1, \dots, e_i) = (x_1, x_2 - \alpha_1(x_1), \dots, x_i - \alpha_{i-1}(x_1, \dots, x_{i-1}))$

$$\alpha_i(x_1, \dots, x_i) = \frac{1}{g_i}(\dot{\alpha}_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}}g_{i-1} - k_i(x_i - \alpha_{i-1}) - f_i)$$

3. apply  $u = \alpha_k(x_1, \dots, x_k)$

Backstepping and CLFs:

Backstepping is sensitive to uncertainties in  $f_i, g_i$  (due to cancelling terms)

$\implies$  Sontag's formula is more practical  $\implies$  we can use it since  $V$  is CLF.

Error system is input affine (using input transformation)

$$\dot{e} = f(e) + g(e)V$$

$$\text{with } f(e) = \begin{pmatrix} f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dots \\ -\alpha_{k-1} \end{pmatrix}, g(e) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

Claim:

$$V(e) = \sum_{i=1}^k V_i(e_i) \text{ is a CLF.}$$

*Proof.* For input affine systems we need to show  $L_g V = 0 \implies L_f V < 0, \forall e \neq 0$ .

$$\dot{V}(e) = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 + e_{k-1}g_{k-1}(\dots)e_k - e_k \dot{\alpha}_{k-1} + e_k u.$$

Here  $e_k u = L_g V$  and the rest is  $L_f V$ .

Assume  $L_g V = 0 \iff e_k = 0$

$$\implies L_f V = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 < 0 \quad \forall e \neq 0 \text{ with } e_k = 0. \quad \square$$

$\implies$  We can apply Sontag's formula to construct  $V$ .

This theory can be extended to systems with  $x_2, \dots, x_k, u \in R^m$  ("block backstepping").

## 5 Systems with inputs and outputs

Study/control systems  $\dot{x} = f(x, u)$  with "output"  $y(t) = h(x(t))$

## 5.1 Sliding mode control

Motivating example

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 + x_2 \end{cases}$$

$$\Rightarrow \dot{y} = x_2 + u$$

Choose:

$$u = \begin{cases} -x_2 - 1, & y > 0 \\ -x_2 + 1, & y < 0 \\ -x_2, & y = 0 \end{cases}$$

$$\Rightarrow \dot{y} = \begin{cases} -1, & y > 0 \\ +1, & y < 0 \\ 0, & y = 0 \end{cases}$$

Solutions(Laratheodory) are if  $y(0) > 0$ , then

$$y(t) = \begin{cases} y(0) - t, & t \leq y(0) \\ 0, & t > y(0) \end{cases}$$

If  $y(0) < 0$ , then

$$y(t) = \begin{cases} y(0) + t, & t \leq -y(0) \\ 0, & t > -y(0) \end{cases}$$

Key property: choose  $u$  s.t.  $y(t)$  goes to zero in finite time  $\Rightarrow x(t)$  tends  $S := \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = 0\}$  in finite time

Consider dynamics on  $S$

$$\begin{cases} \dot{x}_1 = x_2 (x_2 = -x_1 \text{ if } y = 1) = -x_1 \\ \dot{x}_2 = u = -x_2 \end{cases}$$

globally as stable

Two "phases"

1. solutions converge to  $S$  in finite time
2. solutions converge to zero ("on  $S$ ") asymptotically

$\rightsquigarrow$  "sliding mode" control

Remark: in (1) "finite time convergence is crucial"

General procedure:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) = s(x) \end{aligned}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $y : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $s : \mathbb{R}^n \rightarrow \mathbb{R}$

$u$  - scalar input,  $s(x)$  - sliding

single input, single output

Assumptions:  $y$  has relative degree 1, well - defined globally, i.e.  $L_g s(x) \neq 0 \forall x \in \mathbb{R}^n$

Two-step approach:

1. Bring  $x(t)$  to  $S := \{x \in \mathbb{R}^n | S(x) = 0\}$  in finite time
2. Have  $x(t)$  going to zero asymptotically (on  $S$ )
  - switching between nodes 1 and 2
  - mode 2 is "sliding mode"

How are 1 + 2 achieved?

- Design of sliding manifolds crucial!

Need: For  $y(t) = 0$  for all  $t \geq 0$ . All solutions converge to the origin, i.e., "zero dynamics" have globally asymptotically stable origin.

How? e.g. systems in "regular form"  $x = [\eta, \xi]'$

$$\begin{aligned}\dot{\eta} &= f_1(\eta, \xi) \\ \dot{\xi} &= f_2(\eta, \xi) + g_2(\eta, \xi)u\end{aligned}$$

Choose  $s(x) = \eta - \phi(\eta)$ , where  $\phi$  asymptotically stabilizes zero dynamics  $\dot{\eta} = f_1(\eta, \phi(\eta))$  (and  $\phi(0) = 0$ ) Ex. 1.9 in Khalil

- Converging to sliding manifold in finite time:  $\rightsquigarrow \dot{s} = L_f s(x) + L_g s(x)u$ , where  $L_g s(x) \neq 0$ . Obvious choice to render  $S$  invariant is  $u = -\frac{L_f s(x)}{L_g s(x)}$  (mode 2, behaviour on  $S$ )

Motivating example, add

$$\begin{cases} -\hat{u}/L_g s(x) & y > 0 \\ \hat{u}/L_g s(x) & y < 0 \end{cases}$$

where  $\hat{u} > 0$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \operatorname{sgn}(s(x)))$$

$$\rightsquigarrow \dot{y} = -\hat{u} \operatorname{sgn}(y)$$

$\rightsquigarrow$  (Carathéodory) solutions converge to zero in finite time

$\rightsquigarrow x(t)$  converges to  $S$  in finite time

Control Lyapunov perspective

$$V(X) = \frac{1}{2}s(x)^2$$

$$\dot{V}(x) = s(x)\dot{s}(x) = s(x)(L_f s(x) + L_g s(x)u) = -s(x)\text{sgn}(s(x))\hat{u} = |s(x)|\hat{u} < 0, \quad s(x) \neq 0$$

$$\text{Consider } w = \sqrt{2v} \rightsquigarrow^{s \neq 0} \dot{w} = \sqrt{2} \frac{1}{2\sqrt{v}} \dot{v} = -\hat{u}$$

$\rightsquigarrow w$  converges to 0 in finite time  $\Rightarrow v$  converges to 0 in finite time  $\Rightarrow S(x(t))$  converges to 0 in finite time.

**Example.**

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \sin(x_2) \\ \dot{x}_2 &= x_2^2 + x_1 + u \end{aligned}$$

Choose  $s(x) = x_2 + 2x_1$ , where  $-2x_1 := \phi(x_1)$  on  $S$ :  $\dot{x}_1 = -2x_1 + x_1 \sin(-2x_1) \rightsquigarrow$  asymptotically stable

$$\begin{aligned} \dot{s} &= x_2^2 + x_1 + u + 2x_2 + 2x_1 \sin(x_2) \rightsquigarrow u = -(x_2^2 + x_1 + 2x_2 + 2x_1 \sin(x_2) + \hat{u} \text{sgn}(x_2 + 2x_1)), \quad \hat{u} > 0 \\ &\rightsquigarrow \text{yields finite-time convergence to } S. \end{aligned}$$

Alternative sliding mode controllers

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \text{sgn}(s(x))), \quad \hat{u} > 0$$

In particular

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u}|L_g s(x)|\text{sgn}(s(x)))$$

$\rightsquigarrow$  ensure robustness w.r.t. "matched uncertainties"

$$\dot{x} = f(x) + g(x)\sigma(x) + g(x)u$$

$\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$ , bounded (i.e.,  $|\sigma(x)| \leq c \quad \forall x \in \mathbb{R}^n$ )

Why?  $V(x) = \frac{1}{2}s(x)^2$

$$\dot{V} = s(x)(L_f s(x) + L_g s(x)u + L_g s(x)\sigma(x)) = -s(x)\text{sgn}(s(x))\hat{u}|L_g s(x)| + s(x)L_g s(x)\sigma(x) \leq -|s(x)||L_g s(x)|(\hat{u} - c)$$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|)\text{sgn}(s(x)))$$

ensures robustness w.r.t. matched uncertainties s.t.  $\sigma(x) \leq \beta(x) \quad \forall x \in \mathbb{R}^n$

**Example.**

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \sin(x_2) \\ \dot{x}_2 &= \theta x_2^2 + x_1 + u \end{aligned}$$

$$|\theta| \leq 2 \rightsquigarrow |\theta x_2^2| \leq 2x_2^2 = \beta(x)$$

$$\dot{s} = \theta x_2^2 + x_1 + u + 2x_1 + 2x_1 \sin x_2$$

$$u = -(x_1 + 2x_1 + 2x_1 \sin x_2 + \hat{u} \operatorname{sign}(s(x)) + 2x_2^2 \operatorname{sgn}(s(x)))$$

$$L_f s(x) = x_1 + 2x_1 + 2x_1 \sin x_2$$

$$\rightsquigarrow \dot{s} = -\hat{u} \operatorname{sgn}(s(x)) + x_2^2(\theta - 2 \operatorname{sgn}(s(x))) \Rightarrow \text{finite-time convergence to } S.$$

Remedy: replace sign-function by saturated slope (continuous approximation)

can be extended to multi-input systems  $u \in \mathbb{R}^m \rightarrow s : \mathbb{R}^n \rightarrow \mathbb{R}^m$

## 5.2 Dissipativity

Dissipativity: Generalization of Lyapunov theory to systems  $w$  inputs and outputs

$$\begin{aligned} \dot{x} &= f(x, u) & x(0) &= x_0 & f &: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \\ y &= h(x) & & & h &: \mathbb{R}^n \rightarrow \mathbb{R}^p \end{aligned} \tag{10}$$

Definition:

- storage function  $S : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \rightarrow S(x)$  nonnegative (i.e.,  $S(x) \geq 0 \forall x \in \mathbb{R}^n$ )
- supply rate  $s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$ ,  $(u, y) \rightarrow s(u, y)$

Definition: System (10) is dissipative w.r.t. the supply rate  $s$  if there exists a storage function  $S$  s.t.  $\forall x_0 \in \mathbb{R}, \forall t \geq 0, \forall u : [0, t] \rightarrow \mathbb{R}^m$

$$S(x(t)) \leq S(x_0) + \int_0^t s(u(\tau), y(\tau)) d\tau \tag{11}$$

First item - storage at time  $t$ , second item - initial storage, the last item - supply delivered over  $[0, t]$

"dissipation inequality" (DIE) is the inequality (11)

Interpretation:

- "Dissipative systems dissipate storage/stored energy"
- "No storage/energy can be created internally"
- positive  $s$  "supplied" energy / storage  
negative  $s$  "extracted" energy / storage

Remark:

- If  $S$  is differentiable, DIE is equivalent to  $\dot{S}(x) \leq s(u, y) \forall u, x$
- Dissipation (rate) is defined as  $d(x, u) = s(u, h(x)) - \dot{S}(x) \geq 0$

Examples of dissipative systems:

	supply rate	input	output	storage function
electrical	$u \cdot i$	voltage	current	energy storage in all capacitors and inductors
mechanical	$F \cdot V$	force	velocity	Hamiltonian = kinetic + potential energy
thermo-dynamics	$Q + W$	rate of heat	rate of work	internal energy
	$-\frac{a}{T}$		temperature	entropy

How do we compute storage functions?

- in general difficult (similar to computing Lyapunov functions)
- characterization via optimization problem

Introduce "available storage"

$$S_a(x) := \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=0} \left( - \int_0^T s(u(\tau), y(\tau)) d\tau \right)$$

the maximum of energy we can extract

**Theorem 5.1.** System (10) is dissipative w.r.t. the supply rate  $s$  iff  $S_a(x) < \infty$  for all  $x \in \mathbb{R}^n$

Moreover, if  $S_a(x) < \infty$  for all  $x \in \mathbb{R}^n$ , then  $S_a$  is a storage function and  $S(x) \geq S_a(x) \forall x \in \mathbb{R}^n$  for all storage functions  $S$ .

*Proof.* Sketch of proof. " $S_a(x) < \infty \Rightarrow$  dissipativity".  $S_a(x) \geq 0 \forall x \in \mathbb{R}^n$  by definition (can take  $T = 0$ )

$$\begin{aligned} S_a(x) &= \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=0} - \int_0^T s(u(\tau), y(\tau)) d\tau \geq^* \\ &- \int_0^t s(u(\tau), y(\tau)) d\tau + \sup_{u: [t, t+T] \rightarrow \mathbb{R}^m, T \geq 0, x(t)=x(t)} - \int_t^{t+T} s(u(\tau), y(\tau)) d\tau \end{aligned}$$

the last item is  $S_a(x(t))$ ,

$$\Rightarrow S_a(x(t)) - \int_0^t s(u(\tau), y(\tau)) d\tau$$

and this is DIE  $\Rightarrow S_a$  is a storage function

Note for (\*): "suboptimal" to first transfer system to  $x(t)$  and then extract maximum energy starting of  $x(t)$

"Dissip.  $\Rightarrow S_a(x) < \infty$ "

From DIE:  $S(x_0) \geq S(x(T)) - \int_0^T s(u(\tau), y(\tau)) d\tau \geq - \int_0^T s(u(\tau), y(\tau)) d\tau$

for all  $x_0$ , for all  $T \geq 0$ , all  $u(\cdot) \Rightarrow S(x_0) \geq \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=x_0} - \int_0^T s(u(\tau), y(\tau)) d\tau = S_a(x)$

$\Rightarrow S_a(x) < \infty \forall x \in \mathbb{R}^n$  and  $S \geq S_a$  for all storage function  $S$ .  $\square$

Another special supply rate: "required supply"

$$S_r(x) := \inf_{u: [-T, 0] \rightarrow \mathbb{R}^m, T \geq 0, x(-T)=x^*, x(0)=x} \int_{-T}^0 s(u(\tau), y(\tau)) d\tau$$

**Theorem 5.2.** Assume that end state  $x \in \mathbb{R}^n$  is readable from  $x^*$ . If system (10) is dissipative w.r.t. the supply rate  $s$ , then for all storage functions  $S$

$$S(x) \leq S_r(x) + S(x^*) \quad \forall x \in \mathbb{R}^n$$

Furthermore,  $S_r(x) + S(x^*)$  is a storage function.

*Proof.* Sketch of proof.

Consider  $u : [-T, 0] \rightarrow \mathbb{R}^n$  which transfers the system from  $x^*$  to  $x$

$$S(x) - S(x^*) \leq [\text{by DIE}] \inf_{u: [-T, 0] \rightarrow \mathbb{R}^n, T \geq 0, x(-T)=x^*, x(0)=x} \int_{-T}^0 s(u(\tau), y(\tau)) d\tau = S_r(x)$$

$\square$

Remark: Set of all storage functions is convex, i.e. ,  $\alpha S_1 + (1 - \alpha) S_2$ ,  $\alpha \in [0, 1]$  is a storage function (for  $S_1, S_2$  storage functions)

Dissipativity widely used in control theory

If system is dissipative with positive definite storage  $S$  and if there exists a (continuous)  $k : \mathbb{R}^n \rightarrow \mathbb{R}^n$  s.t.

$$s(k(x), h(x)) < 0, \quad \forall x \neq 0$$

then  $x = 0$  is asymptotically stable under  $u = k(x)$

Why? Take  $S$  as Lyapunov function

$$\dot{S} \leq s(u, y) \stackrel{u=k(x)}{=} s(k(x), h(x)) < 0, \quad \forall x \neq 0$$

$L_2$  - gain via supply rate

$$s(u, y) = \frac{1}{2} \gamma^2 \|u\|^2 - \frac{1}{2} \|y\|^2$$

$\rightsquigarrow$  from dissipation inequality

$$\begin{aligned}\frac{1}{2} \int_0^t \gamma^2 \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau &\geq S(x(t)) - S(x(0)) \geq -S(x(0)) \\ \Rightarrow \int_0^t \|y(\tau)\|^2 d\tau &\leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau + 2S(x(0))\end{aligned}$$

$\Rightarrow$  system has  $L_2$  - gain  $\gamma$

Classify optional  $l(x, u^{\leftarrow x})$  operating conditions  $s(u, y) = l(x, u) - l(x^*, u^*)$

**Example.**

$$\dot{x} = u, \quad y = x$$

$S(x) = \frac{1}{2}x^2$ ,  $\dot{S} = xu = uy \rightsquigarrow$  system is dissipative w.r.t. supply rate  $s(u, y) = uy$ .

**Example.** "part-Hamiltonian systems"

$$\dot{x} = [F(x) - R(x)]\nabla H(x) + g(x)u$$

$y = g(x)^T \nabla H(x)$ ,  $H$  - Hamiltonian total stored energy in system

$F(x) = -F^T(x)$  internal interconnection structure (power conserving)  $R(x) \geq 0$  dissipation structure

Take  $S(x) = H(x)$

$$\begin{aligned}\dot{S}(x) &= \nabla H(x) \cdot [F(x) - R(x)]\nabla H(x) + \nabla H(x) \cdot g(x)u \\ &= -\nabla H(x) \cdot R(x)\nabla H(x) + yu \leq yu\end{aligned}$$

as far as  $-\nabla H(x) \cdot R(x)\nabla H(x) \leq 0 \Rightarrow$  dissipative w.r.t.  $s(u, y) = u^T y$

### 5.3 Passivity

$$\begin{aligned}\dot{x} &= f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ y &= h(x), \quad y \in \mathbb{R}^m\end{aligned}\tag{12}$$

(same number of inputs and outputs)

**Definition.** System (12) is passive if it is dissipative w.r.t. supply rate  $s(u, y) = u^T y$

Why "passive"? From circuit theory passive compared to "active" ones as diodes or transistors

Examples: electrical, mechanical systems

Stabilization of passive systems

**Definition.** System (12) is zero-state observable (ZSO) if (for  $u(t) = 0$ )  $y(t) = 0$  for all  $t \geq 0 \Rightarrow x(t) = 0$  for all  $t \geq 0$

"trivial solution  $x(t) \equiv 0$  is observable from the output"



Remark: can be related to zero-state detectability

**Theorem 5.3.** Let system (12) be

i) passive in differentiable storage set

ii) ZSO

Then the feedback  $u = -Py$ ,  $P > 0$  renders the origin asymptotically stable

*Proof.* Sketch of proof From passivity

$$\dot{S} \leq u^T y = -y^T P y \leq 0 \quad (13)$$

$$S(x(t)) - S(x(0)) \leq - \int_0^t y(\tau)^T P y(\tau) d\tau, \quad \forall t \geq 0$$

$$S(x(t)) \geq 0$$

$$S(x_0) \geq \int_0^t y(\tau)^T P y(\tau) d\tau, \quad \forall t \geq 0 \quad (14)$$

$y(\tau)^T P y(\tau) \geq 0$ . Want to show  $S(x_0) > 0$  for all  $x_0 \neq 0$ . By contradiction. Suppose  $\exists \bar{x} \neq 0$  with  $S(\bar{x}) = 0$ .

From (14)  $\Rightarrow y(\tau) = 0 \quad \forall \tau \geq 0$

By ZSO  $\Rightarrow x(\tau) = 0 \quad \forall \tau \geq 0 \Rightarrow \bar{x} = 0$

$\Rightarrow S$  is positive definite.  $\Rightarrow$  Lyapunov stability together with (13)

For convergence, use (13) together with La Salle's invariance principle and ZSO □

Advantage. We have (static) output feedback (no observer needed)

Passivity of interconnections

1. Parallel interconnections of two passive systems are passive

Take  $S(x_1, x_2) = S_1(x_1) + S_2(x_2)$ .

$$\dot{S} \leq u_1^T y_1 + u_2^T y_2 = u^T (y_1 + y_2) = u^T y$$

2. Feedback interconnection of passive systems are passive

Take  $S(x_1 + x_2) = S_1(x_1) + S_2(x_2)$

$$S \leq u_1^T y_1 + u_2^T y_2 \stackrel{x_1=u_2=y}{=} y^T (u_1 + y_2) = y^T u$$

Remark:

- does not work for serious interconnections
- can construct possibly large networks of passive systems

Stability if feedback interconnections:

Main idea: "shortage" of passivity of  $H_1$  can be compensated by "excess" of passivity of  $H_2$

**Theorem 5.4.** Consider feedback interconnection (2) with  $u \equiv 0$ . Assume that  $H_1$  and  $H_2$  are (i) ZSO and dissipative with differentiable  $S_1, S_2$  w.r.t. the supply rates

$$s_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i, \quad i = 1, 2, \quad \rho, \nu \in \mathbb{R} \quad (15)$$

Then the origin  $(x_1, x_2) = (0, 0)$  for interconnection is asymptotically stable if  $\nu_1 + \rho_2 > 0$  and  $\nu_2 + \rho_1 > 0$ .

*Proof.* Take  $S(x) = S_1(x_1) + S_2(x_2)$ .

$$\begin{aligned} \dot{S}(x) &\stackrel{(15)}{\leq} u_1^T y_1 - \rho_1 y_1^T y_1 - \nu_1 u_1^T u_1 \\ &\quad + u_2^T y_2 - \rho_2 y_2^T y_2 - \nu_2 u_2^T u_2 \\ &= -(\rho_1 + \nu_2) y_1^T y_1 - (\rho_2 + \nu_1) y_2^T y_2 \end{aligned}$$

$u_1^T y_1$  and  $u_2^T y_2$  can be excluded as  $u_1 = -y_2, u_2 = y_1$ .

$\Rightarrow$  can show as in previous theorem that  $S$  is positive definite  $\Rightarrow$  Lyapunov stability

For using La Salle:

$$\begin{aligned} y_1 \equiv 0 &\Rightarrow u_2 \equiv 0 \Rightarrow^{ZSO} x_2 \equiv 0 \\ y_2 \equiv 0 &\Rightarrow u_1 \equiv 0 \Rightarrow^{ZSO} x_1 \equiv 0 \end{aligned}$$

□

Remark

- If (15) is satisfied with  $v_i = 0$ : "output - feedback passive"  $\Rightarrow p_i > 0$  - "excess" of passivity,  $p_i < 0$  - "shortage" of passivity ( $|p_i|$ ).
- If (15) satisfied with  $p_i = 0$ : "input - feedforward passive"  $\Rightarrow v_i > 0$  - "excess" of passivity,  $v_i < 0$  - "shortage" of passivity ( $|v_i|$ ).
- Comment on terminology "output feedback passive"  $\dot{s} \leq u^T y - \rho y^T y = [\text{output feedback } u = \bar{u} - ky] = \bar{u}^T y - (\rho + k) y^T y \leq \bar{u}^T y$  In other words, system can be made passive by output feedback.
- Similar for feedforward passivity, system can be made passive by feedback forward the input:  $\bar{y} = y + ku$ .

Remark:

Feedforward interconnection 2) can be extended to allow  $h_1$  or  $h_2$  to explicitly depend on  $u \Rightarrow$  includes static systems (controllers) e.g. state output feedback  $y = h(u)$  e.g.  $y_2 = ku_2 \Rightarrow$  input-strictly passive controller.

Extension of output-feedback/input-feedforward passivity:

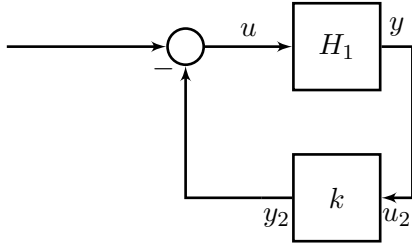
$$s(u, y) = u^T - \rho(y)^T y$$

with  $\rho(y) = [\rho_1(y_1), \dots, \rho_n(y_n)]^T$  with  $\rho_i$  section nonlinearities,  $\rho_i : \mathbb{R} \rightarrow \mathbb{R}$

**Example.**

$$\begin{aligned} \dot{x}_1 &= x_2 \\ H_1 : \quad \dot{x}_2 &= -x_1^3 + x_2 + u \\ y &= x_2 \end{aligned}$$

Take  $S_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \Rightarrow \dot{S}_1 = x_1^3 x_2 - x_1^3 x_2 + x_2^2 + x_2 u$ , define  $y^2 := x_2^2$  then  $yu = x_2 u \Rightarrow$  output - feedback passive with shortage of passivity 1  $\Rightarrow \rho_1 = -1$  and  $v_1 = 0$ .

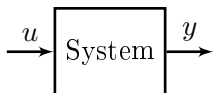


$$y_2 = ku_2 \Rightarrow s(u_2, y_2) = \gamma ku_2^2 + \frac{1-\gamma}{k} y_2^2, 0 < \gamma < 1 \Rightarrow \rho_2 = \frac{1-\gamma}{k}, v_2 = \gamma k$$

$\Rightarrow v_2 + \rho_1 > 0$  for  $k > 1$  and  $\gamma$  close enough to 1,  $v_1 + \rho_2 = \rho_2 > 0$ .  $\Rightarrow$  with ZSO, the origin is a stable.

## 6 Input/Output Methods

References: Desoev, Vidyasagar "Feedback Systes Input-output properties"



$$\begin{aligned} u &: u \rightarrow y \\ u, y &: [0, \infty] \rightarrow \mathbb{R}^m \\ t &\rightarrow u(t), y(t) \end{aligned}$$

## 6.1 Signals and Systems

- How to define "stability" in input/output setting?
- Which signals are "good"?

**Definition.**  $Lp$ -spaces,  $p \in [1, \infty]$ .  $Lp[0, \infty) = \{\Phi : [0, \infty) \rightarrow \mathbb{R}^m, \text{measurable} \mid \int_0^\infty \|\Phi(t)\|^p dt < \infty\}$

Interpretation: "finite energy signal" ( $p=2$ ).

Remark: "measurable" = pointwise limit of a sequence of piecewise constant functions (except on a set of measure 0)

**Example. :**

- continuous function
- functions with "few enough" discontinuities

$Lp$  is a real vector space ("signals  $\Phi(\cdot)$  are vectors") i.e., for  $\Phi, \Phi_1, \Phi_2 \in Lp$ ,  $\alpha \in \mathbb{R}$  vector addition:  $\Phi_1 + \Phi_2 : t \rightarrow \Phi_1(t) + \Phi_2(t) \in Lp$ . Scalar multiplication:  $\alpha\Phi : t \rightarrow \alpha\Phi(t) \in Lp$

Zero element is signal  $\Phi \equiv 0$ .

$Lp$  is a normed vector space with norm  $\|\Phi\|_{Lp} = \sqrt[p]{\int_0^\infty \|\Phi(t)\|^p dt}$  for  $\Phi \in Lp \Rightarrow$

- $\|\Phi\|_{Lp} = 0 \iff \Phi = 0$ , else  $\|\Phi\|_{Lp} > 0$
- for  $\alpha \in \mathbb{R}$ ,  $\|\alpha\Phi\|_{Lp} = |\alpha| \|\Phi\|_{Lp}$
- for  $\Phi_1, \Phi_2 \in Lp$   $\|\Phi_1 + \Phi_2\|_{Lp} \leq \|\Phi_1\|_{Lp} + \|\Phi_2\|_{Lp}$

For  $p \rightarrow \infty$  set of all measurable and (essentially) bounded functions  $L_\infty$ , for continuous  $\phi$

$$\|\phi\|_{L_\infty} = \inf\{c \in \mathbb{R} \mid \|\phi(t)\| \leq c \text{ a.e.}\} = \sup_{t \geq 0} \|\phi(t)\|$$

**Example.**

$$\phi(t) = e^{-\alpha t}, \alpha > 0, p \in [1, \infty)$$

$$\begin{aligned} \|\phi\|_{Lp} &= \sqrt[p]{\int_0^\infty \|e^{-\alpha t}\|^p dt} = \\ &= \sqrt[p]{\left[-\frac{1}{\alpha p} e^{-\alpha p t}\right]_0^\infty} = \sqrt[p]{\frac{1}{\alpha p}} < \infty \\ &\Rightarrow \phi \in Lp \quad \forall p \in [1, \infty) \quad p = \infty : \\ &\sup_{t \geq 0} \phi(t) = \phi(0) = 1 \Rightarrow p \in L_\infty \end{aligned}$$

Special case  $p = 2$

$L_2$  can be equipped with an inner product  $\phi_1, \phi_2 \in L_2$ , we write  $\langle \phi_1, \phi_2 \rangle_{L_2} := \int_0^\infty \phi_1(t)^T \phi_2(t) dt$

symmetry  $\langle \phi_1, \phi_2 \rangle_{L_2} = \langle \phi_2, \phi_1 \rangle_{L_2}$

(bi-)linearity

$$\begin{aligned} \langle \alpha \phi_1, \phi_2 \rangle_{L_2} &= \alpha \langle \phi_1, \phi_2 \rangle_{L_2} \quad \alpha \in \mathbb{R} \\ \langle \phi_1 + \phi_2, \phi_3 \rangle_{L_2} &= \langle \phi_1, \phi_3 \rangle_{L_2} + \langle \phi_2, \phi_3 \rangle_{L_2}, \quad \phi_3 \in L_2 \end{aligned}$$

positive definiteness:  $\langle \phi_1, \phi_1 \rangle_{L_2} = 0$  iff  $\phi_1 = 0$ ,  $\langle \phi_1, \phi_1 \rangle_{L_2} > 0$  else.

$\Rightarrow (L_2, \langle \cdot, \cdot \rangle_{L_2})$  is an inner product space

$\| \phi \|_{L_2}^2 = \langle \phi, \phi \rangle_{L_2}$  "induced norm".

Particularly useful (Cauchy -Schwarz inequality)  $|\langle \phi_1, \phi_2 \rangle_{L_2}| \leq \| \phi_1 \|_{L_2} \| \phi_2 \|_{L_2}$

Original motivation

**Example.**

$$\dot{x} = x + u, \quad y = x, \quad x(0) = 0$$

$$y(t) = \int_0^t e^{(t-\tau)} u(\tau) d\tau$$

$$\text{Take } u(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$$

Clearly,  $u \in L_p$  for any  $p \in [1, \infty)$ . Let  $t \geq 1$ :

$$\begin{aligned} y(t) &= \int_0^1 e^{(t-\tau)} d\tau = e^t \int_0^1 e^{-\tau} d\tau \\ &= e^t [-e^{-\tau}]_0^1 = e^t (1 - e^{-1}) \end{aligned}$$

$\Rightarrow y \notin L_p, \quad p \in [1, \infty) \Rightarrow$  even that  $u \in L_p$ , the output neednot be an  $L_p$  - signal.

Taking  $H : L_p \rightarrow L_p$  does (in general) not make sense? (would be excluded too many "relevant" systems)

Meaningful longer class: extended  $L_p$  spaces

Introduce "truncation operator"

$$\phi_T(t) = \begin{cases} \phi(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

The extension  $L_p^e$  of  $L_p$  is defined as

$$L_p^e = \{ \phi : [0, \infty) \rightarrow \mathbb{R}^n \mid \forall T \geq 0 \quad \phi_T \in L_p \}$$

$L_p^e \setminus L_p$  are "unstable" signals

**Example.**  $\phi(t) = e^t$  ("unstable linear system")

$$\begin{aligned}\|\phi_T\|_{L_p}^p &= \int_0^\infty |\phi_T(t)|^p dt = \int_0^T |\phi(t)|^p dt = \\ &= \int_0^T e^{pt} dt = \frac{1}{p}(e^{Tp} - 1) < \infty, \quad \forall T \geq 0 \\ &\Rightarrow \phi \in L_p^e\end{aligned}$$

We consider systems

$$H : u \mapsto y, L_p^e \mapsto L_p^e$$

and define input-output stability as follows:

**Definition.**  $H$  is  $L_p$ -stable if there exists  $\alpha \in K$ ,  $\beta \geq 0$  s.t.

$$\|(H(u))_T\|_{L_p} \leq \alpha(\|u_T\|_{L_p}) + \beta$$

for all  $u \in L_p^e$  and all  $T \geq 0$ .

$H$  is finite-gain  $L_p$  stable if there exist  $\gamma, \beta \geq 0$  s.t.

$$\|(H(u))_T\|_{L_p} \leq \gamma\|u_T\|_{L_p} + \beta \tag{16}$$

for all  $u \in L_p^e$  and  $T \geq 0$ . Then  $\gamma_p(H) := \{\inf \gamma | \exists \beta \geq 0 \text{ s.t. (16) holds}\}$  is  $L_p$  - gain of  $H$

**Definition.** A map  $H : L_p^e \mapsto L_p^e$  is causal if  $(H(u))_T = (H(u_T))_T$  for all  $u \in L_p^e$  and  $T \geq 0$ .

Interpretation:  $H$  is "nonanticipativity", outputs up to time  $T$  cannot be influenced by inputs after time  $T$ .

Remark: if  $H$  is defined by  $u \mapsto y$ ,  $y = h(x)$ ,  $\dot{x} = f(x, u)$  then it is causal.

- (16)

$$\Rightarrow \|H(u)\|_{L_p} \leq \gamma\|u\|_{L_p} + \beta, \quad \forall u \in L_p \tag{17}$$

- For causal systems, (17) implies (16)
- sometimes slightly different definitions of finite-gain  $L_2$  stability

$$\|(H(u))_T\|_{L_2}^2 \leq \bar{\gamma}^2\|u_T\|_{L_2}^2 + \beta, \quad \forall u \in L_p^e, \quad \forall T \geq 0 \tag{18}$$

One can show

$$\gamma_2(H) := \{\inf \bar{\gamma} | \exists \beta \geq 0 \text{ s.t. (18) holds}\}$$

## 6.2 Input-output stability of state-space systems

**Theorem 6.1.** Consider  $\dot{x} = f(x, u)$ ,  $y = h(x, u)$ . Suppose the system is ISS and there exist  $\alpha_1, \alpha_2 \in K$  and  $\eta \geq 0$  s.t.  $\|L(x, u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$ . Then for each  $x_0 \in \mathbb{R}^n$ , the system is  $L_\infty$  - stable.

*Proof.* From ISS,  $\exists \phi \in KL$  and  $\alpha_3 \in K$  s.t. for all  $t \geq 0$ .

$$\|x(t)\| \leq \phi(\|x_0\|, t) + \alpha_3\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right)$$

Hence

$$\begin{aligned} \|y(t)\| &\leq \alpha_1(\phi(\|x_0\|, t) + \alpha_3(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)) + \alpha_2(\|u(t)\|) + \eta \leq \\ &[\alpha_1(a + b) \leq \alpha_1(2a) + \alpha_2(2b)] \leq \alpha_1(2\phi(\|x_0\|, t)) + \alpha_1(2\alpha_3(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)) \\ &\quad + \alpha_2(\|u(t)\|) + \eta \Rightarrow \|y_T\|_{L_\infty} \leq \gamma(\|u_T\|_{L_\infty}) + \beta \\ &\text{with } \gamma = \alpha_2 \circ 2\alpha_3 + \alpha_2, \quad \beta = \alpha_1(2\phi(\|x_0\|, 0)) + \eta \end{aligned}$$

□

$L_2$  - gain of LTI system: Consider

$$\begin{aligned} x &= Ax + Bu & u, y \in \mathbb{R} &\rightarrow SISO \\ y &= Cx + Du & A \dots Hurwitz \end{aligned} \tag{19}$$

**Lemma 7.** The  $L_2$  gain of (19) is

$$\gamma = \sup_{w \in \mathbb{R}} \sqrt{G(-jw)G(jw)}$$

where  $G(s) = C(SI - A)^{-1}B + D$

*Proof.* Sketch: Due to linearity, we can assume without loss of generality  $x(0) = 0$  ( $x_0 \neq 0$  changes  $\gamma$ ).

$$\begin{aligned} \|y\|_{L_2}^2 &= \int_0^\infty (y(t))^2 dt = [Parseval's theorem] = \frac{1}{2\pi} \int_{-\infty}^\infty Y(-jw)Y(jw)dw = \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty U(-jw)G(-jw)G(jw)U(jw)dw \leq \sup_{w \in \mathbb{R}} (G(-jw)G(jw)) \int_{-\infty}^\infty U(-jw)U(jw)dw = \\ &= [Parseval's th.] = \gamma^2 \|u\|_{L_2}^2 \end{aligned}$$

$\Rightarrow L_2$ -gain of (19) is  $\leq \gamma$ . Can show by contradiction that  $L_2$ -gain equals  $\gamma$ .

□

$L_2$  - gain of input affine system: Consider

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{20}$$

Recall. System has  $L_2$  gain less or equal  $\gamma$  if it is dissipative w.r.t. supply rate  $s(u, y) = \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|y\|_2^2 \Rightarrow$  with diff. storage function  $S$  such that:

$$\frac{\partial S}{\partial x}[f(x) + g(x)u] \leq \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|h(x)\|_2^2 \quad \forall x, u$$

"Worst case" (maximizing)  $u$ :

$$u^* = \frac{1}{\gamma^2}g^T(x) \left( \frac{\partial S}{\partial x} \right)^T$$

$\Rightarrow \frac{\partial S}{\partial x}f(x) + \frac{1}{2\gamma^2}\frac{\partial S}{\partial x}g(x)g^T(x) \left( \frac{\partial S}{\partial x} \right)^T + \frac{1}{2}h^T(x)h(x) \leq 0 \quad \forall x \in \mathbb{R}^n$  (Hamilton-Jacobi equation) If there exist pos. semi-definite  $S$  satisfying (1), then system (20) has  $L_2$ -gain less or equal  $\gamma$ .

## 7 Exercises

### 7.1 Exercise 1

Problem 1:

*Proof.* For any  $t \geq 0$ , we have

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}(V \circ x)(t) = \langle \nabla V(x(t)), \frac{d}{dt}x(t) \rangle = \langle \nabla V(x(t)), f(x(t)) \rangle = L_f V(x(t))$$

□

Problem 2:

*Proof.*

**Lemma 8.** Given the assumptions in Problem 2, if there exists a solution  $x : [0, +\infty] \rightarrow \mathbb{R}^n, t \rightarrow x(t)$ , of  $\dot{x} = f(x)$  s.t.  $x(t) \in K$  for any  $t \geq 0$ , where  $K \subset \mathbb{R}^n$  is a compact with  $O \in K$  ( $O$  - origin), then  $x(t) \xrightarrow{t \rightarrow +\infty} 0$ .

Clearly, for any  $c > 0, lev_{\leq c}V$  is positive invariant w.r.t  $\dot{x} = f(x)$ . Given  $c > 0$ , let  $x_0 \in lev_{\leq c}V$ , i.e.,  $V(x_0) \leq c$ . Then, for any  $t \geq 0$

$$V(x(t)) = V(x_0) + \int_0^t \frac{d}{dt}V(x(\tau))d\tau < V(x_0) \leq c,$$

i.e.  $x(t) \in lev_{\leq c}V$  for any  $t \geq 0$ .

Then, for any  $x_0 \in lev_{\leq c}V$  there exists a solution  $x : [0, +\infty] \rightarrow \mathbb{R}^n$  of  $\dot{x} = f(x)$  s.t.  $x(t) \in lev_{\leq c}V$  for all  $t \geq 0$ . Clearly,  $O \in lev_{\leq c}V$ . We conclude by using the above Lemma ( $K = lev_{\leq c}V$ ). □



Problem 3:

*Proof.* Let  $r > 0$ . By assumption, there exists  $c > 0$  s.t.  $\overline{B(0, r)} \subset \text{lev}_{\leq c} V$ .

Since any bounded set  $\text{lev}_{\leq c} V$  is a subset of the region of attraction, and since the sublevel sets are arbitrary large,  $R^n$  is also the region of attraction.

A condition that ensures that for any  $c > 0$ ,  $\text{lev}_{\leq c} V$  is bounded is  $V(x) \xrightarrow{\|x\| \rightarrow +\infty} +\infty$ .  $\square$

Problem 4:

*Proof.* Let  $P : R^2 \rightarrow R^2$  be continuously differentiable. Consider

$$m\dot{v} = -g\nabla P(q).$$

Consider  $x = (q, v)$ ,  $\dot{q} = v$ ,  $\dot{v} = -\frac{g}{m}\nabla P(q)$ . Let  $H : R^2 \rightarrow R$  be defined by

$$H(q, v) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q).$$

We have

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v)$$

Since  $P$  is positive definite, then  $H$  is positive definite.

Then

$$L \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v) = \langle \nabla H(q, v), \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v) \rangle = 0 \quad \forall (q, v) \in R^2 \times R^2$$

$\implies$  the origin is stable.  $\square$

Problem 5:

*Proof.* For any  $t \geq 0$ , we have

$$\begin{aligned} \frac{d}{dt}V(t, x(t)) &= \frac{d}{dt}(V \circ (id_R, x))(t) = [id_R : R \rightarrow R, t \mapsto t] = \left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \frac{d}{dt}(id_R(t), x(t)) \right\rangle = \\ &= \left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \right\rangle = \frac{\partial}{\partial t}V(t, x(t)) + \left\langle \frac{\partial}{\partial x}V(t, x(t)), f(t, x(t)) \right\rangle = L \begin{pmatrix} 1 \\ f \end{pmatrix} V(x(t)). \end{aligned}$$

$$g(t, x(t)) := \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \quad \square$$

Problem 6:

*Proof.* Consider  $\dot{x} = a \sin(\omega t)$ ,  $x(0) = x_0 \in R$   $a, \omega > 0$ .

This is solved by  $x(t) = -\frac{a}{\omega} \cos(\omega t) + \frac{a}{\omega} + x_0$ .

Clearly,  $x$  is bounded on  $[0, +\infty]$  since  $x(t) \geq x_0$ , and  $x(t) \leq x_0 + 2\frac{a}{\omega}$  for any  $t \geq 0$ .

Choose  $\varepsilon = \frac{a}{\omega}$  and  $t_0 = 0$ . Then  $\forall \delta > 0 \exists x_0 \in B(0, \delta)$ , namely  $x_0$ , s.t.  $\exists t \geq t_0$ , namely  $t = \frac{\pi}{\omega}$ , with  $x(t) \notin B(0, \varepsilon)$  ( $x(\frac{\pi}{\omega}) = 2\frac{a}{\omega} > \varepsilon$ ).  $\square$

Short notes:

Problem 7:

Take  $V(t, x) = \frac{1}{2}x^2$ .

Problem 8:

Take  $V(t, x) = x_1^2 + (1 + e^{-2t})x_2^2$ .

## 7.2 Exercise 2

Problem 1:

*Proof.* a) Since  $\alpha_1$  is continuous and strictly increasing:

$$\forall x, y \in [0, \delta), x < y \implies \alpha_1(x) < \alpha_1(y)$$

$\implies \alpha_1$  is injective, i.e.

$$\forall x, y \in [0, \delta), x \neq y \implies \alpha_1(x) \neq \alpha_1(y).$$

Clearly,  $\alpha_1 : [0, \delta) \rightarrow \alpha_1([0, \delta))$  is surjective, i.e.

$$\forall y \in \alpha_1([0, \delta)) \exists x \in [0, \delta) : \alpha_1(x) = y$$

Thus  $\alpha_1$  is bijective.

Define  $\alpha_1^{-1} : [0, \alpha_1(\delta)) \rightarrow [0, \delta)$  by  $\alpha_1^{-1}(\alpha_1(x)) = x$ .

b) From a) we have  $\alpha_3^{-1} \in K$ . Since  $\alpha_3 \in K_\infty$ ,  $\alpha_3 - 1$  is defined on  $[0, +\infty)$  and  $\alpha_3^{-1}(r) \xrightarrow{r \rightarrow \infty} \infty$

c) Let  $\alpha = \alpha_1 \circ \alpha_2$ . Then we have  $\alpha(0) = \alpha_1(\alpha_2(0)) = 0$  and  $\alpha(r) > 0$  whenever  $r > 0$ . Moreover, for any  $x, y$ :

$$x < y \implies \alpha_2(x) < \alpha_2(y) \implies \alpha(x) = \alpha_1(\alpha_2(x)) < \alpha_1(\alpha_2(y)) = \alpha(y)$$

It is continuous (as composition of continuous functions).

d) From c) we have  $\alpha := \alpha_3 \circ \alpha_4 \in K$ ,  $\alpha$  is defined on  $[0, +\infty)$  since  $\alpha_3, \alpha_4 \in K_\infty$  and

$$r \rightarrow +\infty \implies \alpha_4(r) \rightarrow +\infty \implies \alpha(r) \rightarrow +\infty$$

e) For each  $s, r \mapsto \beta(\alpha_2(r), s)$  is of class  $K$ .

Thus  $r \mapsto \alpha_1(\beta(\alpha_2(r), s)) \in K$ .

For each  $r, s \mapsto \beta(\alpha_2(r), s)$  decreases.

Hence,  $s \mapsto \alpha_1(\beta(\alpha_2(r), s))$  decreases.

Moreover,

$$\alpha_1(\beta(\alpha_2(r), s)) \xrightarrow{s \rightarrow +\infty} 0$$

□

Problem 3:

*Proof.* For  $u = 0$  the origin is UGAS. Consider  $V : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $(t, x) \mapsto \frac{1}{2}x^2$ .

We have

$$\frac{\partial}{\partial t}V(t, x) + \frac{\partial}{\partial x}V(t, x)f(t, x, u) = (\sin(t) - 2)x^2 + xu \leq -x^2 + |x||u| = -(1 - \theta)x^2 - \theta x^2 + |x||u|, \quad \theta \in (0, 1)$$

Hence, whenever  $|x| \geq \frac{|u|}{\theta}$ , the system is ISS with  $\gamma = \frac{r}{\theta}$ .

□

Problem 4:

*Proof.*

$$\dot{x} = -x + (x^2 + 1)d \tag{21}$$

$$\dot{x} = -2x - x^3 + (x^2 + 1)d \tag{22}$$

System (21): Clearly, the system is 0-GAS. However, for  $d = 1$  and  $x > 1$  we have  $x^2 + 1 > x$ .

$$f(x, 1) = -x + (x^2 + 1) > 0$$

and thus  $\dot{x} > 0$ . Hence, if  $x(0) = x_0 > 1$ , the solution diverges (in finite time).

$\implies$  System (21) isn't ISS.

System (22): It is 0-GAS. Moreover, for any finite  $d$  there exists a "large"  $x$  s.t.

$$2x + x^3 > (x^2 + 1)d$$

$$\implies f(x, d) = -2x - x^3 + (x^2 + 1)d < 0$$

and  $\dot{x} < 0 \implies$  System 22 is ISS.

Consider  $V : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \frac{1}{2}x^2$  s.t

$$V'(x)f(x, d) = -2x^2 - x^4 + x(x^2 + 1)d \leq -x^2 - x^2(x^2 + 1) + (x^2 + 1)|x||d|$$

Hence, whenever  $|x| \geq |d|$ ,

$$V'(x)f(x, d) \leq -x^2$$

s.t. system (22) is ISS with  $\gamma(r) = r$ .

□

Problem 5:

*Proof.*

$$\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\|\nabla V(x)\|^2 + |\langle \nabla V(x), \delta u \rangle| \leq [YI] \leq -\|\nabla V(x)\|^2 + \frac{1}{2}\|\nabla V(x)\|^2 + \frac{\delta^2}{2}\|u\|^2$$

Young's inequality:

$$\forall x, y : |\langle x, y \rangle| \leq \varepsilon \frac{\|x\|^p}{p} + \frac{\|y\|^q}{\varepsilon q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \varepsilon > 0$$

Hence, whenever  $\|x\| > \frac{\delta}{\sqrt{c}}\|u\|$ ,  $t \mapsto \|x(t)\|$  is decreasing.

Moreover whenever  $\|x\| \geq \frac{\delta}{\sqrt{c\theta}}\|u\|$ ,  $\theta \in (0, 1)$ , we have  $\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\frac{c}{2}(1-\theta)\|x\|^2 \implies$  ISS.  $\square$

### 7.3 Exercise 3

Motivation: Lyapunov Theory

$$\dot{x} = f(x, u)$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

**Definition.** (CLF) A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a CLF if it is continuous differentiable, positive definite, radially unbounded and  $\forall x \neq 0 \inf_u \langle \nabla V(x), f(x, u) \rangle < 0$

In order to find CLFs, we restrict our analysis to input-affine systems

$$\dot{x} = f(x) + G(x)u$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$

Proposition: A continuous, differentiable, positive definite and radially unbounded.  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a CLF iff

$$\forall x \neq 0 \quad L_G V(x) = 0 \implies L_f V(x) < 0$$

Image to be inserted

Problem 1

Consider  $\dot{x} = \cos(x) + (1 + e^x)u$  where  $f(x) = \cos(x)$ - drift and  $g(x) = 1 + e^x$

Let  $V : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \frac{1}{2}x^2$ . Clearly, continuous differentiable, positive definite and radially unbounded. Moreover, for any nonzero  $x$ , we have  $L_G V(x) \neq 0$ .

Thus, for any  $x \neq 0$ , there exists a control that renders  $\langle \nabla V(x), f(x) + g(x)u \rangle$  negative. Given this CLF, there exists a state feedback  $u = u(x)$ , e.g.

$$u(x) = -\frac{kx + \cos(x)}{1 + e^x}, \quad k > 0$$

Problem

Consider

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 e^{x_1} \cos(x_2) \\ \dot{x}_2 &= x_1^5 \sin(x_2) + u \end{aligned}$$

Take  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2)$

For any  $x \neq 0$ , we have

$$\inf_{u \in \mathbb{R}} (L_f V(x) + L_G V(x)u) = \begin{cases} L_f V(x), & \text{if } L_G V(x) = 0 \\ -\infty & \text{else} \end{cases}$$

In particular,

$$\begin{aligned} L_f V(x) &= \dots = x_1(-x_1^3 + x_2 e^{x_1} \cos(x_2)) + x_2 x_1^5 \sin(x_2) \\ L_G V(x) &= \dots = x_2 \end{aligned}$$

However,

$$L_f V(x)|_{x_2=0} = -x_1^4 < 0 \quad \forall x_1 \neq 0$$

Image to be inserted

Concluding that  $V$  is a CLF.

Problem 2:

$\dot{x} = Ax + Bu$ , input defined system where  $(A, B)$  is stabilizable, there exists  $K \in \mathbb{R}^{m \times n}$  s.t.  $A + BK$  is Hurwitz (cf. KRT). The latter is equivalent to the existence  $P = P^T > 0$  s.t.  $P(A + BK) + (A + BK)^T P < 0$  (cf. Khalil theorem 4,6)

Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $x \mapsto \langle x, Px \rangle$ . Moreover,  $\forall x \neq 0 \exists u = Kx$  s.t.  $\langle \nabla V(x), Ax + Bu \rangle < 0$ , since

$$\langle \nabla V(x), Ax + Bu \rangle = \langle u = Kx, (P(A + BK) + (A + BK)^T P)x \rangle < 0$$

In addition,

$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\|K\|} > 0 \quad \forall x \neq 0, \|x\| < \delta \quad \exists u = Kx \quad \|u\| < \epsilon$$

s.t.  $L_f V(x) + L_G V(x)u < 0$  since  $\|u\| = \|Kx\| \leq \|K\|\|x\| < \|K\|\delta = \epsilon$

Problem 3

Let  $P : \mathbb{R}^2 \rightarrow \mathbb{R}$  be continuous, differentiable consider

$$m\dot{v} = -g\nabla P(q) + F, \quad m, g > 0$$

a) Hamiltonian form. Let  $x := (q, v)$ . Then  $\dot{x} = (-\frac{g}{m}\nabla P(q) + \frac{1}{m}F) = \begin{bmatrix} I \\ -I \end{bmatrix} \begin{bmatrix} \frac{g}{m}\nabla P(q) \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{m}I \\ 0 \end{bmatrix} F = \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F$  given  $H(x) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q)$

b) "CLF". Take  $H$  as a CLF candidate. Then, for any  $x$

$$\begin{aligned} \langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle &= \langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle + \langle \nabla H(x), G(x)F \rangle = \\ &= [\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle = L_f H(x) = 0] = \frac{1}{m} \langle v, F \rangle \end{aligned}$$

Strictly speaking,  $H$  is no CLF, but it reveals how to choose  $F$  s.t. the origin is GAS.

For any point  $x$  for which there exists no control  $F$  s.t.  $\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle < 0$

Choose  $F = 0$ . Why? Using the Krasovskiy-Lasalle inv. principle, we conclude that the origin is GAS, since any solution in  $\{\dot{H}(x) = 0\}$  verifies  $v(t) \equiv 0$ , implying  $\dot{v}(t) \equiv 0$  s.t.

$$0 = -\frac{g}{m}\nabla P(q(t)) + \frac{1}{m}P(t)$$

The last part equals 0. Since  $F = 0$  (by choice) and  $\nabla P(q) = 0$  iff  $q = 0$  we conclude that  $\dot{H}(x) = 0$  can only be "maintained" at the origin.

Problem 4

Consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ux_2 + u^3 \end{aligned}$$

show that  $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$  is CLF and let  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by

$$\ddot{x} + u\dot{x} - u^3 = 0$$

For any  $x$  and  $u$ , we have  $\langle \nabla V(x), f(x, u) \rangle = \dots = x_1(2x_2 - ux_2 + u^3) + x_2(x_2 - ux_2 + u^3) = x_1h_1 + x_2h_2$

Image to be inserted

Hence if  $u < 0$  and  $-u$  "large", then we can render  $\langle \nabla V(x), f(x, u) \rangle < 0$ .

## 7.4 Exercise 4

Consider

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1) + g_2(x_1, x_2)u \end{cases} \quad (23)$$

Using the "preliminary control"

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \check{u} \end{cases} \quad (24)$$

$$u = \frac{1}{g_2(x_1, x_2)}(\check{u} - f_2(x_1, x_2))$$

Idea: Look at the upper(-most) system only and consider  $x_2$  as a "virtual control".

Assumptions: Suppose

- $\exists$  CLF  $V_1$ ;
- $\exists$  (smooth) feedback  $\alpha_1$  s.t.  $L_{f_1+g_1\alpha_1}V_1 < 0$ .

Now, add and subtract  $g_1\alpha_1$  in 24 s.t.

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1)) \\ \dot{x}_2 = \check{u} \end{cases} \quad (25)$$

Next, introduce  $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$  s.t.

$$\begin{cases} \dot{e}_1 = f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 = \check{u} - \dot{\alpha}_1(e_1) \end{cases} \quad (26)$$

Problem 1:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

*Proof.* 1. Choose "virtual control":

$$x_2 = -(k+1)x_1 =: \alpha_1(x_1), \quad k > 0$$

The origin of  $\dot{x}_1 = -kx_1$  is GAS.

(Take  $V_1 : R \rightarrow R$ ,  $x_1 \mapsto \frac{1}{2}x_1^2$  s.t.  $\dot{V}_1(x_1) = -kx_1^2 < 0$  for all  $x_1 \neq 0$ )

2. Error coordinates:

Let  $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$  s.t.

$$\dot{e}_1 = -ke_1 + e_2$$

$$\dot{e}_2 = u + (k+1)(-ke_1 + e_2)$$

3. "Composite CLF":

Define  $V : R \times R \rightarrow R$ ,  $(e_1, e_2) \mapsto V_1(e_1) + \frac{1}{2}e_2^2$  s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(u + (k+1)(-ke_1 + e_2) + e_1)$$

4. Choose control:

Let  $u = -e_1 - (k+1)(e_2 - ke_1) - ke_2$

s.t.  $\dot{V}(e_1, e_2) = -ke_1^2 - ke_2^2 < 0$  for all  $(e_1, e_2) \neq (0, 0)$

Remark: The closed-loop system reads:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

□

Problem 2:

$$\dot{x}_1 = x_1(x_2 - k), \quad k > 0$$

$$\dot{x}_2 = u$$

*Proof.* 1.  $x_2 = 0 =: \alpha_1(x_1)$

The origin of  $\dot{x}_1 = -kx_1$  is GAS ( $V_1(x_1) = \frac{1}{2}x_1^2$ )

2.  $(e_1, e_2) := (x_1, x_2)$  s.t.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = u$$

3.  $V(e_1, e_2) = V_1(e_1) + \frac{1}{2}e_2^2$  s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(e_1^2 + u)$$

4.  $u = -e_1^2 - ke_2$

□

Problem 3:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = u$$

*Proof.* 1. From problem 2:

$\dot{x}_2 = x_2(x_3 - k) - x_1^2 = -x_1^2 - kx_2 = u$  in Problem 2.

The origin of

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

is GAS.

And this is true for  $x_3 = 0 =: \alpha_2(x_1, x_2)$ .



$$2. (e_1, e_2, e_3) := (x_1 - 0, x_2 - \alpha_1(x_1), x_3 - \alpha_2(x_1, x_2)) \text{ s.t.}$$

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = u$$

$$3. V(e_1, e_2, e_3) = V_1(e_1) + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 \text{ s.t.}$$

$$4. u = -e_2^2 - ke_3$$

□

Problem 4:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = x_3(x_4 - k) - x_2^2$$

$$\dot{x}_4 = u$$

*Proof.* 1. Is GAS for

$$x_3(x_4 - k) - x_2^2 = -x_2^2 - kx_3$$

which is attained for  $x_4 = 0 =: \alpha_3(x_1, x_2, x_3)$ .

2.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = e_3(e_4 - k) - e_2^2$$

$$\dot{e}_4 = u$$

...

$$3. u = -e_3^2 - ke_4$$

□

Problem 5:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

...

$$\dot{x}_i = x_i(x_{i+1} - k) - x_{i-1}^2$$

...

$$\dot{x}_n = u$$

*Proof.* We will always have  $u = -e_{n-1}^2 - ke_n$ .

Let  $V : R \times \dots \times R \rightarrow R$ ,  $(e_1, \dots, e_n) \mapsto \sum_{i=1}^n V_i(e_i)$ , where  $V_i(e_i) = \frac{1}{2}e_i^2$ ,  $i = 2, \dots, n$ .

We have  $\dot{V}(e_1, \dots, e_n) = L_{f_1+g_1\alpha_1}V_1(e_1) - k \sum_{i=2}^{n-1} e_i^2 + e_n u + e_{n-1}g_{n-1}(x_1, \dots, x_{n-1})e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1})$ .

We observe that for  $\alpha_i$  being zero, the inequality

$$e_{n-1}g_{n-1}(x_1, \dots, x_{n-1})e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1}) + e_n u < 0$$

hence  $e_{n-1}^2 e_n + e_n u < 0$  for non-zero  $e$ .

It is solved by  $u = -e_{n-1}^2 - ke_n$ ,  $k > 0$ . □

## 7.5 Exercise 5

Consider the SISO system

$$\dot{x} = f(x) + g(x)(u + \sigma(x))$$

$$y = s(x)$$

$f, g : R^n \rightarrow R^n$ ,  $\sigma : R^n \rightarrow R$  and bounded,  $s : R^n \rightarrow R$

Design steps for SMC:

1. If no output is provided, design a sliding surface  $S := \{x \in R^n | s(x) = 0\}$  s.t.
  - (a) the system has rel. degree one;
  - (b) for  $y(t) \equiv 0$ , all solutions converge to the origin ("zero dynamics" have GAS origin)
2. Choose a control s.t. the sliding surface is reached (in finite time), e.g.

$$v(x) = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot \text{sgn}(s(x))), \quad \hat{u} > 0$$

Problem 1:

$$\dot{x}_1 = (x_2 - x_1)x_1^2$$

$$\dot{x}_2 = x_2 + u$$

Sliding surface  $S$ ,  $s : R^2 \rightarrow R$ ,  $(x_1, x_2) \mapsto x_2$

*Proof.* (a) For the given  $S$ , we have  $L_g s(x) = 1$  for any  $x \in R^2$ .

Moreover, from

$$\dot{s}(x) = L_f s(x) + L_g s(x)u$$

(we want  $\dot{s} = 0$ ) we have that for

$$u = -\frac{L_f s(x)}{L_g s(x)} = -x_2$$

the "dynamics on  $S$ " (i.e.  $x_2 = 0$ ) reduced to

$$\dot{\eta} = -\eta^3$$

whose origin is GAS.

(b) Consider

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot \text{sgn}(s(x))) = -x_2 - \hat{u} \cdot \text{sgn}(x_2), \quad \hat{u} > 0$$

such that  $x(t)$  "tends to  $S$ " in finite time (phase 1). Moreover, "on  $S$ ",  $x(t)$  converges to the origin  $t \rightarrow +\infty$  (phase 2).

□

Remark: Given a system in regular form

$$x = (\eta, \xi)^T$$

$$\dot{\eta} = f_1(\eta, \xi)$$

$$\dot{\xi} = f_2(\eta, \xi) + g_2(\eta, \xi)u$$

choose  $s(x) = \xi - \Phi(\eta)$ , s.t.  $\Phi$  as. stabilizes  $\dot{\eta} = f_1(\eta, \Phi(\eta))$ .

Problem 2:

$$\dot{x}_1 = -x_1 \cos x_2 + x_1 x_2$$

$$\dot{x}_2 = x_1 \cos x_1 + \sigma(x) + u$$

*Proof.* (a) (For the design of sliding surface pretend that uncertainty  $\sigma(x) = 0$ )

Let  $S := \{x \in R^2 | s(x) = 0\}$  be def. by  $s : R^2 \rightarrow R, (x_1, x_2) \mapsto x_2(-\Phi(x_1) = 0)$ . We have  $L_g s(x) = 1$  for all  $x \in R^2$ .

From

$$\dot{s}(x) = L_f s(x) + L_g s(x)u$$

(we want  $\dot{s} = 0$ ) s.t. for  $u = -\frac{L_f s(x)}{L_g s(x)} (= -x_1 \cos x_1)$  the "dynamics on  $S$ " (i.e.  $x_2 = 0$ ) reads

$$\dot{\eta} = -\eta$$

whose origin is GAS.

(b) Take

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|) \cdot \text{sgn}(s(x))) (= -x_1 \cos x_1 - (\hat{u} + (x_1^2 + x_2^2)) \cdot \text{sgn}(x_2)), \quad \hat{u} > 0$$

Consider the Lyapunov(-like) function  $V(x) = \frac{1}{2}s(x)^2$  s.t.

$$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x)))$$

Choosing  $u$  as above

$$\begin{aligned} \dot{V}(x) &= s(x)(-(\hat{u} + \beta(x)|L_g s(x)|) \cdot \text{sgn}(s(x)) + \sigma(x)L_g s(x)) \leq -(\hat{u} + \beta(x)|L_g s(x)|)|s(x)| + \\ &|\sigma(x)||L_g s(x)||s(x)| \leq -\hat{u}|s(x)| < 0 \text{ for } s(x) \neq 0 \end{aligned}$$

□

Problem 3:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^3 + \sigma(x) + u$$

$$s(x) = x_2 + x_1, \quad u = -x_2 + x_1^3 - 2 \cdot \operatorname{sgn}(s(x))$$

*Proof.* (a) Given  $S$ , we have  $L_g s(x) = 1$  for all  $x \in R^2$ . The "dynamics on  $S$ " (i.e.  $x_1 + x_2 = 0$ ) reads

$$\dot{\eta}_1 = -\eta_1$$

$$\dot{\eta}_2 = -\eta_2$$

whose origin is GAS.

(b) Take  $V(x) = \frac{1}{2}s(x)^2$  s.t.

$$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x))) \leq -\hat{u}|L_g s(x)||s(x)| + |\sigma(x)||L_g s(x)||s(x)| \leq [|\sigma(x)| \leq c] \leq -(\hat{u} - c)|L_g s(x)||s(x)|.$$

Hence, for  $c < \hat{u} = 2$  there exists  $\varepsilon > 0$  s.t.  $\dot{V}(x) \leq -\varepsilon|s(x)| < 0$  for  $s(x) \neq 0$

□

## 7.6 Exercise 6

Problem 1:

$$\dot{x} = xu(x^2 + u)$$

$$y = h(x)$$

$$s : R \times R \rightarrow R, \quad (u, y) \mapsto uy^2 + u^2y$$

$$S : R \rightarrow R, \quad x \mapsto \frac{x^2}{2}$$

*Proof.* Clearly,  $S$  is non-negative. Moreover:

$$\dot{S}(x) = x^2 u(x^2 + u) = x^4 u + x^2 u^2 = [h(x) = x^2] = s(u, x^2)$$

for all  $x, u \in R$  with  $h : R \rightarrow R, x \mapsto x^2$ .

□

Problem 2:

$$\dot{x} = u, \quad x(0) = x_0$$

$$y = x$$

$$s : R^n \times R^n \rightarrow R, \quad (u, y) \mapsto \langle u, y \rangle$$

*Proof.* For any  $x_0 \in R^n$ , we have

$$\begin{aligned} S_a(x_0) &= \sup_{u: [0, t] \rightarrow R^n, t \geq 0, x(0) = x_0} \left( - \int_0^t \langle u(\tau), y(\tau) \rangle d\tau \right) = \\ &= \sup_{-// -} \left( - \frac{1}{2} \int_0^t \frac{d}{d\tau} \|x(\tau)\|^2 d\tau \right) = \sup_{-// -} \left( - \frac{1}{2} \|x(t)\|^2 + \frac{1}{2} \|x(0)\|^2 \right) \leq \frac{1}{2} \|x_0\|^2 \end{aligned}$$

$\implies$  av. storage is finite  $\implies$  system is dissipative. Moreover, we have for any  $x_0 \in R^n$ ,

$$S_r(x_0) = \inf_{u: [-t, 0] \rightarrow R^n, t \geq 0, x(-t)=0, x(0)=x_0} \int_{-t}^0 \langle u(\tau), y(\tau) \rangle d\tau = \inf_{-// -} \left( \frac{1}{2} \|x_0\|^2 - \frac{1}{2} \|x(-t)\|^2 \right) = \frac{1}{2} \|x_0\|^2$$

( $S_a = S_r \implies$  this is a unique stor. func.)

Hence the (lossless) system is reachable (from 0 to any  $x_0$ ).  $\square$

Problem 3:

*Proof.* Consider the Lyapunov func. cand.  $V(x) = S_1(x_1) + S_2(x_2)$  s.t.

$$\dot{V}(x) \leq s_1(u_1, y_1) + s_2(u_2, y_2) = s_1(u_1, y_1) + s_2(y_1, -u_1) = 0 \implies \text{origin is stable.}$$

$\square$

Remark: the above problem captures many stability results (in the frequency domain). Particular choices of supply rates are:

- $s_i(u_i, y_i) = \|u_i\|^2 - \|y_i\|^2, i = 1, 2$  (small-gain theorem);
- $s_i(u_i, y_i) = \langle u_i, y_i \rangle, i = 1, 2$  (positive operator theorem);
- $s_1(u_1, y_1) = \langle u_1 + ay_1, u_1 + by_1 \rangle$   
 $s_2(u_2, y_2) = -ab \langle u_2 - \frac{1}{a}y_2, u_2 - \frac{1}{b}y_2 \rangle$  (conic operator theorem).

Problem 4:

$$\dot{x} = f(x) + G(x)u$$

$$y = h(x)$$

$$s : R^m \times R^m \rightarrow R, (u, y) \mapsto \|u\|^2 - \|y\|^2$$

*Proof.* Take  $V = S$  s.t.

$$\dot{V}(x) \leq \|u\|^2 - \|h(x)\|^2, \forall x \in R^n, \forall u \in R^m$$

Then the (continuous) state feedback  $u = \gamma h(x)$  for some  $|\gamma|^2 < 1$ , s.t.

$$\dot{V}(x) \leq (|\gamma|^2 - 1) \|h(x)\|^2 < 0, \forall x \neq 0$$

$\square$

Problem 5:

*Proof.* Take  $S(x) = \langle x, P_x \rangle$  s.t.

$$\dot{S}(x) = \langle x, (PA + A^T P)x \rangle + 2 \langle x, PBu \rangle$$

Add and subtract  $\gamma^2||u||^2$  and  $\frac{1}{\gamma^2}||B^T Px||^2$ .

$$\dot{S}(x) = \langle x, (PA + A^T P + \frac{1}{\gamma^2} P B B^T P) x \rangle + \gamma^2 ||u||^2 - \gamma^2 ||u - \frac{1}{\gamma^2} B^T P x||^2$$

Add and subtract  $||y||^2$ .

$$\dot{S}(x) = \langle x, (PA + A^T P + \frac{1}{\gamma^2} P B B^T P + C^T C) x \rangle + \gamma^2 ||u||^2 - ||y||^2 - \gamma^2 ||u - \frac{1}{\gamma^2} B^T P x||^2$$

$$\dot{S}(x) \leq \gamma^2 ||u||^2 - ||y||^2$$

□

## 7.7 Exercise 7

**Definition.** A mapping  $\Phi : R \rightarrow R$ ,  $u \mapsto \Phi(u)$ , belongs to the sector

- $[0, +\infty]$  if  $u\Phi(u) \geq 0$ ,  $\forall u \in R$ ;
- $[\alpha, +\infty]$  if  $u(\Phi(u) - \alpha u) \geq 0$ ,  $\forall u \in R$  and some  $\alpha \in R$ ;
- $[0, \beta]$  if  $\Phi(u)(\Phi(u) - \beta u) \leq 0$ ,  $\forall u \in R$  and some  $\beta \in R$ ;
- $[\alpha, \beta]$  if  $(\Phi(u) - \alpha u)(\Phi(u) - \beta u) \leq 0$ ,  $\forall u \in R$  and some  $\alpha, \beta \in R$ ;

Notation: we write, e.g.,  $\Phi \in [0, +\infty]$ .

Problem 1:

$$\begin{aligned} \dot{x} &= x^3 - kx + u, \quad k > 0 \\ y &= x \end{aligned}$$

*Proof.* Take, e.g.,  $S : R \rightarrow R, x \mapsto \frac{x^2}{2}$  ( $S \geq 0$ ) s.t.

$$\dot{S}(x) = x^2(x^2 - k) + yu \leq yu$$

whenever  $x \in [-\sqrt{k}, \sqrt{k}]$ .

Let  $\bar{x} \in R$  and take  $u = -\bar{x}^3 + k\bar{x}$  with init. condition  $x(0) = \bar{x}$ , s.t. we have  $x(t) = \bar{x}$  for all  $t \geq 0$ . If the system is passive, then along this (constant) solution we must have

$$S(x(t)) - S(\bar{x}) \leq \int_0^t u(\tau)y(\tau)d\tau, \quad t \geq 0$$

This inequality, however, is violated for  $\bar{x} \notin [-\sqrt{k}, \sqrt{k}]$  and hence  $[-\sqrt{k}, \sqrt{k}]$  must be the largest interval. □

Problem 2:

$$\begin{aligned}\dot{x} &= -x + \frac{1}{\beta}h(x) + u, \quad \beta > 0 \\ y &= h(x)\end{aligned}$$

$$S(x) = \int_0^x h(\sigma)d\sigma, \quad h \in [0, \beta]$$

*Proof.* Clearly, we have  $S \geq 0$  since  $h \in [0, \beta]$ .

Moreover,

$$\dot{S}(x) = S'(x)\dot{x} = \dot{x} \frac{d}{dx} \int_0^x h(\sigma)d\sigma = h(x)\dot{x} = \frac{1}{\beta}h(x)(h(x) - \beta x) + yu \leq yu$$

since  $h \in [0, \beta]$ . □

Problem 3:

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + kx_2 + u, \quad k > 0 \\ y = x_2 \end{cases}$$

*Proof.* Take  $S : R^2 \rightarrow R$ ,  $(x_1, x_2) \mapsto \frac{x_1^2}{2} + \frac{x_2^2}{2}$  s.t.  $\dot{S}(x) = uy + ky^2$ .

Let  $u = -\Phi(y)$ ,  $\Phi : R \rightarrow R$  satisfying  $\Phi \in [l, +\infty]$  for some  $l > k$  ( $\nu_2 + \rho_1 > 0$ ) s.t.

$$\dot{S}(x) = -y\Phi(y) + ky^2 \leq -(l - k)y^2$$

Since the system  $H_1$  is ZSO the origin is GAS. □

Problem 4:

*Proof.* Take  $S(x) = S_1(x_1) + S_2(x_2)$  s.t.

$$\dot{S}(x) \leq \langle u_1, y_1 \rangle - \rho_1 \|y_1\|^2 - \nu_1 \|u_1\|^2 + \langle u_2, y_2 \rangle - \rho_2 \|y_2\|^2 - \nu_2 \|u_2\|^2$$

Using that

$$\langle u_1, y_1 \rangle + \langle u_2, y_2 \rangle = \langle u - y_2, y_1 \rangle + \langle v + y_1, y_2 \rangle = \langle u, y_1 \rangle + \langle v, y_2 \rangle$$

and

$$\begin{aligned}\|u_1\|^2 &= \|u\|^2 - 2\langle u, y_2 \rangle + \|y_2\|^2 \\ \|u_2\|^2 &= \|v\|^2 + 2\langle v, y_1 \rangle + \|y_1\|^2\end{aligned}$$

we obtain

$$\begin{aligned}\dot{S}(x) &= -\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} (\nu_2 + \rho_1)I_m & \\ & (\nu_1 + \rho_2)I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \nu_1 I_m & \\ & \nu_2 I_m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} I_m & 2\nu_1 I_m \\ -2\nu_2 I_m & I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \\ &\leq [Coshi - Schwarz] \leq -a\|(y_1, y_2)\|^2 + b\|(u, v)\| \|(y_1, y_2)\| + c\|(u, v)\|^2\end{aligned}$$

with  $a = \min\{\nu_2 + \rho_1, \nu_1 + \rho_2\} > 0$ ,  $b = \|N\| \geq 0$  and  $c = \|M\| \geq 0$ .

Hence,

$$\dot{S}(x) \leq -\frac{1}{2a}(b\|(u, v)\| - a\|(y_1, y_2)\|)^2 + \frac{b^2}{2a}\|(u, v)\|^2 - \frac{a}{2}\|(y_1, y_2)\|^2 + c\|(u, v)\|^2 \leq \frac{b^2 + 2ac}{2a}\|(u, v)\|^2 - \frac{a}{2}\|(y_1, y_2)\|^2 \quad \square$$

Problem 5:

*Proof.* Take  $V(x) = \langle x, Px \rangle$  s.t.

$$\dot{V}(x) = \langle x, (PA + A^T P)x \rangle - 2\Phi(y) \langle x, PB \rangle$$

Add and subtract  $2\Phi(y)^2$  and  $2\Phi(y)\beta Cx$  yields

$$\begin{aligned} \dot{V}(x) &= -\varepsilon \langle x, Px \rangle - \langle x, L^T Lx \rangle - 2\Phi(y) \langle x, PB - \beta C^T \rangle - 2\Phi(y)^2 + 2\Phi(y)(\Phi(y) - \beta y) = \\ &= -\varepsilon \langle x, Px \rangle - |Lx - \sqrt{2}\Phi(y)|^2 + 2\Phi(y)(\Phi(y) - \beta y) \leq -\varepsilon \langle x, Px \rangle. \end{aligned} \quad \square$$

## 7.8 Exercise 8

Problem 1:

Show that if  $\Phi : [0, +\infty) \rightarrow \mathbb{R}$  and  $\Phi \in L_1 \cap L_\infty$ , then  $\Phi \in L_p \forall p \in [1, +\infty]$ .

*Proof.* Holder inequality:

Let  $p, q \in [1, +\infty]$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $f \in L_p$  and  $g \in L_q$ , then  $fg \in L_1$  with

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q}$$

Convention: If  $p = +\infty$  ( $q = +\infty$ ), then  $\frac{1}{p} = 0$  ( $\frac{1}{q} = 0$ ).

Let  $p \in (1, +\infty)$ . We have

$$(\|\Phi\|_{L_p}^p =) \int_0^\infty |\Phi(t)|^p dt = \int_0^\infty |\Phi(t)| \Phi(t)^{p-1} dt \leq \|\Phi\|_{L_1} \|\Phi\|_{L_\infty}^{p-1}$$

Hence,  $\Phi \in L_p$  for any  $p \in [1, +\infty)$  □

Problem 2:

*Proof.* b)  $\Phi_2 = \frac{1}{t+1}$

For  $p \in \{1, 2\}$ , we have

$$\int_0^\infty |\Phi_2(t)|^p dt = \lim_{T \rightarrow +\infty} \int_0^T \frac{1}{(t+1)^p} dt = \begin{cases} \lim_{T \rightarrow +\infty} [\ln(t+1)]_0^T = +\infty, & p = 1 \\ \lim_{T \rightarrow +\infty} [-\frac{1}{t+1}]_0^T < +\infty, & p = 2 \end{cases}$$

s.t.  $\Phi_2 \in L_1$ , but  $\Phi_2 \notin L_2$ .

Moreover,

$$\sup_{t \geq 0} |\Phi_2(t)| = \sup_{t \geq 0} \left| \frac{1}{t+1} \right| = 1 < +\infty$$

s.t.  $\Phi_2 \in L_\infty$ . □

Problem 3:



*Proof.* Take  $v = \frac{u}{\|u\|_{L_p}}$ ,  $u \neq 0$ , s.t.  $v \in L_p$  and  $\|v\|_{L_p} = 1$ .

Then

$$\|Hv\|_{L_p} = \|H \frac{u}{\|u\|_{L_p}}\|_{L_p} = [H - \text{linear}] = \|\frac{1}{\|u\|_{L_p}}Hu\|_{L_p} = \frac{u}{\|u\|_{L_p}}\|Hu\|_{L_p}$$

□

Problem 4: Let  $\Phi : [0, +\infty) \rightarrow \mathbb{R}$  be defined by  $\Phi(t) = t$ . Show that  $\Phi \in L_p^e$  for any  $p \in [1, +\infty]$ .

*Proof.* Let  $p \in [1, +\infty)$  and fix some  $T \geq 0$ . Then

$$(\|\Phi_T\|_{L_p}^p =) \int_0^\infty |\Phi_T(t)|^p dt = \int_0^T |\Phi(t)|^p dt = \int_0^T |t|^p dt = \frac{t^{p+1}}{p+1} \Big|_0^T = \frac{T^{p+1}}{p+1} < +\infty$$

Moreover,

$$\|\Phi_T\|_{L_\infty} = \sup_{t \geq 0} |\Phi_T(t)| = \sup_{t \in [0, T]} |\Phi(t)| = T < +\infty$$

$$\implies \Phi \in L_p^e \quad \forall p \in [1, +\infty].$$

□

Problem 5:

*Proof.*  $\implies$ :

Take  $u, v \in L_p^e$  s.t.  $v_T = u_T$  for some  $T \geq 0$ . Then, by causality of  $H$ ,

$$H(u)_T = H(u_T)_T, \quad H(v)_T = H(v_T)_T$$

Since  $u_T = v_T$ , it follows  $H(u)_T = H(v)_T$ .

$\Leftarrow$ :

Take  $u \in L_p^e$  and consider  $v = u_T$  for some  $T \geq 0$ . Noting that  $v_T = (u_T)_T = u_T$ , we have

$$H(u)_T = H(v)_T = H(u_T)_T$$

□

Problem 6:

*Proof.* Let  $u \in L_\infty^e$  and  $T \geq 0$  s.t. for any  $0 \leq t \leq T$

$$\begin{aligned} |y_i(t)| &\leq \int_0^t |h(t-\tau)| |u(\tau)| d\tau \leq \sup_{\tau \in [0, T]} |u(\tau)| \int_0^t |h(t-\tau)| d\tau = [\sigma = t - \tau] = \\ &= \sup_{\tau \in [0, T]} |u(\tau)| \int_0^t |h(\sigma)| d\sigma \leq \sup_{\tau \in [0, T]} |u(\tau)| \left( \int_0^t |h| + \int_t^\infty |h| \right) \dots \end{aligned}$$

□

## 7.9 Exercise 9

Problem 1:

*Proof.* Let  $H_2 : L_P^e \rightarrow L_P^e$  be defined by  $H_2(e_2) = \frac{1}{2\gamma}e_2$ .  
Using that  $y_2 = H_2(e_2)$ , it follows

$$\|(y_2)_T\|_{L_P} = \frac{1}{2\gamma} \|(e_2)_T\|_{L_2}, \quad \forall e_2 \in L_P^e, \quad \forall T \geq 0$$

Since  $\gamma_1\gamma_2 = \gamma\frac{1}{2\gamma} = \frac{1}{2} < 1$ , the SGT reveals that  $y \in L_P$  whenever  $u \in L_P$ .  $\square$

Problem 2:

*Proof.* Let  $e_2 \in L_\infty^e$  and  $T \geq 0$  s.t.  $y_2 = H_2(e_2)$ . Then

$$(\|(y_2)_T\|_{L_2}^2 =) \int_0^T |y_2(t)|^2 dt \leq \text{ess sup}_{t \in [0, T]} |y_2(t)| \int_0^T |y_2(t)| dt = \|(y_2)_T\|_{L_\infty} \|(y_2)_T\|_{L_1} \leq \delta\varepsilon \|(e_2)_T\|_{L_\infty}^2$$

Hence,

$$\|(y_2)_T\|_{L_2} \leq \sqrt{\delta\varepsilon} \|(e_2)_T\|_{L_\infty}, \quad \forall e_2 \in L_\infty^e, \quad \forall T \geq 0$$

for any  $u \in L_2^e$  and  $T \geq 0$  (with  $y = y_2$ ),

$$\|y_T\|_{L_2} \leq \sqrt{\delta\varepsilon} \|(y_1)_T\|_{L_\infty} \leq \gamma\sqrt{\delta\varepsilon} \|(e_1)_T\|_{L_2} \leq \gamma\sqrt{\delta\varepsilon} \|u_T\|_{L_2} + \gamma\sqrt{\delta\varepsilon} \|y_T\|_{L_2}$$

s.t.

$$(1 - \gamma\sqrt{\delta\varepsilon}) \|y_T\|_{L_2} \leq \gamma\sqrt{\delta\varepsilon} \|u_T\|_{L_2}$$

If  $\gamma\sqrt{\delta\varepsilon} < 1$ , then the SGT reveals that  $y \in L_2$  whenever  $u \in L_2$ .  $\square$

Problem 3:

*Proof.* For any  $u \in L_2^e$  and  $T \geq 0$ , we have

$$\langle u_T, y_T \rangle_{L_2} = \langle u_T, h(x)_T \rangle_{L_2} = \langle a\dot{x}_T + x_T, h(x)_T \rangle_{L_2} = a \int_0^T h(x(t))\dot{x}(t) dt + \int_0^T x(t)h(x(t)) dt$$

Since  $h \in [0, +\infty]$ , i.e.  $xh(x) \geq 0$ ,  $\forall x \in R$ , we have  $\int_0^T x(t)h(x(t)) dt \geq 0$ .

Now,  $\Phi : R \rightarrow R$  be defined by  $\Phi(x) = \int_0^x h(\sigma) d\sigma$ .

Clearly,  $\Phi(x) \geq 0$  for all  $x \in R$ .

It follows that

$$a \int_0^T h(x(t))\dot{x}(t) dt = a \int_{x(0)}^{x(T)} h(\sigma) d\sigma = a(\Phi(x(T)) - \Phi(x(0))) \geq -a\Phi(x(0)) =: \beta$$

Hence, the system is passive ( $\langle u_T, y_T \rangle_{L_2} \geq -\beta$ ,  $\forall u \in L_2^e$ ,  $\forall T \geq 0$ ).  $\square$

Problem 4:

*Proof.* Since  $H_1$  is passive (with zero bias ( $\beta = 0$ )), it follows

$$\langle (e_1)_T, (y_1)_T \rangle_{L_2} \geq 0, \quad \forall e_1 \in L_2^e, \quad \forall T \geq 0.$$

Using that

$$\langle (e_2)_T, (y_2)_T \rangle_{L_2} \geq \delta \|(e_2)_T\|_{L_2}^2, \quad \forall e_2 \in L_2^e, \quad \forall T \geq 0,$$

we have for  $u \in L_2^e$  and  $T \geq 0$  with  $y = y_1$ ,

$$\langle u_T, y_T \rangle_{L_2} = \langle (e_1)_T, (y_1)_T \rangle_{L_2} + \langle (e_2)_T, (y_2)_T \rangle_{L_2} \geq \delta \|y_T\|_{L_2}^2.$$

Using Caschy-Schwarz

$$\|y_T\|_{L_2} \|u_T\|_{L_2} \geq |\langle u_T, y_T \rangle_{L_2}| \geq \langle u_T, y_T \rangle_{L_2} \geq \delta \|y_T\|_{L_2}^2$$

s.t.

$$\|y_T\|_{L_2} \leq \frac{1}{\delta} \|u_T\|_{L_2}$$

Since  $u \in L_2$ , passing to the limit, as  $T \rightarrow \infty$ , reveals  $y \in L_2$ . □