

Nonlinear Control

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Intro

Goals of Course

- overview over modern nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control methods learn the mathematical basis

Differential equations $\dot{x} = f(x)$

Nonlinear differential equation $\dot{x} = f(t, x)$

System with input $\dot{x} = f(x, u)$

System with input and output $\dot{x} = f(x, u), \quad y = g(x, u)$

Input-output methods

Scope

[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997

1 Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \quad (1)$$

Where $f : D \rightarrow \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is open.

Solution to $x : I_{x_0} \rightarrow D$, $t \rightarrow x(t)$ is differentiable

Interval existence solution

Questions:

- # existence of solution
- # "how large" is I_{x_0}
- # uniqueness of solution

Usually we will add some restrictions on f functions, like continuous.

1.1 Existence of solutions

Definition. Function $f : D \rightarrow R^n$ is continuous at $x' \in D$ if for $\forall \epsilon > 0 \exists \delta > 0$ such that for $\forall x \in D, \|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < \epsilon$

Function $f : D \rightarrow R^n$ is continuous on D if it's continuous at $\forall x' \in D$

Theorem 1.1 (Piano). If $f : D \rightarrow R^n$ continuous, then for each $x_0 \in D \exists x : (-\epsilon, \epsilon) \rightarrow D, \epsilon > 0$ satisfying (1).

Further, given a compact set $U \subset D$, then $\exists \alpha > 0$ s.t. $\forall x_0 \in U \exists x : (-\alpha, \alpha) \rightarrow D$ satisfying (1).

Example. Consider equation $\dot{x}(t) = x(t)^2, x(0) = x_0 = 0$. Solution $x(t) = -\frac{1}{t-c}, c = \frac{1}{x_0}$. In this example solution exist in interval $(-c, c)$.

But, what about the number of solutions? Which conditions we should add to guarantee uniqueness of solution?

1.2 Uniqueness of solutions

Definition. Function $f : D \rightarrow R^n$ is locally Lipschitz on D if $\forall x \in D$ there is a neighbourhood $N(x) \subset D$ and $\exists L > 0$ s.t.

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \quad (2)$$

For all $x_1, x_2 \in N$.

- Lipschitz on $W \in D$ if (2) holds $\forall x_1, x_2 \in W$ (with same L)
- globally Lipschitz if (2) holds $\forall x_1, x_2 \in R^n$ (with same L)

We have

- # locally Lipschitz functions are continuous
- # differentiable functions are locally Lipschitz

locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

Lemma 1 (Cromwall). Suppose that $0 \leq \phi(t) \leq c + L \int_0^t \phi(\tau) d\tau$, $c, L > 0$, ϕ - continuous. Then $\phi(t) \leq ce^{Lt}$.

Proof. $c + L \int_0^t \phi(\tau) d\tau := \psi(t)$, $\dot{\psi}(t) = L\phi(t) \leq L\psi(t)$.

Consider $\frac{d}{dt} (\psi(t)e^{-Lt}) = e^{-Lt} (\dot{\psi}(t) - L\psi(t)) \leq 0$, thus $\psi(t)e^{-Lt}$ is decreased, and as a result we have $\phi(t)e^{-Lt} \leq \psi(t)e^{-Lt} \leq \psi(0) = c$

□

Theorem 1.2 (Picard Lindelof). If function $f : D \rightarrow R^n$ is locally Lipschitz then for $\forall x_0 \in D$ $\exists ! x : (-\epsilon, \epsilon) \rightarrow D$, $\epsilon > 0$ satisfying (1).

Proof. * existence from Piano theorem

Proof of uniqueness

Consider two solutions $x_1(\cdot)$ and $x_2(\cdot)$ to (1). $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$, $x_1(0) = x_2(0)$. Then we can integrate equality: $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau)) d\tau$. $|x_1(t) - x_2(t)| \leq \int_0^t |f(x_1(\tau)) - f(x_2(\tau))| d\tau \leq L \int_0^t |x_1(\tau) - x_2(\tau)| d\tau$. Now we can apply Cromwall's lemma with $c = 0$ and $\phi(t) = |x_1(t) - x_2(t)|$, then $\phi(t) \leq 0$, then $x_1(t) = x_2(t)$, $\forall t \in (-\epsilon, \epsilon)$ □

Example.

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$\text{Solutions } x(t) = \begin{cases} \frac{1}{4}(t - c)^2, & \text{if } t \geq c \geq 0 \\ 0, & \text{else} \end{cases}$$

Here we don't have unique solution, we have a family of solutions.

Global existence & uniqueness

- sufficient condition: f globally Lipschitz
- another sufficient condition: solution entirely lies in a compact set
- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff \exists solution $V : R^n \rightarrow R \geq 0$ s.t. $\frac{\partial V}{\partial x} f(x) \leq -V(x)$, $\forall x \in R^n$

1.3 Lyapunov stability

If functions $\dot{V}(x) < 0, \forall x \in D \setminus \{0\}$, then x^* is asymptotically stable.

Definition. Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0 \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \leq \epsilon, \forall t \geq 0$.

Definition. Equilibrium point $x = 0$ is asymptotically stable if it is stable and exist $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t \rightarrow \infty} x(t) \rightarrow 0$.

Theorem 1.3 (Lyapunov's direct method). Let $x^* = 0 \in D$ be an equilibrium point of (1), i.e., $f(0) = 0$. Let $f : D \rightarrow R^n$ is continuous. If there exists a differentiable $V : D \rightarrow R$ s.t.

1. $V(x^*) = 0, V(x) > 0, \forall x \in D \setminus \{0\}$
2. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0, \forall x \in D$

then $x^* = 0$ is stable.

Proof. Fix compact $U = \{x : V(x) \leq c\}$ s.t. $U \in D$. By Piano: exist $\alpha > 0$ s.t. any solution x with $x_0 \in U$ exists at least on the interval $[0, \alpha)$.

TODO proof is not full □

Lyapunov's direct method gives us:

- stability
- convergence (if $V < 0$)
- subset of the region of attraction (all compact $U = \{x : V(x) \leq c\} \in D$)
- existence of solution for all times

2 Nonlinear systems

In this section we consider function $f : R \times D \rightarrow R^n$, where $D \subseteq R^n$, and D is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \tag{3}$$

The origin $x^* \in D$ is an equilibrium point for (3), if $f(t, 0) = 0, \forall t \geq 0$.

Remark: EP (equilibrium point) $x^* = 0$ can be translation of a nonzero solution.

Suppose \bar{y} is a solution of $\dot{y} = g(t, y)$.

Change of coordinates: $x(t) = y(t) - \bar{y}(t)$, then $\dot{x}(t) = \dot{y}(t) - \dot{\bar{y}}(t) = g(t, x(t) + \bar{y}(t)) - \dot{\bar{y}}(t) := f(t, x(t))$. Since $\dot{\bar{y}}(t) = g(t, \bar{y}(t))$, then $f(t, 0) = 0, \forall t \geq 0$.

Existence and uniqueness of solution to (3):

- if f continuous, then exists a local solution
- if f continuous and locally Lipschitz in x^* , then exists a local unique solution

Now we need new stability definitions.

Definition. Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \geq 0, \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| < \epsilon, \forall t \geq t_0$.

Definition. Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0 \exists \delta > 0$, s.t $\forall t_0 \geq 0$, from $\|x_0\| < \delta$ follows $\|x(t)\| < \epsilon, \forall t \geq t_0$.

Definition. Point $x^* = 0$ asymptotically stable if it is stable and $\forall t_0 \geq 0 \exists c > 0$, s.t from $\|x_0\| < c$ follows $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$.

Definition. Point $x^* = 0$ uniformly asymptotically stable if it is uniformly stable and $\exists c > 0$, s.t $\forall t_0 \geq 0$ from $\|x_0\| < c$ follows $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$.

Definition. Convergence: $\forall \eta > 0 \forall t_0 \geq 0, \exists T > 0$ such that $\forall t \geq t_0 + T$ follows $\|x(t)\| < \eta$.

Definition. Uniform convergence: $\forall \eta > 0 \exists T > 0$ such that $\forall t_0 \geq 0$ and $\forall t \geq t_0 + T$ follows $\|x(t)\| < \eta$.

Example. Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0$$

Solution $x(t) = x(t_0) \frac{1+t_0}{1+t}$. It is uniformly stable, because we can choose $\delta = \epsilon$. But does $x(t)$ convergence uniformly? Answer is no.

Definition. Point $x^* = 0$ is globally uniformly asymptotically stable if it is uniformly stable with $\delta \rightarrow \infty$ for $\epsilon \rightarrow \infty$ and $\forall c, \eta \exists T > 0$ such that $\forall t_0 \geq 0$ from $\|x_0\| < c$ follows $\|x(t)\| < \eta, \forall t \geq t_0 + T$.

2.1 Lyapunov's direct method

Consider some function $V : [0, \infty) \times D \rightarrow R, (t, x) \rightarrow V(t, x)$ such that $\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x)$.

Theorem 2.1 (Lyapunov's direct method). Let $f : [0, \infty) \times D \rightarrow R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \rightarrow R$ with:

- $W_1(x) \leq V(t, x) \leq W_2(x), \forall t \geq 0, x \in D$

- $\dot{V}(t, x) \leq 0, \forall t \geq 0, x \in D$

where $W_1, W_2 : D \rightarrow R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.

If further $\dot{V}(t, x) \leq -W_3(x), \forall t \geq 0, x \in D$ with $W_3 : D \rightarrow R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.

If $D = R^n$ and W_1 is radially unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Check function $V(t, x) = \frac{1}{2}x^2$ as candidate for Lyapunov's function. Then $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and $\dot{V}(t, x) = -(1+t)x^2 \leq -x^2(t) =: W_3(x)$. Then from theorem we have, that $X^* = 0$ is globally uniformly asymptotically stable.

2.2 Exponential stability

Definition. Point $X^* = 0$ is an exponentially stable EP of (3) if $\exists \lambda, c, k > 0$ s.t. $t \geq t_0 \geq 0$ and all $\|x_0\| < c$ follows $\|x(t)\| \leq k\|x(t_0)\|e^{\lambda(t-t_0)}$.

Remark: from exponential stability follows uniformly asymptotically stability.

Lemma 2 (Auxiliary result). Let $\dot{x}(t) = f(t, x(t))$, f scalar and $\dot{\xi}(t) \leq f(t, \xi(t))$ with $\xi(t_0) \leq x(t_0)$. Then $\xi(t) \leq x(t) \quad \forall t \geq t_0$.

Theorem 2.2. Let $f : [0, \infty) \times D \rightarrow R^n$ be continuous and $x^* = 0 \in D$ be an EP.

If there is a differentiable function $V : [0, \infty) \times D \rightarrow R$ and constants $k_1, k_2, k_3, a > 0$ s.t.

1. $k_1\|x\|^a \leq V(t, x) \leq k_2\|x\|^a, \forall t \geq 0, x \in D$
2. $\dot{V}(t, x) \leq -k_3\|x\|^a$

then $x^* = 0$ is exponentially stable.

If $D = R^n$, then X^* is globally exponential stable.

Proof. For $c > 0$ small enough, trajectories initialized in $\{x : k_2\|x\|^a < c\}$ remain bounded and in D . From 1) and 2) we can conclude $\dot{V} \leq -\frac{k_3}{k_2}V$. Then from previous Lemma $V(t, x(t)) \leq$

$$V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}. \quad \text{Then } \|x(t)\| \leq [from(1)] \leq \left(\frac{V(t, x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} \leq$$

$$\left(\frac{k_2\|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(t_0)\| e^{-\frac{k_3}{k_2 a}(t-t_0)} \quad \square$$

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Here $V(t, x) = \frac{1}{2}x^2$ then X^* is exponentially stable.

2.3 Comparison function

Definition. A function $\alpha : [0, \delta) \rightarrow [0, \infty)$ is (of) "class K " if it is continuous, strictly increasing, and $\alpha(0) = 0$.

Definition. A function $\alpha : [0, \delta) \rightarrow [0, \infty)$ is "class K_∞ " if $\alpha \in K$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$.

Example. Function $\alpha(r) = \tan^{-1}(r)$ – class K

Function $\alpha(r) = r^k$ – class K_∞

Definition. A function $\beta : [0, \delta) \times [0, \delta) \rightarrow [0, \infty)$ is "class KL " if it is continuous, $\beta(\cdot, s) \in K$ for all fixed s , and for each fixed r , $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s \rightarrow \infty} \beta(r, s) = 0$

Example. Function $\beta(x, s) = \max(r, r^2)e^{-s}$ belongs to class KL .

Properties of comparison functions:

- If $\alpha \in K$ on $[0, \delta)$, then α^{-1} is defined on $[0, \alpha(\delta))$ and $\alpha^{-1} \in K$.
- If $\alpha \in K_\infty$, then $\alpha^{-1} \in K_\infty$
- If $\alpha_1, \alpha_2 \in K$, then $\alpha_1 \circ \alpha_2 \in K$ (same for K_∞)
- If $\alpha_1, \alpha_2 \in K$, $\beta \in KL$ then $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we consider comparison functions and stability definitions.

Lemma 3. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \geq t_0, \forall \|x(t_0)\| < c$ and $\|x(t)\| \leq \alpha(\|x(t_0)\|)$.

(only sufficiency). Given $\epsilon > 0$ choose $\delta < \min(c, \alpha^{-1}(\epsilon))$. Then from $\|x(t_0)\| < \delta$ follows $\|x(t)\| \leq \alpha(\|x(t_0)\|) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$. \square

Lemma 4. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0, \forall \|x(t_0)\| < c$ and $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

(only sufficiency). Let $\|x(t_0)\| < c$. Then $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$. This means uniform convergence. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x(t_0)\|, 0)$. This gives us uniform stability. \square

Lemma 5. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable iff previous lemma holds for all $x_0 \in \mathbb{R}^n$.

Now consider comparison functions and Lyapunov functions

If $W : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous and positive definite, then $\forall r > 0 \exists \alpha_1, \alpha_2 \in K$ s.t. $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$ for all $x \in B_r(0) = \{x \mid \|x\| \leq r\}$.

If W is radially unbounded, then $\exists \alpha_1, \alpha_2 \in K_\infty$ s.t. $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{R}^n$.

Lemma 6 (Auxiliary). Consider $\dot{y} = \alpha(y)$, $y(t_0) = y_0 > 0$, $\alpha \in K$. Then $\exists \beta \in KL$ s.t. $y(t) = \beta(y_0, t - t_0)$.

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t, x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where W_1, W_2, W_3 – continuous and positive defined.

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in K$ such that $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$ and $\dot{V}(t, x) \leq -\alpha_3(\|x\|)$.

Proof uniform stability:

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq [\alpha_1 \in K] \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)).$$

Proof uniform convergence

$\dot{V} \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V))$. We know, that $\alpha_3 \circ \alpha_2^{-1} \in K$. By comparison lemma, $V(t, x(t)) \leq W(t)$, where W solves $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$ with $W(t_0) = V(t_0, x(t_0))$. By auxiliary lemma $\exists \beta \in KL$ s.t. $V(t, x(t)) \leq \beta(V(t_0, x(t_0)), t - t_0)$, then $\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\beta(V(t_0, x(t_0)), t - t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)) =: \bar{\beta}(\|x(t_0)\|, t - t_0)$. From this follows uniform asymptotic stability since $\bar{\beta} \in KL$.

2.4 Converse theorems

Theorem 2.3. Let $X^* = 0$ be an EP of $\dot{x}(t) = f(t, x(t))$ with $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuously differentiable and $\frac{\partial f}{\partial x}$ bounded in \mathbb{R}^n , uniformly in t ($\|\frac{\partial f}{\partial x}(t, x)\| \leq L$ for all $x \in \mathbb{R}^n$, $t \geq 0$, $L > 0$).

If $x^* = 0$ is globally exponentially stable, then exists differentiable $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $c_1, c_2, c_3, c_4 > 0$ s.t. $c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2$, $\dot{V}(t, x) \leq -c_3\|x\|^2$ and $\|\frac{\partial V}{\partial x}\| \leq c_4\|x\|$.

Proof. Let $\Phi(\tau; t, x)$ – solution to $\dot{x}(t) = f(t, x(t))$ which is static at (t, x) .

$V(t, x) = \int_t^{t+\delta} \Phi^T(\tau; t, x) \Phi(\tau; t, x) d\tau$, $\delta > 0$. Upper bound: $V(t, x) = \int_t^{t+\delta} \|\Phi(\tau; t, x)\|_2^2 d\tau \leq [exponential\ stability] \leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau \|x\|_2^2 = \frac{k^2}{2\lambda} (1 - e^{-2\lambda\delta}) \|x\|_2^2$.

Lower bound: since $\|\frac{\partial V}{\partial x}\| \leq L$, then $\|f(t, x)\|_2 \leq L\|x\|_2$. Thus by comparison lemma $\|\Phi(\tau; t, x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$. Set it in $V(t, x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau \|x\|_2^2 = \frac{1}{2L} (1 - e^{-2L\delta}) \|x\|_2^2$.

Decrease conditions: $\dot{V}(t, x) = \dots \leq -(1 - k^2 e^{-2\lambda\delta}) \|x\|_2^2$. □

3 System with inputs

Consider equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (4)$$

where $f : R^n \rightarrow R^n$.

Assumption: f is locally Lipschitz.

Exogenous signal $u : R \rightarrow R^n$.

Input can be "bad" (disturbance) or "good" (control).

3.1 Input-to-state stability

Motivation: LTI system $\dot{x} = Ax + Bu$, $x(0) = x_0$.

Solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$. If A is Hurwitz, then $\|e^{At}\| \leq ce^{-\lambda t}$ for some $c, \lambda > 0$.

How large can x grow for some bounded u ? $\|x(t)\| \leq \|e^{At}\| \|x_0\| + \int_0^t \|e^{A(t-\tau)}\| \|B\| \|u(\tau)\| d\tau \leq e^{-\lambda t} c \|x_0\| + \int_0^t e^{-\lambda(t-\tau)} c \|B\| \|u(\tau)\| d\tau = ce^{-\lambda t} \|x_0\| + (1 - e^{-\lambda t}) \frac{c}{\lambda} \|B\| \sup_{\tau \in [0, t]} \|u(\tau)\|$.

- $ce^{-\lambda t} \|x_0\|$ class KL in $(\|x_0\|, t)$
- $(1 - e^{-\lambda t})$ less than 1
- $\frac{c}{\lambda} \|B\| \sup \|u(\tau)\|$ class K

If $\sup_{\tau \in [0, t]} \|u(\tau)\|$ is bounded then \dot{x} remains bounded. Even more: the smaller $\sup_{\tau \in [0, t]} \|u(\tau)\|$, the smaller $\|x(t)\|$.

Definition. System (4) is input-to-state stable (ISS) if $\exists \beta \in KL, \gamma \in K$ s.t. $\forall x_0 \in R^n, \forall t \geq 0$ follows $\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\sup_{\tau \in [0, t]} \|u(\tau)\|)$.

Remarks:

- From ISS follows O-GAS (global asymptotical stability of $x = 0$ for $\dot{x} = f(x, 0)$)
- γ can be interpreted as "gain" w.r.t. u
- if $\lim_{t \rightarrow \infty} u(t) = 0$ then $\lim_{t \rightarrow \infty} x(t) = 0$

Example. Consider equation $\dot{x} = -x + xu$. System is O-GAS, not ISS (for example $u \equiv \alpha \Rightarrow \dot{x} = x(\alpha - 1)$ all solution diverge).

Example. Consider equation $\dot{x} = -3x + (1 + 2x^2)u$. System is O-GAS, not ISS (for example $u \equiv 1$, $x_0 = 2$, $x(t) = \frac{3-e^t}{3-2e^t}$ has a finite escape time).

Theorem 3.1. Suppose that there exists a continuously differentiable function $V : R^n \rightarrow R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x\|)$, $\forall x : \|x\| \geq \rho(\|u\|)$. Then (4) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof. Idea: same as Lyapunovs direct method when x is "outside" of ball $\{x : \|x\| \leq \rho(\|u\|)\}$

TODO Picture □

Example. Consider equality $\dot{x} = -x^3 + u$. Let $V(x) = \frac{1}{2}x^2$, then $\dot{V} = -x^4 + xu = [0 < \Theta < 1] = -(1 - \Theta)x^4 - \Theta x^4 + xu \leq -(1 - \Theta)x^4$ for all $x : \|x\| \geq \left(\frac{\|u\|}{\Theta}\right)^{\frac{1}{3}}$. Thus, system is ISS with $\gamma(v) = \rho(v) = \left(\frac{v}{\Theta}\right)^{\frac{1}{3}}$.

Remarks:

- Existence of V is both necessary and sufficient for ISS;
- (??) is equivalent to $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_4(\|x\|) + \alpha_5(\|u\|)$, $\forall x, u$ for some $\alpha_4, \alpha_5 \in K$;
- If $x_1 = 0$ is a globally asymptotically stable EP of Σ_1 and Σ_2 is ISS w.r.t. "input" x_1 , then $(x_1, x_2) = (0, 0)$ is a globally asymptotically stable EP for the cascaded system.

Theorem 3.2. Assume that:

- f is globally Lipschitz;
- $x = 0$ is a globally exponentially stable EP for $\dot{x} = f(x, 0)$

Then the system (4) is ISS.

Proof. Sketch: \exists continuous differentiable V :

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x, 0) \leq -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

Then:

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) &= \frac{\partial V}{\partial x} f(x, 0) + \frac{\partial V}{\partial x} (f(x, u) - f(x, 0)) \leq -c_3\|x\|^2 + c_4\|x\|L\|u\| = -c_3(1 - \theta)\|x\|^2 - \theta\|x\|^2 + \\ c_4L\|x\|\|u\| &\leq -c_3(1 - \theta)\|x\|^2 \\ \text{if } \|x\| &\geq \frac{c_4L}{\theta c_3}\|u\|. \end{aligned} \quad \square$$

3.2 Control Lyapunov functions

Motivation: Lyapunov theory for control systems.

(input affine systems)

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i = f(x) + G(x)u,$$

$$f : R^n \rightarrow R^n, g : R^n \rightarrow R^n, G : R^n \rightarrow R^{n \times m}$$

$u : t \rightarrow u(t), R \rightarrow R^m$ is a control signal (decision variable).

Definition. A function $V : R^n \rightarrow R$ is a control Lyapunov function (CLF) if it's differentiable positive definite, radially unbounded and

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) < 0 \quad (5)$$

Remark:

Concept can be generalized to systems $\dot{x} = f(x, u)$. Then 5 becomes

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot f(x, u)) < 0$$

Theorem 3.3 (Artstein). There exists $k : R^n \rightarrow R^m$ (state feedback) which is continuous on $R^n \setminus \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff there exists a CLF.

How to find CLFs?

Proposition:

Condition (5) is equivalent to

$$\forall x \neq 0, \quad \frac{\partial V}{\partial x} G(x) = 0 \implies L_f V(x) < 0 \quad (6)$$

Remark:

$$\frac{\partial V}{\partial x} G(x) = (\nabla V(x)g_1(x), \dots, \nabla V(x)g_m(x)) =: L_G V(x)$$

$$(6) \iff \forall x \neq 0, \quad L_f V(x) \geq 0 \implies L_G V(x) \neq 0$$

Proof. \Leftarrow :

Assume (6) holds. Then:

$$\inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) = \inf_u L_f V(x) + L_G V(x)u < 0$$

Why?

- If $L_G V(x) = 0$, then by (6) $L_f V(x) < 0$;
- If $L_G V(x) \neq 0$, then (at least) for one i we have $\nabla V(x) \cdot g_i(x) \neq 0 \implies$ set $u_i = -c \nabla V(x) \cdot g_i(x)$.

\implies :

If (5) holds for some x with $L_G V(x) = 0$, then we must have $L_f V(x) < 0$. □

Example (discontinuous control):

$$\dot{x} = \begin{cases} 1 - u, & u \geq 1 \\ -1 - u, & u \leq -1 \\ 0, & \text{else} \end{cases}$$

If you want to move the system you need to apply control $|u| \geq 1$.

Using

$$u(x) = \begin{cases} x + 1, & x > 0 \\ x - 1, & x \leq 0 \end{cases}$$

results in closed loop $\dot{x} = -x$ - asymptotically stable.

$V(x) = x^2$ is a CLF.

Theorem 3.4. There exists a continuous $k : R^n \rightarrow R^m$, smooth on $R^n \setminus \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff:

- there exists a (smooth)CLF V ;
- $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall x : 0 < \|x\| < \delta$
 $\exists u \in R^m : \|u\| < \varepsilon$ s.t. $L_f V(x) + L_G V(x)u < 0$

How to construct a globally stabilizing state feedback k from knowledge of a CLF?

"Sontag's formula"

Fix $c \geq 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$k(x) = \begin{cases} -cb(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T, & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases}$$

Proposition: Let $V : R^n \rightarrow R$ be a CLF and k as above. Then $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$

Proof. $\dot{V} = L_f V(x) + L_G V(x)k(x) = a(x) - cb(x)^T b(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T b(x) = -cb(x)^T b(x) - \sqrt{a(x)^2 + (b(x)^T b(x))^2} < 0 \quad \forall x \neq 0$ s.t. $L_G V(x) \neq 0$

$\dot{V} = L_f V(x) + L_G V(x) \cdot 0 < 0 \quad \forall x \neq 0$ s.t. $L_G V(x) = 0$ (since V is CLF)

$\implies V$ - Lyapunov function $\implies \dots$ □

Remarks:

- Sontag's formula is smooth on $R^n \setminus \{0\}$;
- Sontag's formula is continuous at $x = 0$ iff small control property holds.

$$\forall x \neq 0 : \inf_u \frac{\partial V}{\partial x} f(x, u) < 0 \quad \dot{x} = f(x) + G(x)u$$

So this leads to

$$\forall x \neq 0 L_G V(x) = 0 \Rightarrow L_f V(x) \neq 0$$

Remark: The last formula is "optimal" if minimize:

$$\int_0^\infty \frac{1}{2} p(x) b(x)^T b(x) + \frac{1}{2p(x)} u^T u dt$$

$$b(x) := (L_G V(x))^T$$

where $c > 0$

$$p(x) = \begin{cases} c + \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} & b(x) \neq 0 \\ c & b(x) = 0 \end{cases}$$

It still works if $u = \lambda h(x)$ with $\lambda \in [\frac{1}{2}; \infty)$ is applied (large "gain margin")

4 Backstepping

Integrator backstepping

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{7}$$

where $x_1 \in \mathbb{R}^m$, $x_2, u \in \mathbb{R}$ (single input)

image to be inserted

Assumption: we know (smooth) "feedback" $\alpha_1 : \mathbb{R}^n \rightarrow \mathbb{R}$, and positive definite, differentiable $v_1 : \mathbb{R}^m \rightarrow \mathbb{R}$

s.t. $L_{f_1+g_1\alpha_1} V_1(x)$ is negative definite \Rightarrow origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1)$ is asymptotically stable

Goal: Compute feedback $u = k(x)$ which stabilises (7). Backstepping constructs $u = \alpha_2(x_1, x_2)$ s.t. $(e_1, e_2) = (x_1 - 0, x_2 - \alpha_1(x_1)) = 0$ error coordinates

Rewrite (7) :

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1))$$

$$\dot{x}_2 = u$$

image to be inserted

In error coordinates

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 &= u - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial e_1} \dot{e}_1\end{aligned}\tag{8}$$

"backstepping" α_1 through the integrator

Define $V_2(e_2) := \frac{1}{2}e_2^2$, and

$$\begin{aligned}V(e_1, e_2) &= V_1(e_1) + V_2(e_2) \\ \dot{V}(e_1, e_2) &= \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)\alpha_1(e_1)) + \frac{\partial V_1}{\partial e_1}g_1(e_1)e_2 + \frac{\partial V_2}{\partial e_2}(u - \dot{\alpha}_1)\end{aligned}$$

as far as $L_{f_1+g_1\alpha_1}V_1$ -negative definite and $\frac{\partial V_2}{\partial e_2} \rightarrow e_2$

Choose

$$u = (-\frac{\partial V_1}{\partial e_1}g_1(e_1) + \dot{\alpha}_1)(\text{"canaling terms"}) - k_2 e_2(\text{"stabilizing term"}), \quad k_2 > 0 \tag{9}$$

\Rightarrow Then $\dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2 e_2^2 < 0, \forall (e_1, e_2) \neq 0$

$\Rightarrow (e_1, e_2) = (0, 0)$ is an asymptotically stable EP for (8) with u as in (9)

Remark: $(e_1, e_2) \rightarrow (0, 0)$ doesnot necessarily imply that $(x_1, x_2) \rightarrow 0$ for $u = \alpha_2(x_1, x_2) = -\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1))$

where $u \leftarrow$ (9) the original coordinates and $\dot{\alpha}_1 \leftarrow \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + g_1(x_1)x_2)$

But $(x_1, x_2) = (0, 0)$ is asymptotically stable if $\alpha_1(0) = 0$ why? $(e_1, e_2) \rightarrow 0 \Rightarrow x_1 \rightarrow 0 \quad x_2 \rightarrow \alpha_1(0) = 0$

Example.

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= u\end{aligned}$$

Choose $\alpha_1(x_1) = -k \ (k > 0) \rightarrow \dot{x}_1 = -kx_1 \Rightarrow V_1(x_1) = \frac{1}{2}x_1^2$

Then:

$$\begin{aligned}e_1 &= x_1 - 0, \quad \dot{e}_1 = e_1(e_2 - k) \\ e_2 &= x_2 + k, \quad \dot{e}_2 = u\end{aligned}$$

Backstepping yields: $u = -e_1^2 - k_2 e_2$, $k_2 > 0 \Rightarrow (e_1, e_2) = (0, 0)$ is asymptotically stabilized

$(x_1, x_2) = (0, -k)$ is asymptotically stabilized

Can we choose different α_1 s.t. $(x_1, x_2) = (0, 0)$ is stabilized?

Yes, e.g.

$$\alpha_1(x_1) = -x_1^2 \Rightarrow \dot{x}_1 = -x_1^3, \quad V_1(x_1) = \frac{1}{2}x_1^2$$

So we have equations

$$\begin{aligned} e_1 &= x_1 - 0, \quad \dot{e}_1 = e_1(e_2 - e_1^2) \\ e_2 &= x_2 + x_1^2, \quad \dot{e}_2 = u + 2e_1^2(e_2 - e_1^2) \end{aligned}$$

Backstepping results in

$$u = -e_1^2 - 2e_1^2(e_2 - e_1^2) - k_2 e_2, \quad k_2 > 0 \Rightarrow (e_1, e_2) \rightarrow (0, 0), \quad (x_1, x_2) \rightarrow (0, 0)$$

Generalization-1

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

Assumption: $g_2(x_1, x_2) \neq 0, \forall x_1, x_2 \Rightarrow$ Input transformation: $u = \frac{1}{g_2(x_1, x_2)}(V - f_2(x_1, x_2)) \Rightarrow \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad \dot{x}_2 = V \Rightarrow$ can apply integrator backstepping to determine V results in

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)} \left(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2) \right)$$

Generalization 2: (Backstepping through 2 integrators)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1} \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3, \quad x_2, x_3 \in R \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u, \quad u \in R \end{aligned}$$

Assumption: g_2, g_3 nowhere zero.

Shown before: $\exists \alpha_2$: for $x_3 = \alpha_2(x_1, x_2) \quad (e_1, e_2) \rightarrow 0$

Thus $e_3 := x_3 - \alpha_2(x_1, x_2)$

Input transformation:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(V - f_3(x_1, x_2, x_3))$$

$\Rightarrow \dot{x}_1 = \dots, \dot{x}_2 = \dots, \dot{x}_3 = V \Rightarrow$ can apply backstepping once more.

In "error" coordinates:

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dot{e}_2 &= f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1 \\ \dot{e}_3 &= V - \dot{\alpha}_2\end{aligned}$$

Define $V_3(e_3) = \frac{1}{2}e_3^2, V(e_1, e_2, e_3) = \sum_{i=1}^3 V_i(e_i)$

$$\dot{V}(e_1, e_2, e_3) = \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) + e_2(f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1) + e_3(V - \dot{\alpha}_2))$$

All the underlined terms were designed (previously) to be $= L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 < 0$

So: $\dot{V}(e_1, e_2, e_3) = L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 + e_2g_2(e_1, e_2 + \alpha_1(e_1))e_3 + e_3(V - \dot{\alpha}_2)$

Structurally it is exactly the same as it was in backstepping through 1.

Choose:

$$V = \dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1)) - k_3e_3$$

$\dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1))$ - "cancelling terms".

k_3e_3 - "stabilizing term".

In original coordinates:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(\dot{\alpha}_2 - (x_2 - \alpha_1(x_1))g_2(x_1, x_2) - k_3(x_3 - \alpha_2(x_1, x_2)) - f_3(x_1, x_2, x_3))$$

We need α_1, α_2 to compute u .

General backstepping recursion:

Systems in "strict feedback form":

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1}$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

\dots

$$\dot{x}_k = f_k(x_1, \dots, x_k) + g_k(x_1, \dots, x_k)u, \quad x_2, \dots, x_k, u \in R$$

g_2, \dots, g_k nowhere zero, f_i, g_i (sufficiently) smooth, as it is needed in α_i .

Backstepping recursion:

1. "Input data": a CLF V_1 for $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$ with a (smooth) feedback $u_1 = \alpha_1x_1$ which as. stabilizes the origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$.

2. for $i = 2, \dots, k$:

construct a CLF $V_i(e_i) = \frac{1}{2}e_i^2$, $V = \sum_{j=1}^i V_j(e_j)$ and a feedback α_1 which as. stabilizes origin of $(e_1, \dots, e_i) = (x_1, x_2 - \alpha_1(x_1), \dots, x_i - \alpha_{i-1}(x_1, \dots, x_{i-1}))$

$$\alpha_i(x_1, \dots, x_i) = \frac{1}{g_i}(\dot{\alpha}_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}}g_{i-1} - k_i(x_i - \alpha_{i-1}) - f_i)$$

3. apply $u = \alpha_k(x_1, \dots, x_k)$

Backstepping and CLFs:

Backstepping is sensitive to uncertainties in f_i, g_i (due to cancelling terms)

\implies Sontag's formula is more practical \implies we can use it since V is CLF.

Error system is input affine (using input transformation)

$$\dot{e} = f(e) + g(e)V$$

$$\text{with } f(e) = \begin{pmatrix} f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dots \\ -\alpha_{k-1} \end{pmatrix}, g(e) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

Claim:

$$V(e) = \sum_{i=1}^k V_i(e_i) \text{ is a CLF.}$$

Proof. For input affine systems we need to show $L_g V = 0 \implies L_f V < 0, \forall e \neq 0$.

$$\dot{V}(e) = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 + e_{k-1} g_{k-1}(\dots) e_k - e_k \dot{\alpha}_{k-1} + e_k u.$$

Here $e_k u = L_g V$ and the rest is $L_f V$.

Assume $L_g V = 0 \iff e_k = 0$

$$\implies L_f V = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 < 0 \quad \forall e \neq 0 \text{ with } e_k = 0. \quad \square$$

\implies We can apply Sontag's formula to construct V .

This theory can be extended to systems with $x_2, \dots, x_k, u \in R^m$ ("block backstepping").

5 Systems with inputs and outputs

Study/control systems $\dot{x} = f(x, u)$ with "output" $y(t) = h(x(t))$

5.1 Sliding mode control

Motivating example

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ y = x_1 + x_2 \end{cases}$$

$$\Rightarrow \dot{y} = x_2 + u$$

Choose:

$$u = \begin{cases} -x_2 - 1, & y > 0 \\ -x_2 + 1, & y < 0 \\ -x_2, & y = 0 \end{cases}$$

$$\Rightarrow \dot{y} = \begin{cases} -1, & y > 0 \\ +1, & y < 0 \\ 0, & y = 0 \end{cases}$$

Solutions(Laratheodory) are if $y(0) > 0$, then

$$y(t) = \begin{cases} y(0) - t, & t \leq y(0) \\ 0, & t > y(0) \end{cases}$$

If $y(0) < 0$, then

$$y(t) = \begin{cases} y(0) + t, & t \leq -y(0) \\ 0, & t > -y(0) \end{cases}$$

Key property: choose u s.t. $y(t)$ goes to zero in finite time $\Rightarrow x(t)$ tends $S := \{(x_1, x_2) \in \mathbb{R}^2 | x_1 + x_2 = 0\}$ in finite time

Consider dynamics on S

$$\begin{cases} \dot{x}_1 = x_2 (x_2 = -x_1 \text{ if } y = 1) = -x_1 \\ \dot{x}_2 = u = -x_2 \end{cases}$$

globally as stable

Two "phases"

1. solutions converge to S in finite time
2. solutions converge to zero ("on S ") asymptotically

\rightsquigarrow "sliding mode" control

Remark: in (1) "finite time convergence is crucial"

General procedure:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) = s(x) \end{aligned}$$

$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $y : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $s : \mathbb{R}^n \rightarrow \mathbb{R}$

u - scalar input, $s(x)$ - sliding

single input, single output

Assumptions: y has relative degree 1, well - defined globally, i.e. $L_g s(x) \neq 0 \forall x \in \mathbb{R}^n$

Two-step approach:

1. Bring $x(t)$ to $S := \{x \in \mathbb{R}^n | S(x) = 0\}$ in finite time
2. Have $x(t)$ going to zero asymptotically (on S)
 - switching between nodes 1 and 2
 - mode 2 is "sliding mode"

How are 1 + 2 achieved?

- Design of sliding manifolds crucial!

Need: For $y(t) = 0$ for all $t \geq 0$. All solutions converge to the origin, i.e., "zero dynamics" have globally asymptotically stable origin.

How? e.g. systems in "regular form" $x = [\eta, \xi]'$

$$\begin{aligned}\dot{\eta} &= f_1(\eta, \xi) \\ \dot{\xi} &= f_2(\eta, \xi) + g_2(\eta, \xi)u\end{aligned}$$

Choose $s(x) = \eta - \phi(\eta)$, where ϕ asymptotically stabilizes zero dynamics $\dot{\eta} = f_1(\eta, \phi(\eta))$ (and $\phi(0) = 0$) Ex. 1.9 in Khalil

- Converging to sliding manifold in finite time: $\rightsquigarrow \dot{s} = L_f s(x) + L_g s(x)u$, where $L_g s(x) \neq 0$. Obvious choice to render S invariant is $u = -\frac{L_f s(x)}{L_g s(x)}$ (mode 2, behaviour on S)

Motivating example, add

$$\begin{cases} -\hat{u}/L_g s(x) & y > 0 \\ \hat{u}/L_g s(x) & y < 0 \end{cases}$$

where $\hat{u} > 0$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \operatorname{sgn}(s(x)))$$

$$\rightsquigarrow \dot{y} = -\hat{u} \operatorname{sgn}(y)$$

\rightsquigarrow (Carathéodory) solutions converge to zero in finite time

$\rightsquigarrow x(t)$ converges to S in finite time

Control Lyapunov perspective

$$V(X) = \frac{1}{2}s(x)^2$$

$$\dot{V}(x) = s(x)\dot{s}(x) = s(x)(L_f s(x) + L_g s(x)u) = -s(x)\text{sgn}(s(x))\hat{u} = |s(x)|\hat{u} < 0, \quad s(x) \neq 0$$

$$\text{Consider } w = \sqrt{2v} \rightsquigarrow^{s \neq 0} \dot{w} = \sqrt{2} \frac{1}{2\sqrt{v}} \dot{v} = -\hat{u}$$

$\rightsquigarrow w$ converges to 0 in finite time $\Rightarrow v$ converges to 0 in finite time $\Rightarrow S(x(t))$ converges to 0 in finite time.

Example.

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \sin(x_2) \\ \dot{x}_2 &= x_2^2 + x_1 + u \end{aligned}$$

Choose $s(x) = x_2 + 2x_1$, where $-2x_1 := \phi(x_1)$ on S : $\dot{x}_1 = -2x_1 + x_1 \sin(-2x_1) \rightsquigarrow$ asymptotically stable

$$\dot{s} = x_2^2 + x_1 + u + 2x_2 + 2x_1 \sin(x_2) \rightsquigarrow u = -(x_2^2 + x_1 + 2x_2 + 2x_1 \sin(x_2) + \hat{u} \text{sgn}(x_2 + 2x_1)), \quad \hat{u} > 0$$

\rightsquigarrow yields finite-time convergence to S .

Alternative sliding mode controllers

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \text{sgn}(s(x))), \quad \hat{u} > 0$$

In particular

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u}|L_g s(x)|\text{sgn}(s(x)))$$

\rightsquigarrow ensure robustness w.r.t. "matched uncertainties"

$$\dot{x} = f(x) + g(x)\sigma(x) + g(x)u$$

$\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$, bounded (i.e., $|\sigma(x)| \leq c \quad \forall x \in \mathbb{R}^n$)

Why? $V(x) = \frac{1}{2}s(x)^2$

$$\dot{V} = s(x)(L_f s(x) + L_g s(x)u + L_g s(x)\sigma(x)) = -s(x)\text{sgn}(s(x))\hat{u}|L_g s(x)| + s(x)L_g s(x)\sigma(x) \leq -|s(x)||L_g s(x)|(\hat{u} - c)$$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|)\text{sgn}(s(x)))$$

ensures robustness w.r.t. matched uncertainties s.t. $\sigma(x) \leq \beta(x) \quad \forall x \in \mathbb{R}^n$

Example.

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \sin(x_2) \\ \dot{x}_2 &= \theta x_2^2 + x_1 + u \end{aligned}$$

$$|\theta| \leq 2 \rightsquigarrow |\theta x_2^2| \leq 2x_2^2 = \beta(x)$$

$$\dot{s} = \theta x_2^2 + x_1 + u + 2x_1 + 2x_1 \sin x_2$$

$$u = -(x_1 + 2x_1 + 2x_1 \sin x_2 + \hat{u} \operatorname{sign}(s(x)) + 2x_2^2 \operatorname{sgn}(s(x)))$$

$$L_f s(x) = x_1 + 2x_1 + 2x_1 \sin x_2$$

$$\rightsquigarrow \dot{s} = -\hat{u} \operatorname{sgn}(s(x)) + x_2^2 (\theta - 2 \operatorname{sgn}(s(x))) \Rightarrow \text{finite-time convergence to } S.$$

Remedy: replace sign-function by saturated slope (continuous approximation)

can be extended to multi-input systems $u \in \mathbb{R}^m \rightarrow s : \mathbb{R}^n \rightarrow \mathbb{R}^m$

5.2 Dissipativity

Dissipativity: Generalization of Lyapunov theory to systems with inputs and outputs

$$\begin{aligned} \dot{x} &= f(x, u) & x(0) &= x_0 & f &: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \\ y &= h(x) & & & h &: \mathbb{R}^n \rightarrow \mathbb{R}^p \end{aligned} \quad (10)$$

Definition:

- storage function $S : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \rightarrow S(x)$ nonnegative (i.e., $S(x) \geq 0 \forall x \in \mathbb{R}^n$)
- supply rate $s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$, $(u, y) \rightarrow s(u, y)$

Definition: System (10) is dissipative w.r.t. the supply rate s if there exists a storage function S s.t. $\forall x_0 \in \mathbb{R}, \forall t \geq 0, \forall u : [0, t] \rightarrow \mathbb{R}^m$

$$S(x(t)) \leq S(x_0) + \int_0^t s(u(\tau), y(\tau)) d\tau \quad (11)$$

First item - storage at time t , second item - initial storage, the last item - supply delivered over $[0, t]$

"dissipation inequality" (DIE) is the inequality (11)

Interpretation:

- "Dissipative systems dissipate storage/stored energy"
- "No storage/energy can be created internally"
- positive s "supplied" energy / storage
negative s "extracted" energy / storage

Remark:

- If S is differentiable, DIE is equivalent to $\dot{S}(x) \leq s(u, y) \forall u, x$
- Dissipation (rate) is defined as $d(x, u) = s(u, h(x)) - \dot{S}(x) \geq 0$

Examples of dissipative systems:

	supply rate	input	output	storage function
electrical	$u \cdot i$	voltage	current	energy storage in all capacitors and inductors
mechanical	$F \cdot V$	force	velocity	Hamiltonian = kinetic + potential energy
thermo-dynamics	$Q + W$	rate of heat	rate of work	internal energy
	$-\frac{a}{T}$		temperature	entropy

How do we compute storage functions?

- in general difficult (similar to computing Lyapunov functions)
- characterization via optimization problem

Introduce "available storage"

$$S_a(x) := \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=0} \left(- \int_0^T s(u(\tau), y(\tau)) d\tau \right)$$

the maximum of energy we can extract

Theorem 5.1. System (10) is dissipative w.r.t. the supply rate s iff $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$

Moreover, if $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$, then S_a is a storage function and $S(x) \geq S_a(x) \forall x \in \mathbb{R}^n$ for all storage functions S .

Proof. Sketch of proof. " $S_a(x) < \infty \Rightarrow$ dissipativity". $S_a(x) \geq 0 \forall x \in \mathbb{R}^n$ by definition (can take $T = 0$)

$$\begin{aligned} S_a(x) &= \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=0} - \int_0^T s(u(\tau), y(\tau)) d\tau \geq^* \\ &- \int_0^t s(u(\tau), y(\tau)) d\tau + \sup_{u: [t, t+T] \rightarrow \mathbb{R}^m, T \geq 0, x(t)=x(t)} - \int_t^{t+T} s(u(\tau), y(\tau)) d\tau \end{aligned}$$

the last item is $S_a(x(t))$,

$$\Rightarrow S_a(x(t)) - \int_0^t s(u(\tau), y(\tau)) d\tau$$

and this is DIE $\Rightarrow S_a$ is a storage function

Note for (*): "suboptimal" to first transfer system to $x(t)$ and then extract maximum energy starting of $x(t)$

"Dissip. $\Rightarrow S_a(x) < \infty$ "

From DIE: $S(x_0) \geq S(x(T)) - \int_0^T s(u(\tau), y(\tau)) d\tau \geq - \int_0^T s(u(\tau), y(\tau)) d\tau$

for all x_0 , for all $T \geq 0$, all $u(\cdot) \Rightarrow S(x_0) \geq \sup_{u: [0, T] \rightarrow \mathbb{R}^m, T \geq 0, x(0)=x_0} - \int_0^T s(u(\tau), y(\tau)) d\tau = S_a(x)$

$\Rightarrow S_a(x) < \infty \forall x \in \mathbb{R}^n$ and $S \geq S_a$ for all storage function S . \square

Another special supply rate: "required supply"

$$S_r(x) := \inf_{u: [-T, 0] \rightarrow \mathbb{R}^m, T \geq 0, x(-T)=x^*, x(0)=x} \int_{-T}^0 s(u(\tau), y(\tau)) d\tau$$

Theorem 5.2. Assume that end state $x \in \mathbb{R}^n$ is readable from x^* . If system (10) is dissipative w.r.t. the supply rate s , then for all storage functions S

$$S(x) \leq S_r(x) + S(x^*) \forall x \in \mathbb{R}^n$$

Furthermore, $S_r(x) + S(x^*)$ is a storage function.

Proof. Sketch of proof.

Consider $u : [-T, 0] \rightarrow \mathbb{R}^n$ which transfers the system from x^* to x

$$S(x) - S(x^*) \leq [by \text{ DIE}] \inf_{u: [-T, 0] \rightarrow \mathbb{R}^n, T \geq 0, x(-T)=x^*, x(0)=x} \int_{-T}^0 s(u(\tau), y(\tau)) d\tau = S_r(x)$$

\square

Remark: Set of all storage functions is convex, i.e. , $\alpha S_1 + (1 - \alpha) S_2, \alpha \in [0, 1]$ is a storage function (for S_1, S_2 storage functions)

Dissipativity widely used in control theory

If system is dissipative with positive definite storage S and if there exists a (continuous) $k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t.

$$s(k(x), h(x)) < 0, \forall x \neq 0$$

then $x = 0$ is asymptotically stable under $u = k(x)$

Why? Take S as Lyapunov function

$$\dot{S} \leq s(u, y) \stackrel{u=k(x)}{=} s(k(x), h(x)) < 0, \forall x \neq 0$$

L_2 - gain via supply rate

$$s(u, y) = \frac{1}{2} \gamma^2 \|u\|^2 - \frac{1}{2} \|y\|^2$$

\rightsquigarrow from dissipation inequality

$$\begin{aligned}\frac{1}{2} \int_0^t \gamma^2 \|u(\tau)\|^2 - \|y(\tau)\|^2 d\tau &\geq S(x(t)) - S(x(0)) \geq -S(x(0)) \\ \Rightarrow \int_0^t \|y(\tau)\|^2 d\tau &\leq \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau + 2S(x(0))\end{aligned}$$

\Rightarrow system has L_2 - gain γ

Classify optional $l(x, u^{\leftarrow x})$ operating conditions $s(u, y) = l(x, u) - l(x^*, u^*)$

Example.

$$\dot{x} = u, \quad y = x$$

$S(x) = \frac{1}{2}x^2$, $\dot{S} = xu = uy \rightsquigarrow$ system is dissipative w.r.t. supply rate $s(u, y) = uy$.

Example. "part-Hamiltonian systems"

$$\dot{x} = [F(x) - R(x)]\nabla H(x) + g(x)u$$

$y = g(x)^T \nabla H(x)$, H - Hamiltonian total stored energy in system

$F(x) = -F^T(x)$ internal interconnection structure (power conserving) $R(x) \geq 0$ dissipation structure

Take $S(x) = H(x)$

$$\begin{aligned}\dot{S}(x) &= \nabla H(x) \cdot [F(x) - R(x)]\nabla H(x) + \nabla H(x) \cdot g(x)u \\ &= -\nabla H(x) \cdot R(x)\nabla H(x) + yu \leq yu\end{aligned}$$

as far as $-\nabla H(x) \cdot R(x)\nabla H(x) \leq 0 \Rightarrow$ dissipative w.r.t. $s(u, y) = u^T y$

5.3 Passivity

$$\begin{aligned}\dot{x} &= f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ y &= h(x), \quad y \in \mathbb{R}^m\end{aligned} \tag{12}$$

(same number of inputs and outputs)

Definition. System (12) is passive if it is dissipative w.r.t. supply rate $s(u, y) = u^T y$

Why "passive"? From circuit theory passive compared to "active" ones as diodes or transistors

Examples: electrical, mechanical systems

Stabilization of passive systems

Definition. System (12) is zero-state observable (ZSO) if (for $u(t) = 0$) $y(t) = 0$ for all $t \geq 0 \Rightarrow x(t) = 0$ for all $t \geq 0$

"trivial solution $x(t) \equiv 0$ is observable from the output"

Remark: can be related to zero-state detectability

Theorem 5.3. Let system (12) be

i) passive in differentiable storage set

ii) ZSO

Then the feedback $u = -Py$, $P > 0$ renders the origin asymptotically stable

Proof. Sketch of proof From passivity

$$\dot{S} \leq u^T y = -y^T P y \leq 0 \quad (13)$$

$$S(x(t)) - S(x(0)) \leq - \int_0^t y(\tau)^T P y(\tau) d\tau, \quad \forall t \geq 0$$

$$S(x(t)) \geq 0$$

$$S(x_0) \geq \int_0^t y(\tau)^T P y(\tau) d\tau, \quad \forall t \geq 0 \quad (14)$$

$y(\tau)^T P y(\tau) \geq 0$. Want to show $S(x_0) > 0$ for all $x_0 \neq 0$. By contradiction. Suppose $\exists \bar{x} \neq 0$ with $S(\bar{x}) = 0$.

From (14) $\Rightarrow y(\tau) = 0 \quad \forall \tau \geq 0$

By ZSO $\Rightarrow x(\tau) = 0 \quad \forall \tau \geq 0 \Rightarrow \bar{x} = 0$

$\Rightarrow S$ is positive definite. \Rightarrow Lyapunov stability together with (13)

For convergence, use (13) together with La Salle's invariance principle and ZSO □

Advantage. We have (static) output feedback (no observer needed)

Passivity of interconnections

1. Parallel interconnections of two passive systems are passive

Take $S(x_1, x_2) = S_1(x_1) + S_2(x_2)$.

$$\dot{S} \leq u_1^T y_1 + u_2^T y_2 = u^T (y_1 + y_2) = u^T y$$

2. Feedback interconnection of passive systems are passive

Take $S(x_1 + x_2) = S_1(x_1) + S_2(x_2)$

$$S \leq u_1^T y_1 + u_2^T y_2 \stackrel{y_1=u_2=y}{=} y^T (u_1 + y_2) = y^T u$$

Remark:

- does not work for serious interconnections
- can construct possibly large networks of passive systems

Stability if feedback interconnections:

Main idea: "shortage" of passivity of H_1 can be compensated by "excess" of passivity of H_2

Theorem 5.4. Consider feedback interconnection (2) with $u \equiv 0$. Assume that H_1 and H_2 are (i) ZSO and dissipative with differentiable S_1, S_2 w.r.t. the supply rates

$$s_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i, \quad i = 1, 2, \quad \rho, \nu \in \mathbb{R} \quad (15)$$

Then the origin $(x_1, x_2) = (0, 0)$ for interconnection is asymptotically stable if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$.

Proof. Take $S(x) = S_1(x_1) + S_2(x_2)$.

$$\begin{aligned} \dot{S}(x) &\stackrel{(15)}{\leq} u_1^T y_1 - \rho_1 y_1^T y_1 - \nu_1 u_1^T u_1 \\ &\quad + u_2^T y_2 - \rho_2 y_2^T y_2 - \nu_2 u_2^T u_2 \\ &= -(\rho_1 + \nu_2) y_1^T y_1 - (\rho_2 + \nu_1) y_2^T y_2 \end{aligned}$$

$u_1^T y_1$ and $u_2^T y_2$ can be excluded as $u_1 = -y_2, u_2 = y_1$.

\Rightarrow can show as in previous theorem that S is positive definite \Rightarrow Lyapunov stability

For using La Salle:

$$\begin{aligned} y_1 \equiv 0 &\Rightarrow u_2 \equiv 0 \Rightarrow^{ZSO} x_2 \equiv 0 \\ y_2 \equiv 0 &\Rightarrow u_1 \equiv 0 \Rightarrow^{ZSO} x_1 \equiv 0 \end{aligned}$$

□

Remark

- If (15) is satisfied with $v_i = 0$: "output - feedback passive" $\Rightarrow p_i > 0$ - "excess" of passivity, $p_i < 0$ - "shortage" of passivity ($|p_i|$).
- If (15) satisfied with $p_i = 0$: "input - feedforward passive" $\Rightarrow v_i > 0$ - "excess" of passivity, $v_i < 0$ - "shortage" of passivity ($|v_i|$).
- Comment on terminology "output feedback passive" $\dot{s} \leq u^T y - \rho y^T y = [\text{output feedback } u = \bar{u} - ky] = \bar{u}^T y - (\rho + k) y^T y \leq \bar{u}^T y$ In other words, system can be made passive by output feedback.
- Similar for feedforward passivity, system can be made passive by feedback forward the input: $\bar{y} = y + ku$.

Remark:

Feedforward interconnection 2) can be extended to allow h_1 or h_2 to explicitly depend on $u \Rightarrow$ includes static systems (controllers) e.g. state output feedback $y = h(u)$ e.g. $y_2 = ku_2 \Rightarrow$ input-strictly passive controller.

Extension of output-feedback/input-feedforward passivity:

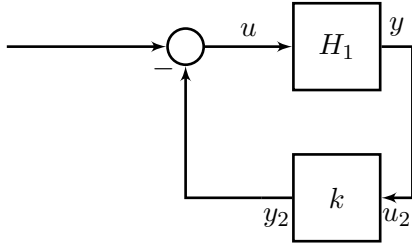
$$s(u, y) = u^T y - \rho(y)^T y$$

with $\rho(y) = [\rho_1(y_1), \dots, \rho_n(y_n)]^T$ with ρ_i section nonlinearities, $\rho_i : \mathbb{R} \rightarrow \mathbb{R}$

Example.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ H_1 : \quad \dot{x}_2 &= -x_1^3 + x_2 + u \\ y &= x_2 \end{aligned}$$

Take $S_1(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \Rightarrow \dot{S}_1 = x_1^3 x_2 - x_1^3 x_2 + x_2^2 + x_2 u$, define $y^2 := x_2^2$ then $yu = x_2 u \Rightarrow$ output - feedback passive with shortage of passivity $1 \Rightarrow \rho_1 = -1$ and $v_1 = 0$.

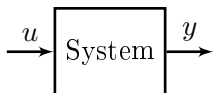


$$y_2 = u_2 * y_2 + ku_2 \Rightarrow s(u_2, y_2) = \gamma k u_2^2 + \frac{1-\gamma}{k} y_2^2, 0 < \gamma < 1 \Rightarrow \rho_2 = \frac{1-\gamma}{k}, v_2 = \gamma k$$

$\Rightarrow v_2 + \rho_1 > 0$ for $k > 1$ and γ close enough to 1, $v_1 + \rho_2 = \rho_2 > 0$. \Rightarrow with ZSO, the origin is asymptotically stable.

6 Input/Output Methods

References: Desoev, Vidyasagar "Feedback Systes Input-output properties"



$$\begin{aligned} u &: u \rightarrow y \\ u, y &: [0, \infty] \rightarrow \mathbb{R}^m \\ t &\rightarrow u(t), y(t) \end{aligned}$$

6.1 Signals and Systems

- How to define "stability" in input/output setting?
- Which signals are "good"?

Definition. Lp -spaces, $p \in [1, \infty]$. $Lp[0, \infty) = \{\Phi : [0, \infty) \rightarrow \mathbb{R}^m, \text{measurable} \mid \int_0^\infty \|\Phi(t)\|^p dt < \infty\}$

Interpretation: "finite energy signal" ($p=2$).

Remark: "measurable" = pointwise limit of a sequence of piecewise constant functions (except on a set of measure 0)

Example. :

- continuous function
- functions with "few enough" discontinuities

Lp is a real vector space ("signals $\Phi(\cdot)$ are vectors") i.e., for $\Phi, \Phi_1, \Phi_2 \in Lp$, $\alpha \in \mathbb{R}$ vector addition: $\Phi_1 + \Phi_2 : t \rightarrow \Phi_1(t) + \Phi_2(t) \in Lp$. Scalar multiplication: $\alpha\Phi : t \rightarrow \alpha\Phi(t) \in Lp$

Zero element is signal $\Phi \equiv 0$.

Lp is a normed vector space with norm $\|\Phi\|_{Lp} = \sqrt[p]{\int_0^\infty \|\Phi(t)\|^p dt}$ for $\Phi \in Lp \Rightarrow$

- $\|\Phi\|_{Lp} = 0 \iff \Phi = 0$, else $\|\Phi\|_{Lp} > 0$
- for $\alpha \in \mathbb{R}$, $\|\alpha\Phi\|_{Lp} = |\alpha| \|\Phi\|_{Lp}$
- for $\Phi_1, \Phi_2 \in Lp$ $\|\Phi_1 + \Phi_2\|_{Lp} \leq \|\Phi_1\|_{Lp} + \|\Phi_2\|_{Lp}$

For $p \rightarrow \infty$ set of all measurable and (essentially) bounded functions L_∞ , for continuous ϕ

$$\|\phi\|_{L_\infty} = \inf\{c \in \mathbb{R} \mid \|\phi(t)\| \leq c \text{ a.e.}\} = \sup_{t \geq 0} \|\phi(t)\|$$

Example.

$$\phi(t) = e^{-\alpha t}, \alpha > 0, p \in [1, \infty)$$

$$\begin{aligned} \|\phi\|_{Lp} &= \sqrt[p]{\int_0^\infty \|e^{-\alpha t}\|^p dt} = \\ &= \sqrt[p]{\left[-\frac{1}{\alpha p} e^{-\alpha p t}\right]_0^\infty} = \sqrt[p]{\frac{1}{\alpha p}} < \infty \\ &\Rightarrow \phi \in Lp \quad \forall p \in [1, \infty) \quad p = \infty : \\ &\sup_{t \geq 0} \phi(t) = \phi(0) = 1 \Rightarrow p \in L_\infty \end{aligned}$$

Special case $p = 2$

L_2 can be equipped with an inner product $\phi_1, \phi_2 \in L_2$, we write $\langle \phi_1, \phi_2 \rangle_{L_2} := \int_0^\infty \phi_1(t)^T \phi_2(t) dt$

symmetry $\langle \phi_1, \phi_2 \rangle_{L_2} = \langle \phi_2, \phi_1 \rangle_{L_2}$

(bi-)linearity

$$\begin{aligned} \langle \alpha \phi_1, \phi_2 \rangle_{L_2} &= \alpha \langle \phi_1, \phi_2 \rangle_{L_2} \quad \alpha \in \mathbb{R} \\ \langle \phi_1 + \phi_2, \phi_3 \rangle_{L_2} &= \langle \phi_1, \phi_3 \rangle_{L_2} + \langle \phi_2, \phi_3 \rangle_{L_2}, \quad \phi_3 \in L_2 \end{aligned}$$

positive definiteness: $\langle \phi_1, \phi_1 \rangle_{L_2} = 0$ iff $\phi_1 = 0$, $\langle \phi_1, \phi_1 \rangle_{L_2} > 0$ else.

$\Rightarrow (L_2, \langle \cdot, \cdot \rangle_{L_2})$ is an inner product space

$\| \phi \|_{L_2}^2 = \langle \phi, \phi \rangle_{L_2}$ "induced norm".

Particularly useful (Cauchy -Schwarz inequality) $|\langle \phi_1, \phi_2 \rangle_{L_2}| \leq \| \phi_1 \|_{L_2} \| \phi_2 \|_{L_2}$

Original motivation

Example.

$$\dot{x} = x + u, \quad y = x, \quad x(0) = 0$$

$$y(t) = \int_0^t e^{(t-\tau)} u(\tau) d\tau$$

$$\text{Take } u(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$$

Clearly, $u \in L_p$ for any $p \in [1, \infty)$. Let $t \geq 1$:

$$\begin{aligned} y(t) &= \int_0^1 e^{(t-\tau)} d\tau = e^t \int_0^1 e^{-\tau} d\tau \\ &= e^t [-e^{-\tau}]_0^1 = e^t (1 - e^{-1}) \end{aligned}$$

$\Rightarrow y \notin L_p, \quad p \in [1, \infty) \Rightarrow$ even that $u \in L_p$, the output neednot be an L_p - signal.

Taking $H : L_p \rightarrow L_p$ does (in general) not make sense? (would be excluded too many "relevant" systems)

Meaningful longer class: extended L_p spaces

Introduce "truncation operator"

$$\phi_T(t) = \begin{cases} \phi(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

The extension L_p^e of L_p is defined as

$$L_p^e = \{ \phi : [0, \infty) \rightarrow \mathbb{R}^n \mid \forall T \geq 0 \quad \phi_T \in L_p \}$$

$L_p^e \setminus L_p$ are "unstable" signals

Example. $\phi(t) = e^t$ ("unstable linear system")

$$\begin{aligned}\|\phi_T\|_{L_p}^p &= \int_0^\infty |\phi_T(t)|^p dt = \int_0^T |\phi(t)|^p dt = \\ &= \int_0^T e^{pt} dt = \frac{1}{p}(e^{Tp} - 1) < \infty, \quad \forall T \geq 0 \\ &\Rightarrow \phi \in L_p^e\end{aligned}$$

We consider systems

$$H : u \mapsto y, L_p^e \mapsto L_p^e$$

and define input-output stability as follows:

Definition. H is L_p -stable if there exists $\alpha \in K$, $\beta \geq 0$ s.t.

$$\|(H(u))_T\|_{L_p} \leq \alpha(\|u_T\|_{L_p}) + \beta$$

for all $u \in L_p^e$ and all $T \geq 0$.

H is finite-gain L_p stable if there exist $\gamma, \beta \geq 0$ s.t.

$$\|(H(u))_T\|_{L_p} \leq \gamma\|u_T\|_{L_p} + \beta \tag{16}$$

for all $u \in L_p^e$ and $T \geq 0$. Then $\gamma_p(H) := \{\inf \gamma | \exists \beta \geq 0 \text{ s.t. (16) holds}\}$ is L_p - gain of H

Definition. A map $H : L_p^e \mapsto L_p^e$ is causal if $(H(u))_T = (H(u_T))_T$ for all $u \in L_p^e$ and $T \geq 0$.

Interpretation: H is "nonanticipativity", outputs up to time T cannot be influenced by inputs after time T .

Remark: if H is defined by $u \mapsto y$, $y = h(x)$, $\dot{x} = f(x, u)$ then it is causal.

- (16)

$$\Rightarrow \|H(u)\|_{L_p} \leq \gamma\|u\|_{L_p} + \beta, \quad \forall u \in L_p \tag{17}$$

- For causal systems, (17) implies (16)
- sometimes slightly different definitions of finite-gain L_2 stability

$$\|(H(u))_T\|_{L_2}^2 \leq \bar{\gamma}^2\|u_T\|_{L_2}^2 + \beta, \quad \forall u \in L_p^e, \quad \forall T \geq 0 \tag{18}$$

One can show

$$\gamma_2(H) := \{\inf \bar{\gamma} | \exists \beta \geq 0 \text{ s.t. (18) holds}\}$$

6.2 Input-output stability of state-space systems

Theorem 6.1. Consider $\dot{x} = f(x, u)$, $y = h(x, u)$. Suppose the system is ISS and there exist $\alpha_1, \alpha_2 \in K$ and $\eta \geq 0$ s.t. $\|h(x, u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$. Then for each $x_0 \in \mathbb{R}^n$, the system is L_∞ - stable.

Proof. From ISS, $\exists \phi \in KL$ and $\alpha_3 \in K$ s.t. for all $t \geq 0$.

$$\|x(t)\| \leq \phi(\|x_0\|, t) + \alpha_3\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right)$$

Hence

$$\begin{aligned} \|y(t)\| &\leq \alpha_1(\phi(\|x_0\|, t) + \alpha_3(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)) + \alpha_2(\|u(t)\|) + \eta \leq \\ &[\alpha_1(a + b) \leq \alpha_1(2a) + \alpha_1(2b)] \leq \alpha_1(2\phi(\|x_0\|, t)) + \alpha_1(2\alpha_3(\sup_{0 \leq \tau \leq t} \|u(\tau)\|)) \\ &\quad + \alpha_2(\|u(t)\|) + \eta \Rightarrow \|y_T\|_{L_\infty} \leq \gamma(\|u_T\|_{L_\infty}) + \beta \\ &\text{with } \gamma = \alpha_1 \circ 2\alpha_3 + \alpha_2, \quad \beta = \alpha_1(2\phi(\|x_0\|, 0)) + \eta \end{aligned}$$

□

L_2 - gain of LTI system: Consider

$$\begin{aligned} x &= Ax + Bu & u, y \in \mathbb{R} &\rightarrow SISO \\ y &= Cx + Du & A \dots Hurwitz \end{aligned} \tag{19}$$

Lemma 7. The L_2 gain of (19) is

$$\gamma = \sup_{w \in \mathbb{R}} \sqrt{G(-jw)G(jw)}$$

where $G(s) = C(sI - A)^{-1}B + D$

Proof. Sketch: Due to linearity, we can assume without loss of generality $x(0) = 0$ ($x_0 \neq 0$ changes γ).

$$\begin{aligned} \|y\|_{L_2}^2 &= \int_0^\infty (y(t))^2 dt = [Parseval's theorem] = \frac{1}{2\pi} \int_{-\infty}^\infty Y(-jw)Y(jw)dw = \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty U(-jw)G(-jw)G(jw)U(jw)dw \leq \sup_{w \in \mathbb{R}} (G(-jw)G(jw)) \int_{-\infty}^\infty U(-jw)U(jw)dw = \\ &= [Parseval's th.] = \gamma^2 \|u\|_{L_2}^2 \end{aligned}$$

$\Rightarrow L_2$ -gain of (19) is $\leq \gamma$. Can show by contradiction that L_2 -gain equals γ .

□

L_2 - gain of input affine system: Consider

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}\tag{20}$$

Recall. System has L_2 gain less or equal γ if it is dissipative w.r.t. supply rate $s(u, y) = \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|y\|_2^2 \Rightarrow$ with diff. storage function S such that:

$$\frac{\partial S}{\partial x}[f(x) + g(x)u] \leq \frac{1}{2}\gamma^2\|u\|_2^2 - \frac{1}{2}\|h(x)\|_2^2 \quad \forall x, u$$

"Worst case" (maximizing) u :

$$u^* = \frac{1}{\gamma^2}g^T(x) \left(\frac{\partial S}{\partial x} \right)^T$$

$\Rightarrow \frac{\partial S}{\partial x}f(x) + \frac{1}{2\gamma^2} \frac{\partial S}{\partial x}g(x)g^T(x) \left(\frac{\partial S}{\partial x} \right)^T + \frac{1}{2}h^T(x)h(x) \leq 0 \quad \forall x \in \mathbb{R}^n$ (Hamilton-Jacobi equation) If there exist pos. semi-definite S satisfying (1), then system (20) has L_2 -gain less or equal γ .

7 Exercises

7.1 Exercise 1

Problem 1:

Proof. For any $t \geq 0$, we have

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}(V \circ x)(t) = \langle \nabla V(x(t)), \frac{d}{dt}x(t) \rangle = \langle \nabla V(x(t)), f(x(t)) \rangle = L_f V(x(t))$$

□

Problem 2:

Proof.

Lemma 8. Given the assumptions in Problem 2, if there exists a solution $x : [0, +\infty] \rightarrow \mathbb{R}^n, t \rightarrow x(t)$, of $\dot{x} = f(x)$ s.t. $x(t) \in K$ for any $t \geq 0$, where $K \subset \mathbb{R}^n$ is a compact with $O \in K$ (O - origin), then $x(t) \xrightarrow{t \rightarrow +\infty} 0$.

Clearly, for any $c > 0, lev_{\leq c}V$ is positive invariant w.r.t $\dot{x} = f(x)$. Given $c > 0$, let $x_0 \in lev_{\leq c}V$, i.e., $V(x_0) \leq c$. Then, for any $t \geq 0$

$$V(x(t)) = V(x_0) + \int_0^t \frac{d}{dt}V(x(\tau))d\tau < V(x_0) \leq c,$$

i.e. $x(t) \in lev_{\leq c}V$ for any $t \geq 0$.

Then, for any $x_0 \in lev_{\leq c}V$ there exists a solution $x : [0, +\infty] \rightarrow \mathbb{R}^n$ of $\dot{x} = f(x)$ s.t. $x(t) \in lev_{\leq c}V$ for all $t \geq 0$. Clearly, $O \in lev_{\leq c}V$. We conclude by using the above Lemma ($K = lev_{\leq c}V$). □

Problem 3:

Proof. Let $r > 0$. By assumption, there exists $c > 0$ s.t. $\overline{B(0, r)} \subset \text{lev}_{\leq c} V$.

Since any bounded set $\text{lev}_{\leq c} V$ is a subset of the region of attraction, and since the sublevel sets are arbitrary large, R^n is also the region of attraction.

A condition that ensures that for any $c > 0$, $\text{lev}_{\leq c} V$ is bounded is $V(x) \xrightarrow{\|x\| \rightarrow +\infty} +\infty$. \square

Problem 4:

Proof. Let $P : R^2 \rightarrow R^2$ be continuously differentiable. Consider

$$m\dot{v} = -g\nabla P(q).$$

Consider $x = (q, v)$, $\dot{q} = v$, $\dot{v} = -\frac{g}{m}\nabla P(q)$. Let $H : R^2 \rightarrow R$ be defined by

$$H(q, v) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q).$$

We have

$$\begin{pmatrix} \dot{q} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v)$$

Since P is positive definite, then H is positive definite.

Then

$$L \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v) = \langle \nabla H(q, v), \begin{pmatrix} I \\ -I \end{pmatrix} \nabla H(q, v) \rangle = 0 \quad \forall (q, v) \in R^2 \times R^2$$

\implies the origin is stable. \square

Problem 5:

Proof. For any $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}V(t, x(t)) &= \frac{d}{dt}(V \circ (id_R, x))(t) = [id_R : R \rightarrow R, t \mapsto t] = \left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \frac{d}{dt}(id_R(t), x(t)) \right\rangle = \\ &= \left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \right\rangle = \frac{\partial}{\partial t}V(t, x(t)) + \left\langle \frac{\partial}{\partial x}V(t, x(t)), f(t, x(t)) \right\rangle = L \begin{pmatrix} 1 \\ f \end{pmatrix} V(x(t)). \end{aligned}$$

$$g(t, x(t)) := \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \quad \square$$

Problem 6:

Proof. Consider $\dot{x} = a \sin(\omega t)$, $x(0) = x_0 \in R$ $a, \omega > 0$.

This is solved by $x(t) = -\frac{a}{\omega} \cos(\omega t) + \frac{a}{\omega} + x_0$.

Clearly, x is bounded on $[0, +\infty]$ since $x(t) \geq x_0$, and $x(t) \leq x_0 + 2\frac{a}{\omega}$ for any $t \geq 0$.

Choose $\varepsilon = \frac{a}{\omega}$ and $t_0 = 0$. Then $\forall \delta > 0 \exists x_0 \in B(0, \delta)$, namely x_0 , s.t. $\exists t \geq t_0$, namely $t = \frac{\pi}{\omega}$, with $x(t) \notin B(0, \varepsilon)$ ($x(\frac{\pi}{\omega}) = 2\frac{a}{\omega} > \varepsilon$). \square

Short notes:

Problem 7:

Take $V(t, x) = \frac{1}{2}x^2$.

Problem 8:

Take $V(t, x) = x_1^2 + (1 + e^{-2t})x_2^2$.

7.2 Exercise 2

Problem 1:

Proof. a) Since α_1 is continuous and strictly increasing:

$$\forall x, y \in [0, \delta), x < y \implies \alpha_1(x) < \alpha_1(y)$$

$\implies \alpha_1$ is injective, i.e.

$$\forall x, y \in [0, \delta), x \neq y \implies \alpha_1(x) \neq \alpha_1(y).$$

Clearly, $\alpha_1 : [0, \delta) \rightarrow \alpha_1([0, \delta))$ is surjective, i.e.

$$\forall y \in \alpha_1([0, \delta)) \exists x \in [0, \delta) : \alpha_1(x) = y$$

Thus α_1 is bijective.

Define $\alpha_1^{-1} : [0, \alpha_1(\delta)) \rightarrow [0, \delta)$ by $\alpha_1^{-1}(\alpha_1(x)) = x$.

b) From a) we have $\alpha_3^{-1} \in K$. Since $\alpha_3 \in K_\infty$, $\alpha_3 - 1$ is defined on $[0, +\infty)$ and $\alpha_3^{-1}(r) \xrightarrow{r \rightarrow \infty} \infty$

c) Let $\alpha = \alpha_1 \circ \alpha_2$. Then we have $\alpha(0) = \alpha_1(\alpha_2(0)) = 0$ and $\alpha(r) > 0$ whenever $r > 0$. Moreover, for any x, y :

$$x < y \implies \alpha_2(x) < \alpha_2(y) \implies \alpha(x) = \alpha_1(\alpha_2(x)) < \alpha_1(\alpha_2(y)) = \alpha(y)$$

It is continuous (as composition of continuous functions).

d) From c) we have $\alpha := \alpha_3 \circ \alpha_4 \in K$, α is defined on $[0, +\infty)$ since $\alpha_3, \alpha_4 \in K_\infty$ and

$$r \rightarrow +\infty \implies \alpha_4(r) \rightarrow +\infty \implies \alpha(r) \rightarrow +\infty$$

e) For each $s, r \mapsto \beta(\alpha_2(r), s)$ is of class K .

Thus $r \mapsto \alpha_1(\beta(\alpha_2(r), s)) \in K$.

For each $r, s \mapsto \beta(\alpha_2(r), s)$ decreases.

Hence, $s \mapsto \alpha_1(\beta(\alpha_2(r), s))$ decreases.

Moreover,

$$\alpha_1(\beta(\alpha_2(r), s)) \xrightarrow{s \rightarrow +\infty} 0$$

□

Problem 3:

Proof. For $u = 0$ the origin is UGAS. Consider $V : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, x) \mapsto \frac{1}{2}x^2$.

We have

$$\frac{\partial}{\partial t}V(t, x) + \frac{\partial}{\partial x}V(t, x)f(t, x, u) = (\sin(t) - 2)x^2 + xu \leq -x^2 + |x||u| = -(1 - \theta)x^2 - \theta x^2 + |x||u|, \quad \theta \in (0, 1)$$

Hence, whenever $|x| \geq \frac{|u|}{\theta}$, the system is ISS with $\gamma = \frac{r}{\theta}$.

□

Problem 4:

Proof.

$$\dot{x} = -x + (x^2 + 1)d \tag{21}$$

$$\dot{x} = -2x - x^3 + (x^2 + 1)d \tag{22}$$

System (21): Clearly, the system is 0-GAS. However, for $d = 1$ and $x > 1$ we have $x^2 + 1 > x$.

$$f(x, 1) = -x + (x^2 + 1) > 0$$

and thus $\dot{x} > 0$. Hence, if $x(0) = x_0 > 1$, the solution diverges (in finite time).

\implies System (21) isn't ISS.

System (22): It is 0-GAS. Moreover, for any finite d there exists a "large" x s.t.

$$2x + x^3 > (x^2 + 1)d$$

$$\implies f(x, d) = -2x - x^3 + (x^2 + 1)d < 0$$

and $\dot{x} < 0 \implies$ System 22 is ISS.

Consider $V : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \frac{1}{2}x^2$ s.t

$$V'(x)f(x, d) = -2x^2 - x^4 + x(x^2 + 1)d \leq -x^2 - x^2(x^2 + 1) + (x^2 + 1)|x||d|$$

Hence, whenever $|x| \geq |d|$,

$$V'(x)f(x, d) \leq -x^2$$

s.t. system (22) is ISS with $\gamma(r) = r$.

□

Problem 5:

Proof.

$$\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\|\nabla V(x)\|^2 + |\langle \nabla V(x), \delta u \rangle| \leq [YI] \leq -\|\nabla V(x)\|^2 + \frac{1}{2}\|\nabla V(x)\|^2 + \frac{\delta^2}{2}\|u\|^2$$

Young's inequality:

$$\forall x, y : |\langle x, y \rangle| \leq \varepsilon \frac{\|x\|^p}{p} + \frac{\|y\|^q}{\varepsilon q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \varepsilon > 0$$

Hence, whenever $\|x\| > \frac{\delta}{\sqrt{c}}\|u\|$, $t \mapsto \|x(t)\|$ is decreasing.

Moreover whenever $\|x\| \geq \frac{\delta}{\sqrt{c\theta}}\|u\|$, $\theta \in (0, 1)$, we have $\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\frac{c}{2}(1-\theta)\|x\|^2 \implies$ ISS. \square

7.3 Exercise 3

Motivation: Lyapunov Theory

$$\dot{x} = f(x, u)$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Definition. (CLF) A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF if it is continuous differentiable, positive definite, radially unbounded and $\forall x \neq 0 \inf_u \langle \nabla V(x), f(x, u) \rangle < 0$

In order to find CLFs, we restrict our analysis to input-affine systems

$$\dot{x} = f(x) + G(x)u$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$

Proposition: A continuous, differentiable, positive definite and radially unbounded. $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF iff

$$\forall x \neq 0 \quad L_G V(x) = 0 \implies L_f V(x) < 0$$

Image to be inserted

Problem 1

Consider $\dot{x} = \cos(x) + (1 + e^x)u$ where $f(x) = \cos(x)$ - drift and $g(x) = 1 + e^x$

Let $V : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \frac{1}{2}x^2$. Clearly, continuous differentiable, positive definite and radially unbounded. Moreover, for any nonzero x , we have $L_G V(x) \neq 0$.

Thus, for any $x \neq 0$, there exists a control that renders $\langle \nabla V(x), f(x) + g(x)u \rangle$ negative. Given this CLF, there exists a state feedback $u = u(x)$, e.g.

$$u(x) = -\frac{kx + \cos(x)}{1 + e^x}, \quad k > 0$$

Problem

Consider

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 e^{x_1} \cos(x_2) \\ \dot{x}_2 &= x_1^5 \sin(x_2) + u \end{aligned}$$

Take $V : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2)$

For any $x \neq 0$, we have

$$\inf_{u \in \mathbb{R}} (L_f V(x) + L_G V(x)u) = \begin{cases} L_f V(x), & \text{if } L_G V(x) = 0 \\ -\infty & \text{else} \end{cases}$$

In particular,

$$\begin{aligned} L_f V(x) &= \dots = x_1(-x_1^3 + x_2 e^{x_1} \cos(x_2)) + x_2 x_1^5 \sin(x_2) \\ L_G V(x) &= \dots = x_2 \end{aligned}$$

However,

$$L_f V(x)|_{x_2=0} = -x_1^4 < 0 \quad \forall x_1 \neq 0$$

Image to be inserted

Concluding that V is a CLF.

Problem 2:

$\dot{x} = Ax + Bu$, input defined system where (A, B) is stabilizable, there exists $K \in \mathbb{R}^{m \times n}$ s.t. $A + BK$ is Hurwitz (cf. KRT). The latter is equivalent to the existence $P = P^T > 0$ s.t. $P(A + BK) + (A + BK)^T P < 0$ (cf. Khalil theorem 4,6)

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \langle x, Px \rangle$. Moreover, $\forall x \neq 0 \exists u = Kx$ s.t. $\langle \nabla V(x), Ax + Bu \rangle < 0$, since

$$\langle \nabla V(x), Ax + Bu \rangle = \langle u = Kx, x \rangle = \langle P(A + BK) + (A + BK)^T P \rangle x < 0$$

In addition,

$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\|K\|} > 0 \quad \forall x \neq 0, \|x\| < \delta \quad \exists u = Kx \quad \|u\| < \epsilon$$

s.t. $L_f V(x) + L_G V(x)u < 0$ since $\|u\| = \|Kx\| \leq \|K\|\|x\| < \|K\|\delta = \epsilon$

Problem 3

Let $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous, differentiable consider

$$m\dot{v} = -g\nabla P(q) + F, \quad m, g > 0$$

a) Hamiltonian form. Let $x := (q, v)$. Then $\dot{x} = (-\frac{g}{m}\nabla P(q) + \frac{1}{m}F) = \begin{bmatrix} I \\ -I \end{bmatrix} \begin{bmatrix} \frac{g}{m}\nabla P(q) \\ v \end{bmatrix} + \begin{bmatrix} \frac{1}{m}I \\ 0 \end{bmatrix} F = \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F$ given $H(x) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q)$

b) "CLF". Take H as a CLF candidate. Then, for any x

$$\begin{aligned} \langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle &= \langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle + \langle \nabla H(x), G(x)F \rangle = \\ &= [\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle = L_f H(x) = 0] = \frac{1}{m} \langle v, F \rangle \end{aligned}$$

Strictly speaking, H is no CLF, but it reveals how to choose F s.t. the origin is GAS.

For any point x for which there exists no control F s.t. $\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle < 0$

Choose $F = 0$. Why? Using the Krasovskiy-Lasalle inv. principle, we conclude that the origin is GAS, since any solution in $\{\dot{H}(x) = 0\}$ verifies $v(t) \equiv 0$, implying $\dot{v}(t) \equiv 0$ s.t.

$$0 = -\frac{g}{m}\nabla P(q(t)) + \frac{1}{m}P(t)$$

The last part equals 0. Since $F = 0$ (by choice) and $\nabla P(q) = 0$ iff $q = 0$ we conclude that $\dot{H}(x) = 0$ can only be "maintained" at the origin.

Problem 4

Consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ux_2 + u^3 \end{aligned}$$

show that $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$ is CLF and let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$\ddot{x} + u\dot{x} - u^3 = 0$$

For any x and u , we have $\langle \nabla V(x), f(x, u) \rangle = \dots = x_1(2x_2 - ux_2 + u^3) + x_2(x_2 - ux_2 + u^3) = x_1h_1 + x_2h_2$

Image to be inserted

Hence if $u < 0$ and $-u$ "large", then we can render $\langle \nabla V(x), f(x, u) \rangle < 0$.

7.4 Exercise 4

Consider

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1) + g_2(x_1, x_2)u \end{cases} \quad (23)$$

Using the "preliminary control"

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \check{u} \end{cases} \quad (24)$$

$$u = \frac{1}{g_2(x_1, x_2)}(\check{u} - f_2(x_1, x_2))$$

Idea: Look at the upper(-most) system only and consider x_2 as a "virtual control".

Assumptions: Suppose

- \exists CLF V_1 ;
- \exists (smooth) feedback α_1 s.t. $L_{f_1+g_1\alpha_1}V_1 < 0$.

Now, add and subtract $g_1\alpha_1$ in 24 s.t.

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1)) \\ \dot{x}_2 = \check{u} \end{cases} \quad (25)$$

Next, introduce $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

$$\begin{cases} \dot{e}_1 = f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 = \check{u} - \dot{\alpha}_1(e_1) \end{cases} \quad (26)$$

Problem 1:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Proof. 1. Choose "virtual control":

$$x_2 = -(k+1)x_1 =: \alpha_1(x_1), \quad k > 0$$

The origin of $\dot{x}_1 = -kx_1$ is GAS.

(Take $V_1 : R \rightarrow R$, $x_1 \mapsto \frac{1}{2}x_1^2$ s.t. $\dot{V}_1(x_1) = -kx_1^2 < 0$ for all $x_1 \neq 0$)

2. Error coordinates:

Let $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

$$\dot{e}_1 = -ke_1 + e_2$$

$$\dot{e}_2 = u + (k+1)(-ke_1 + e_2)$$

3. "Composite CLF":

Define $V : R \times R \rightarrow R$, $(e_1, e_2) \mapsto V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(u + (k+1)(-ke_1 + e_2) + e_1)$$

4. Choose control:

Let $u = -e_1 - (k+1)(e_2 - ke_1) - ke_2$

s.t. $\dot{V}(e_1, e_2) = -ke_1^2 - ke_2^2 < 0$ for all $(e_1, e_2) \neq (0, 0)$

Remark: The closed-loop system reads:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

□

Problem 2:

$$\dot{x}_1 = x_1(x_2 - k), \quad k > 0$$

$$\dot{x}_2 = u$$

Proof. 1. $x_2 = 0 =: \alpha_1(x_1)$

The origin of $\dot{x}_1 = -kx_1$ is GAS ($V_1(x_1) = \frac{1}{2}x_1^2$)

2. $(e_1, e_2) := (x_1, x_2)$ s.t.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = u$$

3. $V(e_1, e_2) = V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(e_1^2 + u)$$

4. $u = -e_1^2 - ke_2$

□

Problem 3:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = u$$

Proof. 1. From problem 2:

$\dot{x}_2 = x_2(x_3 - k) - x_1^2 = -x_1^2 - kx_2 = u$ in Problem 2.

The origin of

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

is GAS.

And this is true for $x_3 = 0 =: \alpha_2(x_1, x_2)$.

$$2. (e_1, e_2, e_3) := (x_1 - 0, x_2 - \alpha_1(x_1), x_3 - \alpha_2(x_1, x_2)) \text{ s.t.}$$

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = u$$

$$3. V(e_1, e_2, e_3) = V_1(e_1) + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 \text{ s.t.}$$

$$4. u = -e_2^2 - ke_3$$

□

Problem 4:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = x_3(x_4 - k) - x_2^2$$

$$\dot{x}_4 = u$$

Proof. 1. Is GAS for

$$x_3(x_4 - k) - x_2^2 = -x_2^2 - kx_3$$

which is attained for $x_4 = 0 =: \alpha_3(x_1, x_2, x_3)$.

2.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = e_3(e_4 - k) - e_2^2$$

$$\dot{e}_4 = u$$

...

$$3. u = -e_3^2 - ke_4$$

□

Problem 5:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

...

$$\dot{x}_i = x_i(x_{i+1} - k) - x_{i-1}^2$$

...

$$\dot{x}_n = u$$

Proof. We will always have $u = -e_{n-1}^2 - ke_n$.

Let $V : R \times \dots \times R \rightarrow R$, $(e_1, \dots, e_n) \mapsto \sum_{i=1}^n V_i(e_i)$, where $V_i(e_i) = \frac{1}{2}e_i^2$, $i = 2, \dots, n$.

We have $\dot{V}(e_1, \dots, e_n) = L_{f_1+g_1\alpha_1}V_1(e_1) - k \sum_{i=2}^{n-1} e_i^2 + e_n u + e_{n-1}g_{n-1}(x_1, \dots, x_{n-1})e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1})$.

We observe that for α_i being zero, the inequality

$$e_{n-1}g_{n-1}(x_1, \dots, x_{n-1})e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1}) + e_n u < 0$$

hence $e_{n-1}^2 e_n + e_n u < 0$ for non-zero e .

It is solved by $u = -e_{n-1}^2 - ke_n$, $k > 0$. □

7.5 Exercise 5

Consider the SISO system

$$\dot{x} = f(x) + g(x)(u + \sigma(x))$$

$$y = s(x)$$

$f, g : R^n \rightarrow R^n$, $\sigma : R^n \rightarrow R$ and bounded, $s : R^n \rightarrow R$

Design steps for SMC:

1. If no output is provided, design a sliding surface $S := \{x \in R^n | s(x) = 0\}$ s.t.
 - (a) the system has rel. degree one;
 - (b) for $y(t) \equiv 0$, all solutions converge to the origin ("zero dynamics" have GAS origin)
2. Choose a control s.t. the sliding surface is reached (in finite time), e.g.

$$v(x) = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot \text{sgn}(s(x))), \quad \hat{u} > 0$$

Problem 1:

$$\dot{x}_1 = (x_2 - x_1)x_1^2$$

$$\dot{x}_2 = x_2 + u$$

Sliding surface S , $s : R^2 \rightarrow R$, $(x_1, x_2) \mapsto x_2$

Proof. (a) For the given S , we have $L_g s(x) = 1$ for any $x \in R^2$.

Moreover, from

$$\dot{s}(x) = L_f s(x) + L_g s(x)u$$

(we want $\dot{s} = 0$) we have that for

$$u = -\frac{L_f s(x)}{L_g s(x)} = -x_2$$

the "dynamics on S " (i.e. $x_2 = 0$) reduced to

$$\dot{\eta} = -\eta^3$$

whose origin is GAS.

(b) Consider

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat{u} \cdot \text{sgn}(s(x))) = -x_2 - \hat{u} \cdot \text{sgn}(x_2), \quad \hat{u} > 0$$

such that $x(t)$ "tends to S " in finite time (phase 1). Moreover, "on S ", $x(t)$ converges to the origin $t \rightarrow +\infty$ (phase 2).

□

Remark: Given a system in regular form

$$x = (\eta, \xi)^T$$

$$\dot{\eta} = f_1(\eta, \xi)$$

$$\dot{\xi} = f_2(\eta, \xi) + g_2(\eta, \xi)u$$

choose $s(x) = \xi - \Phi(\eta)$, s.t. Φ as. stabilizes $\dot{\eta} = f_1(\eta, \Phi(\eta))$.

Problem 2:

$$\dot{x}_1 = -x_1 \cos x_2 + x_1 x_2$$

$$\dot{x}_2 = x_1 \cos x_1 + \sigma(x) + u$$

Proof. (a) (For the design of sliding surface pretend that uncertainty $\sigma(x) = 0$)

Let $S := \{x \in R^2 | s(x) = 0\}$ be def. by $s : R^2 \rightarrow R$, $(x_1, x_2) \mapsto x_2(-\Phi(x_1) = 0)$. We have $L_g s(x) = 1$ for all $x \in R^2$.

From

$$\dot{s}(x) = L_f s(x) + L_g s(x)u$$

(we want $\dot{s} = 0$) s.t. for $u = -\frac{L_f s(x)}{L_g s(x)} (= -x_1 \cos x_1)$ the "dynamics on S " (i.e. $x_2 = 0$) reads

$$\dot{\eta} = -\eta$$

whose origin is GAS.

(b) Take

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + (\hat{u} + \beta(x)|L_g s(x)|) \cdot \text{sgn}(s(x))) (= -x_1 \cos x_1 - (\hat{u} + (\hat{u} + (x_1^2 + x_2^2)) \cdot \text{sgn}(x_2))), \quad \hat{u} > 0$$

Consider the Lyapunov(-like) function $V(x) = \frac{1}{2}s(x)^2$ s.t.

$$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x)))$$

Choosing u as above

$$\begin{aligned} \dot{V}(x) &= s(x)(-(\hat{u} + \beta(x)|L_g s(x)|) \cdot \text{sgn}(s(x)) + \sigma(x)L_g s(x)) \leq -(\hat{u} + \beta(x)|L_g s(x)|)|s(x)| + \\ &|\sigma(x)||L_g s(x)||s(x)| \leq -\hat{u}|s(x)| < 0 \text{ for } s(x) \neq 0 \end{aligned}$$

□

Problem 3:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1^3 + \sigma(x) + u$$

$$s(x) = x_2 + x_1, \quad u = -x_2 + x_1^3 - 2 \cdot \operatorname{sgn}(s(x))$$

Proof. (a) Given S , we have $L_g s(x) = 1$ for all $x \in R^2$. The "dynamics on S " (i.e. $x_1 + x_2 = 0$) reads

$$\dot{\eta}_1 = -\eta_1$$

$$\dot{\eta}_2 = -\eta_2$$

whose origin is GAS.

(b) Take $V(x) = \frac{1}{2}s(x)^2$ s.t.

$$\dot{V}(x) = s(x)(L_f s(x) + L_g s(x)(u + \sigma(x))) \leq -\hat{u}|L_g s(x)||s(x)| + |\sigma(x)||L_g s(x)||s(x)| \leq [|\sigma(x)| \leq c] \leq -(\hat{u} - c)|L_g s(x)||s(x)|.$$

Hence, for $c < \hat{u} = 2$ there exists $\varepsilon > 0$ s.t. $\dot{V}(x) \leq -\varepsilon|s(x)| < 0$ for $s(x) \neq 0$

□

7.6 Exercise 6

Problem 1:

$$\dot{x} = xu(x^2 + u)$$

$$y = h(x)$$

$$s : R \times R \rightarrow R, \quad (u, y) \mapsto uy^2 + u^2y$$

$$S : R \rightarrow R, \quad x \mapsto \frac{x^2}{2}$$

Proof. Clearly, S is non-negative. Moreover:

$$\dot{S}(x) = x^2 u(x^2 + u) = x^4 u + x^2 u^2 = [h(x) = x^2] = s(u, x^2)$$

for all $x, u \in R$ with $h : R \rightarrow R, x \mapsto x^2$.

□

Problem 2:

$$\dot{x} = u, \quad x(0) = x_0$$

$$y = x$$

$$s : R^n \times R^n \rightarrow R, \quad (u, y) \mapsto \langle u, y \rangle$$

Proof. For any $x_0 \in R^n$, we have

$$\begin{aligned} S_a(x_0) &= \sup_{u: [0, t] \rightarrow R^n, t \geq 0, x(0) = x_0} \left(- \int_0^t \langle u(\tau), y(\tau) \rangle d\tau \right) = \\ &= \sup_{-// -} \left(- \frac{1}{2} \int_0^t \frac{d}{d\tau} \|x(\tau)\|^2 d\tau \right) = \sup_{-// -} \left(- \frac{1}{2} \|x(t)\|^2 + \frac{1}{2} \|x(0)\|^2 \right) \leq \frac{1}{2} \|x_0\|^2 \end{aligned}$$

\implies av. storage is finite \implies system is dissipative. Moreover, we have for any $x_0 \in R^n$,

$$S_r(x_0) = \inf_{u: [-t, 0] \rightarrow R^n, t \geq 0, x(-t)=0, x(0)=x_0} \int_{-t}^0 \langle u(\tau), y(\tau) \rangle d\tau = \inf_{-// -} \left(\frac{1}{2} \|x_0\|^2 - \frac{1}{2} \|x(-t)\|^2 \right) = \frac{1}{2} \|x_0\|^2$$

($S_a = S_r \implies$ this is a unique stor. func.)

Hence the (lossless) system is reachable (from 0 to any x_0). \square

Problem 3:

Proof. Consider the Lyapunov func. cand. $V(x) = S_1(x_1) + S_2(x_2)$ s.t.

$$\dot{V}(x) \leq s_1(u_1, y_1) + s_2(u_2, y_2) = s_1(u_1, y_1) + s_2(y_1, -u_1) = 0 \implies \text{origin is stable.}$$

\square

Remark: the above problem captures many stability results (in the frequency domain). Particular choices of supply rates are:

- $s_i(u_i, y_i) = \|u_i\|^2 - \|y_i\|^2, i = 1, 2$ (small-gain theorem);
- $s_i(u_i, y_i) = \langle u_i, y_i \rangle, i = 1, 2$ (positive operator theorem);
- $s_1(u_1, y_1) = \langle u_1 + ay_1, u_1 + by_1 \rangle$
 $s_2(u_2, y_2) = -ab \langle u_2 - \frac{1}{a}y_2, u_2 - \frac{1}{b}y_2 \rangle$ (conic operator theorem).

Problem 4:

$$\dot{x} = f(x) + G(x)u$$

$$y = h(x)$$

$$s : R^m \times R^m \rightarrow R, (u, y) \mapsto \|u\|^2 - \|y\|^2$$

Proof. Take $V = S$ s.t.

$$\dot{V}(x) \leq \|u\|^2 - \|h(x)\|^2, \forall x \in R^n, \forall u \in R^m$$

Then the (continuous) state feedback $u = \gamma h(x)$ for some $|\gamma|^2 < 1$, s.t.

$$\dot{V}(x) \leq (|\gamma|^2 - 1) \|h(x)\|^2 < 0, \forall x \neq 0$$

\square

Problem 5:

Proof. Take $S(x) = \langle x, P_x \rangle$ s.t.

$$\dot{S}(x) = \langle x, (PA + A^T P)x \rangle + 2 \langle x, PBu \rangle$$

Add and subtract $\gamma^2 \|u\|^2$ and $\frac{1}{\gamma^2} \|B^T Px\|^2$.

$$\dot{S}(x) = \langle x, (PA + A^T P + \frac{1}{\gamma^2} PBB^T P)x \rangle + \gamma^2 \|u\|^2 - \gamma^2 \|u - \frac{1}{\gamma^2} B^T Px\|^2$$

Add and subtract $\|y\|^2$.

$$\dot{S}(x) = \langle x, (PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C)x \rangle + \gamma^2 \|u\|^2 - \|y\|^2 - \gamma^2 \|u - \frac{1}{\gamma^2} B^T Px\|^2$$

$$\dot{S}(x) \leq \gamma^2 \|u\|^2 - \|y\|^2$$

□

7.7 Exercise 7

Definition. A mapping $\Phi : R \rightarrow R$, $u \mapsto \Phi(u)$, belongs to the sector

- $[0, +\infty]$ if $u\Phi(u) \geq 0$, $\forall u \in R$;
- $[\alpha, +\infty]$ if $u(\Phi(u) - \alpha) \geq 0$, $\forall u \in R$ and some $\alpha \in R$;
- $[0, \beta]$ if $\Phi(u)(\Phi(u) - \beta) \leq 0$, $\forall u \in R$ and some $\beta \in R$;
- $[\alpha, \beta]$ if $(\Phi(u) - \alpha)(\Phi(u) - \beta) \leq 0$, $\forall u \in R$ and some $\alpha, \beta \in R$;

Notation: we write, e.g., $\Phi \in [0, +\infty]$.

Problem 1:

$$\begin{aligned} \dot{x} &= x^3 - kx + u, \quad k > 0 \\ y &= x \end{aligned}$$

Proof. Take, e.g., $S : R \rightarrow R, x \mapsto \frac{x^2}{2}$ ($S \geq 0$) s.t.

$$\dot{S}(x) = x^2(x^2 - k) + yu \leq yu$$

whenever $x \in [-\sqrt{k}, \sqrt{k}]$.

Let $\bar{x} \in R$ and take $u = -\bar{x}^3 + k\bar{x}$ with init. condition $x(0) = \bar{x}$, s.t. we have $x(t) = \bar{x}$ for all $t \geq 0$. If the system is passive, then along this (constant) solution we must have

$$S(x(t)) - S(\bar{x}) \leq \int_0^t u(\tau)y(\tau)d\tau, \quad t \geq 0$$

This inequality, however, is violated for $\bar{x} \notin [-\sqrt{k}, \sqrt{k}]$ and hence $[-\sqrt{k}, \sqrt{k}]$ must be the largest interval. □

Problem 2:

$$\begin{aligned}\dot{x} &= -x + \frac{1}{\beta}h(x) + u, \quad \beta > 0 \\ y &= h(x)\end{aligned}$$

$$S(x) = \int_0^x h(\sigma)d\sigma, \quad h \in [0, \beta]$$

Proof. Clearly, we have $S \geq 0$ since $h \in [0, \beta]$.

Moreover,

$$\dot{S}(x) = S'(x)\dot{x} = \dot{x} \frac{d}{dx} \int_0^x h(\sigma)d\sigma = h(x)\dot{x} = \frac{1}{\beta}h(x)(h(x) - \beta x) + yu \leq yu$$

since $h \in [0, \beta]$. □

Problem 3:

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + kx_2 + u, \quad k > 0 \\ y = x_2 \end{cases}$$

Proof. Take $S : R^2 \rightarrow R$, $(x_1, x_2) \mapsto \frac{x_1^2}{2} + \frac{x_2^2}{2}$ s.t. $\dot{S}(x) = uy + ky^2$.

Let $u = -\Phi(y)$, $\Phi : R \rightarrow R$ satisfying $\Phi \in [l, +\infty]$ for some $l > k$ ($\nu_2 + \rho_1 > 0$) s.t.

$$\dot{S}(x) = -y\Phi(y) + ky^2 \leq -(l - k)y^2$$

Since the system H_1 is ZSO the origin is GAS. □

Problem 4:

Proof. Take $S(x) = S_1(x_1) + S_2(x_2)$ s.t.

$$\dot{S}(x) \leq \langle u_1, y_1 \rangle - \rho_1 \|y_1\|^2 - \nu_1 \|u_1\|^2 + \langle u_2, y_2 \rangle - \rho_2 \|y_2\|^2 - \nu_2 \|u_2\|^2$$

Using that

$$\langle u_1, y_1 \rangle + \langle u_2, y_2 \rangle = \langle u - y_2, y_1 \rangle + \langle v + y_1, y_2 \rangle = \langle u, y_1 \rangle + \langle v, y_2 \rangle$$

and

$$\|u_1\|^2 = \|u\|^2 - 2\langle u, y_2 \rangle + \|y_2\|^2$$

$$\|u_2\|^2 = \|v\|^2 + 2\langle v, y_1 \rangle + \|y_1\|^2$$

we obtain

$$\begin{aligned}\dot{S}(x) &= -\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} (\nu_2 + \rho_1)I_m & \\ & (\nu_1 + \rho_2)I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \nu_1 I_m & \\ & \nu_2 I_m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} I_m & 2\nu_1 I_m \\ -2\nu_2 I_m & I_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \\ &\leq [Coshi - Schwarz] \leq -a\|(y_1, y_2)\|^2 + b\|(u, v)\| \|(y_1, y_2)\| + c\|(u, v)\|^2\end{aligned}$$

with $a = \min\{\nu_2 + \rho_1, \nu_1 + \rho_2\} > 0$, $b = \|N\| \geq 0$ and $c = \|M\| \geq 0$.

Hence,

$$\dot{S}(x) \leq -\frac{1}{2a}(b\|(u, v)\| - a\|(y_1, y_2)\|)^2 + \frac{b^2}{2a}\|(u, v)\|^2 - \frac{a}{2}\|(y_1, y_2)\|^2 + c\|(u, v)\|^2 \leq \frac{b^2 + 2ac}{2a}\|(u, v)\|^2 - \frac{a}{2}\|(y_1, y_2)\|^2 \quad \square$$

Problem 5:

Proof. Take $V(x) = \langle x, Px \rangle$ s.t.

$$\dot{V}(x) = \langle x, (PA + A^T P)x \rangle - 2\Phi(y) \langle x, PB \rangle$$

Add and subtract $2\Phi(y)^2$ and $2\Phi(y)\beta Cx$ yields

$$\begin{aligned} \dot{V}(x) &= -\varepsilon \langle x, Px \rangle - \langle x, L^T Lx \rangle - 2\Phi(y) \langle x, PB - \beta C^T \rangle - 2\Phi(y)^2 + 2\Phi(y)(\Phi(y) - \beta y) = \\ &= -\varepsilon \langle x, Px \rangle - |Lx - \sqrt{2}\Phi(y)|^2 + 2\Phi(y)(\Phi(y) - \beta y) \leq -\varepsilon \langle x, Px \rangle. \end{aligned} \quad \square$$

7.8 Exercise 8

Problem 1:

Show that if $\Phi : [0, +\infty) \rightarrow \mathbb{R}$ and $\Phi \in L_1 \cap L_\infty$, then $\Phi \in L_p \forall p \in [1, +\infty]$.

Proof. Holder inequality:

Let $p, q \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and $g \in L_q$, then $fg \in L_1$ with

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_q}$$

Convention: If $p = +\infty$ ($q = +\infty$), then $\frac{1}{p} = 0$ ($\frac{1}{q} = 0$).

Let $p \in (1, +\infty)$. We have

$$(\|\Phi\|_{L_p}^p =) \int_0^\infty |\Phi(t)|^p dt = \int_0^\infty |\Phi(t)\Phi(t)^{p-1}| dt \leq [HE] \leq \|\Phi\|_{L_1} \|\Phi\|_{L_\infty}^{p-1}$$

Hence, $\Phi \in L_p$ for any $p \in [1, +\infty)$ □

Problem 2:

Proof. b) $\Phi_2 = \frac{1}{t+1}$

For $p \in \{1, 2\}$, we have

$$\int_0^\infty |\Phi_2(t)|^p dt = \lim_{T \rightarrow +\infty} \int_0^T \frac{1}{(t+1)^p} dt = \begin{cases} \lim_{T \rightarrow +\infty} [\ln(t+1)]_0^T = +\infty, & p = 1 \\ \lim_{T \rightarrow +\infty} [-\frac{1}{t+1}]_0^T < +\infty, & p = 2 \end{cases}$$

s.t. $\Phi_2 \in L_1$, but $\Phi_2 \notin L_2$.

Moreover,

$$\sup_{t \geq 0} |\Phi_2(t)| = \sup_{t \geq 0} \left| \frac{1}{t+1} \right| = 1 < +\infty$$

s.t. $\Phi_2 \in L_\infty$. □

Problem 3:

Proof. Take $v = \frac{u}{\|u\|_{L_p}}$, $u \neq 0$, s.t. $v \in L_p$ and $\|v\|_{L_p} = 1$.

Then

$$\|Hv\|_{L_p} = \|H \frac{u}{\|u\|_{L_p}}\|_{L_p} = [H - \text{linear}] = \|\frac{1}{\|u\|_{L_p}} Hu\|_{L_p} = \frac{1}{\|u\|_{L_p}} \|Hu\|_{L_p}$$

□

Problem 4: Let $\Phi : [0, +\infty) \rightarrow \mathbb{R}$ be defined by $\Phi(t) = t$. Show that $\Phi \in L_p^e$ for any $p \in [1, +\infty]$.

Proof. Let $p \in [1, +\infty)$ and fix some $T \geq 0$. Then

$$(\|\Phi_T\|_{L_p}^p =) \int_0^\infty |\Phi_T(t)|^p dt = \int_0^T |\Phi(t)|^p dt = \int_0^T |t|^p dt = \frac{t^{p+1}}{p+1} \Big|_0^T = \frac{T^{p+1}}{p+1} < +\infty$$

Moreover,

$$\|\Phi_T\|_{L_\infty} = \sup_{t \geq 0} |\Phi_T(t)| = \sup_{t \in [0, T]} |\Phi(t)| = T < +\infty$$

$$\implies \Phi \in L_p^e \forall p \in [1, +\infty].$$

□

Problem 5:

Proof. \implies :

Take $u, v \in L_p^e$ s.t. $v_T = u_T$ for some $T \geq 0$. Then, by causality of H ,

$$H(u)_T = H(u_T)_T, \quad H(v)_T = H(v_T)_T$$

Since $u_T = v_T$, it follows $H(u)_T = H(v)_T$.

\Leftarrow :

Take $u \in L_p^e$ and consider $v = u_T$ for some $T \geq 0$. Noting that $v_T = (u_T)_T = u_T$, we have

$$H(u)_T = H(v)_T = H(u_T)_T$$

□

Problem 6:

Proof. Let $u \in L_\infty^e$ and $T \geq 0$ s.t. for any $0 \leq t \leq T$

$$\begin{aligned} |y_T(t)| &\leq \int_0^t |h(t-\tau)| |u(\tau)| d\tau \leq \sup_{\tau \in [0, T]} |u(\tau)| \int_0^t |h(t-\tau)| d\tau = [\sigma = t - \tau] = \\ &= \sup_{\tau \in [0, T]} |u(\tau)| \int_0^t |h(\sigma)| d\sigma \leq \sup_{\tau \in [0, T]} |u(\tau)| \left(\int_0^t |h| + \int_t^\infty |h| \right) \dots \end{aligned}$$

□

7.9 Exercise 9

Problem 1:

Proof. Let $H_2 : L_P^e \rightarrow L_P^e$ be defined by $H_2(e_2) = \frac{1}{2\gamma}e_2$.
Using that $y_2 = H_2(e_2)$, it follows

$$\|(y_2)_T\|_{L_P} = \frac{1}{2\gamma} \|(e_2)_T\|_{L_2}, \quad \forall e_2 \in L_P^e, \quad \forall T \geq 0$$

Since $\gamma_1\gamma_2 = \gamma\frac{1}{2\gamma} = \frac{1}{2} < 1$, the SGT reveals that $y \in L_P$ whenever $u \in L_P$. \square

Problem 2:

Proof. Let $e_2 \in L_\infty^e$ and $T \geq 0$ s.t. $y_2 = H_2(e_2)$. Then

$$(\|(y_2)_T\|_{L_2}^2 =) \int_0^T |y_2(t)|^2 dt \leq \text{ess sup}_{t \in [0, T]} |y_2(t)| \int_0^T |y_2(t)| dt = \|(y_2)_T\|_{L_\infty} \|(y_2)_T\|_{L_1} \leq \delta\varepsilon \|(e_2)_T\|_{L_\infty}^2$$

Hence,

$$\|(y_2)_T\|_{L_2} \leq \sqrt{\delta\varepsilon} \|(e_2)_T\|_{L_\infty}, \quad \forall e_2 \in L_\infty^e, \quad \forall T \geq 0$$

for any $u \in L_2^e$ and $T \geq 0$ (with $y = y_2$),

$$\|y_T\|_{L_2} \leq \sqrt{\delta\varepsilon} \|(y_1)_T\|_{L_\infty} \leq \gamma\sqrt{\delta\varepsilon} \|(e_1)_T\|_{L_2} \leq \gamma\sqrt{\delta\varepsilon} \|u_T\|_{L_2} + \gamma\sqrt{\delta\varepsilon} \|y_T\|_{L_2}$$

s.t.

$$(1 - \gamma\sqrt{\delta\varepsilon}) \|y_T\|_{L_2} \leq \gamma\sqrt{\delta\varepsilon} \|u_T\|_{L_2}$$

If $\gamma\sqrt{\delta\varepsilon} < 1$, then the SGT reveals that $y \in L_2$ whenever $u \in L_2$. \square

Problem 3:

Proof. For any $u \in L_2^e$ and $T \geq 0$, we have

$$\langle u_T, y_T \rangle_{L_2} = \langle u_T, h(x)_T \rangle_{L_2} = \langle a\dot{x}_T + x_T, h(x)_T \rangle_{L_2} = a \int_0^T h(x(t))\dot{x}(t) dt + \int_0^T x(t)h(x(t)) dt$$

Since $h \in [0, +\infty]$, i.e. $xh(x) \geq 0$, $\forall x \in R$, we have $\int_0^T x(t)h(x(t)) dt \geq 0$.

Now, $\Phi : R \rightarrow R$ be defined by $\Phi(x) = \int_0^x h(\sigma) d\sigma$.

Clearly, $\Phi(x) \geq 0$ for all $x \in R$.

It follows that

$$a \int_0^T h(x(t))\dot{x}(t) dt = a \int_{x(0)}^{x(T)} h(\sigma) d\sigma = a(\Phi(x(T)) - \Phi(x(0))) \geq -a\Phi(x(0)) =: \beta$$

Hence, the system is passive ($\langle u_T, y_T \rangle_{L_2} \geq -\beta$, $\forall u \in L_2^e$, $\forall T \geq 0$). \square

Problem 4:

Proof. Since H_1 is passive (with zero bias ($\beta = 0$)), it follows

$$\langle (e_1)_T, (y_1)_T \rangle_{L_2} \geq 0, \quad \forall e_1 \in L_2^e, \quad \forall T \geq 0.$$

Using that

$$\langle (e_2)_T, (y_2)_T \rangle_{L_2} \geq \delta \|(e_2)_T\|_{L_2}^2, \quad \forall e_2 \in L_2^e, \quad \forall T \geq 0,$$

we have for $u \in L_2^e$ and $T \geq 0$ with $y = y_1$,

$$\langle u_T, y_T \rangle_{L_2} = \langle (e_1)_T, (y_1)_T \rangle_{L_2} + \langle (e_2)_T, (y_2)_T \rangle_{L_2} \geq \delta \|y_T\|_{L_2}^2.$$

Using Caschy-Schwarz

$$\|y_T\|_{L_2} \|u_T\|_{L_2} \geq |\langle u_T, y_T \rangle_{L_2}| \geq \langle u_T, y_T \rangle_{L_2} \geq \delta \|y_T\|_{L_2}^2$$

s.t.

$$\|y_T\|_{L_2} \leq \frac{1}{\delta} \|u_T\|_{L_2}$$

Since $u \in L_2$, passing to the limit, as $T \rightarrow \infty$, reveals $y \in L_2$. □