L (Cromwall). Supp. $0 \le \phi(t) \le c + L\int_0^t \phi(\tau)d\tau$, $c, L > 0$, ϕ – continuous. Then $\phi(t) \le ce^{Lt}$.

Nonlinear systems

Def. Pt $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \ge 0$, $\exists \delta > 0$ s.t. from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

Def. Point $x^* = 0$ is unif. stable if $\forall \epsilon > 0$ $\exists \delta > 0$, s.t $\forall t_0 \ge 0$, from $||x_0|| < \delta$ follows $||x(t)|| < \epsilon$, $\forall t \ge t_0$.

Def. Point $x^* = 0$ asympt. stable if it is stable and $\forall t_0 \ge 0$ $\exists c > 0$, s.t from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

Def. Point $x^* = 0$ unif. asympt. stable if it is unif. stable and $\exists c > 0$, s.t $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\lim_{t\to\infty} ||x(t)|| \to 0$.

Def. Convergence: $\forall \eta > 0$ $\forall t_0 \ge 0$, $\exists T > 0$ such that $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

Def. Unif. convergence: $\forall \eta > 0$ $\exists T > 0$ such that $\forall t_0 \ge 0$ and $\forall t \ge t_0 + T$ follows $||x(t)|| < \eta$.

Def. Pt $x^* = 0$ is glob. unif. asympt. stable if it is unif. stable with $\delta \to \infty$ for $\epsilon \to \infty$ and $\forall c, \eta$ $\exists T > 0$ s.t. $\forall t_0 \ge 0$ from $||x_0|| < c$ follows $\|x(t)\| < \eta$, $\forall t \ge t_0 + T$.

Th. Let $f : [0,\infty) \times D \to R^n$ is contin. and let $x^* = 0$ be EP. If there is a diff. $V : [0,\infty) \times D \to R$ with:

- $W_1(x) \le V(t,x) \le W_2(x)$, $\forall t \ge 0,\ x \in D$
- $\dot V(t,x) \le 0$, $\forall t \ge 0,\ x \in D$

where $W_1, W_2 : D \to R$ contin. and posit. def., then $x^* = 0$ is unif. stable.
If $\dot V(t,x) \le -W_3(x)$, $\forall t \ge 0,\ x \in D$ with $W_3 : D \to R$ contin. and pos. def., the $x^* = 0$ is unif. asympt. stable.
If $D = R^n$ and W_1 is radialy unbounded then $X^* = 0$ is glob. unif. asympt. stable.

L. EP $x^* = 0$ of $\dot x(t) = f(t, x(t))$ is unif. stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \alpha(||x(t_0)||)$.

L. EP $x^* = 0$ of $\dot x(t) = f(t, x(t))$ is unif asympt stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \ge t_0$, $\forall ||x(t_0)|| < c$ and $||x(t)|| \le \beta(||x(t_0)||, t - t_0)$.

System with inputs

Def. System is ISS if $\exists \beta \in KL$, $\gamma \in K$ s.t. $\forall x_0 \in R^n$, $\forall t \ge 0$ follows $||x(t)|| \le \beta(||x_0||, t) + \gamma(\sup_{\tau\in[0,t]} ||u(\tau)||)$.

Th. Suppose that there exists a cont. diff. func. $V : R^n \to R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ s.t. $\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$, $\forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x,u) \le -\alpha_3(||x||)$, $\forall x : ||x|| \ge \rho(||u||)$. Then is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Th. Assume: f is glob. Lipschitz; $x = 0$ is a glob. exp. stable EP for $\dot x = f(x, 0)$ Then ISS.

Th (Artstein). There exists $k : R^n \to R^m$ which is cont. on $R^n \backslash \{0\}$ s.t. $x^* = 0$ is glob. asympt. stable EP for $\dot x = f(x) + G(x)k(x)$ iff there exists a CLF.

Sontag's formula"
Fix $c \ge 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$-cb(x) - \frac{a(x)+\sqrt{a(x)^2+(b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T$$
$$0,\ \ b(x) = 0$$

Backstepping

Integrator backstepping

$$\dot x_1 = f_1(x_1) + g_1(x_1)x_2$$
$$\dot x_2 = u$$
$$u = (-\frac{\partial V_1}{\partial e_1} g_1(e_1) + \dot\alpha_1) - k_2 e_2,\ k_2 > 0$$
$$\dot x_1 = f_1(x_1) + g_1(x_1)x_2$$
$$\dot x_2 = f_2(x_1,x_2) + g_2(x_1,x_2)u$$
$$u = \alpha_2(x_1,x_2) = \frac{1}{g_2(x_1,x_2)}(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot\alpha_1 - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1,x_2))$$
$$\alpha_i(x_1,\dots x_i) = \frac{1}{g_i}(\dot\alpha_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}} g_{i-1} - k_i(x_i - \alpha_{i-1}) - f_i)$$

Systems with inputs and outputs

Two-step approach:

1. Bring $x(t)$ to $S := \{x \in \mathbb{R}^n | S(x) = 0\}$ in finite time
2. Have $x(t)$ going to zero asymptotically (on S)

$$V(X) = \frac{1}{2}s(x)^2$$

$$u = -\frac{1}{L_g s(x)}(L_f s(x) + \hat u sgn(s(x))),\ \hat u > 0$$
$$\dot x = f(x) + g(x)\sigma(x) + g(x)u$$

If $|\sigma(x)| \le \beta(x)$

$$u = -\frac{L_f s(x)}{L_g s(x)} - \frac{1}{L_g s(x)}(\hat u + \beta(x))|L_g s(x)| sgn(s(x))$$

Def (dissipativity).

$$S(x(t)) \le S(x_0) + \int_0^t s(u(\tau), y(\tau))d\tau \quad (1)$$

Introduce "available storage" $S_a(x)$

$$sup_{u:[0,T]\to\mathbb{R}^m, T\ge0, x(0)=0}(-\int_0^T s(u(\tau), y(\tau)))$$

Th. System is dissipative w.r.t. the supply rate s iff $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$
If $S_a(x) < \infty$ for all $x \in \mathbb{R}^n$, then S_a is a storage function and $S(x) \ge S_a(x)$ $\forall x \in \mathbb{R}^n$ for all storage functions S.

If system is dissipative then $x = 0$ is asympt. stable.

$$\dot x = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$
$$y = h(x),\ y \in \mathbb{R}^m \quad (2)$$

Def. System is passive if it is dissipative w.r.t. supply rate $s(u,y) = u^T y$

Def. System is zero-state observable (ZSO) if (for $u(t) = 0$) $y(t) = 0$ for all $t \ge 0 \Rightarrow x(t) = 0$ for all $t \ge 0$

Th. Let system (2) be i) passive in differentiable storage set ii)ZSO. Then the feedback $u = -Py$, $P > 0$ renders the origin asymptotically stable.

Th. Feedback interconnection with $u \equiv 0$. H_1 and H_2 are ZSO and dissipative with S_1, S_2 w.r.t.

$$s_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i,\ i = 1, 2,\ \rho, \nu \in \mathbb{R}$$

The origin $(x_1, x_2) = (0, 0)$ for interconnection is asymptotically stable if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$.

If is satisfied with $v_i = 0$: "output - feedback passive". If (**??**) satisfied with $p_i = 0$: "input - feadforward passive".

Input/Output Methods

Def. *Lp*-spaces, $p \in [1, \infty]$.
$Lp[0,\infty) = \{\Phi : [0,\infty) \to \mathbb{R}^m, measurable | \int_0^\infty ||\Phi(t)||^p dt < \infty\}$

(Cauchy-Schwarz inequality)
$| < \phi_1, \phi_2 >_{L_2} | \le \|\phi_1\|_{L_2} \|\phi_2\|_{L_2}$

Def. H is L_p-stable if there exists $\alpha \in K,\ \beta \ge 0$ s.t.
H is finite-gain L_p stable if there exist $\gamma,\ \beta \ge 0$ s.t.

$$\|(H(u))_T\|_{L_p} \le \gamma \|u_T\|_{L_p} + \beta$$

Def. A map $H : L_p^e \mapsto L_p^e$ is *causal* if $(H(u))_T = (H(u_T))_T$ for all $u \in L_p^e$ and $T \ge 0$.

Th. Consider $\dot x = f(x,u),\ y = h(x,u)$. Suppose the system is ISS and there exist $\alpha_1, \alpha_2 \in K$ and $\eta \ge 0$ s.t. $\|h(x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$. Then for each $x_0 \in \mathbb{R}^n$, the system is L_∞ - stable.

$$x = Ax + Bu \qquad u, y \in \mathbb{R} \to SISO$$
$$y = Cx + Du \qquad A...Hurwitz \quad (3)$$

L. The L_2 gain of (3) is

$$\gamma = \sup_{w\in\mathbb{R}} \sqrt{G(-jw)G(jw)}$$

where $G(s) = C(sI - A)^{-1}B + D$

$$\dot x = f(x) + g(x)u$$
$$y = h(x) \quad (4)$$

Recall. System has L_2 gain less or equal γ if it is dissipative w.r.t. supply rate $s(u,y) = \frac{1}{2}\gamma^2||u||_2^2 - \frac{1}{2}||y||_2^2$

Th. Suppose that H_1 and H_2 are finite-gain L_p stable (with gains γ_1, γ_2). Then the feedback interconnection is finite-gain L_p stable if $\gamma_1\gamma_2 < 1$.

Def. $H : L_p^e \to L_p^e$ is
passive if there exist $B \in \mathbb{R}$ s.t. $\forall u \in L_p^e$, $\forall T \ge 0$, $< u_T, y^T > \ge -B$
output-strictly passive if there exists $B \in \mathbb{R}$ and $\epsilon > 0$ s.t. $\forall u \in L_p^e$, $\forall T \ge 0$ follows $< u_T, y^T > \ge -B + \epsilon||y_T||_{L_2}^2$

L. Let $H : L_p^e \to L_p^e$ be output strictly passive with excess ϵ. Then H has L_2-gain $\le \frac{1}{\epsilon}$.

Th. Suppose exist $\epsilon_i, \delta_i, \beta_i$; $i = 1, 2$ s.t.

$$\langle (e_i)_T, (H_i(e_i))_T \rangle \ge \epsilon_i ||(H_i(e_i))_T||^2 + \delta_i ||(e_i)_T||^2 - \beta_i$$

for all $T \ge 0,\ e_i \in L_2^e,\ i = 1, 2$. If $\epsilon_1 + \delta_2 > 0$ and $\epsilon_2 + \delta_1 > 0$ then the feedback interconnection has finite L_2-gain from $(u_1, u_2) \to (y_1, y_2)$.