

Nonlinear Control

Stuttgart university, May 2018

Intro

Goals of Course

- overview over modern nonlinear analyses and control concepts
- modern methodologies to analyze nonlinear systems
- apply a wide range of nonlinear control methods learn the mathematic basis

Differential equations $\dot{x} = f(x)$

Nonlinear differential equation $\dot{x} = f(t, x)$

System with input $\dot{x} = f(x, u)$

System with input and output $\dot{x} = f(x, u), \quad y = g(x, u)$

Input-output methods

Scope

[1] Khalil Nonlinear System, Prentice Hall, 2002

[2] Sepulchre Constructive Nonlinear Control, Springer-Verlag 1997

1 Differential equations

Consider differential equality

$$\frac{d}{dt}x(t) = \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \quad (1)$$

Where $f : D \rightarrow R^n$, $D \subset R^n$ is open, [here we should explain, what means open set].

Solution to 1 $x : I_{x_0} \rightarrow D$, $t \rightarrow x(t)$ is differentiable

Interval existence solution

Questions:

existence of solution

"how large" is I_{x_0}

uniqueness of solution

Usually we will add some restrictions on f functions, like continuous.

1.1 Existence of solutions

Definition. Function $f : D \rightarrow R^n$ is continuous at $x' \in D$ if for $\forall \epsilon > 0 \exists \delta > 0$ such that for $\forall x \in D$, $\|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < \epsilon$

Function $f : D \rightarrow R^n$ is continuous on D if it's continuous at $\forall x' \in D$

Theorem 1.1 (Peano). If $f : D \rightarrow R^n$ continuous, then for each $x_0 \in D \exists x : (-\epsilon, \epsilon) \rightarrow D$, $\epsilon > 0$ satisfying (1).

Further, given a compact set $U \subset D$, then $\exists \alpha > 0$ s.t. $\forall x_0 \in U \exists x : (-\epsilon, \epsilon) \rightarrow D$ satisfying (1).

Example. Consider equation $\dot{x}(t) = x(t)^2$, $x(0) = x_0 = 0$. Solution $x(t) = -\frac{1}{t-c}$, $c = \frac{1}{x_0}$. In this example solution exist in interval $(-c, c)$.

But, what about the number of solutions? Which conditions we should add to guaranty uniqueness of solution?

1.2 Uniqueness of solutions

Definition. Function $f : D \rightarrow R^n$ is locally Lipschitz (continuous???) on D if $\forall x \in D$ there is a neighborhood $N(x) \subset D$ and $\exists L > 0$ s.t.

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\| \quad (2)$$

For all $x_1, x_2 \in N$.

- Lipschitz on $W \subset D$ if (2) holds $\forall x_1, x_2 \in W$ (with same L)
- globally Lipschitz if (2) holds $\forall x_1, x_2 \in R^n$ (with same L)

We have

locally Lipschitz functions are continuous

differentiable functions are locally Lipschitz

locally Lipschitz functions are Lipschitz on each compact subset of D (Khalil Ex 3.19)

Lemma 1 (Cromwall). Suppose that $0 \leq \phi(t) \leq c + L \int_0^t \phi(\tau) d\tau$, $c, L > 0$, ϕ - continuous. Then $\phi(t) \leq ce^{Lt}$.

Proof. $c + L \int_0^t \phi(\tau) d\tau := \psi(t)$, $\dot{\psi}(t) = L\phi(t) \leq L\psi(t)$.

Consider $\frac{d}{dt} (\psi(t)e^{-Lt}) = e^{-Lt}\dot{\psi}(t) - L\psi(t) \leq 0$, thus $\psi(t)e^{-Lt}$ is decreased, and as a result we have $\phi(t)e^{-Lt} \leq \psi(t)e^{-Lt} \leq \psi(0) = c$

□

Theorem 1.2 (Picard Lindelof). If function $f : D \rightarrow R^n$ is locally Lipschitz then for $\forall x_0 \in D$ $\exists ! x : (-\epsilon, \epsilon) \rightarrow D$, $\epsilon > 0$ satisfying (1).

Proof. * existence from Piano theorem

Proof of uniqueness

Consider two solutions $x_1(\cdot)$ and $x_2(\cdot)$ to (1). $\dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2)$, $x_1(0) = x_2(0)$. Then we can integrate equality: $x_1(t) - x_2(t) = \int_0^t f(x_1(\tau)) - f(x_2(\tau)) d\tau$. $|x_1(t) - x_2(t)| \leq \int_0^t |f(x_1(\tau)) - f(x_2(\tau))| d\tau \leq L \int_0^t |x_1(\tau) - x_2(\tau)| d\tau$. Now we can apply Cromwall's lemma with $c = 0$ and $\phi(t) = |x_1(t) - x_2(t)|$, then $\phi(t) \leq 0$, then $x_1(t) = x_2(t)$, $\forall t \in (-\epsilon, \epsilon)$ □

Example.

$$\dot{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ 0, & \text{else } x < 0 \end{cases}$$

$$\text{Solutions } x(t) = \begin{cases} \frac{1}{4}(t - c)^2, & \text{if } t \geq c \geq 0 \\ 0, & \text{else} \end{cases}$$

Global existance & uniqueness

- sufficient condition: f globally Lipschitz
- another sufficient condition: solution entirely lies in a coplex set
- forward completeness has equivalent Lyapunov-like characterization: system is forward-complete iff \exists solution $V : R^n \rightarrow R \geq 0$ s.t. $\frac{\partial V}{\partial x} f(x) \leq -V(x)$, $\forall x \in R^n$

1.3 Lyapunov stability

If functions $\dot{V}(x) < 0$, $\forall x \in D \setminus \{0\}$, then x^* is asymptotically stable.

Definition. Equilibrium point $x = 0$ is stable if $\forall \epsilon > 0 \exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| \leq \epsilon, \forall t \geq 0$.

Definition. Equilibrium point $x = 0$ is asymptotically stable if it is stable and there exists $\delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\lim_{t \rightarrow \infty} x(t) \rightarrow 0$.

Theorem 1.3 (Lyapunov's direct method). Let $x^* = 0 \in D$ be an equilibrium point of (1), i.e., $f(0) = 0$. Let $f : D \rightarrow R^n$ be continuous. If there exists a differentiable $V : D \rightarrow R$ s.t.

1. $V(x^*) = 0, V(x) > 0, \forall x \in D \setminus \{0\}$
2. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0, \forall x \in D$

then $x^* = 0$ is stable.

Proof. Fix compact $U = \{x : V(x) \leq c\}$ s.t. $U \in D$. By Poincaré: exist $\alpha > 0$ s.t. any solution x with $x_0 \in U$ exists at least on the interval $[0, \alpha)$.

TODO proof is not full □

Lyapunov's direct method gives us:

- stability
- convergence (if $V < 0$)
- subset of the region of attraction (all compact $U = \{x : V(x) \leq c\} \in D$)
- existence of solution for all times

2 Nonlinear systems

In this section we consider function $f : R \times D \rightarrow R^n$, where $D \subseteq R^n$, and D is open.

$$\dot{x}(t) = f(t, x(t)), \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0 \quad (3)$$

The origin $x^* \in D$ is an equilibrium point for (3), if $f(t, 0) = 0, \forall t \geq 0$.

Remark: EP (equilibrium point) $x^* = 0$ can be translation of a nonzero solution.

Suppose \bar{y} is a solution of $\dot{y} = g(t, y)$.

Change of coordinates: $x(t) = y(t) - \bar{y}(t)$, then $\dot{x}(t) = \dot{y}(t) - \dot{\bar{y}}(t) = g(t, x(t) + \bar{y}(t)) - \dot{\bar{y}}(t) := f(t, x(t))$. Since $\dot{\bar{y}}(t) = g(t, \bar{y}(t))$, then $f(t, 0) = 0, \forall t \geq 0$.

Existence and uniqueness of solution to (3):

- if f continuous, then exist local solution
- if f continuous and locally Lipschitz in x^* , then exist local unique solution

Now we need new stability definitions.

Definition. Point $x^* = 0$ is stable if $\forall \epsilon > 0$ and $\forall t_0 \geq 0$, $\exists \delta > 0$ s.t. from $\|x_0\| < \delta$ follows $\|x(t)\| < \epsilon$, $\forall t \geq t_0$.

Definition. Point $x^* = 0$ is uniformly stable if $\forall \epsilon > 0$ $\exists \delta > 0$, s.t. $\forall t_0 \geq 0$, from $\|x_0\| < \delta$ follows $\|x(t)\| < \epsilon$, $\forall t \geq t_0$.

Definition. Point $x^* = 0$ asymptotically stable if it is stable and $\forall t_0 \geq 0$ $\exists c > 0$, s.t. from $\|x_0\| < c$ follows $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$.

Definition. Point $x^* = 0$ uniformly asymptotically stable if it is uniformly stable and $\exists c > 0$, s.t. $\forall t_0 \geq 0$ from $\|x_0\| < c$ follows $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow 0$.

Definition. Convergence: $\forall \eta > 0$ $\forall t_0 \geq 0$, $\exists T > 0$ such that $\forall t \geq t_0 + T$ follows $\|x(t)\| < \eta$.

Definition. Uniform convergence: $\forall \eta > 0$ $\exists T > 0$ such that $\forall t_0 \geq 0$ and $\forall t \geq t_0 + T$ follows $\|x(t)\| < \eta$.

Example. Consider next equation:

$$\dot{x}(t) = -\frac{x(t)}{1+t}, \quad t_0 \geq 0$$

Solution $x(t) = x(t_0) \frac{1+t_0}{1+t}$. It is uniformly stable, because we can choose $\delta = \epsilon$. But does $x(t)$ converge uniformly? Answer is no.

Definition. Point $x^* = 0$ is globally uniformly asymptotically stable if it is uniformly stable with $\delta \rightarrow \infty$ for $\epsilon \rightarrow \infty$ and $\forall c, \eta$ $\exists T > 0$ such that $\forall t_0 \geq 0$ from $\|x_0\| < c$ follows $\|x(t)\| < \eta$, $\forall t \geq t_0 + T$.

2.1 Lyapunov's direct method

Consider some function $V : [0, \infty) \times D \rightarrow R$, $(t, x) \rightarrow V(t, x)$ such that $\dot{V}(t, x) = \frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x)$.

Theorem 2.1 (Lyapunov's direct method). Let $f : [0, \infty) \times D \rightarrow R^n$ is continuous and let $x^* = 0$ be equilibrium point. If there is a differentiable function $V : [0, \infty) \times D \rightarrow R$ with:

- $W_1(x) \leq V(t, x) \leq W_2(x)$, $\forall t \geq 0, x \in D$
- $\dot{V}(t, x) \leq 0$, $\forall t \geq 0, x \in D$

where $W_1, W_2 : D \rightarrow R$ continuous and positive definite, then $x^* = 0$ is uniformly stable.

If further $\dot{V}(t, x) \leq -W_3(x)$, $\forall t \geq 0$, $x \in D$ with $W_3 : D \rightarrow R$ continuous and positive definite, the $x^* = 0$ is uniformly asymptotically stable.

If $D = R^n$ and W_1 is radially unbounded then $X^* = 0$ is globally uniformly asymptotically stable.

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Check function $V(t, x) = \frac{1}{2}x^2$ as candidat for Lyapunov's function. Then $W_1(x) = W_2(x) = \frac{1}{2}x^2$ and $\dot{V}(t, x) = -(1+t)x^2 \leq -x^2(t) =: W_3(x)$. Then from theorem we have, that $X^* = 0$ is globally uniformly asymptotically stable.

2.2 Exponential stability

Definition. Point $X^* = 0$ is an exponentially stable EP of (3) if $\exists \lambda, c, k > 0$ s.t. $t \geq t_0 \geq 0$ and all $\|x_0\| < c$ follows $\|x(t)\| \leq K\|x(t_0)\|e^{\lambda(t-t_0)}$.

Remark: from exponential stability follows uniformly asymptotical stability.

Lemma 2 (Auxilarity result). Let $\dot{x}(t) = f(t, x(t))$, f scalar and $\dot{\xi}(t) \leq f(t, \xi(t))$ with $\xi(t_0) \leq x(t_0)$. Then $\xi(t) \leq x(t) \quad \forall t \geq t_0$.

Theorem 2.2. Let $f : [0, \infty) \times D \rightarrow R^n$ be continuous and $x^* = 0 \in D$ be an EP.

If there is a differentiable function $V : [0, \infty) \times D \rightarrow R$ and constants $k_1, k_2, k_3, a > 0$ s.t.

1. $k_1\|x\|^a \leq V(t, x) \leq k_2\|x\|^a, \forall t \geq 0, x \in D$
2. $\dot{V}(t, x) \leq -k_3\|x\|^a$

then $x^* = 0$ is exponentially stable.

If $D = R^n$, then X^* is globally exponential stable.

Proof. For $c > 0$ small enough, trajectories initialized in $\{x : k_2\|x\|^a < c\}$ remain bounded and in D . From 1) and 2) we can conclude $\dot{V} \leq -\frac{k_3}{k_2}V$. Then from previous Lemma $V(t, x(t)) \leq$

$$V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}. \text{ Then } \|x(t)\| \leq [from 1)] \leq \left(\frac{V(t, x(t))}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{V(t_0, x(t_0))e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}} \leq \left(\frac{k_2\|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right)^{\frac{1}{a}}$$

$$\left(\frac{k_2}{k_1}\right)^{\frac{1}{a}} \|x(t_0)\| e^{-\frac{k_3}{k_2 a}(t-t_0)} \quad \square$$

Example. Consider the equation $\dot{x}(t) = -(1+t)x(t)$.

Here $V(t, x) = \frac{1}{2}x^2$ then X^* is exponentially stable.

2.3 Comparsion function

Definition. A function $\alpha : [0, \delta) \rightarrow [0, \infty)$ is (of) "class K " if it is continous, strictly increasing, and $\alpha(0) = 0$.

Definition. A function $\alpha : [0, \delta) \rightarrow [0, \infty)$ is "class K_∞ if $\alpha \in K$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$.

Example. Function $\alpha(r) = \tan^{-1}(r)$ – class K

Function $\alpha(r) = r^k$ – class K_∞

Definition. A function $\beta : [0, \delta) \times [0, \delta) \rightarrow [0, \infty)$ is "class KL if it is continuous , $\beta(\cdot, s) \in K$ for all fixed s , and for each fixed r , $\beta(r, \cdot)$ is strictly decreasing: $\lim_{s \rightarrow \infty} \beta(r, s) = 0$

Example. Function $\beta(x, s) = \max(r, r^2)e^s$ belong class KL .

Properties of comparsion functions:

- If $\alpha \in K$ on $[0, \delta)$, then α^{-1} is defined on $[0, \alpha(\delta))$ and $\alpha^{-1} \in K$.
- If $\alpha \in K_\infty$, then $\alpha^{-1} \in K_\infty$
- If $\alpha_1, \alpha_2 \in K$, then $\alpha_1 \circ \alpha_2 \in K$ (same for K_∞
- If $\alpha_1, \alpha_2 \in K$, $\beta \in KL$ then $\alpha_1(\beta(\alpha_2(r), s)) \in KL$

Now we conseider comparsion functions and stability definitions.

Lemma 3. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly stable iff $\exists \alpha \in K$ and $c > 0$ s.t. $\forall t \geq t_0, \forall \|x(t_0)\| < c$ and $\|x(t)\| \leq \alpha(\|x(t_0)\|)$.

(only sufficiency). Given $\epsilon > 0$ choose $\delta < \min(c, \alpha^{-1}(\epsilon))$. Then from $\|x(t_0)\| < \delta$ follows $\|x(t)\| \leq \alpha(\|x(t_0)\|) < \alpha(\alpha^{-1}(\epsilon)) = \epsilon$. \square

Lemma 4. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0, \forall \|x(t_0)\| < c$ and $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

(only sufficiency). Let $\|x(t_0)\| < c$. Then $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x(t_0)\|, 0)$. This gives us uniform stability. \square

Lemma 5. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is uniformly asymptotically stable iff $\exists \beta \in KL$ and $c > 0$ s.t. $\forall t \geq t_0, \forall \|x(t_0)\| < c$ and $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$.

(only sufficiency). Let $\|x(t_0)\| < c$. Then $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(c, t - t_0)$. This mean uniform convergence. $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) < \beta(\|x(t_0)\|, 0)$. This gives us uniform stability. \square

Lemma 6. The equilibrium $x^* = 0$ of $\dot{x}(t) = f(t, x(t))$ is globally uniformly asymptotically stable iff previous lemma holds for all $x_0 \in R^n$.

Now consider comparison functions and Lyapunov functions

If $W : R^n \rightarrow R$ is continuous and positive definite, then $\forall r > 0 \quad \exists \alpha_1, \alpha_2 \in K$ s.t. $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$ for all $x \in B_r(0) = \{x \mid \|x\| \leq r\}$.

If W is radially unbounded, then $\exists \alpha_1, \alpha_2 \in K_\infty$ s.t. $\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$ for all $x \in R^n$.

Lemma 7 (Auxiliary). Consider $\dot{y} = \alpha(y)$, $y(t_0) = y_0 > 0$, $\alpha \in K$. Then $\exists \beta \in KL$ s.t. $y(t) = \beta(y_0, t - t_0)$.

Sketch of proof of Lyapunov's direct method:

$$\begin{cases} W_1(x) \leq V(t, x) \leq W_2(x) \\ \dot{V} \leq -W_3(x) \end{cases}$$

Where W_1, W_2, W_3 – continuous and positive defined.

Then $\exists \alpha_1, \alpha_2, \alpha_3 \in K$ such that $\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$ and $\dot{V}(t, x) \leq -\alpha_3(\|x\|)$.

Proof uniform stability:

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq [\alpha_1 \text{ in } K] \leq \alpha_1^{-1}(V(t_0, x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)).$$

Proof uniform convergence

$\dot{V} \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V))$. We know, that $\alpha_3 \circ \alpha_2^{-1} \in K$. By comparison lemma, $V(t, x(t)) \leq W(t)$, where W solves $\dot{W} = -\alpha_3(\alpha_2^{-1}(W))$ with $W(t_0) = V(t_0, x(t_0))$. By auxiliary lemma $\exists \beta \in KL$ s.t. $V(t, x(t)) \leq \beta(V(t_0, x(t_0)), t - t_0)$, then $\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(\beta(V(t_0, x(t_0)), t - t_0)) \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(t_0)\|), t - t_0)) =: \bar{\beta}(\|x(t_0)\|, t - t_0)$. From this follows uniform asymptotic stability since $\bar{\beta} \in KL$.

2.4 Converse theorems

Theorem 2.3. Let $X^* = 0$ be an EP of $\dot{x}(t) = f(t, x(t))$ with $f : [0, \infty) \times R^n \rightarrow R^n$ continuously differentiable and $\frac{\partial f}{\partial x}$ bounded in R^n , uniformly in t ($\|\frac{\partial f}{\partial x}(t, x)\| \leq L$ for all $x \in R^n$, $t \geq 0$, $L > 0$).

If $x^* = 0$ is globally exponentially stable, then exists differentiable $V : [0, \infty) \times R^n \rightarrow R$ and $c_1, c_2, c_3, c_4 > 0$ s.t. $c_1\|x\|^2 \leq V(t, x) \leq c_2\|x\|^2$, $\dot{V}(t, x) \leq -c_3\|x\|^2$ and $\|\frac{\partial V}{\partial x}\| \leq c_4\|x\|$.

Proof. Let $\Phi(\tau; t, x)$ – solution to $\dot{x}(t) = f(t, x(t))$ which static at (t, x) .

$$V(t, x) = \int_t^{t+\delta} \Phi^T(\tau; t, x) \Phi(\tau; t, x) d\tau, \quad \delta > 0. \quad \text{Upper bound: } V(t, x) = \int_t^{t+\delta} \|\Phi(\tau; t, x)\|_2^2 d\tau \leq [\text{exponential stability}] \leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau \|x\|_2^2 = \frac{k^2}{2\lambda} (1 - e^{-2\lambda\delta}) \|x\|_2^2.$$

Lower bound: since $\|\frac{\partial V}{\partial x}\| \leq L$, then $\|f(t, x)\|_2 \leq L\|x\|_2$. Thus by comparison lemma $\|\Phi(\tau; t, x)\|_2^2 \geq \|x\|_2^2 e^{-2L(\tau-t)}$. Set it in $V(t, x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau \|x\|_2^2 = \frac{1}{2L} (1 - e^{-2L\delta}) \|x\|_2^2$.

Decrease conditions: $\dot{V}(t, x) = \dots \leq -(1 - k^2 e^{-2\lambda\delta}) \|x\|_2^2$. □

3 System with inputs

Consider equation:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (4)$$

where $f : R^n \rightarrow R^n$.

Assumption: f in localy Lipschitz.

Exogeneous signa $u : R \rightarrow R^n$.

Input can be "bad" (disturbance) or "good" (control).

3.1 Input-to-state stability

Motivation: LTI system $\dot{x} = Ax + Bu$, $x(0) = x_0$.

Solution: $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$. If A is Hurwitz, then $\|e^{At}\| \leq ce^{-\lambda t}$ for some $c, \lambda > 0$.

How large can x grow for some bounded u ? $\|x(t)\| \leq \|e^{At}\| \|x_0\| + \int_0^t \|e^{A(t-\tau)}\| \|B\| \|u(\tau)\| d\tau \leq e^{-\lambda t} c \|x_0\| + \int_0^t e^{-\lambda(t-\tau)} c \|B\| \|u(\tau)\| d\tau = ce^{-\lambda t} \|x_0\| + (1 - e^{-\lambda t}) \frac{c}{\lambda} \|B\| \sup_{\tau \in [0, t]} \|u(\tau)\|$.

- $ce^{-\lambda t} \|x_0\|$ class KL in $(\|x_0\|, t)$
- $(1 - e^{-\lambda t})$ less than 1
- $\frac{c}{\lambda} \|B\| \sup \|u(\tau)\|$ class K

If $\sup_{\tau \in [0, t]} \|u(\tau)\|$ is bounded than \dot{x} remains bounded. Even more: the smaller $\sup_{\tau \in [0, t]} \|u(\tau)\|$, the smaller $\|x(t)\|$.

Definition. System (4) is input-to-state stable (ISS) if $\exists \beta \in KL, \gamma \in K$ s.t. $\forall x_0 \in R^n, \forall t \geq 0$ follows $\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\sup_{\tau \in [0, t]} \|u(\tau)\|)$.

Remarks:

- From ISS follows O-GAS (global asymptotical stability of $x = 0$ for $\dot{x} = f(x, 0)$)
- γ can be interpreted as "gain" w.r.t. u

- if $\lim_{t \rightarrow \infty} u(t) = 0$ then $\lim_{t \rightarrow \infty} x(t) = 0$

Example. Consider equation $\dot{x} = -x + xu$. System is O-GASS, not ISS (for example $u \equiv \alpha \Rightarrow \dot{x} = x(\alpha - 1)$ all solution diverge).

Example. Consider equation $\dot{x} = -3x + (1 + 2x^2)u$. System is O-GASS, not ISS (for example $u \equiv 1, x_0 = 2, x(t) = \frac{3-e^t}{3-2e^t}$ has a finite escape time).

Theorem 3.1. Suppose that there exists a continuously differentiable function $V : R^n \rightarrow R$ and $\alpha_1, \alpha_2 \in K_\infty$ and $\alpha_3, \rho \in K$ such that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in R^n$ and $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_3(\|x\|), \forall x : \|x\| \geq \rho(\|u\|)$. Then (4) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof. Idea: same as Lyapunovs direct method when x is "outside" of ball $\{x : \|x\| \leq \rho(\|u\|)\}$

TODO Picture □

Example. Consider equality $\dot{x} = -x^3 + u$. Let $V(x) = \frac{1}{2}x^2$, then $\dot{V} = -x^4 + xu = [0 < \Theta < 1] = -(1 - \Theta)x^4 - \Theta x^4 + xu \leq -(1 - \Theta)x^4$ for all $x : \|x\| \geq \left(\frac{\|u\|}{\Theta}\right)^{\frac{1}{3}}$. Thus, system is ISS with $\gamma(v) = \rho(v) = \left(\frac{v}{\Theta}\right)^{\frac{1}{3}}$.

Remarks:

- Existence of V is both necessary and sufficient for ISS;
- (??) is equivalent to $\frac{\partial V}{\partial x} f(x, u) \leq -\alpha_4(\|x\|) + \alpha_5(\|u\|), \forall x, u$ for some $\alpha_4, \alpha_5 \in K$;
- If $x_1 = 0$ is a globally asymptotically stable EP of Σ_1 and Σ_2 is ISS w.r.t. "input" x_1 , then $(x_1, x_2) = (0, 0)$ is a globally asymptotically stable EP for the cascaded system.

Theorem 3.2. Assume that:

- f is globally Lipschitz;
- $x = 0$ is a globally exponentially stable EP for $\dot{x} = f(x, 0)$

Then the system (4) is ISS.

Proof. Sketch: \exists continuous differentiable V :

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x, 0) \leq -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

Then:

$$\begin{aligned} \frac{\partial V}{\partial x} f(x, u) &= \frac{\partial V}{\partial x} f(x, 0) + \frac{\partial V}{\partial x} (f(x, u) - f(x, 0)) \leq -c_3\|x\|^2 + c_4\|x\|L\|u\| = -c_3(1 - \theta)\|x\|^2 - \theta\|x\|^2 + \\ c_4L\|x\|\|u\| &\leq -c_3(1 - \theta)\|x\|^2 \\ \text{if } \|x\| &\geq \frac{c_4L}{\theta c_3}\|u\|. \end{aligned} \quad \square$$

3.2 Control Lyapunov functions

Motivation: Lyapunov theory for control systems.

(input affine systems)

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i = f(x) + G(x)u,$$

$$f : R^n \rightarrow R^n, g : R^n \rightarrow R^n, G : R^n \rightarrow R^{n \times m}$$

$u : t \rightarrow u(t), R \rightarrow R^m$ is a control signal (decision variable).

Definition. A function $V : R^n \rightarrow R$ is a control Lyapunov function (CLF) if it's differentiable positive definite, radially unbounded and

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) < 0 \quad (5)$$

Remark:

Concept can be generalized to systems $\dot{x} = f(x, u)$. Then 5 becomes

$$\forall x \neq 0 \quad \inf_u (\nabla V(x) \cdot f(x, u)) < 0$$

Theorem 3.3 (Artstein). There exists $k : R^n \rightarrow R^m$ (state feedback) which is continuous on $R^n \setminus \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff there exists a CLF.

How to find CLFs?

Proposition:

Condition (5) is equivalent to

$$\forall x \neq 0, \quad \frac{\partial V}{\partial x} G(x) = 0 \implies L_f V(x) < 0 \quad (6)$$

Remark:

$$\frac{\partial V}{\partial x} G(x) = (\nabla V(x)g_1(x), \dots, \nabla V(x)g_m(x)) =: L_G V(x)$$

$$(6) \iff \forall x \neq 0, \quad L_f V(x) \geq 0 \implies L_G V(x) \neq 0$$

Proof. \Leftarrow :

Assume (6) holds. Then:

$$\inf_u (\nabla V(x) \cdot (f(x) + G(x)u)) = \inf_u L_f V(x) + L_G V(x)u < 0$$

Why?

- If $L_G V(x) = 0$, then by (6) $L_f V(x) < 0$;
- If $L_G V(x) \neq 0$, then (at least) for one i we have $\nabla V(x) \cdot g_i(x) \neq 0 \implies$ set $u_i = -c \nabla V(x) \cdot g_i(x)$.

\implies :

If (5) holds for some x with $L_G V(x) = 0$, then we must have $L_f V(x) < 0$. □

Example (discontinuous control):

$$\dot{x} = \begin{cases} 1 - u, & u \geq 1 \\ -1 - u, & u \leq -1 \\ 0, & \text{else} \end{cases}$$

If you want to move the system you need to apply control $|u| \geq 1$.

Using

$$V(x) = \begin{cases} x + 1, & x > 0 \\ x - 1, & x \leq 0 \end{cases}$$

results in closed loop $\dot{x} = -x$ - asymptotically stable.

$V(x) = x^2$ is a CLF.

Theorem 3.4. There exists a continuous $k : R^n \rightarrow R^m$, smooth on $R^n \setminus \{0\}$ s.t. $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$ iff:

- there exists a (smooth)CLF V ;
- $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall x : 0 < \|x\| < \delta$
 $\exists u \in R^m : \|u\| < \varepsilon$ s.t. $L_f V(x) + L_G V(x)u < 0$

How to construct a globally stabilizing state feedback k from knowledge of a CLF?

"Sontag's formula"

Fix $c \geq 0, a(x) := L_f V(x), b(x) := (L_G V(x))^T$

$$k(x) = \begin{cases} -cb(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)}, & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases}$$

Proposition: Let $V : R^n \rightarrow R$ be a CLF and k as above. Then $x^* = 0$ is globally asymptotically stable EP for $\dot{x} = f(x) + G(x)k(x)$

Proof. $\dot{V} = L_f V(x) + L_G V(x)k(x) = a(x) - cb(x)^T b(x) - \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} b(x)^T b(x) = -cb(x)^T b(x) - \sqrt{a(x)^2 + (b(x)^T b(x))^2} < 0 \quad \forall x \neq 0 \text{ s.t. } L_G V(x) \neq 0$

$\dot{V} = L_f V(x) + L_G V(x) \cdot 0 < 0 \quad \forall x \neq 0 \text{ s.t. } L_G V(x) = 0$ (since V is CLF)

$\implies V$ - Lyapunov function $\implies \dots$ □

Remarks:

- Sontag's formula is smooth on $R^n \setminus \{0\}$;
- Sontag's formula is continuous at $x = 0$ iff small control property holds.

$$\forall x \neq 0 : \inf_u \frac{\partial V}{\partial x} f(x, u) < 0 \quad \dot{x} = f(x) + G(x)u$$

So this leads to

$$\forall x \neq 0 L_G V(x) = 0 \Rightarrow L_f V(x) \neq 0$$

Remark: The last formula is "optimal" if minimize:

$$\int_0^\infty \frac{1}{2} p(x) b(x)^T b(x) + \frac{1}{2p(x)} u^T u dt$$

$$b(x) := (L_G V(x))^T$$

where $c > 0$

$$p(x) = \begin{cases} c + \frac{a(x) + \sqrt{a(x)^2 + (b(x)^T b(x))^2}}{b(x)^T b(x)} & b(x) \neq 0 \\ c & b(x) = 0 \end{cases}$$

It still works if $u = \lambda h(x)$ with $\lambda \in [\frac{1}{2}; \infty)$ is applied (large "gain margin")

4 Backstepping

Integrator backstepping

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= u \end{aligned} \tag{7}$$

where $x_1 \in \mathbb{R}^m$, $x_2, u \in \mathbb{R}$ (single input)

image to be inserted

Assumption: we know (smooth) "feedback" $\alpha_1 : \mathbb{R}^n \rightarrow \mathbb{R}$, and positive definite, differentiable $v_1 : \mathbb{R}^m \rightarrow \mathbb{R}$

s.t. $L_{f_1+g_1\alpha_1} V_1(x)$ is negative definite \Rightarrow origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1)$ is asymptotically stable

Goal: Compute feedback $u = k(x)$ which stabilises (7). Backstepping constructs $u = \alpha_2(x_1, x_2)$ s.t. $(e_1, e_2) = (x_1 - 0, x_2 - \alpha_1(x_1)) = 0$ error coordinates

Rewrite (7) :

$$\dot{x}_1 = f_1(x_1) + g_1\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1))$$

$$\dot{x}_2 = u$$

image to be inserted

In error coordinates

$$\begin{aligned}\dot{e}_1 &= f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 &= u - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial e_1} \dot{e}_1 = u - \frac{\partial \alpha_1}{\partial e_1}\end{aligned}\tag{8}$$

"backstepping" α_1 through the integrator

Define $V_2(e_2) := \frac{1}{2}e_2^2$, and

$$\begin{aligned}V(e_1, e_2) &= V_1(e_1) + V_2(e_2) \\ \dot{V}(e_1, e_2) &= \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)\alpha_1(e_1)) + \frac{\partial V_1}{\partial e_1}g_1(e_1)e_2 + \frac{\partial V_2}{\partial e_2}(u - \dot{\alpha}_1)\end{aligned}$$

as far as $L_{f_1+g_1\alpha_1}V_1$ -negative definite and $\frac{\partial V_2}{\partial e_2} \rightarrow e_2$

Choose

$$u = \left(-\frac{\partial V_1}{\partial e_1}g_1(e_1) + \dot{\alpha}_1\right)(\text{"canaling terms"}) - k_2e_2(\text{"stabilizing term"})k_2 > 0\tag{9}$$

$$\Rightarrow \text{Then } \dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0, \forall (e_1, e_2) \neq 0$$

$$\Rightarrow \text{Then } \dot{V}(e_1, e_2) = L_{f_1+g_1\alpha_1}V_1(e_1) - k_2e_2^2 < 0 \forall (e_1, e_2) \neq 0$$

$$\Rightarrow (e_1, e_2) = (0, 0) \text{ is an asymptotically stable EP for (8) with } u \text{ as in (9)}$$

Remark: $(e_1, e_2) \rightarrow (0, 0)$ doesnot necessarily imply that $(x_1, x_2) \rightarrow 0$ for $u = \alpha_2(x_1, x_2) = -\frac{\partial V_1}{\partial x_1}g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1))$

where $u \leftarrow$ (9) the original coordinates and $\dot{\alpha}_1 \leftarrow \frac{\partial \alpha_1}{\partial x_1}(f_1(x_1) + g_1(x_1)x_2)$

But $(x_1, x_2) = (0, 0)$ is asymptotically stable if $\alpha_1(0) = 0$ why? $(e_1, e_2) \rightarrow 0 \Rightarrow x_1 \rightarrow 0 \ x_2 \rightarrow \alpha_1(0) = 0$

Example.

$$\dot{x}_1 = x_1x_2$$

$$\dot{x}_2 = u$$

$$\text{Choose } \alpha_1(x_1) = -k \ (k > 0) \rightarrow \dot{x}_1 = -kx_1 \Rightarrow V_1(x_1) = \frac{1}{2}x_1^2$$

Then:

$$e_1 = x_1 - 0 \ \dot{e}_1 = e_1(e_2 - k)$$

$$e_2 = x_2 + k \quad e_2 = u$$

Backstepping yields: $u = -e_1^2 - k_2 e_2$ $k_2 > 0 \Rightarrow (e_1, e_2) = (0, 0)$ is asymptotically stabilized

$(x_1, x_2) = (0, -k)$ is asymptotically stabilized

Can we choose different α_1 s.t. $(x_1, x_2) = (0, 0)$ is stabilized?

Yes, e.g.

$$\alpha_1(x_1) = -x_1^2 \Rightarrow \dot{x}_1 = -x_1^3 \quad V_1(x_1) = \frac{1}{2}x_1^2$$

So we have equations

$$\begin{aligned} e_1 &= x_1 - 0 \quad \dot{e}_1 = e_1(e_2 - e_1^2) \\ e_2 &= x_2 + x_1^2 \quad \dot{e}_2 = u + 2e_1^2(e_2 - e_1^2) \end{aligned}$$

Backstepping results in

$$u = -e_1^2 - 2e_1^2(e_2 - e_1^2) - k_2 e_2, \quad k_2 > 0 \Rightarrow (e_1, e_2) \rightarrow (0, 0), \quad (x_1, x_2) \rightarrow (0, 0)$$

Generalization-1

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \end{aligned}$$

Assumption: $g_2(x_1, x_2) \neq 0 \forall x_1, x_2 \Rightarrow$ Input transformation: $u = \frac{1}{g_2(x_1, x_2)}(V - f_2(x_1, x_2)) \Rightarrow \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \quad \dot{x}_2 = V \Rightarrow$ can apply integrator backstepping to determine V results in

$$u = \alpha_2(x_1, x_2) = \frac{1}{g_2(x_1, x_2)} \left(-\frac{\partial V_1}{\partial x_1} g_1(x_1) + \dot{\alpha}_1 - k_2(x_2 - \alpha_1(x_1)) - f_2(x_1, x_2) \right)$$

Generalization 2: (Backstepping through 2 integrators)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1} \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3, \quad x_2, x_3 \in R \\ \dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u, \quad u \in R \end{aligned}$$

Assumption: g_2, g_3 nowhere zero.

Shown before: $\exists \alpha_2$: for $x_3 = \alpha_2(x_1, x_2)$ $(e_1, e_2) \rightarrow 0$

Thus $e_3 := x_3 - \alpha_2(x_1, x_2)$

Input transformation:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(V - f_3(x_1, x_2, x_3))$$

$\Rightarrow \dot{x}_1 = \dots, \dot{x}_2 = \dots, \dot{x}_3 = V \Rightarrow$ can apply backstepping once more.

In "error" coordinates:

$$\dot{e}_1 = f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1))$$

$$\dot{e}_2 = f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1$$

$$\dot{e}_3 = V - \dot{\alpha}_2$$

Define $V_3(e_3) = \frac{1}{2}e_3^2$, $V(e_1, e_2, e_3) = \sum_{i=1}^3 V_i(e_i)$

$$\dot{V}(e_1, e_2, e_3) = \frac{\partial V_1}{\partial e_1}(f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) + e_2(f_2(e_1, e_2 + \alpha_1(e_1)) + g_2(e_1, e_2 + \alpha_1(e_1))(e_3 + \alpha_2(e_1, e_2 + \alpha_1(e_1))) - \dot{\alpha}_1) + e_3(V - \dot{\alpha}_2))$$

All the underlined terms were designed (previously) to be $= L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 < 0$

So: $\dot{V}(e_1, e_2, e_3) = L_{f_1+g_1\alpha_1}V_1(e_1) - k^2e_2^2 + e_2g_2(e_1, e_2 + \alpha_1(e_1))e_3 + e_3(V - \dot{\alpha}_2)$

Structurally it is exactly the same as it was in backstepping through 1.

Choose:

$$V = \dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1)) - k_3e_3$$

$\dot{\alpha}_2 - e_2g_2(e_1, e_2 + \alpha_1(e_1))$ - "cancelling terms".

k_3e_3 - "stabilizing term".

In original coordinates:

$$u = \frac{1}{g_3(x_1, x_2, x_3)}(\dot{\alpha}_2 - (x_2 - \alpha_1(x_1))g_2(x_1, x_2) - k_3(x_3 - \alpha_2(x_1, x_2)) - f_3(x_1, x_2, x_3))$$

We need α_1, α_2 to compute u .

General backstepping recursion:

Systems in "strict feedback form":

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2, \quad x_1 \in R^{n_1}$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

\dots

$$\dot{x}_k = f_k(x_1, \dots, x_k) + g_k(x_1, \dots, x_k)u, \quad x_2, \dots, x_k, u \in R$$

g_2, \dots, g_k nowhere zero, f_i, g_i (sufficiently) smooth, as it is needed in α_i .

Backstepping recursion:

1. "Input data": a CLF V_1 for $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$ with a (smooth) feedback $u_1 = \alpha_1 x_1$ which as. stabilizes the origin of $\dot{x}_1 = f_1(x_1) + g_1(x_1)u_1$.
2. for $i = 2, \dots, k$:
 construct a CLF $V_i(e_i) = \frac{1}{2}e_i^2$, $V = \sum_{j=1}^i V_j(e_j)$ and a feedback α_i which as. stabilizes origin of
 $(e_1, \dots, e_i) = (x_1, x_2 - \alpha_1(x_1), \dots, x_i - \alpha_{i-1}(x_1, \dots, x_{i-1}))$

$$\alpha_i(x_1, \dots, x_i) = \frac{1}{g_i}(\dot{\alpha}_{i-1} - \frac{\partial V_{i-1}}{\partial e_{i-1}}g_{i-1} - k_i(x_i - \alpha_{i-1} - f_i))$$
3. apply $u = \alpha_k(x_1, \dots, x_k)$

Backstepping and CLFs:

Backstepping is sensitive to uncertainties in f_i, g_i (due to cancelling terms)

\implies Sontag's formula is more practical \implies we can use it since V is CLF.

Error system is input affine (using input transformation)

$$\dot{e} = f(e) + g(e)V$$

$$\text{with } f(e) = \begin{pmatrix} f_1(e_1) + g_1(e_1)(e_2 + \alpha_1(e_1)) \\ \dots \\ -\alpha_{k-1} \end{pmatrix}, g(e) = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}$$

Claim:

$$V(e) = \sum_{i=1}^k V_i(e_i) \text{ is a CLF.}$$

Proof. For input affine systems we need to show $L_g V = 0 \implies L_f V < 0, \forall e \neq 0$.

$$\dot{V}(e) = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 + e_{k-1} g_{k-1}(\dots) e_k - e_k \dot{\alpha}_{k-1} + e_k u.$$

Here $e_k u = L_g V$ and the rest is $L_f V$.

Assume $L_g V = 0 \iff e_k = 0$

$$\implies L_f V = L_{f_1+g_1\alpha_1} V_1(e_1) - \sum_{i=2}^{k-1} k_i e_i^2 < 0 \quad \forall e \neq 0 \text{ with } e_k = 0. \quad \square$$

\implies We can apply Sontag's formula to construct V .

This theory can be extended to systems with $x_2, \dots, x_k, u \in \mathbb{R}^m$ ("block backstepping").

5 Exercises

5.1 Exercise 1

Problem 1:

Proof. For any $t \geq 0$, we have

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}(V \circ x)(t) = \langle \nabla V(x(t)), \frac{d}{dt}x(t) \rangle = \langle \nabla V(x(t)), f(x(t)) \rangle = L_f V(x(t))$$

□

Problem 2:

Proof.

Lemma 8. Given the assumptions in Problem 2, if there exists a solution $x : [0, +\infty] \rightarrow R^n, t \rightarrow x(t)$, of $\dot{x} = f(x)$ s.t. $x(t) \in K$ for any $t \geq 0$, where $K \subset R^n$ is a compact with $O \in K$ (O - origin), then $x(t) \xrightarrow{t \rightarrow +\infty} 0$.

Clearly, for any $c > 0$, $lev_{\leq c}V$ is positive invariant w.r.t $\dot{x} = f(x)$. Given $c > 0$, let $x_0 \in lev_{\leq c}V$, i.e., $V(x_0) \leq c$. Then, for any $t \geq 0$

$$V(x(t)) = V(x_0) + \int_0^t \frac{d}{dt}V(x(\tau))d\tau < V(x_0) \leq c,$$

i.e. $x(t) \in lev_{\leq c}V$ for any $t \geq 0$.

Then, for any $x_0 \in lev_{\leq c}V$ there exists a solution $x : [0, +\infty] \rightarrow R^n$ of $\dot{x} = f(x)$ s.t. $x(t) \in lev_{\leq c}V$ for all $t \geq 0$. Clearly, $O \in lev_{\leq c}V$. We conclude by using the above Lemma ($K = lev_{\leq c}V$). □

Problem 3:

Proof. Let $r > 0$. By assumption, there exists $c > 0$ s.t. $\overline{B(0, r)} \subset lev_{\leq c}V$.

Since any bounded set $lev_{\leq c}V$ is a subset of the region of attraction, and since the sublevel sets are arbitrary large, R^n is also the region of attraction.

A condition that ensures that for any $c > 0$, $lev_{\leq c}V$ is bounded is $V(x) \xrightarrow{\|x\| \rightarrow +\infty} +\infty$. □

Problem 4:

Proof. Let $P : R^2 \rightarrow R^2$ be continuously differentiable. Consider

$$m\dot{v} = -g\nabla P(q).$$

Consider $x = (q, v)$, $\dot{q} = v$, $\dot{v} = -\frac{g}{m}\nabla P(q)$. Let $H : R^2 \rightarrow R$ be defined by

$$H(q, r) = \frac{1}{2}||v||^2 + \frac{g}{m}P(q).$$

We have

$$\begin{pmatrix} \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla H(q, r)$$

Since P is positive definite, then H is positive definite.

Then

$$L \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla H(q, r) = \langle \nabla H(q, r), \begin{pmatrix} & I \\ -I & \end{pmatrix} \nabla H(q, r) \rangle = 0 \quad \forall (q, r) \in R^2 \times R^2$$

\implies the origin is stable. □

Problem 5:

Proof. For any $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}V(t, x(t)) &= \frac{d}{dt}(V \circ (id_R, x))(t) = [id_R : R \rightarrow R, t \mapsto t] = \left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \frac{d}{dt}(id_R(t), x(t)) \right\rangle = \\ &\left\langle \begin{pmatrix} \frac{\partial}{\partial t}V(t, x(t)) \\ \frac{\partial}{\partial x}V(t, x(t)) \end{pmatrix}, \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \right\rangle = \frac{\partial}{\partial t}V(t, x(t)) + \left\langle \frac{\partial}{\partial t}V(t, x(t)), f(t, x(t)) \right\rangle = L \begin{pmatrix} 1 \\ f \end{pmatrix} V(x(t)). \end{aligned}$$

$$g(t, x(t)) := \begin{pmatrix} 1 \\ f(t, x(t)) \end{pmatrix} \quad \square$$

Problem 6:

Proof. Consider $\dot{x} = a \sin(\omega t)$, $x(0) = x_0 \in R$ $a, \omega > 0$.

This is solved by $x(t) = -\frac{a}{\omega} \cos(\omega t) + \frac{a}{\omega} + x_0$.

Clearly, x is bounded on $[0, +\infty]$ since $x(t) \geq x_0$, and $x(t) \leq x_0 + 2\frac{a}{\omega}$ for any $t \geq 0$.

Choose $\varepsilon = \frac{a}{\omega}$ and $t_0 = 0$. Then $\forall \delta > 0 \quad \exists x_0 \in B(0, \delta)$, namely x_0 , s.t. $\exists t \geq t_0$, namely $t = \frac{\pi}{\omega}$, with $x(t) \notin B(0, \varepsilon)$ ($x(\frac{\pi}{\omega}) = 2\frac{a}{\omega} > \varepsilon$). □

Short notes:

Problem 7:

Take $V(t, x) = \frac{1}{2}x^2$.

Problem 8:

Take $V(t, x) = x_1^2 + (1 + e^{-2t})x_2^2$.

5.2 Exercise 2

Problem 1:

Proof. a) Since α_1 is continuous and strictly increasing:

$$\forall x, y \in [0, \delta), x < y \implies \alpha_1(x) < \alpha_1(y)$$

$\implies \alpha_1$ is injective, i.e.

$$\forall x, y \in [0, \delta), x \neq y \implies \alpha_1(x) \neq \alpha_1(y).$$

Clearly, $\alpha_1 : [0, \delta) \rightarrow \alpha_1([0, \delta))$ is surjective, i.e.

$$\forall y \in \alpha_1([0, \delta)) \exists x \in [0, \delta) : \alpha_1(x) = y$$

Thus α_1 is bijective.

Define $\alpha_1^{-1} : [0, \alpha_1(\delta)) \rightarrow [0, \delta)$ by $\alpha_1^{-1}(\alpha_1(x)) = x$.

b) From a) we have $\alpha_3^{-1} \in K$. Since $\alpha_3 \in K_\infty$, $\alpha_3 - 1$ is defined on $[0, +\infty)$ and $\alpha_3^{-1}(r) \xrightarrow{r \rightarrow \infty} \infty$

c) Let $\alpha = \alpha_1 \circ \alpha_2$. Then we have $\alpha(0) = \alpha_1(\alpha_2(0)) = 0$ and $\alpha(r) > 0$ whenever $r > 0$. Moreover, for any x, y :

$$x < y \implies \alpha_2(x) < \alpha_2(y) \implies \alpha(x) = \alpha_1(\alpha_2(x)) < \alpha_1(\alpha_2(y)) = \alpha(y)$$

It is continuous (as composition of continuous functions).

d) From c) we have $\alpha := \alpha_3 \circ \alpha_4 \in K$, α is defined on $[0, +\infty)$ since $\alpha_3, \alpha_4 \in K_\infty$ and

$$r \rightarrow +\infty \implies \alpha_4(r) \rightarrow +\infty \implies \alpha(r) \rightarrow +\infty$$

e) For each $s, r \mapsto \beta(\alpha_2(r), s)$ is of class K .

Thus $r \mapsto \alpha_1(\beta(\alpha_2(r), s)) \in K$.

For each $r, s \mapsto \beta(\alpha_2(r), s)$ decreases.

Hence, $s \mapsto \alpha_1(\beta(\alpha_2(r), s))$ decreases.

Moreover,

$$\alpha_1(\beta(\alpha_2(r), s)) \xrightarrow{s \rightarrow +\infty} 0$$

□

Problem 3:

Proof. For $u = 0$ the origin is UGAS. Consider $V : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, x) \mapsto \frac{1}{2}x^2$.

We have

$$\frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) f(t, x, u) = (\sin(t) - 2)x^2 + xu \leq -x^2 + |x||u| = -(1 - \theta)x^2 - \theta x^2 + |x||u|, \quad \theta \in (0, 1)$$

Hence, whenever $|x| \geq \frac{|u|}{\theta}$, the system is ISS with $\gamma = \frac{r}{\theta}$.

□

Problem 4:

Proof.

$$\dot{x} = -x + (x^2 + 1)d \quad (10)$$

$$\dot{x} = -2x - x^3 + (x^2 + 1)d \quad (11)$$

System (10): Clearly, the system is 0-GAS. However, for $d = 1$ and $x > 1$ we have $x^2 + 1 > x$.

$$f(x, 1) = -x + (x^2 + 1) > 0$$

and thus $\dot{x} > 0$. Hence, if $x(0) = x_0 > 1$, the solution diverges (in finite time).
 \implies System (10) isn't ISS.

System (11): It is 0-GAS. Moreover, for any finite d there exists a "large" x s.t.

$$2x + x^3 > (x^2 + 1)d$$

$$\implies f(x, d) = -2x - x^3 + (x^2 + 1)d < 0$$

and $\dot{x} < 0 \implies$ System 11 is ISS.

Consider $V : R \rightarrow R, x \mapsto \frac{1}{2}x^2$ s.t

$$V'(x)f(x, d) = -2x^2 - x^4 + x(x^2 + 1)d \leq -x^2 - x^2(x^2 + 1) + (x^2 + 1)|x||d|$$

Hence, whenever $|x| \geq |d|$,

$$V'(x)f(x, d) \leq -x^2$$

s.t. system (11) is ISS with $\gamma(r) = r$. □

Problem 5:

Proof.

$$\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\|\nabla V(x)\|^2 + |\langle \nabla V(x), \delta u \rangle| \leq [YI] \leq -\|\nabla V(x)\|^2 + \frac{1}{2}\|\nabla V(x)\|^2 + \frac{\delta^2}{2}\|u\|^2$$

Young's inequality:

$$\forall x, y : |\langle x, y \rangle| \leq \varepsilon \frac{\|x\|^p}{p} + \frac{\|y\|^q}{\varepsilon q}, \quad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \varepsilon > 0$$

Hence, whenever $\|x\| > \frac{\delta}{\sqrt{c}}\|u\|$, $t \mapsto \|x(t)\|$ is decreasing.

Moreover whenever $\|x\| \geq \frac{\delta}{\sqrt{c\theta}}\|u\|$, $\theta \in (0, 1)$, we have $\langle \nabla V(x), -\nabla V(x) + \delta u \rangle \leq -\frac{c}{2}(1-\theta)\|x\|^2 \implies$
ISS. □

5.3 Exercise 4

Consider

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = f_2(x_1) + g_2(x_1, x_2)u \end{cases} \quad (12)$$

Using the "preliminary control"

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 = \check{u} \end{cases} \quad (13)$$

$$u = \frac{1}{g_2(x_1, x_2)}(\check{u} - f_2(x_1, x_2))$$

Idea: Look at the upper(-most) system only and consider x_2 as a "virtual control".

Assumptions: Suppose

- \exists CLF V_1 ;
- \exists (smooth) feedback α_1 s.t. $L_{f_1+g_1\alpha_1}V_1 < 0$.

Now, add and subtract $g_1\alpha_1$ in 13 s.t.

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha_1(x_1) + g_1(x_1)(x_2 - \alpha_1(x_1)) \\ \dot{x}_2 = \check{u} \end{cases} \quad (14)$$

Next, introduce $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

$$\begin{cases} \dot{e}_1 = f_1(e_1) + g_1(e_1)\alpha_1(e_1) + g_1(e_1)e_2 \\ \dot{e}_2 = \check{u} - \dot{\alpha}_1(e_1) \end{cases} \quad (15)$$

Problem 1:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Proof. 1. Choose "virtual control":

$$x_2 = -(k+1)x_1 =: \alpha_1(x_1), \quad k > 0$$

The origin of $\dot{x}_1 = -kx_1$ is GAS.

(Take $V_1 : R \rightarrow R$, $x_1 \mapsto \frac{1}{2}x_1^2$ s.t. $\dot{V}_1(x_1) = -kx_1^2 < 0$ for all $x_1 \neq 0$)

2. Error coordinates:

Let $(e_1, e_2) := (x_1 - 0, x_2 - \alpha_1(x_1))$ s.t.

$$\dot{e}_1 = -ke_1 + e_2$$

$$\dot{e}_2 = u + (k+1)(-ke_1 + e_2)$$

3. "Composite CLF":

Define $V : R \times R \rightarrow R$, $(e_1, e_2) \mapsto V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(u + (k+1)(-ke_1 + e_2) + e_1)$$

4. Choose control:

Let $u = -e_1 - (k+1)(e_2 - ke_1) - ke_2$

s.t. $\dot{V}(e_1, e_2) = -ke_1^2 - ke_2^2 < 0$ for all $(e_1, e_2) \neq (0, 0)$

Remark: The closed-loop system reads:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} -k & 1 \\ -1 & -k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

□

Problem 2:

$$\dot{x}_1 = x_1(x_2 - k), \quad k > 0$$

$$\dot{x}_2 = u$$

Proof. 1. $x_2 = 0 =: \alpha_1(x_1)$

The origin of $\dot{x}_1 = -kx_1$ is GAS ($V_1(x_1) = \frac{1}{2}x_1^2$)

2. $(e_1, e_2) := (x_1, x_2)$ s.t.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = u$$

3. $V(e_1, e_2) = V_1(e_1) + \frac{1}{2}e_2^2$ s.t.

$$\dot{V}(e_1, e_2) = -ke_1^2 + e_2(e_1^2 + u)$$

4. $u = -e_1^2 - ke_2$

□

Problem 3:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = u$$

Proof. 1. From problem 2:

$\dot{x}_2 = x_2(x_3 - k) - x_1^2 = -x_1^2 - kx_2 = u$ in Problem 2.

The origin of

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

is GAS.

And this is true for $x_3 = 0 =: \alpha_2(x_1, x_2)$.

$$2. (e_1, e_2, e_3) := (x_1 - 0, x_2 - \alpha_1(x_1), x_3 - \alpha_2(x_1, x_2)) \text{ s.t.}$$

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = u$$

$$3. V(e_1, e_2, e_3) = V_1(e_1) + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 \text{ s.t.}$$

$$4. u = -e_2^2 - ke_3$$

□

Problem 4:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

$$\dot{x}_3 = x_3(x_4 - k) - x_2^2$$

$$\dot{x}_4 = u$$

Proof. 1. Is GAS for

$$x_3(x_4 - k) - x_2^2 = -x_2^2 - kx_3$$

which is attained for $x_4 = 0 =: \alpha_3(x_1, x_2, x_3)$.

2.

$$\dot{e}_1 = e_1(e_2 - k)$$

$$\dot{e}_2 = e_2(e_3 - k) - e_1^2$$

$$\dot{e}_3 = e_3(e_4 - k) - e_2^2$$

$$\dot{e}_4 = u$$

...

$$3. u = -e_3^2 - ke_4$$

□

Problem 5:

$$\dot{x}_1 = x_1(x_2 - k)$$

$$\dot{x}_2 = x_2(x_3 - k) - x_1^2$$

...

$$\dot{x}_i = x_i(x_{i+1} - k) - x_{i-1}^2$$

...

$$\dot{x}_n = u$$

Proof. We will always have $u = e_{n-1}^2 - ke_n$.

Let $V : R \times \dots \times R \rightarrow R$, $(e_1, \dots, e_n) \mapsto \sum_{i=1}^n V_i(e_i)$, where $V_i(e_i) = \frac{1}{2}e_i^2$, $i = 2, \dots, n$.

We have $\dot{V}(e_1, \dots, e_n) = L_{f_1+g_1\alpha_1} V_1(e_1) - k \sum_{i=2}^{n-1} e_i^2 + e_n u + e_{n-1} g_{n-1}(x_1, \dots, x_{n-1}) e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1})$.

We observe that for α_i being zero, the inequality

$$e_{n-1} g_{n-1}(x_1, \dots, x_{n-1}) e_n - e_n \dot{\alpha}_{n-1}(x_1, \dots, x_{n-1}) + e_n u < 0$$

hence $e_{n-1}^2 e_n + e_n u < 0$ for non-zero e .

It is solved by $u = e_{n-1}^2 - ke_n$, $k > 0$. □

5.4 Exercise 3

Motivation: Lyapunov Theory

$$\dot{x} = f(x, u)$$

$$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Definition. (CLF) A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF if it is continuous differentiable, positive definite, radially unbounded and $\forall x \neq 0 \inf_u < \nabla V(x), f(x, u) > < 0$

In order to find CLFs, we restrict our analysis to input -affine systems

$$\dot{x} = f(x) + G(x)u$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$

Proposition: A continuous, differentiable, positive definite and radially unbounded. $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a CLF iff

$$\forall x \neq 0 \ L_G V(x) = 0 \Rightarrow L_f V(x) < 0$$

Image to be inserted

Problem 1

Consider $\dot{x} = \cos(x) + (1 + e^x)u$ where $f(x) = \cos(x)$ - drift and $g(x) = 1 + e^x$

Let $V : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto \frac{1}{2}x^2$. Clearly, continuous differentiable, positive definite and radially unbounded. Moreover, for any nonzero x , we have $L_G V(x) \neq 0$.

Thus, for any $x \neq 0$, there exists a control that renders $< \nabla V(x), f(x) + g(x)u >$ negative. Given this CLF, there exists a state feedback $u = u(x)$, e.g.

$$u(x) = -\frac{kx + \cos(x)}{1 + e^x}, \ k > 0$$

Problem

Consider

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2 e^{x_1} \cos(x_2) \\ \dot{x}_2 &= x_1^5 \sin(x_2) + u\end{aligned}$$

Take $V : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2)$

For any $x \neq 0$, we have

$$\inf_{u \in \mathbb{R}} (L_f V(x) + L_G V(x)u) = \begin{cases} L_f V(x), & \text{if } L_G V(x) = 0 \\ -\infty & \text{else} \end{cases}$$

In particular,

$$\begin{aligned}L_f V(x) &= \dots = x_1(-x_1^3 + x_2 e^{x_1} \cos(x_2)) + x_2 x_1^5 \sin(x_2) \\ L_G V(x) &= \dots = x_2\end{aligned}$$

However,

$$L_f V(x)|_{x_2=0} = -x_1^4 < 0 \quad \forall x_1 \neq 0$$

Image to be inserted

Concluding that V is a CLF.

Problem 2:

$\dot{x} = Ax + Bu$, input defined system where (A, B) is stabilizable, there exists $K \in \mathbb{R}^{m \times n}$ s.t. $A + BK$ is Hurwitz (cf. KRT). The latter is equivalent to the existence $P = P^T > 0$ s.t. $P(A + BK) + (A + BK)^T P < 0$ (cf. Khalil theorem 4,6)

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \langle x, Px \rangle$. Moreover, $\forall x \neq 0 \exists u = Kx$ s.t. $\langle \nabla V(x), Ax + Bu \rangle < 0$, since

$$\langle \nabla V(x), Ax + Bu \rangle = \langle u = Kx, (P(A + BK) + (A + BK)^T P)x \rangle < 0$$

In addition,

$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{\|K\|} > 0 \quad \forall x \neq 0, \|x\| < \delta \exists u = Kx \quad \|u\| < \epsilon$$

s.t. $L_f V(x) + L_G V(x)u < 0$ since $\|u\| = \|Kx\| \leq \|K\|\|x\| < \|K\|\delta = \epsilon$

Problem 3

Let $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous, differentiable consider

$$m\dot{v} = -g\nabla P(q) + F, \quad m, g > 0$$

a) Hamiltonian form. Let $x := (q, v)$. Then $\dot{x} = (-\frac{g}{m}\nabla P(q) + \frac{1}{m}F) = \begin{bmatrix} -I & I \end{bmatrix} \begin{bmatrix} \frac{g}{m}\nabla P(q) \\ V \end{bmatrix} + \begin{bmatrix} \frac{1}{m}I \end{bmatrix} F = \begin{bmatrix} -I & I \end{bmatrix} \nabla H(x) + G(x)$ given $H(x) = \frac{1}{2}\|v\|^2 + \frac{g}{m}P(q)$

b) "CLF". Take H as a CLF candidate. Then, for any x

$$\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle = \langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle + \langle \nabla H(x), G(x)F \rangle = [\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) \rangle + \langle \nabla H(x), G(x)F \rangle]$$

Strictly speaking, H is no CLF, but it reveals how to choose F s.t. the origin is GAS.

For any point x for which there exists no control F s.t. $\langle \nabla H(x), \begin{bmatrix} I \\ -I \end{bmatrix} \nabla H(x) + G(x)F \rangle < 0$

Choose $F = 0$. Why? Using the Krasovskiy-Lasalle inv. principle, we conclude that the origin is GAS, since any solution in $\{x | \dot{H}(x) = 0\}$ verifies $v(t) \equiv 0$, implying $\dot{v}(t) \equiv 0$ s.t.

$$0 = -\frac{g}{m} \nabla P(q(t)) + \frac{1}{m} P(t)$$

The last part equals 0. Since $F = 0$ (by choice) and $\nabla P(q) = 0$ iff $q = 0$ we conclude that $\dot{H}(x) = 0$ can only be "maintained" at the origin.

Problem 4

Consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ux_2 + u^3 \end{aligned}$$

show that $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$ is CLF and let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$\ddot{x} + u\dot{x} - u^3 = 0$$

For any x and u , we have $\langle \nabla V(x), f(x, u) \rangle = \dots = x_1(2x_2 - ux_2 + u^3) + x_2(x_2 - ux_2 + u^3) = x_1h_1 + x_2h_2$

Image to be inserted

Hence if $u < 0$ and $-u$ "large", then we can render $\langle \nabla V(x), f(x, u) \rangle < 0$.