ОГЛАВЛЕНИЕ

ГЛАВА 1

MPC

Formulation of control problem

$$\begin{cases} \dot{x} = f(x, u) \\ \quad x(0) = x_0 \\ \quad y = h(x) \end{cases} \tag{1.1}$$

Objective of MPC:

Find stabilizing control strategy that:

- minimize objective function: $J = \int_t^\infty F(x(\tau), u(\tau)) d\tau$

- satisfies constraints: $u(\tau) \in U \ x(\tau) \in X$

  Closed-loop optimal control vs Open-loop optimal control
  Closed-loop:
  Feedback $u = k(x)$

- + Feedback present

- + suit for uncertainty disturbances

- - Finding closed solution hardly possible

  Open-loop optimal control:
  Input trajectory $u = u(t, x_0)$

- + Computation often feasible

- - No feedback

- - Don't know much about system

  MPC - repeated open-loop optima; control in feedback fashion.

ГЛАВА 2

ZERO-TERMINAL CONSTRAINT MPC

Mathematical formulation of NMPC problem:

System dynamics: $\dot{x} = f(x, u)$ $x(0) = x_0$ $x, u \in \mathbb{R}^n$

Constraints: $x(t) \in X$ $u \in U$ $\forall t \geq 0$

Assumptions:

- $f(0,0) \Rightarrow x_1 = 0$- equilibrium point for $u_1 = 0$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ - twice continuously differentiable

- U is a compact set (closed and bounded)

- X is a connected and closed set

- $(0,0) \in int(X \times U)$

MPC optimization problem:

At time t, given initial state $x(t)$

$$\min_{\bar{u}(\cdot,t)} J(x(t), \bar{u}(\cdot; t))$$

with $J(x(t), \bar{u}(\cdot; t)) = \int_t^{t+T} L(\bar{x}(\tau; t), \bar{u}(\tau; t)) d\tau$

s.t.

$$\dot{\bar{x}} = f(x, u), \bar{x}(t; t) = x(t)$$

$$\bar{u}(\tau; t) \in U, \bar{x}(\tau; t) \in X, \ \forall \tau \in [t, t + T]$$

$$\bar{x}(t + T; t) = 0$$

Optimal open-loop solution:

$$\bar{u}^*(\cdot; t) = argmin_{\bar{u}(\cdot; t)} J(x(t), \bar{u}(\cdot; t))$$

Notation:

- Quantities without bar: real system trajectories

- Quantities with bar: predicted trajectories

- L-stage cost

- $(\cdot ; t)$ - predicted at time t

- T - prediction horizon

- Optimal value function $J^*(x(t)) = J(x(t), \bar{u}^*(t))$

Real trajectories deviate from predicted one!
Assumptions:

- $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is continuous and

$$
\begin{cases}
\quad L(0,0) = 0 \\
\quad L(x,u) > 0 \\
\forall (x,u) \neq (0,0)
\end{cases}
\qquad (2.1)
$$

- $J^*(x)$ is continuous at $x = 0$

MPC algorithm:

1. At sampling time t, measure $x(t)$ and solve MPC optimization problem

2. Apply $u_{MPC}(\tau) = \bar{u}^*(\tau, t) \forall t \in [t, t+\delta)$ with sampling time δ

3. Set $t := t + \delta$ and go to step 1

Feasibilty: The MPC problem is feasible at time t if there exists at least one $\bar{u}(\cdot ; t)$ s.t. constraints satisfied.
Theorem:
Suppose that

- (i) assumptions are satisfied

- (ii) and that zero-terminal constraint MPC problem is feasible at $t = 0$

Then:

- MPC problem is recursively feasible

- resulting closed-loop system is asymptotically stable

Let $D \subset \mathbb{R}^n$ be the et of all points for whih (ii) holds. The D is a region of attraction for the closed loop.
Proof.

1. recursive feasibility: by induction

2. • feasible at $t = 0$ by assumption

   • assume: feasibility at t. Consider the candidate solution:

$$\bar{u}(\tau; t + \delta) = \begin{cases} \bar{u}^*(\tau; t) \ \tau \in [t + \delta, t + T] \\ 0 \ \tau \in [t + T, t + \delta + T] \end{cases}$$

3. asymptotic stability

   Idea: use $J^*(x(t))$ as "Lyapunov function"

   Consider:

$$J(x(t + \delta), \bar{u}(\cdot; t + \delta)) = \int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), \bar{u}(\tau; t + \delta))d\tau =$$

$$= \int_{t+\delta}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau + \int_{t+T}^{t+\delta+T} L(0, 0)d\tau (= 0) =$$

$$= J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

by optimality

$$J^*(x(t+\delta)) \leq J(x(t+\delta), \bar{u}(\cdot; t+\delta)) \leq J^*(x(t)) - \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

by induction

$$J^*(x(\infty))(\geq 0) \leq J^*(x(0))(finite) - \int_{0}^{\infty} L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$$

Barbalat's lemma:
ϕ uniformly continuous $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$$\lim_{t \to \infty} \int_{0}^{t} \phi(\tau)d\tau < \infty \Rightarrow \phi(t) \to 0, t \to \infty$$

From Barbalat's lemma $L \to 0$ when $t \to \infty \Rightarrow$ L pos.def. $\|x_{MPC}(t)\| \to 0$ when $t \to \infty \Rightarrow$ convergence

Lyapunov stability: using standard arguments (J^* is continuous at $x = 0$)

Lessons learned:

• feasibility $\Rightarrow$ stability

- value function is Lyapunov function

- have to prove recursive feasibility

- suboptimal solution is sufficient for stability

ГЛАВА 3

QUASI - INFINITE HORIZON MPC

Goal: Relax (restrictive) zero-terminal zero-terminal constraint
Idea: terminal region + local CLF(controller Lyapunov functiom)
MPC optimization problem: At time t

$$\min_{\bar{u}(\cdot;t)} J(x(t), \bar{u}(\cdot;t)) = \int_t^{t+T} L(\bar{x}(\tau;t), \bar{u}(\tau;t))d\tau + F(\bar{x}(t+T;t))$$

$F(\bar{x}(t+T;t))$ - terminal cost
s.t.

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}), \bar{x}(t;t) = x(t)$$
$$\bar{x}(t;t) \in X \ \bar{u}(t;t) \in U \ \forall \tau \in [t, t+T]$$
$$\bar{x}(t+T;t) \in X^f$$

X^f - terminal region
Optimal solution: $\bar{u}^*(\cdot, t)$, $J^*(x(t))$
Assumption 1: Terminal region + terminal controller
There exists an auxilary local controller $u = k^{loc}(x)$ s.t.

1. X^f is positively invariant $\dot{x} = f(x, k^{loc}(x))$

2. $k^{loc}(x) \in U \ \forall x \in X^f$

3. $\dot{F}(x) + L(x, k^{loc}(x)) \leq 0 \ \forall x \in X^f$

$\Rightarrow$ F is local control-Lyapunov function.
Theorem.
Suppose Assumption 1 holds and MPC problem is feasible at $t = 0$. Then:

• recursive feasibility

• closed-loop is asymptotically stable

Proof.

1. Recursive feasibility by induction

    • feasible at $t = 0$ by assumption

- assume feasibility at t

candidate

$$\bar{u}(\tau; t + \delta) = \begin{cases} \bar{u}^*(\tau; t) \ \tau \in [t, t + T] \\ k^{loc}(\bar{x}(\tau; t + \delta)) \ \tau \in [t + T, t + \delta + T] \end{cases}$$

$\Rightarrow$ this is a feasible solution at $t + \delta$

2. asymptotic stability

$$J^*(x(t + \delta)) - J^*(x(t)) \leq J(x(t + \delta), \bar{u}(\cdot; t + \delta)) - J^*(x(t)) =$$

$$\int_{t+\delta}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), \bar{u}(\tau; t + \delta))d\tau + F(\bar{x}(t + \delta + T; t + \delta)) -$$

$$- \int_{t}^{t+T} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau - F(\bar{x}^*(t + T; t)) =$$

$$= \int_{t+T}^{t+\delta+T} L(\bar{x}(\tau; t + \delta), k^{loc}(\bar{x}(\tau; t + \delta)))d\tau + F(\bar{x}(t + \delta + T; t + \delta)) -$$

$$- \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau - F(\bar{x}^*(t + T; t)) \leq$$

As far as from Assumption 1.3 we have the sum of three terms is ≤ 0

$$- \int_{t}^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau$$

$\Rightarrow J^*(x(\infty)) \leq J^*(x(0)) - \int_0^\infty L(x_{MPC}(\tau), u_{MPC}(\tau))d\tau$ From here: some steps as in zero-terminal constraint rose.

How can Assumption 1 be satisfied?
Assume:

- quadratic state cost $L(x, u) = x^T Q x + u^T R u, \ Q, R > 0$

- linearization at the origin is stabilizable $\dot{x} = Ax + Bu$ $A = \frac{\partial F}{\partial x}(0, 0)$ $B = \frac{\partial F}{\partial u}(0, 0)$

Approach:

- Linear auxiliary controller $k^{loc}(x) = Kx$

- Quadratic terminal cost function $F(x) = x^T P x, \ P > 0$

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- Terminal region $X_\alpha^f = \{x \in \mathbb{R}^n | x^T P x \leq \alpha\}$ for some $\alpha > 0$

- Determine P, K, α s.t. Assumption 1.1-1.3 hold:

For (Assumption 1.3):

$$\frac{d}{dt} x(t)^T P x(t) \leq -x(t)^t (Q + K^T R K) x(t) = -x(t)^T Q^* x(t)$$

$$[x^T Q x + u^T R u = [u = Kx] = x^T (Q + K^T R K) x)]$$

$$\frac{d}{dt} x(t)^T P x(t) = f(x, Kx)^T P x + x^T P f(x, Kx)$$

$[f(x, Kx) = (A + BK)x + \phi(x), A + BK = A_K, K$ is chosen s.t. $A_B K$ is Hurwitz$]$
Upper bound for $x^T P \phi(x)$: $L_\phi := sup\{\frac{|\phi(x)|}{|x|}, x \in X_\alpha^f, x \neq 0\}$

$$x^T P \phi(x) \leq |x^T P||\phi(x)| \leq \|P\| L_\phi |x|^2 \leq \frac{\|P\| L_\phi}{\lambda_{min}(P)} x^T P x \qquad (3.1)$$

We choose α small enough s.t.

$$L_\phi \leq \frac{k \lambda_{min}(P)}{\|P\|} \qquad (3.2)$$

for some $k > 0$. Plug this into (3.1): $x^T P \phi(x) \leq k x^T P x$. Insert this into $\frac{d}{dt} x^T P x \leq x^T (A_K P + P A_K) x + 2k x^T P x$

$$= x^T ((A_K + kI)^T P + P(A_K + kI)) x$$

ensure that it $\leq -x^T Q^* x$
$\Rightarrow$ Lyapunov equation which can be solved if any only if $A_K + kI$ is Hurwitz

$$\Leftrightarrow k < -max\ Re\{\lambda(A_K)\} \qquad (3.3)$$

$$\Rightarrow (A_K + kI)^T P + P(A_K + kI) = -Q^* \qquad (3.4)$$

Design procedure

1. Compute K s.t. $(A + BK)$ is Hurwitz

2. Choose $k > 0$ s.t. (3.3) and solve (3.4)

3. Find largest possible α_1 s.t. $Kx \in U, \forall x \in X_\alpha^f$

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4. Find the largest $\alpha \in (0, \alpha_1]$ s.t. (3.2) holds

Alternative to the (4) step
Solve optimization problem

$$\max_x x^T P \phi(x) - k x^T P x \; s.t. \; x^T P x \leq \alpha \qquad (3.5)$$

Iterate this by reducing α from α_1 until optimal value of (4) is nonpositive
Degrees of freedom in design

- calculation of K

- choice of k - tradeoff between "large"terminal region and "large"P

ГЛАВА 4

UNCONSTRAINED MPC

Goal:Guarantee stability + degree of suboptimality without stabilizing terminal constraint + cost

Setup:

- $\dot{x} = f(x, u), x(0) = x_0$

- input constraints $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m \forall t \geq 0$

Infinite-horizon cost function: $J_\infty(x_0, \bar{u}(\cdot; 0)) = \int_0^\infty L(\bar{x}(\tau; 0), \bar{u}(\tau; 0))d\tau \Rightarrow$ optimal value function $J_\infty^*(x_0)$

Assumption: $J_\infty^*(x_0) < \infty, \forall x_0 \Rightarrow$ system is asymptotically stabilizable

Finite-horizon cost function: $J_\infty(x(t), \bar{u}(\cdot; t)) = \int_0^\infty L(\bar{x}(\tau; t), \bar{u}(\tau; t))d\tau$

Infinite-horizon performance resulting from application of MPC controller: $J_\infty^{MPC}(x_0) = \int_0^\infty L(\bar{x}_{MPC}(\tau), \bar{u}_{MPC}(\tau))d\tau$

Definition. Suboptimality index α: $\alpha J_\infty^{MPC}(x_0) \leq J_\infty^*(x_0) \forall x_0$

- $\alpha \leq 1$ by optimality of J_∞^*

- $\alpha > 0$ implies closed-loop stability (Barb.lemma)

Proposition 1: Relaxed dynamic programming
Assume $\exists \alpha \in (0, 1] s.t. \forall x \in \mathbb{R}^n$
$J_T^*(x(t + \delta)) \leq J_T^*(x(t)) - \alpha \int_t^{t+\delta} L(\bar{x}^*(\tau; t), \bar{u}^*(\tau; t))d\tau (*)$
Then the estimate

$$\alpha J_\infty^*(x(t)) \leq \alpha J_\infty^{MPC}(x(t)) \leq J_T^*(x(t)) \leq J_\infty^*(x(t)) \qquad (4.1)$$

holds for all $x \in \mathbb{R}^n$
Proof

- 1 and 3 inequalities follow from optimality (by definition)

- 2 inequality follows from summing up (*) over all sampling instances

$$J_T^*(x(N\delta)) \leq J_T^*(x_0) - \alpha \int_0^{N\delta} L(x_{MPC}(t), u_{MPC}(t))dt \qquad (4.2)$$

$$N \to \infty : J_T^*(x_0) \geq \alpha J_\infty^{MPC}(x_0) \qquad (4.3)$$

Central idea (image to be inserted)
$$L^*(t;t) = L(\bar{x}^*(\tau;t), \bar{u}^*(\tau;t))$$

$$(c): J_T^*(x(t+\delta)) \le \frac{1}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau : (b) \tag{4.4}$$

$$(b): \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau \le \gamma \int_{t}^{t+\delta} L^*(\tau;t)d\tau : (a) \tag{4.5}$$

Theorem 1:

Assume $\exists c \in (0;1]$ and $\gamma > 0$ s.t. 4.4 - 4.5 holds. Then $(*)$ holds with $\alpha = 1 - \gamma\frac{1-\epsilon}{\epsilon}$

Proof.

$$J_T^*(x(t+\delta)) - J_T^*(x(t)) = J_T^*(x(t+\delta)) - \int_{t}^{t+T} L^*(\tau;t)d\tau \le^{(1)}$$

$$\le \frac{1-\epsilon}{\epsilon} \int_{t+\delta}^{t+T} L^*(\tau;t)d\tau - \int_{t}^{t+\delta} L^*(\tau;t)d\tau \le^{(2)}$$

$$\le (\gamma\frac{1-\epsilon}{\epsilon} - 1) \int_{t}^{t+\delta} L^*(\tau;t)d\tau$$

$-\alpha := \gamma\frac{1-\epsilon}{\epsilon} - 1$

Assumption 1: Asymptotic Controlability

For all x, $\exists$ some input trajectory $\hat{u}_x(\cdot)$ with $\hat{u}_x(t) \in \mathbb{U}, \forall t \ge 0$ s.t.

$$L(\hat{x}(t), \hat{u}(t)) \le \beta(t)min_u L(x,u), \forall t > 0$$

with $\beta: \mathbb{R} \to \mathbb{R}_{\ge 0}$ - continuous, positive, strictly decreasing with $\lim_{t\to 0}\beta(t) = 0$ $\Rightarrow \int_0^\infty \beta(\tau)d\tau < \infty$ $B(t) = \int_0^t \beta(\tau)d\tau$

Typical example: (image to be inserted)

How to compute ϵ and γ: Lemma 1: Let Assumption 1 hold. Then the inequality

$$J_T^*(x(t+\delta)) \le \int_{t+\delta}^{t+t'} L^*(\tau;t)d\tau + B(T+\delta-t')L^*(t+t';t) \tag{4.6}$$

holds for all $t' \in [\delta, T]$

(image to be inserted)

Proof.

Consider

$$\bar{u}(\tau; t + \delta) = \begin{cases} \bar{u}^*(\tau; t), \tau \in [t + \delta, t + t'] \\ \hat{u}_{x'}(\tau - t - t'), \tau \in [t + t', t + \delta + T] \end{cases}$$

$$J_T^*(x(t + \delta)) \leq J_T(x(t + \delta), \bar{u}(\cdot; t + \delta)) =$$

$$= \int_{t+\delta}^{t+t'} L^*(\delta; t) d\delta + \int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau - t - t'), \hat{(\tau - t - t')}) d\tau \leq$$

by Assumption 1

$$\int_{t+t'}^{t+\delta+T} L(\hat{x}(\tau - t - t'), \hat{(\tau - t - t')}) d\tau \leq L^*(t + t'; t) \int_0^{T+\delta-t'} \beta(\tau) d\tau$$

as far as $B(t) = \int_0^t \beta(\tau) d\tau$

$$\leq \int_{t+\delta}^{t+t'} L^*(\tau; t) d\tau + B(T + \delta - t') L^*(t + t'; t)$$

Calculation of ϵ from (4.6):

$$J_T^*(x(t + \delta)) \leq \min_{t' \in [\delta, T]} \left(\int_{t+\delta}^{t+t'} L^*(\tau; t) d\tau + B(T + \delta - t') L^*(t + t'; t) \right) \leq$$

$$\int_{t+\delta}^{t+T} L^*(\tau; t) d\tau + B(T) \min_{t' \in [\delta, T]} L^*(t + t'; t)$$

as far as $\min_{t' \in [\delta, T]} L^*(t + t'; t) \leq \frac{1}{T-\delta} \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau$ minimum is less or equal that the average

$$= (1 + \frac{B(T)}{T - \delta}) \int_{t+\delta}^{t+T} L^*(\tau; t) d\tau$$

$(1 + \frac{B(T)}{T-\delta}) = \frac{1}{\epsilon}$
Lemma 2:

$$\int_{t+t'}^{t+T} L^*(\tau; t) d\tau \leq B(T - t') L^*(t + t'; t) \forall t' \in [0; T]$$

Proof. Analogues to lemma 1.
Calculation of γ:

$$\int_{t+\delta}^{t+T} L^*(\tau; t) d\tau \leq \int_{t+\hat{t}}^{t+T} L^*(\tau; t) d\tau (\forall \hat{t} \in [0, \delta]) \leq$$

$$\leq \min_{\hat{t}\in[0,\delta]} \left(B(T-\hat{t})L^*(t+\hat{t};t)\right) \leq$$

$$\leq B(T) \min_{\hat{t}\in[0,\delta]} L^*(t+\hat{t};t) \leq \frac{B(T)}{\delta} \int_t^{t+\delta} L^*(\tau;t)d\tau$$

Denote $\gamma = \frac{B(T)}{\delta}$

$\alpha = 1 - \gamma\frac{1-\epsilon}{\epsilon} = 1 - \frac{B(T)}{\delta}\left(\frac{B(T)}{T-\delta}\right)$

Alternative computation of ϵ (less conservative):

We want to compute ϵ s.t.

$$\epsilon \leq \frac{\int_{t+\delta}^{t+T} L^*(\tau;t)d\tau}{J_T^*(x(t+\delta))}$$

Idea: Minimize

$$\epsilon = \min_{L_t,J_T^*} \frac{\int_\delta^T L_t(\tau;t)d\tau}{J_T^*(x(t+\delta))} \tag{4.7}$$

$J_T^* = 1$ - without loss of generality s.t. $0 \leq L_t \forall \tau \in [\delta, T]$

$$J_T^*(x(t+\delta)) \leq \int_\delta^{t'} L_t(\tau)d\tau + B(T+\delta-t')L_t(t')\forall t' \in [\delta,T]$$

Due to linearity in L_t, without loss of generality we can set $J_T^* = 1$.

$\Rightarrow$ infinite dimensional linear problem

Idea for solution: second constraint has to be active for all times

Differentiate (4.6) with relation to t'

$$0 = L_t(t') + \frac{dB(T+\delta-t')}{dt'}L_t(t') + B(\tau+\delta-t')\dot{L}_t(t')$$

as far as $\frac{dB(T+\delta-t')}{dt'} = \beta(T+\delta-t')$

$$\begin{cases} \dot{L}_t(t') = \frac{\beta(T+\delta-t')-1}{B(T+\delta-t')}L_t(t') \\ \text{initial condition } L_t(t) = \frac{1}{B(\tau)} \end{cases}$$

Solution:

$$\bar{L}_t(t') = \frac{1}{B(T+\delta-t')}e^{-\int_\delta^{t'} \frac{1}{B(T+\delta-\tau)}d\tau}$$

Have to show: $\bar{L}_t$ is a minimizer of (4.7)

$$\int_\delta^T \bar{L}_t(\tau)d\tau \leq \int_{delta}^T L_t(\tau)d\tau$$

for all feasible L_t

Proof.

Assume $\exists L_t$ s.t.

$$\int_\delta^T L_t(\tau)d\tau < \int_\delta^T \bar{L}_t(\tau)d\tau$$

Then $\exists \hat{t} \in [\delta, T]$ s.t. $\int_\delta^{\hat{t}} L_t(\tau)d\tau \leq \int_\delta^{\hat{t}} \bar{L}_t(\tau)d\tau$ and $\bar{L}_t(\hat{t}) > L_t(\hat{t})$

But then

$$1 = \int_\delta^{\hat{t}} \bar{L}_t(\tau)d\tau + B(T+\delta-\hat{t})\bar{L}_t(\hat{t}) > \int_\delta^{\hat{t}} L_t(\tau)d\tau + B(T+\delta-\hat{t})L_t(\hat{t}) \quad (4.8)$$

the sign equality from (4.7) with equality.

Contradiction:

L_t cannot be a feasible solution of (4.7) $\Rightarrow$

$$\epsilon = \int_\delta^T \bar{L}_t(\tau)d\tau = 1 - e^{-\int_0^{T-\delta} \frac{1}{B(T-\tau)}d\tau}$$

Similarly, better estimate for γ can be obtained

$\alpha = 1 - \gamma\frac{1-\epsilon}{\epsilon}$ For $T \to \infty$: both estimates for $\epsilon \to 1 \Rightarrow \alpha \to 1$ as $T \to \infty \Rightarrow$ closed-loop stability for T large enough

ГЛАВА 5

ROBUST MPC

Consider linear (discrete-time) sytem: $x(t+1) = Ax(t) + Bu(t) + w(t)$ in short $x^+ = Ax + Bu + w$

Constraints: $x(t) \in X, u(t) \in U, \forall t = 0, 1...$

Bound on w: W is a compact,convex set which contains 0. $w(t) \in W \forall t = 0, 1, ...$

Main idea: Use additional error feeedback s.t. real systems state contained in a "tube"around some nominal system state.

Repetition of QI-MPC in discrete time: Nominal system:

$$z^+ = Az + Bv$$

At time t, given $z(t)$, solve

$$\min_{v(\cdot|t)} \hat{J}(z(t), v(\cdot|t)) = \sum_{i=t}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

s.t.

$$z(i+1|t) = Az(i|t) + Bv(i|t), z(t|t) = z(t)$$

$$z(i|t) \in Z, v(i|t) \in V, t \le i \le t+N-1$$

$$z(t+N|t) \in Z^f \subseteq Z$$

$\Rightarrow$ optimizer $V^*(\cdot|t)$, optimal value function $\hat{J}^*(z(t))$

Assumption 1:

- Cost is quadratic $L(z, v) = z^T Q z + v^t R v, Q, R > 0$

- There exists a local auxiliary controller $k^{loc} = Kx$ s.t.

  1. Z^f is invariant with $Z^+ = (A+BK)z$, $A_k = A+BK$, i.e. $A_k Z^f \subseteq Z^f$
  2. $Kz \in V \forall z \in Z^f$
  3. $F(A_k z) - F(z) \le -L(z, Kz) \forall z \ in Z^f$

From Assumption 1 it follows (as in continuous time) that

$$\hat{J}^*(z(t+1)) - \hat{J}^*(z(t)) \le -L(z(t), v_{MPC}(t))$$

Since L is quadratic, there exists constants $c_2 > c_1 > 0$ s.t. $\forall z \in Z_N$ - feasible set

1. $c_1|z|^2 \leq \hat{J}^*(z)$

2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1|z|^2$

3. $\hat{J}^*(z) \leq c_2|z|^2$

Why is (3) true?
From Assumption 1.3 $\forall z \in Z^f$

$$\hat{J}^*(z) \leq \hat{J}(z, Kz(\cdot)) = \sum_{i=1}^{N-1} L(z(i), Kz(i)) + F(z(N)) \leq$$

N times apply Assumption 1.3

$$\leq F(z) = z^T P z \leq \lambda_{max}(P)|z|^2$$

Influence of disturbance: Definition.
Mainkowski set addition:

$$A, B \subseteq \mathbb{R}^n A \oplus B = \{a + b | a \in A, b \in B\}$$

Pontryagin set difference:

$$A, B \subseteq \mathbb{R}^n A \ominus B = \{a \in \mathbb{R}^n | a + b \in A, \forall b \in B\}$$

$$(A \ominus B) \oplus B \subseteq A$$

$$A \subseteq (A \oplus B) \ominus B$$

Definition. Robust positively invariant set (RPI set):
S is RPI set for $x^+ = Ax + w$ if $AS \oplus W \subseteq S$ (or equivalently $Ax + w \in S \forall x \in S, \forall w \in W$)
Example:
$x^+ = 0.5x + w$. $w \in [-5, 5]$. So RPI set: $S = [-20, 20]$, minimal RPI set: $S = [-10, 10]$
Minimal RPI set:

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

(Minkowski set addition), min. RPI set exists and is bounded if A is Schour table.
Why?
Current state at time t is x,

possible states at time $t + 1$: $Ax \oplus W$

$t + 2$: $A(Ax \oplus W) \oplus W = A^2 x \oplus AW + w$

....

$t + j$: $A^j x \oplus \sum_{k=0}^{j-1} A^k w$

$\Rightarrow$ by choosing j large enough we can reach any state in S_∞

$\Rightarrow$ any RPI set must satisfy $S_\infty \subseteq S$

Remains to show: S_∞ is an RPI set

$$AS_\infty \oplus W = A \sum_{i=0}^{\infty} A^i w \oplus W = \sum_{i=1}^{\infty} A^i w \oplus W = S_\infty$$

S_∞ in general is difficult to compute

$\Rightarrow$ can compute invariant outer approximations of S_∞ (with bounded complexity)

Example.

Calculate RPI

$$S_\infty = \sum_{i=0}^{\infty} A^i w$$

For the system given and bounded disturbances

$$x^+ = \frac{1}{2} x + w, \ w \in [-5, 5]$$

$$S_\infty = \sum_{i=0}^{\infty} (\frac{1}{2})^i [-5, 5] = [-10, 10]$$

Central idea in tube-based MPC

Use additional error feedback around some nominal input:

$$u_{MPC} = v_{MPC}(x) + K(x - z)$$

Proposition 1

Let $x^+ = Ax + Bu + w$ and $z^+ = Az + Bv$. If $x \in Z \oplus S$ and $u = v + K(x - z)$, then $X^+ \in Z^+ \oplus S$ (RPI set for $x^+ = (A + BK)x + w$)

image to be inserted

Proof:

Let $e(t) := x(t) - x(t) \rightarrow$

$$e^+ = x^+ - z^+ = Ax + B(v + K(x - z)) + w - Az - Bv = (A + BK)e + w$$

As S is RPI for $e^+ = A_k e + w$, we obtain $e \in S \Rightarrow e \ inS \forall w \in W$

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Hence $x \in Z \oplus S \Rightarrow x^+ \in Z^+ \oplus S \forall w \in W$

Robust MPC scheme

MPC problem for robust tube-based MPC: At time t, given $x(t)$, solve

$$\min_{z(t|t), v(\cdot|t)} J(x(t), v(\cdot|t)) = \sum_{i=1}^{t+N-1} L(z(i|t), v(i|t)) + F(z(t+N|t))$$

$$s.t. z(i+1|t) = Az(i|t) + Bv(i|t)$$

$$z(i|t) \in Z = X \ominus S$$

$$v(i|t) \; inV = U \ominus KS$$

$$t \le i \le t + N - 1$$

$$z(t+N|t) \in Z^+ \subseteq Z$$

Initial condition $x(t) \in z(t|t) \oplus S$

$\rightarrow$ optimizer: $z^*(t|t), v^*(\cdot|t) \rightarrow$ optimal value function $J^*(x(t))$

$\rightarrow$ applied input: $u(t) = v^*(t|t) + K(x(t) - z^*(t|t))$

Important: Tightened input/state constraints for the nominal predictions ensure fulfilment of original input/state constraints for real (disturbed) closed-loop system.

Properties of robust MPC scheme (in the following $z^*(x(t)) := z^*(t|t)$)

- a feasible set $X_N = Z_N \oplus S \subseteq X$

- $J^*(x) = \hat{J}^*(z^*(x))$ by definition of J^* and $\hat{J}^*$

- $J^*(x) = 0 \; \forall x \in S$

Why?

If $x \in S$, then $z(x) = 0$ and $v(\cdot|t) = 0$ is a feasible solution. Hence $J^*(x) \le \hat{J}(0,0) = 0$

$\Rightarrow J^*(x) = 0$ and $z^*(X) = 0$

"S serves an origin for the disturbed system"

Theorem: Suppose that Assumption 1 holds and the robust MPC problem is feasible at $t = 0$.

Then:

(i) robust MPC problem is recursively feasible

(ii) closed-loop system robustly exponentially converges to S

(iii) closed-loop system satisfies input/state constraints, i.e. $x(t) \in X$, $u(t) \in U$ $\forall t = 0, 1...$

Proof:

i) Consider candidate solution at time $t + 1$

$$\tilde{V}(i|t+1) = \begin{cases} v^*(i|t) & t+1 \leq i \leq t+N-1 \\ k^{loc}(z^*(t+N|t)) & i = t+N \end{cases}$$

$$\tilde{z}(t+1|t+1) = z^*(t+1|t)$$

it is feasible because $x(t+1) \in z^*(t+1|t) \oplus S$ by proposition 1
image to be inserted
iii) follows from Proposition 1 + definition of tightened constraints
ii) from (1-3) inequalities described below

1. $\hat{J}^*(z) \geq C_1 |z|^2$

2. $\hat{J}^*(z^+) - \hat{J}^*(z) \leq -c_1 |z|^2$

3. $\hat{J}^*(z) \leq c_2 |z|^2$

$J^*(x) = \hat{J}^*(z^*(x))$
we obtain the following $\forall x \in X_N$

4. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(1)} |z^*(x)|^2$

5. $J^*(x) = \hat{J}^*(z^*(x)) \leq^{(3)} c_2 |z^*(x)|^2$

So now we will show convergence to 0

$$J^*(x(t+1)) - J^*(x(t)) = \hat{J}^*(z^*(x(t+1))) - \hat{J}^*(z^*(x(t))) \leq$$

$$\leq \hat{J}^*(z^*(x(t+1|t))) - \hat{J}^*(z^*(x(t))) \leq^{(2)}$$

$$-c_1 |z^*(x(t))|^2 \leq -\frac{c_1}{c_2} J^*(x(t))$$

$\Rightarrow$

$$J^*(x(t+1)) \leq (1 - \frac{c_1}{c_2}) J^*(x(t))$$

where $\gamma := 1 - \frac{c_1}{c_2}$, $\gamma \in (0, 1)$

$$J^*(x(i)) = \gamma^i J^*(x(0)) \leq^{(5)} c_2 \gamma^i |z^*(x(0))|^2$$

$\Rightarrow^{(4)}$

$$|z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}} \sqrt{\gamma}^i |z^*(x(0))|$$

$\Rightarrow z^*(x(t))$ exponentially converges to 0.

   Recall: $x(i) \in z^*(x(i)) \oplus S \Rightarrow$

$$|x(i)|_S \leq |z^*(x(i))| \leq \sqrt{\frac{c_2}{c_1}} \sqrt{\gamma}^i |z^*(x(0))|$$

$|x(i)|_S$ - point-to-set distance
Extensions:

- Nonlinear systems: difficult to compute RPI sets

    – approaches based on input-to-state stability(ISS)

    – approaches which apply MPC two times:

        * first for nominal input
        * to determine local error feedback(Rawlings and Mayne chapter 3-6)

- Linear systems with parametric uncertainties

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

$$(A(t), B(t)) \in \rho : con(A_j, B_j), j = 1, ..., J \; \forall \geq 0$$

Note. co- convex
   Define: $\bar{A} := \frac{1}{J} \sum_{i=0}^{J} A_i$, $\bar{B} := \frac{1}{J} \sum_{i=0}^{J} B_i$

$$x(t+1) = \bar{A}x(t) + \bar{B}(t) + w(t)$$

$$w(t) \in W := (A - \bar{A})x + (B - \bar{B})u | (A, B) \in \rho, x \in X, u \in U$$

W is compact if X,U are compact
   $\rightarrow$ can apply tube MPC as before but: can slow down more!
   If ρ is "small enough closed-loop asymptotically to zero
   Intuition: x converges to the RPI set $S \rightarrow W$ gets smaller
   $\rightarrow x$ converges to RPI set
   Invariant approximations of the minimal RPI set S_∞ is difficult to compute

$$S_\infty := \sum_{i=0}^{\infty} A^i w$$

Define $S_k := \sum_{i=0}^{k-1} A^i w \quad k \geq 1$

In general, S_k for a finite k are not RPI sets (this is the case if only if A is nilpotent)

Theorem:

If $0 \in int(W)$ and A is Schur, then there exists an integer $k > 0$ and $\alpha \in [0, 1)$ s.t.

$$A^k W \subseteq \alpha W \tag{5.1}$$

If (5.1) holds, then

$$S(\alpha, k) := (1 - \alpha)^{-1} S_k$$

is an RPI set for the system $x^+ = Ax + w$

Proof:

i)(5.1) is a direct consequence of our assumptions ii) want to show that $AS(\alpha, k) \oplus W \subseteq S(\alpha, k)$

$$AS(\alpha, k) \oplus W = (1 - \alpha)^{(} - 1) \sum_{i=1}^{k} A^i W \oplus W =$$

$$= (1 - \alpha)^{-1} A^k W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1} \oplus W$$

As far as $A^k W \subseteq \alpha W$ by (5.1)

$$(1 - \alpha)^{-1} \alpha W \oplus W \oplus \sum_{i=1}^{k-1} A^i W (1 - \alpha)^{-1}$$

As $(1 - \alpha)^{-1} \alpha W \oplus W = [(1 - \alpha)^{-1} \alpha + 1] W = (1 - \alpha)^{-1} W$

Then we get

$$= (1 - \alpha)^{-1} \sum_{i=0}^{k-1} A^i W = S(\alpha, k)$$

Remark:

- for a given k s.t. (5.1) can be satisfied, we want to find the smallest possible α ("small scaling factor")

- for a given α, one wants to find the smallest possible k s.t. (5.1) holds ("low complexity"of RPI set)

    $\Rightarrow$ tradeoff between small α and small k needs to be found

- one can determine "how good"$S(\alpha, k)$ is compared to S_∞

$\Rightarrow$ can specify suboptimality degree of approximation a priori

Possible algorithm to determine RPI set

1. fix $\alpha \in (0, 1)$ and $k > 0$ (integer)
2. check whether (5.1) holds:
   - if yes: $S(\alpha, k)$ is a RPI set
   - if not: set $k := k + 1$ and go to (2)