(20) $m=4, \quad k=16, \quad x(0)=1, \quad x'(0)=0$

Sol:- To find $x(t)$ we must solve the IVP

$$\begin{cases} m\frac{d^2x}{dt^2}+kx=0 \\ x(0)=1, \; x'(0)=0 \end{cases} \quad \text{or} \quad 4x''(t)+16x(t)=0$$

Taking laplace transform on b sides we get:

$$\mathcal{L}\{x''(t)+16x(t)\}=\mathcal{L}\{0\}$$

$$4\mathcal{L}\{x''(t)\}+16\mathcal{L}\{x(t)\}=0$$

$$4[s^2\mathcal{L}\{x(t)\}-s^1x(0)-x'(0)]+16X(s)=0$$

$$4s^2X(s)-4s$$

Now inverse Laplace Transform

(iv) $\Rightarrow$ $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s+7} + \frac{1}{s-5}\right\}$

$$" = 3\,\mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

$$" = 3e^{-7t} + e^{5t}$$

$$" = y(t)$$

(viii) $\Rightarrow$ $\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-(-7)}\right\}$

$$- \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$x(t) = -e^{2t} + e^{-7t} - e^{5t} + e^{2t} = \boxed{e^{-7t} - e^{5t}}$$

$\therefore$ Sol. of ~~in~~ IVP

$$\begin{cases} x(t) = e^{7t} - e^{5t} \\ y(t) = 3e^{-7t} + e^{5t} \end{cases}$$

Exercises 8.6

Q Use the given initial cond. to determine the displacement of the object of mass m attached to a spring with constant k, Put $s=2$ in (vi) we get.

$$-9 = A(9) + 0$$

$$\therefore \boxed{A = -1}$$

Put $s = -7$ in (vi) we get

$$-9 = 0 + B(-7-2)$$

$$-9 = -9B$$

$$\boxed{B = 1}$$

$$\therefore \frac{-9}{(s-2)(s+7)} = \frac{-1}{s-2} + \frac{1}{s+7}$$

Also $$\frac{3}{(-s+5)(s-2)} = \frac{A}{-s+5} + \frac{B}{s-2}$$

$$3 = A(s-2) + B(-s+5) \text{ —— (vii)}$$

Put $s = 2$

$$3 = 0 + B(-2+5)$$

$$3 = 3B$$

$$B = 1$$

Put $s = 5$

$$3 = A(5-2) + 0$$

$$\therefore A = 1$$

$$\frac{3}{(-s+5)(s-2)} = \frac{1}{-s+5} + \frac{1}{s-2}$$

(v) $\Rightarrow$ $$X(s) = \frac{-1}{s-2} + \frac{1}{s+7} - \frac{1}{s-5} + \frac{1}{s-2} \text{ (viii)}$$

$$\cancel{Y(s)} = \frac{-4(s-2)}{(s+7)(-s+5)} = \frac{A}{s+7} + \frac{B}{-s+5}$$

$$\rightarrow \quad -4(s-2) = A(-s+5) + B(s+7) \quad \text{---(iii)}$$

Put $s = -7$ in (iii) we get

$$-4(-7-2) = A(+7+5) + 0$$

$$-4(-9) = 12A$$

$$\boxed{A = \frac{36}{12} = 3}$$

Put $s = 5$ in (iii) we get

$$-4(5-2) = 0 + B(5+7)$$

$$-4(3) = 12B$$

$$\boxed{B = -1}$$

$$\therefore Y(s) = \frac{3}{s+7} - \frac{1}{-s+5} \quad \text{---(iv)}$$

$$(i) \rightarrow (s-2)X(s) + \frac{9}{s+7} - \frac{3}{-s+5} = 0$$

$$(s-2)X(s) = -\frac{9}{s+7} + \frac{3}{-s+5}$$

$$X(s) = \frac{-9}{(s-2)(s+7)} + \frac{3}{(-s+5)(s-2)} \quad \text{---(v)}$$

Let $\frac{-9}{(s-2)(s+7)} = \frac{A}{s-2} + \frac{B}{s+7}$

$s' \mathcal{L}\{y(t)\} - y(0) + 9X(s) + 4Y(s) = 0$

Applying initial contion we get

$sY(s) - \cancel{0}4 + 9X(s) + 4Y(s) = 0$

$9X(s) + (s+4)Y(s) = \cancel{0}4$ —— (ii)

Now solving (i) and (ii) by method of elimination we get

$$9(s-2)\cancel{X(s)} + 27Y(s) = 0$$
$$-9(s-2)\cancel{X(s)} - (s-2)(s+4)Y(s) = \cancel{0} - 4(s-2)$$

$$[-(s-2)(s+4) + 27]Y(s) = \cancel{0} - 4(s-2)$$

$$[-s^2 - 4s + 2s + 8 + 27]Y(s) = \cancel{0} - 4(s-2)$$

$$(-s^2 - 2s + 35)Y(s) = -4(s-2)$$

$$Y(s) = \frac{-4(s-2)}{-s^2 - 2s + 35}$$

$$" = \frac{-4(s-2)}{-s^2 - 7s + 5s + 35}$$

$$" = \frac{-4(s-2)}{-s(s+7) + 5(s+7)}$$

$$" = \frac{-4(s-2)}{(s+7)(-s+5)}$$

Laplace Transform Methods for solving System.

Exercises 8.5:

Use Laplace transform to solve each initial value problem.

(1) $\begin{cases} x' - 2x + 3y = 0 & x(0) = 0 \\ y' + 9y + 4y = 0 & y(0) = 4 \end{cases}$

Sol: Taking Laplace transform of both sides of each eqution yields the system.

$$\mathcal{L}\{x' - 2x + 3y\} = \mathcal{L}\{0\}$$

$$\Rightarrow \mathcal{L}\{x'\} - 2\mathcal{L}\{x\} + 3\mathcal{L}\{y\} = 0$$

$$\cancel{s'\mathcal{L}\{x\} - 2X(s) + 3Y(s) = 0}$$

$$\cancel{s' X(s) - 2X(s) + 3Y(s) = 0}$$

$$s'\mathcal{L}\{x\} - x(0) - 2X(s) + 3Y(s) = 0$$

Applying Initial condition we get

$$s'X(s) - 0 - 2X(s) + 3Y(s) = 0$$

$$(s' - 2)X(s) + 3Y(s) = 0 \quad \text{--- (1)}$$

And

$$\mathcal{L}\{y' + 9x + 4y\} = \mathcal{L}\{0\}$$

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$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$4. \quad = e^{-3t}$$

According to convolulation theorem

$$(f*g)(t) = \int_0^t f(t-v)\, g(v)\, dv$$

$$= \int_0^t \frac{1}{2}(t-v)^2 \cdot e^{-3v}\, dv$$

$$=$$

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$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{(f*g)\}\}$$

$$= (f*g)(t)$$

$$= \int_0^t$$

Q. Find the inverse Laplace Transform of each f(tion).

(14) $\dfrac{1}{s^3(s+3)}$

Sol: Let $\dfrac{1}{s^3(s+3)} = \dfrac{1}{s^3} \cdot \dfrac{1}{s+3}$

$$= F(s) \cdot G(s)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{2!}\cdot\frac{2!}{s^3}\right\}$$

$$= \frac{1}{2!}\cdot t^2$$

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$$\text{"} = 3\left[e^{-8s}\cdot\frac{1}{s} - e^{-4s}\frac{1}{s}\right]$$

Theorem (8-9)

Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists, for $s > b > 0$. if a is +ve constant and $f(t)$ is Continous on $[0, \infty)$ then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-$$

Exercise 8.3

(23) $\frac{-3}{s \cdot e^{\pi s}}$

$F(s) = \frac{1}{s}$

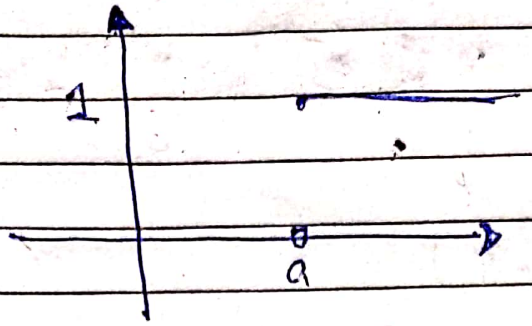
Sol: $-3\,\mathcal{U}(t-\pi)$

8.4 Convolution Theorem

$$\int f(t)\cdot g(t)\,dt \neq \int f(t)\,dt \cdot \int g(t)\,dt$$

$$\mathcal{L}\{f(t)\cdot g(t)\} \neq \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

Theorem (8.13) (Convolution Theorem)



Theorem (8.8). Suppose $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > b > 0$ if a is +ve constant, then

$$\mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}F(s)$$

Exercise 8.3

Find the Laplace Transform of given function.

① $-28\,\mathcal{U}(t-3)$, ③ $3\mathcal{U}(t-8) - \mathcal{U}(t-4)$

Sol: ③

Let $g(t) = 3\mathcal{U}(t-8) - \mathcal{U}(t-4)$

$\therefore G(s) = \mathcal{L}\{g(t)\}$

$= \mathcal{L}\{3\mathcal{U}(t-8) - \mathcal{U}(t-4)\}$

$= 3\mathcal{L}\{\mathcal{U}(t-8)\} - \mathcal{L}\{\mathcal{U}(t-4)\}$

$= 3\mathcal{L}\{1\cdot\mathcal{U}(t-8)\} - \mathcal{L}\{1\,\mathcal{U}(t-4)\}$

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$A = 3, \quad B = 4, \quad C = 2, \quad D = 0$

$$\therefore Y(s) = \frac{3s}{s^2+1} + \frac{4}{s^2+1} + \frac{2s}{(s^2+1)^2}$$

Taking inverse laplace transform.

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3s}{s^2+1} + \frac{4}{s^2+1} + \frac{2s}{(s^2+1)^2}\right\}$$

$$y(t) = 3\,\mathcal{L}^{-1}\left\{\frac{3s}{s^2+1}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} +$$

$$2\,\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\text{"} = 3\cos t + 4\sin t + 2 \cdot \frac{1}{2}\, t \sin t$$

$$\text{"} = 3\cos t + 4\sin t + t\sin t$$

Unit Step Function :-

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1 & t \ge a \end{cases}$$

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Sol: Taking Laplace Transform on both side of DE we get.

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\cos t\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{s}{s^2+1}$$

$$\left[s^2 \mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right] + Y(s) = \frac{s}{s^2+1}$$

$$s^2 Y(s) - s \cdot 3 - 4 + Y(s) = \frac{s}{s^2+1}$$

$$(s^2+1)\,Y(s) = 3s + 4 + \frac{s}{s^2+1}$$

$$= \frac{3s(s^2+1) + 4(s^2+1) + s}{s^2+1}$$

$$= \frac{3s^3 + 4s^2 + 5s + 4}{(s^2+1)^2}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$$

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Theorem 8.6:-

Laplace Transform of 1^{st} Derivate

$$\mathcal{L}\{f'(t)\} = s^1 \mathcal{L}\{f(t)\} - f(0)$$

Corollary (8.7) Laplace Transform of Higher Derivatives

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \dots f^{(n-1)}(0)$$

Proof:-

$=$ $\cancel{\mathcal{L}\{f'(t)\}} =$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underset{\mathrm{I}}{e^{-st}} \underset{\mathrm{II}}{f'(t)}\, dt = \lim_{M \to \infty} \int_0^M e^{-st} f'(t)\, dt$$

Exercise 8.2:-

Solve the IVP:-

(4) $y'' - y' - 2y = 0, \quad y(0) = 8, \quad y'(0) = 7$

(6) $y'' + y = \cos t, \quad y(0) = 3, \quad y'(0) = 4$

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INVERSE LAPLACE TRANSFORM

The inverse laplace Transform of the ftion $F(s)$ is the ftion $f(t)$ if such a ftion exists, that satisfies.

$$\mathcal{L}\{f(t)\} = F(s)$$

We write the inverse Laplace transform of $F(s)$ with

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Theorem 8.3:- If $\mathcal{L}\{f(t)\} = F(s)$ exist, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{st} \cdot f(t)$

Theorem 8.4:

$$\mathcal{L}^{-1}\{a F(s) + b G(s)\} = a \mathcal{L}^{-1}\{F(s)\} + b \mathcal{L}^{-1}\{G(s)\}$$

$$= a f(t) + b g(t)$$

Exe 8.1

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Exe 8.1

use the properties of laplace transform and table 8.1 Compute the laplace transform of each ftion.

(30)

$$f(t) = 1 + \cos 2t$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 + \cos 2t\}$$

$$" = \mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}$$

$$" = \frac{1}{s} + \frac{s}{s^2+4}$$

(28) $f(t) = 28e^t$

(19) $f(t) = e^{-2t} \cdot \cos 4t$

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$$\parallel = -\frac{1}{S}\left(\frac{1}{\infty} - \frac{1}{e^{s}}\right) = -\frac{1}{S}\left(0 - \frac{1}{e^{s}}\right)$$

$$\parallel = \frac{1}{s\,e^{s}}$$

③ $f(t) = 2e^{t}$

⑧ $f(t) = \begin{cases} 1 - t, & 0 < t \leq 1 \\ 0, & t > 1 \end{cases}$

Table 8.1

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
$t^{n}, n = 1, 2, \ldots$	$\frac{n!}{n+1}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^{n} e^{at}, n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\{\cos kt\}$	$= \frac{s}{s^2 + k^2}$

Q2 : Use the definition of L. — — —

⑦ $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$

Sol:

we know that

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$'' = \int_0^1 e^{-st} \cdot 0\, dt + \int_1^{\infty} e^{-st}\, dt$$

$$'' = 0 + \lim_{M \to \infty} \int_1^M e^{-st}\, dt$$

$$'' = \lim_{M \to \infty} \left[\frac{e^{-st}}{-s} \right]_1^M$$

$$'' = '' \left(\frac{e^{-sM}}{-s} - \frac{e^{-s}}{-s} \right)$$

$$'' = -\frac{1}{s} \lim_{M \to \infty} \left(\frac{1}{e^{sM}} - \frac{1}{e^{s}} \right)$$

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Exmple ①:

Compute $\mathcal{L}\{f(t)\}$

if $f(t) = 1$

Sol:

See $= \int_0^\infty e$

Example ②

Compute $\mathcal{L}\{f(t)\}$, if $f(t) = e^{at}$

See book

Example ④

Compute $\mathcal{L}\{\sin t\}$

Sol: H.W

Example ⑤

Compute

$\mathcal{L}\{f(t)\}$, if $f(t) = t$

Sol: H.W

Theorem (8.1) (Linearity Property of the Laplace Transform)

Let a and b be constants and suppose that the Laplace transform of the ftion. $f(t)$ and $g(t)$ exist, then

$$\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

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$$\text{OR} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_2 \\ a_2 & b_1 - b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where the initial population of the two territories $x(0) = x_0$, $y(0) = y_0$ are given

$$X' = AX, \quad X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Example (3)

Chapter 8

Introduction to Laplace Transform

Laplace Transform :- Let $f(t)$ be a function defined on the interval $(0, \infty)$. The Laplace Transform of $f(t)$ is the ftion (of s).

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

provided that the improper integral exists.

7.3 Population Problems

We begin by determining the population in two neighboring territories whose populations x and y of territories depends on several factors. The birth rate of x is a_1 and that of y is b_1. The rate at which citizens of x move to y is a_2, and that at which citizens move from y to x is b_2. After assuming the mortality rate of each territory is disregarded we determine the respective population of these two territories for any time t.

We know that

$$\frac{dP}{dt} = (\text{rate entering}) - (\text{rate leaving})$$

Add

So $$\frac{dx}{dt} = +a_1 x - a_2 x + b_2 y$$

decrease

$$\boxed{\frac{dx}{dt} = x(a_1 - a_2) + b_2 y}$$

$$\frac{dy}{dt} = b_1 y - b_2 y + a_2 x = \boxed{y(b_1 - b_2) + a_2 x}$$

Mortality rate if given will be subtracted

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$$\rightarrow \frac{dI}{dt} = \frac{1}{LC}Q - \frac{R}{L}I + \frac{E(t)}{L}$$

and $$\frac{dQ}{dt} = I$$

So $$\begin{cases} \dfrac{dQ}{dt} = I \\ \dfrac{dI}{dt} = \dfrac{1}{LC}Q - \dfrac{R}{L}I + \dfrac{E(t)}{L} \end{cases}$$

OR $$\begin{pmatrix} Q' \\ I' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ E(t) \end{pmatrix}$$

$$X' = AX + F$$

Example 1 Determine the charge and current in the L-R-C circuit with $L = 1\,H$, $R = 2\Omega$, $C = \frac{4}{3}F$ and $E(t) = e^{-t}$, if $Q(0) = 1$, and $I(0) = 1$

Sol: See book

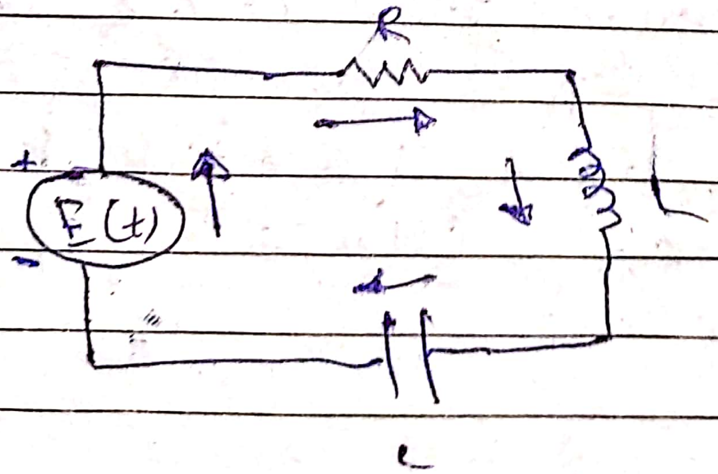
$$\therefore X_p(t) = \begin{pmatrix} \frac{173}{1225} \\ -\frac{1}{49} \end{pmatrix} + \begin{pmatrix} 6/35 \\ 1/7 \end{pmatrix} t$$

$\therefore$ Gen. Sol. is

$$X(t) = X_h(t) + X_p(t)$$

Chapter 7

7.1 L.R.C with loops.



$$(RI) + L\left(\frac{dI}{dt}\right) + \frac{1}{C}Q = E(t)$$