QUESTION #1: DETERMINE A GENERAL SOLUTION OF EACH EQUATION.

(a) $\frac{dy}{dx} = \frac{-4x+3y+15}{2x+y+7}$

Ansr. Let $x = X+h$ and $y = Y+K$

differentiate with respect to x.

$dx = dX$ and $dy = dY$

So $\frac{dy}{dx} = \frac{dY}{dX} = \frac{-4(X+h)+3(Y+K)+15}{2(X+h)+(Y+K)+7}$

$\frac{dY}{dX} = \frac{-4X-4h+3Y+3K+15}{2X+2h+Y+K+7}$

$\frac{dY}{dX} = \frac{-4X+3Y-4h+3K+15}{2X+Y+2h+K+7}$ —— (A)

Let $-4h+3K+15=0$ and $2h+K+7=0$

$-4h+3K=-15$ —(i) and $2h+K=-7$ —(ii)

Multiplying 2 with eq(ii) and adding with eq(i)

$$\begin{aligned} -4h+3K &= -15 \\ 4h+2K &= -14 \\ \hline 5K &= -29 \end{aligned}$$

$$\boxed{K = -29/5}$$

Put in eq (i)

$$-4h+3(-29/5) = -15$$

$$-4h - 87/5 = -15$$

$$-(4h + 87/5) = -15$$

$$4h + 87/5 = 15$$

$$4h = \frac{15}{1} - 87/5$$

$$4h = \frac{75-87}{5} = -12/5$$

$$\boxed{h = -3/5}$$

So eq (A) becomes

$$\frac{dy}{dx} = \frac{-4x+3y-0}{2x+y+0}$$

$$\frac{dy}{dx} = \frac{-4x+3y}{2x+y} \quad \text{—— (B)}$$

Let $y = ux \Rightarrow u = y/x$ —— (iii)

differentiating eq (iii) with respect to x.

$$\frac{dy}{dx} = u + x.\frac{du}{dx}$$

Putting values in eq (B)

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$$u + x\frac{du}{dx} = \frac{-4x+3y}{2x+y}$$

$$u + x\frac{du}{dx} = \frac{\cancel{x}(-4+3(y/x))}{\cancel{x}(2+(y/x))}$$

$$u + x.\frac{du}{dx} = \frac{-4+3u}{2+u}$$

$$x\frac{du}{dx} = \frac{-4+3u}{2+u} - u$$

$$x.\frac{du}{dx} = \frac{-4+3u-u(2+u)}{2+u}$$

$$x.\frac{du}{dx} = \frac{-4+3u-2u-u^2}{2+u}$$

$$x.\frac{du}{dx} = \frac{-4+u-u^2}{2+u}$$

OR $$\frac{2+u}{-4+u-u^2}\,du = \frac{1}{x}\,dx$$

Integrating both sides with indicated variables

$$\int \frac{2+u}{-4+u-u^2}\,du = \int \frac{1}{x}\,dx$$

$$\int \frac{2+u}{-(u^2-u+4)}\,du = \ln(x) + C$$

$$-\int \frac{2+u}{u^2-u+4}\,du = \ln(x) + C$$

$$-\frac{1}{2}\int \frac{4+2u}{u^2-u+4}\,du = \ln(x) + C$$

$$-\frac{1}{2}\int \frac{2u+4-5+5}{u^2-u+4} = \ln(x) + C$$

$$-\frac{1}{2}\int\left[\frac{2u-1}{u^2-u+4} + \frac{5}{u^2-u+4}\right]du = \ln(x) + C$$

$$-\frac{1}{2}\left[\ln(u^2-u+4) + \int \frac{5}{u^2-u+4}\,du\right] = \ln(x) + C$$

$$-\frac{1}{2}\left[\ln(u^2-u+4) + \frac{2\sqrt{15}}{3}\tan^{-1}\left(\frac{2\sqrt{15}\,u}{5} + \frac{\sqrt{15}}{5}\right)\right] = \ln(x) + C$$

$$-\frac{1}{2}\left[\ln\left(\frac{y^2}{x^2} - \frac{y}{x} + 4\right) + \frac{2\sqrt{15}}{3}\tan^{-1}\left(\frac{2\sqrt{15}}{5}\frac{y}{x} + \frac{\sqrt{15}}{5}\right)\right] = \ln(x) + C$$

$$-\frac{1}{2}\left[\ln\left(\left(\frac{y-29/5}{x-3/5}\right)^2 - \left(\frac{y-29/5}{x-3/5}\right) + 4\right) + \frac{2\sqrt{15}}{3}\tan^{-1}\left(\frac{2\sqrt{15}}{5}\cdot\frac{(y-29/5)}{(x-3/5)} + \frac{\sqrt{15}}{5}\right)\right.$$

$$= \ln(x-3/5) + C$$

$$-\frac{1}{2}\left[\ln\left(\frac{(y-29/5)^2-(y-29/5)(x-3/5)+4(x-3/5)^2}{(x-3/5)^2}\right) + \frac{2\sqrt{15}}{3}\tan^{-1}\left(\frac{2\sqrt{15}(y-29/5)}{5x-3} + \frac{\sqrt{15}}{5}\right)\right.$$

$$= \ln(x-3/5) + C$$

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$-\cancel{1}\ \cancel{\ln}$

(b) $\frac{dy}{dx} = \frac{y-x-1}{y-x+5}$

Ans:.

$$\frac{dy}{dx} = \frac{-x+y-1}{-x+y+5}$$

$$\frac{a_1}{a_2} = \frac{-1}{-1} = \frac{b_1}{b_2} = \frac{1}{1}$$

Let $\frac{a_1}{a_2} = 1/k = 1$

$\therefore k = 1$

$$\frac{dy}{dx} = \frac{-x+y-\cancel{k}1}{-xk+yk+5} \quad \text{—— (i)}$$

$$\frac{dy}{dx} = \frac{-x+y-1}{k(-x+y)+5}$$

Let $-x+y = z$

Differentiate with respect to x

$$-1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$\frac{dy}{dx} = \frac{dz}{dx} + 1$ Put in (i)

$$\frac{dz}{dx} + 1 = \frac{-x + y - 1}{-xk + yk + 5}$$

$$\frac{dz}{dx} + 1 = \frac{-x + y - 1}{k(-x+y)+5}$$

$$\frac{dz}{dx} + 1 = \frac{z - 1}{k(z)+5}$$

$$\frac{dz}{dx} = \frac{z-1}{z+5} - 1$$

$$\frac{dz}{dx} = \frac{z-1-(z+5)}{z+5}$$

$$\frac{dz}{dx} = \frac{\cancel{z} - 1 - \cancel{z} - 5}{z+5} = \frac{-6}{z+5}$$

$$\int (z+5)\,dz = \int -6\,dx$$

$$\frac{z^2}{2} + 5z = -6x + C$$

$$\frac{(-x+y)^2}{2} + 5(-x+y) = -6x + C$$

$$\frac{(-x+y)^2}{2} - 5x + 5y = -6x + C$$

$$\frac{(-x+y)^2}{2} + 5y = -x + C$$

QUESTION #3

A 30-Volt Emf is Applied To An LR Series Circuit In which The Inductance is 0.1H And The Resistance Is 50 ohms. Determine The Current I(t) If I(0). Determine The Current If $t \to \infty$

Ans:- The Current in LR Series Circuit is given by:

$$i(t) = V/R\left(1 - e^{-(R/L)t}\right)$$

Put $V = 30$ Volts
$L = 0.1$ Henry
$R = 50$ ohms

$$i(t) = \frac{30}{50}\left(1 - e^{-\left(\frac{50}{0.1}\right)t}\right)$$

$$i(t) = \frac{3}{5}\left(1 - e^{-500t}\right) \quad \text{— (i)}$$

Put $t = 0$

$$i(0) = \frac{3}{5}\left(1 - e^{-500(0)}\right)$$

$$i(0) = \frac{3}{5}\left(1 - e^{0}\right) = \frac{3}{5}(1 - 1) = \frac{3}{5}(0)$$

$$\boxed{i(0) = 0}$$

Now if $t \to \infty$

(i) $\to$ $i(t) = \frac{3}{5}\left(1 - e^{-500(\infty)}\right)$

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$$i(t) = \frac{3}{5}\left(1 - e^{-\infty}\right) \qquad e^{-\infty} = 0$$

$$i(t) = \frac{3}{5}\left(1 - 0\right)$$

$$\boxed{i(t) = \frac{3}{5}}$$

So as $t \rightarrow \infty$, $i(t) \rightarrow 3/5$

QUESTION # 4

A 100-Volt Emf is applied to an RC Series circuit in which the resistance is 200 ohm and capacitance is 10^{-4} farad. Determine the charge $q(t)$ on the capacitor if $q(0) = 0$. Detetermine the current $i(t)$.

Ans:-

$R = 200\ \Omega$

$E(t) = 100V$

$C = 10^{-4} F$

$q(t) = ?\quad q(0) = 0$

As we know that

$$R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

Putting values

$$200\frac{dq}{dt} + \frac{q}{10^{-4}} = 100$$

~~Multiplying~~ ~~20~~

Dividing 200 both sides

$$\frac{dq}{dt} + \frac{q}{200 \times 10^{-4}} = \frac{10\cancel{0}}{20\cancel{0}}$$

$$\frac{dq}{dt} + \frac{\overset{50}{\cancel{10000}}}{\underset{1}{\cancel{200}}} q = \frac{1}{2}$$

$$\frac{dq}{dt} + 50q = \frac{1}{2}$$

Solving above First Order Differential equation.

$$q e^{\int 50\,dt} = \int \frac{1}{2} e^{\int 50\,dt} dt$$

$$q \cdot e^{50t} = \frac{1}{2} \int e^{50t} + C$$

$$q e^{50t} = \frac{1}{2} \cdot \frac{e^{50t}}{50} + C$$

$$\therefore \quad q(t) = \frac{1}{100} + Ce^{-50t}$$

Put $t = 0$

$$q(0) = \frac{1}{100} + Ce^{-50(0)}$$

$$0 = \frac{1}{100} + Ce^{-0}$$

$$0 = \frac{1}{100} + C(1)$$

$$\boxed{C = -1/100}$$

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$$\boxed{q(t) = \frac{1}{100} - \frac{1}{100} e^{-50t}}$$

Differentiate $q(t)$ to get $i(t)$

$$\frac{dq}{dt} = \frac{d}{dt}\left(\frac{1}{100} - \frac{1}{100} e^{-50t}\right)$$

$$i = 0 + \frac{1}{100} \cdot e^{-50t} \cdot \cancel{+}50$$

$$i = \frac{50}{100} e^{-50t} = \frac{1}{2} e^{-50t}$$

$$\boxed{i(t) = \frac{1}{2} e^{-50t}}$$

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Question #5

After 10 days 800g of a radioactive element remains and after 15 days, 560 g remains. Determine the half life of this element?

Ans:-

$$N(t) = N_0 \cdot e^{-kt}$$

After 10 days, put $t = 10$

$$N(10) = N_0 e^{-k(10)}$$

Also $N(10) = 800$

$$\therefore \quad 800 = N_0 e^{-10k} \quad \text{—(i)}$$

After 15 days, put $t = 15$

$$N(15) = N_0 e^{-k(15)}$$

$$N(15) = N_0 e^{-15k}$$

Also $N(15) = 560$

$$\therefore \quad 560 = N_0 e^{-15k} \quad \text{—(ii)}$$

Dividing equation (i) by equation (ii)

$$\frac{80\cancel{0}}{56\cancel{0}} = \frac{\cancel{N_0} e^{-10k}}{\cancel{N_0} e^{-15k}}$$

$$\frac{80}{56} = e^{-10k+15k}$$

$$\frac{10}{7} = e^{5k}$$

$$e^k = \left(10/7\right)^{1/5}$$

$$e^k = 1.073$$

$\boxed{K = 0.071}$

The half life formula can be derived as:

$$N(t) = N_0 e^{-Kt}$$

Put $t = T$, $N(t) = N_0/2$

$$\frac{\cancel{N_0}}{2} = \cancel{N_0} e^{-KT}$$

$$\frac{1}{2} = e^{-KT}$$

$$2 = e^{KT}$$

$$\ln(2) = \cancel{\ln}(\cancel{e}^{KT})$$

$$\frac{\cancel{K}T}{\cancel{K}} = \frac{\ln(2)}{K}$$

$$T = \frac{\ln 2}{K} = \boxed{9.716}$$

Ans

QUESTION # 2

SOLVE THE GIVEN IVP BY FINDING AN APPROPRIATE INTEGRATING FACTOR.

(a) $x\,dx + (x^2y + 4y)\,dy = 0, \quad y(4) = 0$

$$M(x,y) = x \quad , \quad N(x,y) = x^2y + 4y$$

$$\frac{\partial M}{\partial y} = 0 \quad , \quad \frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given equation is not an exact equation.

The integrating factor of the above equation is given by:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{0 - 2xy}{x^2y + 4y} = \boxed{\frac{-2x}{x^2+4}}$$

$$\therefore I.F = e^{\int \frac{-2x}{x^2+4}} = e^{-\int \frac{2x}{x^2+4}} = e^{-\ln(x^2+4)}$$

$$I.F = \frac{1}{e^{\ln(x^2+4)}} = \frac{1}{x^2+4}$$

Now multiplying Both sides of equation by $\frac{1}{x^2+4}$

$$\frac{1}{x^2+4} \cdot x\, dx + (x^2y+4y) \cdot \frac{1}{x^2+4} dy = 0$$

$$\frac{x}{x^2+4} dx + \frac{x^2y+4y}{x^2+4} dy = 0$$

$$\frac{x}{x^2+4} dx + \frac{y\cancel{(x^2+4)}}{\cancel{x^2+4}} dy = 0$$

$$\frac{x}{x^2+4} dx + y\, dy = 0$$

Now again testing the exactness,

$$\frac{\partial M}{\partial y} = 0 \quad , \quad \frac{\partial N}{\partial x} = 0$$

Hence the given equation is transformed into exact equation.

Assume $M(x,y) = \frac{\partial f}{\partial x}$ and $N(x,y) = \frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+4}$$

Integrating with respect to x.

$$f(x,y) = \int \frac{x}{x^2+4} dx$$

$$f(x,y) = \frac{1}{2} \int \frac{2x}{x^2+4} dx$$

$$f(x,y) = \frac{1}{2} \ln(x^2+4) + g(y) \text{ --- (i)}$$

$f(x,y) = \frac{1}{2}\ln(x^2+4) + g(y)$

Differentiating with respect to y

$$\frac{\partial f}{\partial y} = 0 + g'(y)$$

Also $\frac{\partial f}{\partial y} = N(x,y) = y$

$$\therefore \; g'(y) = y$$

$$\rightarrow \; g(y) = \frac{y^2}{2} + c_1$$

Putting values in (i)

$$f(x,y) = \frac{1}{2}\ln(x^2+4) + \frac{y^2}{2} + c_1$$

The general solution is given by:

$$c_2 = \frac{1}{2}\ln(x^2+4) + \frac{y^2}{2} + c_1$$

$$c_2 - c_1 = \frac{1}{2}\ln(x^2+4) + \frac{y^2}{2}$$

$$c_3 - \frac{1}{2}\ln(x^2+4) = \frac{y^2}{2}$$

$$2c_3 - \ln(x^2+4) = y^2$$

$$y = \sqrt{c - \ln(x^2+4)} \quad \text{—(ii)}$$

$$y(4) = \sqrt{c - \ln(16+4)}$$

$$0 = \sqrt{c - \ln(20)}$$

$$0 = c - \ln(20)$$

$$\boxed{c = \ln(20) = 2.995}$$

Put in (ii)

$$\boxed{y = \sqrt{2.995 - \ln(x^2+4)}}$$

ⓑ $(x^2+y^2-5)\,dx = (y + xy)\,dy = 0 \quad , \quad y(0) = 1$

Ans: $(x^2+y^2-5)\,dx - (y+xy)\,dy = 0$

$(x^2+y^2-5)\,dx + (-y-xy)\,dy = 0$

$M(x,y) = x^2+y^2-5 \quad , \quad N(x,y) = -y-xy$

$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = -0-y = -y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So the given equation is not exact equation.

The integrating factor is given by:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-y)}{-y-xy} = \frac{2y+y}{y(-1-x)} = \frac{3y}{y(-1-x)} = \frac{3}{-1-x}$$

$$\text{"} = \frac{3}{-(1+x)} = \boxed{-\frac{3}{1+x}}$$

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The integrating factor is given by:

$$I.F = e^{\int -\frac{3}{1+x}dx} = e^{-\int \frac{3}{x+1}dx} = e^{-3\int \frac{1}{x+1}dx}$$

$$I.F = e^{-3\ln(x+1)} = \frac{1}{e^{3\ln(x+1)}} = \frac{1}{e^{\ln(x+1)^3}} = \boxed{\frac{1}{(x+1)^3}}$$

Multiplying "$\frac{1}{(x+1)^3}$" both sides of the equation.

$$\frac{1}{(x+1)^3}\cdot(x^2+y^2-5)dx + \frac{1}{(x+1)^3}\cdot(-y-xy)dy = 0$$

$$\frac{x^2+y^2-5}{(x+1)^3}dx + \frac{-y-xy}{(x+1)^3}dy = 0$$

$$\frac{x^2+y^2-5}{(x+1)^3}dx + \frac{-y(1+x)}{(x+1)^3}dy = 0$$

$$\frac{x^2+y^2-5}{(x+1)^3}dx + \frac{-y\cancel{(x+1)}}{(x+1)^2\cancel{(x+1)}}dy = 0$$

$$\frac{x^2+y^2-5}{(x+1)^3}dx + \frac{-y}{(x+1)^2}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2-5}{(x+1)^3}\right)$$

$$\frac{\partial M}{\partial y} = \frac{1}{(x+1)^3}\cdot\frac{\partial}{\partial y}(x^2+y^2-5)$$

$$\frac{\partial M}{\partial y} = \frac{1}{(x+1)^3}\cdot 2y = \frac{2y}{(x+1)^3}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-y}{(x+1)^2}\right)$$

$$\frac{\partial N}{\partial x} = -y\cdot\frac{\partial}{\partial x}(x+1)^{-2}$$

$$\frac{\partial N}{\partial x} = -y\cdot -2(x+1)^{-3}$$

$$\frac{\partial N}{\partial x} = \frac{2y}{(x+1)^3}$$

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Now the equation has been transformed into exact equation. As

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Assume $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$

$$\frac{\partial f}{\partial x} = \frac{x^2+y^2-5}{(x+1)^3} = \frac{x^2}{(x+1)^3} + \frac{y^2-5}{(x+1)^3}$$

Integrating with respect to x.

$$f(x,y) = \int \frac{x^2}{(x+1)^3}\,dx + \int \frac{y^2-5}{(x+1)^3} + g(y)$$

$$f(x,y) = \frac{2}{x+1} - \frac{1}{2(x+1)^2} + \ln(x+1) + (y^2-5) \cdot -\frac{1}{2(x+1)^2} + g(y)$$

$$f(x,y) = \frac{2 \cdot 2(x+1) - 1}{2(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2} + g(y) \quad \text{— (ii)}$$

Now differentiating with respect to y.

$$\frac{\partial f}{\partial y} = 0 + 0 - \frac{\partial}{\partial y}\left(\frac{y^2-5}{2(x+1)^2}\right) + g'(y)$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2(x+1)^2} \cdot 2y + g'(y) = -\frac{2y}{2(x+1)^2} + g'(y) = \boxed{-\frac{y}{(x+1)^2} + g'(y)}$$

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$$\frac{\partial f}{\partial y} = \frac{-y}{(x+1)^2} + g'(y)$$

But $\frac{\partial f}{\partial y} = N(x,y) = \frac{-y}{(x+1)^2}$

$$\therefore \cancel{\frac{-y}{(x+1)^2}} + g'(y) = \cancel{\frac{-y}{(x+1)^2}}$$

$$g'(y) = 0$$

$$\Rightarrow g(y) = c_1$$

put in (i)

$$f(x,y) = \frac{4(x+1)-1}{2(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2} + c_1$$

The general solution is given by

$$c_2 = \frac{\overset{2}{\cancel{4}}(x+1)-1}{\underset{1}{\cancel{2}}(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2} + c_1$$

$$c_2 - c_1 = \frac{2(x+1)-1}{(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2}$$

$$c = \frac{2(x+1)-1}{(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2} \quad \text{—(ii)}$$

For $x = 0$, $y = 1$, put in (ii)

$$c = \frac{2(0+1)-1}{(0+1)^2} + \ln(0+1) - \frac{1^2-5}{2(0+1)^2}$$

$$c = \frac{2-1}{(1)^2} + \ln(1) - \frac{-4}{2}$$

$$c = 1 + 0 + 2 = \boxed{3}$$

Put in (ii)

$$3 = \frac{2(x+1)-1}{(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2}$$

$$3 = \frac{2x+2-1}{(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2}$$

$$3 = \frac{2x+1}{(x+1)^2} + \ln(x+1) - \frac{y^2-5}{2(x+1)^2}$$

$$3 = \frac{4x+2-(y^2-5)}{2(x+1)^2} + \ln(x+1)$$

$$3 = \frac{4x+2-y^2+5}{2(x+1)^2} + \ln(x+1)$$

$$\boxed{3 = \frac{4x-y^2+7}{2(x+1)^2} + \ln(x+1)}$$

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