4

APPLICATIONS OF DIFFERENTIATION

We will see that many of the results of this chapter depend on one central fact—the Mean Value Theorem.

4.2

The Mean Value Theorem

In this section, we will learn about:

The significance of the mean value theorem.

MEAN VALUE THEOREM

To arrive at the theorem, we first need the following result.

ROLLE'S THEOREM

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

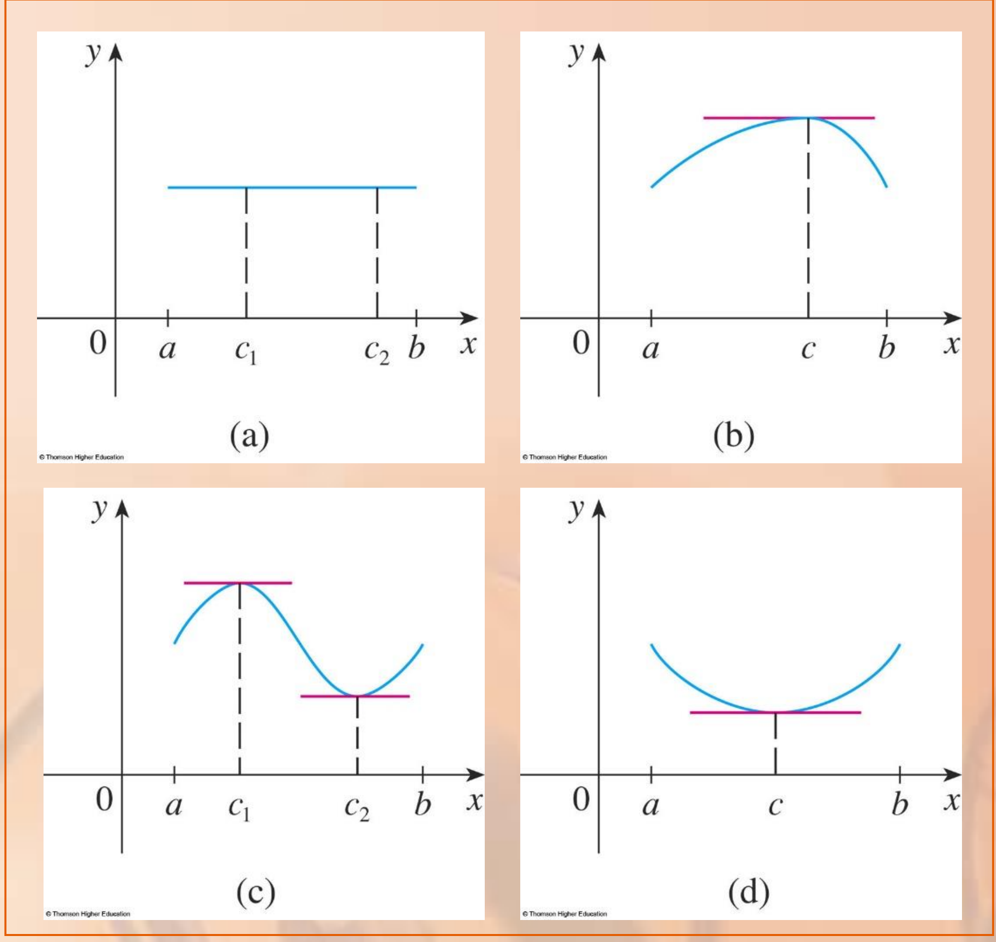
Then, there is a number c in (a, b) such that $f'(c) = 0$.

ROLLE'S THEOREM

Before giving the proof, let's look at the graphs of some typical functions that satisfy the three hypotheses.

ROLLE'S THEOREM

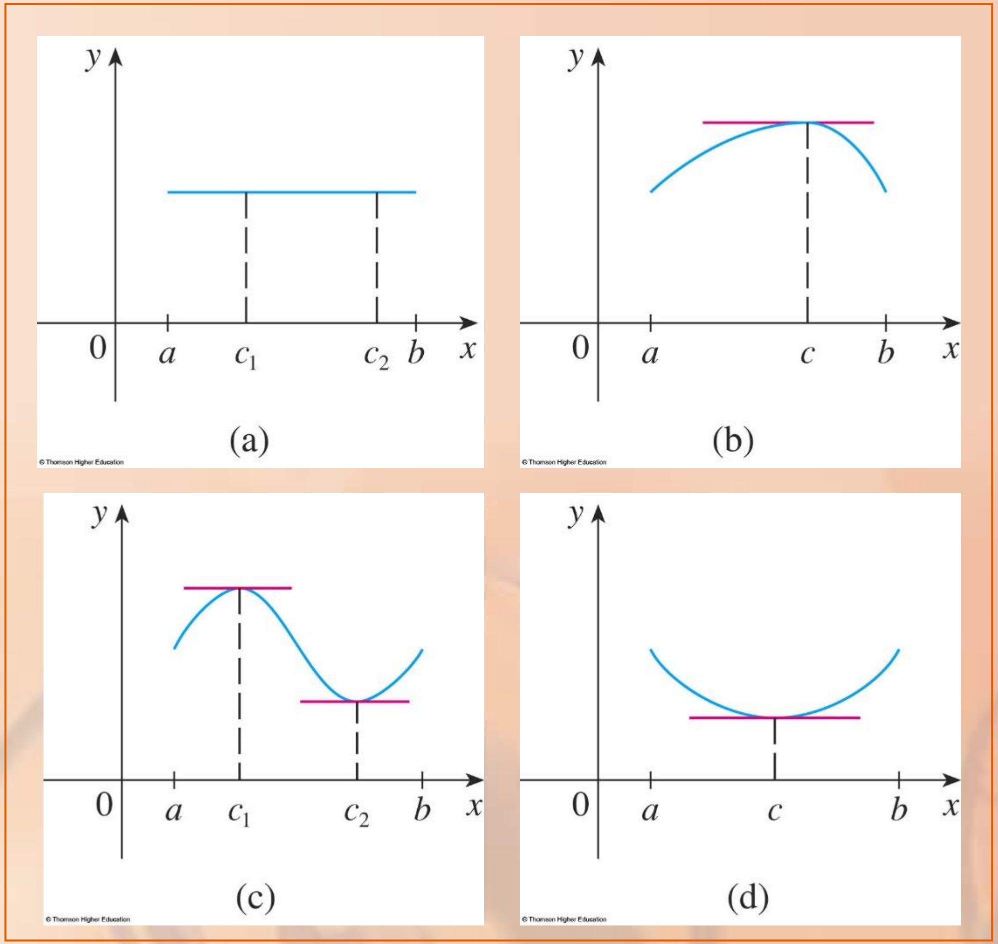
The figures show the graphs of four such functions.



ROLLE'S THEOREM

In each case, it appears there is at least one point (c, $f(c)$) on the graph where the tangent is horizontal and thus $f'(c) = 0$.

- So, Rolle's Theorem is plausible.



There are three cases:

1. $f(x) = k$, a constant

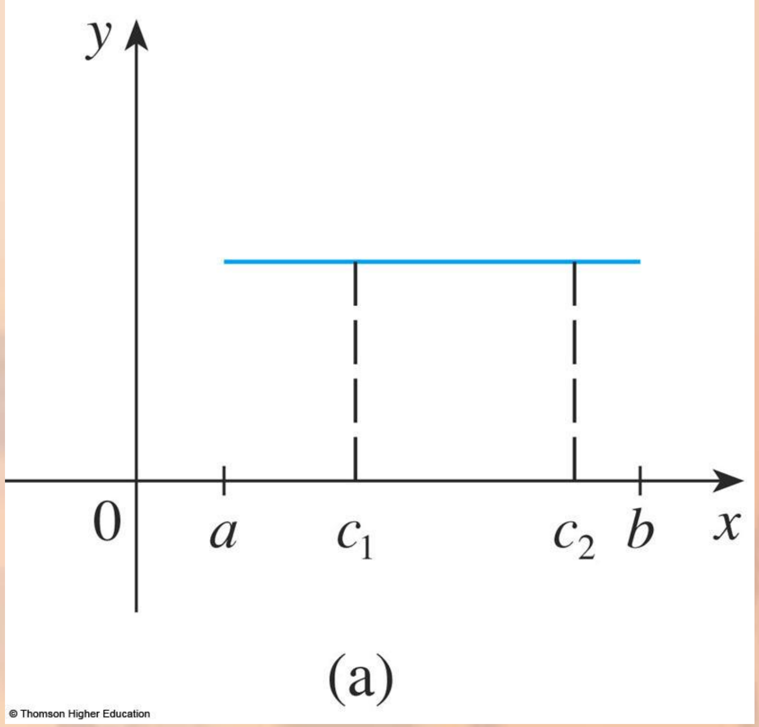
2. $f(x) > f(a)$ for some x in (a, b)

3. $f(x) < f(a)$ for some x in (a, b)

Proof—Case 1

$f(x) = k$, a constant

- Then, $f'(x) = 0$.
- So, the number c can be taken to be any number in (a, b).

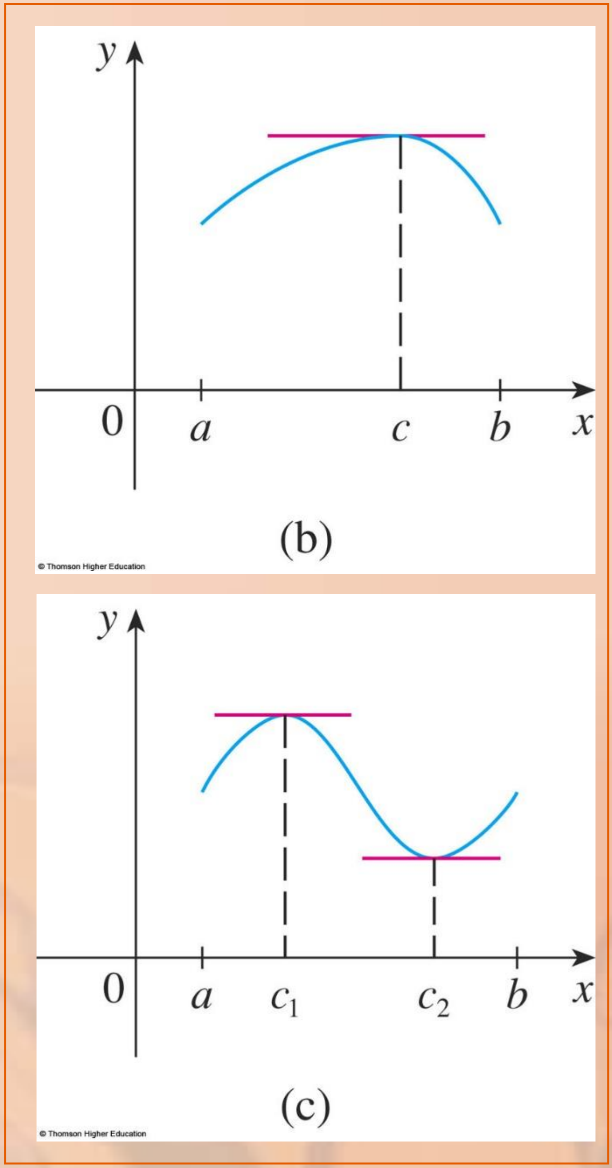


ROLLE'S THEOREM

Proof—Case 2

$f(x) > f(a)$ for some x in (a, b)

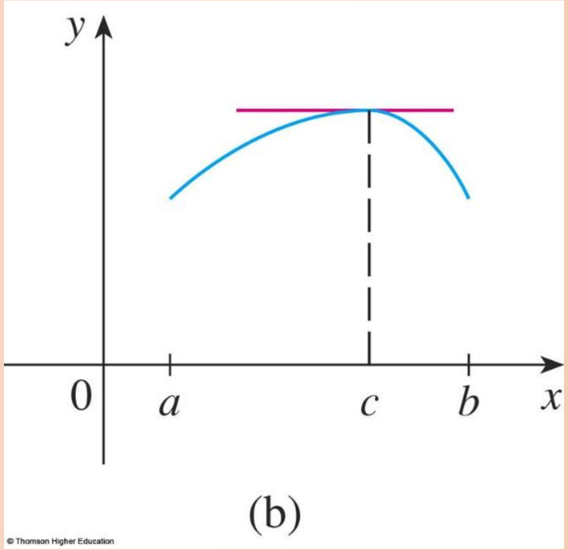
- By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$.



ROLLE'S THEOREM

Proof—Case 2

- As $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b).
- Then, f has a local maximum at c and, by hypothesis 2, f is differentiable at c.
- Thus, $f'(c) = 0$ by Fermat's Theorem.



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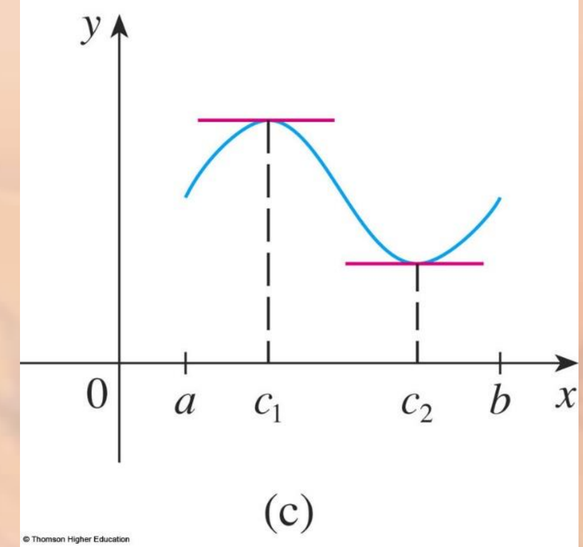
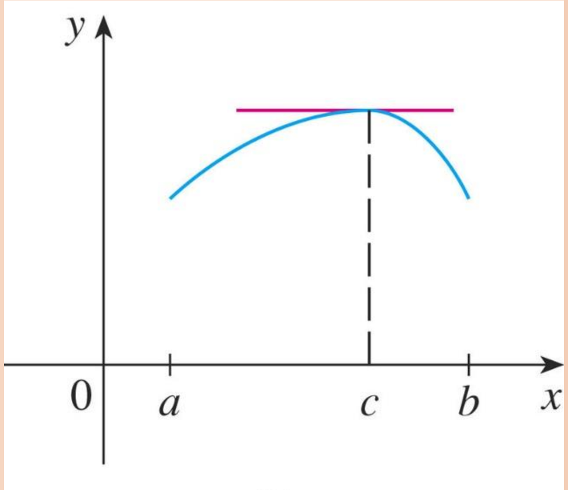


(c)

Proof—Case 3

$f(x) < f(a)$ for some x in (a, b)

- By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b).
- Again, $f'(c) = 0$ by Fermat's Theorem.



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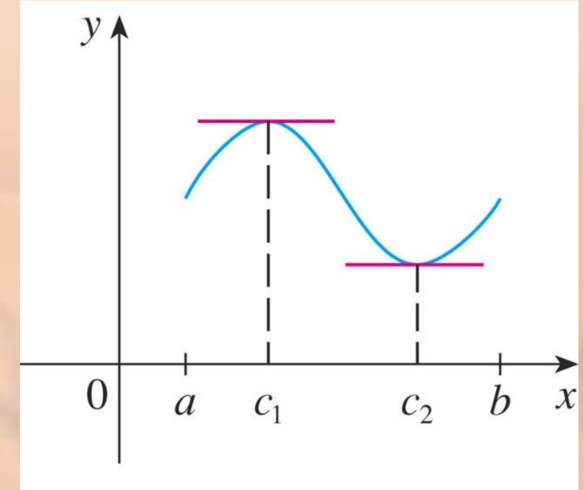


(c)

Example 1

Let's apply the theorem to the position function $s = f(t)$ of a moving object.

- If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$.
- The theorem states that there is some instant of time $t = c$ between a and b when $f'(c) = 0$; that is, the velocity is 0.
- In particular, you can see that this is true when a ball is thrown directly upward.

Example 2

Prove that the equation

$$x^3 + x - 1 = 0$$

has exactly one real root.

Example 2

First, we use the Intermediate Value Theorem (Equation 10 in Section 2.5) to show that a root exists.

- Let $f(x) = x^3 + x - 1$.
- Then, $f(0) = -1 < 0$ and $f(1) = 1 > 0$.
- Since f is a polynomial, it is continuous.
- So, the theorem states that there is a number c between 0 and 1 such that $f(c) = 0$.
- Thus, the given equation has a root.

Example 2

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction.

Suppose that it had two roots *a* and *b*.

- Then, $f(a) = 0 = f(b)$.
- As f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$.
- Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$.
- However, $f'(x) = 3x^2 + 1 \geq 1$ for all x (since $x^2 \geq 0$), so $f'(x)$ can never be 0.

This gives a contradiction.

- So, the equation can't have two real roots.

ROLLE'S THEOREM

Our main use of Rolle's Theorem is in proving the following important theorem—which was first stated by another French mathematician, Joseph-Louis Lagrange.

MEAN VALUE THEOREM

Equations 1 and 2

Let *f* be a function that fulfills two hypotheses:

1. *f* is continuous on the closed interval [*a*, *b*].
2. *f* is differentiable on the open interval (*a*, *b*).

Then, there is a number *c* in (*a*, *b*) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

MEAN VALUE THEOREM

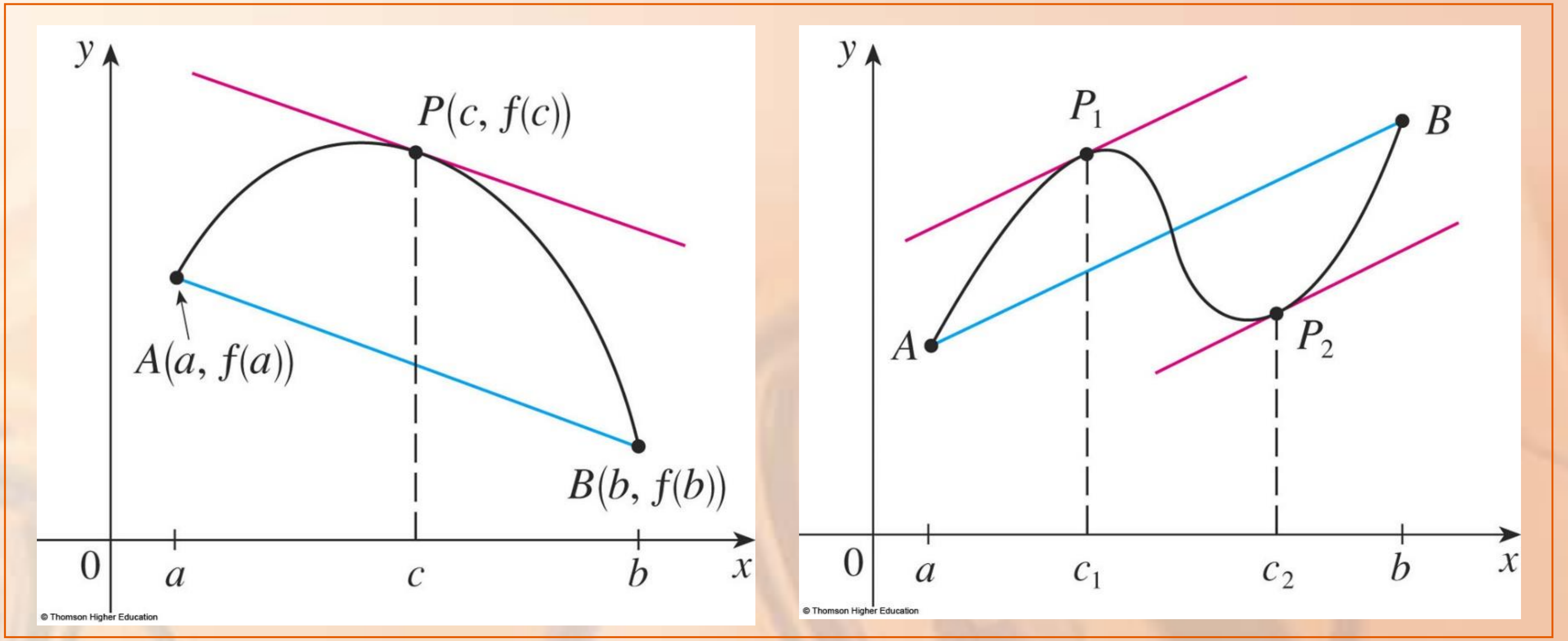
Before proving this theorem,

we can see that it is reasonable by

interpreting it geometrically.

MEAN VALUE THEOREM

The figures show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions.

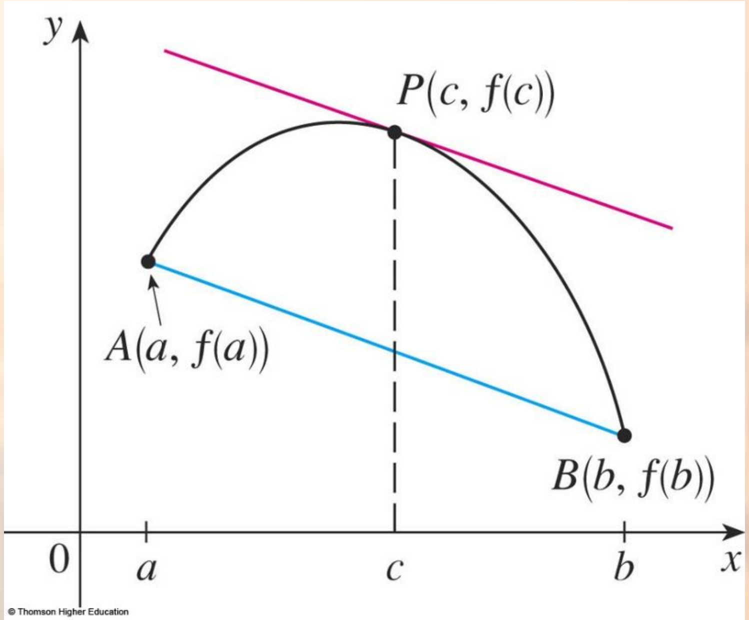


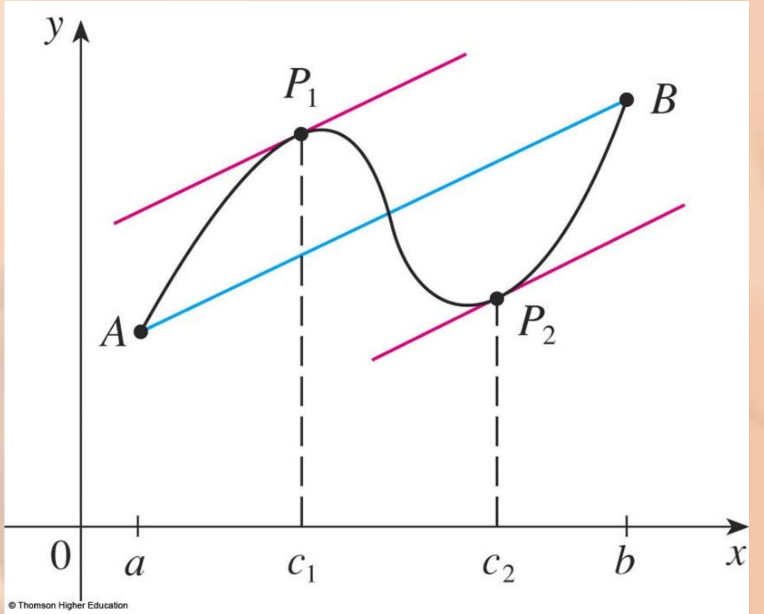
Equation 3

The slope of the secant line *AB* is:

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

- This is the same expression as on the right side of Equation 1.

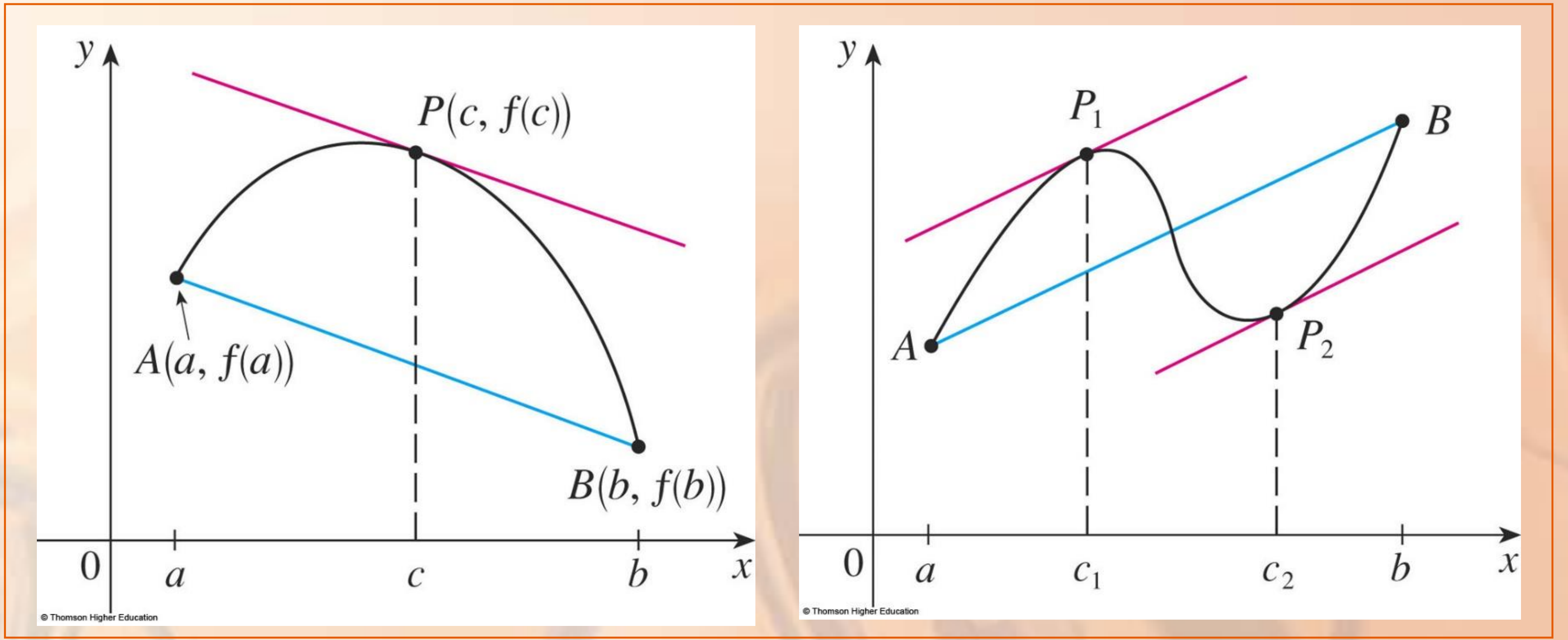




MEAN VALUE THEOREM

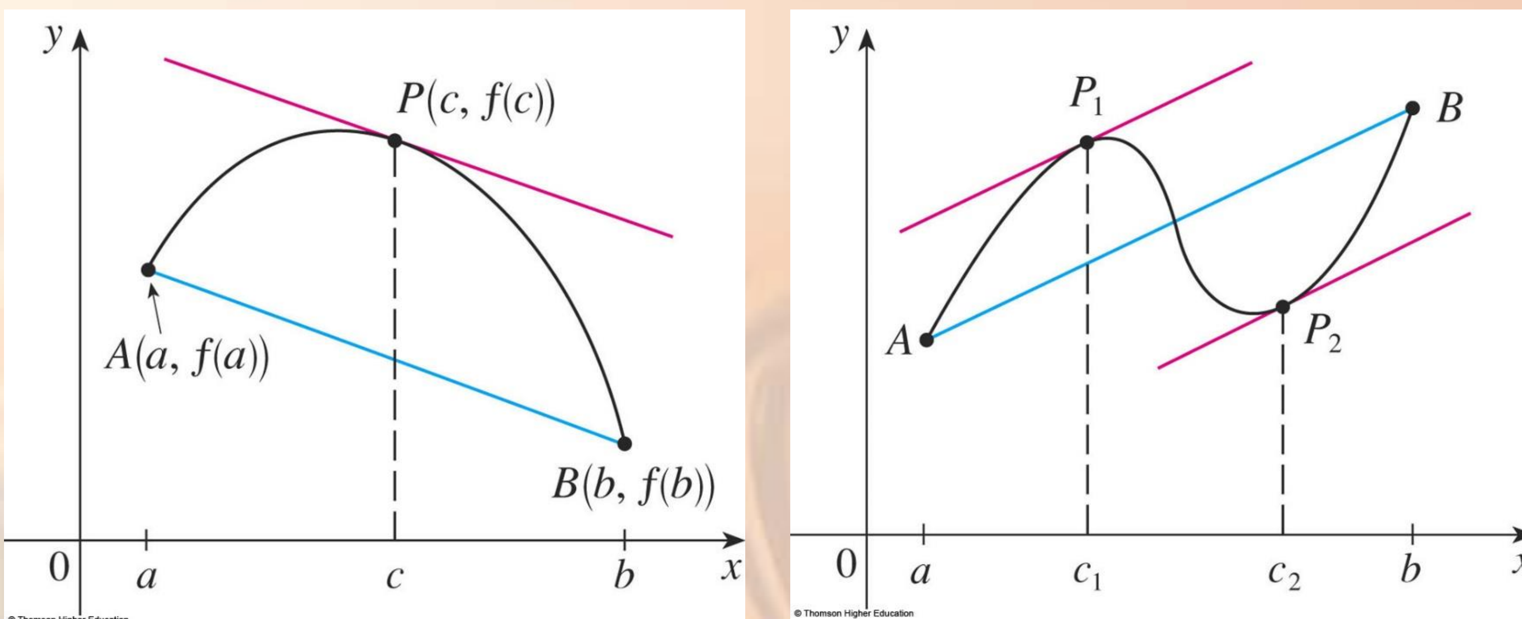
f '(*c*) is the slope of the tangent line at (*c*, *f*(*c*)).

- So, the Mean Value Theorem—in the form given by Equation 1—states that there is at least one point *P*(*c*, *f*(*c*)) on the graph where the slope of the tangent line is the same as the slope of the secant line *AB*.



MEAN VALUE THEOREM

In other words, there is a point *P* where the tangent line is parallel to the secant line *AB*.



PROOF

We apply Rolle's Theorem to a new function *h* defined as the difference between *f* and the function whose graph is the secant line *AB*.

PROOF

Using Equation 3, we see that the equation of the line *AB* can be written as:

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or as:

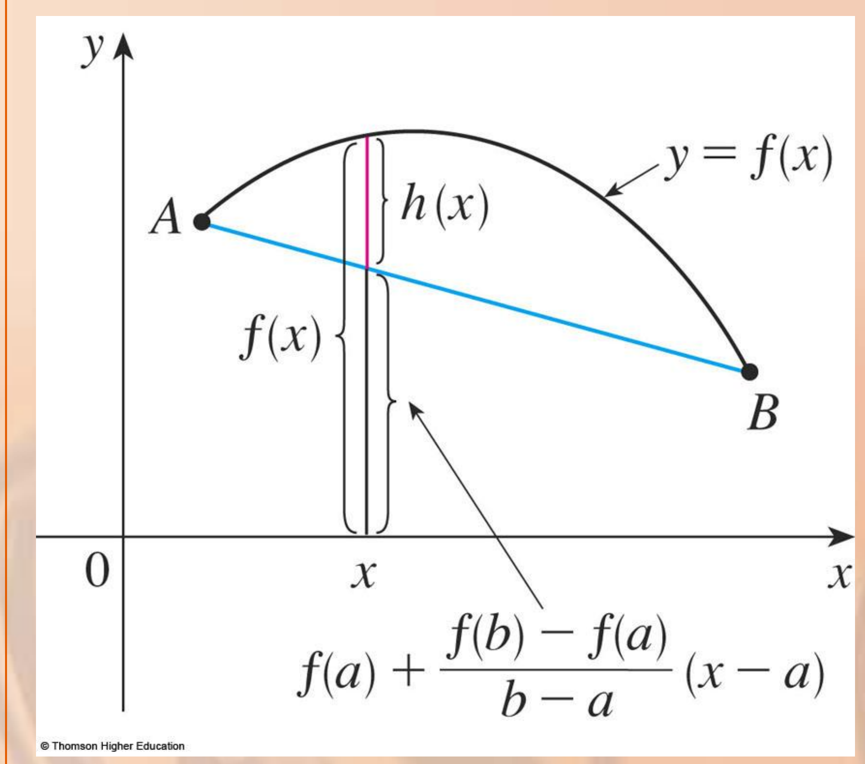
$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

MEAN VALUE THEOREM

Equation 4

So, as shown in the figure,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a)$$



First, we must verify that h satisfies the three hypotheses of Rolle's Theorem—as follows.

HYPOTHESIS 1

The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.

HYPOTHESIS 2

The function *h* is differentiable on (a, b) because both *f* and the first-degree polynomial are differentiable.

- In fact, we can compute *h*' directly from Equation 4:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

- Note that $f(a)$ and $[f(b) - f(a)]/(b - a)$ are constants.

HYPOTHESIS 3

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b-a}(a-a)$$

$$= 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b-a}(b-a)$$

$$= f(b) - f(a) - [f(b) - f(a)]$$

$$= 0$$

Therefore, $h(a) = h(b)$.

MEAN VALUE THEOREM

As *h* satisfies the hypotheses of Rolle's Theorem, that theorem states there is a number *c* in (*a*, *b*) such that *h'*(*c*) = 0.

- Therefore, $0 = h'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a}$
- So, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Example 3

To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$.

Since f is a polynomial, it is continuous and differentiable for all x.

So, it is certainly continuous on [0, 2] and differentiable on (0, 2).

- Therefore, by the Mean Value Theorem, there is a number c in (0,2) such that:

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now, $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$.

So, this equation becomes

$$6 = (3c^2 - 1)^2 = 6c^2 - 2$$

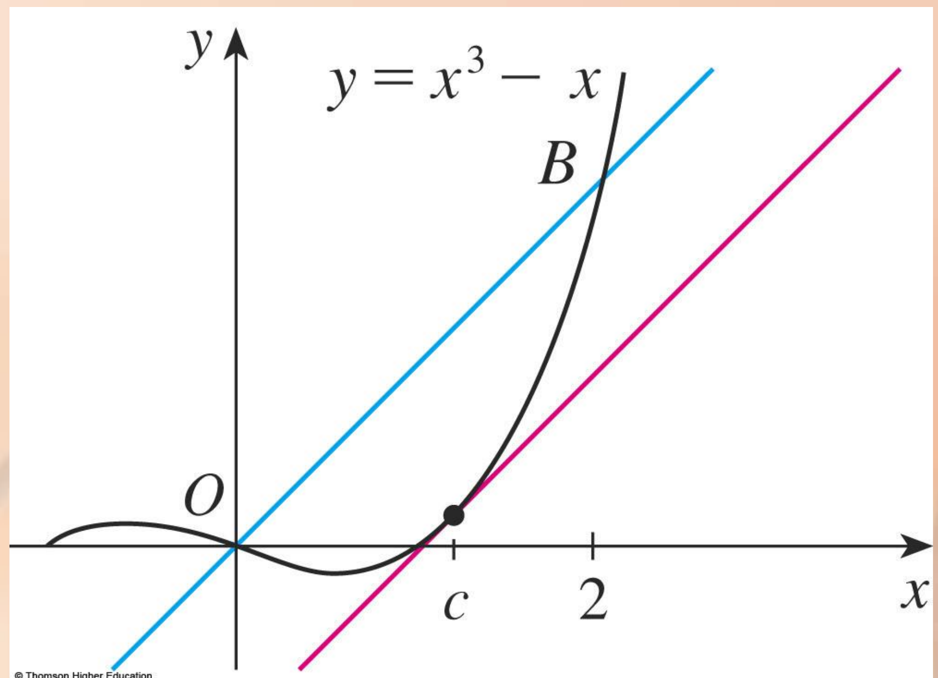
- This gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$.
- However, c must lie in (0, 2), so $c = 2/\sqrt{3}$.

MEAN VALUE THEOREM

Example 3

The figure illustrates this calculation.

- The tangent line at this value of *c* is parallel to the secant line *OB*.



Example 4

If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$.

Thus, the Mean Value Theorem—in the form of Equation 1—tells us that, at some time $t = c$ between *a* and *b*, the instantaneous velocity $f'(c)$ is equal to that average velocity.

- For instance, if a car traveled 180 km in 2 hours, the speedometer must have read 90 km/h at least once.

Example 4

In general, the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval.

MEAN VALUE THEOREM

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative.

- The next example provides an instance of this principle.

Example 5

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x.

How large can $f(2)$ possibly be?

We are given that f is differentiable—and therefore continuous—everywhere.

In particular, we can apply the Mean Value Theorem on the interval [0, 2].

- There exists a number c such that
$$f(2) - f(0) = f'(c)(2 - 0)$$
- So, $f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$

Example 5

We are given that $f\,'(x) \le 5$ for all x.

So, in particular, we know that $f\,'(c) \le 5$.

- Multiplying both sides of this inequality by 2, we have $2\,f\,'(c) \le 10$.
- So, $f(2) = -3 + 2\,f\,'(c) \le -3 + 10 = 7$
- The largest possible value for $f(2)$ is 7.

MEAN VALUE THEOREM

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.

- One of these basic facts is the following theorem.
- Others will be found in the following sections.

Theorem 5

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$.

- Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.

By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$.

So, Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \quad \text{or} \quad f(x_2) = f(x_1)$$

- Therefore, f has the same value at any two numbers x_1 and x_2 in (a, b).
- This means that f is constant on (a, b).

Corollary 7

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b).

That is, $f(x) = g(x) + c$ where c is a constant.

Corollary 7—Proof

Let $F(x) = f(x) - g(x)$.

Then,

$$F'(x) = f\,'(x) - g\,'(x) = 0$$

for all x in (a, b).

- Thus, by Theorem 5, F is constant.
- That is, $f - g$ is constant.

NOTE

Care must be taken in applying Theorem 5.

- Let $f(x) = \dfrac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

- The domain of *f* is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all *x* in *D*.

NOTE

However, *f* is obviously not a constant function.

This does not contradict Theorem 5 because *D* is not an interval.

- Notice that *f* is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

Example 6

Prove the identity

$$\tan^{-1} x + \cot^{-1} x = \pi/2.$$

- Although calculus isn't needed to prove this identity, the proof using calculus is quite simple.

If $f(x) = \tan^{-1} x + \cot^{-1} x$,

then

$$f\,'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x.

- Therefore, $f(x) = C$, a constant.

To determine the value of C, we put $x = 1$ (because we can evaluate $f(1)$ exactly).

Then,

$$C = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

- Thus, $\tan^{-1} x + \cot^{-1} x = \pi/2$.