4

APPLICATIONS OF DIFFERENTIATION

We have already investigated some applications of derivatives.

However, now that we know the differentiation rules, we are in a better position to pursue the applications of differentiation in greater depth.

APPLICATIONS OF DIFFERENTIATION

Here, we learn how derivatives affect

the shape of a graph of a function and,

in particular, how they help us locate

maximum and minimum values of functions.

APPLICATIONS OF DIFFERENTIATION

Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation.

- In particular, we will be able to investigate the optimal shape of a can and to explain the location of rainbows in the sky.

4.1

Maximum and Minimum Values

In this section, we will learn:

How to find the maximum

and minimum values of a function.

Some of the most important applications of differential calculus are optimization problems.

- In these, we are required to find the optimal (best) way of doing something.

Here are some examples of such problems that we will solve in this chapter.

- What is the shape of a can that minimizes manufacturing costs?

- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)

Here are some more examples.

- What is the radius of a contracted windpipe that expels air most rapidly during a cough?

- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function.

- Let's first explain exactly what we mean by maximum and minimum values.

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f.

The number $f(c)$ is called the maximum value of f on D.

Similarly, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D.

The maximum and minimum values of f are called the extreme values of f.

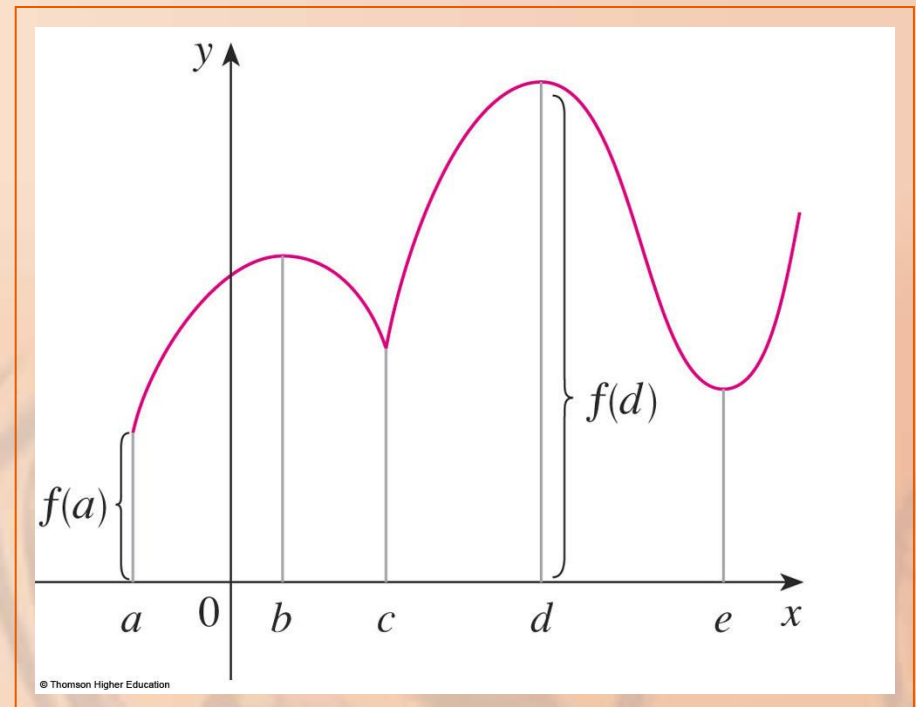The figure shows the graph of a function f with absolute maximum at d and absolute minimum at a.

- Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.
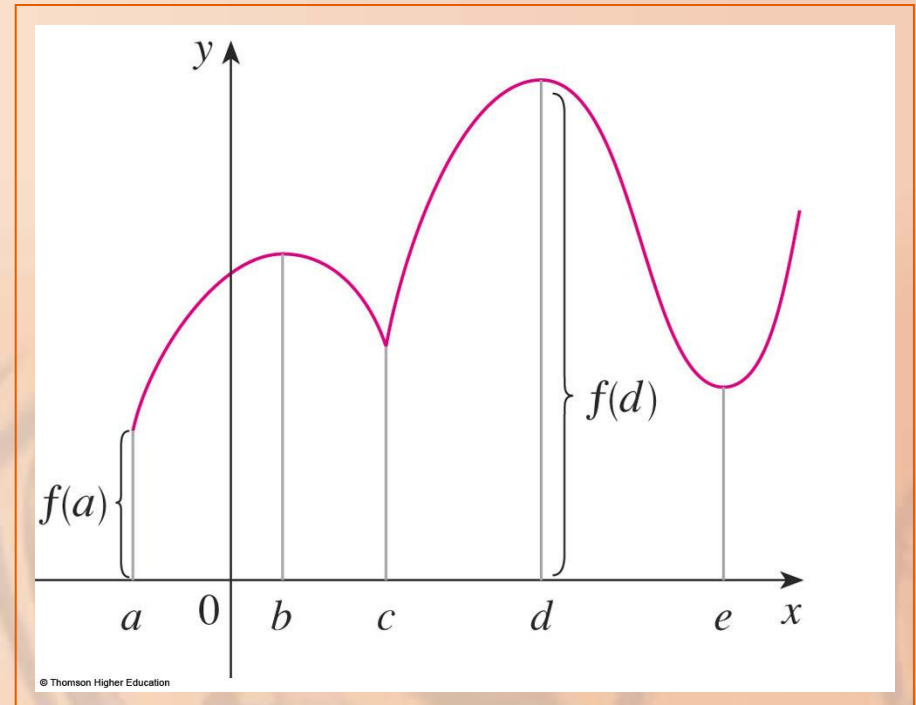
If we consider only values of *x* near *b*—for instance, if we restrict our attention to the interval (*a*, *c*)—then *f*(*b*) is the largest of those values of *f*(*x*).
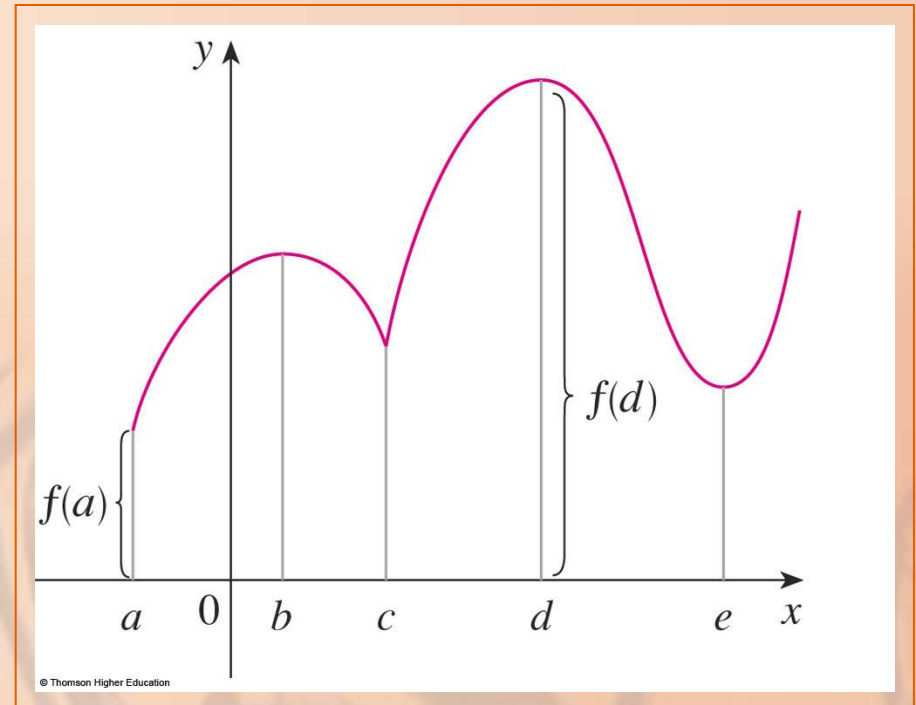
- It is called a local maximum value of *f*.

LOCAL MINIMUM VALUE

Likewise, *f*(*c*) is called a local minimum value of *f* because *f*(*c*) ≤ *f*(*x*) for *x* near *c*—for instance, in the interval (*b*, *d*).

- The function *f* also has a local minimum at *e*.

In general, we have the following definition.

A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c.

- This means that $f(c) \geq f(x)$ for all x in some open interval containing c.

- Similarly, f has a local minimum at c if $f(c) \leq f(x)$ when x is near c.

The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times—since $\cos 2n\pi = 1$ for any integer n and $-1 \leq \cos x \leq 1$ for all x.

Likewise, $\cos (2n + 1)\pi = -1$ is its minimum value—where n is any integer.

If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x.
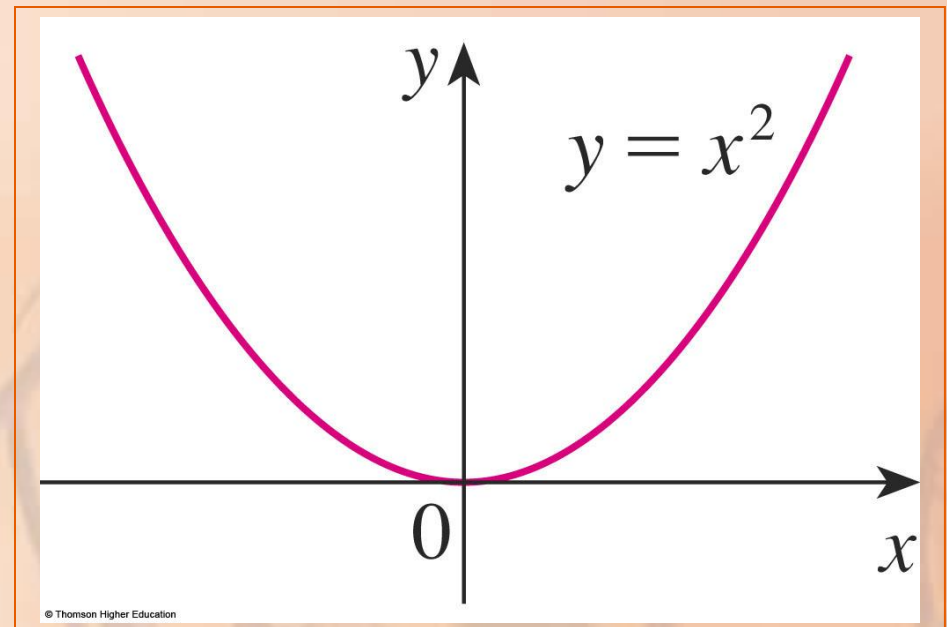
- Therefore, $f(0) = 0$ is the absolute (and local) minimum value of f.

This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$.

- However, there is no highest point on the parabola.
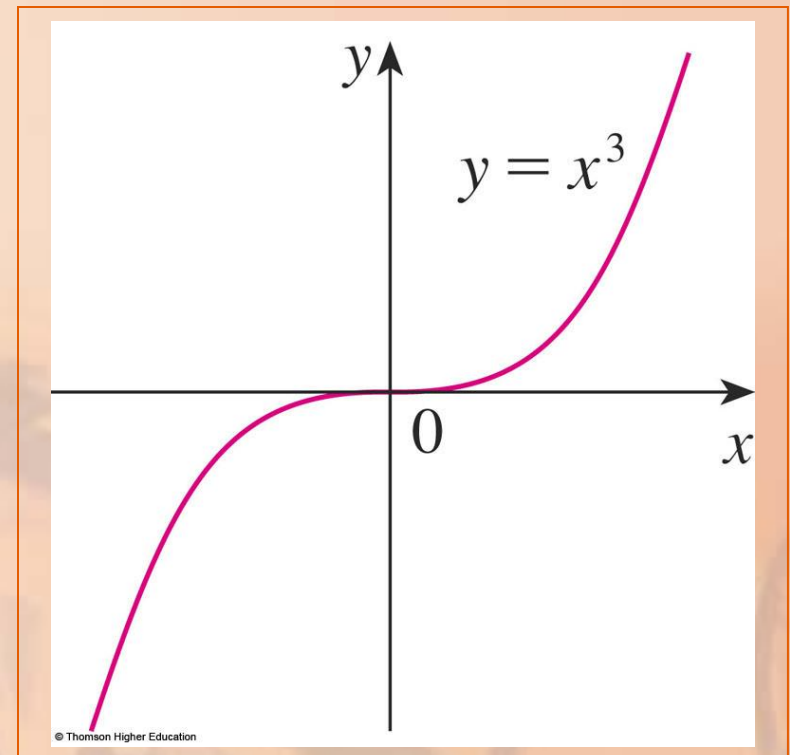- So, this function has no maximum value.

From the graph of the function $f(x) = x^3$, we see that this function has neither an absolute maximum value nor an absolute minimum value.

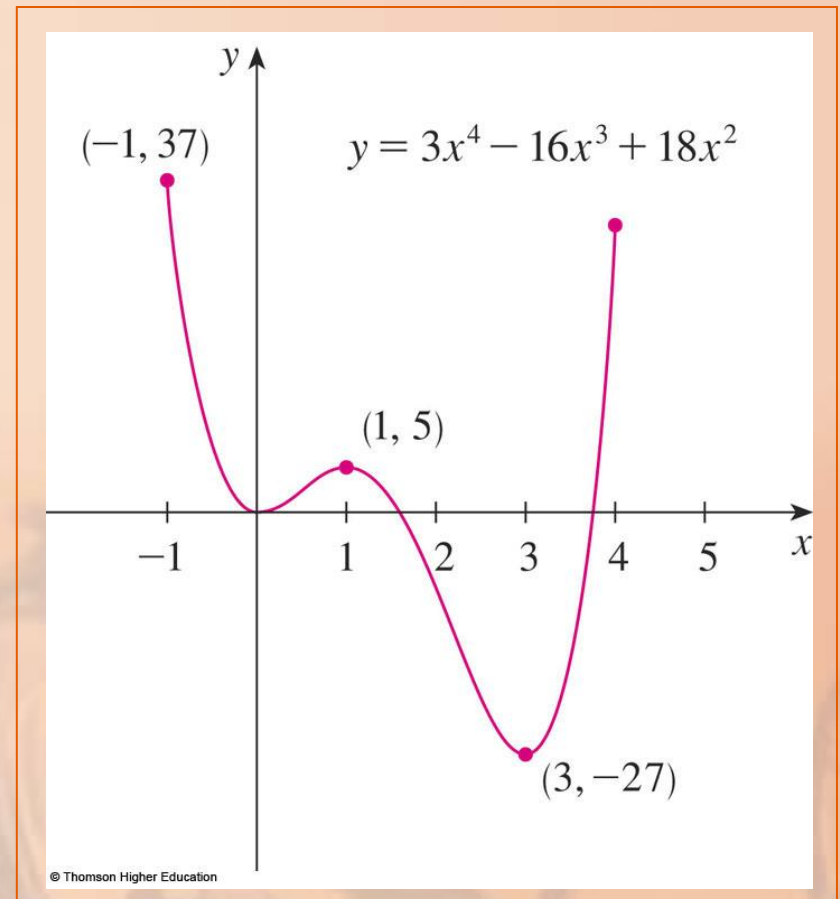- In fact, it has no local extreme values either.

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The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \le x \le 4$$
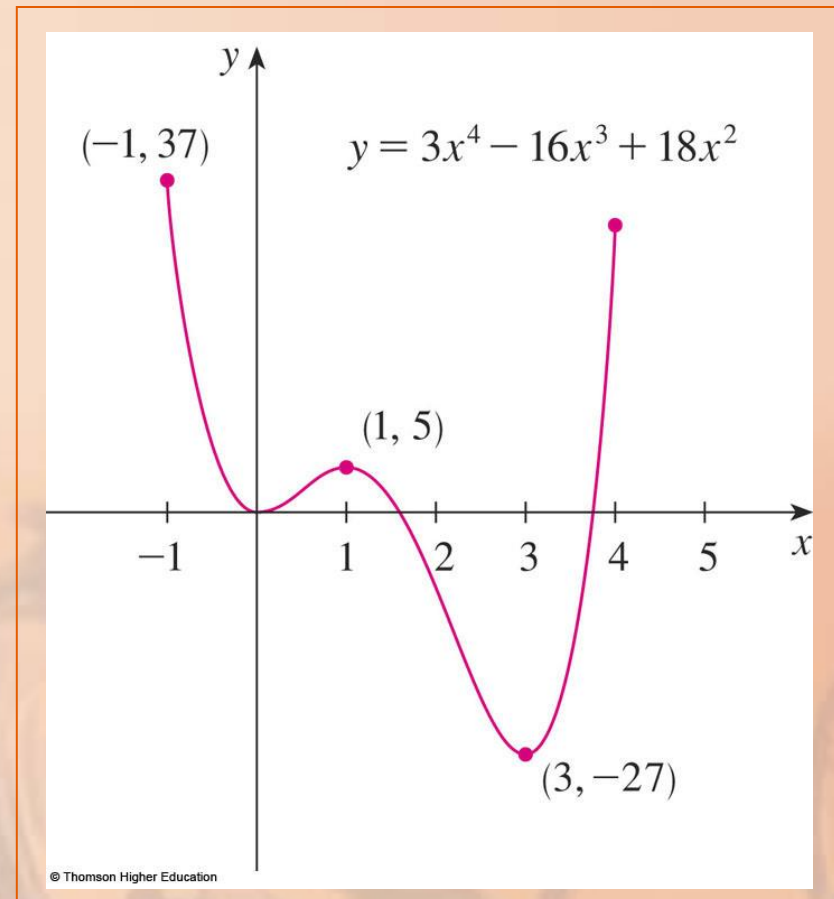
is shown here.

$(-1, 37)$

$y = 3x^4 - 16x^3 + 18x^2$

$(1, 5)$

$(3, -27)$

You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$.

- This absolute maximum is not a local maximum because it occurs at an endpoint.
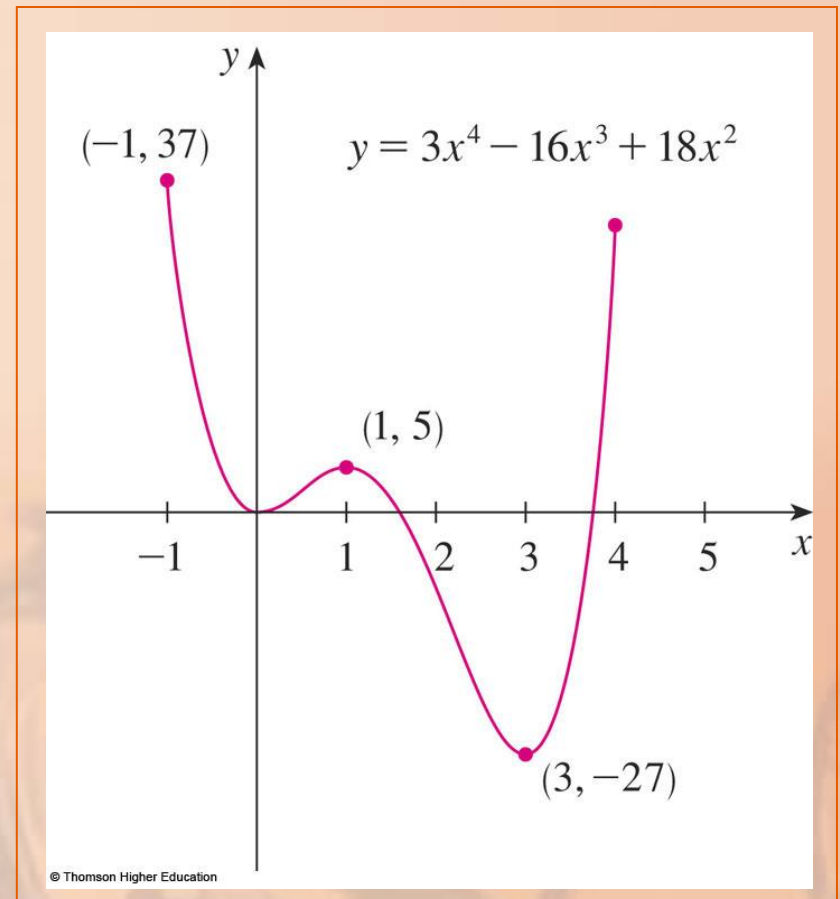

$(-1, 37)$

$y = 3x^4 - 16x^3 + 18x^2$

$(1, 5)$

$(3, -27)$

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Also, $f(0) = 0$ is a local minimum and $f(3) = \text{-}27$ is both a local and an absolute minimum.

- Note that f has neither a local nor an absolute maximum at $x = 4$.

$(-1, 37)$

$y = 3x^4 - 16x^3 + 18x^2$

$(1, 5)$

$(3, -27)$

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We have seen that some functions have extreme values, whereas others do not.

The following theorem gives conditions under which a function is guaranteed to possess extreme values.
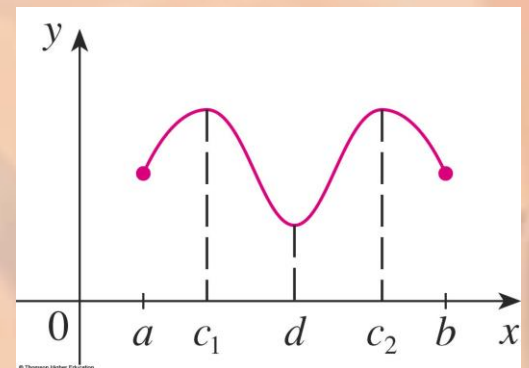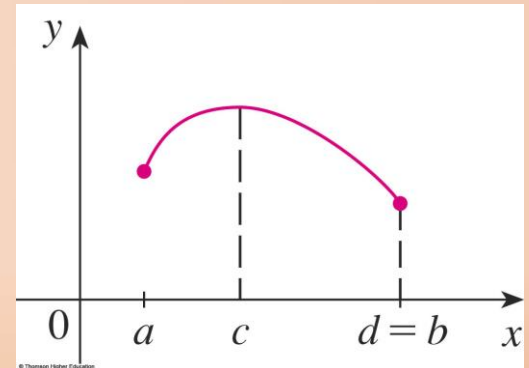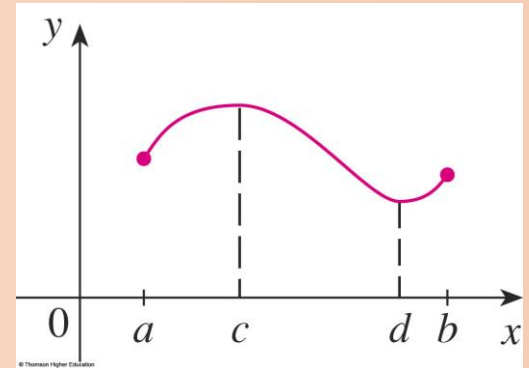
EXTREME VALUE THEOREM   **Theorem 3**

If *f* is continuous on a closed interval [*a*, *b*], then *f* attains an absolute maximum value *f*(*c*) and an absolute minimum value *f*(*d*) at some numbers *c* and *d* in [*a*, *b*].

The theorem is illustrated in the figures.

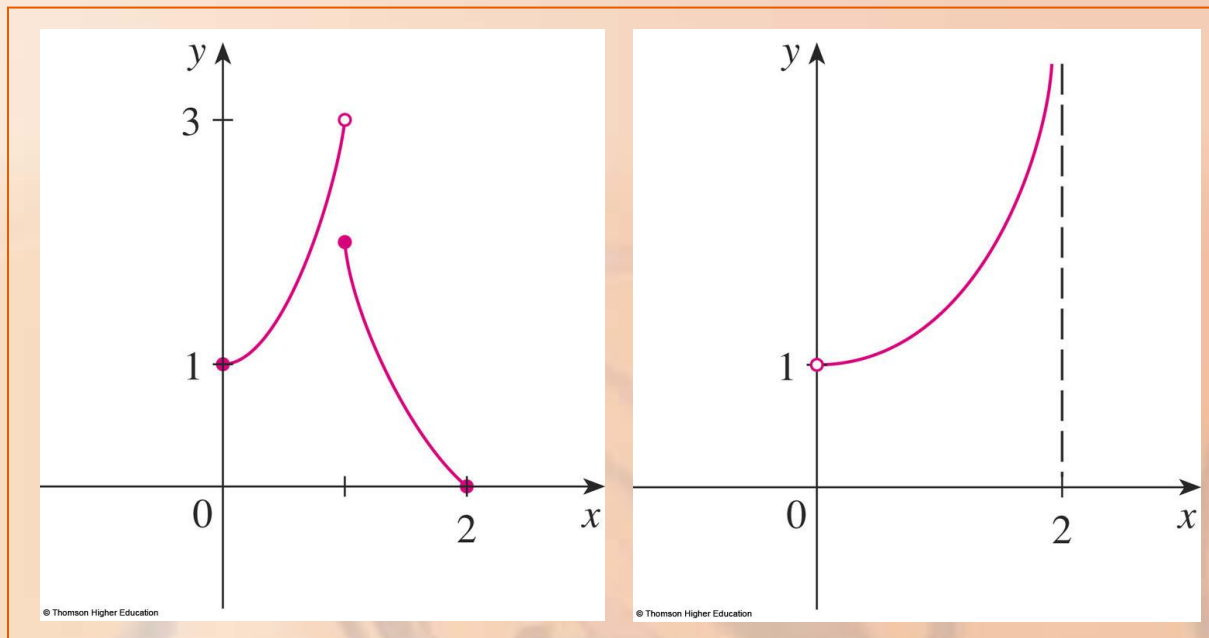- Note that an extreme value can be taken on more than once.

Although the theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

The figures show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the theorem.

The function *f* whose graph is shown is defined on the closed interval [0, 2] but has no maximum value.

- Notice that the range of *f* is [0, 3).

- The function takes on values arbitrarily close to 3, but never actually attains the value 3.

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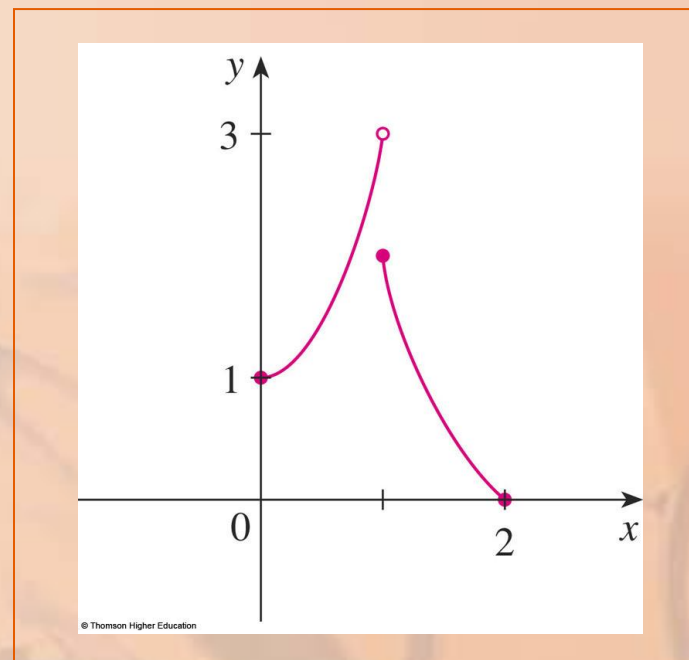
This does not contradict the theorem because *f* is not continuous.
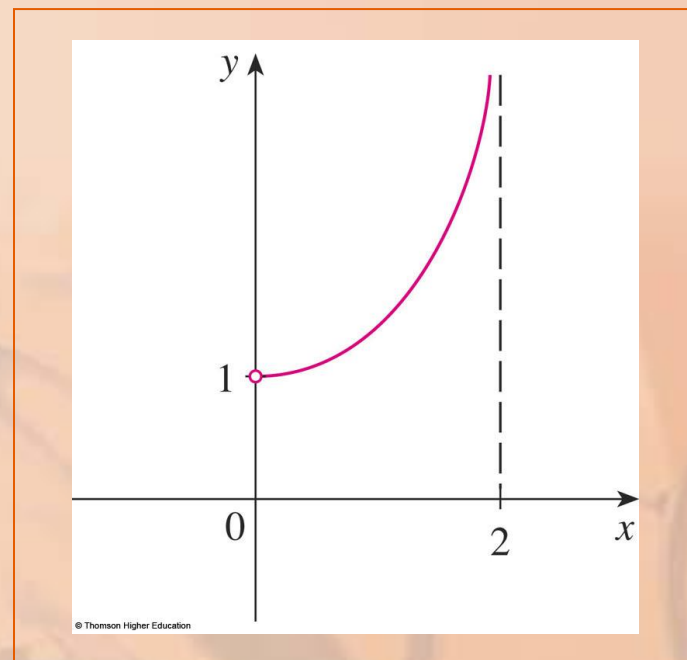
- Nonetheless, a discontinuous function could have maximum and minimum values.
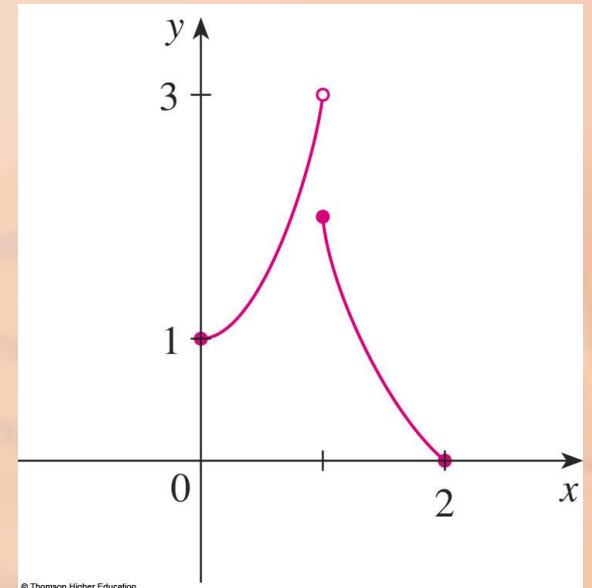
The function *g* shown here is continuous on the open interval (0, 2) but has neither a maximum nor a minimum value.

- The range of *g* is (1, ∞).
- The function takes on arbitrarily large values.
- This does not contradict the theorem because the interval (0, 2) is not closed.

EXTREME VALUE THEOREM

The theorem says that a continuous function on a closed interval has a maximum value and a minimum value.

However, it does not tell us how to find these extreme values.

- We start by looking for local extreme values.

The figure shows the graph of a function f with a local maximum at c and a local minimum at d.
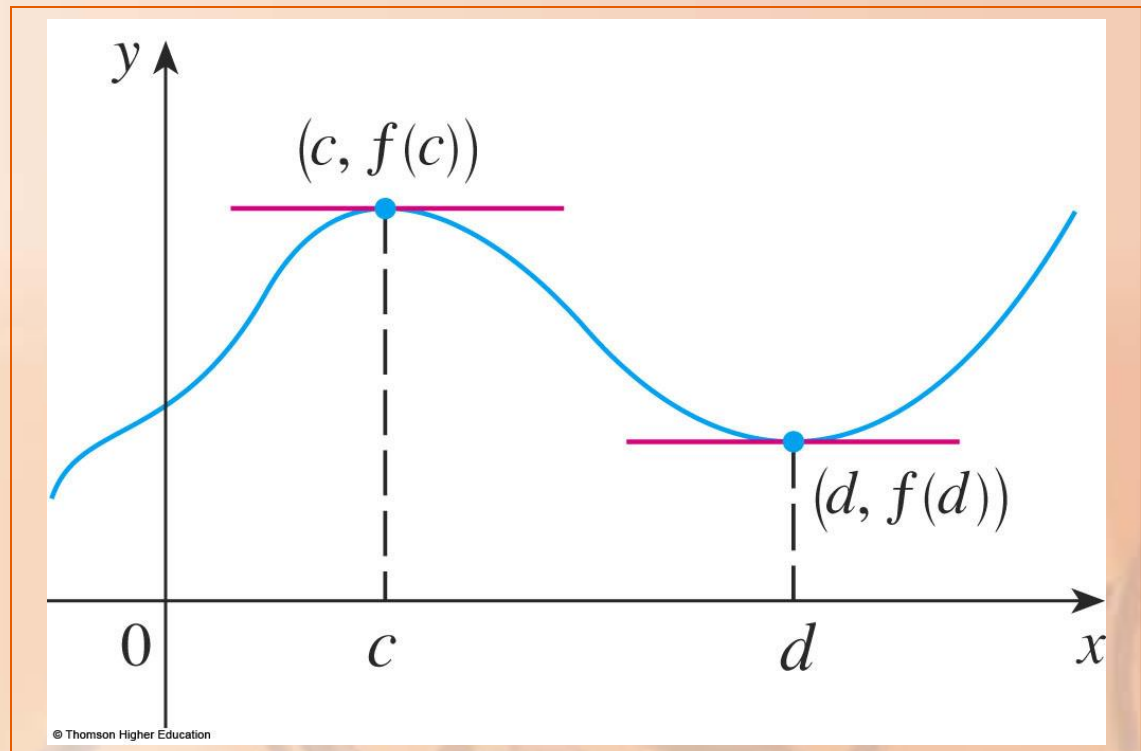
It appears that, at the maximum and minimum points, the tangent lines are horizontal and therefore each has slope 0.
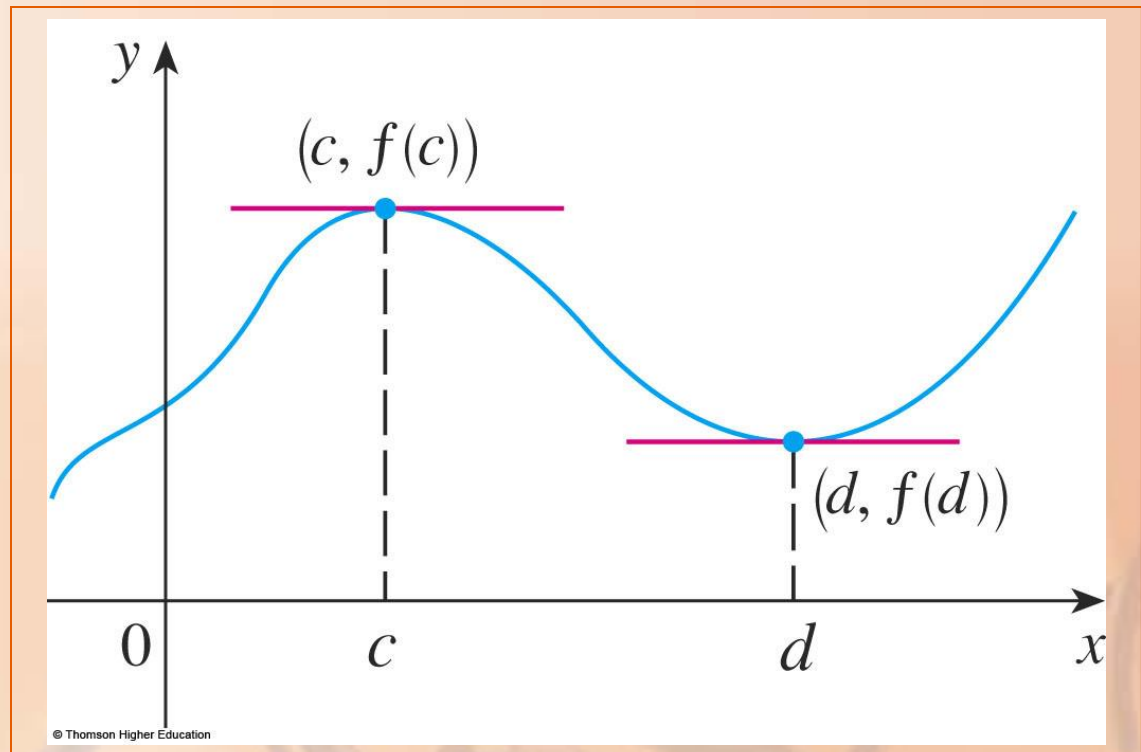
We know that the derivative is the slope of the tangent line.

- So, it appears that $f'(c) = 0$ and $f'(d) = 0$.
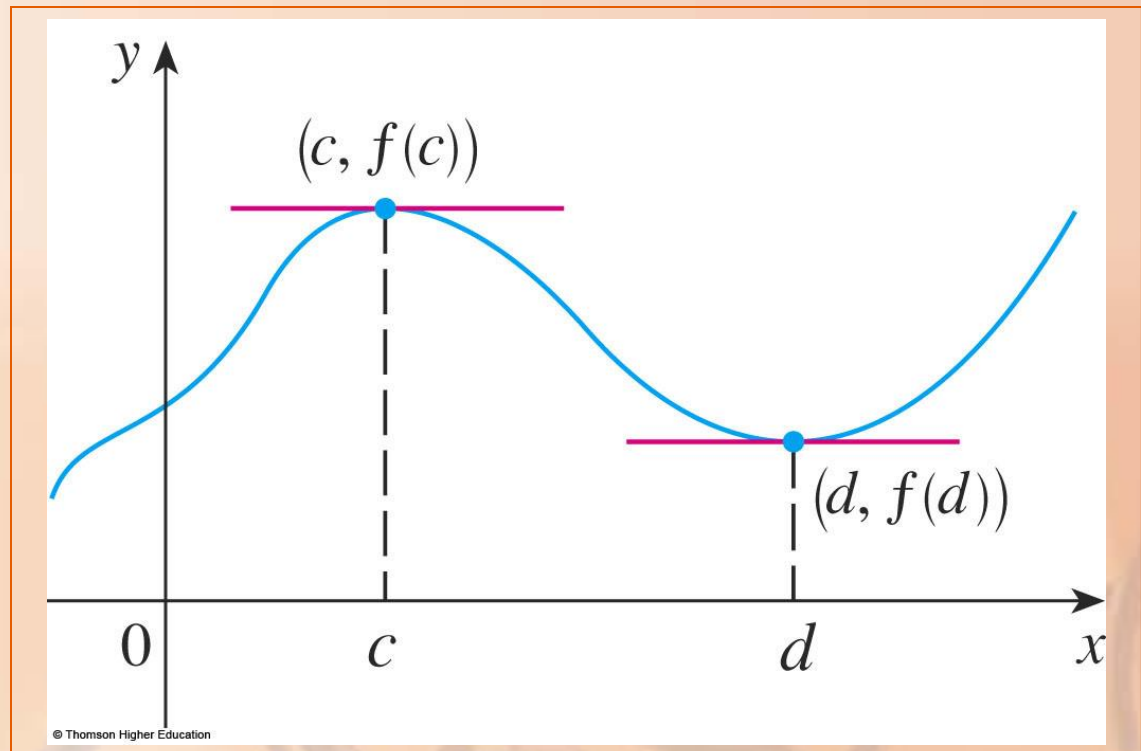
The following theorem says that this is always true for differentiable functions.

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Suppose, for the sake of definiteness, that f has a local maximum at c.

Then, according to Definition 2, $f(c) \geq f(x)$ if x is sufficiently close to c.

This implies that, if *h* is sufficiently close to 0, with *h* being positive or negative, then

$$f(c) \geq f(c + h)$$

and therefore

$$f(c + h) - f(c) \leq 0$$

We can divide both sides of an inequality by a positive number.

- Thus, if $h > 0$ and h is sufficiently small, we have:

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2 in Section 2.3), we get:

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0$$

However, since *f '*(*c*) exists,

we have:

$$f\,'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

- So, we have shown that *f '*(*c*) ≤ 0.

If $h < 0$, then the direction of the inequality in Equation 5 is reversed when we divide by h:

$$\frac{f(c+h) - f(c)}{h} \geq 0 \qquad h < 0$$

So, taking the left-hand limit,

we have:

$$f\,'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

We have shown that $f'(c) \geq 0$ and also that $f'(c) \leq 0$.

Since both these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved the theorem for the case of a local maximum.

- The case of a local minimum can be proved in a similar manner.

- Alternatively, we could use Exercise 76 to deduce it from the case we have just proved.

The following examples caution us against reading too much into the theorem.

- We can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

- However, f has no maximum or minimum at 0—as you can see from the graph.

- Alternatively, observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.

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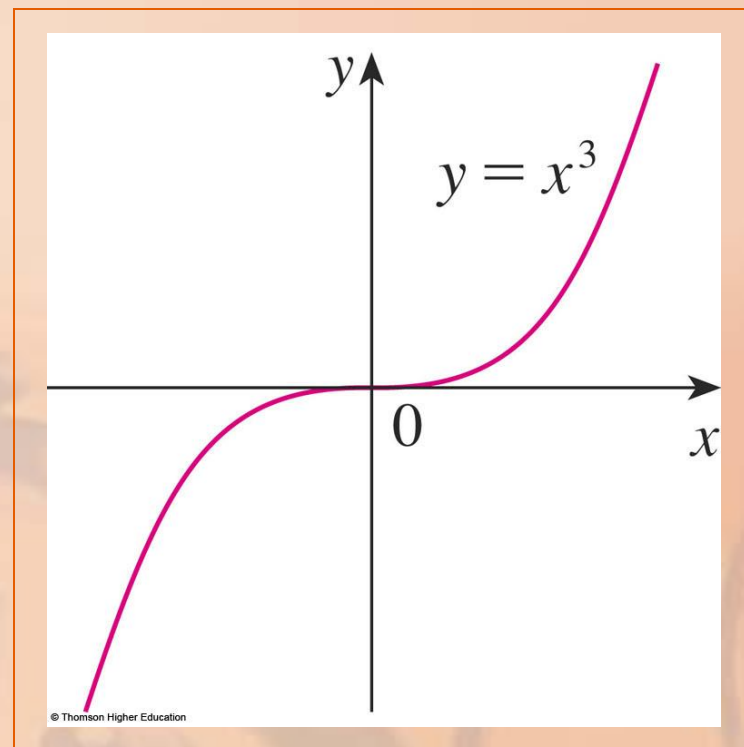
The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $(0, 0)$.

- Instead of having a maximum or minimum at $(0, 0)$, the curve crosses its horizontal tangent there.
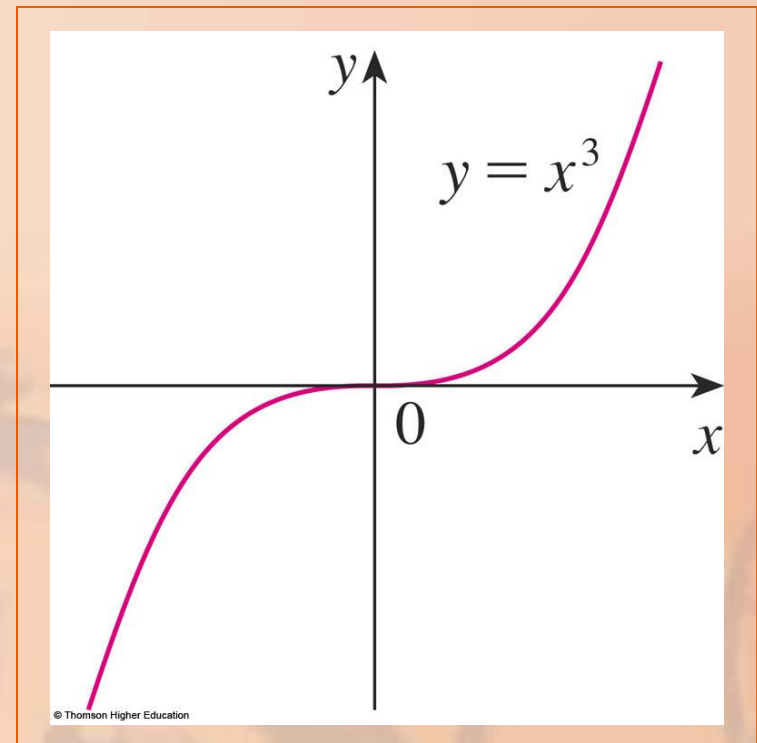
The function $f(x) = |x|$ has its (local and absolute) minimum value at 0.

- However, that value can't be found by setting $f'(x) = 0$.

- This is because—as shown in Example 5 in Section 2.8—$f'(0)$ does not exist.

$y = |x|$

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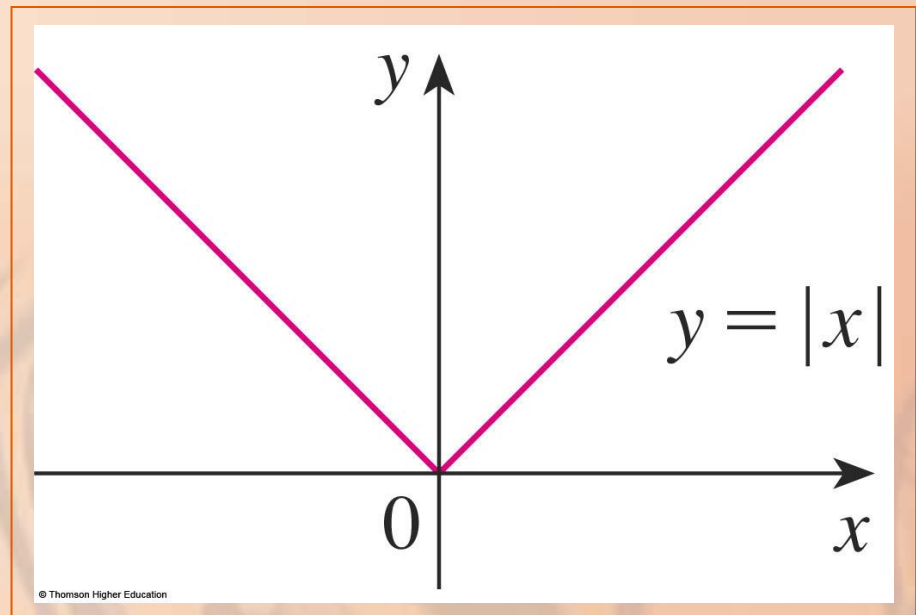
Examples 5 and 6 show that we must be careful when using the theorem.

- Example 5 demonstrates that, even when $f\,'(c) = 0$, there need not be a maximum or minimum at c.

- In other words, the converse of the theorem is false in general.

- Furthermore, there may be an extreme value even when $f\,'(c)$ does not exist (as in Example 6).

The theorem does suggest that we should at least start looking for extreme values of f at the numbers c where either:

- $f\,'(c) = 0$

- $f\,'(c)$ does not exist

Such numbers are given a special name—critical numbers.

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Find the critical numbers of

$$f(x) = x^{3/5}(4 - x).$$

- The Product Rule gives:

$$f'(x) = x^{3/5}(-1) + (4 - x)(\tfrac{3}{5}x^{-2/5})$$

$$= -x^{3/5} + \frac{3(4-x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4-x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

- The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.

- Therefore, $f'(x) = 0$ if $12 - 8x = 0$.

- That is, $x = \dfrac{3}{2}$, and $f'(x)$ does not exist when $x = 0$.

- Thus, the critical numbers are $\dfrac{3}{2}$ and 0.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows (compare Definition 6 with Theorem 4).

If *f* has a local maximum or minimum at *c*, then *c* is a critical number of *f*.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either:

- It is local (in which case, it occurs at a critical number by Theorem 7).

- It occurs at an endpoint of the interval.

Therefore, the following three-step procedure always works.

To find the absolute maximum and minimum values of a continuous function *f* on a closed interval [*a, b*]:

1. Find the values of *f* at the critical numbers of *f* in (*a, b*).
2. Find the values of *f* at the endpoints of the interval.
3. The largest value from 1 and 2 is the absolute maximum value. The smallest is the absolute minimum value.

Find the absolute maximum

and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \le x \le 4$$

As *f* is continuous on [-½, 4], we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

As $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$.

Notice that each of these numbers lies in the interval $(-\frac{1}{2}, 4)$.

The values of f at these critical numbers

are: $f(0) = 1$ $f(2) = -3$

The values of f at the endpoints of the interval
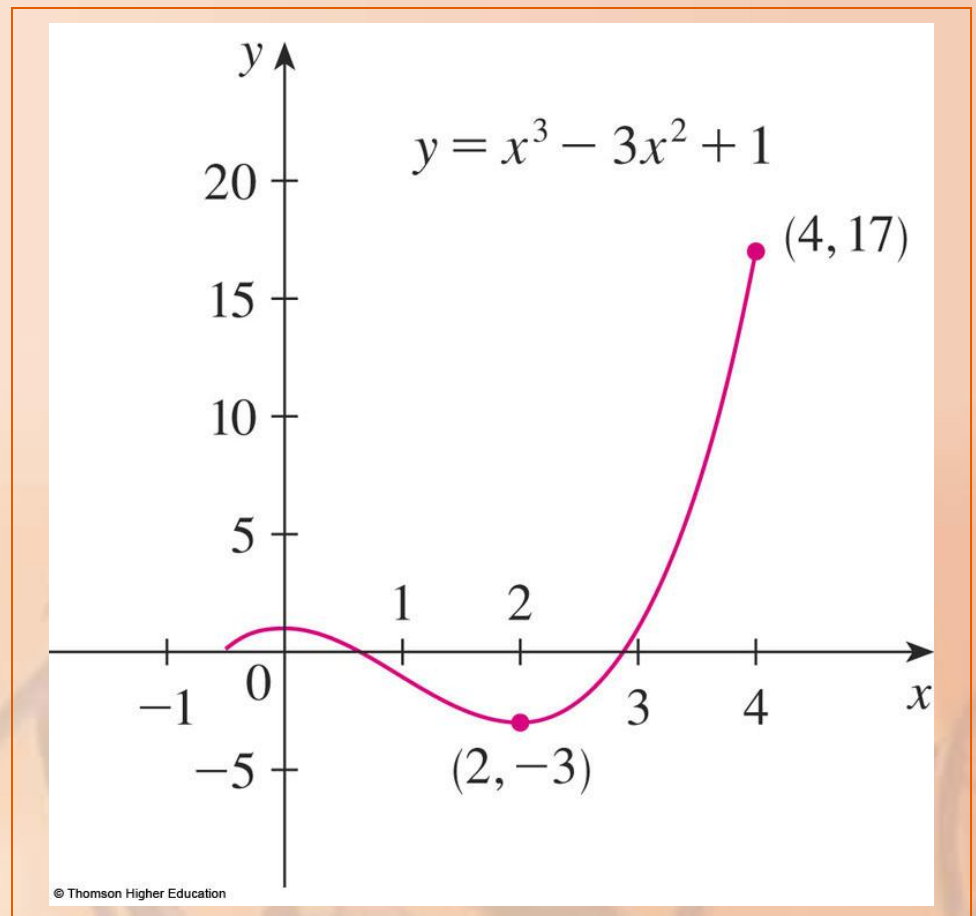
are: $f(-½) = 1/8$ $f(4) = 17$

- Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number.

The graph of *f* is sketched here.

$y = x^3 - 3x^2 + 1$

(4, 17)

(2, −3)

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If you have a graphing calculator or a computer with graphing software, it is possible to estimate maximum and minimum values very easily.

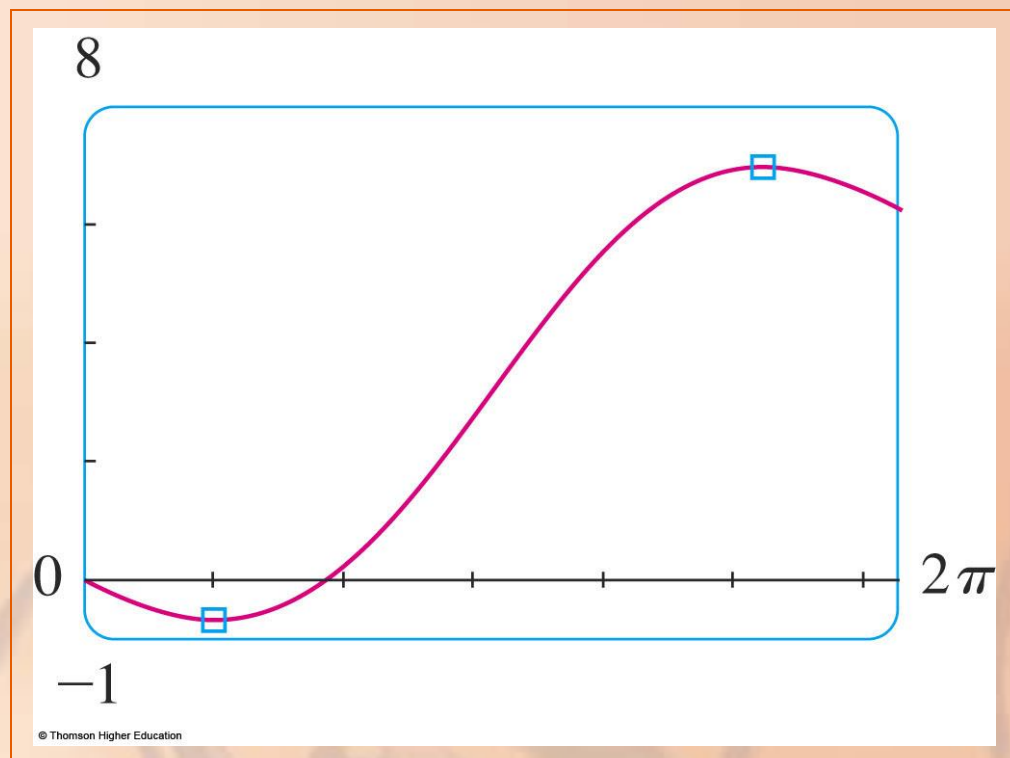- However, as the next example shows, calculus is needed to find the exact values.

a. Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2 \sin x$, $0 \le x \le 2\pi$.

b. Use calculus to find the exact minimum and maximum values.

The figure shows a graph of *f* in
the viewing rectangle [0, 2𝜋] by [-1, 8].
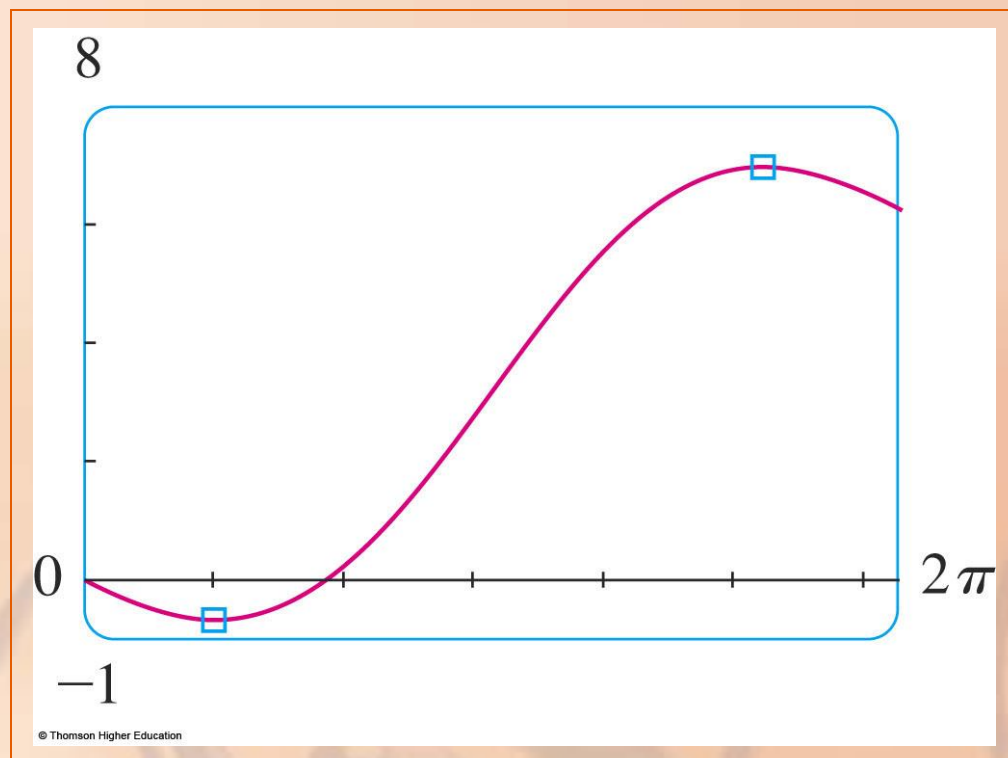
By moving the cursor close to the maximum point, we see the *y*-coordinates don't change very much in the vicinity of the maximum.

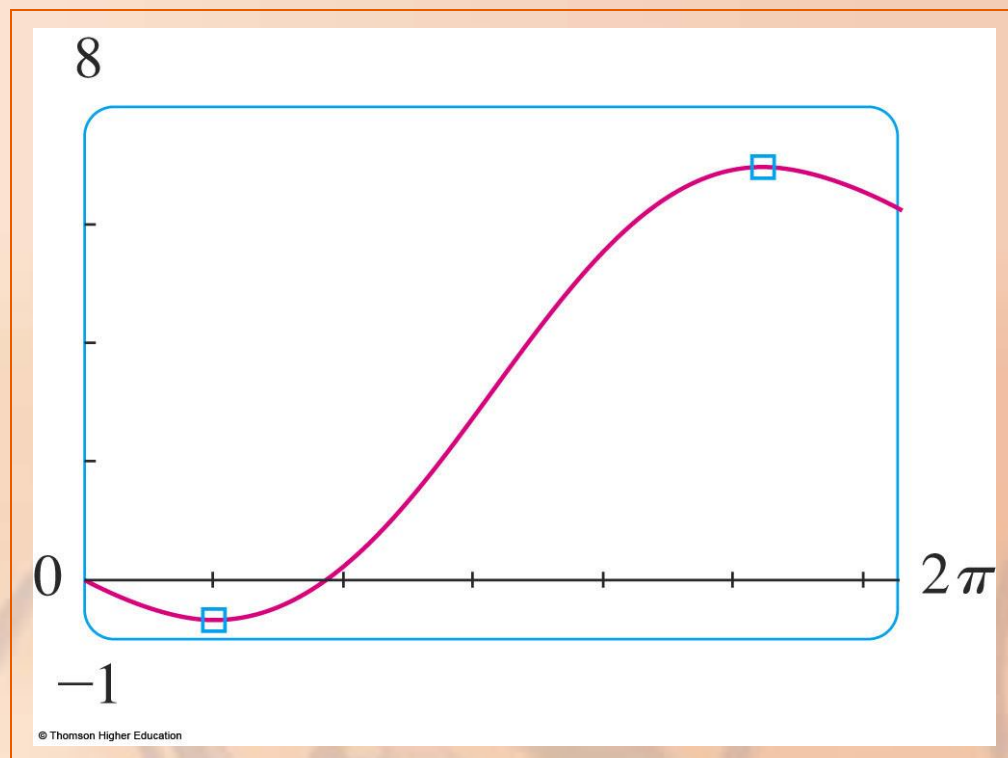- The absolute maximum value is about 6.97

- It occurs when *x* ≈ 5.2

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Similarly, by moving the cursor close to the minimum point, we see the absolute minimum value is about –0.68 and it occurs when $x \approx 1.0$

It is possible to get more accurate estimates by zooming in toward the maximum and minimum points.

However, instead, let's use calculus.

The function $f(x) = x - 2 \sin x$ is continuous on $[0, 2\pi]$.

As $f\,'(x) = 1 - 2 \cos x$, we have $f\,'(x) = 0$ when $\cos x = \frac{1}{2}$.

- This occurs when $x = \pi/3$ or $5\pi/3$.

The values of *f* at these critical points

are

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.684853$$

and

$$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$$

The values of *f* at the endpoints are

$$f(0) = 0$$

and

$$f(2\pi) = 2\pi \approx 6.28$$

Comparing these four numbers and using the Closed Interval Method, we see the absolute minimum value is

$$f(\pi/3) = \pi/3 - \sqrt{3}$$

and the absolute maximum value is

$$f(5\pi/3) = 5\pi/3 + \sqrt{3}$$

- The values from (a) serve as a check on our work.

The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle Discovery.

A model for the velocity of the shuttle during this mission—from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s—is given by:

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second)

Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

We are asked for the extreme values

not of the given velocity function,

but rather of the acceleration function.

So, we first need to differentiate to find the acceleration:

$$a(t) = v'(t)$$

$$= \frac{d}{dt}(0.001302t^3 - 0.09029t^2$$

$$+ 23.61t - 3.083)$$

$$= 0.003906t^2 - 0.18058t + 23.61$$

We now apply the Closed Interval Method to the continuous function *a* on the interval $0 \leq t \leq 126$.

Its derivative is:

$$a'(t) = 0.007812t - 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have:

$a(0) = 23.61$ $a(t_1) \approx 21.52$ $a(126) \approx 62.87$

- The maximum acceleration is about 62.87 ft/s^2.

- The minimum acceleration is about 21.52 ft/s^2.