Q I: SOLVE

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, \; y(p) = 0, \; p > 0 \end{cases}$$

FOR $\lambda > 0$

A Sol:- Writing the characteristic equation of given homogenous differential equation.

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

$$r = \pm\sqrt{-\lambda}$$

$$r = \pm\sqrt{-1 \times \lambda}$$

$$r = \pm i\sqrt{\lambda}$$

$$\alpha = 0, \; \beta = \sqrt{\lambda}$$

The general solution is given by:

$$y(t) = e^{\alpha t}\left[c_1 \cos\beta t + c_2 \sin\beta t\right]$$

$$y(t) = e^{0t}\left[c_1 \cos(\sqrt{\lambda}\,t) + c_2 \sin(\sqrt{\lambda}\,t)\right]$$

$$y(t) = 1\left[c_1 \cos(\sqrt{\lambda}.t) + c_2 \sin(\sqrt{\lambda}.t)\right]$$

$$y(t) = c_1 \cos(\sqrt{\lambda}\,t) + c_2 \sin(\sqrt{\lambda}.t)$$

Now applying initial condition.

$$y(0) = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1(1) + c_2(0)$$

$$\boxed{c_1 = 0}$$

Now $y(P) = c_1 \cos(\sqrt{\lambda}.P) + c_2 \sin(\sqrt{\lambda}.P)$

$$0 = 0 \cos(\sqrt{\lambda}.P) + c_2 \sin(\sqrt{\lambda}.P)$$

$$0 = 0 + c_2 \sin(\sqrt{\lambda}.P)$$

$$c_2 \sin(\sqrt{\lambda}.P) = 0 \quad \text{--- (i)}$$

Now $\sin(\sqrt{\lambda}.P) = 0$

$$\sqrt{\lambda}.P = \sin^{-1}(0)$$

$$\sqrt{\lambda}.P = n\pi$$

$$\sqrt{\lambda} = \frac{n\pi}{P}$$

$$\therefore \lambda = \frac{n^2\pi^2}{P^2}$$

put in (i)

$$c_2 \sin\left(\frac{n\pi}{P}.\right) = 0$$

So general solution can be written as:

$$y(t) = c_2 \sin\left(\frac{n\pi}{P}\right)$$

For $\lambda < 0$

$$\begin{cases} y'' + (-\lambda)y = 0 \\ y(0) = 0,\ y(P) = 0,\ P > 0 \end{cases}$$

$$y'' - \lambda y = 0$$

$$r^2 - \lambda = 0$$

$$r^2 = \lambda$$

$$r = \pm\sqrt{\lambda}$$

$$\therefore\ y(t) = c_1 e^{\sqrt{\lambda}.t} + c_2 t e^{\sqrt{\lambda}.t}$$

Now applying initial conditions.

$$y(0) = c_1 e^0 + c_2 (0) e^0$$

$$0 = c_1 + 0$$

$$\boxed{c_1 = 0}$$

$$y(P) = c_1 e^{\sqrt{\lambda}.P} + c_2 (P) e^{\sqrt{\lambda}.P}$$

$$0 = 0 + c_2 (P) e^{\sqrt{\lambda}.P}$$

$$\therefore \boxed{c_2 = 0}$$

Only trivial solution exists.

So $y = 0$

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For $\lambda = 0$

$$y'' + (0)y = 0$$

$$y'' = 0$$

$$\therefore \; y = 0$$

QUESTION No. II :-

Verify that all singular points of the DE are regular singular points.

a) $y'' - xy = 0$

Ans:- $x = 0$ is the singular point of given differential equation.

$$\lim_{x \to 0} (x-0)P(x) < \infty$$

$$\lim_{x \to 0} (x)(0) < \infty$$

$$0 < \infty$$

$$\lim_{x \to 0} (x-0)^2 Q(x) < \infty$$

$$\lim_{x \to 0} (x)^2(-x) < \infty$$

$$\lim_{x \to 0} -x^3 < \infty$$

$$0 < \infty$$

Since in both case we get a finite number as $x \to 0$ $\therefore$ $x = 0$ is a regular singular point.

$$\frac{1}{2}c_2 = 3$$

$$\boxed{c_2 = 6}$$

So $x(t) = e^{-t}\left[(0)\cos\left(\frac{1}{2}t\right) + (6)\sin\left(\frac{1}{2}t\right)\right]$

$$x(t) = e^{-t}\left[0 + 6\sin\left(\tfrac{1}{2}t\right)\right]$$

$$\boxed{x(t) = 6.e^{-t}.\sin\left(\tfrac{1}{2}t\right)}$$

QUESTION No. VI :

Solve by using the operator method.

a) $X' = \begin{bmatrix} -3 & 7 \\ -1 & 5 \end{bmatrix} X + \begin{bmatrix} \cos t \\ e^{-4t} \end{bmatrix}$

Sol:- $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos t \\ e^{-4t} \end{bmatrix}$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3x & 7y \\ -x & 5y \end{bmatrix} + \begin{bmatrix} \cos t \\ e^{-4t} \end{bmatrix}$$

$$\begin{cases} x' = -3x + 7y + \cos t \\ y' = -x + 5y + e^{-4t} \end{cases}$$

$$\begin{cases} x' + 3x - 7y = \cos t \\ y' - 5y + x = e^{-4t} \end{cases}$$

Writing in operator notation

$$x(D+3) - 7y = \cos t \quad \text{—(i)}$$
$$y(D-5) + x = e^{-4t} \quad \text{—(ii)}$$

Multiplying $(D+3)$ with eq (ii)

$$y(D+3)(D-5) + (D+3)x = (D+3)e^{-4t}$$
$$y(D^2+3D-5D-15) + (D+3)x = -4e^{-4t} + 3e^{-4t}$$
$$y(D^2-2D-15) + (D+3)x = -e^{-4t}$$
$$x(D+3) + y(D^2-2D-15) = -e^{-4t} \quad \text{—(iii)}$$

Subtracting eq (iii) from eq (i)

$$x(D+3) - 7y = \cos t$$
$$\underline{\pm x(D+3) \pm y(D^2-2D-15) = \mp e^{-4t}}$$
$$-7y - y(D^2-2D-15) = \cos t + e^{-4t}$$
$$-7y - y'' - 2y' - 15y = \cos t + e^{-4t}$$
$$-y'' - 2y' - 22y = \cos t + e^{-4t}$$

$y(t) = y_n(t) + y_p(t)$

FINDING $Y_n(t)$

$$-y'' - 2y' - 22y = 0$$
$$-\gamma^2 - 2\gamma - 22 = 0$$
$$\gamma^2 + 2\gamma + 22 = 0$$
$$\frac{-(2) \pm \sqrt{(2)^2 - 4(1)(22)}}{} = \frac{-2 \pm \sqrt{4-88}}{2}$$

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$$r = \frac{-2 \pm \sqrt{-84}}{2} = \frac{-2 \pm \sqrt{-1 \times 4 \times 21}}{2}$$

$$r = \frac{-2 \pm 2i\sqrt{21}}{2}$$

$$r = -1 \pm i\sqrt{21}$$

$$\alpha = -1, \quad \beta = \sqrt{21}$$

$$y_h(t) = e^{-t}\left[c_1 \cos\sqrt{21}t + c_2 \sin\sqrt{21}t\right]$$

FINDING $Y_p(t)$:-

Let

$$y_p(t) = A\cos t + B\sin t + Ce^{-4t}$$

$$y_p'(t) = -A\sin t + B\cos t - 4Ce^{-4t}$$

$$y_p''(t) = -A\cos t - B\sin t + 16Ce^{-4t}$$

$$-(-A\cos t - B\sin t + 16(e^{-4t}) - 2(-A\sin t + B\cos t - 4(e^{-4t}) -$$
$$22(A\cos t + B\sin t + Ce^{-4t}) = \cos t + e^{-4t}$$

$$(A\cos t + B\sin t - 16Ce^{-4t}) + (2A\sin t - 2B\cos t + 8Ce^{-4t}) +$$
$$(22A\cos t + 22B\sin t \; 22Ce^{-4t}) = \cos t + e^{-4t}$$

$$(A - 2B + 22A)\cos t + (B + 2A + 22B)\sin t + (-16C + 8C + 22C)e^{-4t} = \cos t + e^{-4t}$$

$$(-2B + 23A)\cos t + (2A + 23B)\sin t + (14C)e^{-4t} = 1\cos t + e^{-4t} + 0\sin t$$

Comparing co-efficents of $\cos t$, e^{-4t} and $\sin t$

$$-2B + 23A = 1$$
$$2A + 23B = 0 \rightarrow \boxed{A = -\frac{23}{2}B}$$
$$14C = 1 \rightarrow \boxed{C = \frac{1}{14}}$$

$$-2B+23\left(-\frac{23}{2}B\right)=1$$

$$\frac{-2B}{1}-\frac{529B}{2}=1$$

$$\frac{-4B-529B}{2}=1$$

$$-4B-529B=2$$

$$\cancel{-4B=2+529}$$

$$\cancel{-4B=531}$$
$$\cancel{\frac{-4B}{-4}=\frac{531}{-4}}$$

$$\boxed{\cancel{B=\frac{531}{4}}}$$

$$-533B=2$$

$$\boxed{B=-\frac{2}{533}}$$

Also $A=-\frac{23}{2}B$

Put value of B.

$$A=-\frac{23}{\cancel{2}}\left(-\frac{\cancel{2}}{533}\right)$$

$$\boxed{A=\frac{23}{533}}$$

Now $y_p(t)=\frac{23}{533}\cos t-\frac{2}{533}\sin t+\frac{1}{14}e^{-4t}$

The general solution is given by:

$$y(t)=e^{-t}\left[c_1\cos\sqrt{21}t+c_2\sin\sqrt{21}t\right]+\frac{23}{533}\cos t-\frac{2}{533}\sin t+$$

$$r = \frac{-2 \pm \sqrt{-84}}{2} = \frac{-2 \pm \sqrt{-1 \times 4 \times 21}}{2}$$

$\frac{1}{14} e^{-4t}$

Now we shell find $x(t)$ by using operator method.

eq(i) $\Rightarrow$ $x(D+3) - 7y = \cos t$

eq(ii) $\Rightarrow$ $y(D-5) + x = e^{-4t}$

Multipeying $(D-5)$ with eq(i) and 7 with eq(ii)

eq(i) $\Rightarrow$ $x(D-5)(D+3) - 7(D-5)y = (D-5)\cos t$

$x(D^2 - 5D + 3D - 15) - 7(D-5)y = -\sin t - 5\cos t$

$x(D^2 - 2D - 15) - 7(D-5)y = -\sin t - 5\cos t$ — (iv)

eq(ii) $\Rightarrow$ $7y(D-5) + 7x = 7e^{-4t}$ — (v)

Adding eq(iv) and eq(v)

$x(D^2 - 2D - 15) - \cancel{7(D-5)y} = -\sin t - 5\cos t$

$+ 7x + \cancel{7(D-5)y} = 7e^{-4t}$

$x(D^2 - 2D - 15) + 7x = -\sin t - 5\cos t + 7e^{-4t}$

$x'' - 2x' - 15x + 7x = -\sin t - 5\cos t + 7e^{-4t}$

$x'' - 2x' - 8x = -\sin t - 5\cos t + 7e^{-4t}$

Solving in rough work and directly writing the gen. solutoon

$$x(t) = c_1 e^{4t} + c_2 e^{-2t} + \frac{19}{85}\sin(t) + \frac{43}{85}\cos t + \frac{7}{16} e^{-4t}$$

Now the Solution of the System is given by

$$\begin{cases} x(t) = c_1 e^{4t} + c_2 e^{-2t} + \frac{19}{85}\sin(t) + \frac{43}{85}\cos t + \frac{7}{16} e^{-4t} \\ y(t) = e^{-t}\left[c_1 \cos\sqrt{21}\,t + c_2 \sin\sqrt{21}\,t\right] + \frac{28}{533}\cos t - \frac{2}{533}\sin t \\ \quad + \frac{1}{14} e^{-4t} \end{cases}$$

(b) $X' = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} X + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

Ans:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3x - y \\ 4x - y \end{bmatrix} + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$\therefore \quad x'(t) = 3x - y + \cos t$

$\quad y'(t) = 4x - y + \sin t$

$x' - 3x + y = \cos t$

$y' + y - 4x = \sin t$

Writing in operator notation

$x(D-3) + y = \cos t$ —(i)

$y(D+1) - 4x = \sin t$ —(ii)

Multiplying $(D+1)$ with eq(i) and Subtracting eq(ii) from eq(i)

$$x(D+1)(D-3) + \cancel{(D+1)y} = (D+1)\cos t$$
$$\mp 4x \pm \cancel{(D+1)y} = \pm \sin t$$

$$x(D+1)(D-3) + 4x = (D+1)\cos t - \sin t$$

$$x(D^2+D-3D-3) + 4x = -\sin t + \cos t - \sin t$$

$$x(D^2-2D-3) + 4x = -2\sin t + \cos t$$

$$x'' - 2x' - 3x + 4x = -2\sin t + \cos t$$

$$x'' - 2x' + x = -2\sin t + \cos t$$

FINDING $x_h(t)$:-

$$x'' - 2x' + x = 0$$

$$r^2 - 2r + 1 = 0$$

$$r^2 - 2(1)r + (1)^2 = 0$$

$$(r^2 - 1)^2 = 0$$

$$r^2 - 1 = 0$$

$$\boxed{r = 1}$$

$$\therefore \quad x_h(t) = c_1 e^{(1)t} + t c_2 e^{(1)t}$$

$$x_h(t) = c_1 e^t + c_2 t e^t$$

FINDING $x_p(t)$:-

$$x_p(t) = \cancel{x_{p_1}(t)} + \cancel{x_{p_2}(t)}$$

~~Let~~ $\cancel{x_{p_1}(t)} = A \cancel{\sin t + B\sin t}$ $\cos t$ | ~~Let $x_{p_2}(t)$~~ =

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$$x_p(t) = A\sin t + B\cos t$$

$$x_p'(t) = A\cos t - B\sin t$$

$$x_p''(t) = -A\sin t - B\cos t$$

$\therefore$ $(-A\sin t - B\cos t) - 2(A\cos t - B\sin t) + (A\sin t + B\cos t) =$

$-2\sin t + \cos t$

$$-\cancel{A\sin t} - \cancel{B\cos t} - 2A\cos t + 2B\sin t + \cancel{A\sin t} + \cancel{B\cos t} = -2\sin t + \cos t$$

$$-2A\cos t + 2B\sin t = -2\sin t + \cos t$$

Comparing Co-efficients of $\sin t$ and $\cos t$

$-2A = 1$, $2B = -2$

$\boxed{A = -\frac{1}{2}}$, $\boxed{B = -1}$

$\therefore$ $x_p(t) = -\frac{1}{2}\sin t - \cos t$

So general solution is

$$x(t) = c_1 e^t + c_2 t e^t - \frac{1}{2}\sin t - \cos t$$

Now eq(i) $\Rightarrow$ $x(D-3) + y = \cos t$

eq(ii) $\Rightarrow$ $y(D+1) - 4x = \sin t$

Multiplying $(D-3)$ with eq(ii) and 4 with eq(i) and adding.

$$4\cancel{x(D-3)} + 4y = 4\cos t$$

$$y(D+1)(D-3) - \cancel{4(D-3)x} = (D-3)\sin t$$

$$y(D+1)(D-3) + 4y = 4\cos t + (D-3)\sin t$$

$$y(D^2+D-3D-3)+4y=4\cos t+\cos t-3\sin t$$

$$y(D^2-2D-3)+4y=5\cos t-3\sin t$$

$$y''-2y'-3y+4y=5\cos t-3\sin t$$

$$y''-2y'+y=5\cos t-3\sin t$$

After perform same steps in previous equation the general solution is given by:-

$$y(t)=c_1e^t+c_2te^t-\frac{5}{2}\sin t-\frac{3}{2}\cos t$$

$$\begin{cases} x(t)=c_1e^t+c_2te^t-\frac{1}{2}\sin t-\cos t \\ y(t)=c_1e^t+c_2te^t-\frac{5}{2}\sin t-\frac{3}{2}\cos t \end{cases}$$

QUESTION No. VIII :-

$$X' = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} X \quad , \quad X(0) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Ans Sol:- Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

FINDING EIGENVALUES :-

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(1-\lambda)(1-\lambda) - (0)(0)\right] - 0\left[(-2)(1-\lambda) - (0)(-2)\right]$$

$$+1\left[(-2)(0) - (-2)(1-\lambda)\right] = 0$$

$$(4-\lambda)(1-\lambda)^2 - 0 + 1(2(1-\lambda)) = 0$$

$$(4-\lambda)(1+\lambda^2 - 2\lambda) + 2 - 2\lambda = 0$$

$$4 + 4\lambda^2 - 8\lambda - \lambda + \lambda^3 + 2\lambda^2 + 2 - 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

By inspection method, we know that $(\lambda - 1)$ is a factor of above eqation

Now using synthetic division to find other roots

	-1	6	-11	6
1		-1	5	-6
	-1	5	-6	0

$$-\lambda^2 + 5\lambda - 6 = 0$$

OR $\lambda^2 - 5\lambda + 6 = 0$

$$\lambda^2 - 2\lambda - 3\lambda + 6 = 0$$

$$\lambda(\lambda - 2) - 3(\lambda - 2) = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

So all three roots are

$$(\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$$

So $\lambda_1 = 1, \; \lambda_2 = 2, \; \lambda_3 = 3$

Now for each λ we find it's corresponding eigenvector.

FINDING EIGENVECTORS :-

① For $\lambda = 1$:

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It's Augmented matrix B.

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 2/3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 + \frac{2}{3}R_1 \\ R_3 + \frac{2}{3}R_1 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \frac{3}{2}R_2 \\ \frac{3}{2}R_3 \end{array}$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_2 - R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_1 - R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \frac{1}{3}R_1$$

The eigenvector V_1 is now given by:

$$V_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

② For $\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It's Augmented Matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \end{matrix}$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} \frac{1}{2} R_1 \\ -1 R_2 \end{matrix}$$

$$1x_2 + 0y_2 + \frac{1}{2} z_2 = 0$$

$$0x_2 + 1y_2 - z_2 = 0$$

$$0x_2 + 0y_2 + 0z_2 = 0$$

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So $x_2 + \frac{1}{2} z_2 = 0 \Rightarrow x_2 = -\frac{1}{2} z_2$

$y_2 - z_2 = 0 \Rightarrow y_2 = z_2$

$$\therefore \quad V_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

(3) For $\lambda = 3$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It's Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \quad R_2 + \frac{1}{2} R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \quad R_3 - R_1$$

$$\underset{\sim}{R} \left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 + \frac{1}{2}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \frac{1}{2}R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 + 2R_2$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -\frac{1}{2}R_1$$

$\therefore$ $x_3 + z_3 = 0$

$y_3 - z_3 = 0$

So $x_3 = y_3$

$x_3 = -z_3$

$y_3 = z_3$

eigenvector, $V_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

The fundamental matrix $\varphi(t)$ is given by

$$\varphi(t) = \begin{bmatrix} 0e^{t} & -1\,e^{2t} & -1\,e^{3t} \\ 1e^{t} & 2\,e^{2t} & 1\,e^{3t} \\ 0e^{t} & 2\,e^{2t} & 1\,e^{3t} \end{bmatrix}$$

$$\varphi(t) = \begin{bmatrix} 0 & -e^{2t} & -e^{3t} \\ e^{t} & 2e^{2t} & e^{3t} \\ 0 & 2e^{2t} & e^{3t} \end{bmatrix}$$

Now $\varphi(0) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

The inverse of $\varphi(0)$ is :

$$\varphi^{-1}(0) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

Now the solution is given by:

$X(t) = \varphi(t) \cdot \varphi^{-1}(0)\, X(0)$

Putting values
we get

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$$X(t) = \begin{bmatrix} 0 & -e^{2t} & -e^{3t} \\ e^{t} & 2e^{2t} & e^{3t} \\ 0 & 2e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 0 - e^{2t} + 2e^{3t} & 0 + 0 + 0 & 0 - e^{2t} + e^{3t} \\ 0 + 2e^{2t} - 2e^{3t} & e^{t} + 0 + 0 & -e^{t} + 2e^{2t} - e^{3t} \\ 0 + 2e^{2t} - 2e^{3t} & 0 + 0 + 0 & 0 + 2e^{2t} - e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} -e^{2t} + 2e^{3t} & 0 & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & e^{t} & -e^{t} + 2e^{2t} - e^{3t} \\ 2e^{2t} - 2e^{3t} & 0 & 2e^{2t} - e^{3t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} (+e^{2t} - 2e^{3t}) + (0) + (0) \\ (-2\cancel{e^{2t}} + 2e^{3t}) + \cancel{2e^{t}} + 0 \\ -2e^{2t} + 2e^{3t} + 0 + 0 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} e^{2t} - 2e^{3t} \\ 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{bmatrix}$$