4

APPLICATIONS OF DIFFERENTIATION

Many applications of calculus depend on our ability to deduce facts about a function *f* from information concerning its derivatives.

4.3

How Derivatives Affect the Shape of a Graph

In this section, we will learn:

How the derivative of a function gives us the direction in which the curve proceeds at each point.

DERIVATIVES AND GRAPH SHAPE

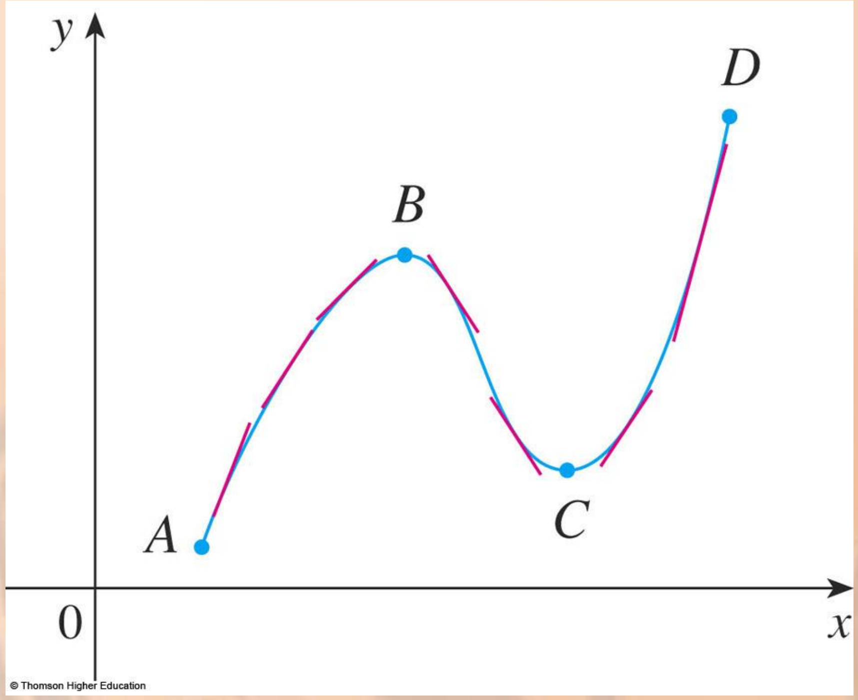
As $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point.

- Thus, it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

WHAT DOES *f'* SAY ABOUT *f* ?

To see how the derivative of *f* can tell us where a function is increasing or decreasing, look at the figure.

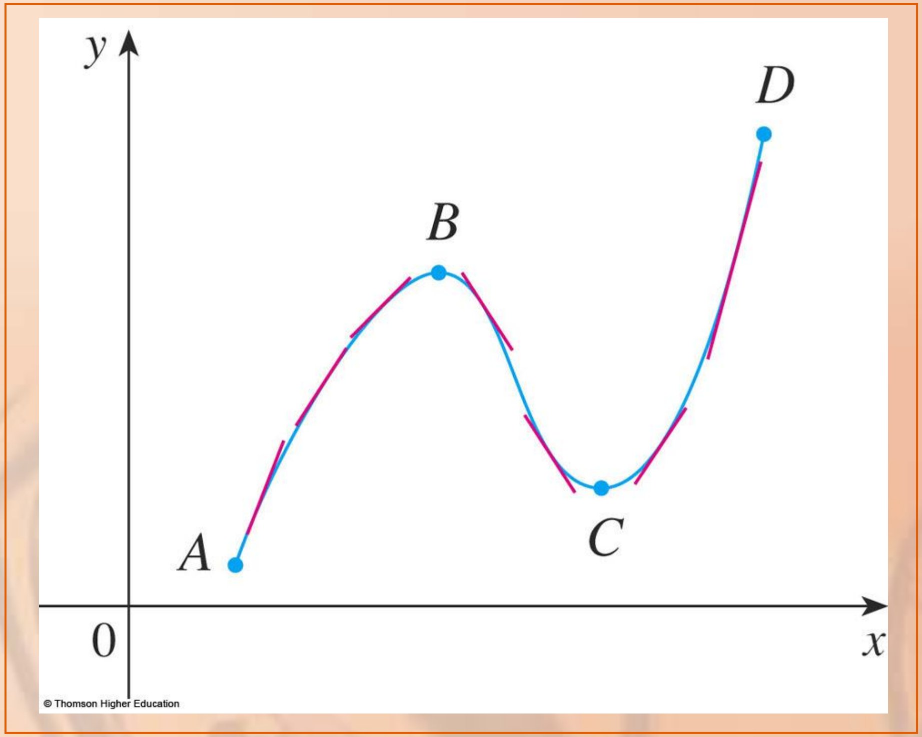
- Increasing functions and decreasing functions were defined in Section 1.1



WHAT DOES f' SAY ABOUT f ?

Between *A* and *B* and between *C* and *D*, the tangent lines have positive slope.

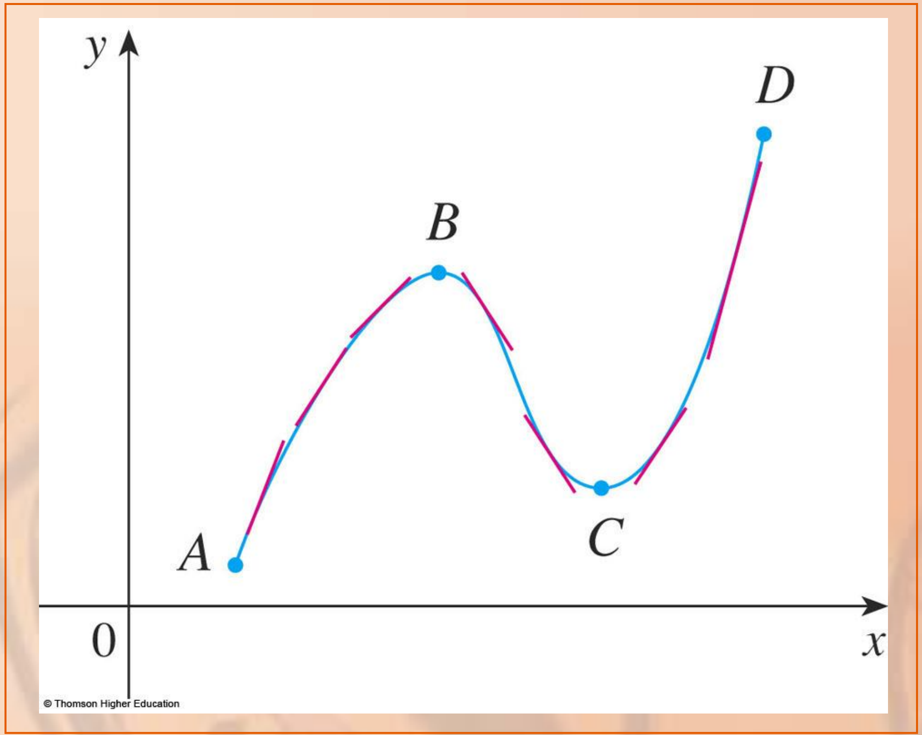
So, $f'(x) > 0$.



WHAT DOES *f'* SAY ABOUT *f* ?

Between *B* and *C*, the tangent lines have negative slope.

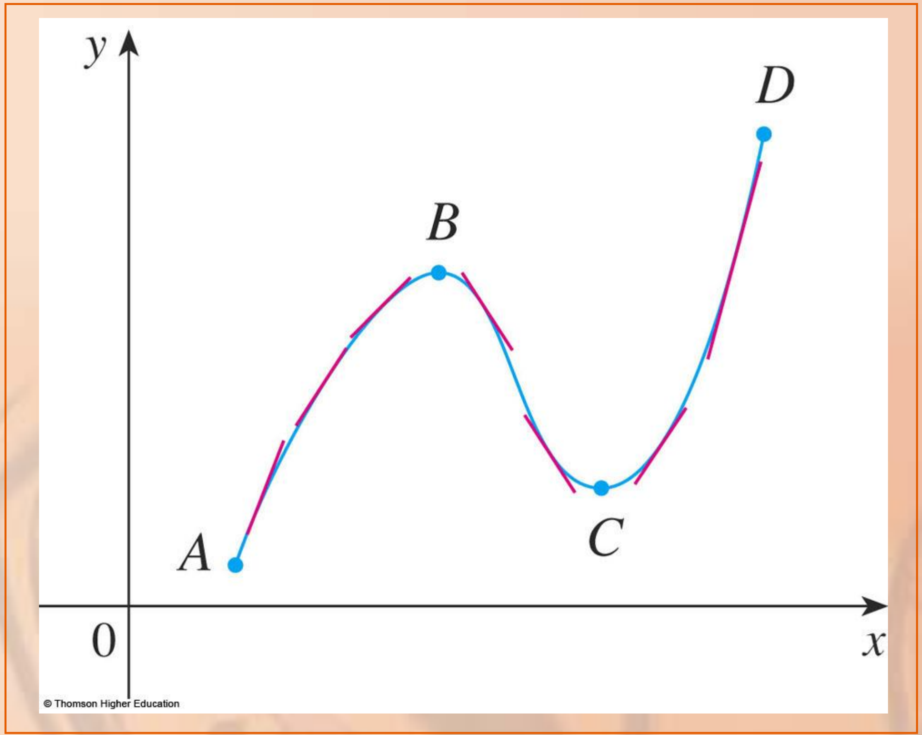
So, $f'(x) < 0$.



WHAT DOES *f'* SAY ABOUT *f* ?

Thus, it appears that *f* increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

- To prove that this is always the case, we use the Mean Value Theorem.



INCREASING/DECREASING TEST (I/D TEST)

a. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

b. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Proof a

Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$.

According to the definition of an increasing function, we have to show that $f(x_1) < f(x_2)$.

Proof a (Equation 1)

Since we are given that $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$.

So, by the Mean Value Theorem, there is a number c between x_1 and x_2 such that:

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Proof a and b

Now, $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$.

Thus, the right side of Equation 1 is positive.

So, $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$

- This shows that *f* is increasing.
- Part (b) is proved similarly.

Example 1

Find where the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and where it is decreasing.

Example 1

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

- To use the ID Test, we have to know where $f'(x) > 0$ and where $f'(x) < 0$.
- This depends on the signs of the three factors of $f'(x)$—namely, $12x$, $x - 2$, and $x + 1$.

I/D TEST

Example 1

We divide the real line into intervals whose endpoints are the critical numbers -1, 0, and 2 and arrange our work in a chart.

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x<-1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1<x<0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0<x<2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x>2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

Example 1

A plus sign indicates the given expression is positive.

A minus sign indicates it is negative.

The last column gives the conclusion based on the I/D Test.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

I/D TEST

Example 1

For instance, $f'(x) < 0$ for $0 < x < 2$.

So, f is decreasing on (0, 2).

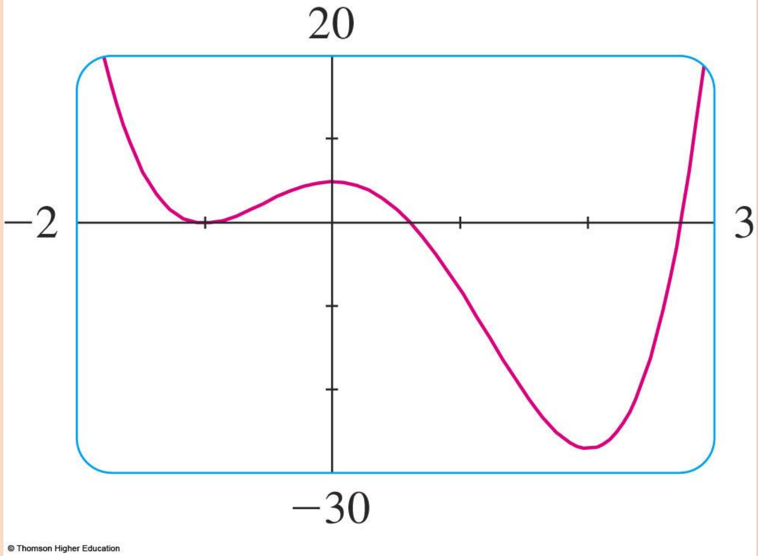
- It would also be true to say that f is decreasing on the closed interval.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	−	−	−	−	decreasing on $(-\infty, -1)$
$-1 < x < 0$	−	−	+	+	increasing on $(-1, 0)$
$0 < x < 2$	+	−	+	−	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	increasing on $(2, \infty)$

I/D TEST

Example 1

The graph of *f* confirms the information in the chart.



Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	−	−	−	−	decreasing on $(-\infty, -1)$
$-1 < x < 0$	−	−	+	+	increasing on $(-1, 0)$
$0 < x < 2$	+	−	+	−	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	increasing on $(2, \infty)$

WHAT DOES *f'* SAY ABOUT *f* ?

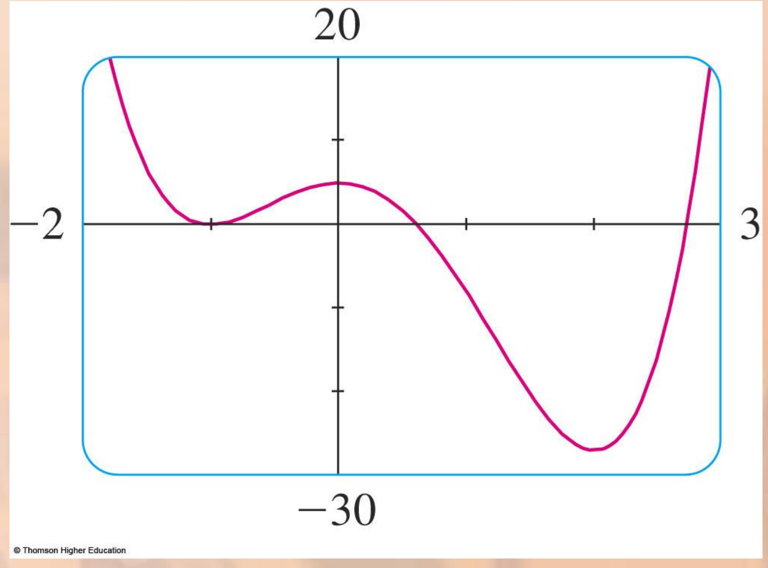
Recall from Section 4.1 that, if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem).

- However, not every critical number gives rise to a maximum or a minimum.
- So, we need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

WHAT DOES *f'* SAY ABOUT *f* ?

You can see from the figure that $f(0) = 5$ is a local maximum value of f because f increases on (-1, 0) and decreases on (0, 2).

- In terms of derivatives, $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$.



WHAT DOES *f'* SAY ABOUT *f* ?

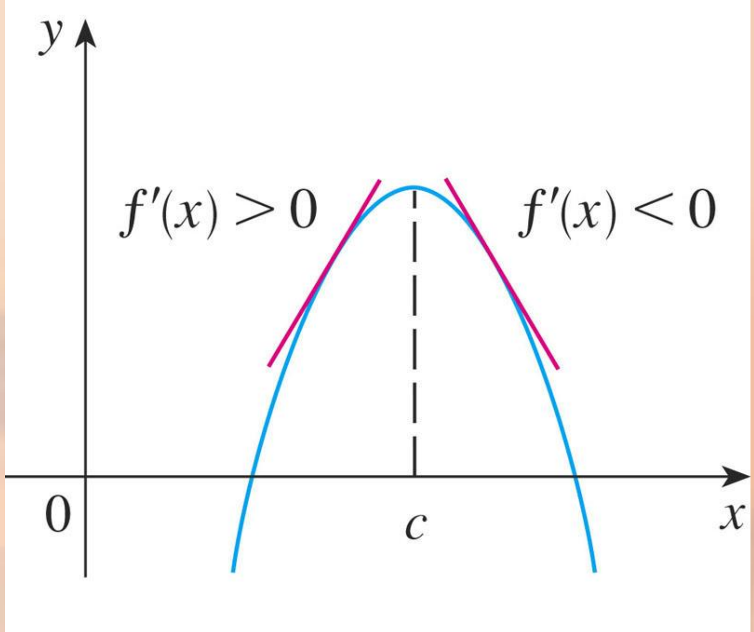
In other words, the sign of $f'(x)$ changes from positive to negative at 0.

- This observation is the basis of the following test.

FIRST DERIVATIVE TEST

Suppose that *c* is a critical number of a continuous function *f*.

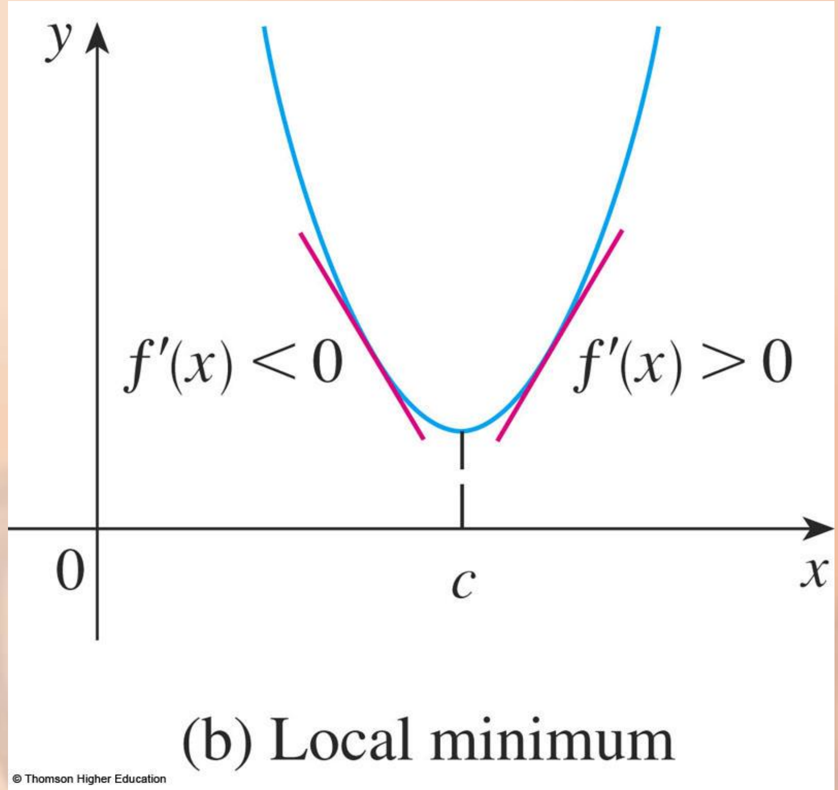
a. If *f'* changes from positive to negative at *c*, then *f* has a local maximum at *c*.



(a) Local maximum

FIRST DERIVATIVE TEST

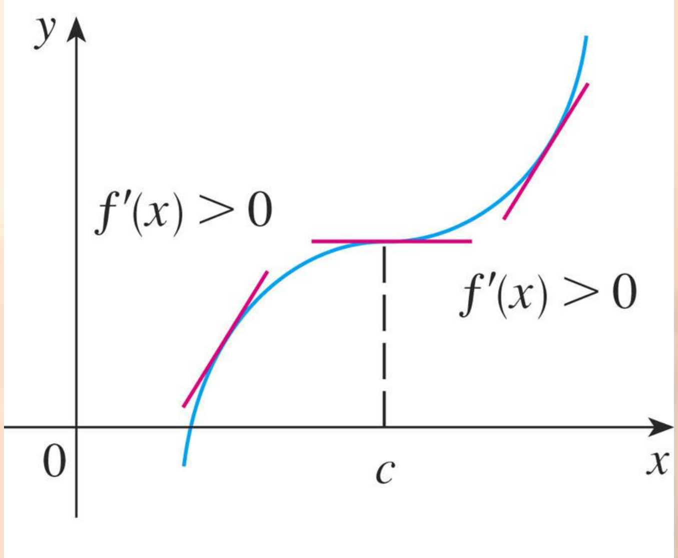
b. If *f'* changes from negative to positive at *c*, then *f* has a local minimum at *c*.



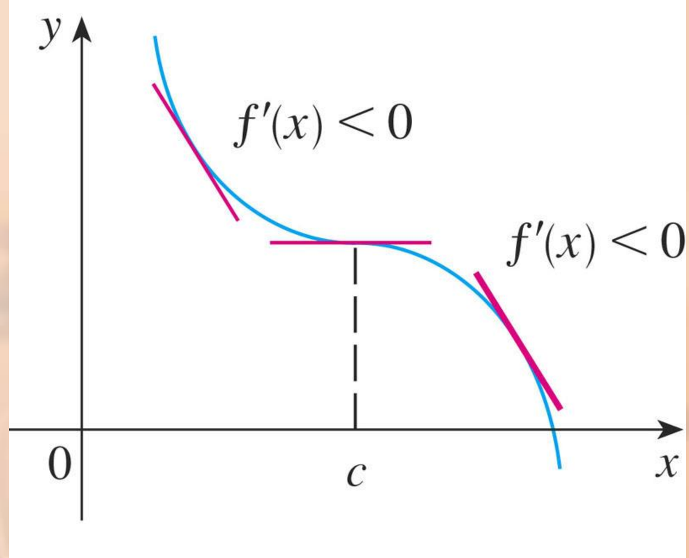
(b) Local minimum

FIRST DERIVATIVE TEST

c. If *f'* does not change sign at *c*—for example, if *f'* is positive on both sides of *c* or negative on both sides—then *f* has no local maximum or minimum at *c*.



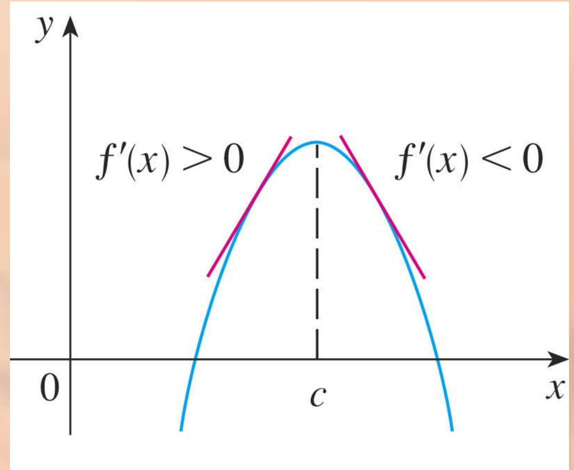
(c) No maximum or minimum



(d) No maximum or minimum

The First Derivative Test is a consequence of the I/D Test.

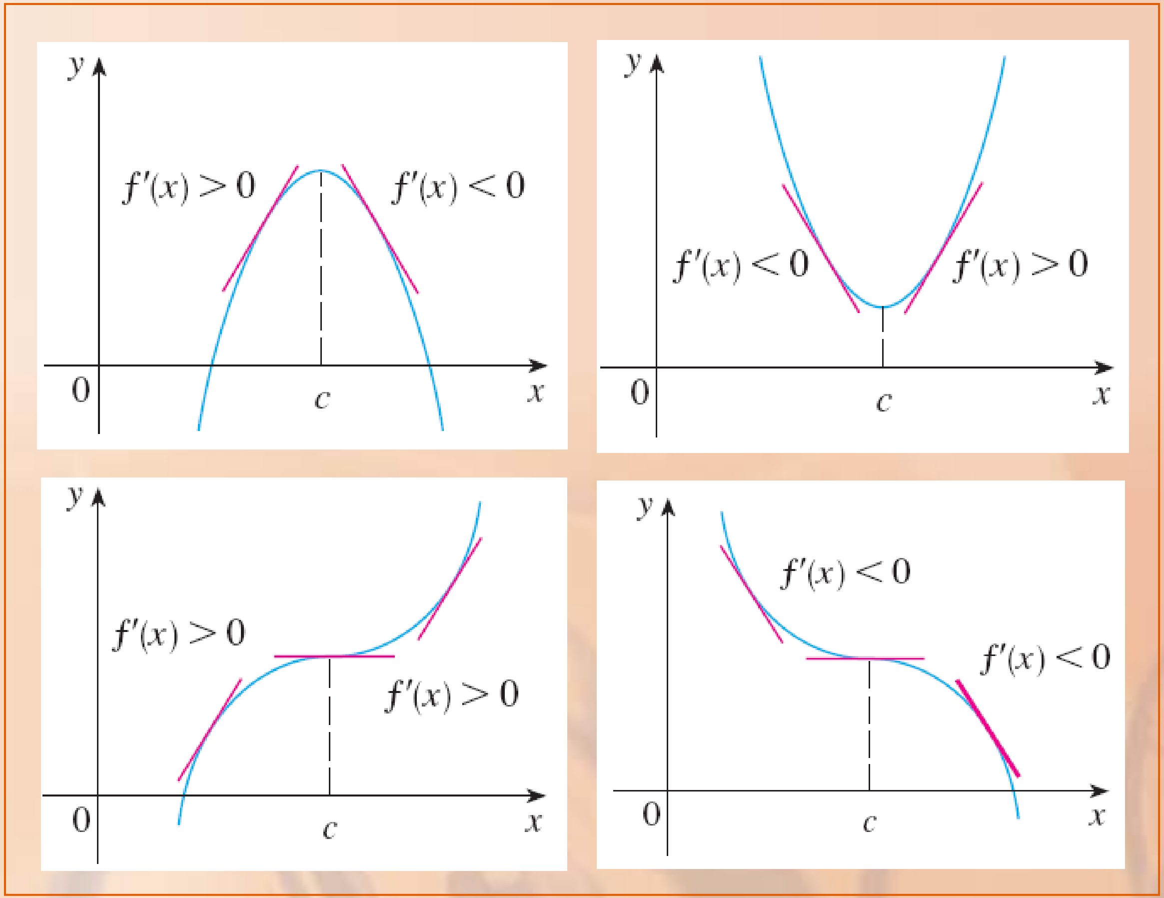
- For instance, in (a), since the sign of $f'(x)$ changes from positive to negative at c, f is increasing to the left of c and decreasing to the right of c.
- It follows that f has a local maximum at c.



(a) Local maximum

FIRST DERIVATIVE TEST

It is easy to remember the test by visualizing diagrams.



Example 2

Find the local minimum and maximum values of the function f in Example 1.

From the chart in the solution to Example 1, we see that $f'(x)$ changes from negative to positive at -1.

- So, $f(-1) = 0$ is a local minimum value by the First Derivative Test.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

 Example 2

Similarly, *f'* changes from negative to positive at 2.

- So, *f*(2) = -27 is also a local minimum value.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

WHAT DOES f' SAY ABOUT f ? Example 2

As previously noted, $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0.

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

Example 3

Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \leq x \leq 2\pi$$

 Example 3

To find the critical numbers of g, we differentiate:

$$g'(x) = 1 + 2\cos x$$

- So, $g'(x) = 0$ when $\cos x = -½$.
- The solutions of this equation are $2\pi/3$ and $4\pi/3$.

Example 3

As *g* is differentiable everywhere,

the only critical numbers are $2\pi/3$ and $4\pi/3$.

So, we analyze *g* in the following table.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

WHAT DOES *f'* SAY ABOUT *f* ? **Example 3**

As $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$.

- The local maximum value is:

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3}$$

$$\approx 3.83$$

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$.

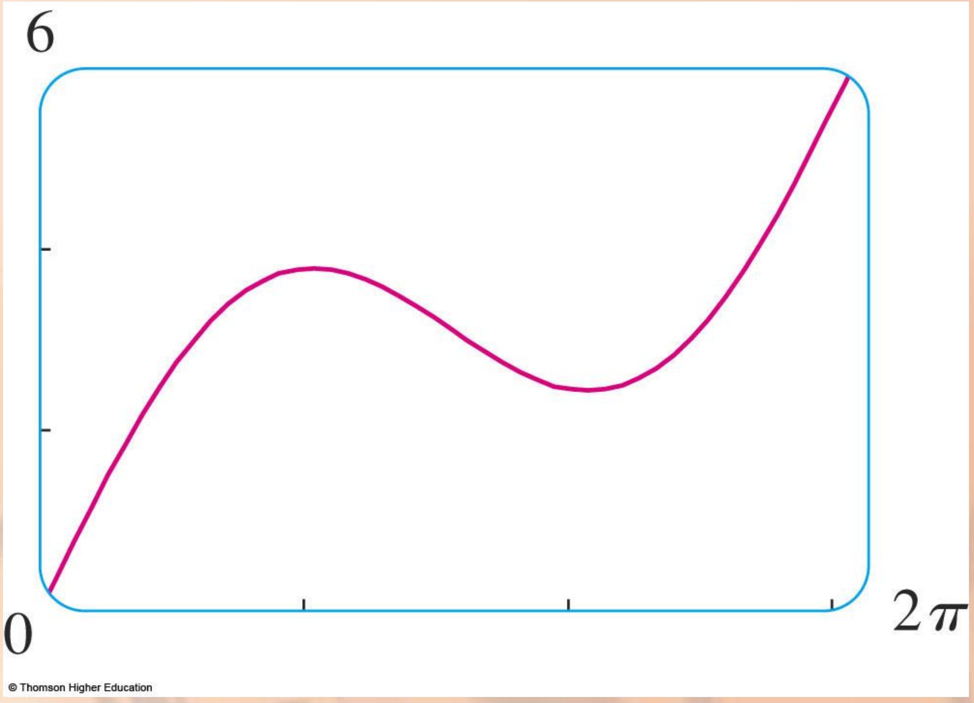
- So, a local minimum value is:

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3}$$

$$\approx 2.46$$

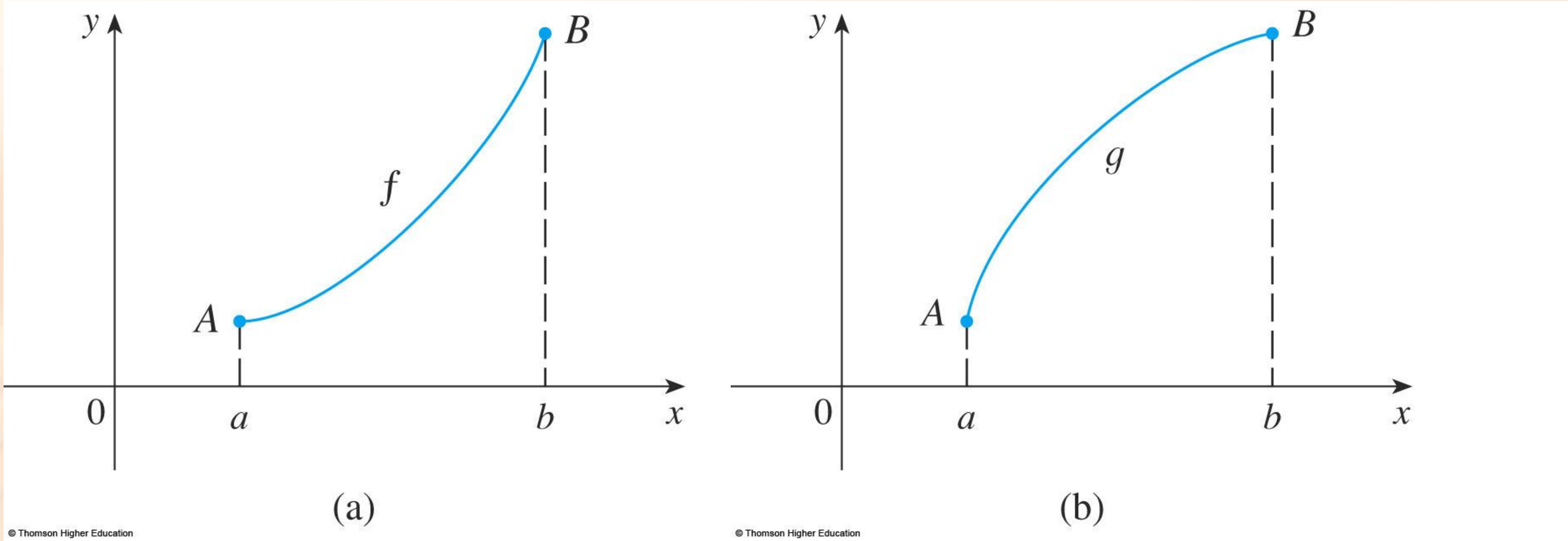
Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

The graph of *g* supports our conclusion.



WHAT DOES *f''* SAY ABOUT *f* ?

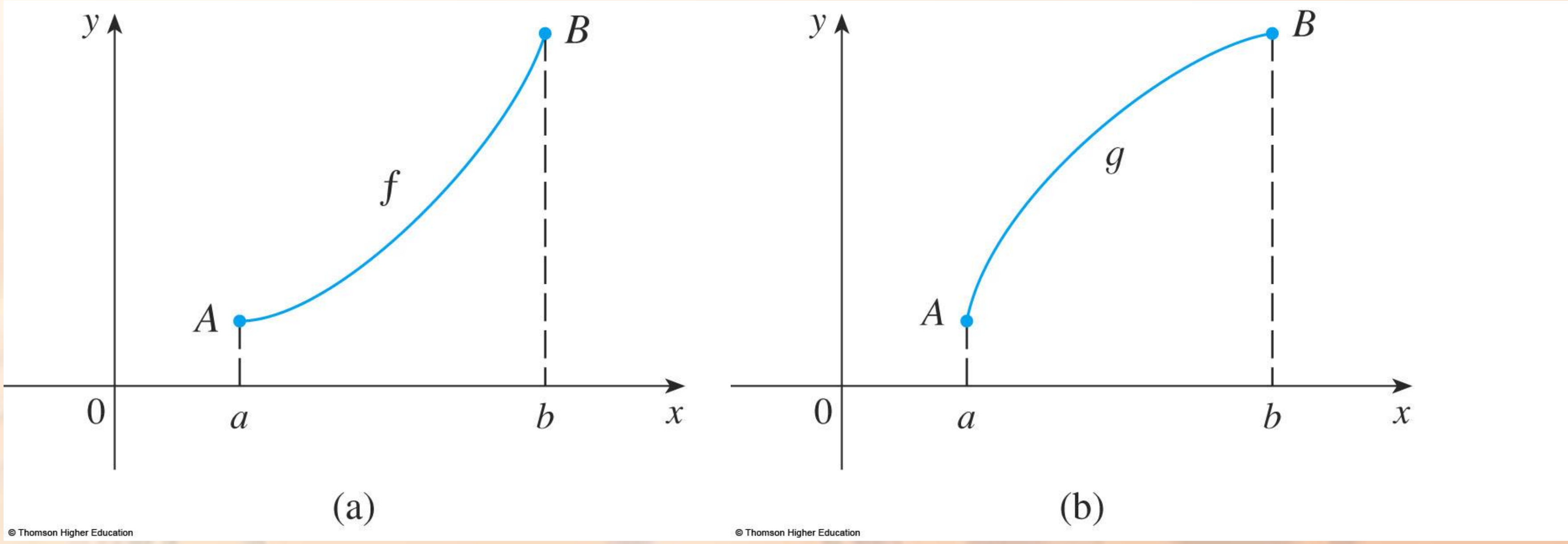
The figure shows the graphs of two increasing functions on (a, b).



WHAT DOES *f''* SAY ABOUT *f* ?

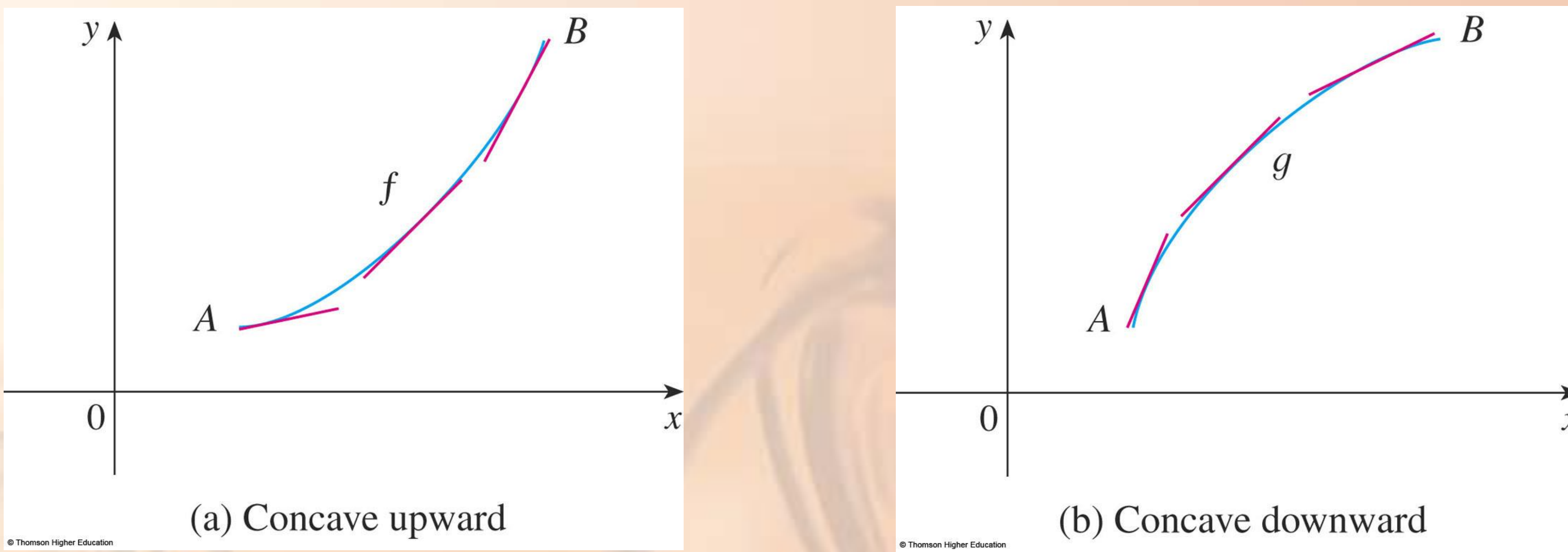
Both graphs join point *A* to point *B*, but they look different because they bend in different directions.

- How can we distinguish between these two types of behavior?



WHAT DOES *f''* SAY ABOUT *f* ?

Here, tangents to these curves have been drawn at several points.

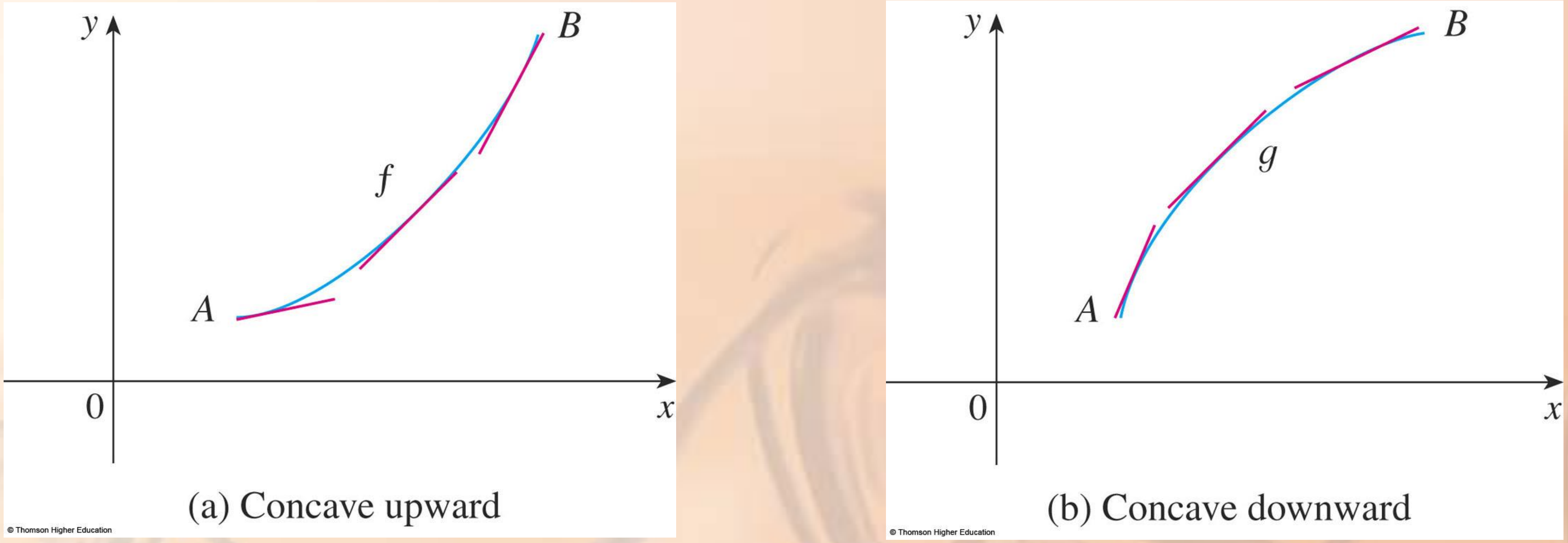


(a) Concave upward

(b) Concave downward

CONCAVE UPWARD

In the first figure, the curve lies above the tangents and *f* is called concave upward on (*a*, *b*).

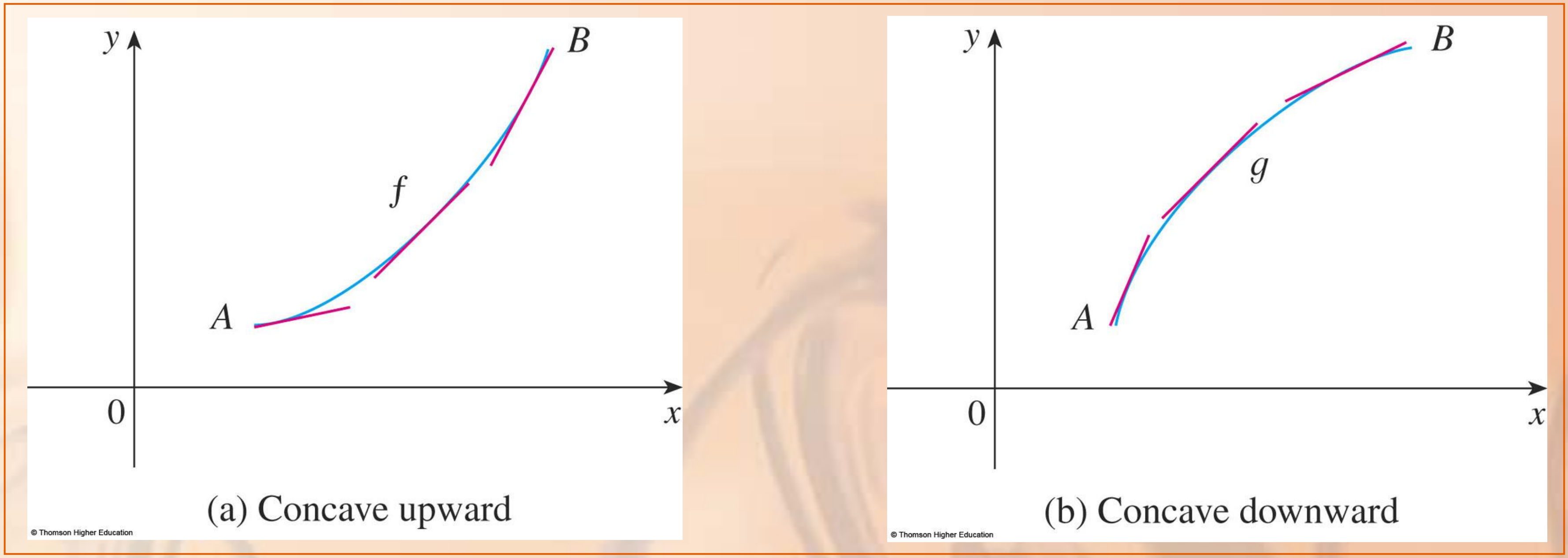


(a) Concave upward

(b) Concave downward

CONCAVE DOWNWARD

In the second figure, the curve lies below the tangents and *g* is called concave downward on (*a*, *b*).



(a) Concave upward

(b) Concave downward

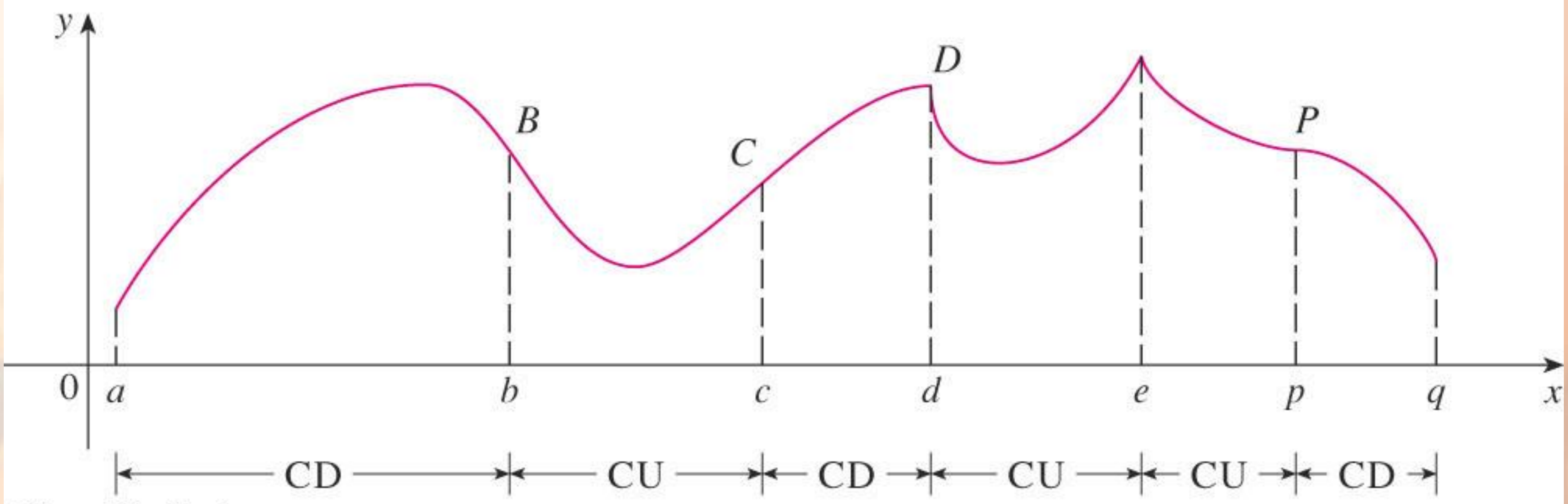
CONCAVITY—DEFINITION

If the graph of *f* lies above all of its tangents on an interval I, it is called concave upward on I.

If the graph of *f* lies below all of its tangents on I, it is called concave downward on I.

CONCAVITY

The figure shows the graph of a function that is concave upward (CU) on the intervals (*b*, *c*), (*d*, *e*), and (*e*, *p*) and concave downward (CD) on the intervals (*a*, *b*), (*c*, *d*), and (*p*, *q*).



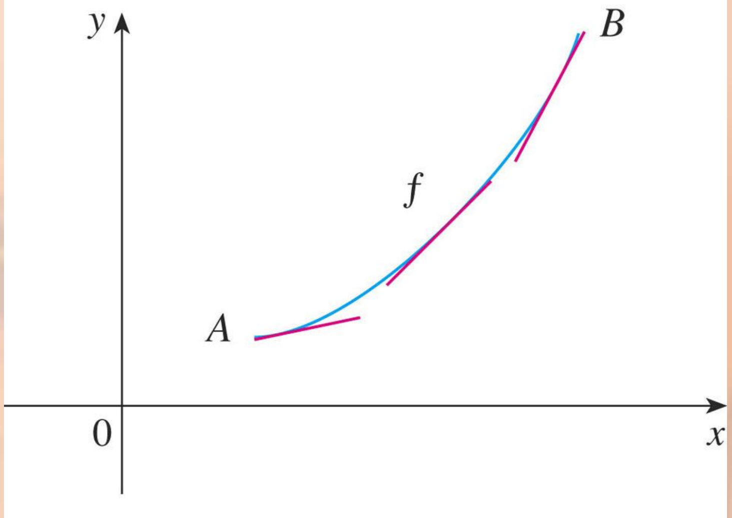
CONCAVITY

Let's see how the second derivative helps determine the intervals of concavity.

CONCAVITY

From this figure, you can see that, going from left to right, the slope of the tangent increases.

- This means that the derivative *f'* is an increasing function and therefore its derivative *f''* is positive.



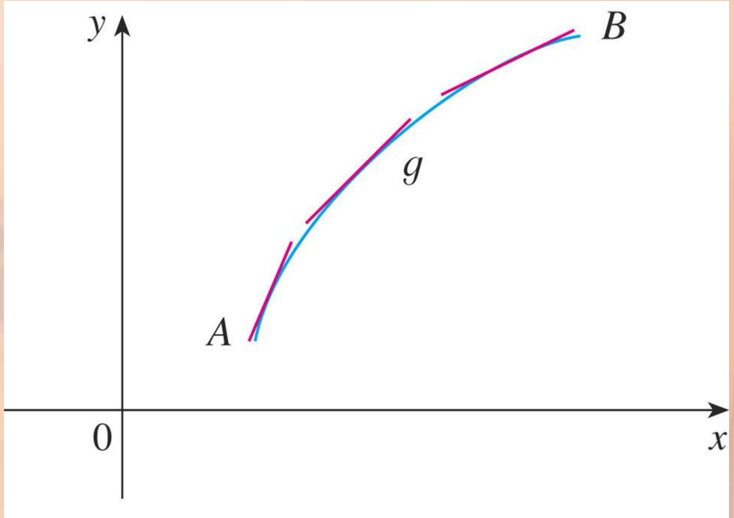
(a) Concave upward

CONCAVITY

Likewise, in this figure, the slope of the tangent decreases from left to right.

So, *f'* decreases and therefore *f''* is negative.

- This reasoning can be reversed and suggests that the following theorem is true.



(b) Concave downward

CONCAVITY TEST

a. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

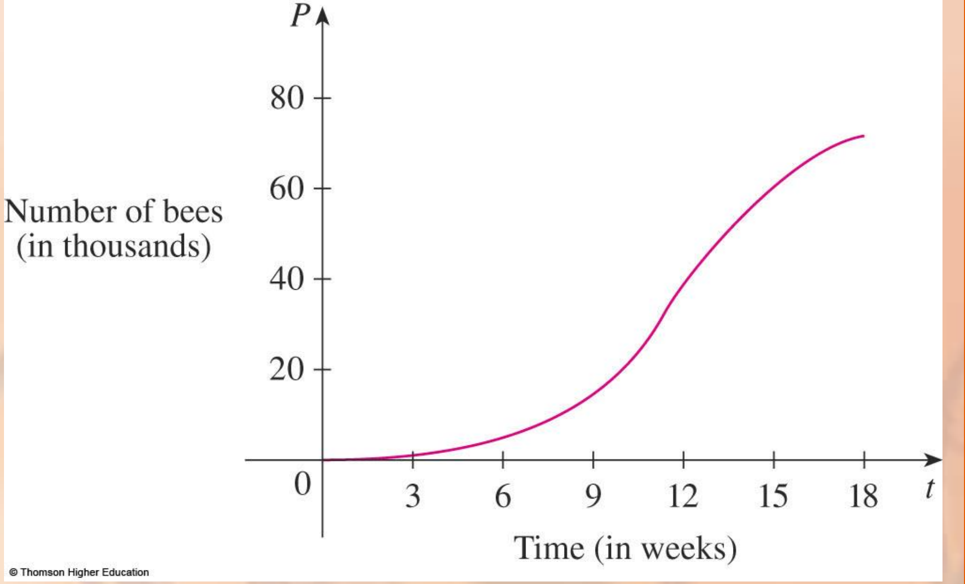
b. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

CONCAVITY

Example 4

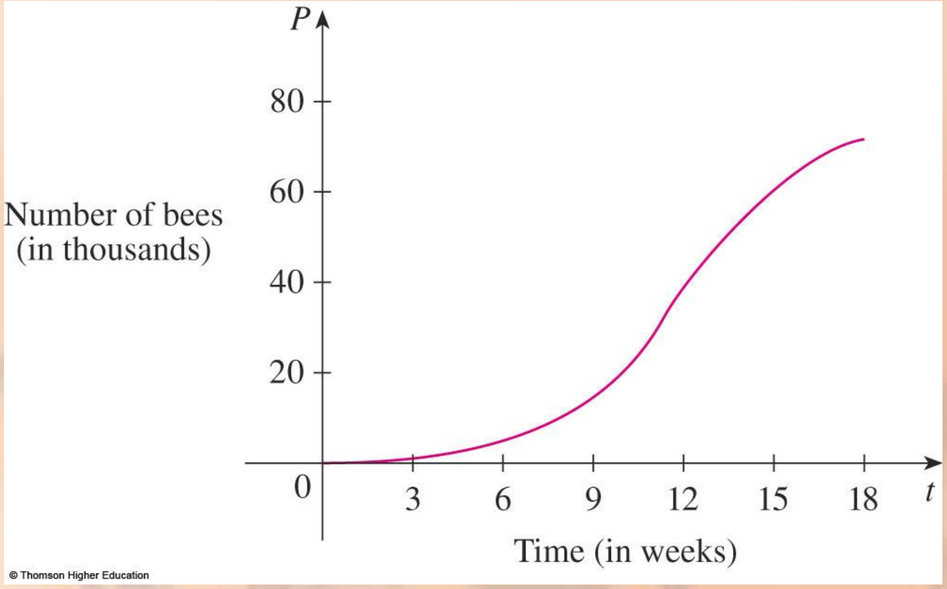
The figure shows a population graph for Cyprian honeybees raised in an apiary.

- How does the rate of population increase change over time?
- When is this rate highest?
- Over what intervals is *P* concave upward or concave downward?



By looking at the slope of the curve as *t* increases, we see that the rate of increase of the population is initially very small.

- Then, it gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off.

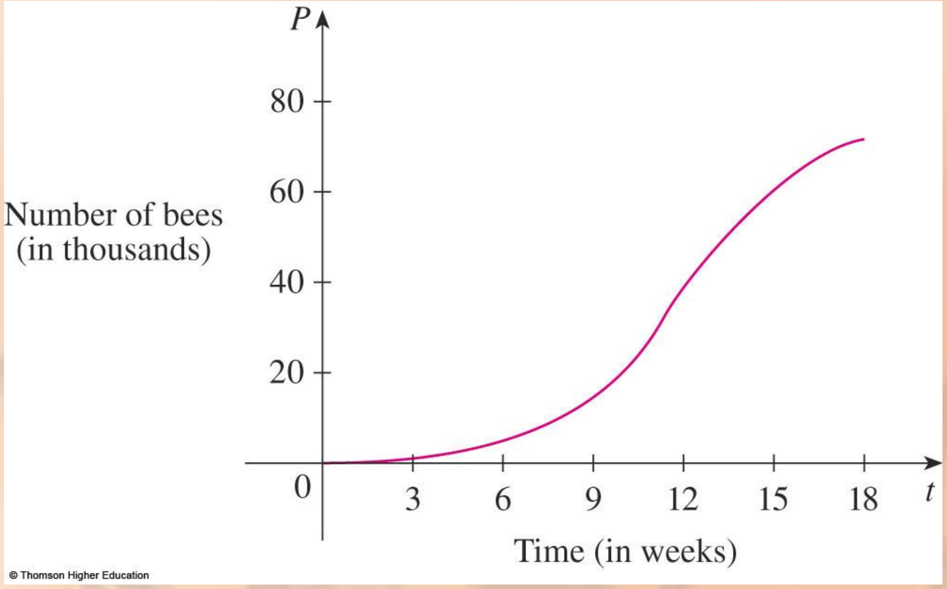


CONCAVITY

Example 4

As the population approaches its maximum value of about 75,000 (called the carrying capacity), the rate of increase, $P'(t)$, approaches 0.

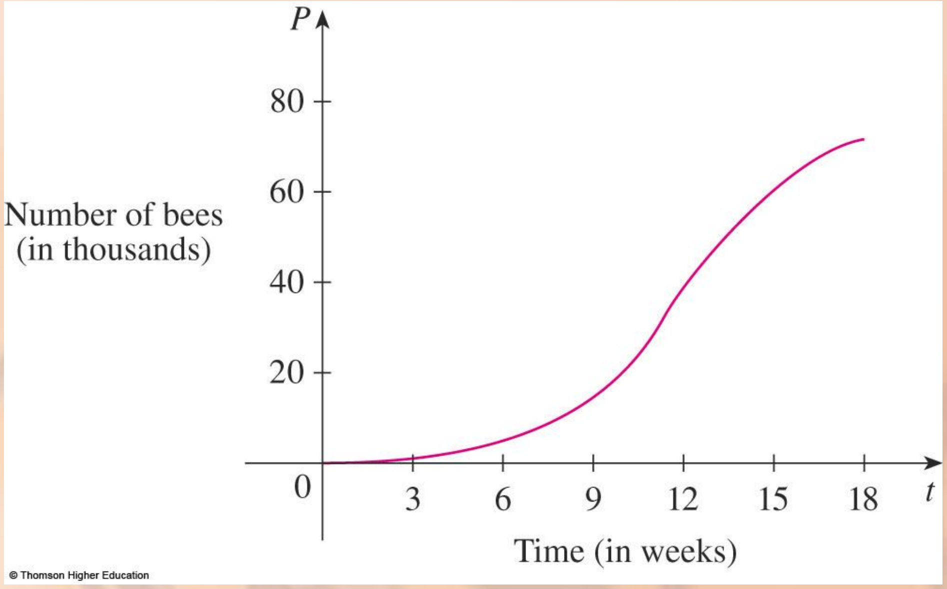
- The curve appears to be concave upward on (0, 12) and concave downward on (12, 18).



INFLECTION POINT

In the example, the curve changed from concave upward to concave downward at approximately the point (12, 38,000).

- This point is called an inflection point of the curve.



INFLECTION POINT

The significance of this point is that the rate of population increase has its maximum value there.

- In general, an inflection point is a point where a curve changes its direction of concavity.

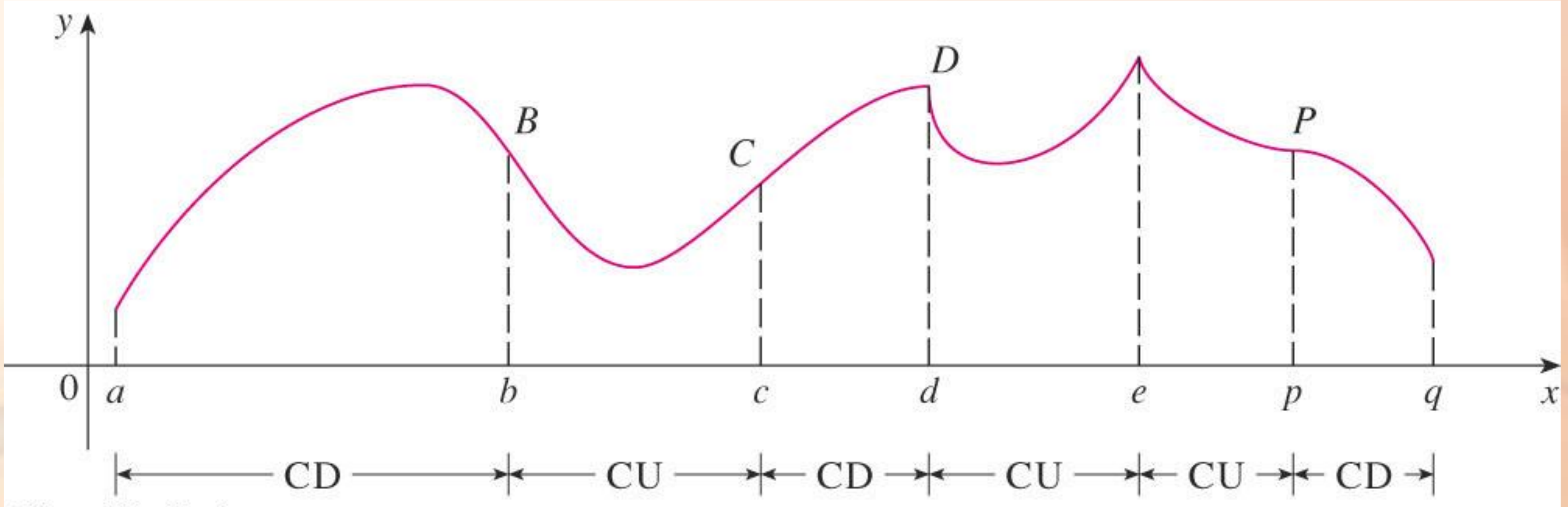
INFLECTION POINT—DEFINITION

A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

INFLECTION POINT

For instance, here, *B*, *C*, *D*, and *P* are the points of inflection.

- Notice that, if a curve has a tangent at a point of inflection, then the curve crosses its tangent there.



INFLECTION POINT

In view of the Concavity Test, there is a point of inflection at any point where the second derivative changes sign.

Example 5

Sketch a possible graph of a function *f* that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$

E. g. 5—Condition i

The first condition tells us that *f* is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

E. g. 5—Condition ii

The second condition says that *f* is concave upward on $(-\infty, -2)$ and $(2, \infty)$, and concave downward on $(-2, 2)$.

E. g. 5—Condition iii

From the third condition, we know that the graph of *f* has two horizontal asymptotes:

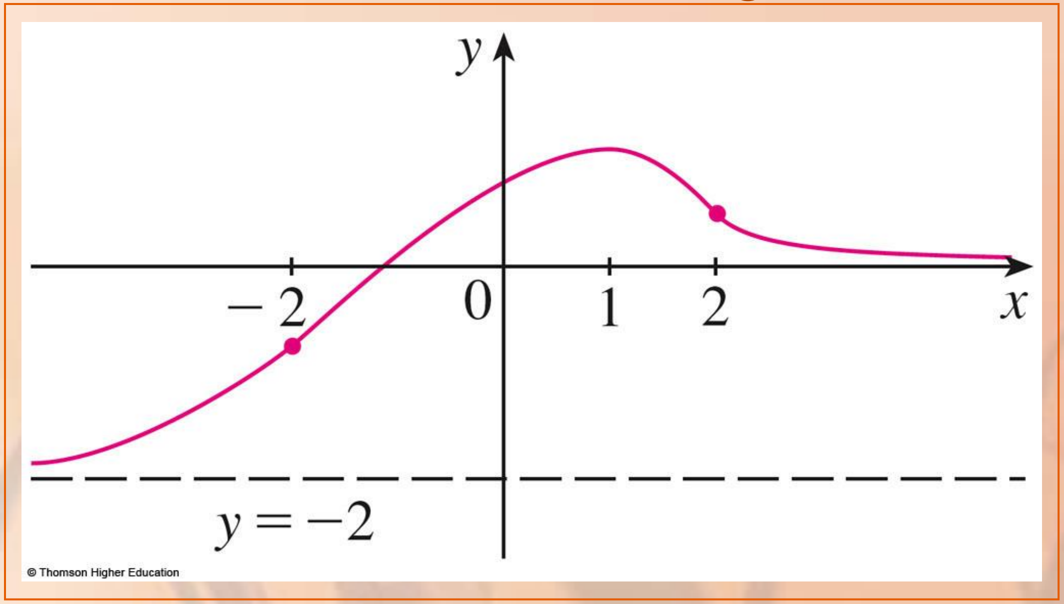
- $y = -2$
- $y = 0$

E. g. 5—Condition iii

We first draw the horizontal asymptote

y = -2 as a dashed line.

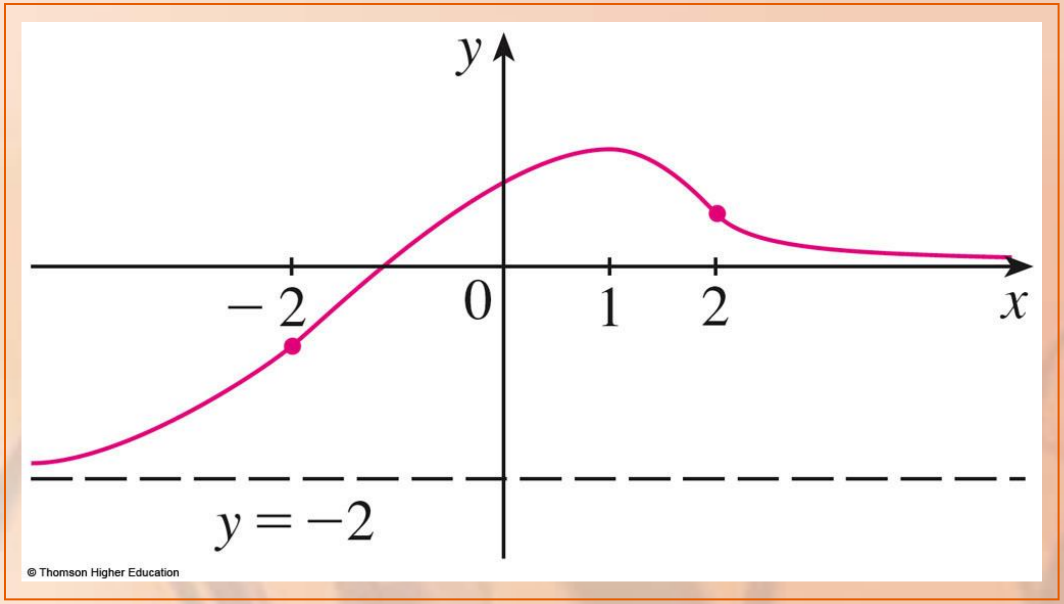
- We then draw the graph of *f* approaching this asymptote at the far left—increasing to its maximum point at $x = 1$ and decreasing toward the *x*-axis at the far right.



E. g. 5—Condition iii

We also make sure that the graph has inflection points when x = -2 and 2.

- Notice that we made the curve bend upward for $x < -2$ and $x > 2$, and bend downward when x is between -2 and 2.



WHAT DOES f'' SAY ABOUT f ?

Another application of the second derivative is the following test for maximum and minimum values.

- It is a consequence of the Concavity Test.

SECOND DERIVATIVE TEST

Suppose f'' is continuous near c.

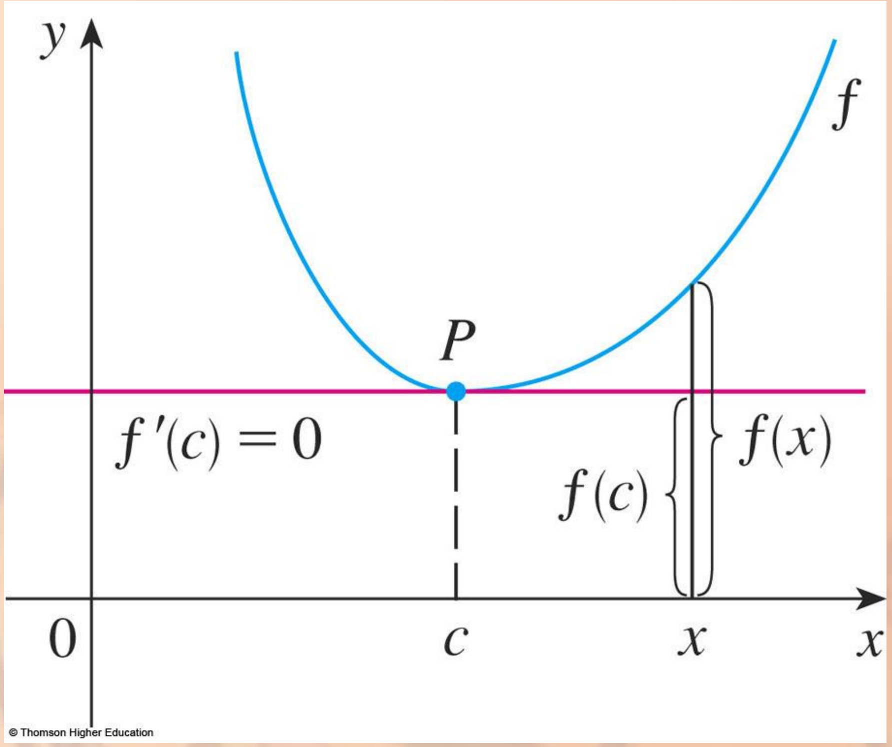
a. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

SECOND DERIVATIVE TEST

For instance, (a) is true because $f''(x) > 0$ near c, and so f is concave upward near c.

- This means that the graph of *f* lies above its horizontal tangent at *c*, and so *f* has a local minimum at *c*.



Example 6

Discuss the curve

$$y = x^4 - 4x^3$$

with respect to concavity, points of inflection, and local maxima and minima.

Use this information to sketch the curve.

Example 6

If $f(x) = x^4 - 4x^3$, then:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers, we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$.

To use the Second Derivative Test, we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Example 6

As $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum.

As $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

WHAT DOES f'' SAY ABOUT f ? **Example 6**

However, since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0.

- In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.

WHAT DOES *f''* SAY ABOUT *f* ?

Example 6

As $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with those numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

WHAT DOES *f''* SAY ABOUT *f* ? Example 6

The point (0, 0) is an inflection point—since the curve changes from concave upward to concave downward there.

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

WHAT DOES *f''* SAY ABOUT *f* ?

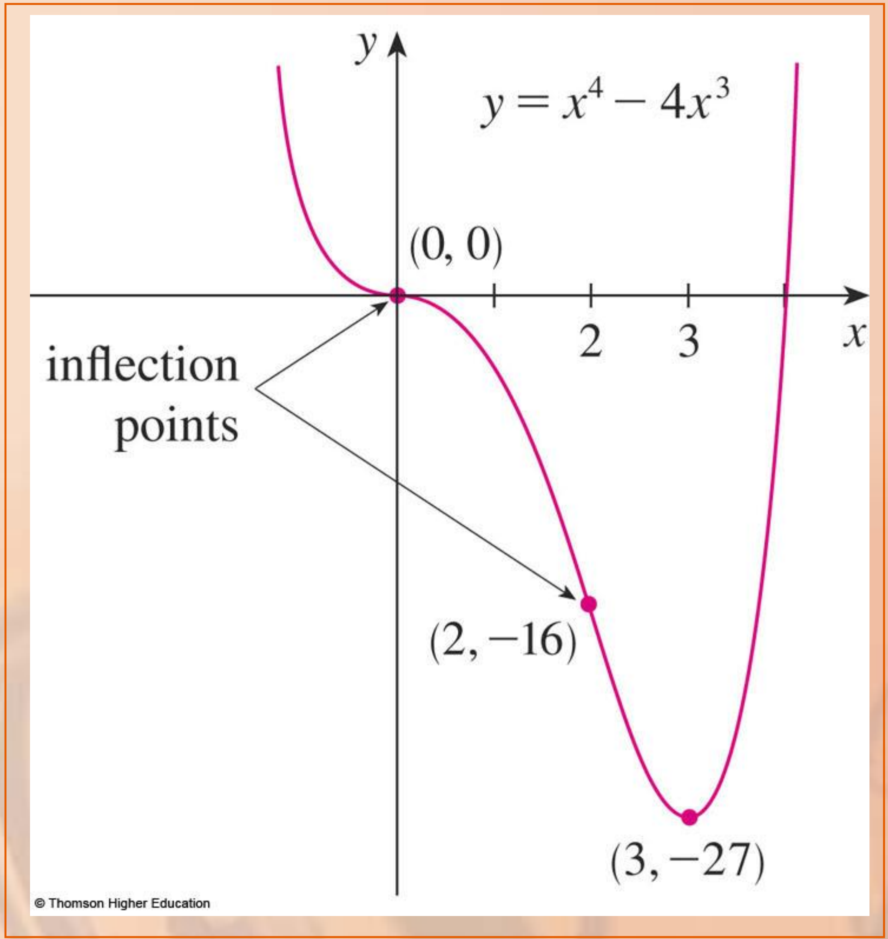
Example 6

Also, (2, -16) is an inflection point—since the curve changes from concave downward to concave upward there.

Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

WHAT DOES f'' SAY ABOUT f ? Example 6

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve.



NOTE

The Second Derivative Test is inconclusive when $f''(c) = 0$.

- In other words, at such a point, there might be a maximum, a minimum, or neither (as in the example).

NOTE

The test also fails when $f''(c)$ does not exist.

In such cases, the First Derivative Test must be used.

- In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 7

Sketch the graph of the function

$$f(x) = x^{2/3}(6 - x)^{1/3}$$

- You can use the differentiation rules to check that the first two derivatives are:

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

- As $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

WHAT DOES *f''* SAY ABOUT *f* ?

Example 7

To find the local extreme values, we use the First Derivative Test.

- As *f'* changes from negative to positive at 0, *f*(0) = 0 is a local minimum.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 4$	$+$	$+$	$+$	$+$	increasing on $(0, 4)$
$4 < x < 6$	$-$	$+$	$+$	$-$	decreasing on $(4, 6)$
$x > 6$	$-$	$+$	$+$	$-$	decreasing on $(6, \infty)$

WHAT DOES f'' SAY ABOUT f ? Example 7

- Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum.

- The sign of f' does not change at 6, so there is no minimum or maximum there.

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

WHAT DOES *f''* SAY ABOUT *f* ?

Example 7

The Second Derivative Test could be used at 4, but not at 0 or 6—since *f''* does not exist at either of these numbers.

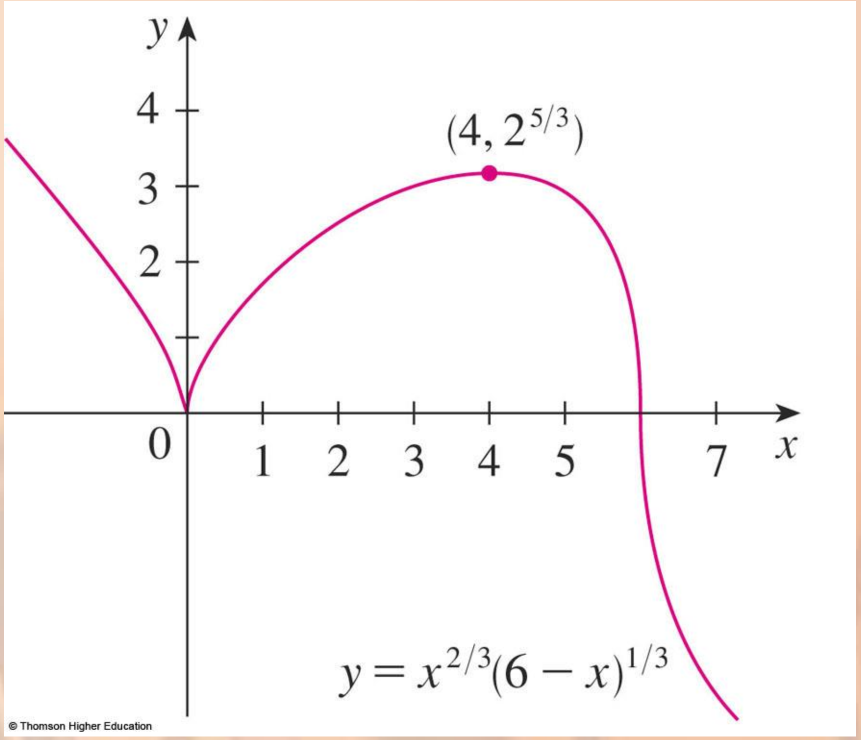
Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geq 0$ for all x, we have:

- $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$
- $f''(x) > 0$ for $x > 6$

So, f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$.

- Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$.



Example 8

Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

- Notice that the domain of f is $\{x \mid x \neq 0\}$.
- So, we check for vertical asymptotes by computing the left and right limits as $x \to 0$.

WHAT DOES *f''* SAY ABOUT *f* ?

As $x \to 0^+$, we know that $t = 1/x \to \infty$.

So, $\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$

- This shows that $x = 0$ is a vertical asymptote.

WHAT DOES *f''* SAY ABOUT *f* ?

As $x \rightarrow 0^-$, we know that $t = 1/x \rightarrow -\infty$.

So, $\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$

As $x \to \pm\infty$, we have $1/x \to 0$.

So, $\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$

- This shows that $y = 1$ is a horizontal asymptote.

Now, let's compute the derivative.

The Chain Rule gives: $$f'(x) = -\frac{e^{1/x}}{x^2}$$

- Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.
- Thus, *f* is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

WHAT DOES *f''* SAY ABOUT *f* ?

There is no critical number.

So, the function has no maximum or minimum.

WHAT DOES *f''* SAY ABOUT *f* ?

The second derivative is:

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4}$$

$$= \frac{e^{1/x}(2x+1)}{x^4}$$

WHAT DOES *f''* SAY ABOUT *f* ?

As $e^{1/x} > 0$ and $x^4 > 0$, we have:

$$f''(x) > 0 \text{ when } x > -\tfrac{1}{2} \ (x \neq 0)$$

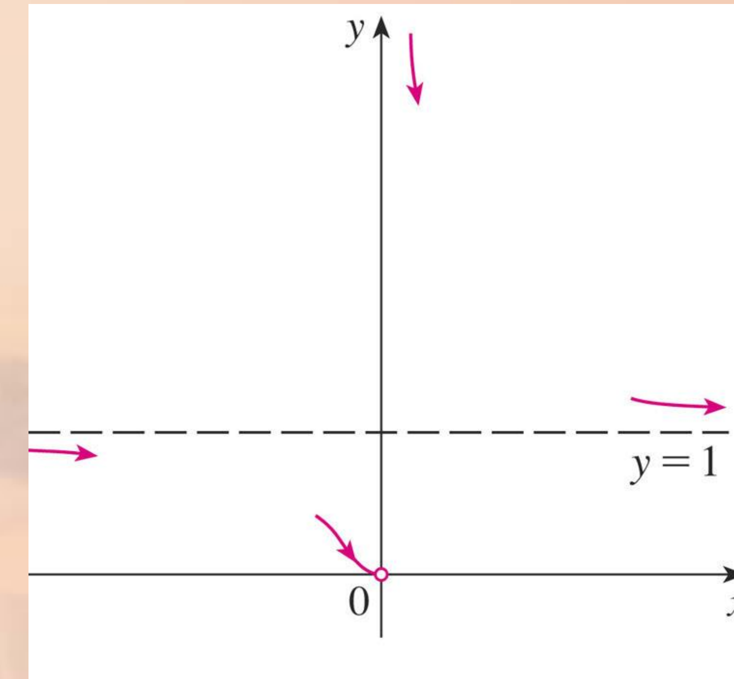
$$f''(x) < 0 \text{ when } x < -\tfrac{1}{2}$$

- So, the curve is concave downward on $(-\infty, -\frac{1}{2})$ and concave upward on $(-\frac{1}{2}, 0)$ and on $(0, \infty)$.

- The inflection point is $(-\frac{1}{2}, e^{-2})$.

WHAT DOES *f''* SAY ABOUT *f* ?

To sketch the graph of *f*, we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch.

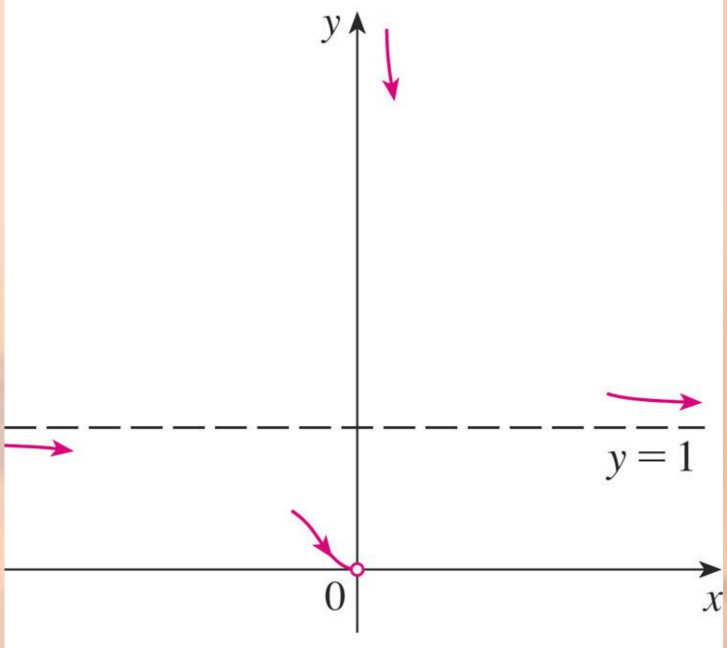


(a) Preliminary sketch

WHAT DOES f'' SAY ABOUT f ?

These parts reflect the information concerning limits and the fact that *f* is decreasing on both $(-\infty, 0)$ and $(0, \infty)$.

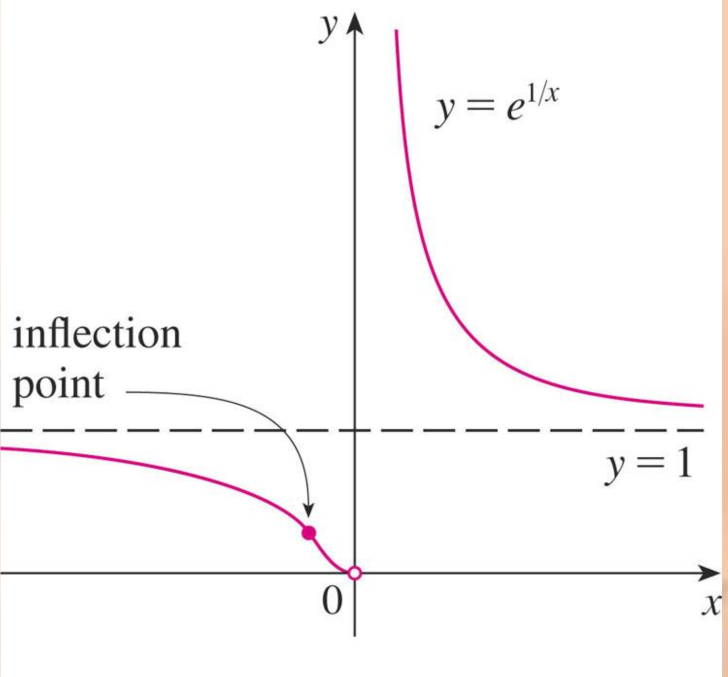
- Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist.



(a) Preliminary sketch

WHAT DOES *f''* SAY ABOUT *f* ?

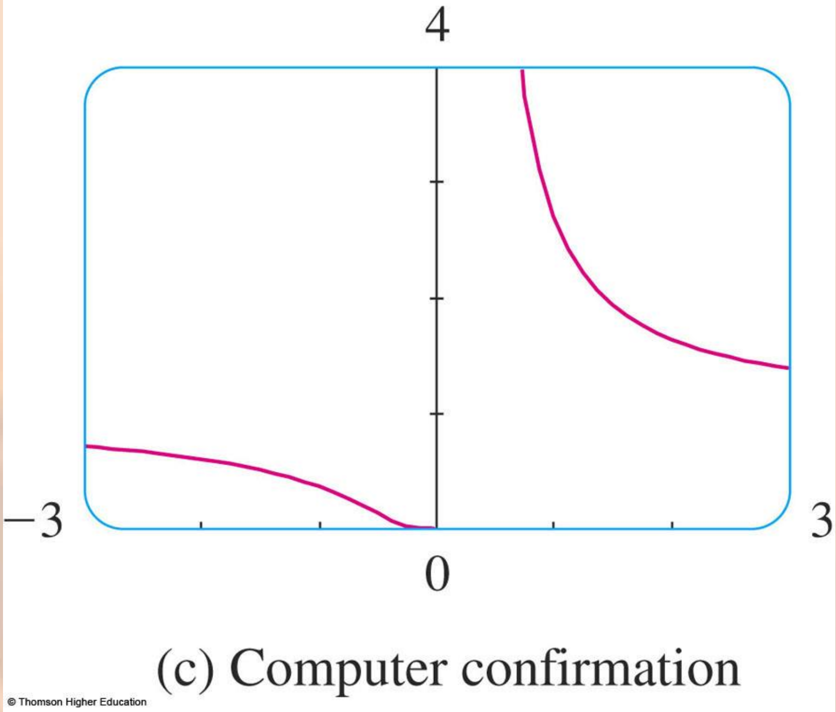
Here, we finish the sketch by incorporating the information concerning concavity and the inflection point.



(b) Finished sketch

WHAT DOES *f''* SAY ABOUT *f* ?

Finally, we check our work with a graphing device.



(c) Computer confirmation