$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$ CHAPTER 7

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EXAMPLE ①:

$L = 1H$

$R = 2\Omega$

$C = 4/3\, F$

$E(t) = e^{-t}$

$Q(0) = 1, \quad I(0) = 1$

Ans:

$$\begin{cases} \frac{dQ}{dt} = I \\ \frac{dI}{dt} = -\frac{1}{LC}Q - \frac{R}{L}I + \frac{E(t)}{L} \end{cases}$$

$$\frac{dQ}{dt} = I$$

$$\frac{dI}{dt} = \frac{1}{(1)(4/3)}Q - \frac{2}{1}I + \frac{e^{-t}}{1}$$

$$\frac{dI}{dt} = \frac{3}{4}Q - 2I + e^{-t}$$

$$\begin{pmatrix} Q' \\ I' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{3}{4} & -2 \end{pmatrix}\begin{pmatrix} Q \\ I \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

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$$\begin{pmatrix} Q' \\ I' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3/4 & -2 \end{pmatrix} \begin{pmatrix} Q \\ I \end{pmatrix}$$

Eigenvalues & Eigenvectors

Let $V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\left| \begin{pmatrix} 0-\lambda & 1 \\ -3/4 & -2-\lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -3/4 & -2-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-2-\lambda) - 1(-3/4) = 0$$

$$2\lambda + \lambda^2 + 3/4 = 0$$

$$\lambda^2 + 2\lambda + 3/4 = 0$$

$$4\lambda^2 + 8\lambda + 3 = 0$$

$$\lambda = \frac{-(8) \pm \sqrt{8^2 - 4(4)(3)}}{2(4)} = \frac{-8 \pm \sqrt{64 - 48}}{8}$$

Date: $\frac{}{2} = \frac{}{2}$ $\frac{\cancel{2}}{\cancel{4}_2} \times \frac{2}{3}$ $\frac{-2}{1} + \frac{3}{2}$ $\frac{-4+3}{2} = \frac{-1}{2}$

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$$\lambda = \frac{-8 \pm 4}{8} = -1 \pm \frac{1}{2}$$

$$\lambda_1 = -1 + \frac{1}{2} \quad , \quad \lambda_2 = -1 - \frac{1}{2}$$

$$\lambda_1 = \frac{-2+1}{2} \quad , \quad \lambda_2 = \frac{-2-1}{2}$$

$$\lambda_1 = -\frac{1}{2} \quad , \quad \lambda_2 = -\frac{3}{2}$$

For $\lambda_1 = -\frac{1}{2}$

$$\begin{pmatrix} -(-\frac{1}{2}) & 1 \\ -\frac{3}{4} & -2-(-\frac{1}{2}) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{3}{4} & -2+\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{3}{4} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

$$\underset{\sim}{R} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ -\frac{3}{4} & -\frac{3}{2} & 0 \end{array} \right) 2R_1$$

For $\lambda_2 = -\frac{3}{2}$

$$\begin{pmatrix} -(-\frac{3}{2}) & 1 \\ -\frac{3}{4} & -2-(-\frac{3}{2}) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{3}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} =$$

$$\underset{\sim}{R} \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ -\frac{3}{4} & -\frac{1}{2} & 0 \end{array} \right) \frac{2}{3}R$$

$$\underset{\sim}{R} \left(\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array} \right) R_2 + \frac{3}{4}$$

$$x_2 + \frac{2}{3} y_2 = 0$$

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$$\sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad R_2 + \frac{3}{4}R_1$$

$x_1 + 2y_1 = 0$

$x_2 = -2y_1$

$$\therefore V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$e^{\frac{1}{2}4}$

$\frac{3}{2}4$

$\rightarrow x_2 = -\frac{2}{3} y_2$

$$V_2 = \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} \text{ OR } \Rightarrow V_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} Q' \\ I' \end{pmatrix} = c_1 e^{-\frac{1}{2}t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-\frac{3}{2}t} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\varphi(t) = \begin{pmatrix} -2e^{-\frac{1}{2}t} & -2e^{-\frac{3}{2}t} \\ e^{-\frac{1}{2}t} & 3e^{-\frac{3}{2}t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{|\varphi(t)|} \cdot \text{Adj. } \varphi(t)$$

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$+\frac{1}{2}-\frac{3}{2} \quad \frac{-4}{2}=-2$

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$$\varphi^{-1}(t)=\frac{1}{(-2e^{-\frac{1}{2}t})(3e^{-\frac{3}{2}t})-(-2e^{-\frac{3}{2}t})(e^{-\frac{1}{2}t})}$$

$$\begin{pmatrix} 3e^{-\frac{3}{2}t} & 2e^{-\frac{3}{2}t} \\ -e^{-\frac{1}{2}t} & -2e^{-\frac{1}{2}t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\frac{1}{-6e^{-2t}+2e^{-2t}}\cdot\begin{pmatrix} 3e^{-\frac{3}{2}t} & 2e^{-\frac{3}{2}t} \\ -e^{-\frac{1}{2}t} & -2e^{-\frac{1}{2}t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\frac{1}{-4e^{-2t}}\begin{pmatrix} 3e^{-\frac{3}{2}t} & 2e^{-\frac{3}{2}t} \\ -e^{-\frac{1}{2}t} & -2e^{-\frac{1}{2}t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\begin{pmatrix} -\frac{3}{4}e^{-\frac{3}{2}t+2t} & -\frac{1}{2}e^{\frac{3}{2}t+2t} \\ \frac{1}{4}e^{-\frac{1}{2}t+2t} & \frac{1}{2}e^{-\frac{1}{2}t+2t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\begin{pmatrix} -\frac{3}{4}e^{\frac{1}{2}t} & -\frac{1}{2}e^{\frac{1}{2}t} \\ \frac{1}{4}e^{\frac{3}{2}t} & \frac{1}{2}e^{\frac{3}{2}t} \end{pmatrix}$$

see book

$$X(t)=\varphi(t)\cdot\varphi^{-1}(0)\,X(0)+\varphi(t)\int_0^t\varphi^{-1}(u)F(u)\,du$$

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$$X(t) = \begin{pmatrix} -2e^{-\frac{1}{2}t} & -2e^{-\frac{3}{2}t} \\ e^{-\frac{1}{2}t} & 3e^{-\frac{3}{2}t} \end{pmatrix} \begin{pmatrix} -\frac{3}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$$

$$\begin{pmatrix} -2e^{-\frac{1}{2}t} & -2e^{-\frac{3}{2}t} \\ e^{-\frac{1}{2}t} & 3e^{-\frac{3}{2}t} \end{pmatrix} \left(\int_0^t \begin{pmatrix} -\frac{3}{4}e^{u/2} & -\frac{1}{2}e^{\frac{1}{2}u} \\ +\frac{1}{4}e^{\frac{3}{2}u} & \frac{1}{2}e^{\frac{3}{2}u} \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} 0 \\ e^{-u} \end{pmatrix} du \right)$$

$$X(t) = \begin{pmatrix} \frac{3}{2}e^{-\frac{1}{2}t} + -\frac{1}{2}e^{-\frac{3}{2}t} & e^{-\frac{1}{2}t} - e^{-\frac{3}{2}t} \\ -\frac{3}{4}e^{-\frac{1}{2}t} + \frac{3}{4}e^{-\frac{3}{2}t} & -\frac{1}{2}e^{-\frac{1}{2}t} + \frac{3}{2}e^{-\frac{3}{2}t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$$

$$\begin{pmatrix} -2e^{-\frac{1}{2}t} & -2e^{-\frac{3}{2}t} \\ e^{-\frac{1}{2}t} & 3e^{-\frac{3}{2}t} \end{pmatrix}$$

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POPULATION PROBLEMS

$$\frac{dx}{dt} = +a_1 x - a_2 x + b_2 y$$

$$\frac{dy}{dt} = +b_1 y - b_2 y + a_2 x$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & b_2 \\ a_2 & b_1 - b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

EXAMPLE 3 :-

$a_1 = 5$, $a_2 = 4$

$b_1 = 5$, $b_2 = 3$

$x(0) = 14$, $y(0) = 7$

Sol :-

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Finding eigenvalues and eigenvectors

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$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 12 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = -2$$

For $\lambda_1 = 5$

$$\begin{pmatrix} 1-5 & 3 \\ 4 & 2-5 \end{pmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -2$

$$\begin{pmatrix} 1+2 & 3 \\ 4 & 2+2 \end{pmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$$

$$3x_2 + 3y_2 = 0$$

$$3x = -3y$$

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$-4x_1 + 3y_1 = 0$

$\frac{-4x_1}{-4} = \frac{-3y_1}{-4}$

$x_1 = \frac{3}{4}y_1$

$V_1 = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

OR

$V_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$x_2 = -y_2$

$\therefore V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$X(t) = C_1 e^{5t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3C_1 e^{5t} - C_2 e^{-2t} \\ 4C_1 e^{5t} + C_2 e^{-2t} \end{pmatrix}$$

$x(0) = 3C_1 - C_2$

$14 = 3C_1 - C_2$

$y(0) = 4C_1 + C_2$

$7 = 4C_1 + C_2$

$3C_1 - \cancel{C_2} = 14$

$4C_1 + \cancel{C_2} = 7$

$7C_1 = 21$

$\boxed{C_1 = 3}$

$14 = 3(3) - C_2$

$14 = 9 - C_2$

$14 - 9 = -C_2$

$C_2 = -5$

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$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3(3)e^{5t} - (-5)e^{-2t} \\ 4(3)e^{5t} + (-5)e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 9e^{5t} + 5e^{-2t} \\ 12e^{5t} - 5e^{-2t} \end{pmatrix}$$