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Chapter 6

Exercises 6.1

(1) $\begin{cases} x' = t, \\ y' = \cos t \end{cases}$

Sol:- Solving 1st equation

$$\int x' = \int t\,dt$$

$$x = \frac{t^2}{2} + c_1$$

Now 2nd Equation

$$\int y' = \int \cos t\,dt$$

$$y = \sin t + C_2$$

General Solution is

$$\begin{cases} x(t) = \frac{1}{2}t^2 + c_1 \\ y(t) = \sin t + C_2 \end{cases}$$

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(2) $\begin{cases} x' = x \\ y' = 1 \end{cases}$

Solving 1^{st} eqtion.

$\int \cancel{x' =} x$

$x' - x = 0$

$r - 1 = 0$

$r = 1$

$\therefore x = c_1 e^t$

Solving 2nd eqtion.

$\int y' = \int 1 \, dt$

$y = t + c_2$

General Sol.

$\begin{cases} x(t) = c_1 e^t \\ y(t) = t + c_2 \end{cases}$

Solve each of the following IVP's.

(6) $x_1' = -x_1 + 1$, $x_1(0) = 0$

$x_2' = x_2$, $x_2(0) = 1$

Ans:

$$x_1' + x_1 = 1$$

$$x_1 e^{\int 1\,dt} = \int (1)\, e^{\int 1\,dt}$$

$$x_1 e^{t} = \int e^{t} + c_1$$

$$\frac{x_1 \cancel{e^{t}}}{\cancel{e^{t}}} = \frac{e^{t} + c_1}{e^{t}}$$

$$x_1 = 1 + e^{-t} c_1$$

$$x_2' = x_2$$

$$\cancel{\int x_2' = \int x_2\,dt}$$

$$x_2' - x_2 = 0$$

$$x_2 e^{\int -1\,dt} = \int (0) \cdot e^{\int -1\,dt}$$

$$x_2 e^{-t} = \int 0 + c_2$$

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$$x_2 e^{-t} = \cancel{t}\, c_2$$

$$\boxed{x_2 = e^{t} c_2}$$

$$\begin{cases} x_1(t) = 1 + e^{-t} c_1 \\ x_2(t) = e^{t} c_2 \end{cases}$$

$x_1(0) = 0$	$x_2(0) = 1$
$x_1(0) = 1 + e^{-0} c_1$	$x_2(0) = e^{0} c_2$
$0 = 1 + c_1$	$1 = 1\, c_2$
$\boxed{c_1 = -1}$	$\boxed{c_2 = 1}$

$$\begin{cases} x_1(t) = 1 - e^{-t} \\ x_2(t) = e^{t} \end{cases}$$

Q: Solve the system using operator method.

(23)
$$x' = -3x + 2y - e^{t}$$
$$y' = -3y + 1$$

Writing in operator Notation

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$$x' + 3x - 2y = -e^{t}$$

$$y' + 3y = 1$$

$$(D+3)x - 2y = -e^{t}$$

$$(D+3)y = 1$$

Solving eq (ii)

$$y' + 3y = 1$$

$$y e^{\int 3\,dt} = \int (1) e^{\int 3\,dt}$$

$$y e^{3t} = \int e^{3t} + c_1$$

$$y e^{3t} = \frac{e^{3t}}{3} + c_1$$

$$y = \frac{1}{3} + e^{-3t} c_1$$

Put in (i)

$$\cancel{y = \ldots}$$

$$\cancel{(D+3)\left(\frac{1}{3} + e^{-3t} c_1\right) = 1}$$

$$(D+3)x - 2\left(\frac{1}{3} + e^{-3t} c_1\right) = -e^{t}$$

$$Dx + 3x - \frac{2}{3} - 2e^{-3t} c_1 = -e^{t}$$

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$$Dx + 3x = \frac{2}{3} + 2e^{-3t}c_1 - e^{t}$$

$$x' + 3x = \frac{2 + 6e^{-3t}c_1 - 3e^{t}}{3}$$

$$x\, e^{\int 3\,dt} = \int \left(\frac{2 + 6e^{-3t}c_1 - 3e^{t}}{3}\right) \cdot e^{\int 3\,dt}$$

$$x\, e^{3t} = \int \left(\frac{2 + 6e^{-3t}c_1 - 3e^{t}}{3}\right) \cdot e^{3t}$$

$$x\, e^{3t} = \int \frac{2e^{3t} + 6 - 3e^{4t}}{3}$$

$$x\, e^{3t} = \frac{1}{3}\int (2e^{3t} - 3e^{4t} + 6)\, dt$$

$$x\, e^{3t} = \frac{1}{3}\left(\left(\frac{2}{3}e^{3t} - \frac{3e^{4t}}{4} + 6t\right) + c_2\right)$$

$$\boxed{x = \left(\frac{2}{9}e^{3t} - \frac{3}{12}e^{4t} + 2t + \frac{1}{3}c_2\right)e^{-3t}}$$

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Exercise 6.3

(9) $\begin{pmatrix} e^t \\ 2e^t \end{pmatrix}, \begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}$

Ans

Let $X_1(t) = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$, $X_2(t) = \begin{pmatrix} 3e^{-t} \\ e^{-t} \end{pmatrix}$

$$W(X_1(t), X_2(t)) = \begin{vmatrix} e^t & 3e^{-t} \\ 2e^t & e^{-t} \end{vmatrix}$$

$$\begin{aligned} // &= (1) - 6(1) \\ // &= \boxed{-5} \end{aligned}$$

$\therefore$ $X_1(t)$ and $X_2(t)$ are linearly indep. for $I = (-\infty, \infty)$.

(10) $\begin{pmatrix} t \\ 1 \end{pmatrix}, \begin{pmatrix} -t \\ t \end{pmatrix}$

Ans Let $X_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $X_2(t) = \begin{pmatrix} -t \\ t \end{pmatrix}$

$$W(X_1(t), X_2(t)) = \begin{vmatrix} t & -t \\ 1 & t \end{vmatrix}$$

$t(t-1)=0$

$t=0,\ t=1$

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$= t^2-(-t) = t^2-t$

Linearly independant for $I = \mathbb{R}-\{0,1\}$.

(11) $\begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$

Let $X_1(t) = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}, X_2(t) = \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$

$W(X_1(t), X_2(t)) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$

$= (\cos 2t)(2\cos 2t) - (\sin 2t)(-2\sin 2t)$

$= 2\cos^2 2t + 2\sin^2 2t$

$= 2(\cos^2 2t + \sin^2 2t)$

$= 2(1) = \boxed{2}$

(14) $\begin{pmatrix} 6e^{4t} \\ 6e^{4t} \\ 3e^{4t} \end{pmatrix}, \begin{pmatrix} e^{-2t} \\ e^{-2t} \\ -e^{-2t} \end{pmatrix}, \begin{pmatrix} 2e^{4t} \\ 2e^{4t} \\ e^{4t} \end{pmatrix}$

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$$W(x_1(t), x_2(t), x_3(t)) = \begin{vmatrix} 6e^{4t} & e^{-2t} & 2e^{4t} \\ 6e^{4t} & e^{-2t} & 2e^{4t} \\ 3e^{4t} & -e^{-2t} & e^{4t} \end{vmatrix}$$

$$= 6e^{4t}\left[e^{2t} - 2e^{2t}\right] - e^{-2t}\left[6e^{8t} - 6e^{8t}\right]$$

$$+ 2e^{4t}\left[-6e^{2t} - 3e^{2t}\right]$$

$$= 6e^{4t}\left[-e^{2t}\right] - e^{-2t}\left[0\right] + 2e^{4t}\left[-9\right.$$

$$= -6e^{6t} - 18e^{6t} = -24e^{6t}$$

Q15: $\varphi(t) = \begin{pmatrix} -2e^{-8t} & 5e^{-t} \\ e^{-8t} & e^{-t} \end{pmatrix}$ and

$$X'(t) = \begin{pmatrix} -3 & 10 \\ 1 & -6 \end{pmatrix} X(t)$$

$$X_1(t) = \begin{pmatrix} -2e^{-8t} \\ e^{-8t} \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} 5e^{-t} \\ e^{-t} \end{pmatrix}$$

For $X_1(t)$

$$X_1'(t) = \begin{pmatrix} 16e^{-8t} \\ -8e^{-8t} \end{pmatrix}$$

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$$\begin{pmatrix} 16e^{-8t} \\ -8e^{-8} \end{pmatrix} = \begin{pmatrix} -3 & 10 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} -2e^{-8t} \\ e^{-8t} \end{pmatrix}$$

$$11 = \begin{pmatrix} 6e^{-8t} + 10e^{-8t} \\ -2e^{-8t} - 6e^{-8t} \end{pmatrix}$$

$$11 = \begin{pmatrix} 16e^{-8t} \\ -8e^{-8t} \end{pmatrix}$$

For $X_2(t)$

$$x_2'(t) = \begin{pmatrix} -3 & 10 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} 5e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} -5e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -15e^{-t} + 10e^{-t} \\ 5e^{-t} - 6e^{-t} \end{pmatrix}$$

$$a = \begin{pmatrix} -10e^{-t} \\ -e^{-t} \end{pmatrix}$$