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CHAPTER 8

EIGENVALUES, EIGENVECTORS, AND DIAGONALIZATION

In the first seven chapters of this book we have used real numbers as entries of matrices and as scalars. Accordingly, we have only dealt with real vector spaces and with the vector space B^n, where the scalars and entries in a vector are the bits 0 and 1. In this chapter, we deal with matrices that have complex entries and with complex vector spaces. You may consult Appendix A for an introduction to complex numbers and for linear algebra with complex numbers.

8.1 EIGENVALUES AND EIGENVECTORS

In this chapter every matrix considered is a square matrix. Let A be an $n \times n$ matrix. Then, as we have seen in Sections 1.5 and 4.3, the function $L: R^n \to R^n$ defined by $L(\mathbf{x}) = A\mathbf{x}$, for $\mathbf{x}$ in R^n, is a linear transformation. A question of considerable importance in a great many applied problems is the determination of vectors $\mathbf{x}$, if there are any, such that $\mathbf{x}$ and $A\mathbf{x}$ are parallel (see Examples 1 and 2). Such questions arise in all applications involving vibrations; they arise in aerodynamics, elasticity, nuclear physics, mechanics, chemical engineering, biology, differential equations, and others. In this section we shall formulate this problem precisely; we also define some pertinent terminology. In the next section we solve this problem for symmetric matrices and briefly discuss the situation in the general case.

DEFINITION Let A be an $n \times n$ matrix. The number λ is called an **eigenvalue** of A if there exists a *nonzero* vector $\mathbf{x}$ in R^n such that

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{1}$$

Every nonzero vector $\mathbf{x}$ satisfying (1) is called an **eigenvector of A associated with the eigenvalue λ**. The word *eigenvalue* is a hybrid one (*eigen* in German means "proper"). Eigenvalues are also called **proper values, characteristic values,** and **latent values**; and eigenvectors are also called **proper vectors**, and so on, accordingly.

Note that $\mathbf{x} = \mathbf{0}$ always satisfies (1), but $\mathbf{0}$ is not an eigenvector, since we insist that an eigenvector be a nonzero vector.

Remark In the preceding definition, the number λ can be real or complex and the vector $\mathbf{x}$ can have real or complex components.

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EXAMPLE 1 If A is the identity matrix I_n, then the only eigenvalue is $\lambda = 1$; every nonzero vector in R^n is an eigenvector of A associated with the eigenvalue $\lambda = 1$:

$$I_n\mathbf{x} = 1\mathbf{x}. \quad \blacksquare$$

EXAMPLE 2 Let

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Then

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_1 = \frac{1}{2}$. Also,

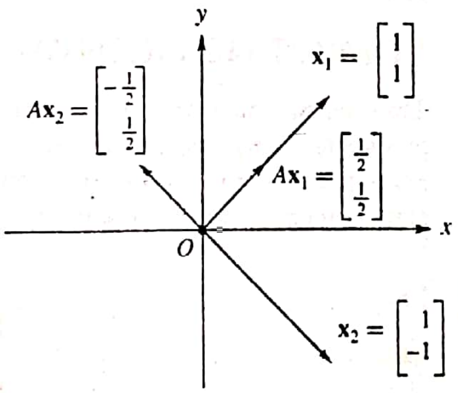
$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so that

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_2 = -\frac{1}{2}$. Figure 8.1 shows that $\mathbf{x}_1$ and $A\mathbf{x}_1$ are parallel, and $\mathbf{x}_2$ and $A\mathbf{x}_2$ are parallel also. This illustrates the fact that if $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ and $A\mathbf{x}$ are parallel. $\blacksquare$

Figure 8.1 ▶



Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. In Figure 8.2 we show $\mathbf{x}$ and $A\mathbf{x}$ for the cases $\lambda > 1$, $0 < \lambda < 1$, and $\lambda < 0$.

An eigenvalue λ of A can have associated with it many different eigenvectors. In fact, if $\mathbf{x}$ is an eigenvector of A associated with λ (i.e., $A\mathbf{x} = \lambda\mathbf{x}$) and r is any nonzero real number, then

$$A(r\mathbf{x}) = r(A\mathbf{x}) = r(\lambda\mathbf{x}) = \lambda(r\mathbf{x}).$$

Thus $r\mathbf{x}$ is also an eigenvector of A associated with λ.

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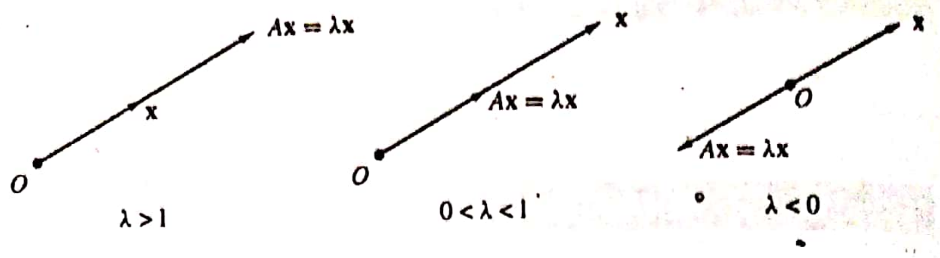
Figure 8.2 ▶



Remark Note that two eigenvectors associated with the same eigenvalue need not be in the same direction. They must only be parallel. Thus, in Example 2, it can be verified easily that $\mathbf{x}_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is another eigenvector associated with the eigenvalue $\lambda_1 = \frac{1}{2}$.

EXAMPLE 3 Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_1 = 0$. Also,

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_2 = 1$ (verify). ■

Example 3 points out the fact that although the zero vector, by definition, cannot be an eigenvector, the number zero can be an eigenvalue.

COMPUTING EIGENVALUES AND EIGENVECTORS

Thus far we have found the eigenvalues and associated eigenvectors of a given matrix by inspection, geometric arguments, or very simple algebraic approaches. In the following example, we compute the eigenvalues and associated eigenvectors of a matrix by a somewhat more systematic method.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

We wish to find the eigenvalues of A and their associated eigenvectors. Thus we wish to find all real numbers λ and all nonzero vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

satisfying (1), that is,

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

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Hence all eigenvectors associated with the eigenvalue $\lambda_1 = 2$ are given by $\begin{bmatrix} r \\ r \end{bmatrix}$, r any nonzero real number. In particular,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector associated with $\lambda_1 = 2$. Similarly, for $\lambda_2 = 3$ we obtain, from (3),

$$\begin{aligned} (3-1)x_1 - \quad\quad x_2 &= 0 \\ 2x_1 + (3-4)x_2 &= 0 \end{aligned}$$

or

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 2x_1 - x_2 &= 0. \end{aligned}$$

All solutions to this last homogeneous system are given by

$$\begin{aligned} x_1 &= \tfrac{1}{2}x_2 \\ x_2 &= \text{any real number } r. \end{aligned}$$

Hence all eigenvectors associated with the eigenvalue $\lambda_2 = 3$ are given by $\begin{bmatrix} \frac{1}{2}r \\ r \end{bmatrix}$, r any nonzero real number. In particular,

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 3$. ∎

In Examples 1, 2, and 3 we found eigenvalues and eigenvectors by inspection, whereas in Example 4 we proceeded in a more systematic fashion. We use the procedure of Example 4 as our standard method, as follows.

DEFINITION Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ matrix. The determinant

$$f(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} \tag{4}$$

is called the **characteristic polynomial of** A. The equation

$$f(\lambda) = \det(\lambda I_n - A) = 0$$

is called the **characteristic equation of** A.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is (verify)

$$\begin{aligned} f(\lambda) = \det(\lambda I_3 - A) &= \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda - 0 & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix} \\ &= \lambda^3 - 6\lambda^2 + 11\lambda - 6. \end{aligned}$$ ∎

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Recall from Chapter 3 that each term in the expansion of the determinant of an $n \times n$ matrix is a product of n elements of the matrix, containing exactly one element from each row and exactly one element from each column. Thus, if we expand $f(\lambda) = \det(\lambda I_n - A)$, we obtain a polynomial of degree n. A polynomial of degree n with real coefficients has n roots (counting repeats), some of which may be complex numbers. The expression involving λ^n in the characteristic polynomial of A comes from the product

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

so the coefficient of λ^n is 1. We can then write

$$f(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.$$

If we let $\lambda = 0$ in $\det(\lambda I_n - A)$ as well as in the expression on the right, then we get $\det(-A) = c_n$, which shows that the constant term c_n is $(-1)^n \det(A)$. This result can be used to establish the following theorem.

THEOREM 8.1 *An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.*

Proof Exercise T.7(b). ■

We now extend our list of nonsingular equivalences.

List of Nonsingular Equivalences

The following statements are equivalent for an $n \times n$ matrix A.

1. A is nonsingular.
2. $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.
6. A has rank n.
7. A has nullity 0.
8. The rows of A form a linearly independent set of n vectors in R^n.
9. The columns of A form a linearly independent set of n vectors in R^n.
10. Zero is *not* an eigenvalue of A.

We now connect the characteristic polynomial of a matrix with its eigenvalues in the following theorem.

THEOREM 8.2 *The eigenvalues of A are the roots of the characteristic polynomial of A.*

Proof Let λ be an eigenvalue of A with associated eigenvector $\mathbf{x}$. Then $A\mathbf{x} = \lambda\mathbf{x}$, which can be rewritten as

$$A\mathbf{x} = (\lambda I_n)\mathbf{x}$$

or

$$(\lambda I_n - A)\mathbf{x} = \mathbf{0}, \tag{5}$$

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Key Terms

Eigenvalue
Eigenvector
Proper value
Characteristic value
Latent value
Characteristic polynomial
Characteristic equation
Roots of the characteristic polynomial
Eigenspace
Leslie matrix
Stable age distribution
Invariant subspace

8.1 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.
   (a) Verify that $\lambda_1 = 1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} r \\ 2r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.
   (b) Verify that $\lambda_1 = 4$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} r \\ -r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.

2. Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.
   (a) Verify that $\lambda_1 = -1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an associated eigenvector.
   (b) Verify that $\lambda_2 = 2$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ is an associated eigenvector.
   (c) Verify that $\lambda_3 = 4$ is an eigenvalue of A and $\mathbf{x}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$ is an associated eigenvector.

In Exercises 3 through 7, find the characteristic polynomial of each matrix.

3. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$
5. $\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
6. $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$
7. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

In Exercises 8 through 15, find the characteristic polynomial, eigenvalues, and eigenvectors of each matrix.

8. $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$
9. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$
10. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
11. $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$
12. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$
13. $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$
14. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$
15. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

16. Find the characteristic polynomial, the eigenvalues and associated eigenvectors of each of the following matrices.
   (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   (b) $\begin{bmatrix} -2 & -4 & -8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 2-i & 2i & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
   (d) $\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$

17. Find all the eigenvalues and associated eigenvectors of each of the following matrices.
   (a) $\begin{bmatrix} -1 & -1+i \\ 1 & 0 \end{bmatrix}$
   (b) $\begin{bmatrix} i & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
   (d) $\begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

In Exercises 18 and 19, find bases for the eigenspaces (see Exercise T.1) associated with each eigenvalue.

18. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
19. $\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 20 through 23, find a basis for the eigenspace (see Exercise T.1) associated with λ.

20. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$
21. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $\lambda = 2$
22. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$, $\lambda = 3$
23. $\begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\lambda = 2$

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Exercise 8.1

Q1 Let $A=\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

a) verify that $\lambda_1=1$ is an eigenvalue of A and $X_1=\begin{bmatrix} r \\ 2r \end{bmatrix}, r\neq 0,$ is an eigenvector associated to $\lambda_1=1$

(5) By definition $AX=\lambda X \Rightarrow AX_1=\lambda_1 X_1$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} r \\ 2r \end{bmatrix} = \begin{bmatrix} 3r-2r \\ -2r+4r \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} = (1)\begin{bmatrix} r \\ 2r \end{bmatrix} = \lambda_1 X_1$$

$\Rightarrow \lambda_1 = 1$ is an eigen value of A and $X_1=\begin{bmatrix} r \\ 2r \end{bmatrix}, r\neq 0,$ is an associated eigen~~value~~ vector.

Q3 – Q7: Find the characteristic polynomial of each matrix

Q3) $A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$. The characteristic polynomial of A is defined as $f(\lambda)=\det(\lambda I_3 - A)$ — (1), where λ is an eigenvalue of A.

$$\lambda I_3 - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 1 & -3 & \lambda-2 \end{bmatrix}$$

$$(1) \Rightarrow f(\lambda) = \begin{vmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & -2 \\ 1 & -3 & \lambda-2 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-2 \end{vmatrix} - 0 + 1\begin{vmatrix} -2 & -1 \\ \lambda-1 & -2 \end{vmatrix}$$

$\Rightarrow f(\lambda)=(\lambda-1)[(\lambda-1)(\lambda-2)-6]+(4+\lambda-1)$

$\Rightarrow f(\lambda)=(\lambda-1)(\lambda^2-3\lambda-4)+4+\lambda-1$

$\Rightarrow f(\lambda)=\lambda^3-4\lambda^2 - \lambda+8+\lambda-1$

$\Rightarrow f(\lambda)=\lambda^3-4\lambda^2+7$ is the desired characteristic polynomial of A.

Exercise 8.1

Q8 — Q15: Find the characteristic polynomial, eigenvalues and eigenvectors of each matrix.

Q14) $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

(S) The characteristic polynomial of A is gin by

$f(\lambda) = \det(\lambda I_3 - A)$ — ①, where λ is an eigenvalue of A. $(\lambda I_3 - A) = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-2 & 0 & 0 \\ -3 & \lambda+1 & 0 \\ 0 & -4 & \lambda-3 \end{bmatrix}$

① $\Rightarrow f(\lambda) = \begin{vmatrix} \lambda-2 & 0 & 0 \\ -3 & \lambda+1 & 0 \\ 0 & -4 & \lambda-3 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda+1 & 0 \\ -4 & \lambda-3 \end{vmatrix} - 0 + 0$

$\Rightarrow f(\lambda) = (\lambda-2)(\lambda+1)(\lambda-3) + 0$

$\Rightarrow f(\lambda) = (\lambda-2)(\lambda^2-2\lambda-3)$

$\Rightarrow f(\lambda) = \lambda^3-4\lambda^2+\lambda+6$ is the desired characteristic polynomial of A.

Now $f(\lambda) = \lambda^3-4\lambda^2+\lambda+6 = 0$, is called the characteristic equation of A.

$\lambda^3-4\lambda^2+\lambda+6 = 0 \Rightarrow \lambda^3+\lambda^2-5\lambda^2-5\lambda+6\lambda+6 = 0$

$\Rightarrow \lambda^2(\lambda+1) - 5\lambda(\lambda+1) + 6(\lambda+1) = 0$

$\Rightarrow (\lambda+1)(\lambda^2-5\lambda+6) = 0 \Rightarrow (\lambda+1)(\lambda-2)(\lambda-3) = 0$

$\Rightarrow \lambda = -1, 2, 3$ are the desired eigen values of A.

Note: The eigenvalues of a diagonal, lower triangular and upper triangular matrix lie on the main diagonal i.e. they are the diagonal entries. Like here A is lower triangular, so its eigen values are $2, -1, 3$.

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Exercise 8.1

Eigenvectors of A: Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A associated with eigenvalue $\lambda_1 = -1$, then by definition, we have

$$AX_1 = \lambda_1 X_1 \Rightarrow \lambda_1 X_1 - AX_1 = 0 \Rightarrow (\lambda_1 I_3 - A)X_1 = 0$$

$$\Rightarrow (-I_3 - A)X_1 = 0 \Rightarrow \begin{bmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -3x \\ -3x \\ -4y-4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -3x &= 0 \\ -3x &= 0 \\ -4y-4z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 0 \\ y+z &= 0 \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} x &= 0 \\ y &= -z \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 0 \\ y &= -\gamma \\ z &= \gamma \end{aligned}, \quad z = \gamma(\neq 0) \in \mathbb{R}$$

Thus $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma \\ \gamma \end{bmatrix}, \gamma \neq 0$, are all eigenvectors associated with $\lambda_1 = -1$. For $\gamma = 1$, $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigen vector associated with $\lambda_1 = -1$.

(*) The eigenspace associated with eigenvalue $\lambda_1 = -1$, is

$$\left\{ \begin{bmatrix} 0 \\ -\gamma \\ \gamma \end{bmatrix} / \gamma \in \mathbb{R} \right\}.$$

(*)' The basis set of the eigenspace associated with $\lambda_1 = -1$, is

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(*)'' The dimension of the eigenspace associated with $\lambda_1 = -1$, is 1, as there is only one eigenvector in the basis set.

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Exercise 8.1

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A associated with the eigenvalue $\lambda_2 = 2$, then by definition

$$A X_2 = \lambda_2 X_2 \Rightarrow \lambda_2 X_2 = A X_2$$

$$\Rightarrow \lambda_2 X_2 - A X_2 = 0 \Rightarrow (\lambda_2 I_3 - A) X_2 = 0$$

$$\Rightarrow (2 I_3 - A) X_2 = 0 \quad \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 0 \\ -3x+3y \\ -4y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} 0=0 \\ -3x+3y=0 \\ -4y-z=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x-y=0 - (i) \\ 4y+z=0 - (ii) \end{matrix}$$

Let $z = r$, $r (\neq 0) \in \mathbb{R}$, then (ii) $\Rightarrow$ $y = -\frac{1}{4} r$

(i) $\Rightarrow$ $x = y \Rightarrow x = -\frac{1}{4} r$.

Thus $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} r \\ -1/4 r \\ r \end{bmatrix}$, $r \neq 0$ all eigen vectors

For $r = 4$, $X_2 = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$ is the eigenvector of A associated with eigenvalue $\lambda_2 = 2$.

similarly eigenspace of A associated with $\lambda_2 = 2$, is $\left\{ \begin{bmatrix} -1/4 r \\ -1/4 r \\ r \end{bmatrix} \middle/ r \in \mathbb{R} \right\}$. Basis of eigenspace associated with eigenvalue $\lambda_2 = 2$, is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\}$

Dimension of eigenspace here is 1, as there is only one vector in the basis set.

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Exercise 8.1

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be ~~an~~ ~~the~~ eigenvector of A associated with eigenvalue $\lambda_3 = 3$, Then by definition $AX_3 = \lambda_3 X_3 \Rightarrow$

$$\lambda_3 I_3 - A X_3 = 0 \Rightarrow$$

$$(3I_3 - A)X_3 = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ -3x+4y \\ -4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} x=0 \\ -3x+4y=0 \\ -4y=0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$$

Thus $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$, $z = r \ (\neq 0) \in \mathbb{R}$

For $r=1$, $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A associated with $\lambda_3 = 3$.

The eigenspace associated with eigen value $\lambda_3 = 3$, is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \middle/ r \in \mathbb{R} \right\}$$

The basis for the eigenspace associated with $\lambda_3 = 3$, is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension of the eigenspace associated with $\lambda_3 = 3$, is 1, as there is only one vector in the basis set.

Q16 — Q24 : similar to Q8 — Q15 .

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T.14. Show that if A is a matrix all of whose columns add up to 1, then $\lambda = 1$ is an eigenvalue of A. (*Hint*: Consider the product $A^T\mathbf{x}$, where $\mathbf{x}$ is a vector all of whose entries are 1 and use Exercise T.4.)

T.15. Let A be an $n \times n$ matrix and consider the linear operator on R^n defined by $L(\mathbf{u}) = A\mathbf{u}$, for $\mathbf{u}$ in R^n. A subspace W of R^n is called **invariant** under L if for any $\mathbf{w}$ in W, $L(\mathbf{w})$ is also in W. Show that an eigenspace of A is invariant under L (see Exercise T.1).

MATLAB Exercises

MATLAB has a pair of commands that can be used to find the characteristic polynomial and eigenvalues of a matrix. Command **poly(A)** *gives the coefficients of the characteristic polynomial of matrix A, starting with the highest-degree term. If we set* $\mathbf{v} =$ **poly(A)** *and then use command* **roots(v)**, *we obtain the roots of the characteristic polynomial of A. This process can also find complex eigenvalues, which are discussed in Appendix A.2.*

Once we have an eigenvalue λ of A, we can use **rref** *or* **homsoln** *to find a corresponding eigenvector from the linear system* $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

ML.1. Find the characteristic polynomial of each of the following matrices using MATLAB.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

ML.2. Use the **poly** and **roots** commands in MATLAB to find the eigenvalues of the following matrices:

(a) $A = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 0 & 1 \\ 4 & 1 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

ML.3. In each of the following cases, λ is an eigenvalue of A. Use MATLAB to find a corresponding eigenvector.

(a) $\lambda = 3$, $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

(b) $\lambda = -1$, $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

(c) $\lambda = 2$, $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$

ML.4. Consider a living organism that can live to a maximum age of two years and whose Leslie matrix is

$$\begin{bmatrix} 0.2 & 0.8 & 0.3 \\ 0.9 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}.$$

Find a stable age distribution.

8.2 DIAGONALIZATION

In this section we show how to find the eigenvalues and associated eigenvectors of a given matrix A by finding the eigenvalues and eigenvectors of a related matrix B that has the same eigenvalues and eigenvectors as A. The matrix B has the helpful property that its eigenvalues are easily obtained. Thus, we will have found the eigenvalues of A. In Section 8.3, this approach will shed much light on the eigenvalue-eigenvector problem. For convenience, we only work with matrices all of whose entries and eigenvalues are real numbers.

SIMILAR MATRICES

DEFINITION A matrix B is said to be **similar** to a matrix A if there is a nonsingular matrix P such that

$$B = P^{-1}AP.$$

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

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be the matrix of Example 4 in Section 8.1. Let

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$B = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus B is similar to A. ■

We shall let the reader (Exercise T.1) show that the following elementary properties hold for similarity:

1. A is similar to A.
2. If B is similar to A, then A is similar to B.
3. If A is similar to B and B is similar to C, then A is similar to C.

By property 2 we replace the statements "A is similar to B" and "B is similar to A" by "A and B are similar."

DEFINITION We shall say that the matrix A is **diagonalizable** if it is similar to a diagonal matrix. In this case we also say that A **can be diagonalized**.

EXAMPLE 2 If A and B are as in Example 1, then A is diagonalizable, since it is similar to B. ■

THEOREM 8.3 *Similar matrices have the same eigenvalues.*

Proof Let A and B be similar. Then $B = P^{-1}AP$, for some nonsingular matrix P. We prove that A and B have the same characteristic polynomials, $f_A(\lambda)$ and $f_B(\lambda)$, respectively. We have

$$\begin{aligned} f_B(\lambda) &= \det(\lambda I_n - B) = \det(\lambda I_n - P^{-1}AP) \\ &= \det(P^{-1}\lambda I_n P - P^{-1}AP) = \det(P^{-1}(\lambda I_n - A)P) \\ &= \det(P^{-1})\det(\lambda I_n - A)\det(P) \qquad (1) \\ &= \det(P^{-1})\det(P)\det(\lambda I_n - A) \\ &= \det(\lambda I_n - A) = f_A(\lambda). \end{aligned}$$

Since $f_A(\lambda) = f_B(\lambda)$, it follows that A and B have the same eigenvalues. ■

It follows from Exercise T.3 in Section 8.1 that the eigenvalues of a diagonal matrix are the entries on its main diagonal. The following theorem establishes when a matrix is diagonalizable.

THEOREM 8.4 *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.*

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Proof Suppose that A is similar to D. Then

$$P^{-1}AP = D,$$

a diagonal matrix, so

$$AP = PD. \tag{2}$$

Let

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

and let $\mathbf{x}_j$, $j = 1, 2, \ldots, n$, be the jth column of P. From Exercise T.9 in Section 1.3, it follows that the jth column of the matrix AP is $A\mathbf{x}_j$, and the jth column of PD is $\lambda_j\mathbf{x}_j$.

Thus from (2) we have

$$A\mathbf{x}_j = \lambda_j\mathbf{x}_j. \tag{3}$$

Since P is a nonsingular matrix, by Theorem 6.13 in Section 6.6 its columns are linearly independent and so are all nonzero. Hence λ_j is an eigenvalue of A and $\mathbf{x}_j$ is a corresponding eigenvector.

Conversely, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigenvalues of A and that the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent. Let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$ be the matrix whose jth column is $\mathbf{x}_j$. Since the columns of P are linearly independent, it follows from Theorem 6.13 in Section 6.6 that P is nonsingular. From (3) we obtain (2), which implies that A is diagonalizable. This completes the proof. ∎

Remark If A is a diagonalizable matrix, then $P^{-1}AP = D$, where D is a diagonal matrix. It follows from the proof of Theorem 8.4 that the diagonal elements of D are the eigenvalues of A. Moreover, P is a matrix whose columns are, respectively, n linearly independent eigenvectors of A. Observe also that in Theorem 8.4, the order of the columns of P determines the order of the diagonal entries in D.

EXAMPLE 3 Let A be as in Example 1. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. (See Example 4 in Section 8.1.) The corresponding eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are linearly independent. Hence A is diagonalizable. Here

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus, as in Example 1,

$$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

On the other hand, if we let $\lambda_1 = 3$ and $\lambda_2 = 2$, then

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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Then

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

Hence

$$P^{-1}AP = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

■

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 1$. Eigenvectors associated with λ_1 and λ_2 are vectors of the form

$$\begin{bmatrix} r \\ 0 \end{bmatrix},$$

where r is any nonzero real number. Since A does not have two linearly independent eigenvectors, we conclude that A is not diagonalizable. ■

The following is a useful theorem because it identifies a large class of matrices that can be diagonalized.

THEOREM 8.5 *If the roots of the characteristic polynomial of an $n \times n$ matrix A are all distinct (i.e., different from each other), then A is diagonalizable.*

Proof Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the distinct eigenvalues of A and let $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be a set of associated eigenvectors. We wish to show that S is linearly independent.

Suppose that S is linearly dependent. Then Theorem 6.4 of Section 6.3 implies that some vector $\mathbf{x}_j$ is a linear combination of the preceding vectors in S. We can assume that $S_1 = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{j-1}\}$ is linearly independent, for otherwise one of the vectors in S_1 is a linear combination of the preceding ones, and we can choose a new set S_2, and so on. We thus have that S_1 is linearly independent and that

$$\mathbf{x}_j = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{j-1}\mathbf{x}_{j-1}, \tag{4}$$

where $c_1, c_2, \dots, c_{j-1}$ are scalars. Premultiplying (multiplying on the left) both sides of Equation (4) by A, we obtain

$$\begin{aligned} A\mathbf{x}_j &= A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{j-1}\mathbf{x}_{j-1}) \\ &= c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_{j-1}A\mathbf{x}_{j-1}. \end{aligned} \tag{5}$$

Since $\lambda_1, \lambda_2, \dots, \lambda_j$ are eigenvalues of A and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_j$, its associated eigenvectors, we know that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ for $i = 1, 2, \dots, j$. Substituting in (5), we have

$$\lambda_j\mathbf{x}_j = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_{j-1}\lambda_{j-1}\mathbf{x}_{j-1}. \tag{6}$$

Multiplying (4) by λ_j, we obtain

$$\lambda_j\mathbf{x}_j = \lambda_jc_1\mathbf{x}_1 + \lambda_jc_2\mathbf{x}_2 + \cdots + \lambda_jc_{j-1}\mathbf{x}_{j-1}. \tag{7}$$

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Subtracting (7) from (6), we have

$$\begin{aligned}0 &= \lambda_j \mathbf{x}_j - \lambda_j \mathbf{x}_j \\ &= c_1(\lambda_1 - \lambda_j)\mathbf{x}_1 + c_2(\lambda_2 - \lambda_j)\mathbf{x}_2 + \cdots + c_{j-1}(\lambda_{j-1} - \lambda_j)\mathbf{x}_{j-1}.\end{aligned}$$

Since S_1 is linearly independent, we must have

$$c_1(\lambda_1 - \lambda_j) = 0, \quad c_2(\lambda_2 - \lambda_j) = 0, \ldots, \quad c_{j-1}(\lambda_{j-1} - \lambda_j) = 0.$$

Now

$$\lambda_1 - \lambda_j \neq 0, \quad \lambda_2 - \lambda_j \neq 0, \ldots, \quad \lambda_{j-1} - \lambda_j \neq 0$$

(because the λ's are distinct), which implies that

$$c_1 = c_2 = \cdots = c_{j-1} = 0.$$

From (4) we conclude that $\mathbf{x}_j = \mathbf{0}$, which is impossible if $\mathbf{x}_j$ is an eigenvector. Hence S is linearly independent, and from Theorem 8.4 it follows that A is diagonalizable. ■

Remark In the proof of Theorem 8.5, we have actually established the following somewhat stronger result: Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be k distinct eigenvalues of A with associated eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent (Exercise T.11).

If the roots of the characteristic polynomial of A are not all distinct, then A may or may not be diagonalizable. The characteristic polynomial of A can be written as the product of n factors, each of the form $\lambda - \lambda_j$, where λ_j is a root of the characteristic polynomial and the eigenvalues of A are the roots of the characteristic polynomial of A. Thus the characteristic polynomial can be written as

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of A, and $k_1, k_2, \ldots, k_r$ are integers whose sum is n. The integer k_i is called the **multiplicity** of λ_i. Thus in Example 4, $\lambda = 1$ is an eigenvalue of

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

of multiplicity 2. It can be shown that A can be diagonalized if and only if for each eigenvalue λ_j of multiplicity k_j we can find k_j linearly independent eigenvectors. This means that the solution space of the linear system $(\lambda_j I_n - A)\mathbf{x} = \mathbf{0}$ has dimension k_j. It can also be shown that if λ_j is an eigenvalue of A of multiplicity k_j, then we can never find more than k_j linearly independent eigenvectors associated with λ_j. We consider the following examples.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is $f(\lambda) = \lambda(\lambda - 1)^2$, so the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 1$. Thus $\lambda_2 = 1$ is an eigenvalue of multiplicity 2. We now consider the eigenvectors associated with the eigenvalues $\lambda_2 = \lambda_3 = 1$. They are obtained by solving the linear system $(1I_3 - A)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

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A solution is any vector of the form

$$\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix},$$

where r is any number, so the dimension of the solution space of the linear system $(1I_3 - A)\mathbf{x} = \mathbf{0}$ is 1. There do not exist two linearly independent eigenvectors associated with $\lambda_2 = 1$. Thus A cannot be diagonalized. ■

EXAMPLE 6 Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is $f(\lambda) = \lambda(\lambda - 1)^2$, so the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$; $\lambda_2 = 1$ is again an eigenvalue of multiplicity 2. Now we consider the solution space of $(1I_3 - A)\mathbf{x} = \mathbf{0}$, that is, of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is any vector of the form

$$\begin{bmatrix} 0 \\ r \\ s \end{bmatrix}$$

for any numbers r and s. Thus we can take as eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$ the vectors

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now we look for an eigenvector associated with $\lambda_1 = 0$. We have to solve the homogeneous system $(0I_3 - A)\mathbf{x} = \mathbf{0}$, or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is any vector of the form

$$\begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}$$

for any number t. Thus

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is an eigenvector associated with $\lambda_1 = 0$. Since $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent, A can be diagonalized. ■

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Thus an $n \times n$ matrix will fail to be diagonalizable only if it does not have n linearly independent eigenvectors.

The procedure for diagonalizing a matrix A is as follows.

Step 1. Form the characteristic polynomial $f(\lambda) = \det(\lambda I_n - A)$ of A.

Step 2. Find the roots of the characteristic polynomial of A.

Step 3. For each eigenvalue λ_j of A of multiplicity k_j, find a basis for the solution space of $(\lambda_j I_n - A)\mathbf{x} = \mathbf{0}$ (the eigenspace associated with λ_j). If the dimension of the eigenspace is less than k_j, then A is not diagonalizable. We thus determine n linearly independent eigenvectors of A. In Section 6.5 we solved the problem of finding a basis for the solution space of a homogeneous system.

Step 4. Let P be the matrix whose columns are the n linearly independent eigenvectors determined in Step 3. Then $P^{-1}AP = D$, a diagonal matrix whose diagonal elements are the eigenvalues of A that correspond to the columns of P.

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Key Terms

Similar matrices
Diagonalizable
Diagonalized
Distinct eigenvalues
Multiplicity of an eigenvalue
Defective matrix

8.2 Exercises

In Exercises 1 through 8, determine whether the given matrix is diagonalizable.

1. $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} -2 & 2 \\ 5 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 3 & 3 & 5 \\ 3 & 2 & 2 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

9. Find a 2×2 nondiagonal matrix whose eigenvalues are 2 and -3, and associated eigenvectors are

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

respectively.

10. Find a 3×3 nondiagonal matrix whose eigenvalues are -2, -2, and 3, and associated eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

respectively.

In Exercises 11 through 22, find, if possible, a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

11. $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 0 \\ 8 & 0 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

21. $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

23. Let A be a 2×2 matrix whose eigenvalues are 3 and 4, and associated eigenvectors are

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

respectively. Without computation, find a diagonal matrix D that is similar to A and nonsingular matrix P such that $P^{-1}AP = D$.

24. Let A be a 3×3 matrix whose eigenvalues are -3, 4, and 4, and associated eigenvectors are

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

respectively. Without computation, find a diagonal matrix D that is similar to A and nonsingular matrix P such that $P^{-1}AP = D$.

In Exercises 25 through 28, find two matrices that are similar to A.

25. $A = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

26. $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

27. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

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In Exercises 29 through 32, determine whether the given matrix is similar to a diagonal matrix.

29. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **30.** $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

31. $\begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix}$ **32.** $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}$

In Exercises 33 through 36, show that each matrix is diagonalizable and find a diagonal matrix similar to the given matrix.

33. $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ **34.** $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

35. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ **36.** $\begin{bmatrix} 0 & -2 & 1 \\ 1 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

In Exercises 37 through 40, show that the given matrix is not diagonalizable.

37. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **38.** $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

39. $\begin{bmatrix} 10 & 11 & 3 \\ -3 & -4 & -3 \\ -8 & -8 & -1 \end{bmatrix}$ **40.** $\begin{bmatrix} 2 & 3 & 3 & 5 \\ 3 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

*A matrix A is called **defective** if A has an eigenvalue λ of multiplicity $m > 1$ for which the associated eigenspace has a basis of fewer than m vectors; that is, the dimension of the eigenspace associated with λ is less than m. In Exercises 41 through 44, use the eigenvalues of the given matrix to determine if the matrix is defective.*

41. $\begin{bmatrix} 8 & 7 \\ 0 & 8 \end{bmatrix}, \lambda = 8, 8$

42. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}, \lambda = 3, 3, 5$

43. $\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ -3 & -3 & -3 \end{bmatrix}, \lambda = 0, 0, 3$

44. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \lambda = 1, 1, -1, -1$

45. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Compute D^9.

46. Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$. Compute A^9. (*Hint*: Find a matrix P such that $P^{-1}AP$ is a diagonal matrix D and show that $A^9 = PD^9P^{-1}$.)

Theoretical Exercises

T.1. Show that:

(a) A is similar to A.

(b) If B is similar to A, then A is similar to B.

(c) If A is similar to B and B is similar to C, then A is similar to C.

T.2. Show that if A is nonsingular and diagonalizable, then A^{-1} is diagonalizable.

T.3. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Find necessary and sufficient conditions for A to be diagonalizable.

T.4. Let A and B be nonsingular $n \times n$ matrices. Show that AB^{-1} and $B^{-1}A$ have the same eigenvalues.

T.5. Prove or disprove: Every nonsingular matrix is similar to a diagonal matrix.

T.6. If A and B are nonsingular, show that AB and BA are similar.

T.7. Show that if A is diagonalizable, then:

(a) A^T is diagonalizable.

(b) A^k is diagonalizable, where k is a positive integer.

T.8. Show that if A and B are similar matrices, then A^k and B^k, for any nonnegative integer k, are similar.

T.9. Show that if A and B are similar matrices, then $\det(A) = \det(B)$.

T.10. Let A be an $n \times n$ matrix and let $B = P^{-1}AP$ be similar to A. Show that if $\mathbf{x}$ is an eigenvector of A associated with the eigenvalue λ of A, then $P^{-1}\mathbf{x}$ is an eigenvector of B associated with the eigenvalue λ of the matrix B.

T.11. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a matrix A with associated eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. Show that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent. (*Hint*: See the proof of Theorem 8.5.)

T.12. Show that if A and B are similar matrices, then they have the same characteristic polynomial.

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Exercise 8.2

Q1 —— Q8: Determine whether the given matrix is diagonalizable

Q5) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(S) As A is upper triangular matrix, hence $\lambda = 3, 3, 3$ are its eigen values of A.

By Theorem 8.5; if the roots of the characteristic polynomial of $n \times n$ matrix A are all distinct (i.e different from each other), then A is diagonalizable "

since roots of the characteristic polynomial, being the eigenvalues, are not distict, so theorem 8.5 fails here, and we will proceed for Theorem 8.4 which states that

" An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors "

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A associated with eigenvalue $\lambda = 3$, Then by definition we have,

$$AX = \lambda X \Rightarrow \lambda X - AX = 0 \Rightarrow (\lambda I_3 - A)X = 0$$

$$\Rightarrow (3I_3 - A)X = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -y \\ -z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -y = 0 \\ -z = 0 \\ 0 = 0 \end{matrix}\right\} \Rightarrow y = z = 0$$

Thus $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ For $x = r$, $X = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$, $r (\neq 0) \in \mathbb{R}$

For $r = 1$, $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is eigenvector of A associated with $\lambda = 3$

i.e only one eigenvector exists and by Theorem 8.4, not 3 eigen vectors exist, hence this theorem also fails and A is not diagonalizable.

Exercise 8.2

Q4) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(S) As A is upper triangular, so its eigen values are the entries on its main diagonal i.e $\lambda = 1, -1, 2$
Since the eigenvalues, being the roots of the characteristic polynomials, are distinct, hence by Theorem 8.5, A is diagonalizable.

Q9: Find a 2×2 nondiagonal matrix whose eigenvalues are 2 and -3, and associated eigenvectors are $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(S) Let A be the desired 2×2 nondiagonal matrix.
Since the eigenvalues are distinct, so A is diagonalizable and there exists a nonsingular matrix

$P = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ such that $P^{-1}AP = B$ — ①

where $B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is the diagonal matrix whose diagonal entries are the eigenvalues of A.

$P^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$. ① $\Rightarrow AP = PB \Rightarrow A = PBP^{-1}$

$\Rightarrow A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2/3 & 2/3 \\ -2 & -1 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 2/3 - 2 & -2/3 - 1 \\ -4/3 - 2 & 4/3 - 1 \end{bmatrix} = \begin{bmatrix} -4/3 & -5/3 \\ -10/3 & 1/3 \end{bmatrix}$

is the desired 2×2 nondiagonal matrix.

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Exercise 8.2

Q10: similar to Q9

Q11 — Q22: Find, if possible, a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

Q12) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(S) we need to find first eigenvalues of A.
The characteristic equation of A is given by
$f(\lambda) = \det(\lambda I_3 - A) = 0 \Rightarrow$

$\lambda I_3 - A = \begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-3 \end{bmatrix} \Rightarrow f(\lambda) = 0 \Rightarrow$

$\begin{vmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2(\lambda-3) = 0 \Rightarrow \lambda = 1, 1, 3$

since eigenvalues are repeated, so theorem 8.5 fails.
we proceed for theorem 8.4.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigenvector associated with $\lambda = 1$, then

$AX = \lambda X \Rightarrow (\lambda I_3 - A)X = 0 \Rightarrow (I_3 - A)X = 0$

$\Rightarrow \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -y-2z \\ 0 \\ -y-2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} -y-2z = 0 \\ 0 = 0 \\ -y-2z = 0 \end{matrix}\right\} \Rightarrow y + 2z = 0 \Rightarrow y = -2z$

$\Rightarrow y = -2r$, where $z = r \in \mathbb{R}$.

Thus $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -2r \\ r \end{bmatrix} = \begin{bmatrix} t \\ -2r \\ r \end{bmatrix}$, for $x = t \in \mathbb{R}$.

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Exercise 8.2

The eigenspace associated to $\lambda = 1$ of multiplicity $m=2$, has dimension 2, thus A is diagonalizable

For $r=0$ and $t=1$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $r=1$ and $t=0$, $X_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow$ For $\lambda=1$ of multiplicity 2 has two linearly independent eigenvectors.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigenvector associated with eigenvalue $\lambda_3 = 3$, then $AX_3 = \lambda_3 X_3 \Rightarrow \lambda_3 I_3 - AX_3 = 0$

$$\Rightarrow (\lambda_3 I_3 - A) X_3 = 0 \Rightarrow (3I_3 - A) X_3 = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2x-y-2z \\ y \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 2x-y-2z=0 \\ y=0 \\ -y=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2x-2z=0 \\ y=0 \end{array}\right\} \Rightarrow \begin{array}{l} x=z \\ y=0 \end{array}$$

$$\Rightarrow \begin{array}{l} x=r \\ y=0 \\ z=r \end{array}, \quad \text{where } z = r\,(\neq 0) \in \mathbb{R}.$$

Thus $X_3 = \begin{bmatrix} r \\ 0 \\ r \end{bmatrix}$, for $r=1$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

By theorem 8.4, there exist three linearly independent eigenvectors, hence A is diagonalizable and there exists a nonsingular matrix $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ such that

$P^{-1}AP = B$, where $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

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Exercise 8.2

Q19) $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) since A is lower triangular, so its eigenvalues are the entries on its main diagonal i.e $\lambda = 3, 3, 3$

As the eigenvalues are repeated i.e $\lambda = 3$ of multiplicity 3.

So Theorem 8.5 fails. we Proceed to solve the question by Theorem 8.4.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector of A associated with $\lambda = 3$, then $AX = \lambda X \Rightarrow \lambda X = AX$

$$\Rightarrow \lambda X - AX = 0 \quad \Rightarrow (\lambda I_3 - A)X = 0$$

$$\Rightarrow (3I_3 - A)X = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 0 &= 0 \\ -2x &= 0 \\ 0 &= 0 \end{aligned}\right\} \Rightarrow x = 0$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ t \end{bmatrix}$, where $r, t \in \mathbb{R}$, not simultaneously zero. The eigenspace associated with $\lambda = 3$ is as

$\left\{ \begin{bmatrix} 0 \\ r \\ t \end{bmatrix} / r, t \in \mathbb{R} \right\}$ of dimension 2 less than the multiplicity of $\lambda = 3$. Hence A is not diagonalizable and the ~~nondi~~ nonsingular matrix P is not possible.

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Exercise 8.2

Q23 – Q24: Similar to Q11 – Q22.

Q25 – Q28: – Find two matrices that are similar to A.

Q25) $A = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$. Since A is upper triangular, so its eigenvalues are as $\lambda = 3, 0$, which are distinct, so by Theorem 8.5, A is diagonalizable and similar to diagonal matrices given by

$$B_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

Q28) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. characteristic equation of A is given as

$$\det(\lambda I_3 - A) = 0 \Rightarrow \begin{vmatrix} \lambda-2 & 0 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -1 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2) \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0 \Rightarrow (\lambda-2)[(\lambda-2)^2 - 1] = 0$$

$$\Rightarrow (\lambda-2)(\lambda-2-1)(\lambda-2+1) = 0 \Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

$\Rightarrow \lambda = 1, 2, 3$ are distinct eigenvalues of A, so by Theorem 8.5, A is diagonalizable and similar to a diagonal matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, the other similar matrix to A is $C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, similarly other similar matrices to A are $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

i.e. there exist $3! = 3 \times 2 \times 1 = 6$ similar matrices to A.

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Exercise 8.2

CP29 — CP32 :- similar to CP25 — CP28 .

CP33 — CP36 :- Similar to CP11 — CP22

CP37 — CP40 :-

CP37: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (S) since A is upper triangular, so its eigenvalues are the entries on the main diagonal i.e. $\lambda = 1, 1$, which are repeated so ~~the~~ theorem 8.5 fails. we proceed for theorem 8.4

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigenvector associated with $\lambda = 1$, then

$$AX = \lambda X \Rightarrow \lambda X - AX = 0 \Rightarrow (\lambda I_2 - A)X = 0$$

$$\Rightarrow (I_2 - A)X = 0 \Rightarrow \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} -y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} -y = 0 \\ 0 = 0 \end{matrix}\right\} \Rightarrow y = 0$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}, \; r (\neq 0) \in \mathbb{R}$$

Eigenspace associated with eigenvalue $\lambda = 1$, is

$\left\{ \begin{bmatrix} r \\ 0 \end{bmatrix} / r \in \mathbb{R} \right\}$ of dimension 1 which is less than multiplicity $m = 2$ of $\lambda = 1$ ~~[illegible]~~, i.e theorem 8.4 also fails i.e there do not exist two linearly independent eigenvectors of A.

Hence A is not diagonalizable

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Exercise 8.2

Q41 — Q44: (Q42) $A=\begin{bmatrix}3 & 0 & 0\\ -2 & 3 & -2\\ 2 & 0 & 5\end{bmatrix}$, $\lambda=3,3,5$

we need to find the eigenspace ~~of~~ associated with $\lambda=3$ of multiplicity $m=2$.

Let $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ be the eigen vector of A associated with $\lambda=3$, Then by definition $AX=\lambda X \Rightarrow \lambda X = AX$

$\Rightarrow \lambda X - AX = 0 \Rightarrow (\lambda I_3 - A)X = 0$

$\Rightarrow (3I_3 - A)X = 0 \Rightarrow \begin{bmatrix}0 & 0 & 0\\ 2 & 0 & 2\\ -2 & 0 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}0\\ 2x+2z\\ -2x-2z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \Rightarrow \left.\begin{matrix}0=0\\ 2x+2z=0\\ -2x-2z=0\end{matrix}\right\} \Rightarrow x+z=0$

$\Rightarrow x=-z \Rightarrow x=-\gamma$, where $z=\gamma\in\mathbb{R}$

Thus $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-\gamma\\ y\\ \gamma\end{bmatrix}=\begin{bmatrix}-\gamma\\ t\\ \gamma\end{bmatrix}$, where $y=t\in\mathbb{R}$

Eigenspace associated with $\lambda=3$, is $\left\{\begin{bmatrix}-\gamma\\ t\\ \gamma\end{bmatrix} / \gamma, t\in\mathbb{R}\right\}$

For $\gamma=1, t=0$, $X_1=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$ and for $\gamma=0, t=1$, $X_2=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$

Basis for eigenspace associated with $\lambda=3$, is

$\left\{\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\right\}$ $\Rightarrow$ Dimension of eigenspace $=2$ which is not fewer than $m=2$

Hence A is not defective.

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