1

In MATLAB, the right side of the preceding expression is computed as

sqrt(dot(u, u))

Verify this alternative procedure on the vectors in Exercise ML.2.

ML.7. In MATLAB, if the n-vectors $\mathbf{u}$ and $\mathbf{v}$ are entered as columns, then

$$\mathbf{u}' * \mathbf{v} \quad \text{or} \quad \mathbf{v}' * \mathbf{u}$$

gives the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$. Verify this using the vectors in Exercise ML.5.

ML.8. Use MATLAB to find the angle between each of the following pairs of vectors. (To convert the angle from radians to degrees, multiply by 180/pi.)

(a) $\mathbf{u} = (3, 2, 4, 0)$, $\mathbf{v} = (0, 2, -1, 0)$

(b) $\mathbf{u} = (2, 2, -1)$, $\mathbf{v} = (2, 0, 1)$

(c) $\mathbf{u} = (1, 0, 0, 2)$, $\mathbf{v} = (0, 3, -4, 0)$

ML.9. Use MATLAB to find a unit vector in the direction of the vectors in Exercise ML.2.

4.3 LINEAR TRANSFORMATIONS

In Section 1.5, we introduced matrix transformations, functions that map R^n into R^m. In this section, we present an alternative approach to matrix transformations. We shall now denote a function mapping R^n into R^m by L. In Chapter 10 we consider linear transformations from a much more general point of view and we study their properties in some detail.

DEFINITION A **linear transformation** L of R^n into R^m is a function assigning a unique vector $L(\mathbf{u})$ in R^m to each $\mathbf{u}$ in R^n such that

(a) $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$, for every $\mathbf{u}$ and $\mathbf{v}$ in R^n.

(b) $L(k\mathbf{u}) = kL(\mathbf{u})$, for every $\mathbf{u}$ in R^n and every scalar k.

A function T of R^n into R^m is said to be **nonlinear** if it is not a linear transformation.

The vector $L(\mathbf{u})$ in R^m is called the **image of u.** The set of all images in R^m of the vectors in R^n is called the **range of** L. Since R^n can be viewed as consisting of points or vectors, $L(\mathbf{u})$, for $\mathbf{u}$ in R^n, can be considered as a point or a vector in R^m.

We shall write the fact that L maps R^n into R^m, even if it is not a linear transformation, as

$$L\colon R^n \to R^m.$$

If $n = m$, a linear transformation $L\colon R^n \to R^n$ is also called a **linear operator** on R^n.

Let A be an $m \times n$ matrix. In Section 1.5, we defined a matrix transformation as a function $L\colon R^n \to R^m$ defined by $L(\mathbf{u}) = A\mathbf{u}$. We now show that every matrix transformation is a linear transformation by verifying that properties (a) and (b) in the preceding Definition hold.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$L(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = L(\mathbf{u}) + L(\mathbf{v}).$$

Moreover, if c is a scalar, then

$$L(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cL(\mathbf{u}).$$

Hence, every matrix transformation is a linear transformation.

2

For convenience we now summarize the matrix transformations that have already been presented in Section 1.5.

Reflection with respect to the x-axis: $L: R^2 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}.$$

Projection into the xy-plane: $L: R^3 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Dilation: $L: R^3 \to R^3$ is defined by $L(\mathbf{u}) = r\mathbf{u}$ for $r > 1$.

Contraction: $L: R^3 \to R^3$ is defined by $L(\mathbf{u}) = r\mathbf{u}$ for $0 < r < 1$.

Rotation counterclockwise through an angle ϕ: $L: R^2 \to R^2$ is defined by

$$L(\mathbf{u}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\mathbf{u}.$$

Shear in the x-direction: $L: R^2 \to R^2$ is defined by

$$L(\mathbf{u}) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\mathbf{u},$$

where k is a scalar.

Shear in the y-direction: $L: R^2 \to R^2$ is defined by

$$L(\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\mathbf{u},$$

where k is a scalar.

EXAMPLE 1 Let $L: R^3 \to R^2$ be defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 1 \\ u_2 - u_3 \end{bmatrix}.$$

To determine whether L is a linear transformation, let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} (u_1 + v_1) + 1 \\ (u_2 + v_2) - (u_3 + v_3) \end{bmatrix}.$$

On the other hand,

$$L(\mathbf{u}) + L(\mathbf{v}) = \begin{bmatrix} u_1 + 1 \\ u_2 - u_3 \end{bmatrix} + \begin{bmatrix} v_1 + 1 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} (u_1 + v_1) + 2 \\ (u_2 - u_3) + (v_2 - v_3) \end{bmatrix}.$$

Since the first coordinates of $L(\mathbf{u} + \mathbf{v})$ and $L(\mathbf{u}) + L(\mathbf{v})$ are different, $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$, so we conclude that the function L is not a linear transformation; that is, L is nonlinear. ∎

3

It can be shown that $L: R^n \to R^m$ is a linear transformation if and only if $L(a\mathbf{u}+b\mathbf{v}) = aL(\mathbf{u})+bL(\mathbf{v})$ for any real numbers a and b and any vectors $\mathbf{u}$, $\mathbf{v}$ in R^n (see Exercise T.4 in the Supplementary Exercises).

The following two theorems give some additional basic properties of linear transformations from R^n to R^m. The proofs will be left as exercises. Moreover, a more general version of the second theorem below will be proved in Section 10.1.

THEOREM 4.6 *If $L: R^n \to R^m$ is a linear transformation, then*

$$L(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1L(\mathbf{u}_1) + c_2L(\mathbf{u}_2) + \cdots + c_kL(\mathbf{u}_k)$$

for any vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in R^n and any scalars $c_1, c_2, \ldots, c_k$.

Proof Exercise T.1. ■

THEOREM 4.7 *Let $L: R^n \to R^m$ be a linear transformation. Then:*

(a) $L(\mathbf{0}_{R^n}) = \mathbf{0}_{R^m}$

(b) $L(\mathbf{u}-\mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$, *for $\mathbf{u}$ and $\mathbf{v}$ in R^n*

Proof Exercise T.2. ■

COROLLARY 4.1 *Let $T: R^n \to R^m$ be a function. If $T(\mathbf{0}_{R^n}) \neq \mathbf{0}_{R^m}$, then T is a nonlinear transformation.*

Proof Let $T(\mathbf{0}_{R^n}) = \mathbf{w} \neq \mathbf{0}_{R^m}$. Then

$$T(\mathbf{0}_{R^n}) = T(\mathbf{0}_{R^n} + \mathbf{0}_{R^n}) = T(\mathbf{0}_{R^n}) + T(\mathbf{0}_{R^n}).$$

However,

$$T(\mathbf{0}_{R^n}) + T(\mathbf{0}_{R^n}) = \mathbf{w} + \mathbf{w} = 2\mathbf{w}.$$

Since $\mathbf{w} \neq 2\mathbf{w}$, T is nonlinear. ■

Remark Example 1 could be solved more easily by using Corollary 4.1 as follows: Since

$$L\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix},$$

L is nonlinear.

These theorems can be used to compute the image of a vector $\mathbf{u}$ in R^2 or R^3 under a linear transformation $L: R^2 \to R^n$ once we know $L(\mathbf{i})$ and $L(\mathbf{j})$, where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. Similarly, we can compute $L(\mathbf{u})$ for $\mathbf{u}$ in R^3 under the linear transformation $L: R^3 \to R^n$ if we know $L(\mathbf{i})$, $L(\mathbf{j})$, and $L(\mathbf{k})$, where $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$. It follows from the observations in Sections 4.1 and 4.2 that if $\mathbf{v} = (v_1, v_2)$ is any vector in R^2 and $\mathbf{u} = (u_1, u_2, u_3)$ is any vector in R^3, then

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} \quad \text{and} \quad \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}.$$

EXAMPLE 2 Let $L: R^3 \to R^2$ be a linear transformation for which we know that

$$L(1,0,0) = (2,-1), \quad L(0,1,0) = (3,1), \quad \text{and} \quad L(0,0,1) = (-1,2).$$

Find $L(-3,4,2)$.

4

Solution Since

$$(-3, 4, 2) = -3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k},$$

we have

$$\begin{aligned} L(-3, 4, 2) &= L(-3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \\ &= -3L(\mathbf{i}) + 4L(\mathbf{j}) + 2L(\mathbf{k}) \\ &= -3(2, -1) + 4(3, 1) + 2(-1, 2) \\ &= (4, 11). \end{aligned}$$

More generally, we pointed out after Example 15 in Section 4.2, that if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is any vector in R^n, then

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n,$$

where

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

This implies that if $L\colon R^n \to R^m$ is a linear transformation for which we know $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, $\ldots$, $L(\mathbf{e}_n)$, then we can compute $L(\mathbf{u})$. Thus, we can easily compute the image of any vector $\mathbf{u}$ in R^n. (See Theorem 10.3 in Section 10.1.)

EXAMPLE 3 Let $L\colon R^2 \to R^3$ be defined by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then L is a linear transformation (verify). Observe that the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix}$$

lies in range L because

$$L\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1-6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} = \mathbf{v}.$$

To find out whether the vector

$$\mathbf{w} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

lies in range L we need to determine whether there is a vector

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$$

so that $L(\mathbf{u}) = \mathbf{w}$. We have

$$L(\mathbf{u}) = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \\ x - 2y \end{bmatrix}.$$

5

Since we want $L(\mathbf{u}) = \mathbf{w}$, we have

$$\begin{bmatrix} x+y \\ y \\ x-2y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}.$$

This means that

$$\begin{aligned} x+y &= 3 \\ y &= 5 \\ x-2y &= 2. \end{aligned}$$

This linear system of three equations in two unknowns has no solution (verify). Hence, $\mathbf{w}$ is not in range L. ■

EXAMPLE 4 (Cryptology) Cryptology is the technique of coding and decoding messages; it goes back to the time of the ancient Greeks. A simple code is constructed by associating a different number with every letter in the alphabet. For example,

$$\begin{array}{cccccccc} A & B & C & D & \cdots & X & Y & Z \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & \cdots & 24 & 25 & 26 \end{array}$$

Suppose that Mark S. and Susan J. are two undercover agents who want to communicate with each other by using a code because they suspect that their phone is being tapped and their mail is being intercepted. In particular, Mark wants to send Susan the message

MEET TOMORROW

Using the substitution scheme given previously, Mark sends the message

13 5 5 20 20 15 13 15 18 18 15 23

A code of this type could be cracked without too much difficulty by a number of techniques, including the analysis of frequency of letters. To make it difficult to crack the code, the agents proceed as follows. First, when they undertook the mission they agreed on a 3×3 nonsingular matrix such as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Mark then breaks the message into four vectors in R^3 (if this cannot be done, we can add extra letters). Thus, we have the vectors

$$\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}, \quad \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}.$$

Mark now defines the linear transformation $L: R^3 \to R^3$ by $L(\mathbf{x}) = A\mathbf{x}$, so the message becomes

$$A\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix}, \quad A\begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix},$$

$$A\begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix}, \quad A\begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix} = \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix}.$$

6

Thus Mark transmits the message

$$38 \quad 28 \quad 15 \quad 105 \quad 70 \quad 50 \quad 97 \quad 64 \quad 51 \quad 117 \quad 79 \quad 61$$

Suppose now that Mark receives the following message from Susan,

$$77 \quad 54 \quad 38 \quad 71 \quad 49 \quad 29 \quad 68 \quad 51 \quad 33 \quad 76 \quad 48 \quad 40 \quad 86 \quad 53 \quad 52$$

which he wants to decode with the same key matrix A as previously. To decode it, Mark breaks the message into five vectors in R^3:

$$\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix}, \quad \begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix}, \quad \begin{bmatrix} 68 \\ 51 \\ 33 \end{bmatrix}, \quad \begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix}, \quad \begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix}$$

and solves the equation

$$L(\mathbf{x}_1) = \begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix} = A\mathbf{x}_1$$

for $\mathbf{x}_1$. Since A is nonsingular,

$$\mathbf{x}_1 = A^{-1}\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 15 \end{bmatrix}.$$

Similarly,

$$\mathbf{x}_2 = A^{-1}\begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 7 \end{bmatrix}, \qquad \mathbf{x}_3 = A^{-1}\begin{bmatrix} 68 \\ 51 \\ 33 \end{bmatrix} = \begin{bmatrix} 18 \\ 1 \\ 16 \end{bmatrix},$$

$$\mathbf{x}_4 = A^{-1}\begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 12 \end{bmatrix}, \qquad \mathbf{x}_5 = A^{-1}\begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ 19 \end{bmatrix}.$$

Using our correspondence between letters and numbers, Mark has received the message

PHOTOGRAPH PLANS

Additional material on cryptology may be found in the references given in Further Readings at the end of this section. ■

We have already seen, in general, that if A is an $m \times n$ matrix, then the matrix transformation $L: R^n \to R^m$ defined by $L(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in R^n is a linear transformation. In the following theorem, we show that if $L: R^n \to R^m$ is a linear transformation, then L must be a matrix transformation.

THEOREM 4.8

Let $L: R^n \to R^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A such that

$$L(\mathbf{x}) = A\mathbf{x} \tag{1}$$

for $\mathbf{x}$ in R^n.

7

Proof If

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is any vector in R^n, then

$$\mathbf{x} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n,$$

so by Theorem 4.6,

$$L(\mathbf{x}) = c_1 L(\mathbf{e}_1) + c_2 L(\mathbf{e}_2) + \cdots + c_n L(\mathbf{e}_n). \tag{2}$$

If we let A be the $m \times n$ matrix whose jth column is $L(\mathbf{e}_j)$ and

$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

then Equation (2) can be written as

$$L(\mathbf{x}) = A\mathbf{x}.$$

We now show that the matrix A is unique. Suppose that we also have

$$L(\mathbf{x}) = B\mathbf{x} \quad \text{for } \mathbf{x} \text{ in } R^n.$$

Letting $\mathbf{x} = \mathbf{e}_j$, $j = 1, \ldots, n$, we obtain

$$L(\mathbf{e}_j) = A\mathbf{e}_j = \mathrm{col}_j(A)$$

and

$$L(\mathbf{e}_j) = B\mathbf{e}_j = \mathrm{col}_j(B).$$

Thus the columns of A and B agree, so $A = B$. ■

The matrix $A = \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \end{bmatrix}$ in Equation (1) is called the **standard matrix representing** L.

EXAMPLE 5 Let $L\colon R^3 \to R^3$ be the linear operator defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y - z \\ x + z \end{bmatrix}.$$

Find the standard matrix representing L and verify Equation (1).

8

Solution The standard matrix A representing L is the 3×3 matrix whose columns are $L(\mathbf{e}_1)$, $L(\mathbf{e}_2)$, and $L(\mathbf{e}_3)$, respectively. Thus

$$L(\mathbf{e}_1) = L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1+0\\0-0\\1+0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} = \operatorname{col}_1(A)$$

$$L(\mathbf{e}_2) = L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0+1\\1-0\\0+0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = \operatorname{col}_2(A)$$

$$L(\mathbf{e}_3) = L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0+0\\0-1\\0+1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix} = \operatorname{col}_3(A).$$

Hence

$$A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\1 & 0 & 1\end{bmatrix}.$$

Thus we have

$$A\mathbf{x} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y\\y-z\\x+z\end{bmatrix} = L(\mathbf{x}),$$

so Equation (1) holds. ∎

Further Readings in Cryptology

Elementary presentation

KOHN, BERNICE. *Secret Codes and Ciphers.* Upper Saddle River, N.J.: Prentice Hall, Inc., 1968 (63 pages).

Advanced presentation

FISHER, JAMES L. *Applications-Oriented Algebra.* New York: T. Harper & Row, Publishers, 1977 (Chapter 9, "Coding Theory").

GARRETT, PAUL. *Making, Breaking Codes.* Upper Saddle River, N.J.: Prentice Hall, Inc., 2001.

HARDY, DAREL W., and CAROL L. WALKER, *Applied Algebra, Codes, Ciphers and Discrete Algorithms.* Upper Saddle River, N.J.: Prentice Hall, Inc., 2002.

KAHN, DAVID. *The Codebreakers.* New York: The New American Library Inc., 1973.

Key Terms

Linear transformation
Nonlinear function
Image
Range
Cryptology

9

4.3 Exercises

1. Which of the following are linear transformations?
 (a) $L(x, y) = (x + 1, y, x + y)$
 (b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y \\ x - z \end{bmatrix}$
 (c) $L(x, y) = (x^2 + x, y - y^2)$

2. Which of the following are linear transformations?
 (a) $L(x, y, z) = (x - y, x^2, 2z)$
 (b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ 3y - 2z \\ 2z \end{bmatrix}$
 (c) $L(x, y) = (x - y, 2x + 2)$

3. Which of the following are linear transformations?
 (a) $L(x, y, z) = (x + y, 0, 2x - z)$
 (b) $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ x^2 + y^2 \end{bmatrix}$
 (c) $L(x, y) = (x - y, 0, 2x + 3)$

4. Which of the following are linear transformations?
 (a) $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_1^2 + u_2 \\ u_1 - u_3 \end{bmatrix}$
 (b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
 (c) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 5 through 12, sketch the image of the given point P or vector $\mathbf{u}$ under the given linear transformation L.

5. $L: R^2 \to R^2$ is defined by
$$L(x, y) = (x, -y);\ P = (2, 3).$$

6. $L: R^2 \to R^2$ is defined by
$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix};\ \mathbf{u} = (1, -2).$$

7. $L: R^2 \to R^2$ is a counterclockwise rotation through 30°; $P = (-1, 3)$.

8. $L: R^2 \to R^2$ is a counterclockwise rotation through $\frac{2}{3}\pi$ radians; $\mathbf{u} = (-2, -3)$.

9. $L: R^2 \to R^2$ is defined by $L(\mathbf{u}) = -\mathbf{u}$; $\mathbf{u} = (3, 2)$.

10. $L: R^2 \to R^2$ is defined by $L(\mathbf{u}) = 2\mathbf{u}$; $\mathbf{u} = (-3, 3)$.

11. $L: R^3 \to R^2$ is defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x - y \end{bmatrix};\ \mathbf{u} = (2, -1, 3).$$

12. $L: R^3 \to R^3$ is defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix};\ \mathbf{u} = (0, -2, 4).$$

13. Let $L: R^3 \to R^3$ be the linear transformation defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + z \\ y + z \\ x + 2y + 2z \end{bmatrix}.$$
Is $\mathbf{w}$ in range L?
 (a) $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (b) $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

14. Let $L: R^3 \to R^3$ be the linear transformation defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
Is $\mathbf{w}$ in range L?
 (a) $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (b) $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

15. Let $L: R^3 \to R^3$ be defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
Find an equation relating a, b, and c so that
$$\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
will lie in range L.

16. Repeat Exercise 15 if $L: R^3 \to R^3$ is defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y + 3z \\ -3x - 2y - z \\ -2x + 2z \end{bmatrix}.$$

17. Let $L: R^2 \to R^2$ be a linear transformation such that
$$L(\mathbf{i}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad L(\mathbf{j}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$
Find $L\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$.

18. Let $L: R^3 \to R^3$ be a linear transformation such that
$$L(\mathbf{i}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\quad L(\mathbf{j}) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\quad \text{and } L(\mathbf{k}) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$
Find $L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)$.

19. Let L be the linear transformation defined in Exercise 11. Find all vectors $\mathbf{x}$ in R^3 such that $L(\mathbf{x}) = \mathbf{0}$.

20. Repeat Exercise 19, where L is the linear transformation defined in Exercise 12.

21. Describe the following linear transformations geometrically.

 (a) $L(x, y) = (-x, y)$

 (b) $L(x, y) = (-x, -y)$ Ref about the origin

 (c) $L(x, y) = (-y, x)$

22. Describe the following linear transformations geometrically.

 (a) $L(x, y) = (y, x)$ Ref about $y = x$

 (b) $L(x, y) = (-y, -x)$

 (c) $L(x, y) = (2x, 2y)$

In Exercises 23 and 24, determine whether L is a linear transformation.

23. $L: R^2 \to R^2$ defined by $L(x, y) = (x + y + 1, x - y)$

24. $L: R^2 \to R^1$ defined by $L(x, y) = \sin x + \sin y$

In Exercises 25 through 30, find the standard matrix representing L.

25. $L: R^2 \to R^2$ is reflection with respect to the y-axis.

26. $L: R^2 \to R^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

27. $L: R^2 \to R^2$ is counterclockwise rotation through $\frac{\pi}{4}$ radians.

28. $L: R^2 \to R^2$ is counterclockwise rotation through $\frac{\pi}{3}$ radians.

29. $L: R^3 \to R^3$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + z \\ y - z \end{bmatrix}.$$

30. $L: R^3 \to R^3$ is defined by $L(\mathbf{u}) = -2\mathbf{u}$.

31. Use the substitution and matrix A of Example 4.

 (a) Code the message SEND HIM MONEY.

 (b) Decode the message 67 44 41 49 39 19 113 76 62 104 69 55.

32. Use the substitution scheme of Example 4 and the matrix

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$

 (a) Code the message WORK HARD.

 (b) Decode the message

 93 36 60 21 159 60 110 43

Theoretical Exercises

T.1. Prove Theorem 4.6.

T.2. Prove Theorem 4.7.

T.3. Show that $L: R^n \to R^n$ defined by $L(\mathbf{u}) = r\mathbf{u}$, where r is a scalar, is a linear operator on R^n.

T.4. Let $\mathbf{u}_0 \neq \mathbf{0}$ be a fixed vector in R^n. Let $L: R^n \to R^n$ be defined by $L(\mathbf{u}) = \mathbf{u} + \mathbf{u}_0$. Determine whether L is a linear transformation. Justify your answer.

T.5. Let $L: R^1 \to R^1$ be defined by $L(\mathbf{u}) = a\mathbf{u} + b$, where a and b are real numbers (of course, $\mathbf{u}$ is a vector in R^1, which in this case means that $\mathbf{u}$ is also a real number). Find all values of a and b such that L is a linear transformation.

T.6. Show that the function $O: R^n \to R^m$ defined by $O(\mathbf{u}) = \mathbf{0}_{R^m}$ is a linear transformation, which is called the **zero linear transformation**.

T.7. Let $I: R^n \to R^n$ be defined by $I(\mathbf{u}) = \mathbf{u}$, for $\mathbf{u}$ in R^n. Show that I is a linear transformation, which is called the **identity operator** on R^n.

T.8. Let $L: R^n \to R^m$ be a linear transformation. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $L(\mathbf{u}) = \mathbf{0}$ and $L(\mathbf{v}) = \mathbf{0}$, then $L(a\mathbf{u} + b\mathbf{v}) = \mathbf{0}$ for any scalars a and b.

T.9. Let $L: R^2 \to R^2$ be the linear transformation defined by $L(\mathbf{u}) = A\mathbf{u}$, where

$$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

For $\phi = 30°$, L defines a counterclockwise rotation by 30°.

 (a) If $T_1(\mathbf{u}) = A^2\mathbf{u}$, describe the action of T_1 on $\mathbf{u}$.

 (b) If $T_2(\mathbf{u}) = A^{-1}\mathbf{u}$, describe the action of T_2 on $\mathbf{u}$.

 (c) What is the smallest positive value of k for which $T(\mathbf{u}) = A^k\mathbf{u} = \mathbf{u}$?

T.10. Let $O: R^n \to R^n$ be the zero linear transformation defined by $O(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v}$ in R^n (see Exercise T.6). Find the standard matrix representing O.

T.11. Let $I: R^n \to R^n$ be the identity linear transformation defined by $I(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v}$ in R^n (see Exercise T.7). Find the standard matrix representing I.

Exercise 4.3

Q1/b which of the following are linear transformations?

$L(x, y) = (x^2+x, y-y^2)$

(S) $L(x, y) = (x^2+x, y-y^2) = L(u)$ — (1)

Let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$, then

a) $L(u_1+u_2) = L(x_1+x_2, y_1+y_2)$

$= \left((x_1+x_2)^2 + (x_1+x_2), (y_1+y_2) - (y_1+y_2)^2\right)$ — (2) By (1)

Also $L(u_1) = L(x_1, y_1) = (x_1^2+x, y_1-y_1^2)$ By (1)

$L(u_2) = L(x_2, y_2) = (x_2^2+x_2, y_2-y_2^2)$ By (1)

Then $L(u_1) + L(u_2) = (x_1^2+x_1, y_1-y_1^2) + (x_2^2+x_2, y_2-y_2^2)$

$= \left(x_1^2+x_2^2+(x_1+x_2), (y_1+y_2) - (y_1^2+y_2^2)\right)$ — (3)

From (2) and (3), we have

$L(u_1+u_2) \neq L(u_1) + L(u_2) \Rightarrow L$ is not linear.

as $(x_1+x_2)^2 \neq x_1^2+x_2^2$ and $(y_1+y_2)^2 \neq y_1^2+y_2^2$

As for linear transformation, the following two conditions must be satisfied

a) $L(u_1+u_2) = L(u_1) + L(u_2)$, $u_1, u_2 \in \mathbb{R}^m$

b) $L(ku) = k\,L(u)$ $k \in \mathbb{R}, u \in \mathbb{R}^n$

if $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Exercise 4.3

$Q_2 + Q_3$: similar to $Q1$

Q_4 which of the following are linear transformations?

b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(b) Here $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$L(u) = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ -y+2z \\ x+y-z \end{bmatrix} \quad - (1)$$

To see whether L is linear, we need to show that

a) $L(u_1+u_2) = L(u_1) + L(u_2) \quad \forall\ u_1, u_2 \in \mathbb{R}^3$

b) $L(ku) = k\,L(u) \quad \forall\ u \in \mathbb{R}^3$ and $k \in \mathbb{R}$

Now a) let $u_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$, Then

$$L(u_1+u_2) = L\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1+x_2)+(y_1+y_2) \\ -(y_1+y_2)+2(z_1+z_2) \\ (x_1+x_2)+(y_1+y_2)-(z_1+z_2) \end{bmatrix}$$

$$= \begin{bmatrix} (x_1+y_1)+(x_2+y_2) \\ (-y_1+2z_1)+(-y_2+2z_2) \\ (x_1+y_1-z_1)+(x_2+y_2-z_2) \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ -y_1+2z_1 \\ x_1+y_1-z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2 \\ -y_2+2z_2 \\ x_2+y_2-z_2 \end{bmatrix} \quad \text{By (1)}$$

$$= L\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

$$= L(u_1) + L(u_2) \Rightarrow L(u_1+u_2) = L(u_1) + L(u_2)$$

so (a) is satisfied.

b) For any $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ and any $k \in \mathbb{R}$, we have

$$L(ku) = L\left(\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}\right) = \begin{bmatrix} kx+ky \\ -ky+2kz \\ kx+ky-kz \end{bmatrix} = \begin{bmatrix} k(x+y) \\ k(-y+2z) \\ k(x+y-z) \end{bmatrix} = k\begin{bmatrix} x+y \\ -y+2z \\ x+y-z \end{bmatrix}$$

$$= k\,L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = k\,L(u) \Rightarrow L(ku) = k\,L(u)$$

Hence L is linear transformation.

Exercise 4.3

Q5 — Q12: Sketch the image of the given point P or vector u under the given linear transformation L.

Q5: $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a counterclockwise rotation through 30°, $P = (-1, 3)$.

(5) L, here is defined by $L(u) = Au$; (1) where

$$A = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} 0.86 & -0.5 \\ 0.5 & 0.86 \end{bmatrix}$$

Let $u = \begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$, then

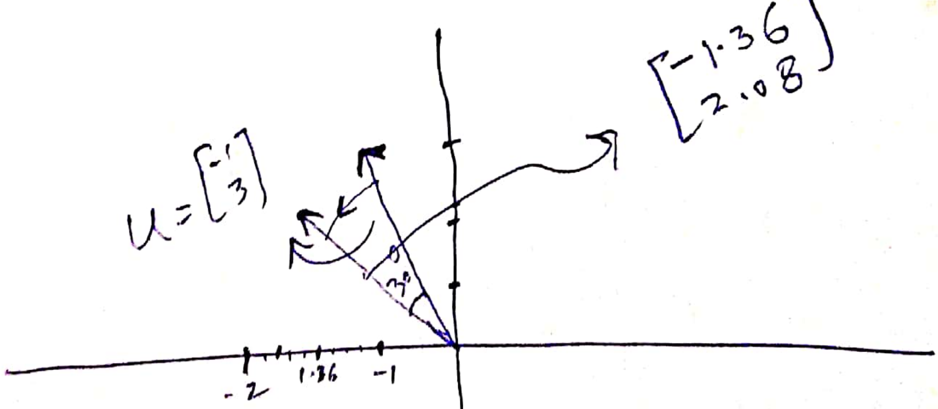
$$(1) \Rightarrow L(u) = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.86 & -0.5 \\ 0.5 & 0.86 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow L(u) = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.86x - 0.5y \\ 0.5x + 0.86y \end{bmatrix} \quad (2)$$

Let $u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ be the position vector of point $P = (-1, 3)$

Then $(2) \Rightarrow L\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} +0.86(-1) - 0.5(3) \\ (0.5)(-1) + (0.86)(3) \end{bmatrix} = \begin{bmatrix} -0.86 - 1.5 \\ -0.5 + 2.58 \end{bmatrix}$

$\Rightarrow L\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -1.36 \\ 2.08 \end{bmatrix}$. Thus image of $u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} -1.36 \\ 2.08 \end{bmatrix}$



Exercise 4.3

Q13/a Let $L:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix}.$$

Is w in range L?

a) $w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

(S) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix}$ — (1)

Let $w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ be in range L, then there exists some $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in the domain $\mathbb{R}^3$ of L such that

$L(u) = w \Rightarrow L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ — (2).

From (1) and (2), we have

$$\begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} x+z &= 1 \quad \text{(i)} \\ y+z &= -1 \quad \text{(ii)} \\ x+2y+2z &= 0 \quad \text{(iii)} \end{aligned}\right\} \to (I)$$

(I) is a linear system.

(ii) in (iii) $\Rightarrow x+2(y+z)=0 \Rightarrow x+2(-1)=0 \Rightarrow \boxed{x=2}$

(i) $\Rightarrow x+z=1 \Rightarrow 2+z=1 \Rightarrow \boxed{z=-1}$ and (ii) $\Rightarrow y-1=-1$

$\Rightarrow \boxed{y=0}$. Thus $u = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ exists which is the solution of (I)

i.e. there exists $u = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ whose image is w under L.

Hence $w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in range L.

Q14 is similar to Q13

Q15 – Q16: similar to Q40 in exercise 1.6

Q15) suppose w is in the range L, then there exists a vector $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $L(u) = w \Rightarrow$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad — (1)$$

But $$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad — (2) \text{ Given}$$

From (1) and (2), we have

$$\underbrace{\begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{d} \Rightarrow AX = d \quad — (3)$$

Augmented matrix of (3) is as

$$[A:d] = \left[\begin{array}{ccc|c} 4 & 1 & 3 & a \\ 2 & -1 & 3 & b \\ 2 & 2 & 0 & c \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 1 & 1/4 & 3/4 & a/4 \\ 2 & -1 & 3 & b \\ 2 & 2 & 0 & c \end{array}\right]$$

$$\xrightarrow{\substack{R_2-2R_1 \\ R_3-2R_1}} \left[\begin{array}{ccc|c} 1 & 1/4 & 3/4 & a/4 \\ 0 & -3/2 & 3/2 & b-a/2 \\ 0 & 3/2 & -3/2 & c-a/2 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1/4 & 3/4 & a/4 \\ 0 & -3/2 & 3/2 & b-a/2 \\ 0 & 0 & 0 & -a+b+c \end{array}\right]$$

solution of (3) exists if $-a+b+c=0$ or $a-b-c=0$, which is the desired equation relating a, b and c so that $w = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ will lie in range L.

Q16: similar to Q15.

Exercise 4.3

$Q_{17}-Q_{18}$: similar to example-2 (Page-3 / book Page-249).

Q_{19}: $L:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x-y \end{bmatrix}$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Find all vectors X in $\mathbb{R}^3$ such that $L(X)=0$.

$L(X)=0 \Rightarrow L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0$ — ①

But $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x-y \end{bmatrix}$ — ②

From ① and ②, we have $\begin{bmatrix} x \\ x-y \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} x \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} x=0 \\ x-y=0 \end{matrix}\right\} \Rightarrow x=y=0$.

Thus $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$, where $z = r \in \mathbb{R}$

Hence $X = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$, $r \in \mathbb{R}$, are all vectors in $\mathbb{R}^3$ for which

$L(X)=0$.

Q_{20}: similar to Q_{19}.

$Q_{21}-Q_{22}$: similar to $Q_{15}-Q_{17}$ in exercise 1.5

$Q_{23}-Q_{24}$: similar to $Q_1 - Q_4$ in this exercise

$Q_{25}-Q_{30}$: similar to example-5 Page-7 (Book Page-253)

$Q_{31}-Q_{32}$: similar to example-4 Page-5 (Book Page-251).