Chapter 13.2, Problem 9P

Step-by-step solution

Step 1 of 3

Principal Argument:

Find the principal value of the argument of a complex number $-1+i$.

A complex number $z = x + iy$ can be represented in polar form as shown below,

$z = r(\cos\theta + i\sin\theta)$ ...... **(1)**

Here $\theta = \arg z, \tan\theta = \frac{y}{x}$.

Principal value of the argument is $\text{Arg } z$ which lies in the interval $-\pi < \text{Arg } z \le \pi$.

Step 2 of 3

Let $z = -1 + i \to x = -1, y = 1$.

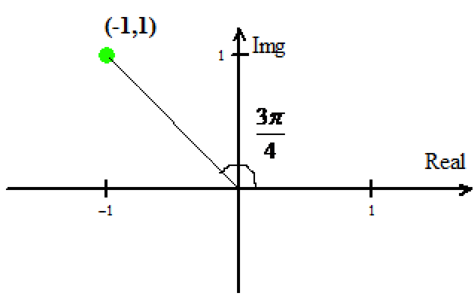
Complex number is in second quadrant.

$$\begin{aligned}\text{Arg } z &= \pi - \tan^{-1}\left|\frac{y}{x}\right| \quad (\text{since } \theta \text{ is in second quadrant}) \\ &= \pi - \tan^{-1}(1) \\ &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4}\end{aligned}$$

Hence, the principal argument is $\boxed{\frac{3\pi}{4}}$.

Step 3 of 3

Graph is as shown below.



Chapter 13.2, Problem 10P

Step-by-step solution

Step 1 of 5

Principal Argument:

Find the principal value of the arguments of a complex numbers $-5, -5-i, -5+i$.

A complex number $z = x + iy$ can be represented in polar form as shown below,

$z = r(\cos\theta + i\sin\theta)$ …… **(1)**

Here $\theta = \arg z, \tan\theta = \frac{y}{x}$.

Principal value of the argument is $\mathrm{Arg}\, z$ which lies in the interval $-\pi < \mathrm{Arg}\, z \le \pi$.

Step 2 of 5

Let $z = -5 \rightarrow x = -5, y = 0$.

Complex number is on negative $x-$ axis.

$$\begin{aligned}\mathrm{Arg}\, z &= \pi - \tan^{-1}\left|\frac{y}{x}\right| \\ &= \pi - \tan^{-1}(0) \\ &= \pi\end{aligned}$$

Hence, the principal argument is $\boxed{\pi}$.

Step 3 of 5

Let $z = -5 - i \rightarrow x = -5, y = -1$.

Complex number is in third quadrant.

$$\begin{aligned}\mathrm{Arg}\, z &= -\left(\pi - \tan^{-1}\left|\frac{y}{x}\right|\right) \\ &= -\left(\pi - \tan^{-1}\left(\frac{1}{5}\right)\right) \\ &\approx -2.94\end{aligned}$$

Hence, the principal argument is $\boxed{-2.94}$.

Step 4 of 5

Let $z = -5 + i \rightarrow x = -5, y = 1$.

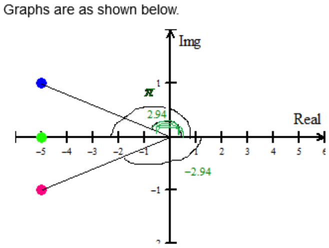
Complex number is in second quadrant.

$$\begin{aligned}\mathrm{Arg}\, z &= \pi - \tan^{-1}\left|\frac{y}{x}\right| \\ &= \pi - \tan^{-1}\left(\frac{1}{5}\right) \\ &\approx 2.94\end{aligned}$$

Hence, the principal argument is $\boxed{2.94}$.

Step 5 of 5

Graphs are as shown below.



Chapter 13.2, Problem 11P

Step-by-step solution

Step 1 of 4

Principal Argument:

Find the principal value of the arguments of a complex numbers $3 \pm 4i$.

A complex number $z = x + iy$ can be represented in polar form as shown below,

$z = r(\cos\theta + i\sin\theta)$ ...... **(1)**

Here $\theta = \arg z, \tan\theta = \frac{y}{x}$.

Principal value of the argument is $\operatorname{Arg} z$ which lies in the interval $-\pi < \operatorname{Arg} z \le \pi$.

Step 2 of 4

Let $z = 3 + 4i \to x = 3, y = 4$.

Complex number is in first quadrant.

$$\begin{aligned}\operatorname{Arg} z &= \tan^{-1}\left|\frac{y}{x}\right| \\ &= \tan^{-1}\left(\frac{4}{3}\right) \\ &\approx 0.927\end{aligned}$$

Hence, the principal argument is $\boxed{0.927}$.

Step 3 of 4

Let $z = 3 - 4i \to x = 3, y = -4$.

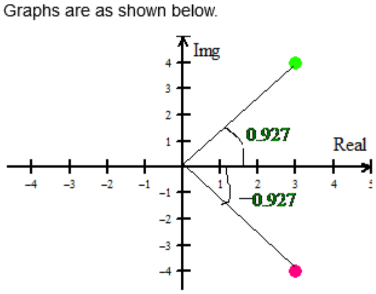
Complex number is in fourth quadrant.

$$\begin{aligned}\operatorname{Arg} z &= -\tan^{-1}\left|\frac{y}{x}\right| \\ &= -\tan^{-1}\left(\frac{4}{3}\right) \\ &\approx -0.927\end{aligned}$$

Hence, the principal argument is $\boxed{-0.927}$.

Step 4 of 4

Graphs are as shown below.



Chapter 13.2, Problem 12P

Step-by-step solution

Step **1** of 3

Principal Argument:

Find the principal value of the arguments of a complex numbers $-\pi-\pi i$.

A complex number $z=x+iy$ can be represented in polar form as shown below,

$z=r(\cos\theta+i\sin\theta)$ ...... **(1)**

Here $\theta=\arg z, \tan\theta=\frac{y}{x}$.

Principal value of the argument is $\operatorname{Arg} z$ which lies in the interval $-\pi<\operatorname{Arg} z\le\pi$.

Step **2** of 3

Let $z=-\pi-\pi i\to x=-\pi, y=-\pi$.

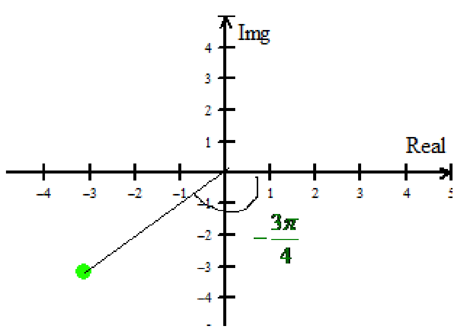
Complex number is in third quadrant.

$$\begin{aligned}\operatorname{Arg} z&=-\left(\pi-\tan^{-1}\left|\frac{y}{x}\right|\right)\\&=-\left(\pi-\tan^{-1}(1)\right)\\&=-\left(\pi-\frac{\pi}{4}\right)\\&=-\frac{3\pi}{4}\end{aligned}$$

Hence, the principal argument is $\boxed{-\frac{3\pi}{4}}$.

Step **3** of 3

Graph is as shown below.



Chapter 13.2, Problem 13P

Step-by-step solution

Step 1 of 4

Principal Argument:

Find the principal value of the argument of a complex number $(1+i)^{20}$.

A complex number $z = x + iy$ can be represented in polar form as shown below,

$z = r(\cos\theta + i\sin\theta)$ ...... **(1)**

Here $\theta = \arg z, \tan\theta = \frac{y}{x}$.

Principal value of the argument is $\text{Arg } z$ which lies in the interval $-\pi < \text{Arg } z \le \pi$.

Step 2 of 4

First convert $(1+i)^{20}$ in the form of $z = x + iy$.

$$\begin{aligned}(1+i)^{20} &= \left[(1+i)^2\right]^{10}\\ &= (2i)^{10}\\ &= 2^{10} i^{10}\\ &= -2^{10} \quad \left(\text{since } i^{10} = -1\right)\end{aligned}$$

Step 3 of 4

Let $z = -2^{10} \to x = -2^{10}, y = 0$.

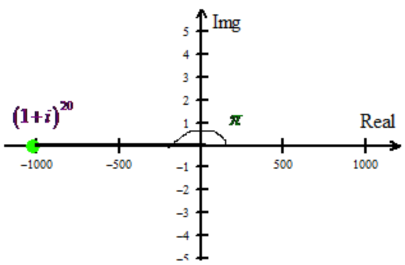
Complex number is on negative $x-$axis.

$$\begin{aligned}\text{Arg } z &= \pi - \tan^{-1}\left|\frac{y}{x}\right|\\ &= \pi - \tan^{-1}(0)\\ &= \pi\end{aligned}$$

Hence, the principal argument is $\boxed{\pi}$.

Step 4 of 4

Graph is as shown below.



Chapter 13.2, Problem 14P

Step-by-step solution

Step 1 of 4

Principal Argument:

Find the principal value of the arguments of a complex numbers $-1 \pm 0.1i$.

A complex number $z = x + iy$ can be represented in polar form as shown below,

$z = r(\cos\theta + i\sin\theta)$ ...... **(1)**

Here $\theta = \arg z, \tan\theta = \frac{y}{x}$.

Principal value of the argument is $\operatorname{Arg} z$ which lies in the interval $-\pi < \operatorname{Arg} z \le \pi$.

Step 2 of 4

Let $z = -1 + 0.1i \to x = -1, y = 0.1$.

Complex number is in second quadrant.

$$\begin{aligned} \operatorname{Arg} z &= \pi - \tan^{-1}\left|\frac{y}{x}\right| \\ &= \pi - \tan^{-1}(0.1) \\ &\approx 3.04 \end{aligned}$$

Hence, the principal argument is $\boxed{3.04}$.

Step 3 of 4

Let $z = -1 - 0.1i \to x = -1, y = -0.1$.

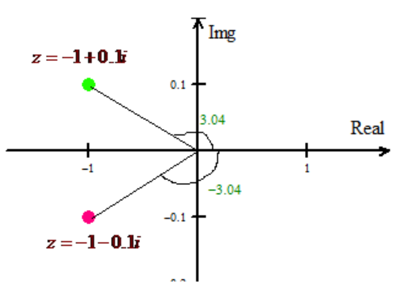
Complex number is in third quadrant.

$$\begin{aligned} \operatorname{Arg} z &= -\left(\pi - \tan^{-1}\left|\frac{y}{x}\right|\right) \\ &= -\left(\pi - \tan^{-1}(0.1)\right) \\ &\approx -3.04 \end{aligned}$$

Hence, the principal argument is $\boxed{-3.04}$.

Step 4 of 4

Graph is as shown below.



Chapter 13.2, Problem 21P

Step-by-step solution

Step 1 of 5

Roots:

Find the roots of $\sqrt[3]{1+i}$.

The n^{th} roots of a complex number $z=r(\cos\theta+i\sin\theta)$ are

$$\sqrt[n]{z}=\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right), k=0,1,\ldots,n-1 \quad \text{...... (1)}$$

Step 2 of 5

Let $z=1+i$.

Find the polar form of z.

Here $x=1, y=1$, so the complex number is in first quadrant.

$$\begin{aligned} r&=\sqrt{x^2+y^2}\\ &=\sqrt{1+1}\\ &=\sqrt{2}\end{aligned}$$

Amplitude θ is such that $\tan\theta=\frac{y}{x}$.

$$\begin{aligned}\theta&=\tan^{-1}\left(\frac{y}{x}\right)\\ &=\tan^{-1}(1)\\ &=\frac{\pi}{4}\end{aligned}$$

Hence, the polar form is $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

Step 3 of 5

From the equation **(1)**, required roots are as shown below.

$$\sqrt[3]{1+i}=\left(\sqrt{2}\right)^{1/3}\left(\cos\frac{2k\pi+\pi/4}{3}+i\sin\frac{2k\pi+\pi/4}{3}\right), k=0,1,2$$

$$\sqrt[3]{1+i}=2^{1/6}\left[\cos\frac{8k\pi+\pi}{12}+i\sin\frac{8k\pi+\pi}{12}\right] \quad \text{...... (2)}$$

Step 4 of 5

Let $k=0$ in the equation **(2)**, it gives

$$z_0=2^{1/6}\left[\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right]$$

Let $k=1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1&=2^{1/6}\left[\cos\frac{8\pi+\pi}{12}+i\sin\frac{8\pi+\pi}{12}\right]\\ &=2^{1/6}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]\end{aligned}$$

Let $k=2$ in the equation **(2)**, it gives

$$\begin{aligned} z_2&=2^{1/6}\left[\cos\frac{16\pi+\pi}{12}+i\sin\frac{16\pi+\pi}{12}\right]\\ &=2^{1/6}\left[\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right]\end{aligned}$$

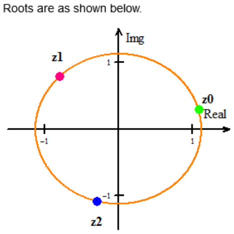
Hence, the required roots are

$$\boxed{2^{1/6}\left[\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right], 2^{1/6}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right], 2^{1/6}\left[\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right]}.$$

Step 5 of 5

All the roots lie on the circle of radius $2^{1/6}$.

Roots are as shown below.



Chapter 13.2, Problem 22P

Step-by-step solution

Step 1 of 4

Roots:

Find the roots of $\sqrt[3]{3+4i}$.

The n^{th} roots of a complex number $z=r(\cos\theta+i\sin\theta)$ are

$$\sqrt[n]{z}=\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right), k=0,1,\dots,n-1 \text{ ...... } \textbf{(1)}$$

Step 2 of 4

Let $z=3+4i$.

Find the polar form of z.

Here $x=3, y=4$, so the complex number is in first quadrant.

$$\begin{aligned} r&=\sqrt{x^2+y^2} \\ &=\sqrt{3^2+4^2} \\ &=\sqrt{25} \\ &=5 \end{aligned}$$

Amplitude θ is such that $\tan\theta=\frac{y}{x}$.

$$\begin{aligned} \theta&=\tan^{-1}\left(\frac{y}{x}\right) \\ &=\tan^{-1}\left(\frac{4}{3}\right)\approx 0.927 \end{aligned}$$

Hence, the polar form is $3+4i=5(\cos(0.927)+i\sin(0.927))$.

Step 3 of 4

From the equation **(1)**, required roots are as shown below.

$$\sqrt[3]{3+4i}=5^{1/3}\left(\cos\frac{2k\pi+0.927}{3}+i\sin\frac{2k\pi+0.927}{3}\right), k=0,1,2 \text{ ...... } \textbf{(2)}$$

Let $k=0$ in the equation **(2)**, it gives

$$\begin{aligned} z_0&=5^{1/3}\left(\cos\frac{0.927}{3}+i\sin\frac{0.927}{3}\right) \\ &=5^{1/3}(\cos(0.309)+i\sin(0.309)) \end{aligned}$$

Let $k=1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1&=5^{1/3}\left(\cos\frac{2\pi+0.927}{3}+i\sin\frac{2\pi+0.927}{3}\right) \\ &=5^{1/3}(\cos(2.403)+i\sin(2.403)) \end{aligned}$$

Let $k=2$ in the equation **(2)**, it gives

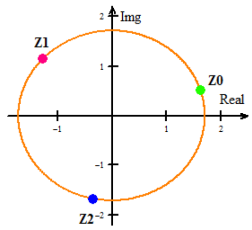
$$\begin{aligned} z_2&=5^{1/3}\left(\cos\frac{4\pi+0.927}{3}+i\sin\frac{4\pi+0.927}{3}\right) \\ &=5^{1/3}(\cos(4.5)+i\sin(4.5)) \end{aligned}$$

Hence, the required roots are $\boxed{\begin{array}{l} 5^{1/3}(\cos(0.309)+i\sin(0.309)), \\ 5^{1/3}(\cos(2.403)+i\sin(2.403)), \\ 5^{1/3}(\cos(4.5)+i\sin(4.5)) \end{array}}$.

Step 4 of 4

All the roots lie on the circle of radius $5^{1/3}$.

Roots are as shown below.



Chapter 13.2, Problem 23P

Step-by-step solution

Step 1 of 5

Roots:

Find the roots of $\sqrt[3]{216}$.

The n^{th} roots of a complex number $z = r(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), k = 0,1,...,n-1 \text{ ...... } \textbf{(1)}$$

Step 2 of 5

Let $z = 216$.

Find the polar form of z.

Here $x = 216, y = 0$, so the complex number is on positive $x-$axis.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{216^2} \\ &= 216 \end{aligned}$$

Amplitude θ is such that $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \theta &= \tan^{-1} 0 \\ &= 0 \end{aligned}$$

Hence, the polar form is $216 = 216(\cos 0 + i\sin 0)$.

Step 3 of 5

From the equation **(1)**, required roots are as shown below.

$$\sqrt[3]{216} = 216^{1/3}\left(\cos\frac{2k\pi+0}{3} + i\sin\frac{2k\pi+0}{3}\right), k = 0,1,2 \text{ ...... } \textbf{(2)}$$

Step 4 of 5

Let $k = 0$ in the equation **(2)**, it gives

$$\begin{aligned} z_0 &= 216^{1/3}(\cos 0 + i\sin 0) \\ &= 6(\cos 0 + i\sin 0) \\ &= 6 \end{aligned}$$

Let $k = 1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1 &= 216^{1/3}\left(\cos\frac{2\pi+0}{3} + i\sin\frac{2\pi+0}{3}\right) \\ &= 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \end{aligned}$$

Let $k = 2$ in the equation **(2)**, it gives

$$\begin{aligned} z_2 &= 216^{1/3}\left(\cos\frac{4\pi+0}{3} + i\sin\frac{4\pi+0}{3}\right) \\ &= 6\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \end{aligned}$$

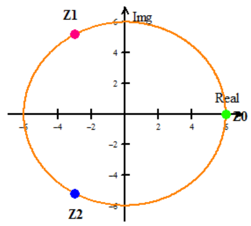
Hence, the required roots are

$$\boxed{6,\ 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right),\ 6\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}.$$

Step 5 of 5

All the roots lie on the circle of radius 6.

Roots are as shown below.



Chapter 13.2, Problem 24P

Step-by-step solution

Step 1 of 4

Roots:

Find the roots of $\sqrt[4]{-4}$.

The n^{th} roots of a complex number $z = r(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), k = 0,1,\ldots,n-1 \text{ ...... } \textbf{(1)}$$

Step 2 of 4

Let $z = -4$.

Find the polar form of z.

Here $x = -4, y = 0$, so the complex number is on negative $x-$axis.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{4^2} \\ &= 4 \end{aligned}$$

Amplitude θ is such that $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \theta &= \pi - \tan^{-1} 0 \\ &= \pi \end{aligned}$$

Hence, the polar form is $-4 = 4(\cos\pi + i\sin\pi)$.

Step 3 of 4

From the equation **(1)**, required roots are as shown below.

$$\sqrt[4]{-4} = 4^{1/4}\left(\cos\frac{2k\pi+\pi}{4} + i\sin\frac{2k\pi+\pi}{4}\right), k = 0,1,2,3 \text{ ...... } \textbf{(2)}$$

Let $k = 0$ in the equation **(2)**, it gives

$$\begin{aligned} z_0 &= 4^{1/4}\left(\cos\frac{0+\pi}{4} + i\sin\frac{0+\pi}{4}\right) \\ &= 4^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \end{aligned}$$

Let $k = 1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1 &= 4^{1/4}\left(\cos\frac{2\pi+\pi}{4} + i\sin\frac{2\pi+\pi}{4}\right) \\ &= 4^{1/4}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \end{aligned}$$

Let $k = 2$ in the equation **(2)**, it gives

$$\begin{aligned} z_2 &= 4^{1/4}\left(\cos\frac{4\pi+\pi}{4} + i\sin\frac{4\pi+\pi}{4}\right) \\ &= 4^{1/4}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \end{aligned}$$

Let $k = 3$ in the equation **(2)**, it gives

$$\begin{aligned} z_3 &= 4^{1/4}\left(\cos\frac{6\pi+\pi}{4} + i\sin\frac{6\pi+\pi}{4}\right) \\ &= 4^{1/4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \end{aligned}$$

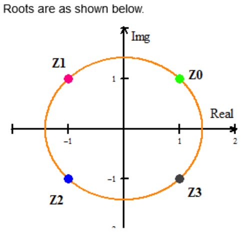
Hence, the required roots are

$$\boxed{\begin{aligned} &4^{1/4}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), 4^{1/4}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \\ &4^{1/4}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), 4^{1/4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \end{aligned}}.$$

Step 4 of 4

All the roots lie on the circle of radius $4^{1/4}$.

Roots are as shown below.



Chapter 13.2, Problem 25P

Step-by-step solution

Step 1 of 4

Roots:

Find the roots of $\sqrt[4]{i}$.

The n^{th} roots of a complex number $z = r(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), k = 0,1,\ldots,n-1 \text{ ...... } \textbf{(1)}$$

Step 2 of 4

Let $z = i$.

Find the polar form of z.

Here $x = 0, y = 1$, so the complex number is on positive $y-$axis.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{1^2} \\ &= 1 \end{aligned}$$

Amplitude θ is such that $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \theta &= \tan^{-1}\frac{1}{0} \\ &= \frac{\pi}{2} \end{aligned}$$

Hence, the polar form is $i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Step 3 of 4

From the equation **(1)**, required roots are as shown below.

$$\sqrt[4]{i} = 1^{1/4}\left(\cos\frac{2k\pi+\pi/2}{4} + i\sin\frac{2k\pi+\pi/2}{4}\right), k = 0,1,2,3 \text{ ...... } \textbf{(2)}$$

Let $k = 0$ in the equation **(2)**, it gives

$$\begin{aligned} z_0 &= 1^{1/4}\left(\cos\frac{0+\pi/2}{4} + i\sin\frac{0+\pi/2}{4}\right) \\ &= \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} \end{aligned}$$

Let $k = 1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1 &= 1^{1/4}\left(\cos\frac{2\pi+\pi/2}{4} + i\sin\frac{2\pi+\pi/2}{4}\right) \\ &= \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8} \end{aligned}$$

Let $k = 2$ in the equation **(2)**, it gives

$$\begin{aligned} z_2 &= 1^{1/4}\left(\cos\frac{4\pi+\pi/2}{4} + i\sin\frac{4\pi+\pi/2}{4}\right) \\ &= \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8} \end{aligned}$$

Let $k = 3$ in the equation **(2)**, it gives

$$\begin{aligned} z_3 &= 1^{1/4}\left(\cos\frac{6\pi+\pi/2}{4} + i\sin\frac{6\pi+\pi/2}{4}\right) \\ &= \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8} \end{aligned}$$

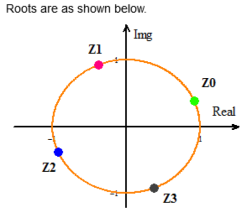
Hence, the required roots are

$$\boxed{\begin{aligned} &\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}, \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}, \\ &\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}, \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8} \end{aligned}}.$$

Step 4 of 4

All the roots lie on the circle of radius 1.

Roots are as shown below.



Chapter 13.2, Problem 26P

Step-by-step solution

Step 1 of 5

Roots:

Find the roots of $\sqrt[8]{1}$.

The n^{th} roots of a complex number $z = r(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), k = 0,1,\ldots,n-1 \quad \text{...... (1)}$$

Step 2 of 5

Let $z = 1$.

Find the polar form of z.

Here $x = 1, y = 0$, so the complex number is on positive $x-$axis.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{1^2+0^2} \\ &= 1 \end{aligned}$$

Amplitude θ is such that $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \theta &= \tan^{-1}\frac{0}{1} \\ &= 0 \end{aligned}$$

Hence, the polar form is $1 = 1(\cos 0 + i\sin 0)$.

Step 3 of 5

From the equation **(1)**, required roots are as shown below.

$$\sqrt[8]{1} = 1^{1/8}\left(\cos\frac{2k\pi+0}{8} + i\sin\frac{2k\pi+0}{8}\right), k = 0,1,\ldots,7 \quad \text{...... (2)}$$

Let $k = 0$ in the equation **(2)**, it gives

$$\begin{aligned} z_0 &= 1^{1/8}(\cos 0 + i\sin 0) \\ &= 1 \end{aligned}$$

Let $k = 1$ in the equation **(2)**, it gives

$$\begin{aligned} z_1 &= 1^{1/8}\left(\cos\frac{2\pi+0}{8} + i\sin\frac{2\pi+0}{8}\right) \\ &= \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{aligned}$$

Let $k = 2$ in the equation **(2)**, it gives

$$\begin{aligned} z_2 &= 1^{1/8}\left(\cos\frac{4\pi+0}{8} + i\sin\frac{4\pi+0}{8}\right) \\ &= i \end{aligned}$$

Let $k = 3$ in the equation **(2)**, it gives

$$\begin{aligned} z_3 &= 1^{1/8}\left(\cos\frac{6\pi+0}{8} + i\sin\frac{6\pi+0}{8}\right) \\ &= -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{aligned}$$

Step 4 of 5

Let $k = 4$ in the equation **(2)**, it gives

$$\begin{aligned} z_4 &= 1^{1/8}\left(\cos\frac{8\pi+0}{8} + i\sin\frac{8\pi+0}{8}\right) \\ &= -1 \end{aligned}$$

Let $k = 5$ in the equation **(2)**, it gives

$$\begin{aligned} z_5 &= 1^{1/8}\left(\cos\frac{10\pi+0}{8} + i\sin\frac{10\pi+0}{8}\right) \\ &= -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \end{aligned}$$

Let $k = 6$ in the equation **(2)**, it gives

$$\begin{aligned} z_6 &= 1^{1/8}\left(\cos\frac{12\pi+0}{8} + i\sin\frac{12\pi+0}{8}\right) \\ &= -i \end{aligned}$$

Let $k = 7$ in the equation **(2)**, it gives

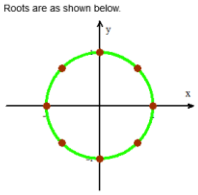
$$\begin{aligned} z_7 &= 1^{1/8}\left(\cos\frac{14\pi+0}{8} + i\sin\frac{14\pi+0}{8}\right) \\ &= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \end{aligned}$$

Hence, the required roots are $\boxed{\pm 1, \pm i, \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}}$.

Step 5 of 5

All the roots lie on the circle of radius 1.

Roots are as shown below.



Chapter 13.2, Problem 27P

Step-by-step solution

Step 1 of 5

The objective is to find and graph all the roots in the complex plane.

Roots:

Find the roots of $\sqrt[5]{-1}$.

The n^{th} roots of a complex number $z = r(\cos\theta + i\sin\theta)$ are,

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), k = 0,1,\ldots,n-1 \quad \text{...... (1)}$$

Step 2 of 5

Take $z = -1$.

Find the polar form of z.

Here $x = -1, y = 0$, so the complex number is on negative $x-$axis.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{1^2+0^2} \\ &= 1 \end{aligned}$$

Amplitude θ is such that $\tan\theta = \frac{y}{x}$.

$$\begin{aligned} \theta &= \tan^{-1}\frac{0}{-1} \\ &= \pi \end{aligned}$$

Hence, the polar form is $-1 = 1(\cos\pi + i\sin\pi)$.

Step 3 of 5

From the equation **(1)**, required roots are as shown below.

$$\sqrt[5]{-1} = 1^{1/5}\left(\cos\frac{2k\pi+\pi}{5} + i\sin\frac{2k\pi+\pi}{5}\right), k = 0,1,2,3,4 \quad \text{...... (2)}$$

Take $k = 0$ in the equation **(2)**, it gives,

$$\begin{aligned} z_0 &= 1^{1/5}\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) \\ &= \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} \end{aligned}$$

Take $k = 1$ in the equation **(2)**, it gives,

$$\begin{aligned} z_1 &= 1^{1/5}\left(\cos\frac{2\pi+\pi}{5} + i\sin\frac{2\pi+\pi}{5}\right) \\ &= \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} \end{aligned}$$

Take $k = 2$ in the equation **(2)**, it gives,

$$\begin{aligned} z_2 &= 1^{1/5}\left(\cos\frac{4\pi+\pi}{5} + i\sin\frac{4\pi+\pi}{5}\right) \\ &= -1 \end{aligned}$$

Step 4 of 5

Take $k = 3$ in the equation **(2)**, it gives,

$$\begin{aligned} z_3 &= 1^{1/5}\left(\cos\frac{6\pi+\pi}{5} + i\sin\frac{6\pi+\pi}{5}\right) \\ &= \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} \\ &= \cos\left(2\pi - \frac{3\pi}{5}\right) + i\sin\left(2\pi - \frac{3\pi}{5}\right) \\ &= \cos\left(\frac{3\pi}{5}\right) - i\sin\left(\frac{3\pi}{5}\right) \left[\text{because } \cos\theta = \cos(2\pi-\theta) \text{ and } (-\sin\theta) = \sin(2\pi-\theta)\right] \end{aligned}$$

Let $k = 4$ in the equation **(2)**, it gives

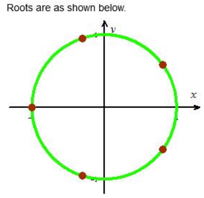
$$\begin{aligned} z_4 &= 1^{1/5}\left(\cos\frac{8\pi+\pi}{5} + i\sin\frac{8\pi+\pi}{5}\right) \\ &= \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} \\ &= \cos\left(2\pi - \frac{\pi}{5}\right) + i\sin\left(2\pi - \frac{\pi}{5}\right) \\ &= \cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right) \left[\text{because } \cos\theta = \cos(2\pi-\theta) \text{ and } (-\sin\theta) = \sin(2\pi-\theta)\right] \end{aligned}$$

Hence, the required roots are $\boxed{\begin{matrix} \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \\ \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5} \end{matrix}}$.

Step 5 of 5

All the roots lie on the circle of radius 1.

Roots are as shown below.



Chapter 13.2, Problem 28P

Step-by-step solution

Step **1** of 2

Roots:

Find the roots of the below equation,

$z^2-(6-2i)z+17-6i=0$.

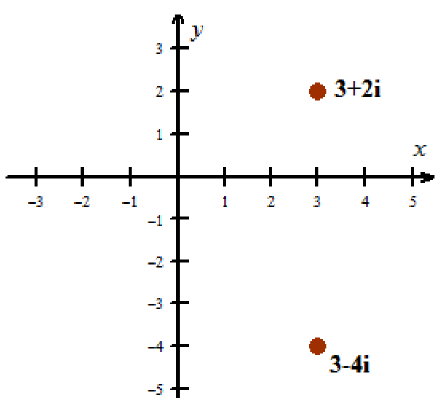
Use the quadratic formula to find the roots of this equation.

$$
\begin{aligned}
z &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
&= \frac{(6-2i)\pm\sqrt{(6-2i)^2-4(17-6i)}}{2} \\
&\quad (\text{since } a=1, b=-(6-2i), c=17-6i) \\
&= \frac{(6-2i)\pm\sqrt{36-4-24i-68+24i}}{2} \\
&= \frac{(6-2i)\pm\sqrt{-36}}{2} \\
&= \frac{(6-2i)\pm 6i}{2} \\
&= 3+2i, 3-4i
\end{aligned}
$$

Hence, the roots of the equation are $\boxed{3+2i, 3-4i}$.

Step **2** of 2

Roots are as shown below.



Chapter 13.2, Problem 29P

Step-by-step solution

Step 1 of 10

Complex equation to be solved is given by,

$$z^2+z+1-i=0$$

Let, the complex number z be represented as $z=x+iy$, where real and imaginary parts of z are given as follows,

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

Step 2 of 10

Hence the complex equation is given by,

$$(x+iy)^2+x+iy+1-i=0$$

Step 3 of 10

Expand and simplify to separate out real and imaginary parts as follows,

$$x^2+2xyi+(yi)^2+x+iy+1-i=0$$
$$x^2+2xyi+y^2i^2+x+iy+1-i=0$$
$$x^2+2xyi+y^2(-1)+x+iy+1-i=0$$
$$x^2+2xyi-y^2+x+iy+1-i=0$$
$$x^2-y^2+x+1+2xyi+iy-i=0$$
$$x^2-y^2+x+1+i(2xy+y-1)=0$$
$$(x^2-y^2+x+1)+i(2xy+y-1)=0$$

Step 4 of 10

0 can be represented as a complex number $0+0i$.

$$(x^2-y^2+x+1)+i(2xy+y-1)=0+0i$$

Equating real and imaginary parts it can be written as,

$$x^2-y^2+x+1=0$$
$$2xy+y-1=0$$

Step 5 of 10

From the 2nd equation it can be written as,

$$2xy=-y+1$$
$$x=\frac{-y+1}{2y}$$

Step 6 of 10

Substitute value of $x=\frac{-y+1}{2y}$ in 1st equation,

$$\left(\frac{-y+1}{2y}\right)^2-y^2+\left(\frac{-y+1}{2y}\right)+1=0$$
$$\frac{(-y+1)^2}{4y^2}-y^2+\left(\frac{-y+1}{2y}\right)+1=0$$
$$\left(\frac{y^2-2y+1}{4y^2}-y^2+\left(\frac{-y+1}{2y}\right)+1\right)4y^2=0\times 4y^2$$
$$y^2-2y+1-4y^4-2y^2+2y+4y^2=0$$
$$-4y^4+3y^2-2y+1+2y=0$$
$$-4y^4+3y^2+1=0$$
$$-4y^4+4y^2-y^2+1=0$$
$$4y^2(-y^2+1)+1(-y^2+1)=0$$
$$(-y^2+1)(4y^2+1)=0$$

This gives $y^2=1$ and $y^2=\frac{-1}{4}$.

Step 7 of 10

$y^2=\frac{-1}{4}$ has no solutions, as y can only be a real number.

$y^2=1$ will give $y=1$ and $y=-1$.

As multiplication by $4y^2$ is carried out, $y=0$ could be one of the solutions. But for $y=0$, x does not exist, therefore $y=0$ cannot be a solution.

Step 8 of 10

At $y=1$,

$$x=\frac{-1+1}{2(1)}$$
$$x=0$$

At $y=-1$

$$x=\frac{1+1}{2(-1)}$$
$$x=-1$$

Step 9 of 10

Using the values of x and y, the solutions of the complex equations are given as follows,

For $x=0$ and $y=1$,

$$z=0+i(1)$$
$$z=i$$

For $x=-1$ and $y=-1$,

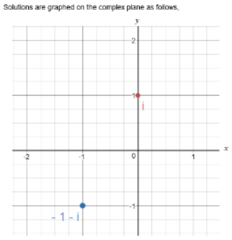
$$z=-1+i(-1)$$
$$z=-1-i$$

Step 10 of 10

Hence the solutions of the complex equation $z^2+z+1-i=0$ are,

$$z=\{i,-1-i\}$$

Solutions are graphed on the complex plane as follows,



Chapter 13.2, Problem 30P

Step-by-step solution

Step 1 of 5

Roots:

Find the roots of the below equation,

$$z^4+324=0 \to z=\sqrt[4]{-324}.$$

There are four roots for this equation.

Step 2 of 5

Let $z=-324$.

Find the polar form of z.

Here $x=-324, y=0$, so the complex number is on negative $x-$axis.

$$r=\sqrt{x^2+y^2}$$
$$=\sqrt{324^2}$$
$$=324$$

Amplitude θ is such that $\tan\theta=\frac{y}{x}$.

$$\theta=\pi-\tan^{-1}0$$
$$=\pi$$

Hence, the polar form is $-324=324(\cos\pi+i\sin\pi)$.

Step 3 of 5

From the equation **(1)**, required roots are as shown below.

$$\sqrt[4]{-324}=324^{1/4}\left(\cos\frac{2k\pi+\pi}{4}+i\sin\frac{2k\pi+\pi}{4}\right), k=0,1,2,3 \text{ ...... } \mathbf{(2)}$$

Let $k=0$ in the equation **(2)**, it gives

$$z_0=324^{1/4}\left(\cos\frac{0+\pi}{4}+i\sin\frac{0+\pi}{4}\right)$$
$$=324^{1/4}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$
$$=324^{1/4}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$
$$=3+3i$$

Let $k=1$ in the equation **(2)**, it gives

$$z_1=324^{1/4}\left(\cos\frac{2\pi+\pi}{4}+i\sin\frac{2\pi+\pi}{4}\right)$$
$$=324^{1/4}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$
$$=324^{1/4}\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$$
$$=-3+3i$$

Let $k=2$ in the equation **(2)**, it gives

$$z_2=324^{1/4}\left(\cos\frac{4\pi+\pi}{4}+i\sin\frac{4\pi+\pi}{4}\right)$$
$$=324^{1/4}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$
$$=324^{1/4}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$
$$=-3-3i$$

Let $k=3$ in the equation **(2)**, it gives

$$z_3=324^{1/4}\left(\cos\frac{6\pi+\pi}{4}+i\sin\frac{6\pi+\pi}{4}\right)$$
$$=324^{1/4}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$
$$=324^{1/4}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)$$
$$=3-3i$$

Hence, the required roots are $\boxed{\pm3\pm3i}$.

Step 4 of 5

Equation can be factored as shown below.

$$(z-(3+3i))(z-(3-3i))(z-(-3+3i))(z-(-3-3i))=0$$
$$\left[(z-3)^2+9\right]\left[(z+3)^2+9\right]=0$$
$$(z^2-6z+18)(z^2+6z+18)=0$$

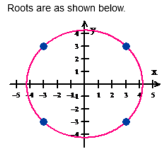
Hence, the equation can be split into product of quadratic factors as

$$\boxed{(z^2-6z+18)(z^2+6z+18)=0}$$

Step 5 of 5

All the roots lie on the circle of radius $324^{1/4}$.

Roots are as shown below.



Chapter 13.2, Problem 31P

Step-by-step solution

Step 1 of 4

Roots:

Find the roots of the below equation,

$z^4 - 6iz^2 + 16 = 0$.

Use the quadratic formula to find the roots of this equation.

$$\begin{aligned}
z^2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{6i \pm \sqrt{(-6i)^2 - 4\times 16}}{2} \\
&\quad (\text{since } a = 1, b = -6i, c = 16) \\
&= \frac{6i \pm \sqrt{-100}}{2} \\
&= \frac{6i \pm 10i}{2} \\
&= 8i, -2i
\end{aligned}$$

Step 2 of 4

Now find the values of z.

$z^2 = 8i, z^2 = -2i$.

Let $z^2 = 8i \rightarrow z = \sqrt{8i}$.

$$\begin{aligned}
z &= \left[\sqrt{8\cos\left(\frac{\pi}{2}\right) + i\sin\frac{\pi}{2}}\right] \\
&= \sqrt{8}\left[\cos\left(\frac{2k\pi + \pi/2}{2}\right) + i\sin\left(\frac{2k\pi + \pi/2}{2}\right)\right], k = 0,1 \\
&= \sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \sqrt{8}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\
&= 2 + 2i, -2 - 2i
\end{aligned}$$

Step 3 of 4

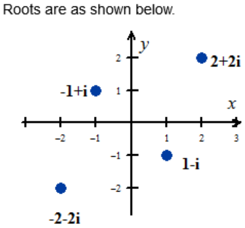
Let $z^2 = -2i \rightarrow z = \sqrt{-2i}$.

$$\begin{aligned}
z &= \left[\sqrt{2\cos\left(\frac{3\pi}{2}\right) + i\sin\frac{3\pi}{2}}\right] \\
&= \sqrt{2}\left[\cos\left(\frac{2k\pi + 3\pi/2}{2}\right) + i\sin\left(\frac{2k\pi + 3\pi/2}{2}\right)\right], k = 0,1 \\
&= \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right), \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\
&= 1 - i, -1 + i
\end{aligned}$$

Hence, the roots of the equation are $\boxed{2 + 2i, -2 - 2i, 1 - i, -1 + i}$.

Step 4 of 4

Roots are as shown below.



Chapter 13.3, Problem 1P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$$|z+1-5i| \le \frac{3}{2}$$

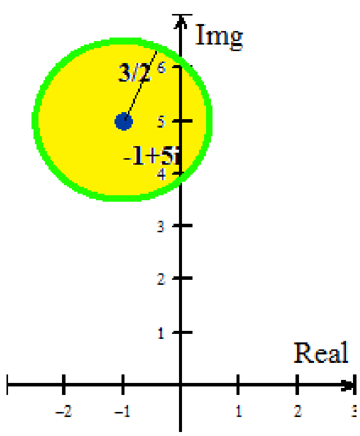
Consider the circle with center $-1+5i$ and radius $\frac{3}{2}$ in a complex plane.

Here $|z+1-5i| \le \frac{3}{2}$ represents all the points inside and on the circle.

Thus the inequality represents the closed disk with centre $-1+5i$ and radius $\frac{3}{2}$.

Step 2 of 2

Sketch is as shown below.



Chapter 13.3, Problem 2P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$0 < |z| < 1$

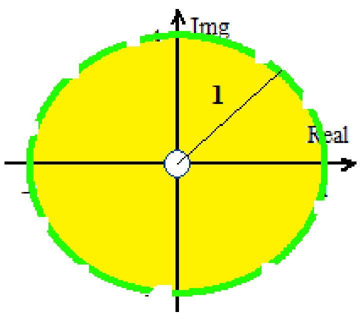
Consider the circles with center 0 and radii $0, 1$ in a complex plane.

Here $0 < |z| < 1$ represents the region between two circles.

Thus the inequality is an annual ring with centre 0 and radius is from $0 \text{ to } 1$.

Step 2 of 2

Sketch is as shown below.



Chapter 13.3, Problem 3P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$$\pi < |z - 4 + 2i| < 3\pi$$

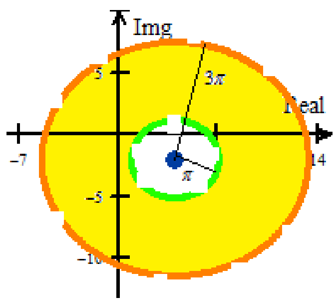
Consider the circles with center $4 - 2i$ and radii $\pi, 3\pi$ in a complex plane.

Here $\pi < |z - (4 - 2i)| < 3\pi$ represents the region between two circles.

Thus the inequality is an annual ring with centre $4 - 2i$ and radius is from π to 3π.

Step 2 of 2

Sketch is as shown below.



Chapter 13.3, Problem 4P

Step-by-step solution

Step 1 of 1

Graph:

Sketch the below inequality,

$-\pi < \operatorname{Im} z < \pi$

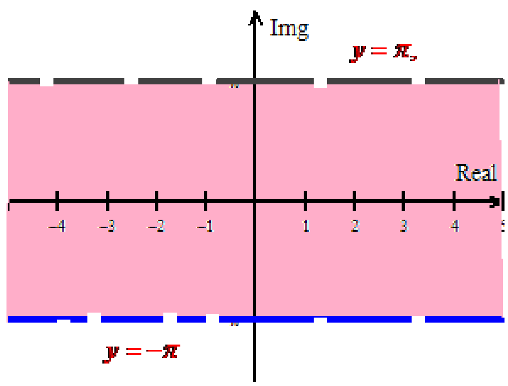
Let $z = x + iy$.

Then the imaginary part of z is y.

$-\pi < \operatorname{Im} z < \pi \to -\pi < y < \pi$

Here $-\pi < y < \pi$ represents the region between two parallel lines $y = \pi, y = -\pi$.

Sketch is as shown below.



Chapter 13.3, Problem 5P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$$|\arg z| < \frac{\pi}{4}$$

Let $z = x + iy$.

Now $\arg z$ is defined as $\theta = \tan^{-1}\frac{y}{x}$.

$$|\theta| < \frac{\pi}{4} \to -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$-\frac{\pi}{4} < \tan^{-1}\frac{y}{x} < \frac{\pi}{4}$$

$$\tan\left(-\frac{\pi}{4}\right) < \frac{y}{x} < \tan\frac{\pi}{4}$$

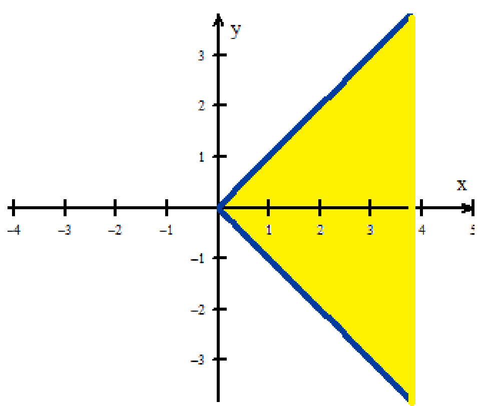
$$-1 < \frac{y}{x} < 1$$

$$-x < y < x$$

Thus the region lies between the lines $y = x, y = -x$ in the first and fourth quadrants.

Step 2 of 2

Sketch is as shown below.



Comments (1)

Anonymous

should it be a dotted line? since -x

Chapter 13.3, Problem 6P

Step-by-step solution

Step 1 of 3

Consider the provided statement to determine and sketch the sets in the complex plane.

The provided inequality is,

$$\mathrm{Re}\left(\frac{1}{z}\right)<1$$

As it is known that in complex plane the number is written in the form of $z=x+iy$.

Therefore,

$$\begin{aligned} z &= x+iy \\ \frac{1}{z} &= \frac{1}{x+iy}\times\frac{x-iy}{x-iy} \\ &= \frac{x-iy}{x^2+y^2} \\ \mathrm{Re}\left(\frac{1}{z}\right) &= \frac{x}{x^2+y^2} \end{aligned}$$

Step 2 of 3

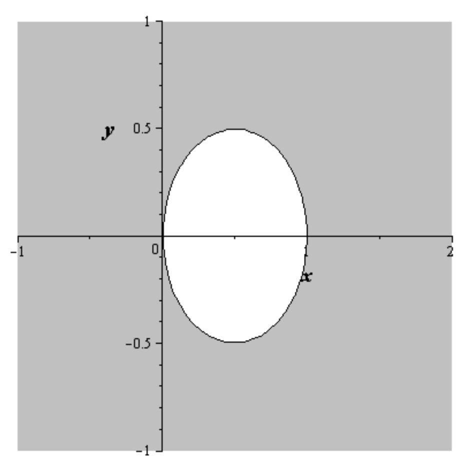
As it provided that,

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{z}\right) &< 1 \\ \frac{x}{x^2+y^2} &< 1 \\ x^2+y^2 &> x \\ \left(x-\frac{1}{2}\right)^2+y^2 &> \frac{1}{4} \end{aligned}$$

The region is represented by the set of points lying outside the circle having centre at $\left(\frac{1}{2},0\right)$ and radius $=\frac{1}{2}$.

Step 3 of 3

Thus the region is $x^2+y^2-x>0$ is sketched is as shown below,



Chapter 13.3, Problem 7P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$$\operatorname{Re}(z) \geq -1$$

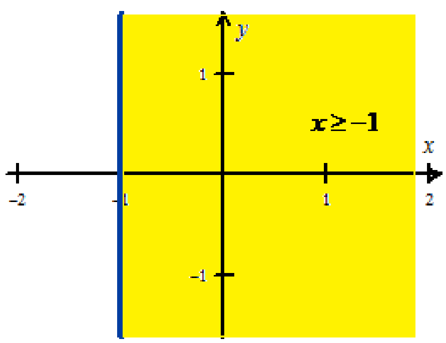
Let $z = x + iy \rightarrow \operatorname{Re} z = x$.

$$\operatorname{Re}(z) \geq -1$$
$$x \geq -1$$

Thus the region is a half plane to the right of $x = -1$.

Step 2 of 2

Sketch is as shown below.



Chapter 13.3, Problem 8P

Step-by-step solution

Step 1 of 2

Graph:

Sketch the below inequality,

$|z+i| \geq |z-i|$

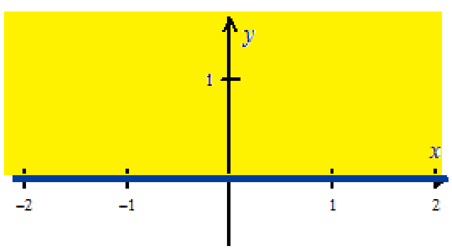
Let $z = x+iy$.

$$|x+iy+i| \geq |x+iy-i|$$
$$|x+i(y+1)| \geq |x+i(y-1)|$$
$$\sqrt{x^2+(y+1)^2} \geq \sqrt{x^2+(y-1)^2}$$
$$x^2+(y+1)^2 \geq x^2+(y-1)^2$$
$$4y \geq 0$$
$$y \geq 0$$

Thus the region is a half plane on and above the $x-\text{axis}$.

Step 2 of 2

Sketch is as shown below.



Chapter 13.3, Problem 14P

Step-by-step solution

Step 1 of 3

Consider the function,

$$f(z)=\begin{cases}0 & \text{if } z=0\\ \dfrac{\operatorname{Re} z^2}{|z|} & \text{if } z\neq 0\end{cases}$$

The objective is to determine whether the function $f(z)$ is continuous at $z=0$ or not.

Step 2 of 3

A function $f(z)$ is said to be continuous at $z=z_0$ if

$$\lim_{z\to z_0} f(z)=f(z_0)$$

Consider the limit,

$$\begin{aligned}\lim_{z\to 0} f(z)&=\lim_{z\to 0}\frac{\operatorname{Re} z^2}{|z|}\\ &=\lim_{(x,y)\to(0,0)}\frac{\operatorname{Re}(x+iy)^2}{|x+iy|}\\ &=\lim_{(x,y)\to(0,0)}\frac{\operatorname{Re}(x^2-y^2+i2xy)}{\sqrt{x^2+y^2}}\\ &=\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\end{aligned}$$

Step 3 of 3

Path I: $x\to 0, y\to 0$

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{\sqrt{x^2+y^2}}&=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\\ &=\lim_{y\to 0}\frac{0^2-y^2}{\sqrt{0^2+y^2}}\\ &=\lim_{y\to 0}\frac{-y^2}{y}\\ &=\lim_{y\to 0}(-y)\end{aligned}$$

$=0$

Path II: $y\to 0, x\to 0$

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{\sqrt{x^2+y^2}}&=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\\ &=\lim_{x\to 0}\frac{x^2-0^2}{\sqrt{x^2+0^2}}\\ &=\lim_{x\to 0}\frac{x^2}{x}\\ &=\lim_{x\to 0}(x)\end{aligned}$$

$=0$

Path III: $y\to mx, x\to 0$

$$\begin{aligned}\lim_{\substack{y\to mx\\ x\to 0}}\frac{x^2-y^2}{\sqrt{x^2+y^2}}&=\lim_{x\to 0}\frac{x^2-m^2x^2}{\sqrt{x^2+m^2x^2}}\\ &=\lim_{x\to 0}\frac{x^2(1-m^2)}{x\sqrt{1+m^2}}\\ &=\lim_{x\to 0}\frac{x(1-m^2)}{\sqrt{1+m^2}}\\ &=0\end{aligned}$$

Path IV: $y\to mx^2, x\to 0$

$$\begin{aligned}\lim_{\substack{y\to mx^2\\ x\to 0}}\frac{x^2-y^2}{\sqrt{x^2+y^2}}&=\lim_{x\to 0}\frac{x^2-m^2x^4}{\sqrt{x^2+m^2x^4}}\\ &=\lim_{x\to 0}\frac{x^2(1-m^2x^2)}{x\sqrt{1+m^2x^2}}\\ &=\lim_{x\to 0}\frac{x(1-m^2x^2)}{\sqrt{1+m^2x^2}}\\ &=0\end{aligned}$$

Thus, the value of the limit in all four paths is 0.

That is,

$$\lim_{z\to 0} f(z)=0=f(0)$$

Hence, the function $f(z)$ is $\boxed{\text{continuous}}$.

Chapter 13.3, Problem 15P

Step-by-step solution

Step 1 of 2

Continuity:

Consider the below function,

$$f(z)=\begin{cases}0 & \text{if } z=0\\ |z|^2\,\mathrm{Im}\left(\frac{1}{z}\right) & \text{if } z\neq 0\end{cases}$$

Determine whether the function is continuous at $z=0$ or not.

A function $f(z)$ is said to be continuous at $z=z_0$ if

$\lim\limits_{z\to z_0} f(z)=f(z_0)$.

Let $z=x+iy$.

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{z}\right)&=\mathrm{Im}\left(\frac{1}{x+iy}\right)\\ &=\mathrm{Im}\left(\frac{x-iy}{x^2+y^2}\right)\\ &=-\frac{y}{x^2+y^2}\end{aligned}$$

$|z|^2=x^2+y^2$.

Step 2 of 2

Consider the limit $\lim\limits_{z\to 0} f(z)$.

$$\begin{aligned}\lim_{z\to 0} f(z)&=\lim_{z\to 0}|z|^2\,\mathrm{Im}\left(\frac{1}{z}\right)\\ &=\lim_{(x,y)\to(0,0)}\left(x^2+y^2\right)\left(-\frac{y}{x^2+y^2}\right)\\ &=\lim_{(x,y)\to(0,0)}(-y)\\ &=0\end{aligned}$$

Thus the value of the limit is 0.

$\lim\limits_{z\to 0} f(z)=f(0)$.

Hence, the function is $\boxed{\text{continuous}}$.

Chapter 13.3, Problem 16P

Step-by-step solution

Step 1 of 2

Continuity:

Consider the below function,

$$f(z)=\begin{cases}0 & \text{if } z=0\\ \operatorname{Im}(z^2)/|z|^2 & \text{if } z\neq 0\end{cases}$$

Determine whether the function is continuous at $z=0$ or not.

A function $f(z)$ is said to be continuous at $z=z_0$ if

$\lim_{z\to z_0} f(z)=f(z_0)$.

Let $z=x+iy$.

$$\begin{aligned}\operatorname{Im}(z^2)&=\operatorname{Im}\left[(x+iy)^2\right]\\&=\operatorname{Im}(x^2-y^2+2ixy)\\&=2xy\end{aligned}$$

$|z|^2=x^2+y^2$.

Step 2 of 2

Consider the limit $\lim_{z\to 0} f(z)$.

$$\begin{aligned}\lim_{z\to 0} f(z)&=\lim_{z\to 0}\operatorname{Im}(z^2)/|z|^2\\&=\lim_{(x,y)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right)\end{aligned}$$

Let (x,y) approaches $(0,0)$ along the line $y=mx$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right)&=\lim_{x\to 0}\frac{2mx^2}{x^2+m^2x^2}\\&=\lim_{x\to 0}\frac{2m}{1+m^2}\\&\neq 0\end{aligned}$$

Thus the value of the limit depends on how (x,y) approaches $(0,0)$.

$\lim_{z\to 0} f(z)\neq f(0)$.

Hence, the function is $\boxed{\text{not continuous}}$.

Chapter 13.3, Problem 17P

Step-by-step solution

Step 1 of 2

Continuity:

Consider the below function,

$$f(z)=\begin{cases}0 & \text{if } z=0\\ \dfrac{\operatorname{Re} z}{1-|z|} & \text{if } z\neq 0\end{cases}$$

Determine whether the function is continuous at $z=0$ or not.

A function $f(z)$ is said to be continuous at $z=z_0$ if

$\lim_{z\to z_0} f(z)=f(z_0)$.

Let $z=x+iy$.

$$\begin{aligned}|z|&=|x+iy|\\&=\sqrt{x^2+y^2}\\ \operatorname{Re} z&=x\end{aligned}$$

Step 2 of 2

Consider the limit $\lim_{z\to 0} f(z)$.

$$\begin{aligned}\lim_{z\to 0} f(z)&=\lim_{z\to 0}\frac{\operatorname{Re} z}{1-|z|}\\&=\lim_{(x,y)\to(0,0)}\frac{x}{1-\sqrt{x^2+y^2}}\end{aligned}$$

Let (x,y) approaches $(0,0)$ along the line $y=mx$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x}{1-\sqrt{x^2+y^2}}&=\lim_{x\to 0}\frac{x}{1-\sqrt{x^2+m^2x^2}}\\&=\lim_{x\to 0}\frac{x}{1-x\sqrt{1+m^2}}\\&=0\end{aligned}$$

Thus the value of the limit is zero.

$\lim_{z\to 0} f(z)=f(0)$.

Hence, the function is $\boxed{\text{continuous}}$.

Chapter 13.3, Problem 18P

Step-by-step solution

Step 1 of 2

Differentiation:

Find the value of the derivative of the function $\frac{z-i}{z+i}$ at i.

Derivative of a complex function f at a point z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Step 2 of 2

Consider the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

$$\begin{aligned}
f'(i) &= \lim_{z \to i} \frac{f(z) - f(i)}{z - i} \\
&= \lim_{z \to i} \frac{\frac{z-i}{z+i} - 0}{z - i} \\
&= \lim_{z \to i} \frac{1}{z+i} \\
&= \frac{1}{2i} \\
&= -\frac{i}{2}
\end{aligned}$$

Hence, the value of the derivative is $\boxed{f'(i) = -\frac{i}{2}}$.

Chapter 13.3, Problem 19P

Step-by-step solution

Step 1 of 2

Differentiation:

Find the value of the derivative of the function $(z-4i)^8$ at $3+4i$.

Derivative of a complex function f at a point z_0 is

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.$$

Step 2 of 2

Consider the limit $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

$$f'(3+4i)=\lim_{z\to 3+4i}\frac{f(z)-f(3+4i)}{z-(3+4i)}$$

$$=\lim_{z\to 3+4i}\frac{(z-4i)^8-(3+4i-4i)^8}{z-(3+4i)}$$

$$=\lim_{z\to 3+4i}\frac{(z-4i)^8-3^8}{z-(3+4i)}$$

$$=\lim_{z\to 3+4i}\frac{(z-4i+3)(z-4i-3)\left[(z-4i)^2+3^2\right]\left[(z-4i)^4+3^4\right]}{z-(3+4i)}$$

$$=\lim_{z\to 3+4i}(z-4i+3)\left[(z-4i)^2+3^2\right]\left[(z-4i)^4+3^4\right]$$

$$=(3+4i-4i+3)\left[(3+4i-4i)^2+3^2\right]\left[(3+4i-4i)^4+3^4\right]$$

$$=6(18)(162)$$

$$=17496$$

Hence, the value of the derivative is $\boxed{f'(3+4i)=17,496}$.

Chapter 13.3, Problem 20P

Step-by-step solution

Step 1 of 3

The objective is to find the derivative of $\frac{1.5z+2i}{3iz-4}$ for any z.

Differentiation:

Derivative of a complex function f at a point z_0 is

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.$$

Step 2 of 3

Consider the limit $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

$$\begin{aligned}
f'(z_0)&=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\\
&=\lim_{z\to z_0}\frac{\frac{1.5z+2i}{3iz-4}-\frac{1.5z_0+2i}{3iz_0-4}}{z-z_0}
\end{aligned}$$

$$=\lim_{z\to z_0}\frac{(1.5z+2i)(3iz_0-4)-(1.5z_0+2i)(3iz-4)}{(3iz-4)(3iz_0-4)(z-z_0)}$$

$$=\lim_{z\to z_0}\frac{4.5izz_0-6z-6z_0-8i-(4.5zz_0i-6z_0-6z-8i)}{(3iz-4)(3iz_0-4)(z-z_0)}$$

$$=\lim_{z\to z_0}\frac{4.5izz_0-6z-6z_0-8i-4.5zz_0i+6z_0+6z+8i}{(3iz-4)(3iz_0-4)(z-z_0)}$$

$$=\lim_{z\to z_0}\frac{0}{(3iz-4)(3iz_0-4)(z-z_0)}$$

$$=\lim_{z\to z_0}0$$

$$=0$$

Hence, the value of the derivative is $\boxed{f'(z)=0}$.

Step 3 of 3

Here z approaches for any z_0 the value of $\frac{f(z)-f(z_0)}{z-z_0}$ is always 0.

Therefore, for any value of z_0, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=0$.

Hence, the derivative of $\frac{1.5z+2i}{3iz-4}$ at any z is always 0.

Chapter 13.3, Problem 21P

Step-by-step solution

Step 1 of 2

Differentiation:

Find the value of the derivative of the function $i(1-z)^n$ at 0.

Derivative of a complex function f at a point z_0 is

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.$$

Step 2 of 2

Consider the limit $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

$$\begin{aligned}
f'(0)&=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}\\
&=\lim_{z\to 0}\frac{i(1-z)^n-i(1-0)^n}{z-0}\\
&=\lim_{z\to 0}\frac{i\left[(1-z)^n-1^n\right]}{z-0}\\
&=\lim_{z\to 0}\frac{i\left[(1-z-1)\left((1-z)^{n-1}+(1-z)^{n-2}+\ldots+1\right)\right]}{z-0}\\
&=\lim_{z\to 0}\frac{i\left[-z\left((1-z)^{n-1}+(1-z)^{n-2}+\ldots+1\right)\right]}{z}\\
&=-i[1+1+\ldots+n\text{ times}]\\
&=-ni
\end{aligned}$$

Hence, the value of the derivative is $\boxed{-ni}$.

Chapter 13.3, Problem 22P

Step-by-step solution

Step 1 of 3

Differentiation:

Find the value of the derivative of the function $\left(iz^3+3z^2\right)^3$ at $2i$.

Derivative of a complex function f at a point z_0 is

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.$$

Step 2 of 3

Let $f(z)=\left(iz^3+3z^2\right)^3$.

$$\begin{aligned}f(2i)&=\left(i(2i)^3+3(2i)^2\right)^3\\&=(8-12)^3\\&=-64\end{aligned}$$

Step 3 of 3

Consider the limit $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

$$\begin{aligned}f'(2i)&=\lim_{z\to 2i}\frac{f(z)-f(2i)}{z-2i}\\&=\lim_{z\to 2i}\frac{\left(iz^3+3z^2\right)^3-(-64)}{z-2i}\\&=\lim_{z\to 2i}\frac{\left(iz^3+3z^2\right)^3+4^3}{z-2i}\\&=\lim_{z\to 2i}\frac{\left(iz^3+3z^2+4\right)\left[\left(iz^3+3z^2\right)^2-4\left(iz^3+3z^2\right)+16\right]}{z-2i}\end{aligned}$$

$$=\lim_{z\to 2i}\frac{(z-2i)\left(iz^2+z+2i\right)\left[\left(iz^3+3z^2\right)^2-4\left(iz^3+3z^2\right)+16\right]}{z-2i}$$

$$=\lim_{z\to 2i}\left(iz^2+z+2i\right)\left[\left(iz^3+3z^2\right)^2-4\left(iz^3+3z^2\right)+16\right]$$

$$=\left(i(2i)^2+2i+2i\right)\left[\left(i(2i)^3+3(2i)^2\right)^2-4\left(i(2i)^3+3(2i)^2\right)+16\right]$$

$$=(-4i+4i)\left[\left(i(2i)^3+3(2i)^2\right)^2-4\left(i(2i)^3+3(2i)^2\right)+16\right]$$

$$=0$$

Hence, the value of the derivative is $\boxed{0}$.

Chapter 13.3, Problem 23P

Step-by-step solution

Step 1 of 3

Quotient rule:

It is the rule formed for differentiation when one function is divided by another. It can be represented in general form like:

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{g(x)f'(x)-f(x)g'(x)}{\left\{g(x)^2\right\}}$$

Step 2 of 3

Use the quotient rule of differentiation to solve the given expression, that is;

$$\begin{aligned} f'(z)&=\frac{(z+i)^3.3z^2-z^3.3(z+i)^2}{(z+i)^6} \\ &=\frac{3z^2(z+i)^2[z+i-z]}{(z+i)^6} \\ &=\frac{3iz^2}{(z+i)^4} \end{aligned}$$

Step 3 of 3

Find the value of the derivative at $z=i$ as follows:

$$\begin{aligned} \left[f'(z)\right]_{z=i}&=\frac{3i(-1)}{(2i)^4} \\ &=-\frac{3}{16}i \end{aligned}$$

Therefore, the value at derivative at $z=i$ is:

$$\boxed{-\frac{3}{16}i}$$

Chapter 13.4, Problem 1P

Step-by-step solution

Step 1 of 6

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Step 2 of 6

In polar co-ordinates

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \;\; and \;\; \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta$$

Now the Cauchy-Riemann equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\; and \;\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since both x and y are functions of r and θ, by chain rule we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{1}{\cos\theta} + \frac{\partial u}{\partial \theta}\cdot\left(\frac{-1}{r\sin\theta}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{1}{\sin\theta} + \frac{\partial u}{\partial \theta}\cdot\left(\frac{1}{r\cos\theta}\right)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{1}{\cos\theta} + \frac{\partial v}{\partial \theta}\cdot\left(-\frac{1}{r\sin\theta}\right)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{1}{\sin\theta} + \frac{\partial v}{\partial \theta}\cdot\left(\frac{1}{r\cos\theta}\right)$$

Step 3 of 6

Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial r}\cdot\frac{1}{\cos\theta} + \frac{\partial u}{\partial \theta}\left(-\frac{1}{r\sin\theta}\right) = \frac{\partial v}{\partial r}\cdot\frac{1}{\sin\theta} + \frac{\partial v}{\partial \theta}\left(\frac{1}{r\cos\theta}\right)$$

$$\sin\theta\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\cdot\cos\theta = \cos\theta\frac{\partial v}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial v}{\partial \theta} \qquad \text{..................(1)}$$

And

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial r}\cdot\frac{1}{\sin\theta} + \frac{\partial u}{\partial \theta}\cdot\frac{1}{r\cos\theta} = -\frac{\partial v}{\partial r}\cdot\frac{1}{\cos\theta} + \frac{\partial v}{\partial \theta}\cdot\frac{1}{r\sin\theta}$$

$$\Rightarrow \cos\theta\frac{\partial u}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial v}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial v}{\partial \theta} \qquad \text{..............(2)}$$

Step 4 of 6

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding gives

$$(1)\times\sin\theta$$

$$\sin^2\theta\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta\cdot\cos\theta = \sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial v}{\partial \theta} \qquad \text{.........(3)}$$

$$(2)\times\cos\theta$$

$$\cos^2\theta\frac{\partial u}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial u}{\partial \theta} = -\sin\theta\cos\theta\frac{\partial v}{\partial r} + \frac{1}{r}\cos^2\theta\frac{\partial v}{\partial \theta} \qquad \text{........(4)}$$

Add the equations (3) and (4) we get

$$\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial u}{\partial r} = \frac{1}{r}\left(\sin^2\theta + \cos^2\theta\right)\frac{\partial v}{\partial \theta}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{...........(5)}$$

Step 5 of 6

Similarly multiplying (1) by $\cos\theta$ (2) by $\sin\theta$ and subtracting gives

$$(1)\times\cos\theta$$

$$\sin\theta\cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\cos^2\theta = \cos^2\theta\frac{\partial v}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial v}{\partial \theta} \qquad \text{........(6)}$$

$$(2)\times\sin\theta$$

$$\sin\theta\cos\theta\frac{\partial u}{\partial r} + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial \theta} = -\sin^2\theta\frac{\partial v}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\frac{\partial v}{\partial \theta} \qquad \text{........(7)}$$

Subtract the equation (7) from equation (6)

$$-\frac{1}{r}\frac{\partial u}{\partial \theta}\left(\cos^2\theta + \sin^2\theta\right) = \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial v}{\partial r}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \text{..........(8)}$$

Step 6 of 6

The Cauchy-Riemann Equations in Polar coordinates

$$\boxed{\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial v}{\partial r} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}}$$

Chapter 13.4, Problem 2P

Step-by-step solution

Step 1 of 5

Consider the function

$f(z) = iz\bar{z}$ ...... (1)

Let $z = x + iy$, then the conjugate of $z = x + iy$ is $\bar{z} = x - iy$.

The product of z and $\bar{z}$

$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2 - (iy)^2 \qquad \text{Since } (a+b)(a-b) = a^2 - b^2 \\ &= x^2 + y^2 \qquad \text{Since } i^2 = -1 \end{aligned}$$

Step 2 of 5

So the complex function (1)

$$\begin{aligned} f(z) &= iz\bar{z} \\ u(x,y) + iv(x,y) &= i(x^2+y^2) \end{aligned} \quad \text{...... (2)}$$

Comparing the real and imaginary parts from both the sides of (2) gives

$$\begin{aligned} u(x,y) &= 0 \\ v(x,y) &= x^2 + y^2 \end{aligned}$$

Step 3 of 5

Recall the Cauchy-Riemann equations,

$u_x = v_y$ and $u_y = -v_x$

The partial derivative of $u(x,y)$ with respect to x and y

$u_x = 0$ and $u_y = 0$

The partial derivative of $v(x,y)$ with respect to x

$$\begin{aligned} \frac{\partial}{\partial x} v(x,y) &= \frac{\partial}{\partial x}(x^2+y^2) \\ &= 2x \end{aligned}$$

The partial derivative of $v(x,y)$ with respect to y

$$\begin{aligned} \frac{\partial}{\partial y} v(x,y) &= \frac{\partial}{\partial y}(x^2+y^2) \\ &= 2y \end{aligned}$$

Step 4 of 5

At $z \neq 0$, the Cauchy-Riemann equations,

$u_x \neq v_y$ and $u_y \neq -v_x$

Thus the function is not analytic for $z \neq 0$.

Step 5 of 5

At $z = 0\left(i.e., (x,y) = (0,0)\right)$, the function $f(z) = iz\bar{z}$ satisfies Cauchy-Riemann equations.

For,

$$\begin{aligned} u_x &= 0 \\ &= 2(0) \\ &= v_y \end{aligned} \qquad \text{and}$$

$$\begin{aligned} u_y &= 0 \\ &= 2(0) \\ &= -v_x \end{aligned}$$

Thus, the function $f(z) = iz\bar{z}$ is analytic at $z = 0$ only.

Chapter 13.4, Problem 3P

Step-by-step solution

Step 1 of 2

Given

$f(z)=e^{-2x}(\cos 2y - i\sin 2y)$ ...... (1)

The given function is rewritten as

$f(z)=e^{-2x}\cos 2y - i\,e^{-2x}\sin 2y$

That is

$f(z)=u+iv=e^{-2x}\cos 2y - i\,e^{-2x}\sin 2y$ ...... (2)

Comparing real and imaginary parts from both the sides of (2) gives

$u=e^{-2x}\cos 2y;$
$v=-e^{-2x}\sin 2y$

Step 2 of 2

Now computing the partial derivatives of u and v with respect to x and y gives

$u_x=-2e^{-2x}\cos 2y;\ u_y=-2e^{-2x}\sin 2y$
$v_x=2e^{-2x}\sin 2y\ ;\ v_y=-2e^{-2x}\cos 2y$

Hence it is evident that

$u_x=v_y\ ;\ u_y=-v_x$

Thus it is concluded that the Cauchy – Riemann equations are satisfied.

Therefore the given function is analytic.

Chapter 13.4, Problem 4P

Step-by-step solution

Step 1 of 5

The complex function is given as,

$$f(z)=e^{x}(\cos y-i\sin y)$$
$$f(z)=e^{x}\cos y-ie^{x}\sin y$$

Comparing with $f(z)=u(x,y)+iv(x,y)$ it can be written as,

$$u(x,y)=e^{x}\cos y$$
$$v(x,y)=-e^{x}\sin y$$

Step 2 of 5

A complex function to be analytic, the necessary and sufficient conditions that must be satisfied by the function are the Cauchy-Riemann equations, which are given as follows,

$$u_x=v_y$$
$$u_y=-v_x$$

Step 3 of 5

To find u_x, partially differentiate $u(x,y)$ with respect to x,

$$\frac{\partial}{\partial x}(u(x,y))=\frac{\partial}{\partial x}(e^{x}\cos y)$$
$$u_x=\cos y\frac{\partial}{\partial x}(e^{x})$$
$$u_x=e^{x}\cos y$$

Step 4 of 5

To find v_y, partially differentiate $v(x,y)$ with respect to y,

$$\frac{\partial}{\partial y}(v(x,y))=\frac{\partial}{\partial y}(-e^{x}\sin y)$$
$$v_y=-e^{x}\frac{\partial}{\partial y}(\sin y)$$
$$v_y=-e^{x}\cos y$$

Step 5 of 5

As $u_x\neq v_y$, that is $f(z)=e^{x}(\cos y-i\sin y)$ does not satisfy the first Cauchy-Riemann equation, it can be concluded that the function $f(z)=e^{x}(\cos y-i\sin y)$ is not analytic.

Chapter 13.4, Problem 5P

Step-by-step solution

Step 1 of 2

Given

$f(z)=\mathrm{Re}\left(z^2\right)-i\,\mathrm{Im}\left(z^2\right)$ ...... (1)

The given function is rewritten as

$$\begin{aligned}
f(z)&=\left[\mathrm{Re}\left(x+iy\right)^2\right]-i\left[\mathrm{Im}\left(x+iy\right)^2\right]\\
&=\mathrm{Re}\left(x^2-y^2+2ixy\right)-i\,\mathrm{Im}\left(x^2-y^2+2ixy\right)\\
&=\left(x^2-y^2\right)-i\,2xy
\end{aligned}$$

So that the given function is represented by

$f(z)=u+iv=\left(x^2-y^2\right)-2ixy$ ...... (2)

Comparing real and imaginary parts from both the sides of (2) gives

$u=x^2-y^2;$
$v=-2xy$

Step 2 of 2

Now computing the partial derivatives of u and v with respect to x and y gives

$u_x=2x;\ u_y=-2y$
$v_x=-2y\ ;\ v_y=-2x$

Hence it is evident that

$u_x\neq v_y\ ;\ u_y\neq -v_x$

Thus it is concluded that the Cauchy – Riemann equations are not satisfied.

Therefore the given function is not analytic.

Chapter 13.4, Problem 6P

Step-by-step solution

Step 1 of 5

Consider the function,

$$f(z)=\frac{1}{z-z^5} \quad \ldots\ldots (1)$$

The objective is to determine whether the function $f(z)$ is analytic or not.

Step 2 of 5

The polar form of z is,

$$z=r(\cos\theta+i\sin\theta).$$

The function $f(z)$ can be written as,

$$f(z)=\frac{1}{z(1-z^4)}$$

$$=\frac{1}{z-z^5}$$

$$=\frac{1}{\left(r(\cos\theta+i\sin\theta)-(r(\cos\theta+i\sin\theta))^5\right)} \text{ Since, } z=r(\cos\theta+i\sin\theta).$$

$$=\frac{1}{\left(r(\cos\theta+i\sin\theta)-r^5(\cos5\theta+i\sin5\theta)\right)} \text{ De Moivre's formula}$$

$$=\frac{1}{(r\cos\theta-r^5\cos5\theta)+i(r\sin\theta-r^5\sin5\theta)}$$

Rationalize the denominator,

$$=\frac{1}{(r\cos\theta-r^5\cos5\theta)+i(r\sin\theta-r^5\sin5\theta)}\times\frac{(r\cos\theta-r^5\cos5\theta)-i(r\sin\theta-r^5\sin5\theta)}{(r\cos\theta-r^5\cos5\theta)-i(r\sin\theta-r^5\sin5\theta)}$$

$$=\frac{(r\cos\theta-r^5\cos5\theta)-i(r\sin\theta-r^5\sin5\theta)}{(r\cos\theta-r^5\cos5\theta)^2+(r\sin\theta-r^5\sin5\theta)^2}$$

$$=\frac{(r\cos\theta-r^5\cos5\theta)}{(r\cos\theta-r^5\cos5\theta)^2+(r\sin\theta-r^5\sin5\theta)^2}-i\frac{(r\sin\theta-r^5\sin5\theta)}{(r\cos\theta-r^5\cos5\theta)^2+(r\sin\theta-r^5\sin5\theta)^2}$$

$$=\frac{(r\cos\theta-r^5\cos5\theta)}{r^2+r^{10}-2r^6(\cos\theta\cos5\theta+\sin\theta\sin5\theta)}-i\frac{(r\sin\theta-r^5\sin5\theta)}{r^2+r^{10}-2r^6(\cos\theta\cos5\theta+\sin\theta\sin5\theta)}$$

$$=\frac{(r\cos\theta-r^5\cos5\theta)}{r^2+r^{10}-2r^6\cos4\theta}-i\frac{(r\sin\theta-r^5\sin5\theta)}{r^2+r^{10}-2r^6\cos4\theta} \qquad \text{Since, cosAcosB+sinAsinB=cos(A-B)}$$

$$=\frac{(\cos\theta-r^4\cos5\theta)}{r+r^9-2r^5\cos4\theta}-i\frac{(\sin\theta-r^4\sin5\theta)}{r+r^9-2r^5\cos4\theta}$$

Step 3 of 5

Here,

$$u=\frac{(\cos\theta-r^4\cos5\theta)}{r+r^9-2r^5\cos4\theta}, \text{ and } v=-\frac{(\sin\theta-r^4\sin5\theta)}{r+r^9-2r^5\cos4\theta}$$

$$u_r=\frac{(r+r^9-2r^5\cos4\theta)(-4r^3\cos5\theta)-(\cos\theta-r^4\cos5\theta)(1+9r^8-10r^4\cos4\theta)}{(r+r^9-2r^5\cos4\theta)^2}$$

$$=\frac{-3r^4\cos5\theta+5r^{12}\cos5\theta-2r^8\cos4\theta\cos5\theta-\cos\theta(1+9r^8-10r^4\cos4\theta)}{(r+r^9-2r^5\cos4\theta)^2}$$

$$v_\theta=\frac{(r+r^9-2r^5\cos4\theta)(5r^4\cos5\theta-\cos\theta)-(r^4\sin5\theta-\sin\theta)(1+9r^8-10r^4\cos4\theta)}{(r+r^9-2r^5\cos4\theta)^2}$$

$$=\frac{\begin{matrix}5r^5\cos5\theta-r\cos\theta+5r^{13}\cos5\theta-r^9\cos\theta-10r^9\cos4\theta\cos5\theta+2r^5\cos4\theta\cos\theta\\-r^4\sin5\theta+\sin\theta-9r^{12}\sin5\theta+9r^8\sin\theta+10r^8\sin5\theta\cos4\theta-10r^4\cos4\theta\sin\theta\end{matrix}}{(r+r^9-2r^5\cos4\theta)^2}$$

Step 4 of 5

Cauchy-Riemann equations in polar form is,

$$u_r=\frac{1}{r}v_\theta,$$

$$v_r=-\frac{1}{r}u_\theta.$$

From the above values of u_r and v_θ it is observed that, $u_r\neq\frac{1}{r}v_\theta$.

Therefore, the function $f(z)=\frac{1}{z-z^5}$ is not analytic.

Step 5 of 5

So that the rational function $f(z)$ is defined for all z except for those values of z for which the denominator is 0.

To find these values of z, the denominator is equated to 0.

This gives,

$$z-z^5=0$$

$$z(1-z^4)=0$$

By fourth roots of unity,

$$z(1-z)(1+z)(z-i)(z+i)=0$$

It follows that,

$$z=0;1;-1;i;-i$$

Hence, the function $f(z)$ is not analytic at $z=0;\pm1;\pm i$.

Chapter 13.4, Problem 7P

Step-by-step solution

Step 1 of 13

The complex function is given as,

$$f(z)=\frac{i}{z^8}$$
$$f(z)=iz^{-8}$$

Step 2 of 13

Let the complex function z be represented in Polar form as follows,

$$z=r(\cos\theta+i\sin\theta)$$

Hence, z^{-8} can be given by,

$$z^{-8}=\left(r(\cos\theta+i\sin\theta)\right)^{-8}$$

Using De Moivre's formula, $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, it can be simplified as,

$$z^{-8}=r^{-8}\left(\cos(-8\theta)+i\sin(-8\theta)\right)$$

Step 3 of 13

But as $\cos(-\phi)=\cos(\phi)$ and $\sin(-\phi)=-\sin(\phi)$ for any ϕ, it can be written as,

$$z^{-8}=r^{-8}\left(\cos(8\theta)-i\sin(8\theta)\right)$$

Step 4 of 13

$$z^{-8}=r^{-8}\cos(8\theta)-ir^{-8}\sin(8\theta)$$
$$z^{-8}=\left[r^{-8}\cos(8\theta)\right]+i\left[-r^{-8}\sin(8\theta)\right]$$

But as $f(z)=iz^{-8}$,

$$f(z)=i\left(\left[r^{-8}\cos(8\theta)\right]+i\left[-r^{-8}\sin(8\theta)\right]\right)$$
$$f(z)=ir^{-8}\cos(8\theta)-i^2r^{-8}\sin(8\theta)$$
$$f(z)=ir^{-8}\cos(8\theta)-(-1)r^{-8}\sin(8\theta)$$
$$f(z)=r^{-8}\sin(8\theta)+ir^{-8}\cos(8\theta)$$

Comparing with $f(z)=u(r,\theta)+iv(r,\theta)$ it can be written as,

$$u(r,\theta)=r^{-8}\sin(8\theta)$$
$$v(r,\theta)=r^{-8}\cos(8\theta)$$

Step 5 of 13

A complex function to be analytic, the necessary and sufficient conditions that must be satisfied by the function are the Cauchy-Riemann equations, which are given in Polar form as follows,

$$u_r=\frac{1}{r}v_\theta$$
$$v_r=-\frac{1}{r}u_\theta$$

Step 6 of 13

To find u_r, partially differentiate $u(r,\theta)$ with respect to r.

$$\frac{\partial}{\partial r}\left(u(r,\theta)\right)=\frac{\partial}{\partial r}\left(r^{-8}\sin(8\theta)\right)$$
$$u_r=\sin(8\theta)\frac{\partial}{\partial x}\left(r^{-8}\right)$$
$$u_r=-8r^{-9}\sin(8\theta)$$

Step 7 of 13

To find v_θ, partially differentiate $v(r,\theta)$ with respect to θ.

$$\frac{\partial}{\partial\theta}\left(v(r,\theta)\right)=\frac{\partial}{\partial\theta}\left(r^{-8}\cos(8\theta)\right)$$
$$v_\theta=r^{-8}\frac{\partial}{\partial\theta}\left(\cos(8\theta)\right)$$
$$v_\theta=-8r^{-8}\sin(8\theta)$$

Step 8 of 13

To find u_θ, partially differentiate $u(r,\theta)$ with respect to θ.

$$\frac{\partial}{\partial\theta}\left(u(r,\theta)\right)=\frac{\partial}{\partial\theta}\left(r^{-8}\sin(8\theta)\right)$$
$$u_\theta=r^{-8}\frac{\partial}{\partial\theta}\left(\sin(8\theta)\right)$$
$$u_\theta=8r^{-8}\cos(8\theta)$$

Step 9 of 13

To find v_r, partially differentiate $v(r,\theta)$ with respect to r.

$$\frac{\partial}{\partial r}\left(v(r,\theta)\right)=\frac{\partial}{\partial r}\left(r^{-8}\cos(8\theta)\right)$$
$$v_r=\cos(8\theta)\frac{\partial}{\partial r}\left(r^{-8}\right)$$
$$v_r=-8r^{-9}\cos(8\theta)$$

Step 10 of 13

As,

$$v_\theta=-8r^{-8}\sin(8\theta)$$
$$\frac{1}{r}v_\theta=\frac{1}{r}\left(-8r^{-8}\sin(8\theta)\right)$$
$$\frac{1}{r}v_\theta=-8r^{-9}\sin(8\theta)$$
$$\frac{1}{r}v_\theta=u_r$$
$$u_r=\frac{1}{r}v_\theta$$

The first Cauchy-Riemann equation is satisfied.

Step 11 of 13

As,

$$u_\theta=8r^{-8}\cos(8\theta)$$
$$-\frac{1}{r}u_\theta=-\frac{1}{r}\left(8r^{-8}\cos(8\theta)\right)$$
$$-\frac{1}{r}u_\theta=-8r^{-9}\cos(8\theta)$$
$$-\frac{1}{r}u_\theta=v_r$$
$$v_r=-\frac{1}{r}u_\theta$$

The second Cauchy-Riemann equation is satisfied.

Step 12 of 13

From the complex function $f(z)=\frac{i}{z^8}$, it can be seen that, $f(z)$ does not exit at $z=0$.

Hence, $f(z)$ cannot be analytic at $z=0$.

Step 13 of 13

Both the Cauchy-Riemann equations are satisfied by the complex function $f(z)=\frac{i}{z^8}$,

hence it can be concluded that, the function $f(z)=\frac{i}{z^8}$ is analytic at $z\neq 0$.

Chapter 13.4, Problem 8P

Step-by-step solution

Step 1 of 2

Given

$f(z) = Arg(2\pi z)$ ...... (1)

Let $z = r\cos\theta + ir\sin\theta$

Then the given function is rewritten as

$$\begin{aligned} f(z) &= Arg\left[2\pi(r\cos\theta + ir\sin\theta)\right] \\ &= Arg\left[2\pi r\cos\theta + i2\pi r\sin\theta\right] \\ &= \theta \end{aligned}$$

Thus the given function is represented as

$f(z) = u + iv = \theta$

Comparison of real and imaginary parts from the two sides gives

$$\begin{aligned} u &= \theta \\ v &= 0 \end{aligned}$$

Step 2 of 2

Now computing the partial derivatives of u and v with respect to r and θ gives

$u_r = 0$; $u_\theta = 1$

$v_r = 0$; $v_\theta = 0$

The above relations evidently imply that

$u_r = \frac{1}{r} v_\theta$;

$v_r \neq -\frac{1}{r} u_\theta$

Hence Cauchy – Riemann equations are only partly satisfied.

Therefore it is concluded that the given function is not an analytic function.

Chapter 13.4, Problem 9P

Step-by-step solution

Step 1 of 1

Generally a rational function $f(z)=\frac{P(z)}{Q(z)}$

Where $P(z)$ and $Q(z)$ are polynomials in z is analytic for all values of z for which $Q(z)\neq 0$

The function is $f(z)=\frac{3\pi^2}{\left(z^3+4\pi^2 z\right)}$

Rewrite the function $f(z)$.

$$\begin{aligned} f(z)&=\frac{3\pi^2}{z\left(z^2+4\pi^2\right)} \\ &=\frac{3\pi^2}{z(z+2i\pi)(z-2i\pi)} \end{aligned}$$

Make the denominator equal to zero to find the values at which the function is undefined.

$$z(z-2\pi i)(z+2\pi i)=0$$

$$z=0,2\pi i,-2\pi i$$

Therefore the given function is analytic for all values of z, except for $z=0;2\pi i;-2\pi i$.

Chapter 13.4, Problem 10P

Step-by-step solution

Step 1 of 12

The complex function is given as,

$$f(z)=\ln|z|+i\ \mathrm{Arg}\ z$$

Comparing with $f(z)=u(x,y)+iv(x,y)$ it can be written as,

$$u(x,y)=\ln|z|$$
$$v(x,y)=\mathrm{Arg}\ z$$

Step 2 of 12

As modulus of z can be given as $|z|=\sqrt{x^2+y^2}$, $u(x,y)$ can be written as follows,

$$u(x,y)=\ln\sqrt{x^2+y^2}$$
$$u(x,y)=\ln\left(x^2+y^2\right)^{\frac{1}{2}}$$
$$u(x,y)=\frac{1}{2}\ln\left(x^2+y^2\right)$$

Step 3 of 12

Principal value of argument of z $(\mathrm{Arg}\ z)$, and hence $v(x,y)$, is given as follows,

$$v(x,y)=\mathrm{Arg}\ z=\begin{cases}\tan^{-1}\left(\frac{y}{x}\right) & \text{if } y>0\\ \pi & \text{if } y=0, x<0\\ 0 & \text{if } y=0, x>0\\ \tan^{-1}\left(\frac{y}{x}\right)-2\pi & \text{if } y<0\end{cases}$$

Comments (1)

Anonymous
Where is this part coming from, or is this always given to us?

Step 4 of 12

A complex function to be analytic, the necessary and sufficient conditions that must be satisfied by the function are the Cauchy-Riemann equations, which are given as follows,

$$u_x=v_y$$
$$u_y=-v_x$$

Step 5 of 12

To find u_x, partially differentiate $u(x,y)$ with respect to x,

$$\frac{\partial}{\partial x}(u(x,y))=\frac{\partial}{\partial x}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)$$
$$u_x=\frac{1}{2}\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right)$$
$$u_x=\frac{1}{2}\frac{1}{x^2+y^2}\frac{\partial}{\partial x}\left(x^2+y^2\right)$$
$$u_x=\frac{1}{2}\frac{1}{x^2+y^2}\left[\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial x}\left(y^2\right)\right]$$
$$u_x=\frac{1}{2}\frac{1}{x^2+y^2}[2x+0]$$
$$u_x=\frac{x}{x^2+y^2}$$

Step 6 of 12

To find u_y, partially differentiate $u(x,y)$ with respect to y,

$$\frac{\partial}{\partial y}(u(x,y))=\frac{\partial}{\partial y}\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)$$
$$u_y=\frac{1}{2}\frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)$$
$$u_y=\frac{1}{2}\frac{1}{x^2+y^2}\frac{\partial}{\partial y}\left(x^2+y^2\right)$$
$$u_y=\frac{1}{2}\frac{1}{x^2+y^2}\left[\frac{\partial}{\partial y}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)\right]$$
$$u_y=\frac{1}{2}\frac{1}{x^2+y^2}[0+2y]$$
$$u_y=\frac{y}{x^2+y^2}$$

Step 7 of 12

Partial derivatives of $v(x,y)$ at $y\neq 0$ are obtained as follows,

To find v_x, partially differentiate $v(x,y)$ with respect to x,

$$\frac{\partial}{\partial x}(v(x,y))=\frac{\partial}{\partial x}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$
$$v_x=\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$
$$v_x=\frac{x^2}{y^2+x^2}\times y\times\frac{\partial}{\partial x}\left(x^{-1}\right)$$
$$v_x=\frac{x^2}{y^2+x^2}\times y\times\frac{-1}{x^2}$$
$$v_x=\frac{-y}{x^2+y^2}$$

Step 8 of 12

To find v_y, partially differentiate $v(x,y)$ with respect to y,

$$\frac{\partial}{\partial y}(v(x,y))=\frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$
$$v_y=\frac{1}{\left(\frac{y}{x}\right)^2+1}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$
$$v_y=\frac{x^2}{y^2+x^2}\times\frac{1}{x}\times\frac{\partial}{\partial x}(y)$$
$$v_y=\frac{x^2}{y^2+x^2}\times\frac{1}{x}$$
$$v_y=\frac{x}{x^2+y^2}$$

Step 9 of 12

As,

$$u_x=\frac{x}{x^2-y^2}$$
$$u_x=v_y$$

The first Cauchy-Riemann equation is satisfied

Step 10 of 12

As,

$$u_y=\frac{y}{x^2+y^2}$$
$$u_y=-\left(-\frac{y}{x^2+y^2}\right)$$
$$u_y=-v_x$$

The second Cauchy-Riemann equation is satisfied

Step 11 of 12

From the partial derivatives, u_x, u_y, v_x and v_y, it can be seen that,

$$x^2+y^2\neq 0$$

Hence,

$$\sqrt{x^2+y^2}\neq 0$$
$$|z|\neq 0$$

Therefore,

$$z\neq 0$$

Step 12 of 12

Both the Cauchy-Riemann equations are satisfied by the complex function $f(z)=\ln|z|+i\ \mathrm{Arg}\ z$, hence it can be concluded that, the function $f(z)=\ln|z|+i\ \mathrm{Arg}\ z$ is analytic when $z\neq 0$.

Chapter 13.4, Problem 11P

Step-by-step solution

Step 1 of 4

Analytic functions:

Consider the below function,

$f(z) = \cos x \cosh y - i \sin x \sinh y$

Determine whether the above function is analytic or not.

Step 2 of 4

A function is analytic if and only if C-R equations are satisfied.

C-R equations in Cartesian form are

$u_x = v_y, u_y = -v_x$. ...... **(1)**

Step 3 of 4

Let $f(z) = \cos x \cosh y - i \sin x \sinh y$.

Real part of the function is $u(x, y) = \cos x \cosh y$.

Imaginary part of the function is $v(x, y) = -\sin x \sinh y$.

Step 4 of 4

Find the partial derivatives of u, v.

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}(\cos x \cosh y) \\ &= -\sin x \cosh y \\ u_y &= \frac{\partial}{\partial y}(\cos x \cosh y) \\ &= \cos x \sinh y \end{aligned}$$

$$\begin{aligned} v_x &= \frac{\partial}{\partial x}(-\sin x \sinh y) \\ &= -\cos x \sinh y \\ v_y &= \frac{\partial}{\partial y}(-\sin x \sinh y) \\ &= -\sin x \cosh y \end{aligned}$$

It is clear that C-R equations are satisfied, since

$u_x = v_y, u_y = -v_x$.

Hence, the function is $\boxed{\text{analytic}}$.

Chapter 13.4, Problem 12P

Step-by-step solution

Step 1 of 3

Harmonic functions:

Consider the below function,

$$u(x,y)=x^2+y^2$$

Determine whether the above function is harmonic or not.

Step 2 of 3

A function is harmonic then it satisfies the equations,

$u_{xx}+u_{yy}=0, v_{xx}+v_{yy}=0$. …… **(1)**

Step 3 of 3

Here $u(x,y)=x^2+y^2$.

$$u_x=2x, u_y=2y$$
$$u_{xx}=2, u_{yy}=2$$

It is clear that $u_{xx}+u_{yy}\neq 0$.

Hence, the function is $\boxed{\text{not harmonic}}$.

Chapter 13.4, Problem 13P

Step-by-step solution

Step 1 of 5

Harmonic functions:

Consider the below function,

$u(x,y)=xy$

Determine whether the above function is harmonic or not.

Step 2 of 5

A function is harmonic then it satisfies the equations,

$u_{xx}+u_{yy}=0, v_{xx}+v_{yy}=0$. ...... **(1)**

Step 3 of 5

Here $u(x,y)=xy$.

$$u_x=y, u_y=x$$
$$u_{xx}=0, u_{yy}=0$$

It is clear that $u_{xx}+u_{yy}=0$.

Hence, the function is $\boxed{\text{harmonic}}$.

Step 4 of 5

Now find the harmonic conjugate.

$$\begin{aligned}
u_x &= y\\
v_y &= y \qquad (\text{since } u_x=v_y)\\
v &= \frac{y^2}{2}+h(x) \quad (\text{integrate with respect to } y)\\
v_x &= h'(x)\\
-x &= h'(x) \quad (\text{since } v_x=-u_y)\\
h(x) &= -\frac{x^2}{2}+c \to v=\frac{y^2}{2}-\frac{x^2}{2}+c
\end{aligned}$$

Here c is a real arbitrary constant.

Step 5 of 5

Now the analytic function is $f(z)=u+iv$.

$$\begin{aligned}
f(z) &= xy+i\left(\frac{y^2}{2}-\frac{x^2}{2}+c\right)\\
&= -\frac{i}{2}\left(-y^2+x^2+2ixy+c\right)\\
&= -\frac{i}{2}\left(z^2+c\right) \quad (\text{here } c \text{ is a real constant})
\end{aligned}$$

Hence, the required analytic function is $\boxed{f(z)=-\frac{i}{2}\left(z^2+c\right)}$.

Chapter 13.4, Problem 14P

Step-by-step solution

Step 1 of 5

Harmonic functions:

Consider the below function,

$v(x, y) = xy$

Determine whether the above function is harmonic or not.

Step 2 of 5

A function is harmonic then it satisfies the equations,

$u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$. ...... **(1)**

Step 3 of 5

Here $v(x, y) = xy$.

$$v_x = y, v_y = x$$
$$v_{xx} = 0, v_{yy} = 0$$

It is clear that $v_{xx} + v_{yy} = 0$.

Hence, the function is $\boxed{\text{harmonic}}$.

Step 4 of 5

Now find the harmonic conjugate.

$$v_x = y$$
$$-u_y = y \qquad (\text{since } u_y = -v_x)$$
$$u = -\frac{y^2}{2} + h(x) \quad (\text{integrate with respect to } y)$$
$$u_x = h'(x)$$
$$x = h'(x) \quad (\text{since } u_x = v_y)$$
$$h(x) = \frac{x^2}{2} + c \to u = -\frac{y^2}{2} + \frac{x^2}{2} + c$$

Here c is a real arbitrary constant.

Step 5 of 5

Now the analytic function is $f(z) = u + iv$.

$$\begin{aligned} f(z) &= \frac{x^2}{2} - \frac{y^2}{2} + c + ixy \\ &= \frac{1}{2}\left(x^2 - y^2 + 2ixy + c\right) \\ &= \frac{1}{2}\left(z^2 + c\right) \quad (\text{here } c \text{ is a real constant}) \end{aligned}$$

Hence, the required analytic function is $\boxed{f(z) = \frac{1}{2}\left(z^2 + c\right)}$.

Chapter 13.4, Problem 15P

Step-by-step solution

Step 1 of 17

The function is given as,

$$u=\frac{x}{x^2+y^2}$$

Step 2 of 17

For the given function to be harmonic, it must satisfy Laplace's equation.

Step 3 of 17

The Laplace's equation for u is given by,

$$\nabla^2 u=0$$
$$u_{xx}+u_{yy}=0$$

Step 4 of 17

To find u_x, partially differentiate $u(x,y)$ with respect to x.

$$\frac{\partial}{\partial x}(u(x,y))=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$
$$u_x=\frac{(x^2+y^2)\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$
$$u_x=\frac{(x^2+y^2)1-x(2x+0)}{(x^2+y^2)^2}$$
$$u_x=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$
$$u_x=\frac{-x^2+y^2}{(x^2+y^2)^2}$$

Step 5 of 17

To find u_y, partially differentiate $u(x,y)$ with respect to y.

$$\frac{\partial}{\partial y}(u(x,y))=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$
$$u_y=x\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$
$$u_y=x\frac{\partial}{\partial y}(x^2+y^2)^{-1}$$
$$u_y=x(-1)(x^2+y^2)^{-2}\frac{\partial}{\partial y}(x^2+y^2)$$
$$u_y=\frac{-x}{(x^2+y^2)^2}(0+2y)$$
$$u_y=\frac{-2xy}{(x^2+y^2)^2}$$

Step 6 of 17

To find u_{xx}, partially differentiate u_x with respect to x.

$$\frac{\partial}{\partial x}(u_x)=\frac{\partial}{\partial x}\left(\frac{-x^2+y^2}{(x^2-y^2)^2}\right)$$
$$u_{xx}=\frac{(x^2+y^2)^2\frac{\partial}{\partial x}[-x^2+y^2]-[-x^2+y^2]\frac{\partial}{\partial x}\left[(x^2+y^2)^2\right]}{\left[(x^2+y^2)^2\right]^2}$$
$$u_{xx}=\frac{(x^2+y^2)^2(-2x+0)-2(-x^2+y^2)(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^4}$$
$$u_{xx}=\frac{-2x(x^2+y^2)^2-2[-x^2+y^2](2x+0)}{(x^2+y^2)^4}$$

Simplification yields,

$$u_{xx}=\frac{-2x(-x^2+3y^2)}{(x^2+y^2)^3}$$

Step 7 of 17

To find u_{yy}, partially differentiate u_y with respect to y.

$$\frac{\partial}{\partial y}(u_y)=\frac{\partial}{\partial y}\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$
$$u_{yy}=\frac{(x^2+y^2)^2\frac{\partial}{\partial y}(-2xy)+2xy\frac{\partial}{\partial y}\left[(x^2+y^2)^2\right]}{\left([x^2+y^2]^2\right)^2}$$
$$u_{yy}=\frac{(x^2+y^2)^2(-2x)+4xy[x^2+y^2]\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^4}$$
$$u_{yy}=\frac{-2x[x^2+y^2]^2+4xy(x^2+y^2)(0+2y)}{(x^2+y^2)^4}$$

Simplification yields,

$$u_{yy}=\frac{-2x(x^2-3y^2)}{(x^2+y^2)^3}$$

Step 8 of 17

Substitute the values of u_{xx} and u_{yy} in Laplace's equation gives,

$$u_{xx}+u_{yy}=0$$
$$\frac{-2x[-x^2+3y^2]}{(x^2+y^2)^3}+\frac{-2x(x^2-3y^2)}{(x^2+y^2)^3}=0$$
$$\frac{-2x[-x^2+3y^2]}{(x^2+y^2)^3}-\frac{-2x[-x^2+3y^2]}{(x^2+y^2)^3}=0$$
$$0=0$$

Step 9 of 17

But also, it must be noted that, second partial derivatives do not exist when,

$$(x^2+y^2)^3=0$$
$$\Rightarrow x^2+y^2=0$$
$$\Rightarrow\sqrt{x^2+y^2}=0$$
$$\Rightarrow|z|=0$$
$$\Rightarrow z=0$$

Step 10 of 17

Therefore, the function $u=\frac{x}{x^2+y^2}$ satisfies Laplace's equation at $z\neq 0$.

Step 11 of 17

Hence, as $u=\frac{x}{x^2+y^2}$ satisfies Laplace's equation at $z\neq 0$, it can be concluded that it is a harmonic function at $z\neq 0$.

Therefore, let corresponding analytic function at $z\neq 0$ be,

$$f(z)=u(x,y)+iv(x,y)$$

Step 12 of 17

As $f(z)$ is analytic at $z\neq 0$, it must satisfy Cauchy Riemann equations at $z\neq 0$.

Therefore,

$$u_x=v_y$$
$$u_y=-v_x$$

Therefore,

$$v_y=\frac{-x^2+y^2}{(x^2+y^2)^2}$$
$$v_x=-\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

Step 13 of 17

Integrate v_x with respect to x.

$$v=\int\frac{2xy}{(x^2+y^2)^2}dx$$
$$v=2y\int\frac{x}{(x^2+y^2)^2}dx$$

Applying u-substitution $u=x^2+y^2$

$$v=2y\frac{1}{2}\int\frac{1}{u^2}du$$

Solving and back substituting the value of $u=x^2+y^2$, we get

$$v=\frac{-y}{x^2+y^2}+h(y)$$

Step 14 of 17

Partially differentiate v with respect to y.

$$v_y=\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)+\frac{dh}{dy}$$

Solving partial derivative and simplifying results in,

$$v_y=\frac{-x^2+y^2}{(x^2+y^2)^2}+\frac{dh}{dy}$$

Step 15 of 17

Comparing $v_y=\frac{-x^2+y^2}{(x^2+y^2)^2}+\frac{dh}{dy}$ with $v_y=\frac{-x^2+y^2}{(x^2+y^2)^2}$, implies that,

$$\frac{dh}{dy}=0$$

This implies,

$$h(y)=C$$

Step 16 of 17

Therefore, harmonic conjugate function of $u=\frac{x}{x^2+y^2}$ at $z\neq 0$ can be written as,

$$v=\frac{-y}{x^2+y^2}+C$$

Step 17 of 17

Hence, the corresponding analytic function at $z\neq 0$ is given by,

$$f(z)=u(x,y)+iv(x,y)$$
$$f(z)=\frac{x}{x^2+y^2}+i\left(\frac{-y}{x^2+y^2}+C\right)$$

Chapter 13.4, Problem 16P

Step-by-step solution

Step 1 of 5

Harmonic functions:

Consider the below function,

$$u(x,y) = \sin x \cosh y$$

Determine whether the above function is harmonic or not.

Step 2 of 5

A function is harmonic then it satisfies the equations,

$u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$. …… **(1)**

Step 3 of 5

Here $u(x,y) = \sin x \cosh y$.

$$u_x = \cos x \cosh y, u_y = \sin x \sinh y$$

$$u_{xx} = -\sin x \cosh y,$$
$$u_{yy} = \sin x \cosh y$$

It is clear that $u_{xx} + u_{yy} = 0$.

Hence, the function is $\boxed{\text{harmonic}}$.

Step 4 of 5

Now find the harmonic conjugate.

$$u_y = \sin x \sinh y$$
$$v_x = -\sin x \sinh y \qquad (\text{since } u_y = -v_x)$$
$$v = \cos x \sinh y + h(y) \quad (\text{integrate with respect to } x)$$
$$v_y = \cos x \cosh y + h'(y)$$
$$\cos x \cosh y = \cos x \cosh y + h'(y) \quad (\text{since } v_y = u_x)$$
$$h'(y) = 0 \to h(y) = c$$
$$v = \cos x \sinh y + c$$

Here c is a real arbitrary constant.

Step 5 of 5

Now the analytic function is $f(z) = u + iv$.

$$\begin{aligned} f(z) &= \sin x \cosh y + i(\cos x \sinh y + c) \\ &= (\sin x \cosh y + i \cos x \sinh y) + ic \\ &= \sin(x+iy) + ic \\ &= \sin z + ic \end{aligned}$$

Hence, the required analytic function is $\boxed{f(z) = \sin z + ic}$.

Chapter 13.4, Problem 17P

Step-by-step solution

Step 1 of 5

Harmonic functions:

Consider the below function,

$v(x,y)=(2x+1)y$

Determine whether the above function is harmonic or not.

Step 2 of 5

A function is harmonic then it satisfies the equations,

$u_{xx}+u_{yy}=0, v_{xx}+v_{yy}=0$. ...... **(1)**

Step 3 of 5

Here $v(x,y)=(2x+1)y$.

$$v_x=2y, v_y=2x+1$$
$$v_{xx}=0, v_{yy}=0$$

It is clear that $v_{xx}+v_{yy}=0$.

Hence, the function is $\boxed{\text{harmonic}}$.

Step 4 of 5

Now find the harmonic conjugate.

$$\begin{aligned} v_x &= 2y \\ -u_y &= 2y \qquad (\text{since } u_y=-v_x) \\ u &= -y^2+h(x) \quad (\text{integrate with respect to } y) \\ u_x &= h'(x) \\ 2x+1 &= h'(x) \quad (\text{since } u_x=v_y) \\ h(x) &= x^2+x+c \rightarrow u=-y^2+x^2+x+c \end{aligned}$$

Here c is a real arbitrary constant.

Step 5 of 5

Now the analytic function is $f(z)=u+iv$.

$$\begin{aligned} f(z) &= (x^2-y^2+x+c)+i(2x+1)y \\ &= (x^2-y^2+i2xy)+(x+iy)+c \quad (\text{here } c \text{ is a real constan}) \\ &= (x+iy)^2+(x+iy)+c \\ &= z^2+z+c \end{aligned}$$

Hence, the required analytic function is $\boxed{f(z)=z^2+z+c}$.

Chapter 13.4, Problem 18P

Step-by-step solution

Step 1 of 3

Consider the following function,

$u(x,y)=x^3-3xy^2$ ...... **(1)**

Check whether the above function is harmonic or not.

Recollect that the harmonic function

A function is harmonic if it satisfies the equations,

$u_{xx}+u_{yy}=0, v_{xx}+v_{yy}=0$.

Step 2 of 3

Differentiate the function **(1)** with respect to *x*, partially,

$u_x=3x^2-3y^2$,

Again differentiate the function **(1)** with respect to *x*, partially,

$u_{xx}=6x$,

Differentiate the function **(1)** with respect to *y*, partially,

$u_y=-6xy$

Again differentiate the function **(1)** with respect to *x*, partially,

$u_{yy}=-6x$

It is clear that $u_{xx}+u_{yy}=0$.

Hence, the function is $\boxed{\text{harmonic}}$.

Step 3 of 3

Find an analytic function:

Recollect that the analytic function

The function $f(z)=u+iv$ is analytic in a domain D if and only if the first partial derivatives of u and v satisfies the two Cauchy-Riemann equations

$u_x=v_y, u_y=-v_x$

Find the value v:

$$u_y=-6xy$$
$$v_x=6xy \qquad \text{Since } u_y=-v_x$$

Integrate the function v_x with respect to *x*

$\int v_x dx=\int 6xydx$

$v=3x^2y+h(y)$ ...... **(2)**

Differentiate the function v with respect to *y* partially,

$$v_y=3x^2+h'(y)$$
$$3x^2-3y^2=3x^2+h'(y) \qquad \text{Since } v_y=u_x$$
$$h'(y)=-3y^2$$

Integrate the function $h'(y)$ with respect to *y*, it becomes

$h(y)=-y^3+c$

Here c is a real arbitrary constant.

Substitute this value in **(2)**, it becomes.

$v=3x^2y-y^3+c$

The analytic function is $f(z)=u+iv$.

$$\begin{aligned} f(z)&=x^3-3xy^2+i(3x^2y-y^3+c)\\ &=(x^3+i3x^2y-3xy^2-iy^3)+ic\\ &=z^3+ic && \text{Since } z=x+iy \end{aligned}$$

Hence, the required analytic function is $f(z)=\boxed{z^3+ic}$.

Chapter 13.4, Problem 19P

Step-by-step solution

Step 1 of 3

Harmonic functions:

Consider the below function,

$v(x, y) = e^x \sin 2y$

Determine whether the above function is harmonic or not.

Step 2 of 3

A function is harmonic then it satisfies the equations,

$u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$. ...... **(1)**

Step 3 of 3

Here $v(x, y) = e^x \sin 2y$.

$v_x = e^x \sin 2y, v_y = 2e^x \cos 2y$

$v_{xx} = e^x \sin 2y, v_{yy} = -4e^x \sin 2y$

It is clear that $v_{xx} + v_{yy} \neq 0$.

Hence, the function is $\boxed{\text{not harmonic}}$.

Chapter 13.4, Problem 20P

Step-by-step solution

Step 1 of 1

Give the details of $v_{xx}+v_{yy}=0$.

The function $f(z)=u+iv$ is analytic then both u, v satisfy the Laplace Equation.

The function $f(z)$ is analytic, so C-R equations are satisfied.

$u_x=v_y, u_y=-v_x$

Differentiate $u_y=-v_x$ with respect to x and $u_x=v_y$ with respect to y.

$$u_{yx}=-v_{xx}, u_{xy}=v_{yy}$$
$$-v_{xx}=v_{yy} \qquad (\text{since } u_{xy}=u_{yx})$$
$$v_{xx}+v_{yy}=0$$

Hence, the result is verified.

Chapter 13.4, Problem 21P

Step-by-step solution

Step 1 of 3

Determine a,b:

Consider the below harmonic function,

$u = e^{\pi x}\cos ay$.

Find the values of a,b and its harmonic conjugate.

Step 2 of 3

Function $u(x,y)$ satisfies the Laplace equation.

$$u_x = \pi e^{\pi x}\cos ay \to u_{xx} = \pi^2 e^{\pi x}\cos ay$$
$$u_y = -ae^{\pi x}\sin ay \to u_{yy} = -a^2 e^{\pi x}\cos ay$$
$$u_{xx} + u_{yy} = 0 \to \pi^2 e^{\pi x}\cos ay - a^2 e^{\pi x}\cos ay = 0$$
$$\left(\pi^2 - a^2\right)e^{\pi x}\cos ay = 0$$
$$\pi^2 - a^2 = 0 \to a = \pm\pi$$

Hence, the required value of a is $\boxed{\pi}$.

Step 3 of 3

Now find the harmonic conjugate.

$$u = e^{\pi x}\cos \pi y$$
$$u_x = \pi e^{\pi x}\cos \pi y$$
$$v_y = \pi e^{\pi x}\cos \pi y \qquad \left(\text{since } u_x = v_y\right)$$
$$v = e^{\pi x}\sin \pi y + h(x) \quad \left(\text{integrate with respect to } y\right)$$
$$v_x = \pi e^{\pi x}\sin \pi y + h'(x)$$
$$\pi e^{\pi x}\sin \pi y = \pi e^{\pi x}\sin \pi y + h'(x) \qquad \left(\text{since } v_x = -u_y\right)$$
$$h'(x) = 0 \to h(x) = c$$
$$v = e^{\pi x}\sin \pi y + c$$

Here c is a real arbitrary constant.

Hence, the required conjugate function is $\boxed{v(x,y) = e^{\pi x}\sin \pi y}$.

Chapter 13.4, Problem 22P

Step-by-step solution

Step 1 of 3

Determine a, b:

Consider the below harmonic function,

$u = \cos ax \cosh 2y$.

Find the values of a, b and its harmonic conjugate.

Step 2 of 3

Function $u(x, y)$ satisfies the Laplace equation.

$$u_x = -a \sin ax \cosh 2y \to u_{xx} = -a^2 \cos ax \cosh 2y$$
$$u_y = 2\cos ax \sinh 2y \to u_{yy} = 4\cos ax \cosh 2y$$

$$u_{xx} + u_{yy} = 0$$
$$-a^2 \cos ax \cosh 2y + 4\cos ax \cosh 2y = 0$$
$$\left(4 - a^2\right) \cos ax \cosh 2y = 0$$
$$a = \pm 2$$

Hence, the required value of a is $\boxed{2}$.

Step 3 of 3

Now find the harmonic conjugate.

$$u = \cos 2x \cosh 2y$$
$$u_x = -2\sin 2x \cosh 2y$$
$$v_y = -2\sin 2x \cosh 2y \qquad \left(\text{since } u_x = v_y\right)$$
$$v = -\sin 2x \sinh 2y + h(x) \quad \left(\text{integrate with respect to } y\right)$$
$$v_x = -2\cos 2x \sinh 2y + h'(x)$$
$$-2\cos 2x \sinh y = -2\cos 2x \sinh 2y + h'(x) \quad \left(\text{since } v_x = -u_y\right)$$
$$h'(x) = 0 \to h(x) = c$$
$$v = -\sin 2x \sinh 2y + c$$

Here c is a real arbitrary constant.

Hence, the required conjugate function is $\boxed{v(x, y) = -\sin 2x \sinh 2y}$.

Chapter 13.4, Problem 23P

Step-by-step solution

Step 1 of 3

Determine a,b:

Consider the below harmonic function,

$u = ax^3 + bxy$.

Find the value of a and its harmonic conjugate.

Step 2 of 3

Function $u(x,y)$ satisfies the Laplace equation.

$$u_x = 3ax^2 + by \to u_{xx} = 6ax$$
$$u_y = bx \to u_{yy} = 0$$
$$u_{xx} + u_{yy} = 0 \to 6ax = 0 \to a = 0$$

Hence, the required value of a is $\boxed{0}$.

Step 3 of 3

Now find the harmonic conjugate.

$$u_x = by$$
$$v_y = by \qquad (\text{since } u_x = v_y)$$
$$v = \frac{by^2}{2} + h(x) \quad (\text{integrate with respect to } y)$$
$$v_x = h'(x)$$
$$-bx = h'(x) \quad (\text{since } v_x = -u_y)$$
$$h(x) = -\frac{bx^2}{2} + c \to v = \frac{by^2}{2} - \frac{bx^2}{2} + c$$

Here c is a real arbitrary constant.

Hence, the required conjugate function is $\boxed{v = \frac{b}{2}\left(y^2 - x^2\right) + c}$.

Chapter 13.4, Problem 24P

Step-by-step solution

Step 1 of 3

Determine a, b:

Consider the below harmonic function,

$u = \cosh ax \cos y$.

Find the value of a and its harmonic conjugate.

Step 2 of 3

Function $u(x, y)$ satisfies the Laplace equation.

$$u_x = a \sinh ax \cos y \to u_{xx} = a^2 \cosh ax \cos y$$
$$u_y = -\cosh ax \sin y \to u_{yy} = -\cosh ax \cos y$$

$$u_{xx} + u_{yy} = 0 \to (a^2 - 1)\cosh ax \cos y = 0$$
$$a^2 - 1 = 0 \to a = \pm 1$$

Hence, the required value of a is $\boxed{1}$.

Step 3 of 3

Now find the harmonic conjugate.

$$u_x = \sinh x \cos y$$
$$v_y = \sinh x \cos y \quad (\text{since } u_x = v_y)$$
$$v = \sinh x \sin y + h(x) \quad (\text{integrate with respect to } y)$$
$$v_x = \cosh x \sin y + h'(x)$$
$$\cosh x \cos y = \cosh x \sin y + h'(x) \quad (\text{since } v_x = -u_y)$$
$$h'(x) = 0 \to h(x) = c$$

Here c is a real arbitrary constant.

Hence, the required conjugate function is $\boxed{v = \sinh x \sin y + c}$.

Chapter 13.5, Problem 3P

Step-by-step solution

Step 1 of 3

Value of the function:

Find the values of $e^z, |e^z|$, if the number z is $z = 2\pi i(1+i)$.

From the Euler's formula,

$e^{iy} = \cos y + i\sin y$ ...... **(1)**

Step 2 of 3

Consider e^z.

$$\begin{aligned} z &= 2\pi i(1+i) \\ &= 2\pi i - 2\pi \end{aligned}$$

$$\begin{aligned} e^z &= e^{2\pi i - 2\pi} \\ &= e^{-2\pi} \cdot e^{2\pi i} \\ &= e^{-2\pi}(\cos 2\pi + i\sin 2\pi) \quad (\text{from } (1)) \\ &= e^{-2\pi}(1+0i) \end{aligned}$$

It is in the form of $u+iv$.

Hence, the value of e^z is $\boxed{e^{-2\pi}}$.

Step 3 of 3

Consider $|e^z|$.

$$\begin{aligned} |e^z| &= |e^{-2\pi} \cdot e^{2\pi i}| \\ &= e^{-2\pi}|e^{2\pi i}| \\ &= e^{-2\pi} \qquad \left(\text{since } |e^{i\theta}| = 1\right) \end{aligned}$$

Hence, the value of $|e^z|$ is $\boxed{e^{-2\pi}}$.

Chapter 13.5, Problem 14P

Step-by-step solution

Step 1 of 1

Real and Imaginary parts:

Find the real and imaginary parts $e^{-\pi z}$.

$$e^{-\pi z} = e^{-\pi(x+iy)}$$
$$= e^{-\pi x} \cdot e^{-i\pi y}$$
$$= e^{-\pi x}\left(\cos \pi y - i \sin \pi y\right)$$

Real part is $\boxed{e^{-\pi x} \cos \pi y}$.

Imaginary part is $\boxed{-e^{-\pi x} \sin \pi y}$.

Chapter 13.5, Problem 15P

Step-by-step solution

Step 1 of 1

Real and Imaginary parts:

Find the real and imaginary parts e^{z^2}.

$$e^{z^2} = e^{(x+iy)^2}$$
$$= e^{x^2-y^2+2ixy}$$
$$= e^{x^2-y^2} \cdot e^{2ixy}$$
$$= e^{x^2-y^2}\left[\cos 2xy + i \sin 2xy\right]$$

Real part is $\boxed{e^{x^2-y^2} \cos 2xy}$.

Imaginary part is $\boxed{e^{x^2-y^2} \sin 2xy}$.

Chapter 13.5, Problem 16P

Step-by-step solution

Step 1 of 1

Real and Imaginary parts:

Find the real and imaginary parts $e^{1/z}$.

$$\begin{aligned} e^{1/z} &= e^{1/(x+iy)} \\ &= e^{(x-iy)/(x^2+y^2)} \\ &= e^{x/(x^2+y^2)} \cdot e^{-iy/(x^2+y^2)} \\ &= e^{x/(x^2+y^2)}\left[\cos\frac{y}{x^2+y^2} - i\sin\frac{y}{x^2+y^2}\right] \end{aligned}$$

Real part is $\boxed{e^{x/(x^2+y^2)}\cos\frac{y}{x^2+y^2}}$.

Imaginary part is $\boxed{-e^{x/(x^2+y^2)}\sin\frac{y}{x^2+y^2}}$.

Chapter 13.5, Problem 17P

Step-by-step solution

Step 1 of 1

Real and Imaginary parts:

Find the real and imaginary parts e^{z^3}.

$$\begin{aligned}
e^{z^3} &= e^{(x+iy)^3} \\
&= e^{x^3-3xy^2+i\left(3x^2y-y^3\right)} \\
&= e^{x^3-3xy^2}\left[\cos\left(3x^2y-y^3\right)+i\sin\left(3x^2y-y^3\right)\right] \\
&= e^{x^3-3xy^2}\cos\left(3x^2y-y^3\right)+ie^{x^3-3xy^2}\sin\left(3x^2y-y^3\right)
\end{aligned}$$

Real part is $\boxed{e^{x^3-3xy^2}\cos\left(3x^2y-y^3\right)}$.

Imaginary part is $\boxed{e^{x^3-3xy^2}\sin\left(3x^2y-y^3\right)}$.

Chapter 13.5, Problem 18P

Step-by-step solution

There is no solution to this problem yet.

Get help from a Chegg subject expert.

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Chapter 13.5, Problem 19P

Step-by-step solution

Step 1 of 2

Solve the equation:

Find all the solutions of the equation $e^{z}=1$.

$$e^{x+iy}=1$$
$$e^{x}\left(\cos y+i\sin y\right)=1+i0$$
$$e^{x}\cos y=1, e^{x}\sin y=0$$

Squaring and adding gives,

$$e^{2x}\left(\cos^{2}y+\sin^{2}y\right)=1^{2}+0^{2}$$
$$e^{2x}=1$$
$$2x=0$$
$$x=0$$

Step 2 of 2

Substitute $x=0$ in $e^{x}\cos y=1, e^{x}\sin y=0$.

$$\cos y=1, \sin y=0$$
$$y=2n\pi, n=0,1,2,\ldots$$

Hence, the solutions of the equation are $\boxed{z=0+2in\pi, n=0,1,\ldots}$.

Chapter 13.5, Problem 20P

Step-by-step solution

Step 1 of 3

The objective is to find all the solutions of the following in complex plane:

$$e^Z = 4+3i$$

Step 2 of 3

The given equation is,

$$e^Z = 4+3i$$

$$e^{x+iy} = 4+3i$$

$$e^x(\cos y + i\sin y) = 4+3i$$

$$e^x \cos y = 4, e^x \sin y = 3$$

Squaring and adding gives,

$$e^{2x}(\cos^2 y + \sin^2 y) = 4^2 + 3^2$$

$$e^{2x} = 25$$

$$2x = 2\ln 5$$

$$x = \ln 5$$

Step 3 of 3

Substitute $x = \ln 5$ in $e^x \cos y = 4, e^x \sin y = 3$.

$$5\cos y = 4, 5\sin y = 3$$

$$\cos y = \frac{4}{5}, \sin y = \frac{3}{5}$$

$$\tan y = \frac{3}{4}$$

$$y = n\pi + \tan^{-1}\left(\frac{3}{4}\right), n = 0,1,2,\ldots$$

Hence, the solutions of the equation are $\boxed{z = \ln 5 + i\left(n\pi + \tan^{-1}\frac{3}{4}\right), n = 0,1,\ldots}$.

Chapter 13.5, Problem 21P

Step-by-step solution

Step 1 of 1

Solve the equation:

Find all the solutions of the equation $e^{z}=0$.

$$e^{x+iy}=0$$

$$e^{x}\left(\cos y+i\sin y\right)=0+i0$$

$$e^{x}\cos y=0, e^{x}\sin y=0$$

Squaring and adding gives,

$$e^{2x}\left(\cos^{2} y+\sin^{2} y\right)=0^{2}+0^{2}$$

$$e^{2x}=0$$

$$x \text{ is not defined}$$

Hence, the equation has $\boxed{\text{no solution}}$.

Chapter 13.5, Problem 22P

Step-by-step solution

Step 1 of 11

The objective is to find all solutions of the equation,

$e^z = -2$

Step 2 of 11

Let the complex number,

$z = x + iy$,

where $x, y \in \mathbb{R}$

Step 3 of 11

Therefore, e^z can be written as follows,

$e^z = e^{x+iy}$

By using the property of the complex exponential function, it can be written as follows,

$e^z = e^x(\cos y + i \sin y)$

$e^z = e^x \cos y + ie^x \sin y$

Step 4 of 11

-2 can be written in complex form as follows,

$-2 = -2 + i0$

Step 5 of 11

Hence, from the given equation,

$e^z = -2$

$e^x \cos y + ie^x \sin y = -2 + i0$

This implies,

$e^x \cos y = -2$

&

$e^x \sin y = 0$

Step 6 of 11

For e^x with all $x \in \mathbb{R}$, it is always true that,

$e^x > 0$

Therefore, $e^x \sin y = 0$ implies that,

$\sin y = 0$

Step 7 of 11

All possible values of y which satisfies $\sin y = 0$ are given by,

$y = k\pi$ where k is an integer.

Step 8 of 11

As $e^x > 0$, $e^x \cos y = -2$ implies that,

$\cos y = -2e^{-x}$

$\cos y < 0$

$\cos y < 0$ is only possible when k is an odd integer in $y = k\pi$.

Or it can also be written as,

$y = (2k+1)\pi$ where k is an integer.

Step 9 of 11

As it is always true that,

$\cos((2k+1)\pi) = -1$

Therefore,

$-1 = -2e^{-x}$

$\frac{1}{e^{-x}} = 2$

$e^x = 2$

This implies,

$x = \ln 2$

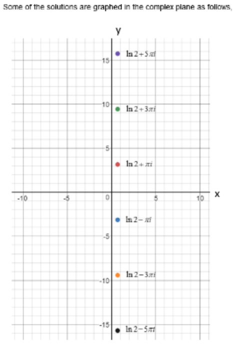
Step 10 of 11

Hence, the solution set of all the possible solutions of the equation $e^z = -2$ is given by,

$z = \{\ln 2 + i(2k+1)\pi, k \in \mathbb{Z}\}$

Step 11 of 11

Some of the solutions are graphed in the complex plane as follows,



Chapter 13.6, Problem 6P

Step-by-step solution

Step 1 of 2

Value of the function:

Find $\sin 2\pi i$ in the form of $u + iv$.

Essential understanding:

$$\begin{aligned} &\cos iz = \cosh z \\ &\sin iz = i \sinh z \rightarrow \sinh z = -i \sin iz \end{aligned} \quad \text{...... (1)}$$

Step 2 of 2

Consider the function $\sin 2\pi i$.

$$\begin{aligned} \sin 2\pi i &= i \sinh 2\pi \qquad (\text{from } (1)) \\ &\approx 267.7i \end{aligned}$$

Hence, the result is $\boxed{i \sinh 2\pi \approx 267.7i}$.

Chapter 13.6, Problem 7P

Step-by-step solution

Step 1 of 2

Value of the function:

Find $\cos i, \sin i$ in the form of $u + iv$.

Essential understanding:

$$\begin{aligned} &\cos iz = \cosh z \\ &\sin iz = i\sinh z \rightarrow \sinh z = -i\sin iz \end{aligned} \quad \text{...... (1)}$$

Step 2 of 2

Consider the function $\cos i$.

$$\begin{aligned} \cos i &= \cos 1i \\ &= \cosh 1 \qquad (\text{from } (1)) \\ &\approx 1.543 \end{aligned}$$

Consider the function $\sin i$.

$$\begin{aligned} \sin i &= \sin 1i \\ &= i\sinh 1 \qquad (\text{from } (1)) \\ &\approx 1.175i \end{aligned}$$

Hence, the result is $\boxed{\cos i \approx 1.543, \sin i \approx 1.175i}$.

Chapter 13.6, Problem 8P

Step-by-step solution

Step 1 of 3

Consider the following functions:

$f(z)=\cos(\pi i),\ \ f(z)=\cosh(\pi i)$

Step 2 of 3

First take the function, $f(z)=\cos(\pi i)$.

$$\begin{aligned} f(z)&=\cos(\pi i)\\ &=\cosh(\pi) \qquad \text{Since } \cos(iz)=\cosh(z)\\ &=\frac{e^{\pi}+e^{-\pi}}{2} \qquad \text{Since } \cosh(\theta)=\frac{e^{\theta}+e^{-\theta}}{2} \end{aligned}$$

$$=\frac{1}{2}[23.1407+0.0432]$$

$$=\frac{1}{2}[23.1839]$$

$$=11.592$$

So, the function $f(z)=\cos(\pi i)$ can be written as follows.

$$\begin{aligned} f(z)&=\cos(\pi i)\\ &=11.592+i(0) \end{aligned}$$

Therefore, the required result is $\boxed{\cos\pi i=11.592}$

Step 3 of 3

Next, take the function, $f(z)=\cosh(\pi i)$.

$$\begin{aligned} \cosh(\pi i)&=\frac{e^{\pi i}+e^{-\pi i}}{2}\\ &=\frac{\cos(\pi)+i\sin(\pi)+\cos(\pi)-i\sin(\pi)}{2}\\ &=\frac{2\cos(\pi)}{2}\\ &=\cos(\pi)\\ &=-1 \end{aligned}$$

So, the function $f(z)=\cosh(\pi i)$ can be written as follows.

$$\begin{aligned} f(z)&=\cosh(\pi i)\\ &=-1+i(0) \end{aligned}$$

Therefore, the required result is $\boxed{\cosh\pi i=-1}$.

Chapter 13.6, Problem 9P

Step-by-step solution

Step 1 of 3

Value of the function:

Find $\cosh(-1+2i), \cos(-2-i)$ in the form of $u+iv$.

Essential understanding:

$$\cosh(z_1+z_2)=\cosh z_1 \cosh z_2+\sinh z_1 \sinh z_2 \quad \ldots\ldots \textbf{(1)}$$

Step 2 of 3

Consider the function $\cosh(-1+2i)$.

$$\begin{aligned}\cosh(-1+2i)&=\cosh(-1)\cosh(2i)+\sinh(-1)\sinh(2i)\\&=\cosh(-1)\cos 2+\sinh(-1)(i\sin 2)\\&\left(\text{since } \cosh iz=\cos z, \sinh iz=i\sin z\right)\\&=\left(\frac{e^{-1}+e^{-(-1)}}{2}\right)\cos 2+\left(\frac{e^{-1}-e^{-(-1)}}{2}\right)(i\sin 2)\\&\left(\text{since } \cosh x=\frac{e^x+e^{-x}}{2}, \sinh x=\frac{e^x+e^{-x}}{2}\right)\\&=(1.543)(-0.642)+(-1.175)(i0.909)\end{aligned}$$

$$=-0.642-1.069i$$

Step 3 of 3

Consider the function $\cos(-2-i)$.

$$\begin{aligned}\cos(-2-i)&=\cos(2+i)\\&=\cos 2\cos(i)-\sin 2\sin(i)\\&=\cos 2(\cosh 1)-\sin 2(i\sinh 1)\\&\left(\text{since } \cos iz=\cosh z, \sin iz=i\sinh z\right)\\&=\left(\frac{e^{1}+e^{-1}}{2}\right)\cos 2-\sin 2\left(i\left(\frac{e^{-1}-e^{-(-1)}}{2}\right)\right)\\&=-0.642-1.069i\end{aligned}$$

Hence, the both results are, $\boxed{-0.642-1.069i}$.

Chapter 13.6, Problem 10P

Step-by-step solution

Step 1 of 3

Value of the function:

Find $\sinh(3+4i), \cosh(3+4i)$ in the form of $u+iv$.

Essential understanding:

$\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$ ...... **(1)**

$\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$ ...... **(2)**

Step 2 of 3

Consider the function $\sinh(3+4i)$.

$$\begin{aligned}\sinh(3+4i) &= \sinh 3 \cosh 4i + \cosh 3 \sinh 4i \qquad (\text{from}(2)) \\ &= \sinh 3 \cos 4 + i \cosh 3 \sin 4 \\ &\quad (\text{since } \cosh iz = \cos z, \sinh iz = i \sin z) \\ &= -6.548 - 7.619i\end{aligned}$$

Step 3 of 3

Consider the function $\cosh(3+4i)$.

$$\begin{aligned}\cosh(3+4i) &= \cosh 3 \cosh 4i + \sinh 3 \sinh 4i \\ &= \cosh 3 \cos 4 + i \sinh 3 \sin 4 \\ &\quad (\text{since } \cosh iz = \cos z, \sinh iz = i \sin z) \\ &= -6.581 - 7.582i\end{aligned}$$

Hence, the both results are $\boxed{\begin{array}{l}\sinh(3+4i) = -6.548 - 7.619i \\ \cosh(3+4i) = -6.581 - 7.582i\end{array}}$.

Chapter 13.6, Problem 11P

Step-by-step solution

Step 1 of 3

Value of the function:

Find $\sin \pi i, \cos\left(\frac{1}{2}\pi - \pi i\right)$ in the form of $u + iv$.

Essential understanding:

$\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$ ...... **(1)**

Step 2 of 3

Consider the function $\sin \pi i$.

$\sin \pi i = i \sinh \pi \approx 11.549i$

$(\text{since } \sin iz = i \sinh z)$

Hence, the result is $\boxed{\sin \pi i \approx 11.549i}$.

Step 3 of 3

Consider the function $\cos\left(\frac{1}{2}\pi - \pi i\right)$.

$$\begin{aligned}\cos\left(\frac{1}{2}\pi - \pi i\right) &= \cos\frac{\pi}{2}\cos \pi i + \sin\frac{\pi}{2}\sin \pi i \\ &= 0 + 1 \times \sin \pi i \\ &= i \sinh \pi \, (\text{since } \sin iz = i \sinh z) \\ &= 11.549i\end{aligned}$$

Hence, the result is $\boxed{\cos\left(\frac{1}{2}\pi - \pi i\right) \approx 11.549i}$.

Chapter 13.6, Problem 12P

Step-by-step solution

Step 1 of 3

Value of the function:

Find $\cos\frac{1}{2}\pi i, \cos\left[\frac{1}{2}\pi(1+i)\right]$ in the form of $u+iv$.

Essential understanding:

$\cos(z_1+z_2)=\cos z_1 \cos z_2 - \sin z_1 \sin z_2$ ...... **(1)**

Step 2 of 3

Consider the function $\cos\frac{\pi}{2}i$.

$$\cos\frac{\pi}{2}i=\cosh\frac{\pi}{2}\approx 2.509$$

$$(\text{since } \cos iz=\cosh z)$$

Hence, the result is $\boxed{\cos\frac{\pi}{2}i\approx 2.509}$.

Step 3 of 3

Consider the function $\cos\left[\frac{1}{2}\pi(1+i)\right]$.

$$\begin{aligned}\cos\left[\frac{1}{2}\pi(1+i)\right]&=\cos\frac{\pi}{2}\cos\frac{\pi i}{2}-\sin\frac{\pi}{2}\sin\frac{\pi i}{2}\\&=0-1\times\sin\frac{\pi i}{2}\\&=-i\sinh\frac{\pi}{2}(\text{since } \sin iz=i\sinh z)\\&\approx -2.3i\end{aligned}$$

Hence, the result is $\boxed{\cos\left(\frac{1}{2}\pi+\frac{\pi i}{2}\right)\approx -2.3i}$.

Chapter 13.6, Problem 16P

Step-by-step solution

Step 1 of 8

Consider the equation,

$\sin z = 100.$

The objective is to solve the equation for the values of z.

Step 2 of 8

Use the formula $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$.

Substitute the formula in the equation.

$$\begin{aligned}
\sin z &= 100 \\
\frac{e^{iz} - e^{-iz}}{2i} &= 100 \\
e^{iz} - e^{-iz} &= 200i \\
e^{iz} - \frac{1}{e^{iz}} &= 200i \\
\frac{\left(e^{iz}\right)^2 - 1}{e^{iz}} &= 200i \\
e^{2iz} - 1 &= 200ie^{iz} \\
e^{2iz} - 200ie^{iz} - 1 &= 0
\end{aligned}$$

Step 3 of 8

Use the quadratic equation formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the equation.

Compare the equation $e^{2iz} - 200ie^{iz} - 1 = 0$ with the general equation $ae^{2iz} + be^{iz} + c = 0$.

Then $a = 1, b = -200i, c = -1$.

Substitute the known values in the formula $e^{iz} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned}
e^{iz} &= \frac{-(-200i) \pm \sqrt{(-200i)^2 - 4(1)(-1)}}{2(1)} \\
&= \frac{200i \pm \sqrt{40000i^2 + 4}}{2} \\
&= \frac{200i \pm \sqrt{40000(-1) + 4}}{2} && \text{Use } i^2 = -1 \\
&= \frac{200i \pm \sqrt{-40000 + 4}}{2}
\end{aligned}$$

Step 4 of 8

Further simplification is as follows.

$$\begin{aligned}
&= \frac{200i \pm \sqrt{-39996}}{2} \\
&= \frac{200i \pm i\sqrt{39996}}{2} && \text{Use } \sqrt{-1} = i \\
&= \frac{200i \pm i6\sqrt{1111}}{2} \\
&= 100i \pm i3\sqrt{1111} \\
&= 100i - i3\sqrt{1111}\ ,\ 100i + i3\sqrt{1111} \\
&\approx 0.005i\ ,\ 200i
\end{aligned}$$

Step 5 of 8

The roots can be rewritten as,

$$\begin{aligned}
e^{iz} &= 0.005i \quad \text{or} \quad e^{iz} = 200i \\
e^{i(x+iy)} &= 0.005i \quad \text{or} \quad e^{i(x+iy)} = 200i \\
e^{ix-y} &= 0.005i \quad \text{or} \quad e^{ix-y} = 200i \\
e^{-y} \cdot e^{ix} &= 0.005i \quad \text{or} \quad e^{-y} \cdot e^{ix} = 200i \\
e^{-y}(\cos x + i\sin x) &= 0.005i \quad \text{or} \quad e^{-y}(\cos x + i\sin x) = 200i \\
e^{-y}\cos x + ie^{-y}\sin x &= 0.005i \quad \text{or} \quad e^{-y}\cos x + ie^{-y}\sin x = 200i
\end{aligned}$$

Step 6 of 8

First consider the equation $e^{-y}\cos x + ie^{-y}\sin x = 0.005i$.

$e^{-y}\cos x + ie^{-y}\sin x = 0.005i.$

Compare the coefficients of like terms on both sides.

$e^{-y}\cos x = 0\ ,\ e^{-y}\sin x = 0.005.$

Now

$$\begin{aligned}
e^{-y}\cos x &= 0 \\
\cos x &= 0 && \text{As } e^{-y} \neq 0 \text{ for any } y
\end{aligned}$$

Know that $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned}
\sin^2 x + 0 &= 1 \\
\sin^2 x &= 1 \\
\sin x &= 1
\end{aligned}$$

Now consider,

$$\begin{aligned}
e^{-y}\sin x &= 0.005 \\
e^{-y} &= 0.005\ , && \text{since } \sin x = 1
\end{aligned}$$

Now solve the equations $\cos x = 0$ and $\sin x = 1$.

$\cos x = 0$ and $\sin x = 1$

$x = 2n\pi + \frac{\pi}{2}$, for all $n \in \square$

Solve the equation $e^{-y} = 0.005$ for the value of y.

$$\begin{aligned}
e^{-y} &= 0.005 \\
e^{y} &= \frac{1}{0.005} \\
e^{y} &= 200 \\
\ln e^{y} &= \ln(200) && \text{Take ln both sides} \\
y &\approx 5.29
\end{aligned}$$

Thus, the number z can be written as

$$\begin{aligned}
z &= x + iy \\
&= \left(2n\pi + \frac{\pi}{2}\right) + 5.29i\ ,\ n \in \square.
\end{aligned}$$

Therefore, solution to the equation is $\boxed{z = \left(2n\pi + \frac{\pi}{2}\right) + 5.29i\ ,\ n \in \square}$.

Step 7 of 8

Now solve the equation $e^{-y}\cos x + ie^{-y}\sin x = 200i$.

Compare the coefficients of like terms on both sides

$e^{-y}\cos x = 0\ ,\ e^{-y}\sin x = 200.$

Now

$$\begin{aligned}
e^{-y}\cos x &= 0 \\
\cos x &= 0 && \text{As } e^{-y} \neq 0 \text{ for any } y
\end{aligned}$$

Know that $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned}
\sin^2 x + 0 &= 1 \\
\sin^2 x &= 1 \\
\sin x &= 1
\end{aligned}$$

Now consider

$$\begin{aligned}
e^{-y}\sin x &= 200 \\
e^{-y} &= 200\ , && \text{since } \sin x = 1
\end{aligned}$$

Step 8 of 8

Now solve the equations $\cos x = 0$ and $\sin x = 1$.

$\cos x = 0$ and $\sin x = 1$

$x = 2n\pi + \frac{\pi}{2}$, for all $n \in \square$

Solve the equation $e^{-y} = 200$ for the value of y

$$\begin{aligned}
e^{-y} &= 200 \\
e^{y} &= \frac{1}{200} \\
e^{y} &= 0.005 \\
\ln e^{y} &= \ln(0.005) && \text{Take ln both sides} \\
y &\approx -5.298
\end{aligned}$$

Thus, the number z can be written as

$$\begin{aligned}
z &= x + iy \\
&= \left(2n\pi + \frac{\pi}{2}\right) - 5.29i\ ,\ n \in \square.
\end{aligned}$$

Therefore, solution to the equation is $\boxed{z = \left(2n\pi + \frac{\pi}{2}\right) - 5.29i\ ,\ n \in \square}$.

Chapter 13.6, Problem 17P

Step-by-step solution

Step 1 of 1

Solve the equation:

Solve the equation $\cosh z = 0$.

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$0 = \frac{e^z + e^{-z}}{2}$$
$$0 = \frac{e^{2z} + 1}{2e^z}$$
$$e^{2z} + 1 = 0$$
$$e^{2z} = -1$$
$$e^{2x} \cdot e^{2iy} = -1$$
$$e^{2x} \cos 2y = -1, e^{2x} \sin 2y = 0$$
$$\cos 2y = -1, \sin 2y = 0 \text{ and } e^{2x} = 1$$
$$2y = \pm(2n+1)\pi \to y = \pm(2n+1)\frac{\pi}{2}, n = 0,1,2,...$$
$$e^{2x} = 1 \to x = 0$$

Hence, the solutions of the equation are $\boxed{0 \pm i(2n+1)\frac{\pi}{2}, n = 0,1,2,...}$.

Chapter 13.6, Problem 19P

Step-by-step solution

Step 1 of 1

Solve the equation:

Solve the equation $\sinh z = 0$.

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$0 = \frac{e^z - e^{-z}}{2}$$

$$0 = \frac{e^{2z} - 1}{2e^z}$$

$$e^{2z} - 1 = 0$$

$$e^{2z} = 1$$

$$e^{2x} \cdot e^{2iy} = 1$$

$$e^{2x} \cos 2y = 1, e^{2x} \sin 2y = 0$$

$$\cos 2y = 1, \sin 2y = 0 \text{ and } e^{2x} = 1$$

$$2y = \pm 2n\pi \to y = \pm n\pi, n = 0, 1, 2, \ldots$$

$$e^{2x} = 1 \to x = 0$$

Hence, the solutions of the equation are $\boxed{0 \pm in\pi, n = 0, 1, 2, \ldots}$.

Chapter 13.7, Problem 12P

Step-by-step solution

Step 1 of 3

All values of $\ln z$:

Find all values of a logarithm $\ln e$.

Essential understanding:

$$\begin{aligned} \ln z &= \operatorname{Ln} z \pm 2n\pi i \\ \operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z \end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\operatorname{Ln} e$.

Consider $z = e$.

$$\begin{aligned} |z| &= |e| \\ &= e \end{aligned}$$

$$\begin{aligned} \operatorname{Arg} z &= \operatorname{Arg}(e) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z \\ &= \ln e \\ &= 1 \end{aligned}$$

Step 3 of 3

Now evaluate $\ln e$.

$$\begin{aligned} \ln z &= \operatorname{Ln} z \pm 2n\pi i \\ &= 1 \pm 2n\pi i \end{aligned}$$

Hence, the required value is $\boxed{1 \pm 2n\pi i, n = 0,1,2...}$.

Chapter 13.7, Problem 13P

Step-by-step solution

Step 1 of 3

All values of $\ln z$:

Find all values of a logarithm $\ln 1$.

Essential understanding:

$$\begin{aligned} \ln z &= \text{Ln } z \pm 2n\pi i \\ \text{Ln } z &= \ln|z| + i\text{Arg } z \end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\text{Ln } 1$.

Consider $z = 1$.

$$\begin{aligned} |z| &= |1| \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{Arg } z &= \text{Arg } (1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Ln } z &= \ln|z| + i\text{Arg } z \\ &= \ln 1 + 0i \\ &= 0 \end{aligned}$$

Step 3 of 3

Now evaluate $\ln 1$.

$$\begin{aligned} \ln z &= \text{Ln } z \pm 2n\pi i \\ &= 0 \pm 2n\pi i \end{aligned}$$

Hence, the required value is $\boxed{\pm 2n\pi i, n = 0, 1, 2...}$.

Chapter 13.7, Problem 14P

Step-by-step solution

Step 1 of 3

All values of $\ln z$:

Find all values of a logarithm $\ln(-7)$.

Essential understanding:

$$\begin{aligned}\ln z &= \mathrm{Ln}\, z \pm 2n\pi i \\ \mathrm{Ln}\, z &= \ln|z| + i\mathrm{Arg}\, z\end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\mathrm{Ln}\ (\text{-}7)$.

Consider $z = -7$.

$$\begin{aligned}|z| &= |-7| \\ &= 7\end{aligned}$$

$$\begin{aligned}\mathrm{Arg}\, z &= \mathrm{Arg}\ (-7) \\ &= \pi\end{aligned}$$

$$\begin{aligned}\mathrm{Ln}\, z &= \ln|z| + i\mathrm{Arg}\, z \\ &= \ln 7 + \pi i\end{aligned}$$

Step 3 of 3

Now evaluate $\ln\ (-7)$.

$$\begin{aligned}\ln z &= \mathrm{Ln}\, z \pm 2n\pi i \\ &= \ln 7 + \pi i \pm 2n\pi i \\ &= \ln 7 + (1 \pm 2n)\pi i \\ &\approx 1.946 + (1 \pm 2n)\pi i\end{aligned}$$

Hence, the required value is $\boxed{1.946 + (1 \pm 2n)\pi i, n = 0, 1, 2, \ldots}$.

Chapter 13.7, Problem 15P

Step-by-step solution

Step 1 of 3

All values of $\ln z$:

Find all values of a logarithm $\ln e^{i}$.

Essential understanding:

$$\begin{aligned}\ln z &= \operatorname{Ln} z \pm 2n\pi i \\ \operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z\end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\operatorname{Ln} e^{i}$.

Consider $z = e^{i}$.

$$\begin{aligned}|z| &= \left|e^{i}\right| \\ &= 1\end{aligned}$$

$$\begin{aligned}\operatorname{Arg} z &= \operatorname{Arg} e^{i} \\ &= 1\end{aligned}$$

$$\begin{aligned}\operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z \\ &= \ln 1 + 1i \\ &= i\end{aligned}$$

Step 3 of 3

Now evaluate $\ln e^{i}$.

$$\begin{aligned}\ln z &= \operatorname{Ln} z \pm 2n\pi i \\ \ln e^{i} &= i \pm 2n\pi i \\ &= (1 \pm 2n\pi)i\end{aligned}$$

Hence, the required value is $\boxed{(1 \pm 2n\pi)i, n = 0, 1, 2, \ldots}$.

Chapter 13.7, Problem 16P

Step-by-step solution

Step 1 of 3

All values of $\ln z$:

Find all values of a logarithm $\ln(4+3i)$.

Essential understanding:

$$\begin{aligned}\ln z &= \operatorname{Ln} z \pm 2n\pi i \\ \operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z\end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\operatorname{Ln}\,(4+3i)$.

Consider $z = 4+3i$.

$$\begin{aligned}|z| &= |4+3i| \\ &= \sqrt{4^2+3^2} \\ &= 5\end{aligned}$$

$$\begin{aligned}\operatorname{Arg} z &= \operatorname{Arg}\,(4+3i) \\ &= \tan^{-1}\left(\frac{3}{4}\right)\end{aligned}$$

$$\begin{aligned}\operatorname{Ln} z &= \ln|z| + i\operatorname{Arg} z \\ &= \ln 5 + i\tan^{-1}\left(\frac{3}{4}\right) \\ &\approx 1.609 + 0.644i\end{aligned}$$

Step 3 of 3

Now evaluate $\ln\,(4+3i)$.

$$\begin{aligned}\ln z &= \operatorname{Ln} z \pm 2n\pi i \\ \ln(4+3i) &= 1.609 + 0.644i \pm 2n\pi i \\ &= 1.609 + (0.644 \pm 2n\pi)i\end{aligned}$$

Hence, the required value is $\boxed{1.609 + (0.644 \pm 2n\pi)i, n = 0,1,2,...}$.

Chapter 13.7, Problem 17P

Step-by-step solution

Step 1 of 3

Proof:

Show that $\ln i^2 \neq 2\ln i$.

Essential understanding:

$$\begin{aligned} \ln z &= \mathrm{Ln}\, z \pm 2n\pi i \\ \mathrm{Ln}\, z &= \ln|z| + i\mathrm{Arg}\, z \end{aligned} \quad \text{...... (1)}$$

Step 2 of 3

First evaluate $\mathrm{Ln}\, i,\ \mathrm{Ln} i^2$.

$$|i| = 1$$
$$|i^2| = 1$$

$$\mathrm{Arg}\, i = \frac{\pi}{2}$$
$$\mathrm{Arg}\, i^2 = \pi$$

$$\mathrm{Ln}\, z = \ln|z| + i\mathrm{Arg}\, z$$
$$\mathrm{Ln}\, i = \ln 1 + \frac{\pi i}{2}$$
$$\mathrm{Ln}\, i^2 = \ln 1 + \pi i$$

Step 3 of 3

Now evaluate $2\ln i, \ln\ i^2$.

$$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$$
$$\ln i = \frac{\pi i}{2} \pm 2n\pi i \rightarrow 2\ln i = \pi i \pm 4n\pi i$$
$$\ln i^2 = \pi i \pm 2n\pi i$$

Set of values of $2\ln i$ are $\{\pi i, 5\pi i, -3\pi i, ...\}$.

Set of values of $\ln i^2$ are $\{\pi i, -\pi i, \pi i, ...\}$.

Hence, the set of values of $2\ln i$ differs from the set of values of $\ln i^2$.

Chapter 13.7, Problem 18P

Step-by-step solution

Step 1 of 1

Equation solving:

Solve the equation $\ln z = -\frac{\pi i}{2}$.

$$\begin{aligned}\ln z &= -\frac{\pi i}{2} \\ z &= e^{-\pi i/2} \\ &= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} \\ &= -i\end{aligned}$$

Hence, the solution of the equation is $\boxed{-i}$.

Chapter 13.7, Problem 19P

Step-by-step solution

Step 1 of 1

Equation solving:

Solve the equation $\ln z = 4 - 3i$.

$$\begin{aligned}\ln z &= 4 - 3i \\ z &= e^{4-3i} \\ &= e^{4}(\cos 3 - i\sin 3) \\ &\approx -54.05 - 7.7i\end{aligned}$$

Hence, the solution of the equation is $\boxed{-54.05 - 7.7i}$.

Chapter 13.7, Problem 20P

Step-by-step solution

Step 1 of 2

Equation solving:

The objective is to solve the equation, $\ln z = e - \pi i$.

Step 2 of 2

Consider,

$$\ln z = e - \pi i$$
$$e^{\ln z} = e^{e-\pi i}$$
$$z = e^{e-\pi i}$$
$$= e^{e}.e^{-\pi i}$$
$$= e^{e}\left(\cos(-\pi) + i\sin(-\pi)\right)$$
$$= e^{e}\left(\cos\pi - i\sin\pi\right)$$

Simplifying further:

$$z = e^{e}(-1 - i0)$$
$$= -e^{e}$$
$$\approx -15.15$$

Hence, the solution of the equation is $\boxed{z = -15.15}$.

Chapter 13.7, Problem 21P

Step-by-step solution

Step 1 of 1

Equation solving:

Solve the equation $\ln z = 0.6 + 0.4i$.

$$\begin{aligned} \ln z &= 0.6 + 0.4i \\ z &= e^{0.6+0.4i} \\ &= e^{0.6}\left(\cos 0.4 + i \sin 0.4\right) \\ &\approx 1.678 + 0.710i \end{aligned}$$

Hence, the solution of the equation is $\boxed{1.678 + 0.710i}$.

Chapter 13.7, Problem 22P

Step-by-step solution

Step 1 of 4

General power:

Find the principal value of $(2i)^{2i}$.

$$(2i)^{2i} = e^{\ln(2i)^{2i}}$$
$$= e^{2i\ln(2i)}$$

Step 2 of 4

First evaluate $\mathrm{Ln}(2i)$.

Consider $z = 2i$.

$$|z| = |2i|$$
$$= 2$$
$$\mathrm{Arg}\, z = \mathrm{Arg}\,(2i)$$
$$= \frac{\pi}{2}$$
$$\mathrm{Ln}\, z = \ln|z| + i\,\mathrm{Arg}\, z$$
$$= \ln 2 + \frac{\pi i}{2}$$

Step 3 of 4

Now evaluate $\ln(2i)$.

$$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$$
$$\ln(2i) = \ln 2 + \frac{\pi i}{2} \pm 2n\pi i$$

Step 4 of 4

Now evaluate $(2i)^{2i}$.

$$(2i)^{2i} = e^{2i\ln(2i)}$$
$$= e^{2i\left[\ln 2 + \frac{\pi}{2} i \pm 2n\pi i\right]}$$

Principal value is obtained for $n = 0$.

$$e^{2i\left[\ln 2 + \frac{\pi}{2} i\right]} = e^{-\pi} \cdot e^{2i\ln 2}$$
$$= e^{-\pi} \cdot e^{2i\ln 2}$$
$$= e^{-\pi}\left[\cos(2\ln 2) + i\sin(2\ln 2)\right]$$
$$\approx 0.008 + 0.042i$$

Hence, the required value is $\boxed{0.008 + 0.042i}$.

Chapter 13.7, Problem 23P

Step-by-step solution

Step 1 of 4

General power:

Find the principal value of $(1+i)^{1-i}$.

$$(1+i)^{1-i} = e^{\ln(1+i)^{1-i}}$$
$$= e^{(1-i)\ln(1+i)}$$

Step 2 of 4

First evaluate $\mathrm{Ln}(1+i)$.

Consider $z = 1+i$.

$$|z| = |1+i|$$
$$= \sqrt{1+1}$$
$$= \sqrt{2}$$

$$\mathrm{Arg}\, z = \mathrm{Arg}\,(1+i)$$
$$= \tan^{-1}(1)$$
$$= \frac{\pi}{4}$$

$$\mathrm{Ln}\, z = \ln|z| + i\mathrm{Arg}\, z$$
$$= \ln\sqrt{2} + \frac{\pi i}{4}$$

Step 3 of 4

Now evaluate $\ln(1+i)$.

$$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$$
$$\ln(1+i) = \ln\sqrt{2} + \frac{\pi i}{4} \pm 2n\pi i$$

Step 4 of 4

Now evaluate $(1+i)^{1-i}$.

$$(1+i)^{1-i} = e^{(1-i)\ln(1+i)}$$
$$= e^{(1-i)\left[\ln\sqrt{2} + \frac{\pi i}{4} \pm 2n\pi i\right]}$$

Principal value is obtained for $n = 0$.

$$e^{(1-i)\left[\ln\sqrt{2} + \frac{\pi i}{4}\right]} = e^{\ln\sqrt{2} + \frac{\pi}{4} + i\left(-\ln\sqrt{2} + \frac{\pi}{4}\right)}$$
$$= e^{\ln\sqrt{2} + \frac{\pi}{4}}\left[\cos\left(-\ln\sqrt{2} + \frac{\pi}{4}\right) + i\sin\left(-\ln\sqrt{2} + \frac{\pi}{4}\right)\right]$$
$$\approx 2.8079 + 1.3179i$$

Hence, the required value is $\boxed{2.8079 + 1.3179i}$.

Chapter 13.7, Problem 24P

Step-by-step solution

Step 1 of 4

General power:

Find the principal value of $(1-i)^{1+i}$.

$$\begin{aligned}(1-i)^{1+i} &= e^{\ln(1-i)^{1+i}} \\ &= e^{(1+i)\ln(1-i)}\end{aligned}$$

Step 2 of 4

First evaluate $\mathrm{Ln}(1-i)$.

Consider $z=1-i$.

$$\begin{aligned}|z| &= |1-i| \\ &= \sqrt{1+1} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathrm{Arg}\, z &= \mathrm{Arg}\,(1-i) \\ &= \tan^{-1}\left(\frac{-1}{1}\right) \\ &= -\frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}\mathrm{Ln}\, z &= \ln|z| + i\mathrm{Arg}\, z \\ &= \ln\sqrt{2} - \frac{\pi i}{4}\end{aligned}$$

Step 3 of 4

Now evaluate $\ln(1-i)$.

$$\ln z = \mathrm{Ln}\, z \pm 2n\pi i$$

$$\ln(1-i) = \ln\sqrt{2} - \frac{\pi i}{4} \pm 2n\pi i$$

Step 4 of 4

Now evaluate $(1-i)^{1+i}$.

$$\begin{aligned}(1-i)^{1+i} &= e^{(1+i)\ln(1-i)} \\ &= e^{(1+i)\left[\ln\sqrt{2} - \frac{\pi i}{4} \pm 2n\pi i\right]}\end{aligned}$$

Principal value is obtained for $n=0$.

$$\begin{aligned}e^{(1+i)\left[\ln\sqrt{2} - \frac{\pi i}{4}\right]} &= e^{\ln\sqrt{2} + \frac{\pi}{4} + i\left(\ln\sqrt{2} - \frac{\pi}{4}\right)} \\ &= e^{\ln\sqrt{2} + \frac{\pi}{4}}\left[\cos\left(\ln\sqrt{2} - \frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2} - \frac{\pi}{4}\right)\right] \\ &\approx 2.8079 - 1.3179i\end{aligned}$$

Hence, the required value is $\boxed{2.8079-1.3179i}$.

Chapter 13.7, Problem 25P

Step-by-step solution

Step 1 of 4

General power:

Find the principal value of $(-3)^{3-i}$.

$$\begin{aligned}(-3)^{3-i} &= e^{\ln(-3)^{3-i}} \\ &= e^{(3-i)\ln(-3)}\end{aligned}$$

Step 2 of 4

First evaluate $\mathrm{Ln}(-3)$.

Consider $z=-3$.

$$\begin{aligned}|z| &= |-3| \\ &= 3\end{aligned}$$

$$\begin{aligned}\mathrm{Arg}\, z &= \mathrm{Arg}\,(-3) \\ &= \pi\end{aligned}$$

$$\begin{aligned}\mathrm{Ln}\, z &= \ln|z| + i\mathrm{Arg}\, z \\ &= \ln 3 + \pi i\end{aligned}$$

Step 3 of 4

Now evaluate $\ln(-3)$.

$$\begin{aligned}\ln z &= \mathrm{Ln}\, z \pm 2n\pi i \\ \ln(-3) &= \ln 3 + \pi i \pm 2n\pi i\end{aligned}$$

Step 4 of 4

Now evaluate $(-3)^{3-i}$.

$$\begin{aligned}(-3)^{3-i} &= e^{(3-i)\ln(-3)} \\ &= e^{(3-i)[\ln 3 + \pi i \pm 2n\pi i]}\end{aligned}$$

Principal value is obtained for $n=0$.

$$\begin{aligned}e^{(3-i)[\ln 3 + \pi i]} &= e^{3\ln 3 + \pi + i(3\pi - \ln 3)} \\ &= e^{3\ln 3 + \pi}\left(\cos(3\pi - \ln 3) + i\sin(3\pi - \ln 3)\right) \\ &\approx -284.2 + 556.4i\end{aligned}$$

Hence, the required value is $\boxed{-284.2 + 556.4i}$.

Chapter 14.1, Problem 21P

Step-by-step solution

Step 1 of 3

Integration:

Find the integration $\int_C \mathrm{Re}\, z dz$.

Here C is the shortest path from $1+i$ to $3+3i$.

Step 2 of 3

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0)$ ...... **(1)**

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$ ...... **(2)**

Step 3 of 3

Here $f(z) = \mathrm{Re}\, z$ is not analytic function, so second method is applicable.

Parametric form of the path is

$$z(t) = x(t) + iy(t)$$
$$z(t) = t + it, 1 \le t \le 3$$
$$\mathrm{Re}\, z = t, z'(t) = 1 + i$$

Now apply the second method.

$$\begin{aligned}\int_C \mathrm{Re}\, z dz &= \int_1^3 t(1+i) dt \\ &= (1+i)\left[\frac{t^2}{2}\right]_1^3 \\ &= (1+i)\left[\frac{9}{2} - \frac{1}{2}\right] \\ &= 4 + 4i\end{aligned}$$

Hence, the result is $\boxed{4+4i}$.

Chapter 14.1, Problem 22P

Step-by-step solution

Step 1 of 4

Integration:

Find the integration $\int_C \operatorname{Re} z\,dz$.

Here C is the parabola $y = 1 + \frac{1}{2}(x-1)^2$ from $1+i$ to $3+3i$.

Step 2 of 4

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$ ...... **(1)**

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$ ...... **(2)**

Step 3 of 4

Here $f(z) = \operatorname{Re} z$ is not analytic function, so second method is applicable.

Let $x - 1 = t$ then $y = 1 + \frac{t^2}{2}, 0 \le t \le 2$.

Parametric form of the path is

$$z(t) = x(t) + iy(t)$$
$$z(t) = (1+t) + \left(1 + \frac{t^2}{2}\right)i, 0 \le t \le 2$$
$$\operatorname{Re} z = 1 + t, z'(t) = 1 + ti$$

Step 4 of 4

Now apply the second method.

$$\begin{aligned}\int_C \operatorname{Re} z\,dz &= \int_0^2 (1+t)(1+ti)\,dt \\ &= \int_0^2 \left[1 + t + i\left(t + t^2\right)\right]dt \\ &= \left[\left(t + \frac{t^2}{2}\right) + i\left(\frac{t^2}{2} + \frac{t^3}{3}\right)\right]_0^2\end{aligned}$$

$$= 2 + \frac{4}{2} + \left(\frac{4}{2} + \frac{8}{3}\right)i$$
$$= 4 + \left(2 + \frac{8}{3}\right)i$$

Hence, the result is $\boxed{4 + \frac{14i}{3}}$.

Chapter 14.1, Problem 23P

Step-by-step solution

Step 1 of 3

The objective is to find the integration $\int_C e^Z dz$.

Here C is the shortest path from πi to $2\pi i$.

Step 2 of 3

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$ ...... **(1)**

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ ...... **(2)**

Step 3 of 3

Here $f(z) = e^Z$ is an analytic function, so first method is applicable.

$$\int_C e^Z dz = \int_{\pi i}^{2\pi i} e^Z dz$$
$$= \left[e^Z\right]_{\pi i}^{2\pi i}$$
$$= e^{2\pi i} - e^{\pi i}$$
$$= 1 - (-1)$$
$$= 2$$

Hence, $\boxed{\int_C e^Z dz = 2}$.

Chapter 14.1, Problem 24P

Step-by-step solution

Step 1 of 3

Integration:

Find the integration $\int_C \cos 2z dz$.

Here C is the semi circle path from $-\pi i$ to πi.

Step 2 of 3

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0)$ **...... (1)**

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$ **...... (2)**

Step 3 of 3

Here $f(z) = \cos 2z$ is an analytic function, so first method is applicable.

$$
\begin{aligned}
\int_C \cos 2z dz &= \int_{-\pi i}^{\pi i} \cos 2z dz \\
&= \left[\frac{\sin 2z}{2}\right]_{-\pi i}^{\pi i} \\
&= \frac{1}{2}\left(\sin 2\pi i - \sin(-2\pi i)\right) \\
&= \sin 2\pi i
\end{aligned}
$$

Hence, the result is $\boxed{\sin 2\pi i}$.

Chapter 14.1, Problem 25P

Step-by-step solution

Step 1 of 4

Integration:

Find the integration $\int_C ze^{z^2}\,dz$.

Here, C is the path from 1 to i.

Step 2 of 4

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$ ...... **(1)**

Here, F is the anti-derivative of *f*.

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$ ...... **(2)**

Step 3 of 4

Here, $f(z) = ze^{z^2}$ is an analytic function, so first method is applicable.

So, first find the anti-derivative of $f(z) = ze^{z^2}$.

Take $u = z^2$ and $du = 2z\,dz$.

Then, $\int_C ze^{z^2}\,dz = \int \frac{e^u}{2}\,du$

$= \frac{e^u}{2}$

$= \frac{e^{z^2}}{2}$

This implies, $F(z) = \frac{e^{z^2}}{2}$ is the anti-derivative of $f(z) = ze^{z^2}$.

Step 4 of 4

Applying the first method as follows:

$$\begin{aligned}\int_C ze^{z^2}\,dz &= \int_1^i ze^{z^2}\,dz \\ &= F(i) - F(1) \\ &= \frac{1}{2}\left(e^{i^2} - e^{1}\right) \\ &= \frac{1}{2}\left(e^{-1} - e\right)\end{aligned}$$

$= -\sinh 1$

Hence, the result is $\boxed{-\sinh 1}$.

Chapter 14.1, Problem 26P

Step-by-step solution

Step 1 of 4

Integration:

Find the integration $\int_C \left(z+z^{-1}\right)dz$.

Here C is the unit circle in counter clockwise direction.

Step 2 of 4

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1)-F(z_0)$ ...... (1)

Second method:

If $z=z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ ...... (2)

Step 3 of 4

Here $f(z)=z+z^{-1}$ is not analytic function, so second method is applicable.

Parametric form of the path is as shown below.

$$\begin{aligned}
z(t) &= x(t)+iy(t) \\
z(t) &= \cos t + i\sin t, 0 \le t \le 2\pi \\
z'(t) &= (-\sin t + i\cos t) \rightarrow z'(t) = ie^{it} \\
\frac{1}{z} &= \cos t - i\sin t
\end{aligned}$$

Step 4 of 4

Now apply the second method.

$$\begin{aligned}
\int_C \left(z+z^{-1}\right)dz &= \int_0^{2\pi} \left(e^{it}+e^{-it}\right)\left(ie^{it}\right)dt \\
&= \int_0^{2\pi} \left(ie^{2it}+i\right)dt \\
&= \left[\frac{e^{2it}}{2}+it\right]_0^{2\pi}
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{4\pi i}-1}{2}+i(2\pi-0) \\
&= 2\pi i
\end{aligned}$$

Hence, the result is $\boxed{\int_C \left(z+z^{-1}\right)dz = 2\pi i}$.

Chapter 14.1, Problem 27P

Step-by-step solution

Step 1 of 3

Integration:

Find the integration $\int_C \sec^2 z dz$.

Here C is the shortest path from $\frac{\pi}{4}$ to $\frac{\pi i}{4}$.

Step 2 of 3

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$ ...... **(1)**

Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ ...... **(2)**

Step 3 of 3

Here $f(z) = \sec^2 z$ is an analytic function in the given domain, so first method is applicable.

$$\begin{aligned}\int_C \sec^2 z dz &= \int_{\pi/4}^{\pi i/4} \sec^2 z dz \\ &= [\tan z]_{\pi/4}^{\pi i/4} \\ &= \tan\frac{\pi i}{4} - \tan\frac{\pi}{4} \\ &= i\tanh\frac{\pi}{4} - 1\end{aligned}$$

Hence, the result is $\boxed{i\tanh\frac{\pi}{4} - 1}$.

Chapter 14.1, Problem 28P

Step-by-step solution

Step 1 of 4

Integration:

Find the integration $\int_C \left(\frac{5}{z-2i}-\frac{6}{(z-2i)^2}\right)dz$.

Here C is the circle $|z-2i|=4$ clockwise.

Step 2 of 4

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1)-F(z_0)$ ...... **(1)**

Second method:

If $z=z(t), a\le t\le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ ...... **(2)**

Step 3 of 4

Here $f(z)=\frac{5}{z-2i}-\frac{6}{(z-2i)^2}$ is not analytic in the region $|z-2i|=4$, so second method is applicable.

Parametric form of the path is

$$z-2i = 4e^{-it}, 0\le t\le 2\pi$$
$$z = 2i+4e^{-it}$$
$$z'(t) = -4ie^{-it}$$

Step 4 of 4

Now apply the second method.

$$\begin{aligned}\int_C \left(\frac{5}{z-2i}-\frac{6}{(z-2i)^2}\right)dz &= \int_C \left(\frac{5}{4e^{-it}}-\frac{6}{16e^{-2it}}\right)\left(-4ie^{-it}\right)dt \\ &= \int_0^{2\pi}\left[-5i+\frac{6ie^{it}}{4}\right]dt \\ &= -5i[t]_0^{2\pi}+\frac{6}{4}\left[e^{it}\right]_0^{2\pi} \\ &= -10\pi i\end{aligned}$$

Hence, the result is $\boxed{-10\pi i}$.

Chapter 14.1, Problem 29P

Step-by-step solution

Step 1 of 5

Integration:

Find the integration $\int_C \operatorname{Im} z^2 dz$.

Here C is the triangle with vertices $0,1,i$ in counter clockwise direction.

Step 2 of 5

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$ ...... **(1)**

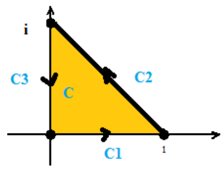
Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ ...... **(2)**

Step 3 of 5

Consider the diagram.



Step 4 of 5

Here $f(z) = \operatorname{Im} z^2$ is not analytic in the region, so second method is applicable.

Parametric forms of the paths are as shown below.

$$\begin{aligned}
&C1: z(t) = t + 0i, \quad 0 \le t \le 1 \\
&C2: z(t) = t + (1-t)i, 1 \le t \le 0 \\
&C3: z(t) = ti, \quad 1 \le t \le 0
\end{aligned}$$

Step 5 of 5

Now apply the second method.

$$\begin{aligned}
\int_C f(z)dz &= \int_{C1} \operatorname{Im} z^2 dz + \int_{C2} \operatorname{Im} z^2 dz + \int_{C3} \operatorname{Im} z^2 dz \\
&= \int_{C1} \operatorname{Im}(t+0i)^2 z'(t)dt + \int_{C2} \operatorname{Im}(t+(1-t)i)^2 z'(t)dt + \int_{C3} \operatorname{Im}(ti)^2 z'(t)dt \\
&= \int_0^1 0dt + \int_1^0 2t(1-t)(1-i)dt + \int_1^0 0 \cdot idt
\end{aligned}$$

$$= -(1-i)\int_0^1 2t(1-t)dt$$

$$= (i-1)\left[t^2 - \frac{2t^3}{3}\right]_0^1$$

$$= (i-1)\left(1 - \frac{2}{3}\right)$$

$$= \frac{i-1}{3}$$

Hence, the result is $\boxed{\frac{i-1}{3}}$.

Chapter 14.1, Problem 30P

Step-by-step solution

Step 1 of 5

Integration:

Find the integration $\int_C \operatorname{Re} z^2 dz$.

Here C is the square with vertices $0, i, 1+i, 1$ in clockwise direction.

Step 2 of 5

Essential understanding:

First method:

If $f(z)$ is analytic in its domain, then

$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$ **...... (1)**

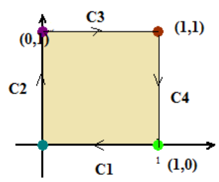
Second method:

If $z = z(t), a \le t \le b$ and C is a smooth path, then

$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$ **...... (2)**

Step 3 of 5

Consider the diagram.



Step 4 of 5

Here $f(z) = \operatorname{Re} z^2$ is not analytic in the region, so second method is applicable.

Parametric forms of the paths are as shown below.

$$
\begin{aligned}
C1 &: z(t) = t + 0i, \quad 1 \le t \le 0 \\
C2 &: z(t) = 0 + ti, \quad 0 \le t \le 1 \\
C3 &: z(t) = t + i, \quad 0 \le t \le 1 \\
C4 &: z(t) = 1 + ti, \quad 1 \le t \le 0
\end{aligned}
$$

Step 5 of 5

Now apply the second method.

$$
\begin{aligned}
\int_C f(z)dz &= \int_{C1} \operatorname{Re} z^2 dz + \int_{C2} \operatorname{Re} z^2 dz + \int_{C3} \operatorname{Re} z^2 dz + \int_{C4} \operatorname{Re} z^2 dz \\
&= \int_{C1} \operatorname{Re} t^2 z'(t)dt + \int_{C2} \operatorname{Re}(ti)^2 z'(t)dt + \int_{C3} \operatorname{Re}(t+i)^2 z'(t)dt + \int_{C4} \operatorname{Re}(1+ti)^2 z'(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_1^0 t^2 dt + \int_0^1 -t^2 \cdot idt + \int_0^1 (t^2 - 1)dt + \int_1^0 (1-t^2)idt \\
&= \left[\frac{t^3}{3}\right]_1^0 - \left[\frac{it^3}{3}\right]_0^1 + \left[\frac{t^3}{3} - t\right]_0^1 + i\left[t - \frac{t^3}{3}\right]_1^0 \\
&= -\frac{1}{3} - \frac{i}{3} + \frac{1}{3} - 1 - i + \frac{i}{3} \\
&= -(1+i)
\end{aligned}
$$

Hence, the result is $\boxed{-(1+i)}$.

Chapter 14.2, Problem 1P

Step-by-step solution

Step 1 of 1

303-14.2-13P

The given problem is that we have to verify Theorem 1 for the integral of the square with vertices $1+i$, $-1+i$, $-1-i$ and $1-i$.

Theorem1 is Cauchy integral theorem which states that if $f(z)$ is analytic in a simply connected domain then for every simple closed path C in D

$$\oint_C f(z)\,dz = 0$$

The graph of curve is given as:

Now integral of $f(z) = z^2$ over curve C is given as

$$\oint_C f(z)\,dz = \int_{1+i}^{-1+i} z^2\,dz + \int_{-1+i}^{-1-i} z^2\,dz + \int_{-1-i}^{1-i} z^2\,dz + \int_{1-i}^{1+i} z^2\,dz$$

$$= \frac{1}{3}\left(z^3\right)_{1+i}^{-1+i} + \frac{1}{3}\left(z^3\right)_{-1+i}^{-1-i} + \frac{1}{3}\left(z^3\right)_{-1-i}^{1-i} + \frac{1}{3}\left(z^3\right)_{1-i}^{1+i}$$

$$= \frac{1}{3}\left[(-1+i)^3 - (1+i)^3\right] + \frac{1}{3}\left[(-1-i)^3 - (-1+i)^3\right] + \frac{1}{3}\left[(1-i)^3 - (-1-i)^3\right] + \frac{1}{3}\left[(1+i)^3 - (1-i)^3\right]$$

$$= \frac{1}{3}\left[(-1+i)^3 - (1+i)^3 + (-1-i)^3 - (-1+i)^3 + (1-i)^3 - (-1-i)^3 + (1+i)^3 - (1-i)^3\right] = 0$$

$$\boxed{\Rightarrow \oint_C f(z)\,dz = 0}$$

Hence it is verified

Chapter 14.2, Problem 2P

Step-by-step solution

Step 1 of 2

(a) Consider the function $f(z)=\frac{1}{z}$. This function is not analytic at $z=0$. Therefore,

Thus by Cauchy's integral theorem

$$\oint_C \frac{dz}{z}=0$$

For all closed path C which does not contains the point $Z=0$

Step 2 of 2

(b) Consider the function $f(z)=\frac{\exp\left(\frac{1}{z^2}\right)}{z^2+16}$. This function is not analytic at $z=0$,

and at $Z=\pm 4i$. Therefore, by Cauchy integral theorem

$$\oint_C \frac{\exp\left(\frac{1}{z^2}\right)}{z^2+16}dz=0$$

for all closed path C which does not contains points $z=0,\pm 4i$

Chapter 14.2, Problem 3P

Step-by-step solution

Step 1 of 2

303-14.2-15P

Given integral is $\oint_c \frac{dz}{z^2}$,

Where c is given as the boundary of the square with vertices

$1+i,-1+i,-1-i,1-i\left(\text{counter clockwise}\right)$

To determine whether the integral is Zero.

Step 2 of 2

From the given path for integration it is obvious that the function

$f(z)=\frac{1}{z^2}$ Is not analytic inside c.

Therefore Cauchy integral theorem can not be applied for the function.

However by using the criteria of independence of path and also using the principle

Of deformation of path, a suitable path can be chosen .so that the point $z=0$ is excluded from with in its boundary, then

$\oint_c \frac{dz}{z^2}$ Will be equal to zero.

Therefore by using deformation of path principle, it can be concluded that

$$\oint_c \frac{dz}{z^2}=0$$

Chapter 14.2, Problem 4P

Step-by-step solution

Step 1 of 1

No, the function cannot be analytic everywhere in the annulus, $1<|z|<3$, Because if the function is analytic in the annulus $1<|z|<3$, then by principle of Deformation, the value of the integration over the unit circle and over the circle of radius 3 will be same.

Chapter 14.2, Problem 5P

Step-by-step solution

Step 1 of 3

The points for which the function $f(z)=\dfrac{\cos z^2}{z^4+1}$ is not analytic are given by the solution of $z^4+1=0$ .i.e.

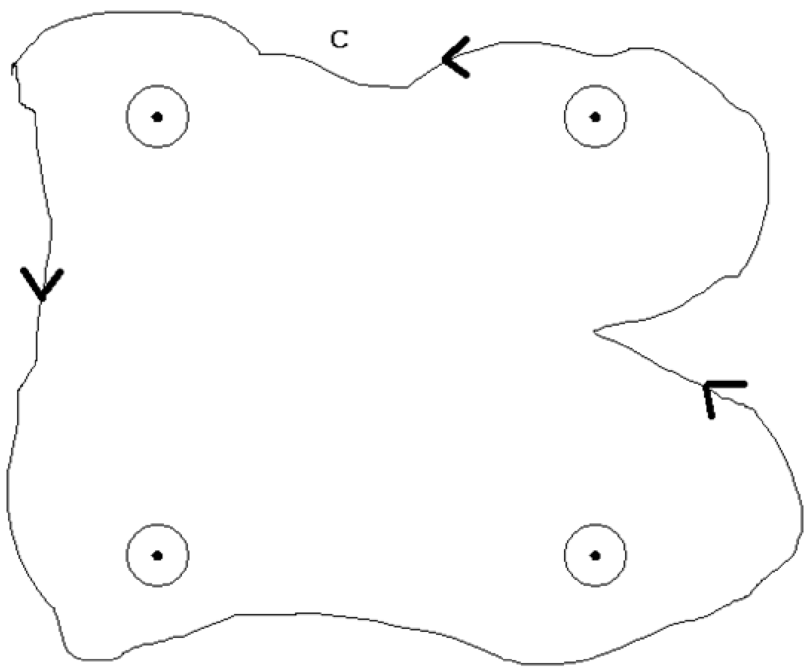
$z^4=-1=\cos(2n+1)\pi+i\sin(2n+1)\pi,\ n=0,1,2,3$

Now by De Moivre's law, we have

$z=\cos(2n+1)\dfrac{\pi}{4}+i\sin(2n+1)\dfrac{\pi}{4},\ n=0,1,2,3$

Step 2 of 3

All these four points are shown in figure.



Step 3 of 3

Therefore, the domain is 5-connected.

Chapter 14.2, Problem 6P

Step-by-step solution

Step 1 of 3

(a) Now

$$\int_0^{1+i} e^z dz = \left[e^z \right]_0^{1+i}$$
$$= e^{1+i} - e^0$$
$$= e^{1+i} - 1$$

Step 2 of 3

(b) Now we divide path of integration as follows:

(i) Over the $x-$ axis from origin to 1

In this case $z = x$ and this implies $dz = dx$ and hence

$$\int_0^1 e^x dx = \left[e^x \right]_0^1$$
$$= e^1 - e^0$$
$$= e - 1$$

(ii) Straight line from 1 to $1+i$

In this case $z = 1 + iy$ where $0 \le y \le 1$ and this implies $dz = idy$ and hence

$$\int_0^1 e^{1+iy} idy = ei\int_0^1 e^{iy} dy$$
$$= ei\left[\frac{e^{iy}}{i} \right]_0^1$$
$$= e\left[e^{iy} \right]_0^1$$
$$= e^{1+i} - e$$

Therefore, the value of the integration will be

$$\int_0^{1+i} e^z dz = (e-1) + \left(e^{1+i} - e \right)$$
$$= e^{1+i} - 1$$

Step 3 of 3

Thus in both the cases value of integration is same and hence it is path independent.

Chapter 14.2, Problem 7P

Step-by-step solution

Step 1 of 3

Consider the following function:

$$f(z)=\frac{1}{z^2+4}$$

Then $f(z)=\frac{1}{(z+2i)(z-2i)}$.

As the function $f(z)$ is not differentiable at $z=\pm 2i$, the function $f(z)$ is not analytic at $z=\pm 2i$.

So, the singular points of $f(z)$ are $z=\pm 2i$.

Step 2 of 3

(a)

Conclude that the integral of $\frac{1}{z^2+4}$ over $|z-2|=2$ is zero or not.

Put $z=\pm 2i$ in $|z-2|$.

$$\begin{aligned}|z-2|&=|\pm 2i-2|\\&=2|\pm i-1|\\&=2\sqrt{2}\\&>2\end{aligned}$$

So, these two points lie outside $|z-2|=2$.

Therefore, by Cauchy's integral theorem,

$$\oint_C f(z)\,dz=0$$

$$\oint_{|z-2|=2}\frac{1}{z^2+4}\,dz=0.$$

Hence, the answer is $\boxed{\text{Yes}}$.

Step 3 of 3

(b)

Conclude that the integral of $\frac{1}{z^2+4}$ over $|z-2|=3$ is zero or not.

Put $z=\pm 2i$ in $|z-2|$.

$$\begin{aligned}|z-2|&=|\pm 2i-2|\\&=2|\pm i-1|\\&=2\sqrt{2}\\&<2\end{aligned}$$

So, these two points lie outside $|z-2|=3$.

Then, these would have to move the contour across $\pm 2i$.

Therefore, by Cauchy's integral theorem,

$$\oint_C f(z)\,dz\neq 0$$

$$\oint_{|z-2|=3}\frac{1}{z^2+4}\,dz\neq 0.$$

Hence, the answer is $\boxed{\text{No}}$.

Chapter 14.2, Problem 8P

Step-by-step solution

There is no solution to this problem yet.

Get help from a Chegg subject expert.

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Chapter 14.2, Problem 9P

Step-by-step solution

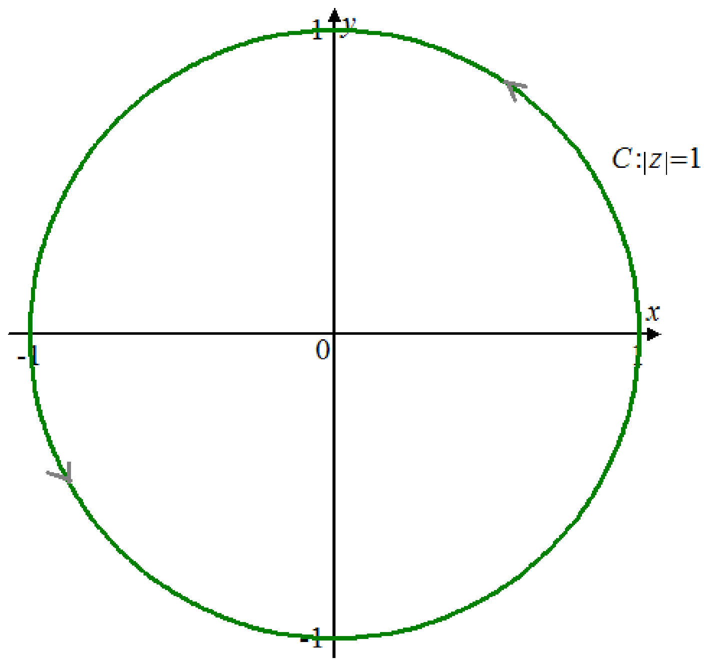
Step 1 of 2

Consider the following function:

$$f(z)=e^{-z^2}$$

Integrate this function around the unit circle, $C:|z|=1$, and verify whether the Cauchy's integral theorem is applicable or not.

The graph of this unit circle is as shown below:



Step 2 of 2

Cauchy's integral theorem:

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D.

$$\oint_C f(z)=0$$

The exponential function $f(z)=e^{-z^2}$ is an analytic everywhere on the plane.

So by Cauchy's integral theorem, for a simple closed curve $C:|z|=1$, $\oint_C e^{-z^2}dz=0.$

So, Cauchy's integral theorem can be applied to this function.

Chapter 14.2, Problem 10P

Step-by-step solution

Step 1 of 2

The objective is to verify Cauchy`s theorem applicable or not.

Consider the function $f(z) = \tan\frac{z}{4}$

$$= \frac{\sin\frac{z}{4}}{\cos\frac{z}{4}}$$

$f(z)$ is not analytic when $\cos\frac{z}{4} = 0$.

$$\frac{z}{4} = (2n+1)\frac{\pi}{2},\ n = 0,1,2,3........$$
$$\Rightarrow z = 2(2n+1)\pi$$

$f(z)$ is not analytic at the points, $z = 2(2n+1)\pi, \qquad n = 0,1,2,3......$

Step 2 of 2

Cauchy`s integral theorem requires only that $f(z)$ is analytic inside and on the closed curve, $C:|z| = 1$.

The point $z = 2(2n+1)\pi, \qquad n = 0,1,2,3......$lies outside the unit circle $|z| = 1$, for each n. So, $f(z)$ is analytic inside and on the closed curve $C:|z| = 1$.

Apply Cauchy integral theorem, $\boxed{\oint_{|z|=1} \tan\frac{z}{4}dz = 0}$

Chapter 14.2, Problem 11P

Step-by-step solution

Step 1 of 2

The function $\begin{aligned} f(z) &= \frac{1}{2z-1} \\ &= \frac{1/2}{z-\frac{1}{2}} \end{aligned}$,

is not analytic at the point for which $z-\frac{1}{2}=0$.

That means $z=\frac{1}{2}$, which lies inside the unit circle $|z|=1$.

Here we cannot apply Cauchy integral theorem.

To find $\oint\limits_{|z|=1} \frac{1}{2z-1}dz$.

Step 2 of 2

Let $z(t)=\frac{1}{2}+e^{it}$ and this implies

$dz = ie^{it}dt$

and we have

$$\begin{aligned} \oint\limits_{|z|=1} \frac{1}{2z-1}dz &= \int_0^{2\pi} \frac{(1/2)}{e^{it}} ie^{it}dt \\ &= \frac{i}{2}(t)_0^{2\pi} \\ &= \pi i \end{aligned}$$

Hence, $\boxed{\oint\limits_{|z|=1} \frac{1}{2z-1}dz = \pi i}$.

Chapter 14.2, Problem 12P

Step-by-step solution

Step 1 of 3

Consider the function $f(z)=(\bar{z})^3$.

The objective is to integrate $f(z)$ counterclockwise around the unit circle. And, it is required to indicate that whether Cauchy's Integral Theorem applies to $f(z)$.

Step 2 of 3

Let $z=re^{it}$, where $0\le t\le 2\pi$

For unit circle, radius $r=1$, then $z=e^{it}$, where $0\le t\le 2\pi$

Therefore,

$$\begin{aligned}\bar{z}&=\overline{e^{it}}\\&=e^{\bar{i}t}\\&=e^{-it}\end{aligned}$$

And, $\bar{z}=e^{-it}$ is not analytic in the connected domain D where the simple closed path, unit circle C in D encloses only points of D.

So, Cauchy's Integral Theorem cannot apply to $f(z)=(\bar{z})^3$.

Step 3 of 3

Since $\bar{z}=e^{-it}$, $(\bar{z})^3=e^{-3it}$

And, $z=e^{it}\Rightarrow dz=ie^{it}dt$

Therefore, the integral of $f(z)$ is

$$\begin{aligned}\int_C f(z)dz&=\int_C(\bar{z})^3\,dz\\&=\int_{t=0}^{2\pi}e^{-3it}\left(ie^{it}dt\right)\\&=i\int_{t=0}^{2\pi}e^{-2it}dt\end{aligned}$$

$$\begin{aligned}&=i\left(\frac{e^{-2it}}{-2i}\right)_{t=0}^{2\pi}\\&=-\frac{1}{2}\left(e^{-2it}\right)_{t=0}^{2\pi}\\&=-\frac{1}{2}\left(e^{-4\pi i}-1\right)\\&=-\frac{1}{2}\left((\cos 4\pi-i\sin 4\pi)-1\right)\\&=-\frac{1}{2}\left((1-i(0))-1\right)\\&=0\end{aligned}$$

Therefore, $\int_C(\bar{z})^3dz=\boxed{0}$

Chapter 14.2, Problem 13P

Step-by-step solution

Step 1 of 2

Cauchy's Theorem:

Find the integral $\int_C f(z)\,dz$ around the unit circle counterclockwise.

Here $f(z) = \frac{1}{z^4 - 1.1}$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)\,dz = 0$.

Step 2 of 2

Integrand $f(z) = \frac{1}{z^4 - 1.1}$ is analytic in the region $|z| = 1$.

Thus the Cauchy's integral theorem is applicable.

Hence, from the Cauchy's theorem the value of the integral is 0.

Value of the integral is $\boxed{0}$.

Answer is $\boxed{\text{Yes}}$.

Chapter 14.2, Problem 14P

Step-by-step solution

Step 1 of 4

Cauchy's Theorem:

Find the integral $\int_C f(z)dz$ around the unit circle counterclockwise.

Here $f(z) = \frac{1}{z}$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)dz = 0$.

Step 2 of 4

Integrand $\frac{1}{z}$ is not analytic at $z = 0$

Integrand is not analytic in the region $|z| = 1$.

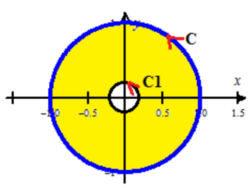
It is because $|0| < 1$.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 3 of 4

Consider the below diagram.



Step 4 of 4

Integrand is not analytic at $z = 0$.

Now take the circle with center 0 and radius ε.

From the Cauchy's theorem for multiply connected domains,

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

Now consider the integral,

$$\int_C \frac{1}{z}dz = \int_{C_1} \frac{1}{z}dz$$

$$= \int_0^{2\pi} \frac{i\varepsilon e^{it}dt}{\varepsilon e^{-it}} \quad \left(\text{since } |z| = \varepsilon \to z = \varepsilon e^{it}\right)$$

$$= \int_0^{2\pi} ie^{2it}dt$$

$$= \left[\frac{e^{2it}}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[e^{4\pi i} - e^0\right]$$

$$= 0$$

Value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 15P

Step-by-step solution

Step 1 of 3

Cauchy's Theorem:

Find the integral $\int_C f(z)\,dz$ around the unit circle counterclockwise.

Here $f(z) = \text{Im}\, z$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)\,dz = 0$.

Step 2 of 3

Integrand $\text{Im}\, z$ is not analytic at $z \neq 0$

Integrand is not analytic in the region $|z| = 1$.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 3 of 3

Now consider the integral,

$$\int_C f(z)\,dz = \int_C \text{Im}\, z\,dz$$
$$= \int_0^{2\pi} (\sin t)(-\sin t + i\cos t)\,dt$$
$$\begin{pmatrix} \text{since } |z| = 1 \to z = \cos t + i\sin t \\ z'(t) = -\sin t + i\cos t \end{pmatrix}$$
$$= \int_0^{2\pi} \left(-\sin^2 t + i\sin t\cos t\right) dt$$
$$= \int_0^{2\pi} \left(-\sin^2 t\right) dt + \int_0^{2\pi} i\sin t\cos t\,dt$$
$$= -\left[2 \times 2 \times \frac{1}{2} \times \frac{\pi}{2}\right] + \frac{i}{2}\int_0^{2\pi} \sin 2t\,dt$$
$$= -\pi + \frac{i}{2}(0)$$
$$= -\pi$$

Value of the integral is $\boxed{-\pi}$.

Chapter 14.2, Problem 16P

Step-by-step solution

Step 1 of 4

Cauchy's Theorem:

Find the integral $\int_C f(z)\,dz$ around the unit circle counterclockwise.

Here $f(z)=\frac{1}{\pi z-1}$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)\,dz = 0$.

Step 2 of 4

Integrand $\frac{1}{\pi z-1}$ is not analytic at $z=\frac{1}{\pi}$

Integrand is not analytic in the region $|z|=1$.

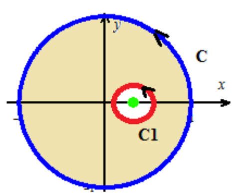
It is because $\left|\frac{1}{\pi}\right|<1$.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 3 of 4

Consider the below diagram.



Step 4 of 4

Integrand is not analytic at $z=\frac{1}{\pi}$.

Now take the circle with center $\frac{1}{\pi}$ and radius ε.

From the Cauchy's theorem for multiply connected domains,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

Now consider the integral,

$$\begin{aligned}\int_C \frac{1}{\pi z-1}\,dz &= \frac{1}{\pi}\int_{C_1}\frac{dz}{z-1/\pi}\\ &= \frac{1}{\pi}\int_0^{2\pi}\frac{i\varepsilon e^{it}\,dt}{\varepsilon e^{it}} \quad \left(\text{since } |z-1/\pi|=\varepsilon\right)\\ &= \frac{1}{\pi}(2\pi i)\\ &= 2i\end{aligned}$$

Value of the integral is $\boxed{2i}$.

Chapter 14.2, Problem 17P

Step-by-step solution

Step 1 of 4

Cauchy's Theorem:

Find the integral $\int_C f(z)\,dz$ around the unit circle counterclockwise.

Here $f(z)=\frac{1}{|z|^2}$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)\,dz = 0.$

Step 2 of 4

Integrand $f(z)=\frac{1}{|z|^2}$ is not analytic at $z=0.$

Integrand is not analytic in the region $|z|=1.$

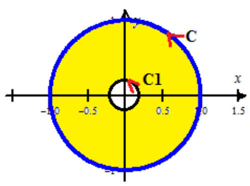
It is because $|0|<1.$

Thus the Cauchy's integral theorem is not applicable.

Hence, the answer is $\boxed{\text{No}}$.

Step 3 of 4

Consider the below diagram.



Step 4 of 4

Integrand is not analytic at $z=0.$

Now take the circle with center 0 and radius $\varepsilon.$

From the Cauchy's theorem for multiply connected domains,

$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz.$

Now consider the integral,

$$\int_C \frac{1}{|z|^2}\,dz = \int_{C_1} \frac{1}{|z|^2}\,dz$$

$$= \int_0^{2\pi} \frac{i\varepsilon e^{it}\,dt}{\varepsilon^2} \qquad \text{as } |z|=\varepsilon \to z=\varepsilon e^{it} \Rightarrow dz = i\varepsilon e^{it}\,dt.$$

$$= \frac{i}{\varepsilon}\left[e^{it}\right]_0^{2\pi}$$

$$= \frac{i}{\varepsilon}\left(e^{2\pi i}-e^{0}\right)$$

$$= \frac{i}{\varepsilon}(1-1) \qquad \text{as } e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1.$$

$$= 0$$

Therefore, the value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 18P

Step-by-step solution

Step 1 of 4

Cauchy's Theorem:

Find the integral $\int_C f(z)\,dz$ around the unit circle counterclockwise.

Here $f(z)=\frac{1}{4z-3}$.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then every simple closed path C in D, then

$\oint_C f(z)\,dz=0$.

Step 2 of 4

Integrand $f(z)=\frac{1}{4z-3}$ is not analytic at $z=\frac{3}{4}$

Integrand is not analytic in the region $|z|=1$.

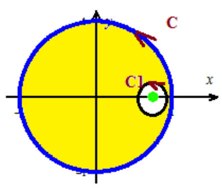
It is because $\left|\frac{3}{4}\right|<1$.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 3 of 4

Consider the below diagram.



Step 4 of 4

Integrand is not analytic at $z=\frac{3}{4}$.

Now take the circle with center $\frac{3}{4}$ and radius ε.

From the Cauchy's theorem for multiply connected domains,

$\int_C f(z)\,dz=\int_{C_1} f(z)\,dz$

Now consider the integral,

$$\begin{aligned}\int_C \frac{1}{4z-3}\,dz &= \frac{1}{4}\int_{C_1}\frac{dz}{z-3/4}\\ &= \frac{1}{4}\int_0^{2\pi}\frac{i\varepsilon e^{it}\,dt}{\varepsilon e^{it}} \quad (\text{since } |z-3/4|=\varepsilon)\\ &= \frac{1}{4}(2\pi i)\\ &= \frac{\pi i}{2}\end{aligned}$$

Value of the integral is $\boxed{\frac{\pi i}{2}}$.

Chapter 14.2, Problem 19P

Step-by-step solution

Step 1 of 3

Consider the function,

$f(z) = z^3 \cot z.$

The objective is to integrate the function *f* counterclockwise around the unit circle and to determine whether Cauchy's integral theorem is applicable to the function or not.

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then $\oint_C f(z)\,dz = 0$ for every simple closed path C in D.

A function $f(z)$ is said to be analytic at a point $z = z_0$ in the domain *D* if $f(z)$ is defined and differentiable in the neighborhood of z_0.

Step 2 of 3

Rewrite the function as,

$$\begin{aligned} f(z) &= z^3 \cot z \\ &= z^3\left(\frac{\cos z}{\sin z}\right) \qquad \text{Use } \cot x = \frac{\cos x}{\sin x}. \end{aligned}$$

Since $\sin z$ is 0 at $z = n\pi, \text{ for } n = 0,1,2,\ldots$, so the function $f(z) = z^3\left(\frac{\cos z}{\sin z}\right)$ is undefined at $z = n\pi, \text{ for } n = 0,1,2,\ldots$.

Therefore, the function $f(z) = z^3\left(\frac{\cos z}{\sin z}\right)$ is not analytic at $z = n\pi, \text{ for } n = 0,1,2,\ldots$.

Step 3 of 3

Here, all the points $z = n\pi, \text{ for } n = 0,1,2,\ldots$ lies outside the unit circle $|z| = 1$ except the point $z = 0.$

That is, the point $z = 0$ lies inside the unit circle $|z| = 1.$

And the function $f(z) = z^3\left(\frac{\cos z}{\sin z}\right)$ is differentiable in the neighbourhood of $z = 0$ in the unit circle $|z| = 1.$

Thus, the function $f(z) = z^3\left(\frac{\cos z}{\sin z}\right)$ is analytic in the unit circle $|z| = 1.$

Therefore, by Cauchy's integral theorem,

$$\begin{aligned} \oint_{|z|=1} f(z)\,dz &= \oint_{|z|=1} z^3\left(\frac{\cos z}{\sin z}\right)dz \\ &= 0 \end{aligned}$$

Therefore, the integral value is $\boxed{\oint_{|z|=1} z^3\left(\frac{\cos z}{\sin z}\right)dz = 0}$.

Hence, the Cauchy's integral theorem $\boxed{\text{is applicable}}$ for this function.

Chapter 14.2, Problem 20P

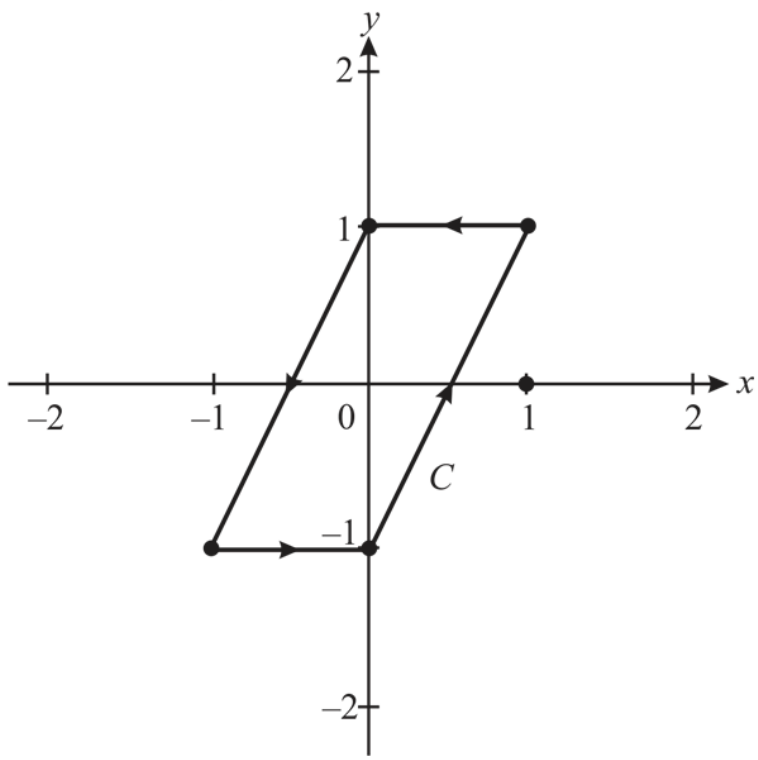
Step-by-step solution

Step 1 of 2

Consider the function:

$f(z) = \mathrm{Ln}(1-z)$.

Sketch the contour parallelogram joining the vertices $\pm i, \pm(1+i)$, that is $(0,1),(0,-1),(1,1),(-1,-1)$.



From the figure, observe that no singularities of $f(z)$ lie inside the contour.

Step 2 of 2

Use Cauchy's Integral Theorem to evaluate the integral $\oint_C \mathrm{Ln}(1-z)\,dz$.

Since $\mathrm{Ln}(z)$ is analytic in $\mathbb{C}\setminus(-\infty,0]$, $f(z)$ is analytic in $\mathbb{C}\setminus(-\infty,1]$.

Clearly, $(-\infty,1]\cap|z|\le 0=\varnothing$.

Thus, $f(z)$ is analytic when $|z|\le 0$.

The function has the singularity at $z=1$, which lies outside the contour.

$\oint_C \mathrm{Ln}(1-z)\,dz = 0$

Hence, the value of the integral $\oint_C \mathrm{Ln}(1-z)\,dz$ is $\boxed{0}$

Chapter 14.2, Problem 21P

Step-by-step solution

Step 1 of 5

Contour integrals:

Evaluate the integral,

$$\int_C \frac{dz}{z-3i}$$

Here C is the boundary by the region $|z| = \pi$.

Step 2 of 5

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0.$$

Step 3 of 5

Here $f(z) = \frac{1}{z-3i}$ is not an analytic function.

Integrand is not analytic in the region $|z| = \pi$.

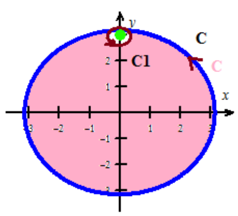
It is because $|3i| < \pi$.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 4 of 5

Consider the below diagram.



Step 5 of 5

Integrand is not analytic at $z = 3i$.

Now take the circle with center $3i$ and radius ε.

From the Cauchy's theorem for multiply connected domains,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

Now consider the integral,

$$\begin{aligned}\int_C \frac{1}{z-3i}\,dz &= \int_{C_1} \frac{dz}{z-3i} \\ &= \int_0^{2\pi} \frac{i\varepsilon e^{it}\,dt}{\varepsilon e^{it}} \quad (\text{since } |z-3i| = \varepsilon) \\ &= i[t]_0^{2\pi} \\ &= 2\pi i\end{aligned}$$

Value of the integral is $\boxed{2\pi i}$.

Chapter 14.2, Problem 22P

Step-by-step solution

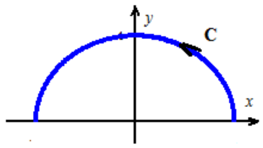
Step 1 of 4

Contour integrals:

Evaluate the integral,

$$\int_C \operatorname{Re} z dz$$

Here C is the boundary as shown below.



Step 2 of 4

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)dz = 0.$$

Step 3 of 4

Integrand $\operatorname{Re} z$ is not analytic in the region.

Thus the Cauchy's integral theorem is not applicable.

Answer is $\boxed{\text{No}}$.

Step 4 of 4

Now consider the integral,

$$\begin{aligned}\int_C f(z)dz &= \int_C \operatorname{Re} z dz \\ &= \int_0^\pi \cos t(-\sin t + i\cos t)dt \\ &\quad \left(\begin{array}{l}\text{since } |z| = 1 \to z = \cos t + i\sin t \\ z'(t) = -\sin t + i\cos t\end{array}\right) \\ &= \int_0^\pi (i\cos^2 t - \sin t\cos t)dt\end{aligned}$$

$$= i\int_0^\pi \cos^2 t dt - \frac{1}{2}\int_0^\pi \sin 2t dt$$

$$= i\int_0^\pi \frac{1+\cos 2t}{2}dt - \frac{1}{2}\int_0^\pi \sin 2t dt$$

$$= \frac{i}{2}\left[t + \frac{\sin 2t}{2}\right]_0^\pi + \frac{1}{4}[\cos 2t]_0^\pi$$

$$= \frac{\pi i}{2}$$

Value of the integral is $\boxed{\frac{\pi i}{2}}$.

Chapter 14.2, Problem 23P

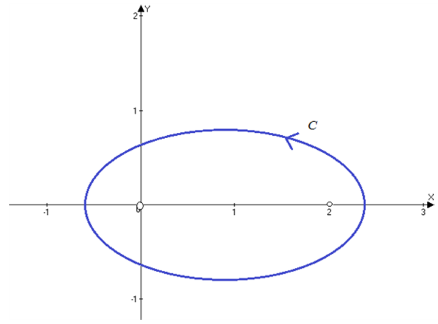
Step-by-step solution

Step 1 of 4

The objective is to evaluate the following integral by using partial fractions:

$$\int_C \frac{2z-1}{z^2-z}\,dz$$

Here C is the boundary as shown below.



Determine whether Cauchy's theorem is applicable to this integral or not.

Step 2 of 4

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0.$$

From the deformation of the path,

$$\int (z-z_0)^m\,dz = \begin{cases} 2\pi i, & m=-1 \\ 0 & m \neq -1 \text{ and integer} \end{cases} \quad \text{...... (1)}$$

Step 3 of 4

Consider the integral,

$$\begin{aligned}\int_C f(z)\,dz &= \int_C \frac{2z-1}{z^2-z}\,dz \\ &= \int_C \frac{2z-1}{z(z-1)}\,dz\end{aligned}$$

Simplify the integrand by using partial fractions as follows:

Consider,

$$\frac{2z-1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} \quad \text{...... (2)}$$

$$\frac{2z-1}{z(z-1)} = \frac{A(z-1)+Bz}{z(z-1)}$$

$$2z-1 = A(z-1)+Bz$$

Put, $z=0$ then $0-1 = A(0-1)+0 \Rightarrow A=1.$

Put, $z=1$ then $2-1 = A(0)+B \Rightarrow B=1.$

Substitute *A* and *B* values in equation (2).

$$\frac{2z-1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

Step 4 of 4

So, the integral becomes,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C \frac{2z-1}{z(z-1)}\,dz \\ &= \oint_C \left[\frac{1}{z} + \frac{1}{z-1}\right] dz \\ &= \int_C \frac{1}{z}\,dz + \int_C \frac{1}{z-1}\,dz \\ &= \int_C \frac{1}{z-0}\,dz + \int_C \frac{1}{z-1}\,dz\end{aligned}$$

$$\oint_C f(z)\,dz = \int_C (z-0)^{-1}\,dz + \int_C (z-1)^{-1}\,dz$$

From (1),

$$\int_C (z-0)^{-1}\,dz = 2\pi i,$$

$$\int_C (z-1)^{-1}\,dz = 2\pi i.$$

Therefore,

$$\begin{aligned}\oint_C f(z)\,dz &= \int_C (z-0)^{-1}\,dz + \int_C (z-1)^{-1}\,dz \\ &= 2\pi i + 2\pi i \\ &= 4\pi i\end{aligned}$$

Hence, the value of the given integral is, $\boxed{4\pi i}$.

Chapter 14.2, Problem 24P

Step-by-step solution

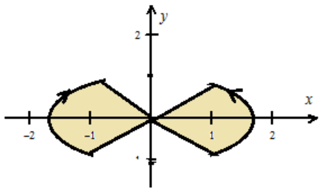
Step 1 of 4

Contour integrals:

The objective is to evaluate the integral,

$$\int_C \frac{dz}{z^2-1}$$

Here C is the boundary as shown below.



Step 2 of 4

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0.$$

From the deformation of the path,

$$\int (z-z_0)^m\,dz = \begin{cases} 2\pi i, & m=-1 \\ 0 & m \neq -1 \text{ and integer} \end{cases} \quad \text{...... (1)}$$

Step 3 of 4

Now consider the integral,

$$\begin{aligned}\int_C \frac{dz}{z^2-1} &= \int_C \frac{dz}{(z+1)(z-1)} \\ &= \frac{1}{2}\int_C \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz \\ &= \frac{1}{2}\int_C \left[\frac{1}{z-1}\right] dz - \frac{1}{2}\int_C \left[\frac{1}{z+1}\right]\end{aligned}$$

Contour in which singularity $z=-1$ lies is in clockwise direction hence, $\int_C dz = -\int_C dz$.

$$\begin{aligned}&= \frac{1}{2}\int_C \left[\frac{1}{z-1}\right] dz - \left(-\frac{1}{2}\int_C \left[\frac{1}{z+1}\right]\right) \\ &= \frac{1}{2}\int_C \left[\frac{1}{z-1}\right] dz + \frac{1}{2}\int_C \left[\frac{1}{z+1}\right] \\ &= \frac{1}{2}[2\pi i + 2\pi i] \quad (\text{from } (1)) \\ &= 4\pi i\end{aligned}$$

Value of the integral is $\boxed{4\pi i}$.

Step 4 of 4

Cauchy's theorem is applicable for this problem.

From the diagram it is visible that the given region is multiple connected.

Chapter 14.2, Problem 25P

Step-by-step solution

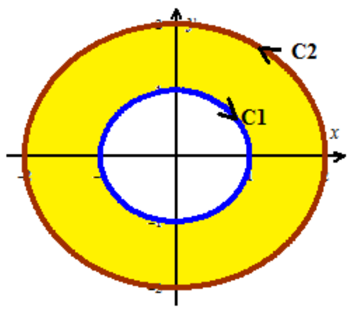
Step 1 of 3

Contour integrals:

Evaluate the integral,

$\int_C \frac{e^z}{z} dz$.

Here C is the boundary as shown below.



Step 2 of 3

Cauchy's Theorem for multi connected regions if $f(z)$ is analytic in domain D that contains the boundary curves of C_1, C_2 then,

$\oint_{C1} f(z) dz = \oint_{C2} f(z) dz$.

From the deformation of the path,

$$\int (z - z_0)^m dz = \begin{cases} 2\pi i, & m = -1 \\ 0 & m \neq -1 \text{ and integer} \end{cases} \quad \text{...... (1)}$$

Step 3 of 3

Now consider the integral $\int_C \frac{e^z}{z} dz$,

$$\int_C \frac{e^z}{z} dz = \int_{C1} \frac{e^z}{z} dz = -\int_{C2} \frac{e^z}{z} dz \quad \left(\begin{array}{l}\text{because orientations are in} \\ \text{opposite directions}\end{array}\right)$$

$$2\int \frac{e^z}{z} dz = 0$$

$$\int \frac{e^z}{z} dz = 0$$

Value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 26P

Step-by-step solution

Step 1 of 2

Given that

$\oint_C \coth\frac{1}{2}z dz, C$ is the circle $\left|z-\frac{1}{2}\pi i=1\right|$

Now since the function $f(z)=\coth\frac{z}{2}$ is analytic at each point inside the circle

$\left|z-\frac{1}{2}\pi i\right|=1$, we can apply Cauchy's Integral theorem

Step 2 of 2

Thus by Cauchy's Integral theorem we get

$$\boxed{\oint_C \coth\frac{1}{2}z\, dz = 0}$$

Chapter 14.2, Problem 27P

Step-by-step solution

Step 1 of 4

Contour integrals:

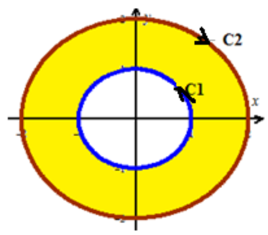
Evaluate the integral,

$$\int_C \frac{\cos z}{z}\,dz.$$

Here C consists of $|z|=1$ counterclockwise and $|z|=3$ clockwise.

Step 2 of 4

Here C is the region as shown below.



Step 3 of 4

Cauchy's Theorem for multi connected regions if $f(z)$ is analytic in domain D that contains the boundary curves of C_1, C_2 then,

$$\oint_{C1} f(z)\,dz = \oint_{C2} f(z)\,dz.$$

From the deformation of the path,

$$\int (z-z_0)^m\,dz = \begin{cases} 2\pi i, & m=-1 \\ 0 & m \neq -1 \text{ and integer} \end{cases} \quad \text{...... (1)}$$

Step 4 of 4

Now consider the integral $\int_C \frac{\cos z}{z}\,dz$,

$$\int_C \frac{\cos z}{z}\,dz = \int_{C1} \frac{\cos z}{z}\,dz = \int_{C2} \frac{\cos z}{z}\,dz \quad \left(\begin{array}{l}\text{because orientations are in}\\ \text{opposite directions}\end{array}\right)$$

$$2\int_C \frac{\cos z}{z}\,dz = 0$$

$$\int_C \frac{\cos z}{z}\,dz = 0$$

Value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 28P

Step-by-step solution

Step 1 of 3

Contour integrals:

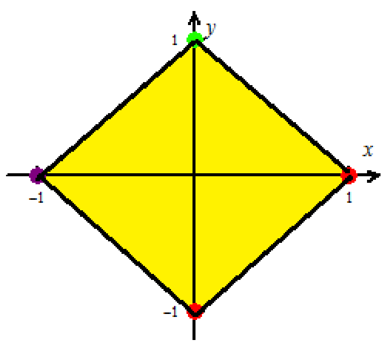
Evaluate the integral,

$$\int_C \frac{\tan\frac{1}{2}z}{z^4-16}\,dz.$$

Here C is the boundary of the square with vertices $\pm 1, \pm i$.

Step 2 of 3

Consider the below region.



Step 3 of 3

Integrand $\frac{\tan\frac{1}{2}z}{z^4-16}$ is not analytic at $z=\pm 2i, \pm 2, (2n+1)\pi, n=0,1,2,\ldots$.

Integrand is analytic in the above region.

Thus the Cauchy's integral theorem is applicable.

From the Cauchy's integral theorem,

$$\int_C \frac{\tan\frac{1}{2}z}{z^4-16}\,dz = 0$$

Hence, the value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 29P

Step-by-step solution

Step 1 of 4

Contour integrals:

Evaluate the integral,

$$\int_C \frac{\sin z}{z+2iz}\,dz$$

Here $C:|z-4-2i|=5.5$ clockwise.

Step 2 of 4

Cauchy's Theorem states that if $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz=0.$$

From the deformation of the path,

$$\int (z-z_0)^m\,dz=\begin{cases}2\pi i, & m=-1\\ 0 & m\neq -1 \text{ and integer}\end{cases} \quad \text{...... } \mathbf{(1)}$$

Step 3 of 4

Integrand $\frac{\sin z}{z+2iz}$ is not analytic at $z=0$

Integrand is not analytic in the region $C:|z-4-2i|=5.5$.

$$\begin{aligned}|0-4-2i|&=\sqrt{4^2+2^2}\\ &=\sqrt{20}\\ &<5.5\end{aligned}$$

Thus the Cauchy's integral theorem is not applicable.

Step 4 of 4

Consider the integral,

$$\begin{aligned}\int_C \frac{\sin z}{z+2iz}\,dz&=\frac{1}{1+2i}\int_C \frac{\sin z}{z}\,dz\\ &=\frac{1}{1+2i}[2\pi i\times \sin 0]\quad \left(\begin{array}{l}\text{from cauchy's integral}\\ \text{theorem}\end{array}\right)\\ &=0\end{aligned}$$

Hence, the value of the integral is $\boxed{0}$.

Chapter 14.2, Problem 30P

Step-by-step solution

Step 1 of 4

Cauchy's Theorem: If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D, $\oint_C f(z)\,dz = 0$.

Step 2 of 4

Consider the integral, $\oint_C f(z)\,dz = \oint_C \frac{2z^3+z^2+4}{z^4+4z^2}dz$ where circle $C:|z-2|=4$ in clockwise direction.

The objective is to Evaluate the integral and check Cauchy's theorem can be applied or not.
Consider the integral,

$$\int_C \frac{2z^3+z^2+4}{z^4+4z^2}dz = \int_C \frac{2z^3+z^2+4}{z^2\left(z^2+4\right)}dz$$

By the partial fraction,

$$\begin{aligned}\int_C \frac{2z^3+z^2+4}{z^2\left(z^2+4\right)}dz &= \int_C \frac{1}{4}\left[\frac{1}{z^2}-\frac{1}{z^2+4}\right]\left(2z^3+z^2+4\right)dz \\ &= \frac{1}{4}\int_C \frac{\left(2z^3+z^2+4\right)}{z^2}dz - \frac{1}{4}\int_C \frac{\left(2z^3+z^2+4\right)}{z^2+4}dz\end{aligned}$$

Step 3 of 4

Consider the integral $\int_C \frac{\left(2z^3+z^2+4\right)}{z^2+4}dz$.

Note that Cauchy's theorem is applicable on $\int_C \frac{\left(2z^3+z^2+4\right)}{z^2+4}dz$, since $z=\pm 2i$ lies outside the region $C:|z-2|=4$.

Therefore,

$$\int_C \frac{\left(2z^3+z^2+4\right)}{z^2+4}dz = 0$$

Next consider the integral,

$$\int_C \frac{\left(2z^3+z^2+4\right)}{z^2}dz = \int_C \left(2z+1+\frac{4}{z^2}\right)dz$$

Step 4 of 4

Assumed that, $z=e^{i\theta}$ then $dz = e^{i\theta}\cdot d\theta$ so that,

$$\begin{aligned}\int_C \left(2z+1+\frac{4}{z^2}\right)dz &= \int_0^{2\pi}\left(2e^{i\theta}+1+\frac{4}{e^{2i\theta}}\right)e^{i\theta}d\theta \\ &= \int_0^{2\pi}\left(2e^{2i\theta}+e^{i\theta}+4e^{-i\theta}\right)d\theta \\ &= 2\int_0^{2\pi}e^{2i\theta}d\theta + \int_0^{2\pi}e^{i\theta}d\theta + 4\int_0^{2\pi}e^{-i\theta}d\theta \\ &= 2\left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi} + \left[\frac{e^{i\theta}}{i}\right]_0^{2\pi} + 4\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}\end{aligned}$$

Put the upper and lower limit to get,

$$\begin{aligned}\int_C \left(2z+1+\frac{4}{z^2}\right)dz &= 2\left[\frac{e^{2i(2\pi)}}{2i}-\frac{e^{2i(0)}}{2i}\right]+\left[\frac{e^{i(2\pi)}}{i}-\frac{e^{i(0)}}{i}\right]+4\left[\frac{e^{-i(2\pi)}}{-i}-\left(\frac{e^{-i(0)}}{-i}\right)\right] \\ &= 2\left[\frac{e^{2i(2\pi)}}{2i}-\frac{1}{2i}\right]+\left[\frac{e^{i(2\pi)}}{i}-\frac{1}{i}\right]+4\left[\frac{e^{-i(2\pi)}}{-i}-\left(\frac{1}{-i}\right)\right] \\ &= 2\left[\frac{1}{2i}-\frac{1}{2i}\right]+\left[\frac{1}{i}-\frac{1}{i}\right]+4\left[-\frac{1}{i}+\frac{1}{i}\right] \\ &= 0\end{aligned}$$

Hence, $\boxed{\oint_C \frac{2z^3+z^2+4}{z^4+4z^2}dz = 0}$.

Chapter 14.3, Problem 1P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{z^2}{z^2-1}$ around the circle $|z+1|=1$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \quad \text{...... (1)}$$

Step 2 of 2

Here integrand $\frac{z^2}{z^2-1}$ is not analytic at $z=\pm 1$.

Region $|z+1|=1$ contains $z=-1$.

$$\oint_C \frac{z^2}{z^2-1}dz = \oint_C \frac{z^2/(z-1)}{(z+1)}dz$$

It is in the form of **(1)**, here $f(z)=\frac{z^2}{z-1}, z_0=-1$.

$$\begin{aligned} f(-1) &= \frac{(-1)^2}{-1-1} \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{z^2/(z-1)}{(z+1)}dz &= 2\pi i \times \left(-\frac{1}{2}\right) \quad (\text{from (1)}) \\ &= -\pi i \end{aligned}$$

Hence, the result is $\boxed{-\pi i}$.

Comments (2)

Anonymous

z=-1 is a typo right? Should be z=1 because the circle never crosses over the imaginary axis.

Anonymous

nevermind, I'm an idiot.

Chapter 14.3, Problem 2P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{z^2}{z^2-1}$ around the circle $|z-1-i|=\frac{\pi}{2}$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \mathbf{(1)}$$

Step 2 of 2

Here integrand $\frac{z^2}{z^2-1}$ is not analytic at $z=\pm 1$.

Region $|z-1-i|=\frac{\pi}{2}$ contains $z=1$.

$$\oint_C \frac{z^2}{z^2-1}dz = \oint_C \frac{z^2/(z+1)}{(z-1)}dz$$

It is in the form of **(1)**, here $f(z)=\frac{z^2}{z+1}, z_0=1$.

$$\begin{aligned} f(1) &= \frac{1^2}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{z^2/(z+1)}{(z-1)}dz &= 2\pi i \times \left(\frac{1}{2}\right) \quad (\text{from } (1)) \\ &= \pi i \end{aligned}$$

Hence, the result is $\boxed{\pi i}$.

Chapter 14.3, Problem 3P

Step-by-step solution

Step 1 of 4

The objective is to integrate the provided expression by use of the Cauchy's formula.

Step 2 of 4

Consider the provided statement to integrate $f(z)=\frac{z^2}{z^2-1}$ by Cauchy's formula counterclockwise around the circle $|z+i|=1.4$.

Step 3 of 4

As it is known that Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0}=2\pi i\times f(z_0) \text{ ...... (1)}$$

As the function $\frac{z^2}{z^2-1}$ is not analytic at $z=\pm 1$ therefore it can be rewrite as below;

$$\frac{z^2}{z^2-1}=\frac{1}{2}\left[\frac{z}{z-1}+\frac{z}{z+1}\right]$$

As it is provided that $|z+i|=1.4$ therefore $z=1,-1$ lies within the circle.

The pole $z=1$ is outside the circles.

$$\begin{aligned}|z+1|&=1\\|1+1|&=1\\2&>1\end{aligned}$$

The pole $z=-1$ is inside the circles.

$$\begin{aligned}|z+1|&=1\\|-1+1|&=1\\0&<1\end{aligned}$$

Now, the residue at $z=-1$.

$$\begin{aligned}r_1&=\lim_{z\to-1}(z+1)\left(\frac{z^2}{z^2-1}\right)\\&=\lim_{z\to-1}(z+1)\left(\frac{z^2}{(z-1)(z+1)}\right)\\&=\lim_{z\to-1}\left(\frac{z^2}{(z-1)}\right)\\&=-\frac{1}{2}\end{aligned}$$

Step 4 of 4

Therefore, the residue is:

$$\begin{aligned}\oint_C \frac{z^2}{z^2-1}dz&=2\pi i\left(-\frac{1}{2}\right)\\&=-\pi i\end{aligned}$$

The pole $z=1$ outside the circle as:

$$\begin{aligned}|z+i|&=1.4\\|1+i|&=1.4\\\sqrt{1^2+1^2}&=1.4\\\sqrt{2}&>1.4\end{aligned}$$

The pole $z=-1$ outside the circle as:

$$\begin{aligned}|z+i|&=1.4\\|-1+i|&=1.4\\\sqrt{(-1)^2+1^2}&=1.4\\\sqrt{2}&>1.4\end{aligned}$$

That is

$$\oint_C \frac{z^2}{z^2-1}dz=0$$

Thus, the provided integral equation with pole $(-1,1)$ at circle $|z+i|=1.4$ is zero.

Hence, the integral value of the provided equation is $\boxed{0}$.

Chapter 14.3, Problem 4P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula: Consider that $f(z)$ is an analytic function within a simply connected domain D and z_0 be any point on the domain D then for any simple closed path C in D that encloses z_0 then,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0)$$

Here, the integration is taken counter-clockwise.

Step 2 of 3

Consider the function, $\frac{z^2}{z^2-1}$

Here the circle is, $C: |z+5-5i| = 7$. Clearly this circle contains poles of the above function $z=-1$ and $z=1$.

The objective is to integrate the above function by Cauchy integral formula.

For this consider,

$$\begin{aligned}\frac{z^2}{z^2-1} &= \frac{1}{2}\int_C \frac{z^2}{z-1}dz - \frac{1}{2}\int_C \frac{z^2}{z+1}dz \\ &= f_1(z) + f_2(z)\end{aligned}$$

Comments (1)

Anonymous

how come z=1 inside |z+5-5i| =7? (|1+5-5i| = sqrt61 >7)

Step 3 of 3

Consider the function,

$$f_1(z) = \frac{1}{2}\int_C \frac{z^2}{z-1}dz$$

By Cauchy integral formula,

$$\begin{aligned}\frac{1}{2}\int_C \frac{z^2}{z-1}dz &= \frac{1}{2} \times 2\pi i \times f(1) \\ &= \pi i \times 1 \\ &= \pi i\end{aligned}$$

Next consider the function,

$$f_2(z) = \frac{1}{2}\int_C \frac{z^2}{z+1}dz$$

Then again, by the Cauchy integral formula,

$$\begin{aligned}\frac{1}{2}\int_C \frac{z^2}{z+1}dz &= \frac{1}{2} \times 2\pi i \times f(-1) \\ &= \pi i \times 1 \\ &= \pi i\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C \frac{z^2}{z^2-1}dz &= \frac{1}{2}\int_C \frac{z^2}{z-1}dz - \frac{1}{2}\int_C \frac{z^2}{z+1}d \\ &= \pi i - \pi i \\ &= 0\end{aligned}$$

Hence, $\boxed{\oint_C \frac{z^2}{z^2-1}dz = 0}$.

Chapter 14.3, Problem 5P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{\cos 3z}{6z}$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0)$$ ...... **(1)**

Step 2 of 2

Here integrand $\frac{\cos 3z}{6z}$ is not analytic at $z=0$.

Region $|z|=1$ contains $z=0$.

$$\oint_C \frac{\cos 3z}{6z} dz = \oint_C \frac{(\cos 3z)/6}{z} dz$$

It is in the form of **(1)**, here $f(z)=\frac{\cos 3z}{6}, z_0=0$.

$$\begin{aligned} f(0) &= \frac{\cos 0}{6} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{(\cos 3z)/6}{z} dz &= 2\pi i \times \left(\frac{1}{6}\right) \quad (\text{from } (1)) \\ &= \frac{\pi i}{3} \end{aligned}$$

Hence, the result is $\boxed{\frac{\pi i}{3}}$.

Chapter 14.3, Problem 6P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{e^{2z}}{\pi z-i}$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i\times f(z_0) \quad \text{...... } \mathbf{(1)}$$

Step 2 of 2

Here integrand $\frac{e^{2z}}{\pi z-i}$ is not analytic at $z=\frac{i}{\pi}$.

Region $|z|=1$ contains $z=\frac{i}{\pi}$.

$$\oint_C \frac{e^{2z}}{\pi z-i}dz = \frac{1}{\pi}\oint_C \frac{e^{2z}}{z-i/\pi}dz$$

It is in the form of **(1)**, here $f(z)=e^{2z}, z_0=\frac{i}{\pi}$.

$$f\left(\frac{i}{\pi}\right)=e^{2i/\pi}$$

$$\begin{aligned}\frac{1}{\pi}\oint_C \frac{e^{2z}}{z-i/\pi}dz &= \frac{1}{\pi}\left[2\pi i\times\left(e^{2i/\pi}\right)\right] \quad (\text{from } (1)) \\ &= 2ie^{2i/\pi}\end{aligned}$$

Hence, the result is $\boxed{2ie^{2i/\pi}}$.

Chapter 14.3, Problem 7P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{z^3}{2z-i}$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \mathbf{(1)}$$

Step 2 of 2

Here integrand $\frac{z^3}{2z-i}$ is not analytic at $z=\frac{i}{2}$.

Region $|z|=1$ contains $z=\frac{i}{2}$.

$$\oint_C \frac{z^3}{2z-i}dz = \frac{1}{2}\oint_C \frac{z^3}{z-i/2}dz$$

It is in the form of **(1)**, here $f(z)=z^3, z_0=\frac{i}{2}$.

$$\begin{aligned} f\left(\frac{i}{2}\right) &= \left(\frac{i}{2}\right)^3 \\ &= \frac{-i}{8} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\oint_C \frac{z^3}{z-i/2}dz &= \frac{1}{2}\left[2\pi i \times \left(\frac{-i}{8}\right)\right] \quad (\text{from } (1)) \\ &= \frac{\pi}{8} \end{aligned}$$

Hence, the result is $\boxed{\frac{\pi}{8}}$.

Chapter 14.3, Problem 8P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration of $\frac{z^2 \sin z}{4z-1}$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \textbf{(1)}$$

Step 2 of 2

Here integrand $\frac{z^2 \sin z}{4z-1}$ is not analytic at $z=\frac{1}{4}$.

Region $|z|=1$ contains $z=\frac{1}{4}$.

$$\oint_C \frac{z^2 \sin z}{4z-1} dz = \frac{1}{4} \oint_C \frac{z^2 \sin z}{z-1/4} dz$$

It is in the form of **(1)**, here $f(z)=z^2 \sin z, z_0=\frac{1}{4}$.

$$\begin{aligned} f\left(\frac{1}{4}\right) &= (1/4)^2 \sin(1/4) \\ &= \frac{1}{16}\sin(1/4) \end{aligned}$$

$$\begin{aligned} \frac{1}{4}\oint_C \frac{z^2 \sin z}{z-1/4} dz &= \frac{1}{4}\left[2\pi i \times \left(\frac{1}{16}\sin(1/4)\right)\right] \quad (\text{from } (1)) \\ &= \frac{\pi i \sin(1/4)}{32} \end{aligned}$$

Hence, the result is $\boxed{\frac{\pi i \sin(1/4)}{32}}$.

Chapter 14.3, Problem 9P

Step-by-step solution

There is no solution to this problem yet.

Get help from a Chegg subject expert.

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Chapter 14.3, Problem 10P

Step-by-step solution

Step 1 of 4

Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing z_0 is

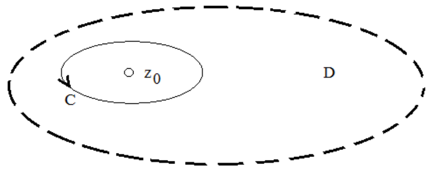


Fig.356. Cauchy's integral formula

Step 2 of 4

(a)

Considering the integral as:

$$\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz$$

Rewriting and simplifying the integral as:

$$\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz = 2\pi i\left[z^3-6\right]\Big|_{z=\frac{i}{2}}$$

$$= 2\pi i\left(\frac{i}{2}\right)^3 - 12\pi i$$

$$\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz = \boxed{\frac{\pi}{4}-12\pi i} \qquad \left(z_0=\frac{1}{2}i \text{ inside C}\right)$$

Step 3 of 4

(b)

Considering the integral as:

$$\oint_C \frac{\sin z}{z-\frac{1}{2}\pi}\,dz$$

Rewriting and simplifying the integral as:

$$\oint_C \frac{\sin z}{z-\frac{1}{2}\pi}\,dz = 2\pi i\left[\sin z\right]\Big|_{z=\frac{\pi}{2}}$$

$$= 2\pi i\left(\sin\left(\frac{\pi}{2}\right)\right)$$

$$= 2\pi i(1) \qquad \left(\text{Since}, \sin\left(\frac{\pi}{2}\right)=1\right)$$

$$\oint_C \frac{z^3-6}{z-\frac{1}{2}i}\,dz = \boxed{2\pi i} \qquad \left(\begin{array}{l} z_0=\frac{\pi}{2} \text{ lies on the boundary of C,} \\ \text{by Cauchy integral theorem} \end{array}\right)$$

Step 4 of 4

(c)

Considering the integral as:

$$\oint_C \frac{z^3-4}{2z-i}\,dz$$

Rewriting and simplifying the integral as:

$$\oint_C \frac{z^3-4}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-2}{z-\frac{1}{2}i}\,dz$$

$$= 2\pi i\left[\frac{1}{2}z^3-2\right]\Big|_{z=\frac{i}{2}}$$

$$= 2\pi i\left(\frac{i}{2}\right)^3 - 4\pi i$$

$$\oint_C \frac{z^3-4}{2z-i}\,dz = \boxed{\frac{\pi}{4}-4\pi i} \qquad \left(z_0=\frac{1}{2}i \text{ inside C}\right)$$

Chapter 14.3, Problem 11P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

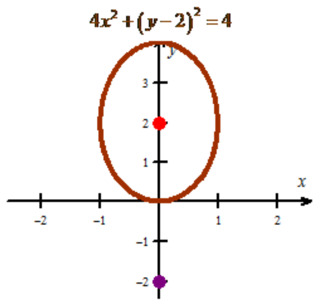
Find the integration $\oint_C \frac{dz}{z^2+4}$ around $C: 4x^2 + (y-2)^2 = 4$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \mathbf{(1)}$$

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{1}{z^2+4}$ is not analytic at $z = \pm 2i$.

$z = \pm 2i \to x = 0, y = \pm 2$

Region $C: 4x^2 + (y-2)^2 = 4$ contains $z = 2i$.

$$\oint_C \frac{dz}{z^2+4} = \oint_C \frac{1/(z+2i)\,dz}{z-2i}$$

It is in the form of **(1)**, here $f(z) = \frac{1}{z+2i}, z_0 = 2i$.

$$\begin{aligned} f(2i) &= \frac{1}{2i+2i} \\ &= \frac{-i}{4} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{1/(z+2i)\,dz}{z-2i} &= 2\pi i \times \left(-\frac{i}{4}\right) \quad (\text{from } (1)) \\ &= \frac{\pi}{2} \end{aligned}$$

Hence, the result is $\boxed{\frac{\pi}{2}}$.

Chapter 14.3, Problem 12P

Step-by-step solution

Step 1 of 4

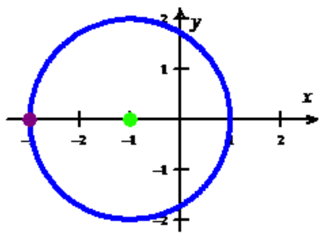
The objective is to find the integration $\oint_C \frac{zdz}{z^2+4z+3}$ around the circle with center (-1) and radius 2.

Step 2 of 4

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \quad \text{...... (1)}$$

Region is as shown below.



Step 3 of 4

Here integrand $\frac{z}{z^2+4z+3}$ is not analytic at $z=-3,-1$.

Region contains $z=-3,-1$.

$$\oint_C \frac{z}{z^2+4z+3}dz = \oint_C \frac{z}{(z+3)(z+1)}dz$$

$$= \frac{1}{2}\oint_C \left(\frac{3}{z+3} - \frac{1}{z+1}\right)dz$$

$$= \frac{3}{2}\oint_C \frac{1}{z+3}dz - \frac{1}{2}\oint_C \frac{1}{z+1}dz$$

$$= \frac{3}{2}(2\pi i) - \frac{1}{2}(2\pi i) \qquad (\text{from } (1))$$

$$= 2\pi i$$

Hence, the result is $\boxed{2\pi i}$.

Step 4 of 4

Chapter 14.3, Problem 13P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

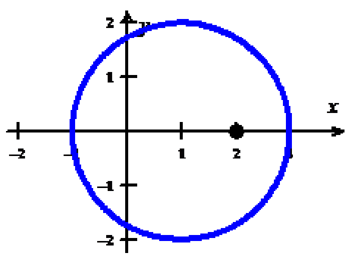
Find the integration $\oint_C \frac{z+2}{z-2}dz$ around $C:|z-1|=2$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \textbf{(1)}$$

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{z+2}{z-2}$ is not analytic at $z=2$.

Region $C:|z-1|=2$ contains $z=2$.

$$\oint_C \frac{z+2}{z-2}dz = \oint_C \frac{(z+2)}{z-2}dz$$

It is in the form of **(1)**, here $f(z)=z+2, z_0=2$.

$$\begin{aligned} f(2) &= 2+2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \oint_C \frac{z+2}{z-2}dz &= 2\pi i \times (4) \quad (\text{from } (1)) \\ &= 8\pi i \end{aligned}$$

Hence, the result is $\boxed{8\pi i}$.

Chapter 14.3, Problem 14P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

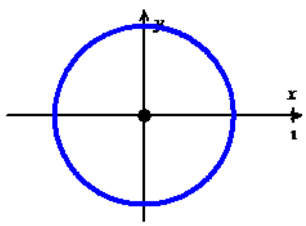
Find the integration $\oint_C \frac{e^z}{ze^z - 2iz} dz$ around $C: |z| = 0.6$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{z - z_0} = 2\pi i \times f(z_0)$ ...... **(1)**

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{e^z}{ze^z - 2iz}$ is not analytic at $z = 0, e^z = 2i$.

Region $C: |z| = 0.6$ contains $z = 0$.

$$\oint_C \frac{e^z}{ze^z - 2iz} dz = \oint_C \frac{e^z/(e^z - 2i)}{z} dz$$

It is in the form of **(1)**, here $f(z) = \frac{e^z}{e^z - 2i}, z_0 = 0$.

$$\begin{aligned} f(0) &= \frac{e^0}{e^0 - 2i} \\ &= \frac{1}{1-2i} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{e^z/(e^z - 2i)}{z} dz &= 2\pi i \times \left(\frac{1}{1-2i}\right) \quad (\text{from } (1)) \\ &= \frac{2\pi i(1+2i)}{5} \\ &= \frac{2\pi i - 4\pi}{5} \end{aligned}$$

Hence, the result is $\boxed{\frac{2\pi i - 4\pi}{5}}$.

Chapter 14.3, Problem 15P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

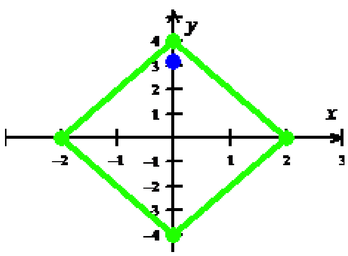
Find the integration $\oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}dz$ around the square with vertices $\pm 2, \pm 4i$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0) \text{ ...... } \mathbf{(1)}$$

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{\cosh(z^2-\pi i)}{z-\pi i}$ is not analytic at $z=\pi i$.

Above region contains $z=\pi i$.

$$\oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}dz = \oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}dz$$

It is in the form of **(1)**, here $f(z)=\cosh(z^2-\pi i), z_0=\pi i$.

$$f(\pi i)=\cosh(-\pi^2-\pi i)$$

$$\oint_C \frac{\cosh(z^2-\pi i)}{z-\pi i}dz = 2\pi i\times\cosh(-\pi^2-\pi i) \quad (\text{from } (1))$$

Hence, the result is $\boxed{2\pi i\times\cosh(-\pi^2-\pi i)}$.

Chapter 14.3, Problem 16P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

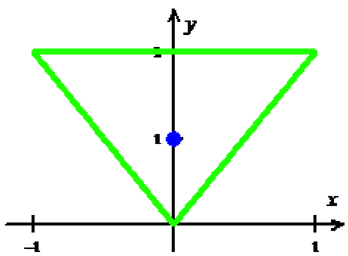
Find the integration $\oint_C \frac{\tan z}{z-i}\,dz$ around the triangle with vertices $0, \pm 1 + 2i$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0)$ ...... **(1)**

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{\tan z}{z-i}$ is not analytic at $z = i$.

Above region contains $z = i$.

$$\oint_C \frac{\tan z}{z-i}\,dz = \oint_C \frac{(\tan z)}{z-i}\,dz$$

It is in the form of **(1)**, here $f(z) = \tan z, z_0 = i$.

$$\begin{aligned} f(i) &= \tan i \\ &= i \tanh 1 \end{aligned}$$

$$\begin{aligned} \oint_C \frac{(\tan z)}{z-i}\,dz &= 2\pi i \times i \tanh 1 \quad (\text{from } (1)) \\ &= -2\pi \tanh 1 \end{aligned}$$

Hence, the result is $\boxed{-2\pi \tanh 1}$.

Chapter 14.3, Problem 17P

Step-by-step solution

Step 1 of 4

Cauchy's Integral formula:

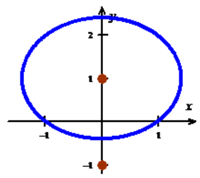
Find the integration $\oint_C \frac{\text{Ln }(z+1)dz}{z^2+1}$ around $C:|z-i|=1.4$.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{z-z_0} = 2\pi i\times f(z_0)$ ...... **(1)**

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here, integrand $\frac{\text{Ln }(z+1)}{z^2+1}$ is not analytic at $z=\pm i$.

Region contains $z=i,-i$.

From the graph, $z=-i(\text{or } y=-1)$ lie outside the circle.

Therefore, the integral becomes,

$$\oint_C \frac{\text{Ln }(z+1)dz}{z^2+1} = \oint_C \frac{\text{Ln }(z+1)/(z+i)dz}{z-i}$$

It is in the form of **(1)**, here $f(z)=\frac{\text{Ln}(z+1)}{z+i}, z_0=i$.

$$f(i)=\frac{\text{Ln}(i+1)}{2i}$$
$$=\frac{\ln\sqrt{2}+i\frac{\pi}{4}}{2i}$$

Since $\text{Ln}(x+iy)=\text{Ln}\, r+i\theta$ such that $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ ...... (2)

As in, $\text{Ln}(1+i)$, $x=1, y=1$.

This implies, $r=\sqrt{1^2+1^2}$

$=\sqrt{2}$

And, $\theta=\tan^{-1}\left(\frac{1}{1}\right)$

$=\tan^{-1}1$

$=\frac{\pi}{4}$

This implies, from (2), $\text{Ln}(1+i)=\text{Ln}\sqrt{2}+i\frac{\pi}{4}$.

Step 4 of 4

Calculate $\oint_C \frac{\text{Ln }(z+1)/(z+i)dz}{z-i}$ as follows:

$$\oint_C \frac{\text{Ln }(z+1)/(z+i)dz}{z-i} = 2\pi i\times\left[\frac{i\frac{\pi}{4}+\ln\sqrt{2}}{2i}\right] \quad (\text{from } (1))$$
$$=i\frac{\pi^2}{4}+\pi\ln\sqrt{2}$$

Hence, the result is $\boxed{\frac{i\pi^2}{4}+\pi\ln\sqrt{2}}$.

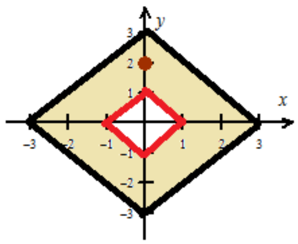
Chapter 14.3, Problem 18P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

Find the integration $\oint_C \frac{\sin z dz}{4z^2 - 8iz}$, where C contains the boundaries of the squares shown below.



Step 2 of 3

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{z - z_0} = 2\pi i \times f(z_0) \text{ ...... } \textbf{(1)}$$

Step 3 of 3

Here $\frac{\sin z}{4z^2 - 8iz}$ is not analytic at $z = 2i, 0$.

Above region contains $z = 2i$.

$$\oint_C \frac{\sin z}{4z^2 - 8iz} dz = \oint_C \frac{\sin z/4z}{z - 2i} dz$$

It is in the form of **(1)**, here $f(z) = \frac{\sin z}{4z}, z_0 = 2i$.

$$\begin{aligned} f(2i) &= \frac{\sin 2i}{4(2i)} \\ &= \frac{\sinh 2}{8} \quad (\text{since } \sin 2i = i \sinh 2) \end{aligned}$$

$$\begin{aligned} \oint_C \frac{\sin z/4z}{z - 2i} dz &= 2\pi i \times \left[\frac{\sinh 2}{8}\right] \quad (\text{from } (1)) \\ &= \frac{i\pi \sinh 2}{4} \end{aligned}$$

Hence, the result is $\boxed{\frac{i\pi \sinh 2}{4}}$.

Chapter 14.3, Problem 19P

Step-by-step solution

Step 1 of 3

Cauchy's Integral formula:

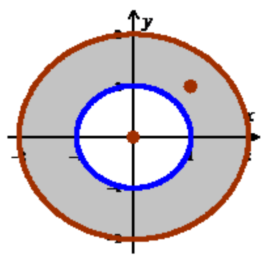
Find the integration $\oint_C \frac{e^{z^2}dz}{z^2(z-1-i)}$ around C contains $|z|=2$ counter-clockwise and $|z|=1$ clockwise.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0)$ ...... **(1)**

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{e^{z^2}}{z^2(z-1-i)}$ is not analytic at $z=0,1+i$.

Above region contains $z=1+i$.

$$\oint_C \frac{e^{z^2}dz}{z^2(z-1-i)} = \oint_C \frac{e^{z^2}/z^2\,dz}{z-1-i}$$

It is in the form of **(1)**, here $f(z)=\frac{e^{z^2}}{z^2}, z_0=1+i$.

$$\begin{aligned} f(1+i) &= \frac{e^{(1+i)^2}}{(1+i)^2} \\ &= \frac{e^{2i}}{2i} \end{aligned}$$

$$\begin{aligned} \oint_C \frac{e^{z^2}/z^2\,dz}{z-1-i} &= 2\pi i \times \left[\frac{e^{2i}}{2i}\right] \quad (\text{from } (1)) \\ &= \pi e^{2i} \end{aligned}$$

Hence, the result is $\boxed{\pi e^{2i}}$.

Chapter 14.3, Problem 20P

Step-by-step solution

Step 1 of 2

Cauchy's Integral formula:

Find the integration $\oint_C \frac{dz}{(z-z_1)(z-z_2)}$, where C is the simple closed path enclosed z_1, z_2.

Cauchy's Integral formula states that if $f(z)$ is analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{z-z_0} = 2\pi i \times f(z_0)$ ...... **(1)**

Step 2 of 2

Here $\frac{1}{(z-z_1)(z-z_2)}$ is not analytic at $z = z_1, z_2$.

Above region contains $z = z_1, z_2$.

$$\begin{aligned}\oint_C \frac{dz}{(z-z_1)(z-z_2)} &= \frac{1}{z_1-z_2}\oint_C \left[\frac{1}{z-z_1} - \frac{1}{z-z_2}\right] dz \\ &= \frac{1}{z_1-z_2}\left[\oint_C \frac{dz}{z-z_1} - \oint_C \frac{dz}{z-z_2}\right] \\ &= \frac{1}{z_1-z_2}[2\pi i - 2\pi i] \quad (\text{from}(1)) \\ &= 0\end{aligned}$$

Hence, the result is $\boxed{0}$.

Chapter 14.4, Problem 1P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{\sin z}{z^4} dz$ around the unit circle $|z| = 1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \text{...... (1)}$$

Step 2 of 3

Here integrand $\frac{\sin z}{z^4}$ is not analytic at $z = 0$.

Region $|z| = 1$ contains $z = 0$.

$$\oint_C \frac{\sin z}{z^4} dz = \oint_C \frac{\sin z dz}{z^4}$$

It is in the form of **(1)**, here $f(z) = \sin z, z_0 = 0, n = 3$.

$$f(z) = \sin z$$
$$f'(z) = \cos z$$
$$f''(z) = -\sin z$$
$$f^{(3)}(z) = -\cos z$$
$$f^{(3)}(0) = -1$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned}\oint_C \frac{\sin z dz}{z^4} &= \frac{2\pi i \times f^{(3)}(0)}{3!} \quad (\text{from } (1)) \\ &= \frac{2\pi i \times -1}{6} \\ &= -\frac{\pi i}{3}\end{aligned}$$

Hence, the result is $\boxed{-\frac{\pi i}{3}}$.

Chapter 14.4, Problem 2P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{z^6}{(2z-1)^6}dz$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i\times f^{(n)}(z_0)}{n!}, n=1,2,...$ ...... **(1)**

Step 2 of 3

Here integrand $\frac{z^6}{(2z-1)^6}$ is not analytic at $z=\frac{1}{2}$.

Region $|z|=1$ contains $z=\frac{1}{2}$.

$$\oint_C \frac{z^6}{(2z-1)^6}dz = \frac{1}{2^6}\oint_C \frac{z^6 dz}{(z-1/2)^6}$$

It is in the form of **(1)**, here $f(z)=z^6, z_0=\frac{1}{2}, n=5$.

$$\begin{aligned} f(z) &= z^6 \\ f^{(5)}(z) &= 6!z \\ f^{(5)}\left(\frac{1}{2}\right) &= \frac{6!}{2} \\ &= 360 \end{aligned}$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned} \frac{1}{2^6}\oint_C \frac{z^6}{(z-1/2)^6} &= \frac{1}{2^6}\left[\frac{2\pi i\times f^{(5)}\left(\frac{1}{2}\right)}{5!}\right] \quad (\text{from }(1)) \\ &= \frac{\pi i\times 360}{2^5\times 5!} \\ &= \frac{3\pi i}{32} \end{aligned}$$

Hence, the result is $\boxed{\frac{3\pi i}{32}}$.

Chapter 14.4, Problem 3P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{e^z}{z^n} dz$ around the unit circle $|z| = 1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \textbf{...... (1)}$$

Step 2 of 3

Here integrand $\frac{e^z}{z^n}$ is not analytic at $z = 0$.

Region $|z| = 1$ contains $z = 0$.

$$\oint_C \frac{e^z}{z^n} dz = \oint_C \frac{e^z dz}{z^n}$$

It is in the form of **(1)**, here $f(z) = e^z, z_0 = 0$.

$$\begin{aligned} f(z) &= e^z \\ f^{(n-1)}(z) &= e^z \\ f^{(n-1)}(0) &= 1 \end{aligned}$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned} \oint_C \frac{e^z}{z^n} dz &= \frac{2\pi i \times f^{(n-1)}(0)}{(n-1)!} \quad (\text{from } (1)) \\ &= \frac{2\pi i}{(n-1)!} \end{aligned}$$

Hence, the result is $\boxed{\frac{2\pi i}{(n-1)!}}$.

Chapter 14.4, Problem 4P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{e^z \cos z}{(z-\pi/4)^3} dz$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n=1,2,... \quad \textbf{...... (1)}$$

Step 2 of 3

Here integrand $\frac{e^z \cos z}{(z-\pi/4)^3}$ is not analytic at $z=\frac{\pi}{4}$.

Region $|z|=1$ contains $z=\frac{\pi}{4}$.

$$\oint_C \frac{e^z \cos z}{(z-\pi/4)^3} dz = \oint_C \frac{e^z \cos z dz}{(z-\pi/4)^3}$$

It is in the form of **(1)**, here $f(z)=e^z \cos z, z_0=\frac{\pi}{4}, n=2$.

$$\begin{aligned} f(z) &= e^z \cos z \\ f'(z) &= e^z(\cos z - \sin z) \\ f''(z) &= e^z(\cos z - \sin z) + e^z(-\sin z - \cos z) \\ &= -2\sin z e^z \\ f''\left(\frac{\pi}{4}\right) &= -\sqrt{2}e^{\pi/4} \end{aligned}$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned} \oint_C \frac{e^z \cos z dz}{(z-\pi/4)^3} &= \frac{2\pi i \times f^{(2)}\left(\frac{\pi}{4}\right)}{2!} \quad (\text{from } (1)) \\ &= \pi i\left(-\sqrt{2}e^{\pi/4}\right) \end{aligned}$$

Hence, the result is $\boxed{\pi i\left(-\sqrt{2}e^{\pi/4}\right)}$.

Chapter 14.4, Problem 5P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{\cosh 2z}{(z-1/2)^4}dz$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \textbf{...... (1)}$$

Step 2 of 3

Here integrand $\frac{\cosh 2z}{(z-1/2)^4}$ is not analytic at $z=\frac{1}{2}$.

Region $|z|=1$ contains $z=\frac{1}{2}$.

$$\oint_C \frac{\cosh 2z}{(z-1/2)^4}dz = \oint_C \frac{\cosh 2z}{(z-1/2)^{3+1}}dz$$

It is in the form of **(1)**, here $f(z)=\cosh 2z, z_0 = \frac{1}{2}, n=3$.

$$f(z) = \cosh 2z$$
$$f'(z) = 2\sinh 2z$$
$$f''(z) = 4\cosh 2z$$
$$f^{(3)}(z) = 8\sinh 2z$$
$$f^{(3)}\left(\frac{1}{2}\right) = 8\sinh 1$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned}\oint_C \frac{\cosh 2z}{(z-1/2)^{3+1}}dz &= \frac{2\pi i \times f^{(3)}\left(\frac{1}{2}\right)}{3!} \quad (\text{from } (1)) \\ &= \frac{2\pi i \times 8\sinh 1}{6} \\ &= \frac{8\pi i \sinh 1}{3}\end{aligned}$$

Hence, the result is $\boxed{\frac{8\pi i \sinh 1}{3}}$.

Chapter 14.4, Problem 6P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2}$ around the unit circle $|z|=1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n=1,2,\ldots \quad \textbf{...... (1)}$$

Step 2 of 3

Here integrand $\frac{1}{(z-2i)^2(z-i/2)^2}$ is not analytic at $z=2i,\frac{i}{2}$.

Region $|z|=1$ contains $z=\frac{i}{2}$.

$$\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2} = \oint_C \frac{dz/(z-2i)^2}{(z-i/2)^2}$$

It is in the form of **(1)**, here $f(z)=\frac{1}{(z-2i)^2}, z_0=\frac{i}{2}, n=1$.

$$\begin{aligned}
f(z) &= \frac{1}{(z-2i)^2} \\
f'(z) &= \frac{-2}{(z-2i)^3} \\
f'\left(\frac{i}{2}\right) &= \frac{-2}{(i/2-2i)^3} \\
&= \frac{16i}{27}
\end{aligned}$$

Step 3 of 3

Now from **(1)**,

$$\begin{aligned}
\oint_C \frac{dz/(z-2i)^2}{(z-i/2)^2} &= \frac{2\pi i \times f^{(1)}\left(\frac{i}{2}\right)}{1!} \quad (\text{from }(1)) \\
&= 2\pi i \times \frac{16i}{27} \\
&= \frac{-32\pi}{27}
\end{aligned}$$

Hence, the result is $\boxed{\frac{-32\pi}{27}}$.

Chapter 14.4, Problem 7P

Step-by-step solution

Step 1 of 3

Contour integration:

Find the integration $\oint_C \frac{\cos z}{z^{2n+1}} dz, n = 0, 1, \ldots$ around the unit circle $|z| = 1$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \text{...... } \mathbf{(1)}$$

Step 2 of 3

Here integrand $\frac{\cos z}{z^{2n+1}}$ is not analytic at $z = 0$.

Region $|z| = 1$ contains $z = 0$.

$$\oint_C \frac{\cos z}{z^{2n+1}} dz = \oint_C \frac{\cos z dz}{z^{2n+1}}$$

It is in the form of **(1)**, here $f(z) = \cos z, z_0 = 0, n = 2n$.

$$f(z) = \cos z$$
$$f^{(2n)}(z) = (-1)^n \cos z$$
$$f^{(2n)}(0) = (-1)^n$$

Step 3 of 3

Now from **(1)**,

$$\oint_C \frac{\cos z dz}{z^{2n+1}} = \frac{2\pi i \times f^{(2n)}(0)}{(2n)!} \quad (\text{from } (1))$$
$$= \frac{2\pi i (-1)^n}{(2n)!}$$

Hence, the result is $\boxed{\frac{(-1)^n 2\pi i}{(2n)!}}$.

Chapter 14.4, Problem 8P

Step-by-step solution

Step 1 of 4

Contour integration:

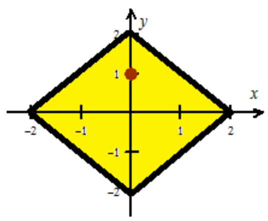
Find the integration $\oint_C \frac{z^3+\sin z}{(z-i)^3}dz$ around the square with vertices $\pm 2, \pm 2i$.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \ldots\ldots \textbf{(1)}$$

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{z^3+\sin z}{(z-i)^3}$ is not analytic at $z = i$.

Region contains $z = i$.

$$\oint_C \frac{z^3+\sin z}{(z-i)^3}dz = \oint_C \frac{z^3+\sin z}{(z-i)^{2+1}}dz$$

It is in the form of **(1)**, here $f(z) = z^3 + \sin z, z_0 = i, n = 2$.

$$\begin{aligned} f(z) &= z^3 + \sin z \\ f'(z) &= 3z^2 + \cos z \\ f''(z) &= 6z - \sin z \\ f''(i) &= 6i - \sin i \end{aligned}$$

Step 4 of 4

Now from **(1)**,

$$\begin{aligned} \oint_C \frac{z^3+\sin z}{(z-i)^{2+1}}dz &= \frac{2\pi i \times f^{(2)}(i)}{2!} \quad (\text{from } (1)) \\ &= \pi i \times (6i - \sin i) \\ &= -6\pi + \pi \sinh 1 \end{aligned}$$

Hence, the result is $\boxed{-6\pi + \pi \sinh 1}$.

Chapter 14.4, Problem 9P

Step-by-step solution

Step 1 of 4

Contour integration:

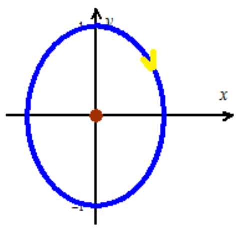
Find the integration $\oint_C \frac{\tan \pi z}{z^2} dz$ around the ellipse $16x^2 + y^2 = 1$ in clockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \text{...... (1)}$$

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{\tan \pi z}{z^2}$ is not analytic at $z = 0$.

Region contains $z = 0$.

$$\oint_C \frac{\tan \pi z}{z^2} dz = \oint_C \frac{\tan \pi z}{z^{1+1}} dz$$

It is in the form of **(1)**, here $f(z) = \tan \pi z, z_0 = 0, n = 1$.

$$f(z) = \tan \pi z$$
$$f'(z) = \pi \sec^2 \pi z$$
$$f'(0) = \pi$$

Step 4 of 4

Now from **(1)**,

$$\oint_C \frac{\tan \pi z}{z^{1+1}} dz = -\frac{2\pi i \times f^{(1)}(0)}{1!} \quad (\text{from } (1) \text{ and clockwise})$$
$$= -2\pi i \times \pi$$
$$= -2\pi^2 i$$

Hence, the result is $\boxed{-2\pi^2 i}$.

Chapter 14.4, Problem 10P

Step-by-step solution

Step 1 of 4

Contour integration:

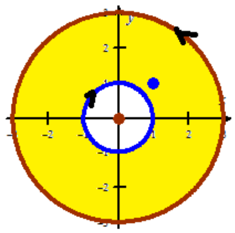
Find the integration $\oint_C \frac{4z^3-6}{z(z-1-i)^2}dz$, where C consists of $|z|=3$ counter clockwise and $|z|=1$ in clockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n=1,2,...$ ...... **(1)**

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{4z^3-6}{z(z-1-i)^2}$ is not analytic at $z=0,1+i$.

Region contains $z=1+i$.

$$\oint_C \frac{4z^3-6}{z(z-1-i)^2}dz = \oint_C \frac{(4z^3-6)/z}{(z-1-i)^{1+1}}dz$$

It is in the form of **(1)**, here $f(z)=\frac{4z^3-6}{z}, z_0=1+i, n=1$.

$$\begin{aligned} f(z) &= \frac{4z^3-6}{z} \\ &= 4z^2-\frac{6}{z} \\ f'(z) &= 8z+\frac{6}{z^2} \\ f'(1+i) &= 8+5i \end{aligned}$$

Step 4 of 4

Now from **(1)**,

$$\begin{aligned} \oint_C \frac{(4z^3-6)/z}{(z-1-i)^{1+1}}dz &= -\frac{2\pi i \times f^{(1)}(1+i)}{1!} \quad (\text{from } (1)) \\ &= 2\pi i(8+5i) \\ &= 16\pi i - 10\pi \end{aligned}$$

Hence, the result is $\boxed{16\pi i-10\pi}$.

Chapter 14.4, Problem 11P

Step-by-step solution

Step 1 of 5

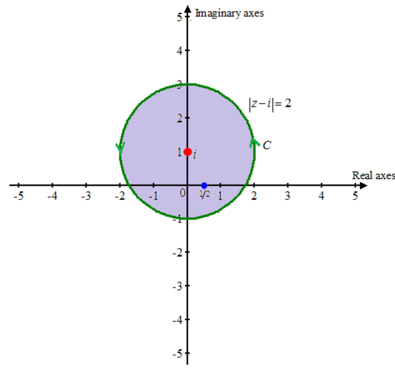
The objective is to evaluate,

$$\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz,\quad C:|z-i|=2 \text{ counterclockwise.}$$

Step 2 of 5

Let $g(z)=\frac{(1+z)\sin z}{(2z-1)^2}$

Sketch of the contour $C:|z-i|=2$ counterclockwise direction is shown below.



From the above figure, observe that, g has a singularity $z=\frac{1}{2}$ that lies inside the circle $C:|z-i|=2$.

Step 3 of 5

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz=\frac{2\pi i\times f^{(n)}(z_0)}{n!},n=1,2,\ldots \quad \ldots\ldots (1)$$

Step 4 of 5

In this case, the integrand $\frac{(1+z)\sin z}{(2z-1)^2}$ is not analytic at $z=\frac{1}{2}$.

Region contains $z=\frac{1}{2}$.

$$\begin{aligned}\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz &= \oint_C \frac{(1+z)\sin z}{2^2\left(z-\frac{1}{2}\right)^2}\,dz\\ &=\frac{1}{4}\oint_C \frac{(1+z)\sin z}{(z-1/2)^{1+1}}\,dz\end{aligned}$$

It is in the form of (1), here $f(z)=(1+z)\sin z, z_0=\frac{1}{2}, n=1$.

$$\begin{aligned}f(z)&=(1+z)\sin z\\ f'(z)&=\sin z+(1+z)\cos z\\ f'\left(\frac{1}{2}\right)&=\sin\frac{1}{2}+\frac{3}{2}\cos\frac{1}{2}\end{aligned}$$

Step 5 of 5

From the formula (1), we have that,

$$\begin{aligned}\oint_C \frac{(1+z)\sin z}{(z-1/2)^{1+1}}\,dz &= \frac{2\pi i}{1!}f^{(1)}\left(\frac{1}{2}\right)\quad \left[\text{With } f(z)=(1+z)\sin z\right]\\ &=(2\pi i)\left(\sin\frac{1}{2}+\frac{3}{2}\cos\frac{1}{2}\right)\end{aligned}$$

And hence,

$$\begin{aligned}\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz &= \frac{1}{4}\oint_C \frac{(1+z)\sin z}{(z-1/2)^{1+1}}\,dz\\ &=\frac{1}{4}(2\pi i)\left(\sin\frac{1}{2}+\frac{3}{2}\cos\frac{1}{2}\right)\\ &=\frac{1}{2}\pi i\left(\sin\frac{1}{2}+\frac{3}{2}\cos\frac{1}{2}\right)\\ &=\boxed{2.821i}\end{aligned}$$

Chapter 14.4, Problem 12P

Step-by-step solution

Step 1 of 3

The objective is to evaluate

$\oint_C \frac{e^{z^2}}{z(z-2i)^2}dz$ around $C:|z-3i|=2$ clockwise.

Step 2 of 3

Assume $f(z)=\frac{e^{z^2}}{z(z-2i)^2}$.

Sketch the counter $C:|z-3i|=2$ clockwise direction as shown below.

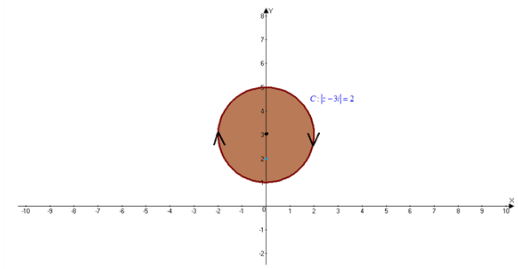


Figure 1

The function $f(z)=\frac{e^{z^2}}{z(z-2i)^2}$ is not analytic at $z=0, z=2i$.

The singularity $z=2i$ lies inside the circle $C:|z-3i|=2$.

Step 3 of 3

Cauchy's integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i\times f^{(n)}(z_0)}{n!}, n=1,2,\ldots \quad \text{...... (1)}$$

Simplify the integral $\oint_C \frac{e^{z^2}}{z(z-2i)^2}dz$ using Cauchy's integral formula.

$$\oint_C \frac{e^{z^2}}{z(z-2i)^2}dz = \oint_{C_1} \frac{e^{z^2}}{z(z-2i)^2}dz + \oint_{C_2} \frac{e^{z^2}}{z(z-2i)^2}dz$$

Here, C_1 is defined for $z=0$ and C_2 is defined for $z=2i$.

Since $z=0$ does not lie in the region $C:|z-3i|=2$.

Therefore, $\oint_{C_1} \frac{e^{z^2}}{z(z-2i)^2}dz = 0$.

and

$$\begin{aligned}\oint_{C_2} \frac{e^{z^2}}{z(z-2i)^2}dz &= -\frac{2\pi i}{1!}\times\left[\frac{d}{dz}\left(\frac{e^{z^2}}{z}\right)\right]_{z=2i} \qquad \{\text{Negative sign due to clockwise}\} \\ &= -2\pi i\times\left[\frac{z\cdot 2ze^{z^2}-e^{z^2}\cdot 1}{z^2}\right]_{z=2i} \\ &= -2\pi i\times\left[\frac{2(2i)^2e^{(2i)^2}-e^{(2i)^2}}{(2i)^2}\right] \\ &= -2\pi i\times\left[\frac{-8e^{-4}-e^{-4}}{-4}\right]\end{aligned}$$

$$= -\frac{9}{2}\pi i e^{-4}$$

This implies as follows:

$$\begin{aligned}\oint_C \frac{e^{z^2}}{z(z-2i)^2}dz &= 0-\frac{9}{2}\pi i e^{-4} \\ &= -\frac{9}{2}\pi i e^{-4}\end{aligned}$$

Hence, the required value of the integral is $\boxed{\oint_C \frac{e^{z^2}}{z(z-2i)^2}dz = -\frac{9}{2}\pi i e^{-4}}$.

Chapter 14.4, Problem 13P

Step-by-step solution

Step 1 of 4

Contour integration:

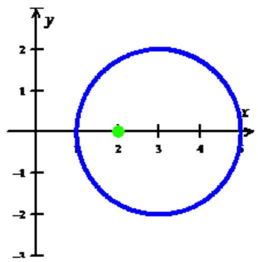
Find the integration $\oint_C \frac{\operatorname{Ln} z}{(z-2)^2} dz$ around $C: |z-3| = 2$ in counterclockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \text{...... (1)}$$

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{\operatorname{Ln} z}{(z-2)^2}$ is not analytic at $z = 2$.

Region contains $z = 2$.

$$\oint_C \frac{\operatorname{Ln} z}{(z-2)^2} dz = \oint_C \frac{\operatorname{Ln} z}{(z-2)^{1+1}} dz$$

It is in the form of **(1)**, here $f(z) = \operatorname{Ln} z, z_0 = 2, n = 1$.

$$f(z) = \operatorname{Ln} z$$
$$f'(z) = \frac{1}{z}$$
$$f'(2) = \frac{1}{2}$$

Step 4 of 4

Now from **(1)**,

$$\oint_C \frac{\operatorname{Ln} z}{(z-2)^{1+1}} dz = \frac{2\pi i \times f^{(1)}(2)}{1!} \quad (\text{from (1)})$$
$$= 2\pi i \left(\frac{1}{2}\right)$$
$$= \pi i$$

Hence, the result is $\boxed{\pi i}$.

Chapter 14.4, Problem 14P

Step-by-step solution

Step 1 of 5

Contour integration:

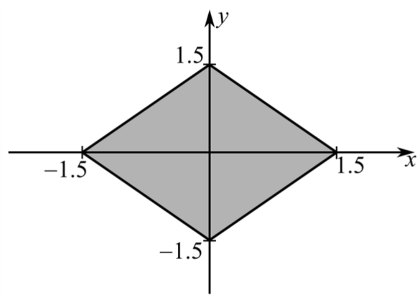
Find the integration $\oint_C \frac{\text{Ln }(z+3)}{(z-2)(z+1)^2}dz$ around the boundary of the square with vertices $\pm 1.5, \pm 1.5i$ counterclockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n = 1, 2, \ldots \quad \ldots\ldots (1)$$

Step 2 of 5

Region is as shown below.



Step 3 of 5

Here integrand $\frac{\text{Ln }(z+3)}{(z-2)(z+1)^2}$ is not analytic at $z = 2, -1$.

Region contains $z = -1$.

$$\oint_C \frac{\text{Ln }(z+3)}{(z-2)(z+1)^2}dz = \oint_C \text{Ln }(z+3)\cdot\left[\frac{1}{9(z-2)} - \frac{1}{9(z+1)} - \frac{1}{3(z+1)^2}\right]dz$$

Step 4 of 5

Refer the below work to split $\frac{1}{(z-2)(z+1)^2}$ into partial fractions.

$$\frac{1}{(z-2)(z+1)^2} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$
$$1 = A(z+1)^2 + B(z-2)(z+1) + C(z-2)$$

Let $z = -1$ in the above equation.

$-3C = 1$

This implies;

$C = -\frac{1}{3}$

Let $z = 2$ in the above equation.

$9A = 1$

This implies;

$A = \frac{1}{9}$

Equate the coefficient of z^2 on both sides.

$A + B = 0$

This implies;

$B = -\frac{1}{9}$

Step 5 of 5

Now from (1),

$$\oint_C \left[\text{Ln }(z+3)\cdot\left[\begin{array}{l}\frac{1}{9(z-2)} - \\ \frac{1}{9(z+1)} - \frac{1}{3(z+1)^2}\end{array}\right]\right]dz = \left\{\begin{array}{l}\oint_C \frac{\text{Ln }(z+3)}{9(z-2)}dz - \frac{1}{9}\oint_C \frac{\text{Ln }(z+3)}{z+1}dz \\ -\frac{1}{3}\oint_C \frac{\text{Ln }(z+3)}{(z+1)^2}dz\end{array}\right\}$$
$$= \left\{\begin{array}{l}0 - \frac{1}{9}\left[2\pi i \times (\text{Ln }(z+3))_{z=-1}\right] \\ -\frac{1}{3}\left[2\pi i \times (\text{Ln }(z+3))'_{z=-1}\right]\end{array}\right\}$$
$$= -\frac{2\pi i}{9}(\text{Ln}2) - \frac{2\pi i}{3}\left(\frac{1}{2}\right)$$
$$= -\frac{2\pi i}{9}(\text{Ln}2) - \frac{\pi i}{3}$$

Hence, the result is $\boxed{-\frac{2\pi i}{9}(\text{Ln}2) - \frac{\pi i}{3}}$.

Chapter 14.4, Problem 15P

Step-by-step solution

Step 1 of 3

Contour integration:

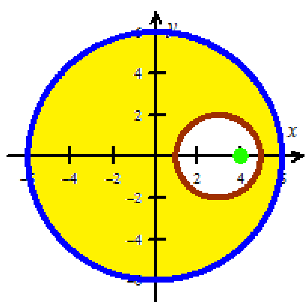
Find the integration $\oint_c \frac{\cosh 4z}{(z-4)^3}dz$, C consists of $|z|=6$ counterclockwise and $|z-3|=2$ clockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n=1,2,\ldots \quad \text{...... (1)}$$

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{\cosh 4z}{(z-4)^3}$ is not analytic at $z=4$.

Region does not contain $z=4$.

Hence, the integrand is analytic in the region.

By Cauchy's integral formula,

$$\oint_c \frac{\cosh 4z}{(z-4)^3}dz = 0$$

Hence, the result is $\boxed{0}$.

Chapter 14.4, Problem 16P

Step-by-step solution

Step 1 of 6

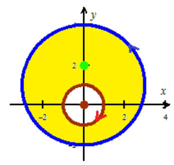
Consider the integral, $\oint_C \frac{e^{4z}}{z(z-2i)^2}\,dz$,

C consists of $|z-i|=3$ counter clockwise and $|z|=1$ clockwise.

The objective is to sketch the contour and then evaluate the integral using it.

Step 2 of 6

Region is shown below.



Step 3 of 6

The integrand $\frac{e^{4z}}{z(z-2i)^2}$ is not analytic at $z=0,2i$

Region contains $z=2i$.

Step 4 of 6

Split $\frac{1}{z(z-2i)^2}$ into partial fractions.

$$\frac{1}{z(z-2i)^2}=\frac{A}{z}+\frac{B}{z-2i}+\frac{C}{(z-2i)^2}$$

$$1=A(z-2i)^2+Bz(z-2i)+Cz$$

Let $z=0$ in the above equation.

$$1=-4A\to A=-\frac{1}{4}$$

Let $z=2i$ in the above equation.

$$1=C(2i)\to C=\frac{-i}{2}$$

Equate the coefficient of z^2 on both sides.

$$A+B=0\to B=\frac{1}{4}.$$

So, $\frac{1}{z(z-2i)^2}=\frac{-1}{4z}+\frac{1}{4(z-2i)}-\frac{i}{2(z-2i)^2}$

Therefore, $\oint_C \frac{e^{4z}}{z(z-2i)^2}\,dz=\oint_C e^{4z}\left(\frac{-1}{4z}+\frac{1}{4(z-2i)}-\frac{i}{2(z-2i)^2}\right)dz$

Step 5 of 6

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D.

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz=\frac{2\pi i\times f^{(n)}(z_0)}{n!},n=1,2,\ldots \quad \ldots\ldots \text{(i)}$$

Step 6 of 6

From (i),

$$\begin{aligned}\oint_C e^{4z}\left(\frac{-1}{4z}+\frac{1}{4(z-2i)}-\frac{i}{2(z-2i)^2}\right)&=\oint_C \frac{-e^{4z}}{4z}\,dz+\oint_C \frac{e^{4z}}{4(z-2i)}\,dz-\oint_C \frac{ie^{4z}}{2(z-2i)^2}\,dz\\&=-\frac{1}{4}(0)+\frac{1}{4}\left[2\pi i\times\left(e^{4z}\right)_{z=2i}\right]-\frac{i}{2}\left[2\pi i\times\left(e^{4z}\right)'_{z=2i}\right]\\&=\frac{\pi i}{2}\left(e^{8i}\right)+\pi\left(4e^{8i}\right)\\&=e^{8i}\left(\frac{\pi i}{2}+4\pi\right)\end{aligned}$$

Hence, the result is $\boxed{e^{8i}\left(\frac{\pi i}{2}+4\pi\right)}$.

Chapter 14.4, Problem 17P

Step-by-step solution

Step 1 of 3

Contour integration:

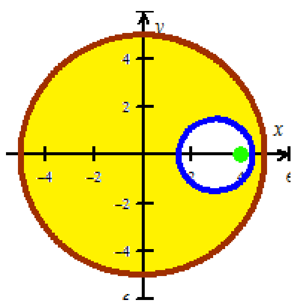
Find the integration $\oint_C \frac{e^{-z}\sin z}{(z-4)^3}dz$, C consists of $|z|=5$ counterclockwise and $|z-3|=\frac{3}{2}$ clockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i\times f^{(n)}(z_0)}{n!}, n=1,2,\ldots \quad \text{...... (1)}$$

Step 2 of 3

Region is as shown below.



Step 3 of 3

Here integrand $\frac{e^{-z}\sin z}{(z-4)^3}$ is not analytic at $z=4$.

Region does not contain $z=4$.

Hence, the integrand is analytic in the region.

By Cauchy's integral formula,

$$\oint_C \frac{e^{-z}\sin z}{(z-4)^3}dz = 0$$

Hence, the result is $\boxed{0}$.

Chapter 14.4, Problem 18P

Step-by-step solution

Step 1 of 4

Contour integration:

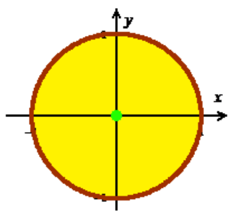
Find the integration $\oint_C \frac{\sinh z}{z^n} dz$ around $C:|z|=1$ in counterclockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i \times f^{(n)}(z_0)}{n!}, n=1,2,\ldots \quad \textbf{...... (1)}$$

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{\sinh z}{z^n}$ is not analytic at $z=0$.

Region contains $z=0$.

$$\oint_C \frac{\sinh z}{z^n} dz = \oint_C \frac{\sinh z}{z^{n-1+1}} dz$$

It is in the form of **(1)**, here $f(z)=\sinh z, z_0=0, n=n-1$.

$$f(z)=\sinh z$$
$$f'(z)=\cosh z$$
$$f''(z)=\sinh z$$
$$\vdots$$
$$f^{(n-1)}(z)=\begin{cases}\sinh z & \text{if } n \text{ is odd}\\ \cosh z & \text{if } n \text{ is even}\end{cases}$$

Step 4 of 4

Now from **(1)**,

$$\oint_C \frac{\sinh z}{z^{n-1+1}} dz = \frac{2\pi i \times f^{(n-1)}(0)}{(n-1)!} \quad (\text{from } (1))$$
$$= \begin{cases}\frac{2\pi i}{(n-1)!} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Hence, the result is $\boxed{\begin{cases}\frac{2\pi i}{(n-1)!} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}}$.

Chapter 14.4, Problem 19P

Step-by-step solution

Step 1 of 4

Contour integration:

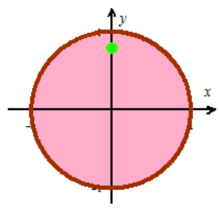
Find the integration $\oint_C \frac{e^{3z}}{(4z-\pi i)^3}dz$ around $C:|z|=1$ in counterclockwise.

Cauchy's Integral formula states that if $f(z)$ and its derivatives are analytic in a simply connected domain D, then for any simple closed curve C that contains a point z_0 in D,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i\times f^{(n)}(z_0)}{n!}, n=1,2,\dots \quad \textbf{...... (1)}$$

Step 2 of 4

Region is as shown below.



Step 3 of 4

Here integrand $\frac{e^{3z}}{(4z-\pi i)^3}$ is not analytic at $z=\frac{\pi i}{4}$.

Region contains $z=\frac{\pi i}{4}$.

$$\oint_C \frac{e^{3z}}{(4z-\pi i)^3}dz = \frac{1}{4^3}\oint_C \frac{e^{3z}}{(z-\pi i/4)^{2+1}}dz$$

It is in the form of **(1)**, here $f(z)=e^{3z}, z_0=\frac{\pi i}{4}, n=2$.

$$\begin{aligned} f(z)&=e^{3z}\\ f'(z)&=3e^{3z}\\ f''(z)&=9e^{3z}\\ f''\left(\frac{\pi i}{4}\right)&=9e^{3\pi i/4} \end{aligned}$$

Step 4 of 4

Now from **(1)**,

$$\begin{aligned} \frac{1}{4^3}\oint_C \frac{e^{3z}}{(z-\pi i/4)^{2+1}}dz &= \frac{1}{4^3}\frac{2\pi i\times f^{(2)}\left(\frac{\pi i}{4}\right)}{2!} \quad (\text{from (1)})\\ &=\frac{1}{64}\times\pi i\times 9e^{3\pi i/4}\\ &=\frac{9\pi i}{64}\left[-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]\\ &=\frac{-9\pi}{64\sqrt{2}}(1+i) \end{aligned}$$

Hence, the result is $\boxed{\frac{-9\pi}{64\sqrt{2}}(1+i)}$.

Chapter 14.4, Problem 20P

Step-by-step solution

There is no solution to this problem yet.

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Chapter 16.1, Problem 1P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z) = \frac{\cos z}{z^4}$$

series expansion of $\cos z$ with centre $z = 0$ is

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \text{ ...... (1)}$$

Step 2 of 3

Now

$$\frac{\cos z}{z^4} = \frac{1}{z^4}[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots]$$

$$= \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \ldots$$

So Laurent series expansion of given function is

$$\frac{\cos z}{z^4} = \frac{1}{z^4} - \frac{1}{2!z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \ldots \text{ ...... (2)}$$

Step 3 of 3

To find the Region of convergence of (2) first find the Radius of convergence of (1)

According to Cauchy-Hadamard formula Radius of Convergence of (1) is

$$R = \sqrt{\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

From the given series

$$a_n = \frac{(-1)^n}{(2n)!}$$

$$R = \sqrt{\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

$$= \sqrt{\lim_{n \to \infty} \left| \frac{\frac{(-1)^n}{(2n)!}}{\frac{(-1)^{n+1}}{(2n+2)!}} \right|}$$

$$= \infty$$

Radius of convergence of (1) is ∞

Hence Region of convergence of (2) is $0 < |z| < \infty$

Chapter 16.1, Problem 2P

Step-by-step solution

Step 1 of 3

Given function is

$f(z) = \frac{1}{z^2} \exp\left(\frac{-1}{z^2}\right)$

series expansion of e^z with centre $z = 0$ is

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \ldots$$

Step 2 of 3

Put $z = \frac{-1}{z^2}$ in above expansion we get

$$e^{\frac{-1}{z^2}} = 1 + \left(\frac{-1}{z^2}\right) + \frac{\left(\frac{-1}{z^2}\right)^2}{2!} + \frac{\left(\frac{-1}{z^2}\right)^3}{3!} + \ldots + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! z^{2n}} \quad \ldots\ldots (1)$$

$$\frac{e^{\frac{-1}{z^2}}}{z^2} = \frac{1}{z^2}\left[1 + \left(\frac{-1}{z^2}\right) + \frac{\left(\frac{-1}{z^2}\right)^2}{2!} + \frac{\left(\frac{-1}{z^2}\right)^3}{3!} + \ldots + \ldots\right]$$

$$= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{2! z^6} - \frac{1}{3! z^8} + \ldots \quad \ldots\ldots (2)$$

Step 3 of 3

So Laurent series expansion of given function is

$$\frac{e^{\frac{-1}{z^2}}}{z^2} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{2! z^6} - \frac{1}{3! z^8} + \ldots$$

To find the Region of convergence of (2) first find the Radius of convergence of (1)

Radius of convergence of (1) is $R = \infty$

Hence Region of convergence of (2) is $0 < |z| < \infty$

Chapter 16.1, Problem 3P

Step-by-step solution

Step 1 of 3

The objective is to expand $f(z)=\frac{e^{z^2}}{z^3}$ in a Laurent series that converges for $0<|z|<R$ and to determine the region of convergence.

The Taylor series expansion of e^x is:

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

Then expansion of e^{z^2} is:

$$\begin{aligned} e^{z^2} &= 1+\frac{z^2}{1!}+\frac{(z^2)^2}{2!}+\frac{(z^2)^3}{3!}+\frac{(z^2)^4}{4!}+\cdots \\ &= 1+z^2+\frac{1}{2}z^4+\frac{1}{6}z^6+\frac{1}{24}z^8\cdots \end{aligned}$$

Step 2 of 3

The Laurent series of $f(z)=\frac{e^{z^2}}{z^3}$ is:

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left[1+z^2+\frac{1}{2}z^4+\frac{1}{6}z^6+\frac{1}{24}z^8+\cdots\right] \\ &= \frac{1}{z^3}+\frac{z^2}{z^3}+\frac{1}{2}\frac{z^4}{z^3}+\frac{1}{6}\frac{z^6}{z^3}+\frac{1}{24}\frac{z^8}{z^3}+\cdots \\ &= \frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^5}{24}+\cdots \end{aligned}$$

Therefore, the Laurent series of $f(z)=\frac{e^{z^2}}{z^3}$ that converges for $0<|z|<R$ is:

$$\boxed{f(z)=\frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^5}{24}+\cdots}.$$

Step 3 of 3

To determine the radius of convergence use the Cauchy Hadamard formula $R=\sqrt{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$.

Take $a_n=\frac{1}{n!}$

Then

$$\begin{aligned} R &= \sqrt{\lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|} \\ &= \sqrt{\lim_{n\to\infty}\left|\frac{(n+1)n!}{n!}\right|} \\ &= \sqrt{\lim_{n\to\infty}|(n+1)|} \\ &= \infty \end{aligned}$$

The radius of convergence is $\boxed{\infty}$.

Hence, the region of convergence is $\boxed{0<|z|<\infty}$.

Therefore, the Laurent series of $f(z)=\frac{e^{z^2}}{z^3}$ that converges for $0<|z|<\infty$ is:

$$\boxed{f(z)=\frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^5}{24}+\cdots}.$$

Chapter 16.1, Problem 4P

Step-by-step solution

Step 1 of 4

Consider the function $f(z)=\frac{\sin \pi z}{z^2}$.

The object is to expand the function in a Laurent series that converges for $0<|z|<R$ and determine the precise region of convergence.

Step 2 of 4

Consider,

$$f(z)=\frac{\sin \pi z}{z^2}.$$

Know the expansion of $\sin x$ is

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$$

Taylor series expansion of $\sin \pi z$ with center $z=0$ is

$$\begin{aligned}\sin \pi z &= \pi z-\frac{\pi^3 z^3}{3!}+\frac{\pi^5 z^5}{5!}-\frac{\pi^7 z^7}{7!}+...\\ &=\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n+1} z^{2n+1}}{(2n+1)!} \quad \text{...... (1)}\end{aligned}$$

Step 3 of 4

Now,

$$\begin{aligned}\frac{\sin \pi z}{z^2} &= \frac{1}{z^2}\left[\pi z-\frac{\pi^3 z^3}{3!}+\frac{\pi^5 z^5}{5!}-\frac{\pi^7 z^7}{7!}+...\right]\\ &=\frac{\pi}{z}-\frac{\pi^3 z}{3!}+\frac{\pi^5 z^3}{5!}-\frac{\pi^7 z^5}{7!}+...\end{aligned}$$

So, Laurent series expansion of given function is

$$\frac{\sin \pi z}{z^2}=\frac{\pi}{z}-\frac{\pi^3 z}{3!}+\frac{\pi^5 z^3}{5!}-\frac{\pi^7 z^5}{7!}+... \quad \text{...... (2)}$$

Step 4 of 4

To find the Region of convergence of (2) first find the Radius of convergence of (1)

According to Cauchy-Hadamard formula Radius of Convergence of (1) is

$$R=\sqrt{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$

From the given series

$$a_n=\frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$$

$$\begin{aligned}R&=\sqrt{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}\\ &=\sqrt{\lim_{n\to\infty}\left|\frac{\frac{(-1)^n \pi^{2n+1}}{(2n+1)!}}{\frac{(-1)^{n+1} \pi^{2n+3}}{(2n+3)!}}\right|}\\ &=\infty\end{aligned}$$

Radius of convergence of (1) is ∞

Hence Region of convergence of (2) is $0<|z|<\infty$

Therefore, the Laurent series expansion of the given function near a singularity at 0 is

$$\frac{\pi}{z}-\frac{\pi^3 z}{3!}+\frac{\pi^5 z^3}{5!}-\frac{\pi^7 z^5}{7!}+...,\ 0<|z|<\infty.$$

Chapter 16.1, Problem 5P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z) = \frac{1}{z^2 - z^3}$$
$$= \frac{1}{z^2(1-z)}$$

series expansion of $\frac{1}{1-z}$ with centre $z = 0$ is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots + \ldots$$
$$= \sum_{n=0}^{\infty} z^n \quad \ldots\ldots (1)$$

Step 2 of 3

Now

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2}[1 + z + z^2 + z^3 + z^4 + \ldots + \ldots]$$
$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + z^4 + \ldots + \ldots$$

So Laurent series expansion of given function is

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + z^4 + \ldots + \ldots \quad \ldots\ldots (2)$$

Step 3 of 3

To find the Region of convergence of (2) first find the Radius of convergence of (1)

According to Cauchy-Hadamard formula Radius of Convergence of (1) is

$$R = \sqrt{\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$

From the given series

$$a_n = 1$$

$$R = \sqrt{\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|}$$
$$= \sqrt{\lim_{n \to \infty} \left| \frac{1}{1} \right|}$$
$$= 1$$

Radius of convergence of (1) is 1

Hence Region of convergence of (2) is $0 < |z| < 1$

Chapter 16.1, Problem 6P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z)=\frac{\sinh 2z}{z^2}$$

series expansion of $\sinh 2z$ with centre $z=0$ is

$$\sinh 2z = 2z+\frac{2^3 z^3}{3!}+\frac{2^5 z^5}{5!}+\frac{2^7 z^7}{7!}+...$$

$$=\sum_{n=0}^{\infty}\frac{2^{2n+1}z^{2n+1}}{(2n+1)!} \text{ ...... (1)}$$

Step 2 of 3

Now

$$\frac{\sinh 2z}{z^2}=\frac{1}{z^2}[2z+\frac{2^3 z^3}{3!}+\frac{2^5 z^5}{5!}+\frac{2^7 z^7}{7!}+...]$$

$$=\frac{2}{z}+\frac{2^3 z}{3!}+\frac{2^5 z^3}{5!}+\frac{2^7 z^5}{7!}+...$$

So Laurent series expansion of given function is

$$\frac{\sinh 2z}{z^2}=\frac{2}{z}+\frac{2^3 z}{3!}+\frac{2^5 z^3}{5!}+\frac{2^7 z^5}{7!}+... \text{ ...... (2)}$$

Step 3 of 3

To find the Region of convergence of (2) first find the Radius of convergence of (1)

According to Cauchy-Hadamard formula Radius of Convergence of (1) is

$$R=\sqrt{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$

From the given series

$$a_n=\frac{2^{2n+1}}{(2n+1)!}$$

$$R=\sqrt{\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|}$$

$$=\sqrt{\lim_{n\to\infty}\left|\frac{\frac{2^{2n+1}}{(2n+1)!}}{\frac{2^{2n+3}}{(2n+3)!}}\right|}$$

$$=\infty$$

Radius of convergence of (1) is ∞

Hence Region of convergence of (2) is $0<|z|<\infty$

Chapter 16.1, Problem 7P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z) = z^3 \cosh\left(\frac{1}{z}\right)$$

series expansion of $\cosh z$ with centre $z = 0$ is

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \ldots$$

Step 2 of 3

Put $z = \frac{1}{z}$ in above expansion we get

$$\cosh(\frac{1}{z}) = 1 + \frac{(\frac{1}{z})^2}{2!} + \frac{(\frac{1}{z^2})^4}{4!} + \ldots + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!z^{2n}} \quad \ldots\ldots \text{ (1)}$$

$$z^3 \cosh(\frac{1}{z}) = z^3[\left[1 + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z^2}\right)^4}{4!} + \ldots + \ldots\right]$$

$$= z^3 + \frac{z}{2!} + \frac{1}{4!z} + \frac{1}{6!z^3} + \ldots \quad \ldots\ldots \text{ (2)}$$

So Laurent series expansion of given function is

$$z^3 \cosh(\frac{1}{z}) = z^3 + \frac{z}{2!} + \frac{1}{4!z} + \frac{1}{6!z^3} + \ldots$$

Step 3 of 3

To find the Region of convergence of (2) first find the Radius of convergence of (1)

Radius of convergence of (1) is $R = \infty$

Hence Region of convergence of (2) is $0 < |z| < \infty$

Chapter 16.1, Problem 10P

Step-by-step solution

Step 1 of 2

Write the function,

$$f(z)=\frac{z^2-3i}{(z-3)^2}$$

The singularity is $z_0=3$

It is required to find the Laurent series that converges for $0<|z-3|<R$.

Step 2 of 2

Let $z-3=u$, then $z=u+3$.

Rewrite the function in terms of *u* as,

$$\begin{aligned}f(z)&=\frac{z^2-3i}{(z-3)^2}\\&=\frac{(u+3)^2-3i}{u^2}\\&=\frac{1}{u^2}(u^2+6u+9-3i)\\&=1+\frac{6}{u}+\frac{9-3i}{u^2} && \text{provided } |u|<\infty\\&=1+\frac{6}{z-3}+\frac{9-3i}{(z-3)^2} && (\text{since } z-3=u)\end{aligned}$$

Thus, the Laurent series expansion of $f(z)$ with respect to $z_0=3$ is,

$$\frac{z^2-3i}{(z-3)^2}=\boxed{1+\frac{6}{z-3}+\frac{9-3i}{(z-3)^2}}$$

Here the “annulus” of convergence is the whole complex plane without $z_0=3$, so the precise region of convergence is, $0<|z-3|<\infty$.

Chapter 16.1, Problem 11P

Step-by-step solution

Step 1 of 2

Consider the following function

$$f(z)=\frac{z^2}{(z-\pi i)^4},\quad z_0=\pi i$$

Its need to write the Laurent series expansion of $f(z)$ about the indicated point

On rewriting the function, we have that

$$f(z)=\frac{z^2}{(z-\pi i)^4}$$

$$=\frac{[\pi i+z-\pi i]^2}{(z-\pi i)^4}$$

$$=\frac{[\pi i+(z-\pi i)]^2}{(z-\pi i)^4}$$

$$=\frac{[\pi i+u]^2}{u^4}\text{, let } z-\pi i=u$$

$$=\frac{(\pi i)^2+u^2+2(\pi i)(u)}{u^4}$$

Step 2 of 2

Continuation to the above

$$\begin{aligned}f(z)&=\frac{(\pi i)^2+u^2+2(\pi i)(u)}{u^4}\\&=\frac{(\pi i)^2}{u^4}+\frac{u^2}{u^4}+\frac{2(\pi i)(u)}{u^4}\\&=\frac{(\pi i)^2}{u^4}+\frac{1}{u^2}+\frac{2(\pi i)}{u^3}\\&=\frac{(\pi i)^2}{u^4}+\frac{1}{u^2}+\frac{2\pi i}{u^3}\\&=\frac{(\pi i)^2}{(z-\pi i)^4}+\frac{1}{(z-\pi i)^2}+\frac{2\pi i}{(z-\pi i)^3}\quad\text{Replace } u \text{ with } z-\pi i\end{aligned}$$

Hence, Laurent series expansion of $f(z)$ is

$$\frac{z^2}{(z-\pi i)^4}=\boxed{\frac{(\pi i)^2}{(z-\pi i)^4}+\frac{1}{(z-\pi i)^2}+\frac{2\pi i}{(z-\pi i)^3}}$$

Chapter 16.1, Problem 12P

Step-by-step solution

Step 1 of 4

Given

$$f(z)=\frac{1}{z^2(z-i)} \quad \text{...... (1)}$$

And singularity is $z_0=i$

We have to find the Laurent series at specified singularity.

For this let $z-i=u$

Then $z=u+i$

Put this value in (1) we get

$$\begin{aligned} f(z)&=\frac{1}{z^2(z-i)}\\ &=\frac{1}{u(u+i)^2}\\ &=\frac{-1}{u(\frac{u}{i}+1)^2}\\ &=\frac{-1}{u(1-iu)^2} \end{aligned}$$

Step 2 of 4

We know that series expansion of $\frac{1}{(1-iu)^2}$ with centre $u=0$ is

$$\begin{aligned} \frac{1}{(1-iu)^2}&=(1-iu)^{-2}\\ &=1+2_{c_1}(iu)+3_{c_2}(iu)^2+4_{c_3}(iu)^3+...\\ &=1+2(iu)+3(iu)^2+4(iu)^3+... \end{aligned}$$

$$=\sum_{n=0}^{\infty}(n+1)i^n u^n \quad \text{...... (2)}$$

Therefore

$$\begin{aligned} \frac{-1}{u(1-iu)^2}&=\frac{-1}{u}[1+2(iu)+3(iu)^2+4(iu)^3+...]\\ &=\frac{-1}{u}-2i+3u+4iu^2+...\\ &=\frac{-1}{z-i}-2i+3(z-i)+4i(z-i)^2+... \end{aligned}$$

Step 3 of 4

Laurent series expansion of given function at given singularity is

$$\frac{1}{z^2(z-i)}=\frac{-1}{z-i}-2i+3(z-i)+4i(z-i)^2+... \quad \text{...... (3)}$$

To find the Region of Convergence of (3) first find the Radius of convergence of (2).From (2)

$$\begin{aligned} \frac{1}{(1-iu)^2}&=\sum_{n=0}^{\infty}(n+1)i^n u^n\\ &=\sum_{n=0}^{\infty}(n+1)i^n(z-i)^n \end{aligned}$$

Step 4 of 4

Here $a_n=(n+1)i^n$

According to Cauchy-Hadamard formula Radius of Convergence of above series is

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$\begin{aligned} R&=\lim_{n\to\infty}\left|\frac{(n+1)i^n}{(n+2)i^{n+1}}\right|\\ &=1 \end{aligned}$$

Hence Radius of convergence of (2) is 1

So Region of convergence of (3) is $0<|z-i|<1$

Chapter 16.1, Problem 13P

Step-by-step solution

Step 1 of 5

Consider the complex function

$f(z)=\frac{1}{z^3(z-i)^2}$ The singularity of the complex function $\frac{1}{z^3(z-i)^2}$ is $z_0=i$

The objective is to find the Laurent series of the function $\frac{1}{z^3(z-i)^2}$ at the singularity $z_0=i$.

Rewrite the complex function $f(z)=\frac{1}{z^3(z-i)^2}$ as below.

$$\begin{aligned}
f(z)&=\frac{1}{z^3(z-i)^2}\\
&=\frac{1}{(z-i+i)^3(z-i)^2}\\
&=\frac{1}{(z-i)^2}\left[\frac{1}{i^3\left(\frac{z-i}{i}+1\right)^3}\right]\\
&=\frac{1}{i^3(z-i)^2}\left[\left(\frac{z-i}{i}+1\right)^{-3}\right]
\end{aligned}$$

Step 2 of 5

Use binomial expansion, $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$, to expand the term $\left(1+\frac{z-i}{i}\right)^{-3}$.

$$\begin{aligned}
\left(1+\frac{z-i}{i}\right)^{-3}&=1+(-3)\left(\frac{z-i}{i}\right)+\frac{(-3)(-4)}{2!}\left(\frac{z-i}{i}\right)^2\\
&\quad+\frac{(-3)(-4)(-5)}{3!}\left(\frac{z-i}{i}\right)^3+....\\
&=1-\frac{3}{2(1!)}\left(\frac{z-i}{i}\right)+\frac{(2)(3)(4)}{2(2!)}\left(\frac{z-i}{i}\right)^2\\
&\quad-\frac{(2)(3)(4)(5)}{2(3!)}\left(\frac{z-i}{i}\right)^3+....\\
&=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)!}{2(n!)}\left(\frac{z-i}{i}\right)^n\\
&=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2}\left(\frac{z-i}{i}\right)^n
\end{aligned}$$

Step 3 of 5

The Laurent series of $f(z)=\frac{1}{z^3(z-i)^2}$ about the singular point $z_0=i$ is

$$\begin{aligned}
f(z)&=\frac{1}{z^3(z-i)^2}\\
&=\frac{1}{i^3(z-i)^2}\left[\left(\frac{z-i}{i}+1\right)^{-3}\right]\\
&=\frac{1}{i^3(z-i)^2}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2}\left(\frac{z-i}{i}\right)^n\right]\\
&=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}(z-i)^{n-2}\\
&=\frac{2}{2(i)^3}(z-i)^{-2}-\frac{6}{2(i)^4}(z-i)^{-1}+\frac{12}{2(i)^5}(z-i)^0-\frac{20}{2(i)^6}(z-i)+...\\
&=i(z-i)^{-2}-3(z-i)^{-1}-6i+10(z-i)+...
\end{aligned}$$

Hence, the Laurent series of the function $f(z)=\frac{1}{z^3(z-i)^2}$ is

$$\boxed{f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}(z-i)^{n-2}}.$$

Step 4 of 5

Use Cauchy-Hadamard formula for Radius of Convergence, $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, to find the radius of convergence of the series.

Consider the series,

$$f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}(z-i)^{n-2}$$

The n^{th} term is

$$a_n=(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}$$

Then the $(n+1)^{th}$ term is

$$a_{n+1}=(-1)^{n+1}\frac{(n+3)(n+2)}{2(i)^{n+4}}$$

Step 5 of 5

Find the ratio $\frac{a_n}{a_{n+1}}$.

$$\begin{aligned}
\frac{a_n}{a_{n+1}}&=(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}\left(\frac{2(i)^{n+4}}{(-1)^{n+1}(n+3)(n+2)}\right)\\
&=-\frac{(n+1)}{(n+3)}i
\end{aligned}$$

The radius of the convergence is

$$\begin{aligned}
R&=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\\
&=\lim_{n\to\infty}\left|-\frac{(n+1)}{(n+3)}i\right|\\
&=\lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)}\right|\\
&=1
\end{aligned}$$

Thus, the radius of convergence of $\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}(z-i)^{n-2}$ is $\boxed{R=1}$.

Hence, the Laurent series of $f(z)=\frac{1}{z^3(z-i)^2}$ that converges for $0<|z-z_0|<1(=R)$ is

$$\boxed{\begin{aligned}&f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2(i)^{n+3}}(z-i)^{n-2}\text{ or}\\&f(z)=i(z-i)^{-2}-3(z-i)^{-1}-6i+10(z-i)+...\end{aligned}}.$$

Chapter 16.2, Problem 1P

Step-by-step solution

Step 1 of 3

The objective is to find the zero of a function $f(z)$ and order as well.

$z = z_0$ is called zero of a function if $f(z_0) = 0$.

Order of the zero:

A zero $z = z_0$ has order n if $f(z_0), f'(z_0), f''(z_0), ..., f^{(n-1)}(z_0)$ are all zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$.

Step 2 of 3

Given function is,

$$f(z) = \sin^4\left(\frac{z}{2}\right)$$

To find zeros of a function,

Assume,

$$f(z) = 0$$
$$\sin^4(\frac{z}{2}) = 0$$

$z = 2n\pi,\quad$ n is an integer.

Therefore, $z = 0, \pm 2\pi, \pm 4\pi, ...$ are zeros of a given function.

Step 3 of 3

To find order,

Consider,

$$f(z) = \sin^4\left(\frac{z}{2}\right)$$
$$f'(z) = 4\sin^3\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right).\frac{1}{2}$$
$$f''(z) = 2\left\{3\sin^2\left(\frac{z}{2}\right)\cos^2\left(\frac{z}{2}\right).\frac{1}{2} - 2\sin^4\left(\frac{z}{2}\right).\frac{1}{2}\right\}$$
$$f'''(z) = 3\sin\left(\frac{z}{2}\right)\cos^3\left(\frac{z}{2}\right).\frac{1}{2} - 2.4\sin^3\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right).\frac{1}{2}$$
$$f^4(z) = \frac{3}{2}\left\{\cos^4\left(\frac{z}{2}\right).\frac{1}{2} - 3\sin^2\left(\frac{z}{2}\right)\cos^2\left(\frac{z}{2}\right).\frac{1}{2}\right\} - 4\left\{3\sin^2\left(\frac{z}{2}\right)\cos^2\left(\frac{z}{2}\right).\frac{1}{2} - \sin^4\left(\frac{z}{2}\right).\frac{1}{2}\right\}$$

According to definition of order of zero, these are zeros of fourth order since $f^4(z) \neq 0$.

Hence, order of the zero is four.

Chapter 16.2, Problem 2P

Step-by-step solution

Step 1 of 2

Consider the following complex expression:

$$\left(z^4-81\right)^3$$

The objective is to determine the location and order of the zeros.

Let $f(z)=\left(z^4-81\right)^3$

Rewrite the function as follows:

$$f(z)=\left[\left(\left(z^2\right)^2-9^2\right)\right]^3$$
$$f(z)=\left[\left(z^2-9\right)\left(z^2+9\right)\right]^3$$
$$f(z)=\left[(z-3)(z+3)(z+3i)(z-3i)\right]^3$$

Step 2 of 2

By definition of zero

$$f(z)=0$$

then

$$\left[(z-3)(z+3)(z+3i)(z-3i)\right]^3=0$$
$$\text{or}\,(z-3)(z+3)(z+3i)(z-3i)=0$$
$$\text{or}\; z=3,-3,3i,-3i$$

Also since f', f'', f''' are all zero for $z=\pm 3$ and $z=\pm 3i$, these are zeros of order three.

Chapter 16.2, Problem 3P

Step-by-step solution

Step 1 of 2

Zero of a function $f(z)$:

$z = z_0$ is called zero of a function if $f(z_0) = 0$.

Order of the zero:

A zero $z = z_0$ has order n if $f(z_0), f'(z_0), f''(z_0), \ldots, f^{(n-1)}(z_0)$ are all zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$.

Step 2 of 2

Given function is

$f(z) = (z + 81i)^4$

To find zeros of a function let

$$f(z) = 0$$
$$\Rightarrow (z + 81i)^4 = 0$$
$$\Rightarrow (z + 81i) = 0$$
$$\Rightarrow z = -81i$$

Therefore $-81i$ is zero of a given function

According to the definition of "order of zero", this is zero of order four.

Chapter 16.2, Problem 4P

Step-by-step solution

Step 1 of 2

Zero of a function $f(z)$:

$z = z_0$ is called zero of a function if $f(z_0) = 0$.

Order of the zero:

A zero $z = z_0$ has order n if $f(z_0), f'(z_0), f''(z_0), \ldots, f^{(n-1)}(z_0)$ are all zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$.

Step 2 of 2

Given function is

$f(z) = \tan^2 2z$

To find zeros of a function let

$f(z) = 0$

$\Rightarrow \tan^2 2z = 0$

$\Rightarrow \tan 2z = 0$

$\Rightarrow 2z = n\pi$

$\Rightarrow z = \dfrac{n\pi}{2}$, where n is an integer.

Therefore $z = 0, \pm\dfrac{\pi}{2}, \pm\pi, \pm\dfrac{3\pi}{2}, \ldots$ are zeros of a given function

According to the definition of "order of zero", these are zeros of order two.

Chapter 16.3, Problem 3P

Step-by-step solution

Step 1 of 2

Given function is

$$f(z) = \frac{\sin 2z}{z^6} \quad \ldots\ldots \text{ (1)}$$

Clearly it has a pole of order six at $z = 0$.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\operatorname{Res} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Step 2 of 2

So Residue of (1) at $z = 0$ is

$$\begin{aligned}\operatorname{Res} f(z) &= \frac{1}{5!} \lim_{z \to 0} \{\frac{d^5}{dz^5}[z^6 \frac{\sin 2z}{z^6}]\} \\ &= \frac{1}{5!} \lim_{z \to 0} \{2^5 \cos 2z\} \\ &= \frac{4}{15}\end{aligned}$$

Hence Residue of a given function at the singular point $z = 0$ is $\frac{4}{15}$.

Chapter 16.3, Problem 4P

Step-by-step solution

Step 1 of 2

Consider the function,

$$f(z)=\frac{\cos z}{z^4} \quad \text{...... (1)}$$

The objective is to determine all singularities in the finite plane, and the corresponding residues.

Step 2 of 2

Here, $z=0$ is the pole of order four.

The residue of $f(z)$ has a pole of order m at $z=z_0$ is,

$$\operatorname{Res} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

So, the residue of $f(z)$ has a pole of order 4 at $z=0$ is,

$$\begin{aligned}\operatorname{Res} f(z)&=\frac{1}{(4-1)!}\lim_{z\to 0}\left\{\frac{d^{4-1}}{dz^{4-1}}\left[(z-0)^4\frac{\cos z}{z^4}\right]\right\}\\ &=\frac{1}{3!}\lim_{z\to 0}\left\{\frac{d^3}{dz^3}\left[z^4\frac{\cos z}{z^4}\right]\right\}\\ &=\frac{1}{6}\lim_{z\to 0}\left\{\frac{d^3}{dz^3}[\cos z]\right\}\\ &=\frac{1}{6}\lim_{z\to 0}\left\{\frac{d^2}{dz^2}[-\sin z]\right\}\end{aligned}$$

$$=-\frac{1}{6}\lim_{z\to 0}\left\{\frac{d}{dz}[\cos z]\right\}$$

$$=\frac{1}{6}\lim_{z\to 0}[\sin z]$$

$$=\frac{1}{6}(0)$$

$$=0$$

Hence, residue of the function $f(z)$ at the singular point $z=0$ is $\boxed{0}$.

Chapter 16.3, Problem 5P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z)=\frac{8}{1+z^2} \quad \ldots\ldots (1)$$

Clearly it has simple poles at $z=i,-i$.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\operatorname{Res} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Step 2 of 3

Residue of (1) at $z=i$ is

$$\begin{aligned}\operatorname{Res} f(z)&=\frac{1}{0!}\lim_{z\to i}\{(z-i)\frac{8}{(z-i)(z+i)}\}\\&=\lim_{z\to i}\{\frac{8}{(z+i)}\}\\&=-4i\end{aligned}$$

Therefore Residue of (1) at $z=i$ is $-4i$.

Step 3 of 3

Residue of (1) at $z=-i$ is

$$\begin{aligned}\operatorname{Res} f(z)&=\frac{1}{0!}\lim_{z\to -i}\{(z+i)\frac{8}{(z-i)(z+i)}\}\\&=\lim_{z\to -i}\{\frac{8}{(z-i)}\}\\&=4i\end{aligned}$$

Therefore Residue of (1) at $z=-i$ is $4i$.

Chapter 16.3, Problem 6P

Step-by-step solution

Step 1 of 2

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The given function is

$$f(z) = \tan z$$

Step 2 of 2

$$f(z) = \tan z$$
$$= \frac{\sin z}{\cos z}$$

For poles denominator equals zero

$$\cos z = 0$$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\underset{z=\pm\pi/2,\pm 3\pi/2,\ldots\ldots}{\operatorname{Res}} f(z) = [\sin z]_{z=\pm\pi/2,\pm 3\pi/2,\ldots\ldots}$$
$$= -1$$

Alternatively

$$\underset{z\to z_0}{\operatorname{Res}} f(z) = \left[\frac{P(z)}{Q'(z)}\right]_{z=z_0}$$

So, that

$$\underset{z\to\pm\frac{\pi}{2},\ldots\ldots}{\operatorname{Res}} \frac{\sin z}{\cos z} = \left[\frac{\sin z}{-\sin z}\right]_{z=\pm\frac{\pi}{2},\pm\frac{3\pi}{2}\ldots\ldots}$$
$$= -1$$

Hence the solution is $\boxed{-1}$

Chapter 16.3, Problem 7P

Step-by-step solution

Step 1 of 2

Given function is

$f(z) = \cot \pi z$ ...... (1)

Clearly it has a simple poles at $z = n$ .where n is an integer.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\mathrm{Re}s\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \{ \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \}$$

Step 2 of 2

So Residue of (1) at $z = n$ is

$$\mathrm{Re}s\, f(z) = \frac{1}{0!} \lim_{z \to n} \{ (z-n) \frac{\cos \pi z}{\sin \pi z} \}$$

Use L'hospital's Rule to find this limit

$$\mathrm{Re}s\, f(z) = \lim_{z \to n} \{ \frac{\cos \pi z - \pi (z-n) \sin \pi z}{\pi \cos \pi z} \}$$
$$= \frac{1}{\pi}$$

Hence Residue of a given function at the singular point $z = n$ is $\frac{1}{\pi}$.

Chapter 16.3, Problem 8P

Step-by-step solution

Step 1 of 3

Given function is

$$f(z)=\frac{\pi}{(z+1)^2(z-1)^2} \quad \ldots\ldots (1)$$

Clearly it has poles of order two at $z=1,-1$.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\operatorname{Res} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Step 2 of 3

Residue of (1) at $z=1$ is

$$\begin{aligned}\operatorname{Res} f(z)&=\frac{1}{1!}\lim_{z\to 1}\{\frac{d}{dz}[(z-1)^2\frac{\pi}{(z+1)^2(z-1)^2}]\}\\&=\lim_{z\to 1}\{\frac{-2\pi}{(z+1)^3}\}\\&=\frac{-\pi}{4}\end{aligned}$$

Therefore Residue of (1) at $z=1$ is $\frac{-\pi}{4}$.

Step 3 of 3

Residue of (1) at $z=-1$ is

$$\begin{aligned}\operatorname{Res} f(z)&=\frac{1}{1!}\lim_{z\to -1}\{\frac{d}{dz}[(z+1)^2\frac{\pi}{(z+1)^2(z-1)^2}]\}\\&=\lim_{z\to -1}\{\frac{-2\pi}{(z-1)^3}\}\\&=\frac{\pi}{4}\end{aligned}$$

Therefore Residue of (1) at $z=-1$ is $\frac{\pi}{4}$.

Chapter 16.3, Problem 9P

Step-by-step solution

Step 1 of 2

Given function is

$$f(z)=\frac{1}{1-e^{z}} \quad \ldots\ldots (1)$$

Clearly it has a simple poles at $z=2n\pi i$ .where n is an integer.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\mathrm{Res}\, f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Step 2 of 2

So Residue of (1) at $z=2n\pi i$ is

$$Res\, f(z)=\frac{1}{0!}\lim_{z\to 2n\pi i}\{(z-2n\pi i)\frac{1}{1-e^{z}}\}$$

Use L'hospital's Rule to find this limit

$$\begin{aligned}\mathrm{Res}\, f(z) &= \lim_{z\to 2n\pi i}\{\frac{1}{-e^{z}}\}\\ &= -1\end{aligned}$$

Hence Residue of a given function at the singular point $z=2n\pi i$ is -1

Chapter 16.3, Problem 12P

Step-by-step solution

Step 1 of 3

Consider the function $f(z)=e^{\frac{1}{1-z}}$ ...... (1)

The object is to find all singularities in the finite plane and the corresponding residues.

Step 2 of 3

Consider, $f(z)=e^{\frac{1}{1-z}}$.

Since $1-z=0$

Clearly $z=1$ is an essential singularity.

If $z=z_0$ is a singular point then the coefficient of $\frac{1}{z-z_0}$ in Laurent series expansion of $f(z)$ is called Residue of $f(z)$.

So, expand (1) in Laurent series about $z=1$.

Step 3 of 3

Let $z-1=u$

Then Laurent series expansion of (1) is

$$\begin{aligned}f(z)&=e^{\frac{1}{1-z}}\\&=e^{\frac{-1}{u}}\end{aligned}$$

Expansion of e^x is

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\begin{aligned}f(z)&=1-\frac{1}{u}+\frac{\frac{1}{u^2}}{2!}-\frac{\frac{1}{u^3}}{3!}+\ldots\\&=1-\frac{1}{u}+\frac{1}{2!u^2}-\frac{1}{3!u^3}+\ldots\\&=1-\frac{1}{(z-1)}+\frac{1}{2!(z-1)^2}-\frac{1}{3!(z-1)^3}+\ldots\end{aligned}$$

Here the Coefficient of $\frac{1}{(z-1)}$ is -1

Thus, $\text{Residue}\left(e^{\frac{1}{1-z}},z=1\right)=-1.$

Therefore, Residue of a given function is -1.

Chapter 16.4, Problem 1P

Step-by-step solution

Step 1 of 7

Given integral is

$$\int_0^{\pi}\frac{2d\theta}{k-\cos\theta} \;\ldots\ldots (1)$$

Now take the integral

$$\int_0^{2\pi}\frac{2d\theta}{k-\cos\theta} \;\ldots\ldots (2)$$

Clearly

$$\int_0^{\pi}\frac{2d\theta}{k-\cos\theta}=\frac{1}{2}\int_0^{2\pi}\frac{2d\theta}{k-\cos\theta} \;\ldots\ldots (3)$$

Convert the integral as a function of z

Step 2 of 7

Let $e^{i\theta}=z$

$$\begin{aligned}
&\Rightarrow ie^{i\theta}d\theta=dz\\
&\Rightarrow d\theta=\frac{dz}{iz}\\
&\Rightarrow d\theta=\frac{-i}{z}dz\\
&\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\\
&\Rightarrow \cos\theta=\frac{z+z^{-1}}{2}\\
&\qquad\quad=\frac{z^2+1}{2z}
\end{aligned}$$

Put all these values in (2), we get integrand as a function of z

Step 3 of 7

Here $\theta=0$ to 2π

$$\begin{aligned}
\int_0^{2\pi}\frac{2d\theta}{k-\cos\theta}&=\oint_C\frac{-2idz}{z(k-\frac{z^2+1}{2z})}\\
&=\oint_C\frac{4idz}{z^2-2kz+1}
\end{aligned}$$

Here C is unit circle $|z|=1$

Now we have to solve the integral

$$\oint_C\frac{4idz}{z^2-2kz+1} \;\ldots\ldots (4)$$

Step 4 of 7

We have to find the singularities of integrand.

$$\begin{aligned}
&z^2-2kz+1=0\\
&\Rightarrow z=\frac{2k\pm\sqrt{4k^2-4}}{2}\\
&\Rightarrow z=k\pm\sqrt{k^2-1}
\end{aligned}$$

So $k\pm\sqrt{k^2-1}$ are singularities of integrand.

Clearly $k\pm\sqrt{k^2-1}$ are simple poles.

$$\left|k-\sqrt{k^2-1}\right|<1$$

$$\left|k+\sqrt{k^2-1}\right|>1$$

So singularity $k-\sqrt{k^2-1}$ with in the contour $C:|z|=1$

Step 5 of 7

Find Residues at this singular point.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\operatorname{Res} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Here $f(z)=\dfrac{4i}{z^2-2kz+1}$

Residue of $f(z)$ at $k-\sqrt{k^2-1}$ is

$$\begin{aligned}
\operatorname{Res} f(z)&=\frac{1}{0!}\lim_{z\to k-\sqrt{k^2-1}}\{(z-k+\sqrt{k^2-1})\frac{4i}{(z-k-\sqrt{k^2-1})(z-k+\sqrt{k^2-1})}\}\\
&=\lim_{z\to k-\sqrt{k^2-1}}\{\frac{4i}{(z-k-\sqrt{k^2-1})}\}\\
&=\frac{-2i}{\sqrt{k^2-1}}
\end{aligned}$$

Therefore Residue of $f(z)$ at $z=k-\sqrt{k^2-1}$ is $\dfrac{-2i}{\sqrt{k^2-1}}$

Step 6 of 7

Now evaluate the integral (4) using Residue Theorem

$$\oint_C\frac{4idz}{z^2-2kz+1}=2\pi i\text{ (sum of residues)}$$

$$\oint_C\frac{4idz}{z^2-2kz+1}=2\pi i(\frac{-2i}{\sqrt{k^2-1}})$$

Therefore $\displaystyle\oint_C\frac{4idz}{z^2-2kz+1}=\frac{4\pi}{\sqrt{k^2-1}}$

$$\int_0^{2\pi}\frac{2d\theta}{k-\cos\theta}=\frac{4\pi}{\sqrt{k^2-1}}$$

Step 7 of 7

From (3)

$$\begin{aligned}
\int_0^{\pi}\frac{2d\theta}{k-\cos\theta}&=\frac{1}{2}\int_0^{2\pi}\frac{2d\theta}{k-\cos\theta}\\
&=\frac{2\pi}{\sqrt{k^2-1}}
\end{aligned}$$

Therefore

$$\boxed{\int_0^{\pi}\frac{2d\theta}{k-\cos\theta}=\frac{2\pi}{\sqrt{k^2-1}}}$$

Chapter 16.4, Problem 3P

Step-by-step solution

Step 1 of 6

Given integral is

$$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta \text{ ...... (1)}$$

Convert the integral as a function of z

Let $e^{i\theta}=z$

$$\Rightarrow ie^{i\theta}d\theta=dz$$

$$\Rightarrow d\theta=\frac{dz}{iz}$$

$$\Rightarrow d\theta=\frac{-i}{z}dz$$

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\Rightarrow \cos\theta=\frac{z+z^{-1}}{2}$$

$$=\frac{z^2+1}{2z}$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$=\frac{z^2-1}{2iz}$$

Step 2 of 6

Put all these values in (1), we get integrand as a function of z

Here $\theta=0$ to 2π

$$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta=\oint_C\frac{1+\frac{z^2-1}{2iz}}{3+\frac{z^2+1}{2z}}\left(\frac{-i}{z}dz\right)$$

$$=-\oint_C\frac{z^2+2iz-1}{z(z^2+6z+1)}dz$$

Here C is unit circle $|z|=1$

Now we have to solve the integral

$$-\oint_C\frac{z^2+2iz-1}{z(z^2+6z+1)}dz \text{ ...... (2)}$$

Step 3 of 6

We have to find the singularities of integrand.

$$z(z^2+6z+1)=0$$

$$\Rightarrow z=0, z=\frac{-6\pm\sqrt{36-4}}{2}$$

$$\Rightarrow z=0,-3\pm\sqrt{8}$$

So $z=0,-3\pm\sqrt{8}$ are singularities of integrand.

Clearly $z=0,-3\pm\sqrt{8}$ are simple poles.

$$|0|<1$$

$$\left|-3+\sqrt{8}\right|<1$$

So singularities $z=0,-3+\sqrt{8}$ lie within the contour $C:|z|=1$

Step 4 of 6

Find Residues at this singular point.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\text{Res}\, f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right\}$$

Here $f(z)=\dfrac{-(z^2+2iz-1)}{z(z^2+6z+1)}$

Residue of $f(z)$ at $z=0$ is

$$\text{Res}\, f(z)=\frac{1}{0!}\lim_{z\to 0}\left\{(z-0)\frac{-(z^2+2iz-1)}{z(z^2+6z+1)}\right\}$$

$$=\lim_{z\to 0}\left\{\frac{-(z^2+2iz-1)}{(z^2+6z+1)}\right\}$$

$$=1$$

Therefore Residue of $f(z)$ at $z=0$ is 1

Step 5 of 6

Residue of $f(z)$ at $z=-3+\sqrt{8}$ is

$$\text{Res}\, f(z)=\frac{1}{0!}\lim_{z\to -3+\sqrt{8}}\left\{(z+3-\sqrt{8})\frac{-(z^2+2iz-1)}{z(z+3-\sqrt{8})(z+3+\sqrt{8})}\right\}$$

$$=\lim_{z\to -3+\sqrt{8}}\left\{\frac{-(z^2+2iz-1)}{z(z+3+\sqrt{8})}\right\}$$

$$=\lim_{z\to -3+\sqrt{8}}\left\{\frac{-((-3+\sqrt{8})^2+2i(-3+\sqrt{8})-1)}{(-3+\sqrt{8})(2\sqrt{8})}\right\}$$

$$=\frac{-(16-6\sqrt{8}-i(6-2\sqrt{8}))}{-6\sqrt{8}+16}$$

Now evaluate the integral (2) using Residue Theorem

$$-\oint_C\frac{z^2+2iz-1}{z(z^2+6z+1)}dz=2\pi i \text{ (sum of residues)}$$

$$-\oint_C\frac{z^2+2iz-1}{z(z^2+6z+1)}dz=2\pi i\left(1-\frac{-(16-6\sqrt{8}-i(6-2\sqrt{8}))}{-6\sqrt{8}+16}\right)$$

$$=2\pi i\left(\frac{i(6-2\sqrt{8})}{16-6\sqrt{8}}\right)$$

$$=\frac{-\pi(6-2\sqrt{8})}{8-3\sqrt{8}}$$

$$=\frac{\pi}{\sqrt{2}}\left(\frac{2\sqrt{16}-6\sqrt{2}}{8-3\sqrt{8}}\right)$$

$$=\frac{\pi}{\sqrt{2}}\left(\frac{8-3\sqrt{8}}{8-3\sqrt{8}}\right)$$

$$=\frac{\pi}{\sqrt{2}}$$

Step 6 of 6

Therefore $-\oint_C\dfrac{z^2+2iz-1}{z(z^2+6z+1)}dz=\dfrac{\pi}{\sqrt{2}}$

$$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta=\frac{\pi}{\sqrt{2}}$$

Therefore

$$\int_0^{2\pi}\frac{1+\sin\theta}{3+\cos\theta}d\theta=\frac{\pi}{\sqrt{2}}$$

Chapter 16.4, Problem 4P

Step-by-step solution

Step 1 of 5

303-16.4-8P

The given function is

$$\int_0^{2\pi} \frac{1+4\cos\theta}{17-8\cos\theta}\,d\theta$$

Step 2 of 5

Here,

$$\begin{aligned}\cos\theta &= \frac{z+\frac{1}{z}}{2}\\ &= \frac{z^2+1}{2z}\end{aligned}$$

Then,

$$\begin{aligned}\int_0^{2\pi} \frac{1+4\cos\theta}{17-8\cos\theta}\,d\theta &= \oint_c \frac{1+4\left(\frac{z^2+1}{2z}\right)}{17-8\left(\frac{z^2+1}{2z}\right)}\frac{dz}{iz}\\ &= \oint_c \frac{1+2\left(\frac{z^2+1}{z}\right)}{17-4\left(\frac{z^2+1}{z}\right)}\frac{dz}{iz}\\ &= \oint_c \frac{z+2z^2+2}{\left(17z-4z^2-4\right)}\frac{dz}{iz}\\ &= \frac{-1}{i}\oint_c \frac{2z^2+z+2}{z\left(4z^2-17z+4\right)}dz\\ &= \frac{-1}{4i}\oint_c \frac{\left(2z^2+z+2\right)}{z\left(z-\frac{1}{4}\right)(z-4)}dz\end{aligned}$$

Here, $z=0,\frac{1}{4},4$ are simple poles of given function $f(z)$.

The poles $z=0$ and $z=\frac{1}{4}$ only lies inside the unit circle $|z|=1$

Step 3 of 5

For $z=0$ poles, we get,

$$\begin{aligned}\operatorname*{Res}_{z=0} f(z) &= \frac{-1}{4i}\left(\frac{2}{\left(-\frac{1}{4}\right)(-4)}\right)\\ &= \frac{-2}{i(4)}\\ &= \frac{-2}{4i}\\ &= \frac{-1}{2i}\end{aligned}$$

Step 4 of 5

$$\begin{aligned}\operatorname*{Res}_{z=4} f(z) &= \frac{-1}{4i}\left[\frac{2\times 4^2+4+2}{4\left(4-\frac{1}{4}\right)}\right]\\ &= \frac{-1}{4i}\left[\frac{32+6}{4\times \frac{15}{4}}\right]\\ &= \frac{-1}{4i}\left[\frac{38}{15}\right]\\ &= \frac{-38}{15\times 4i}\\ &= \frac{-19}{30i}\end{aligned}$$

Step 5 of 5

And For $z=\frac{1}{4}$ poles, we get,

$$\begin{aligned}\operatorname*{Res}_{z=\frac{1}{4}} f(z) &= \frac{-1}{4i}\left(\frac{2\times \frac{1}{16}+\frac{1}{4}+2}{\frac{1}{4}\left(\frac{1}{4}-4\right)}\right)\\ &= \frac{-1}{4i}\left[\frac{\frac{1}{8}+\frac{1}{4}+2}{\frac{1}{4}\left(-\frac{15}{4}\right)}\right]\\ &= \frac{-1}{4i}\frac{\left[\frac{1+2+16}{8}\right]}{-\frac{15}{16}}\\ &= \frac{-1}{4i}\times\frac{16}{-15}\times\frac{19}{8}\\ &= \frac{38}{4i\times 15}\end{aligned}$$

$z=\frac{1}{4}$ is inside the circle and $z=4$ out side the circle

Then,

$$\begin{aligned}\sum R^{*} &= -\frac{1}{2i}+\frac{38}{4i\times 15}\\ &= \frac{-1}{2i}+\frac{19}{30i}\\ &= \frac{-15+18+1}{30i}\\ &= \frac{19-15}{30i}\\ &= \frac{4}{30i}\end{aligned}$$

Then,

$$\begin{aligned}\oint \frac{1+4\cos\theta}{17-8\cos\theta}d\theta &= 2\pi i\left(\frac{4}{30i}\right)\\ &= \frac{4\pi}{15}\end{aligned}$$

The solution is $\boxed{\frac{4\pi}{15}}$

Chapter 16.4, Problem 5P

Step-by-step solution

Step 1 of 6

Given integral is

$$\int_0^{2\pi} \frac{\cos^2\theta}{5-4\cos\theta} d\theta \quad \ldots\ldots (1)$$

Convert the integral as a function of z

Let $e^{i\theta} = z$

$$\Rightarrow ie^{i\theta} d\theta = dz$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\Rightarrow d\theta = \frac{-i}{z} dz$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Rightarrow \cos\theta = \frac{z + z^{-1}}{2}$$

$$= \frac{z^2 + 1}{2z}$$

Put all these values in (1), we get integrand as a function of z

Step 2 of 6

Here $\theta = 0$ to 2π

$$\int_0^{2\pi} \frac{\cos^2\theta}{5-4\cos\theta} d\theta = \oint_C \frac{\left(\frac{z^2+1}{2z}\right)^2}{5-4\left(\frac{z^2+1}{2z}\right)} \left(\frac{-i}{z} dz\right)$$

$$= \frac{i}{4} \oint_C \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} dz$$

Here C is unit circle $|z| = 1$

Step 3 of 6

Now we have to solve the integral

$$\frac{i}{4} \oint_C \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} dz \quad \ldots\ldots (2)$$

We have to find the singularities of integrand.

$$z^2(2z-1)(z-2) = 0$$

$$\Rightarrow z = 0, \frac{1}{2}, 2$$

So $z = 0, \frac{1}{2}, 2$ are singularities of integrand.

Clearly $z = \frac{1}{2}, 2$ are simple poles, and $z = 0,$ is pole of order two.

$$|0| < 1$$

$$\left|\frac{1}{2}\right| < 1$$

So singularities $z = 0, \frac{1}{2}$ lie with in the contour $C : |z| = 1$

Step 4 of 6

Find Residues at this singular point.

Residue of $f(z)$ at a m th-order pole at z_0 is

$$\mathrm{Res}\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}$$

Here $f(z) = \frac{(z^2+1)^2}{z^2(2z-1)(z-2)}$

Residue of $f(z)$ at $z = 0$ is

$$\mathrm{Res}\, f(z) = \frac{1}{1!} \lim_{z \to 0} \left\{ \frac{d}{dz} \left(z^2 \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} \right) \right\}$$

$$= \lim_{z \to 0} \left\{ \frac{d}{dz} \left(\frac{(z^2+1)^2}{(2z-1)(z-2)} \right) \right\}$$

$$= \lim_{z \to 0} \left(\frac{(2z^2-5z+2)(4z^3+4z) - (z^2+1)^2(4z-5)}{(2z^2-5z+2)^2} \right)$$

$$= \frac{5}{4}$$

Therefore Residue of $f(z)$ at $z = 0$ is $\frac{5}{4}$

Step 5 of 6

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\mathrm{Res}\, f(z) = \frac{1}{0!} \lim_{z \to \frac{1}{2}} \left\{ (z - \frac{1}{2}) \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} \right\}$$

$$= \lim_{z \to \frac{1}{2}} \left\{ \left(\frac{(z^2+1)^2}{2z^2(z-2)} \right) \right\}$$

$$= \frac{-25}{12}$$

Step 6 of 6

Now evaluate the integral (2) using Residue Theorem

$$\oint_C \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} dz = 2\pi i \text{ (sum of residues)}$$

$$\frac{i}{4} \oint_C \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} dz = 2\pi i \left(\frac{5}{4} - \frac{25}{12}\right) \frac{i}{4}$$

$$= \frac{5\pi}{12}$$

Therefore $\frac{i}{4} \oint_C \frac{(z^2+1)^2}{z^2(2z-1)(z-2)} dz = \frac{5\pi}{12}$

$$\int_0^{2\pi} \frac{\cos^2\theta}{5-4\cos\theta} d\theta = \frac{5\pi}{12}$$

Therefore

$$\int_0^{2\pi} \frac{\cos^2\theta}{5-4\cos\theta} d\theta = \frac{5\pi}{12}$$

Chapter 16.4, Problem 6P

Step-by-step solution

Step 1 of 7

Consider the integral $\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta$

The objective is to evaluate the trigonometric integral.

If $z=e^{i\theta}$, then

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-z^{-1}}{2i},$$

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z+z^{-1}}{2}$$

$$d\theta=\frac{dz}{iz}$$

Step 2 of 7

The integrand is transformed as follows:

$$\frac{\sin^2\theta}{5-4\cos\theta}=\frac{\left(\frac{z-z^{-1}}{2i}\right)^2}{5-4\left(\frac{z+z^{-1}}{2}\right)}$$

$$=\frac{\left(\frac{z^2-1}{2iz}\right)^2}{5-4\left(\frac{z^2+1}{2z}\right)}$$

$$=\frac{\left(z^2-1\right)^2}{-4z^2}\times\frac{z}{5z-2z^2-2}$$

$$=\frac{\left(z^2-1\right)^2}{8z\left(z-\frac{1}{2}\right)(z-2)}$$

Step 3 of 7

Now rewrite the integral as the following contour integral

$$\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta=\int_0^{2\pi}\frac{\left(z^2-1\right)^2}{8z\left(z-\frac{1}{2}\right)(z-2)}\frac{dz}{iz}$$

$$=\frac{-i}{8}\oint_C\frac{\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z-2)}dz\qquad\left(\text{since }\frac{1}{i}=-i\right)$$

Use the following **Cauchy Residue theorem**:

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of singular points $z_1,z_2,\ldots,z_n$ within C, then $\oint_C f(z)dz=2\pi i\sum_{k=1}^{n}\operatorname{Res}\left(f(z),z_k\right)$

Recall the **following**:

If f has a pole of order n at $z=z_0$, then $\operatorname{Res}\left(f(z),z_0\right)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right)$

Residue at simple Pole $z=z_0$: $\operatorname{Res}\left(f(z),z_0\right)=\lim_{z\to z_0}(z-z_0)f(z)$

Step 4 of 7

Let $f(z)=\frac{\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z-2)}$

Clearly, f is analytic everywhere except at $z=0,1/2,2$ which are the zeroes of the denominator.

Here $z=0$ is a pole of order 2 (number of times repeated) and $2,1/2$ are simple poles

The only poles that lie inside c (unit circle) are $z=0,1/2$

Calculate the residue at $z=0$ as follows:

$$\operatorname{Res}\left(f(z),0\right)=\frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}z^2\frac{\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z-2)}$$

$$=\lim_{z\to 0}\frac{d}{dz}\left\{\frac{\left(z^2-1\right)^2}{\left(z-\frac{1}{2}\right)(z-2)}\right\}$$

$$=\lim_{z\to 0}\frac{d}{dz}\left\{\frac{2z^4-4z^2+2}{2z^2-5z+2}\right\}$$

$$=\lim_{z\to 0}\frac{\left(2z^2-5z+2\right)\left(8z^3-8z\right)-\left(2z^4-4z^2+2\right)(4z-5)}{\left(2z^2-5z+2\right)^2}$$

$$=\frac{(0+2)(0)-(0+2)(0-5)}{(0+2)^2}$$

$$=\frac{5}{2}$$

Step 5 of 7

Calculate the residue at $z=1/2$ as follows:

$$\operatorname{Res}\left(f(z),1/2\right)=\lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\frac{\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z+2)}$$

$$=\lim_{z\to 1/2}\frac{\left(z^2-1\right)^2}{z^2(z-2)}$$

$$=\frac{\left(\frac{1}{4}-1\right)^2}{\frac{1}{4}\left(\frac{1}{2}-2\right)}$$

$$=\frac{-3}{2}$$

Step 6 of 7

By Cauchy Residue theorem,

$$\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta=\frac{-i}{8}2\pi i\left[\operatorname{Res}\left(f(z),0\right)+\operatorname{Res}\left(f(z),1/2\right)\right]$$

$$=-i^2\frac{\pi}{4}\left(\frac{5}{2}-\frac{3}{2}\right)$$

$$=\boxed{\frac{\pi}{4}}$$

The given function is

$$\int_0^{2\pi}\frac{\sin^2\theta}{5-4\cos\theta}d\theta$$

We know that,

$$\sin\theta=\frac{z-1/z}{2i}$$

And

$$\cos\theta=\frac{z+1/z}{2}$$

$$d\theta=\frac{d\theta}{iz}$$

$$\sin^2\theta=\left[\frac{z-\frac{1}{z}}{2i}\right]^2=\frac{-1}{4}\left(z-\frac{1}{z}\right)^2=\frac{-1}{4z^2}\left(z^2-1\right)^2$$

$$5-4\cos\theta=5-2\left(z+\frac{1}{z}\right)=\frac{-2z^2+5z-2}{z}$$

So that

$$\frac{\sin^2\theta}{5-4\cos\theta}=\frac{-1}{4z^2}\cdot\frac{\left(z^2-1\right)^2}{\left(-2z^2+5z-2\right)}\cdot z$$

$$=\frac{-1}{4z}\frac{\left(z^2-1\right)^2}{(-2)\left(z^2-\frac{5}{2}z+1\right)}$$

$$\frac{\sin^2\theta}{5-4\cos\theta}d\theta=\frac{-i}{8}\frac{\left(z^2-1\right)^2}{z^2(z-2)\left(z-\frac{1}{2}\right)}$$

Step 7 of 7

The poles of the function are $z=0, z=\frac{1}{2}$ and $z=2$ out of the these poles only two namely $z=0$ and $z=\frac{1}{2}$ lie inside the contour. The pole $z=0$ of second order and $z=\frac{1}{2}$ is a simple pole.

Now, Residue at $z=0$

$$=\lim_{z\to 0}\frac{d}{dz}\left[z^2\cdot\frac{\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z-2)}\right]$$

$$=\lim_{z\to 0}\frac{\left(z-\frac{1}{2}\right)(z-2)\cdot 2\left(z^2-1\right)\cdot 2z-\left(z^2-1\right)^2\cdot\left(2z-\frac{5}{2}\right)}{\left(z-\frac{1}{2}\right)^2(z-2)^2}$$

$$=\frac{5}{2}$$

And, Residue at $z=\frac{1}{2}$

$$=\lim_{z\to\frac{1}{2}}\left[\frac{\left(z-\frac{1}{2}\right)\left(z^2-1\right)^2}{z^2\left(z-\frac{1}{2}\right)(z-2)}\right]$$

$$=\left[\frac{\left(z^2-1\right)^2}{z^2(z-2)}\right]_{z=\frac{1}{2}}$$

$$=\frac{\frac{9}{16}}{\frac{1}{4}\left(\frac{-3}{2}\right)}=\frac{-3}{2}$$

Hence $\sum\operatorname{Res}=\frac{-i}{8}\left[\frac{5}{2}-\frac{3}{2}\right]=\frac{-i}{8}$

Therefore $\oint_C\frac{\sin^2\theta}{5-4\cos\theta}d\theta=2\pi i\left(\frac{-i}{8}\right)=\frac{\pi}{4}$

Chapter 16.4, Problem 7P

Step-by-step solution

Step 1 of 5

Given integral is

$$\int_0^{2\pi}\frac{a}{a-\sin\theta}d\theta \text{ ...... (1)}$$

Convert the integral as a function of z

Let $e^{i\theta}=z$

$$\Rightarrow ie^{i\theta}d\theta=dz$$

$$\Rightarrow d\theta=\frac{dz}{iz}$$

$$\Rightarrow d\theta=\frac{-i}{z}dz$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\Rightarrow \sin\theta=\frac{z-z^{-1}}{2i}$$

$$=\frac{z^2-1}{2iz}$$

Step 2 of 5

Put all these values in (1), we get integrand as a function of z

Here $\theta=0$ to 2π

$$\int_0^{2\pi}\frac{a}{a-\sin\theta}d\theta=\oint_C\frac{a}{a-(\frac{z^2-1}{2iz})^2}(\frac{-i}{z}dz)$$

$$=\oint_C\frac{-2a}{z^2-2aiz-1}dz$$

Here C is unit circle $|z|=1$

Now we have to solve the integral

$$\oint_C\frac{-2a}{z^2-2aiz-1}dz \text{ ...... (2)}$$

Step 3 of 5

We have to find the singularities of integrand.

$$z^2-2aiz-1=0$$

$$\Rightarrow z=\frac{2ai\pm\sqrt{4-4a^2}}{2}$$

$$=(a\pm\sqrt{a^2-1})i$$

So $z=(a\pm\sqrt{a^2-1})i$ are singularities of integrand.

Clearly $z=(a\pm\sqrt{a^2-1})i$ are simple poles

$$\left|(a-\sqrt{a^2-1})i\right|<1$$

$$\left|(a+\sqrt{a^2-1})i\right|>1$$

So singularities $z=(a-\sqrt{a^2-1})i$ lie with in the contour $C:|z|=1$

Step 4 of 5

Find Residues at this singular point.

Residue of $f(z)$ at a mth-order pole at z_0 is

$$\operatorname{Res} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\{\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\}$$

Here $f(z)=\frac{-2a}{z^2-2aiz-1}$

Residue of $f(z)$ at $z=(a-\sqrt{a^2-1})i$ is

$$\operatorname{Res} f(z)=\lim_{z\to(a-\sqrt{a^2-1})i}\{(z-(a-\sqrt{a^2-1})i)\frac{-2a}{(z-(a+\sqrt{a^2-1})i)(z-(a-\sqrt{a^2-1})i)})\}$$

$$=\lim_{z\to(a-\sqrt{a^2-1})i}\{\frac{-2a}{(z-(a+\sqrt{a^2-1})i)}\}$$

$$=\frac{-ia}{\sqrt{a^2-1}}$$

Therefore Residue of $f(z)$ at $z=a-\sqrt{a^2-1}$ is $\frac{-ia}{\sqrt{a^2-1}}$

Step 5 of 5

Now evaluate the integral (2) using Residue Theorem

$$\oint_C\frac{-2a}{z^2-2aiz-1}dz=2\pi i\text{(sum of residues)}$$

$$\oint_C\frac{-2a}{z^2-2aiz-1}dz=2\pi i(\frac{-ia}{\sqrt{a^2-1}})$$

$$=\frac{2\pi a}{\sqrt{a^2-1}}$$

Therefore $\oint_C\frac{-2a}{z^2-2aiz-1}dz=\frac{2\pi a}{\sqrt{a^2-1}}$

$$\int_0^{2\pi}\frac{a}{a-\sin\theta}d\theta=\frac{2\pi a}{\sqrt{a^2-1}}$$

Therefore $\int_0^{2\pi}\frac{a}{a-\sin\theta}d\theta=\frac{2\pi a}{\sqrt{a^2-1}}$

Chapter 16.4, Problem 8P

Step-by-step solution

Step 1 of 1

303-16.4-4P

The given equation is,

$$\int_0^{2\pi} \frac{d\theta}{8-2\sin\theta}$$

$$\text{Let} \sin\theta = \frac{\left(z-\frac{1}{z}\right)}{2i} \text{and } d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{8-2\sin\theta} = \oint_c \frac{dz}{iz\left[8-\frac{2\left(z-\frac{1}{z}\right)}{2i}\right]}$$

$$= \oint_c \frac{dz}{iz\left[8-\frac{\left(z^2-1\right)}{iz}\right]}$$

$$= \oint_c \frac{dz}{\left(8iz-z^2+1\right)}$$

$$= \oint_c \frac{-dz}{z^2-8iz-1}$$

Now for poles,

$$z^2-8iz-1=0$$

$$z = \frac{8i \pm \sqrt{-64+4}}{2}$$

$$= 4i \pm \frac{\sqrt{-60}}{2}$$

$$= 4i \pm \sqrt{15}.i$$

$$z = 4i+\sqrt{15}i$$

$$z = 4i-\sqrt{15}i$$

out of these two poles only $z=\left(4-\sqrt{15}\right)i$ lies inside the circle $|z|=1$

$$\oint_c \frac{-dz}{\left(z-4i-\sqrt{15}i\right)\left(z-4i+\sqrt{15}i\right)} = 2\pi i \sum R^+$$

Therefore,

$$R^+ = \operatorname*{Res}_{z=+4i-\sqrt{15}i}\left[\frac{-1}{\left(z-4i-\sqrt{15}i\right)\left(z-4i+\sqrt{15}i\right)}\right]$$

$$= \frac{-1}{-2\sqrt{15}i}$$

Then,

$$\int_0^{2\pi} \frac{d\theta}{8-2\sin\theta} = 2\pi i \frac{1}{2\sqrt{15}i}$$

$$= \frac{\pi}{\sqrt{15}}$$

Hence the solution is $\boxed{\frac{\pi}{\sqrt{15}}}$

Chapter 16.4, Problem 9P

Step-by-step solution

Step 1 of 2

303-16.4-7P

It is given that,

$$\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}d\theta$$

Step 2 of 2

$\because$ We know that

$$\cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right)$$
$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$
$$d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi}\frac{\cos\theta}{13-12\cos 2\theta} = \oint_c \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{13-\frac{12}{2}\left(z^2+\frac{1}{z^2}\right)}\frac{dz}{iz}$$
$$= \oint_c \frac{\frac{1}{2}\left(\frac{z^2+1}{z}\right)}{13-6\left(\frac{z^4+1}{z^2}\right)}\frac{dz}{iz}$$
$$= \oint_c \frac{\frac{1}{2}\left(\frac{z^2+1}{z}\right)}{\frac{13z^2-6z^4-6}{z^2}}\frac{dz}{iz}$$
$$= \oint_c \frac{1}{2i}\frac{dz\left(z^2+1\right)}{13z^2-6z^4-6}$$
$$= \frac{1}{2i}\oint_c \frac{-\left(z^2+1\right)}{6z^4-13z^2+6}$$
$$= \frac{1}{2i\times 6}\oint_c \frac{-\left(z^2+1\right)}{z^4-\frac{13}{6}z^2+1}$$
$$= \frac{1}{12i}\oint_c \frac{-\left(z^2+1\right)}{z^4-\frac{13}{6}z^2+1}$$

For poles

$$z^4-\frac{13}{6}z^2+1=0$$
$$z^2 = \frac{\frac{13}{6}\pm\sqrt{\frac{169}{36}-4}}{2}$$
$$= \frac{\frac{13}{6}\pm\frac{5}{6}}{2}$$
$$= \frac{13}{12}\pm\frac{5}{12}$$
$$= \frac{18}{12}, \frac{8}{12}$$
$$= \frac{3}{2}, \frac{2}{3}$$
$$z^2 = \frac{3}{2}, \frac{2}{3}$$
$$\Rightarrow z = \pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{2}{3}}$$

Out of these poles only $z=\pm\sqrt{\frac{2}{3}}$ lie inside the circle $|z|=1$

Then,

$$\frac{1}{12i}\oint_c \frac{-\left(z^2+1\right)}{\left(z^4-\frac{13}{6}z^2+1\right)} = \frac{1}{2i}\oint_c \frac{-\left(z^2+1\right)}{\left(z^2-\frac{3}{2}\right)\left(z^2-\frac{2}{3}\right)}dz$$
$$= \frac{1}{12i}\oint_c \frac{\left(z^2+1\right)}{\left(\frac{3}{2}-z^2\right)\left(z^2-\frac{2}{3}\right)}dz$$

$c: |z|=1$ Unit circle

We can write

$$\frac{1}{12i}\oint_c \frac{\left(z^2+1\right)}{\left(\frac{3}{2}-z^2\right)\left(z-\sqrt{\frac{2}{3}}\right)\left(z+\sqrt{\frac{2}{3}}\right)}dz$$
$$= 2\pi i\sum R^+$$

$$\sum R^+ = \frac{1}{12i}\left[\operatorname*{Res}_{z=\sqrt{\frac{2}{3}}}\frac{\left(\frac{2}{3}+1\right)}{\left(\frac{3}{2}-\frac{2}{3}\right)\left(\sqrt{\frac{2}{3}}+\sqrt{\frac{2}{3}}\right)}\operatorname*{Res}_{z=-\sqrt{\frac{2}{3}}}\frac{\left(\frac{2}{3}+1\right)}{\left(\frac{2}{3}-\frac{2}{3}\right)\left(-\sqrt{\frac{2}{3}}-\sqrt{\frac{2}{3}}\right)}\right]$$
$$= \frac{1}{12i}\left[\frac{\frac{5}{3}}{\left(\frac{5}{6}\right)\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)}-\frac{\frac{5}{3}}{\left(\frac{5}{6}\right)\left(2\sqrt{\frac{2}{3}}\right)}\right]=0$$

Then

$$\sum R^+ = 0$$
$$\int_0^{2\pi}\frac{\cos\theta}{13-12\cos 2\theta}=0$$

The solution is $\boxed{\int_0^{2\pi}\frac{\cos\theta}{13-12\cos 2\theta}=0}$