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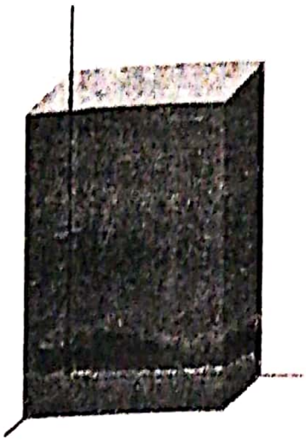
CHAPTER 5



Applications of Vectors in R^2 and R^3 (Optional)

5.1 CROSS PRODUCT IN R^3

Prerequisites. Section 4.1, Vectors in the Plane. Chapter 3.

In this section we discuss an operation that is meaningful only in R^3. Despite this limitation, it has many important applications in a number of different situations. We shall consider several of these applications in this section.

DEFINITION If

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are two vectors in R^3, then their **cross product** is the vector $\mathbf{u} \times \mathbf{v}$ defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \tag{1}$$

The cross product $\mathbf{u} \times \mathbf{v}$ can also be written as a "determinant,"

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{2}$$

The right side of (2) is not really a determinant, but it is convenient to think of the computation in this manner. If we expand (2) along the first row, we obtain

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k},$$

which is the right side of (1). Observe that the cross product $\mathbf{u} \times \mathbf{v}$ is a vector while the dot product $\mathbf{u} \cdot \mathbf{v}$ is a number.

EXAMPLE 1 Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Then expanding along the first row, we have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{vmatrix} = -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}.$$

■

(2)

Some of the algebraic properties of cross product are described in the following theorem. The proof, which follows easily from the properties of determinants, is left to the reader (Exercise T.1).

THEOREM 5.1 (**Properties of Cross Product**) *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^3 *and* c *is a scalar, then:*

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
(d) $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
(e) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
(f) $\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$
(g) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
(h) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$ ∎

EXAMPLE 2 It follows from (1) that

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}; \qquad \mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Also,

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

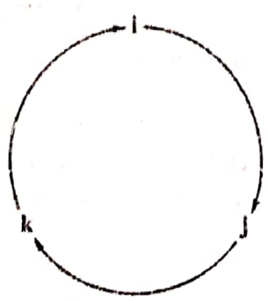


Figure 5.1 ▲

These rules can be remembered by the method indicated in Figure 5.1. Moving around the circle in a clockwise direction, we see that the cross product of two vectors taken in the indicated order is the third vector; moving in a counterclockwise direction, we see that the cross product taken in the indicated order is the negative of the third vector. The cross product of a vector with itself is the zero vector. ∎

Although many of the familiar properties of the real numbers hold for the cross product, it should be noted that two important properties do not hold. The commutative law does not hold, since $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. Also, the associative law does not hold, since $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$ while $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$.

We shall now take a closer look at the geometric properties of the cross product. First, we observe the following additional property of the cross product, whose proof we leave to the reader:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \qquad \text{Exercise T.2.} \tag{3}$$

It is also easy to show (Exercise T.4) that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{4}$$

EXAMPLE 3 Let $\mathbf{u}$ and $\mathbf{v}$ be as in Example 1, and let $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Then

$$\mathbf{u} \times \mathbf{v} = -\mathbf{i} + 12\mathbf{j} - 5\mathbf{k} \quad \text{and} \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 8,$$
$$\mathbf{v} \times \mathbf{w} = 3\mathbf{i} - 12\mathbf{j} + 7\mathbf{k} \quad \text{and} \quad \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 8,$$

which illustrates Equation (3). ∎

3

From the construction of $\mathbf{u} \times \mathbf{v}$, it follows that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$; that is,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0, \tag{5}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0. \tag{6}$$

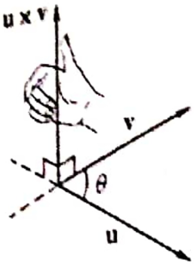


Figure 5.2 ▲

These equations can also be verified directly by using the definitions of $\mathbf{u} \times \mathbf{v}$ and dot product, or by using Equation (3) and properties (a) and (e) of the cross product (Theorem 5.1). Then $\mathbf{u} \times \mathbf{v}$ is also orthogonal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It can be shown that if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the direction of $\mathbf{u} \times \mathbf{v}$ is determined as follows. If we curl the fingers of the right hand in the direction of a rotation through the angle θ from $\mathbf{u}$ to $\mathbf{v}$, then the thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$ (Figure 5.2).

The magnitude of $\mathbf{u} \times \mathbf{v}$ can be determined as follows. From the definition of the length of a vector we have

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) & \\
&= \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{u} \times \mathbf{v})] & \text{by (3)} \\
&= \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{v})\mathbf{u} - (\mathbf{v} \cdot \mathbf{u})\mathbf{v}] & \text{by (g) of Theorem 5.1} \\
&= (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{u}) & \text{by (b), (c), and (d) of Theorem 4.3} \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 & \text{by (b) of Theorem 4.3 and the definition of length of a vector.}
\end{aligned}$$

From Equation (4) of Section 4.2 it follows that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Hence

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1 - \cos^2\theta) \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta.
\end{aligned}$$

Taking square roots, we obtain

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta. \tag{7}$$

Note that in (7) we do not have to write $|\sin\theta|$, since $\sin\theta$ is nonnegative for $0 \le \theta \le \pi$. It follows that vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ (Exercise T.5).

We now consider several applications of cross product.

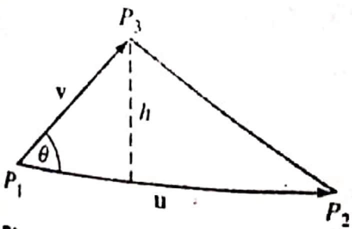


Figure 5.3 ▲

Area of a Triangle Consider the triangle with vertices P_1, P_2, P_3 (Figure 5.3). The area of this triangle is $\frac{1}{2}bh$, where b is the base and h is the height. If we take the segment between P_1 and P_2 to be the base and denote $\overrightarrow{P_1P_2}$ by the vector $\mathbf{u}$, then

$$b = \|\mathbf{u}\|.$$

Letting $\overrightarrow{P_1P_3} = \mathbf{v}$, we find that the height h is given by

$$h = \|\mathbf{v}\| \sin\theta.$$

Hence, by (7), the area A_T of the triangle is

$$A_T = \tfrac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\|.$$

(4)

EXAMPLE 4 Find the area of the triangle with vertices $P_1(2, 2, 4)$, $P_2(-1, 0, 5)$, and $P_3(3, 4, 3)$.

Solution We have

$$\mathbf{u} = \overrightarrow{P_1P_2} = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{v} = \overrightarrow{P_1P_3} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

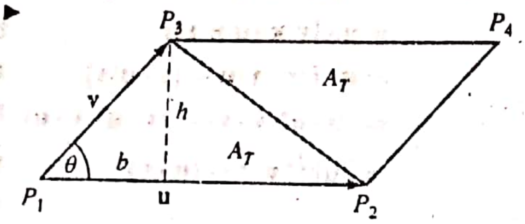
Then

$$A_T = \tfrac{1}{2}\|(-3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\|$$
$$= \tfrac{1}{2}\|-2\mathbf{j} - 4\mathbf{k}\| = \|-\mathbf{j} - 2\mathbf{k}\| = \sqrt{5}.$$ ■

Area of a Parallelogram The area A_P of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ (Figure 5.4) is $2A_T$, so

$$A_P = \|\mathbf{u} \times \mathbf{v}\|.$$

Figure 5.4 ►



EXAMPLE 5 If P_1, P_2, and P_3 are as in Example 4, then the area of the parallelogram with adjacent sides $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ is $2\sqrt{5}$. (Verify.) ■

Volume of a Parallelepiped Consider the parallelepiped with a vertex at the origin and edges $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (Figure 5.5). The volume V of the parallelepiped is the product of the area of the face containing $\mathbf{v}$ and $\mathbf{w}$ and the distance d from this face to the face parallel to it. Now

$$d = \|\mathbf{u}\|\,|\cos\theta|,$$

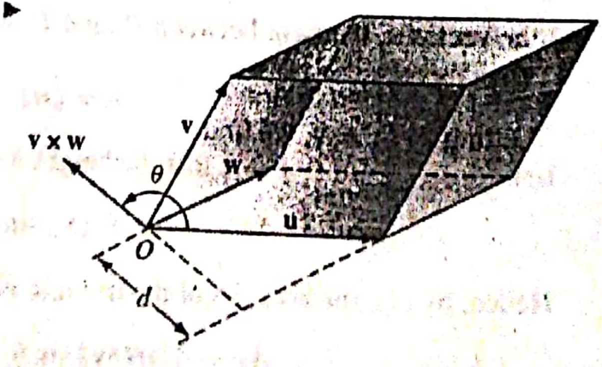
where θ is the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$, and the area of the face determined by $\mathbf{v}$ and $\mathbf{w}$ is $\|\mathbf{v} \times \mathbf{w}\|$. Hence

$$V = \|\mathbf{v} \times \mathbf{w}\|\,\|\mathbf{u}\|\,|\cos\theta| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \tag{8}$$

From Equations (3) and (4), we also have

$$V = \left|\det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right)\right|. \tag{9}$$

Figure 5.5 ►



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EXAMPLE 6 Consider the parallelepiped with a vertex at the origin and edges $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Then

$$\mathbf{v} \times \mathbf{w} = 5\mathbf{i} - 5\mathbf{k}.$$

Hence $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -10$. Thus the volume V is given by (8) as

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |-10| = 10.$$

We can also compute the volume by Equation (9) as

$$V = \left|\det\left(\begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}\right)\right| = |-10| = 10.$$

Key Terms

Cross product
Jacobi identity

5.1 Exercises

In Exercises 1 and 2, compute $\mathbf{u} \times \mathbf{v}$.

1. (a) $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
   (b) $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (2, 3, -1)$
   (c) $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$
   (d) $\mathbf{u} = (2, -1, 1)$, $\mathbf{v} = -2\mathbf{u}$
2. (a) $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (3, 1, 2)$
   (b) $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{k}$
   (c) $\mathbf{u} = 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{u}$
   (d) $\mathbf{u} = (4, 0, -2)$, $\mathbf{v} = (0, 2, -1)$
3. Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{w} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $c = -3$. Verify properties (a) through (d) of Theorem 5.1.
4. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.
   (a) Verify Equation (3).
   (b) Verify Equation (4).
5. Let $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
   (a) Verify Equation (3).
   (b) Verify Equation (4).
6. Verify that each of the cross products $\mathbf{u} \times \mathbf{v}$ in Exercise 1 is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
7. Verify that each of the cross products $\mathbf{u} \times \mathbf{v}$ in Exercise 2 is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
8. Verify Equation (7) for the pairs of vectors in Exercise 1.
9. Find the area of the triangle with vertices $P_1(1, -2, 3)$, $P_2(-3, 1, 4)$, $P_3(0, 4, 3)$.
10. Find the area of the triangle with vertices P_1, P_2, and P_3, where $\overrightarrow{P_1P_2} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\overrightarrow{P_1P_3} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.
11. Find the area of the parallelogram with adjacent sides $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$.
12. Find the volume of the parallelepiped with a vertex at the origin and edges $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$.
13. Repeat Exercise 12 for $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Theoretical Exercises

T.1. Prove Theorem 5.1.

T.2. Show that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

T.3. Show that $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

T.4. Show that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

T.5. Show that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

T.6. Show that $\|\mathbf{u} \times \mathbf{v}\|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2$.

T.7. Prove the **Jacobi identity**:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}.$$

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MATLAB Exercises

There are two MATLAB *routines that apply to the material in this section. They are* **cross**, *which computes the cross product of a pair of 3-vectors; and* **crossdemo**, *which displays graphically a pair of vectors and their cross product. Using routine* **dot** *with* **cross**, *we can carry out the computations in Example 6. (For directions on the use of* MATLAB *routines, use* **help** *followed by a space and the name of the routine.)*

ML.1. Use cross in MATLAB to find the cross product of each of the following pairs of vectors.

(a) $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

(b) $\mathbf{u} = (1, 0, 3)$, $\mathbf{v} = (1, -1, 2)$

(c) $\mathbf{u} = (1, 2, -3)$, $\mathbf{v} = (2, -1, 2)$

ML.2. Use routine cross to find the cross product of each of the following pairs of vectors.

(a) $\mathbf{u} = (2, 3, -1)$, $\mathbf{v} = (2, 3, 1)$

(b) $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{u}$

(c) $\mathbf{u} = (1, -2, 1)$, $\mathbf{v} = (3, 1, -1)$

ML.3. Use **crossdemo** in MATLAB to display the vectors $\mathbf{u}$, $\mathbf{v}$, and their cross product.

(a) $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{u} = (-2, 4, 5)$, $\mathbf{v} = (0, 1, -3)$

(c) $\mathbf{u} = (2, 2, 2)$, $\mathbf{v} = (3, -3, 3)$

ML.4. Use **cross** in MATLAB to check your answers to Exercises 1 and 2.

ML.5. Use MATLAB to find the volume of the parallelepiped with vertex at the origin and edges $\mathbf{u} = (3, -2, 1)$, $\mathbf{v} = (1, 2, 3)$, and $\mathbf{w} = (2, -1, 2)$.

ML.6. The angle of intersection of two planes in 3-space is the same as the angle of intersection of perpendiculars to the planes. Find the angle of intersection of plane P_1 determined by $\mathbf{x}$ and $\mathbf{y}$ and plane P_2 determined by $\mathbf{v}$, $\mathbf{w}$, where

$$\mathbf{x} = (2, -1, 2), \quad \mathbf{y} = (3, -2, 1),$$
$$\mathbf{v} = (1, 3, 1), \quad \mathbf{w} = (0, 2, -1).$$

5.2 LINES AND PLANES

Prerequisites. Section 4.1, Vectors in the Plane. Section 5.1, Cross Product in R^3.

LINES IN R^2

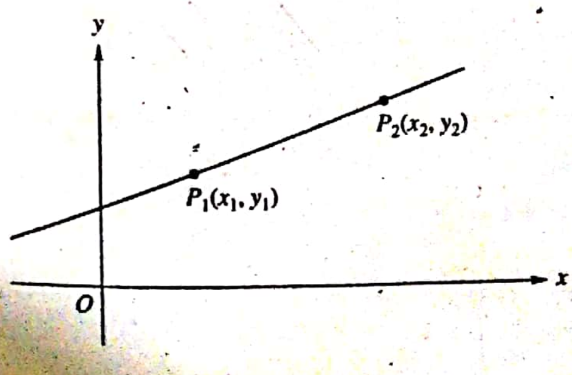
Any two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in R^2 (Figure 5.6) determine a straight line whose equation is

$$ax + by + c = 0, \tag{1}$$

where a, b, and c are real numbers, and a and b are not both zero. Since P_1 and P_2 lie on the line, their coordinates satisfy Equation (1):

$$ax_1 + by_1 + c = 0 \tag{2}$$
$$ax_2 + by_2 + c = 0. \tag{3}$$

Figure 5.6 ▶



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We now write (1), (2), and (3) as a linear system in the unknowns a, b, and c, obtaining

$$\begin{aligned} xa + yb + c &= 0 \\ x_1a + y_1b + c &= 0 \\ x_2a + y_2b + c &= 0. \end{aligned} \tag{4}$$

We seek a condition on the values x and y that allow (4) to have a nontrivial solution a, b, and c. Since (4) is a homogeneous system, it has a nontrivial solution if and only if the determinant of the coefficient matrix is zero, that is, if and only if

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{5}$$

Thus every point $P(x, y)$ on the line satisfies (5) and, conversely, a point satisfying (5) lies on the line.

EXAMPLE 1 Find an equation of the line determined by the points $P_1(-1, 3)$ and $P_2(4, 6)$.

Solution Substituting in (5), we obtain

$$\begin{vmatrix} x & y & 1 \\ -1 & 3 & 1 \\ 4 & 6 & 1 \end{vmatrix} = 0.$$

Expanding this determinant in cofactors along the first row, we have (verify)

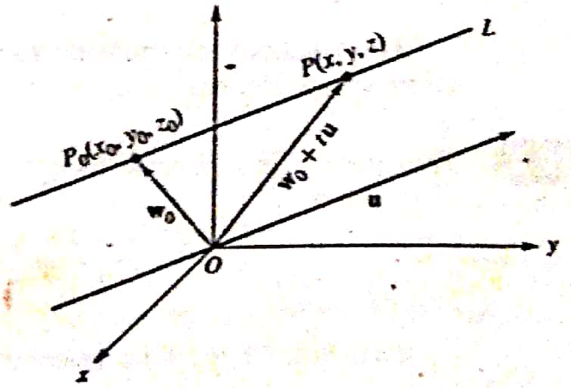
$$-3x + 5y - 18 = 0. \qquad \blacksquare$$

LINES IN R^3

We may recall that in R^2 a line is determined by specifying its slope and one of its points. In R^3 a line is determined by specifying its direction and one of its points. Let $\mathbf{u} = (a, b, c)$ be a nonzero vector in R^3, and let $P_0 = (x_0, y_0, z_0)$ be a point in R^3. Let $\mathbf{w}_0$ be the vector associated with P_0 and let $\mathbf{x}$ be the vector associated with the point $P(x, y, z)$. The line L through P_0 and parallel to $\mathbf{u}$ consists of the points $P(x, y, z)$ (Figure 5.7) such that

$$\mathbf{x} = \mathbf{w}_0 + t\mathbf{u}, \qquad -\infty < t < \infty. \tag{6}$$

Figure 5.7 ▶



Equation (6) is called a **parametric equation** of L, since it contains the parameter t, which can be assigned any real number. Equation (6) can also be written in terms of the components as

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \qquad -\infty < t < \infty \\ z &= z_0 + tc, \end{aligned} \tag{7}$$

which are called **parametric equations** of L.

EXAMPLE 2 Parametric equations of the line through the point $P_0(-3, 2, 1)$, which is parallel to the vector $\mathbf{u} = (2, -3, 4)$, are

$$\begin{aligned} x &= -3 + 2t \\ y &= 2 - 3t \qquad -\infty < t < \infty \\ z &= 1 + 4t. \end{aligned}$$

EXAMPLE 3 Find parametric equations of the line L through the points $P_0(2, 3, -4)$ and $P_1(3, -2, 5)$.

Solution The desired line is parallel to the vector $\mathbf{u} = \overrightarrow{P_0P_1}$. Now

$$\mathbf{u} = (3 - 2, -2 - 3, 5 - (-4)) = (1, -5, 9).$$

Since P_0 lies on the line, we can write parametric equations of L as

$$\begin{aligned} x &= 2 + t \\ y &= 3 - 5t \qquad -\infty < t < \infty \\ z &= -4 + 9t. \end{aligned}$$

In Example 3 we could have used the point P_2 instead of P_1. In fact, we could use any point on the line in parametric equations of L. Thus a line can be represented in infinitely many ways in parametric form. If a, b, and c are all nonzero in (7), we can solve each equation for t and equate the results to obtain equations in **symmetric form** of the line through P_0 and parallel to $\mathbf{u}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The equations in symmetric form of a line are useful in some analytic geometry applications.

EXAMPLE 4 The equations in symmetric form of the line in Example 3 are

$$\frac{x - 2}{1} = \frac{y - 3}{-5} = \frac{z + 4}{9}.$$

Exercises T.2 and T.3 consider the intersection of two lines in R^3.

(9)

PLANES IN R^3

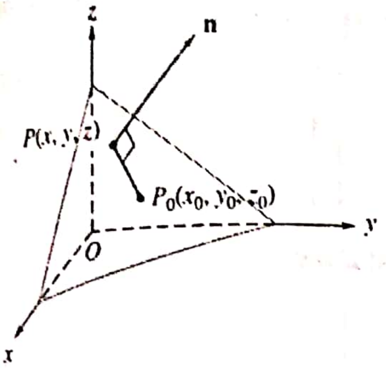


Figure 5.8 ▲

A plane in R^3 can be determined by specifying a point in the plane and a vector perpendicular to the plane. This vector is called a **normal** to the plane.

To obtain an equation of the plane passing through the point $P_0(x_0, y_0, z_0)$ and having the nonzero vector $\mathbf{n} = (a, b, c)$ as a normal, we proceed as follows. A point $P(x, y, z)$ lies in the plane if and only if the vector $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$ (Figure 5.8). Thus $P(x, y, z)$ lies in the plane if and only if

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0. \tag{8}$$

Since

$$\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0),$$

we can write (8) as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{9}$$

EXAMPLE 5 Find an equation of the plane passing through the point $(3, 4, -3)$ and perpendicular to the vector $\mathbf{n} = (5, -2, 4)$.

Solution Substituting in (9), we obtain the equation of the plane as

$$5(x - 3) - 2(y - 4) + 4(z + 3) = 0. \quad \blacksquare$$

If we multiply out and simplify, (9) can be rewritten as

$$ax + by + cz + d = 0. \tag{10}$$

It is not difficult to show (Exercise T.1) that the graph of an equation of the form given in (10), where a, b, c, and d are constants, is a plane with normal $\mathbf{n} = (a, b, c)$ provided a, b, and c are not all zero.

EXAMPLE 6 Find an equation of the plane passing through the points $P_1(2, -2, 1)$, $P_2(-1, 0, 3)$, and $P_3(5, -3, 4)$.

Solution Let an equation of the desired plane be as given by (10). Since P_1, P_2, and P_3 lie in the plane, their coordinates satisfy (10). Thus we obtain the linear system (verify)

$$\begin{aligned} 2a - 2b + c + d &= 0 \\ -a + 3c + d &= 0 \\ 5a - 3b + 4c + d &= 0. \end{aligned}$$

Solving this system, we have (verify)

$$a = \tfrac{8}{17}r, \qquad b = \tfrac{15}{17}r, \qquad c = -\tfrac{3}{17}r, \qquad d = r,$$

where r is any real number. Letting $r = 17$, we obtain

$$a = 8, \qquad b = 15, \qquad c = -3, \qquad d = 17.$$

Thus, an equation for the desired plane is

$$8x + 15y - 3z + 17 = 0. \tag{11}$$ ■

10

EXAMPLE 7 A second solution to Example 6 is as follows. Proceeding as in the case of a line in R^2 determined by two distinct points P_1 and P_2, it is not difficult to show (Exercise T.5) that an equation of the plane through the noncollinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

In our example, the equation of the desired plane is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & -2 & 1 & 1 \\ -1 & 0 & 3 & 1 \\ 5 & -3 & 4 & 1 \end{vmatrix} = 0.$$

Expanding this determinant in cofactors along the first row, we obtain (verify) Equation (11). ∎

EXAMPLE 8 A third solution to Example 6 using Section 5.1, Cross Product in R^3, is as follows. The nonparallel vectors $\overrightarrow{P_1P_2} = (-3, 2, 2)$ and $\overrightarrow{P_1P_3} = (3, -1, 3)$ lie in the plane, since the points P_1, P_2, and P_3 lie in the plane. The vector

$$\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (8, 15, -3)$$

is then perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ and is thus a normal to the plane. Using the vector $\mathbf{n}$ and the point $P_1(2, -2, 1)$ in (9), we obtain an equation of the plane as

$$8(x - 2) + 15(y + 2) - 3(z - 1) = 0,$$

which when expanded agrees with Equation (11). ∎

The equations of a line in symmetric form can be used to determine two planes whose intersection is the given line.

EXAMPLE 9 Find two planes whose intersection is the line

$$\begin{aligned} x &= -2 + 3t \\ y &= 3 - 2t \qquad -\infty < t < \infty \\ z &= 5 + 4t. \end{aligned}$$

Solution First, find equations of the line in symmetric form as

$$\frac{x+2}{3} = \frac{y-3}{-2} = \frac{z-5}{4}.$$

The given line is then the intersection of the planes

$$\frac{x+2}{3} = \frac{y-3}{-2} \quad \text{and} \quad \frac{x+2}{3} = \frac{z-5}{4}.$$

Thus the given line is the intersection of the planes

$$2x + 3y - 5 = 0 \quad \text{and} \quad 4x - 3z + 23 = 0.$$ ∎

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Two planes are either parallel or they intersect in a straight line. They are parallel if their normals are parallel. In the following example we determine the line of intersection of two planes.

EXAMPLE 10 Find parametric equations of the line of intersection of the planes

$$\pi_1: 2x + 3y - 2z + 4 = 0 \quad \text{and} \quad \pi_2: x - y + 2z + 3 = 0.$$

Solution Solving the linear system consisting of the equations of π_1 and π_2, we obtain (verify)

$$\begin{aligned} x &= -\tfrac{13}{5} - \tfrac{4}{5}t \\ y &= \tfrac{2}{5} + \tfrac{6}{5}t \qquad -\infty < t < \infty \\ z &= 0 + \phantom{\tfrac{4}{5}}t \end{aligned}$$

as parametric equations of the line L of intersection of the planes (see Figure 5.9). ■

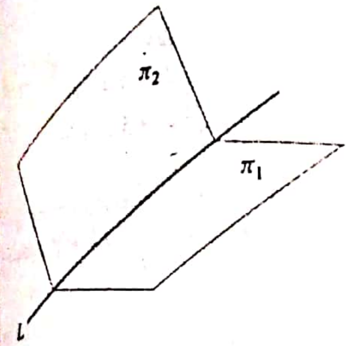

Figure 5.9 ▲

Three planes in R^3 may intersect in a plane, in a line, in a unique point, or have no set of points in common. These possibilities can be detected by solving the linear system consisting of their equations.

Key Terms

Parametric equation(s) of a line
Symmetric form of a line
Normal to a plane
Skew lines

5.2 Exercises

1. In each of the following, find an equation of the line in R^2 determined by the given points.
   (a) $P_1(-2, -3)$, $P_2(3, 4)$
   (b) $P_1(2, -5)$, $P_2(-3, 4)$
   (c) $P_1(0, 0)$, $P_2(-3, 5)$
   (d) $P_1(-3, -5)$, $P_2(0, 2)$

2. In each of the following, find the equation of the line in R^2 determined by the given points.
   (a) $P_1(1, 1)$, $P_2(2, 2)$
   (b) $P_1(1, 2)$, $P_2(1, 3)$
   (c) $P_1(2, -4)$, $P_2(-3, -4)$
   (d) $P_1(2, -3)$, $P_2(3, -2)$

3. State which of the following points are on the line
$$\begin{aligned} x &= 3 + 2t \\ y &= -2 + 3t \qquad -\infty < t < \infty \\ z &= 4 - 3t. \end{aligned}$$
   (a) $(1, 1, 1)$ (b) $(1, -1, 0)$
   (c) $(1, 0, -2)$ (d) $(4, -\frac{1}{2}, \frac{5}{2})$

4. State which of the following points are on the line
$$\frac{x-4}{-2} = \frac{y+3}{2} = \frac{z-4}{-5}.$$
   (a) $(0, 1, -6)$ (b) $(1, 2, 3)$
   (c) $(4, -3, 4)$ (d) $(0, 1, -1)$

5. In each of the following, find the parametric equations of the line through the point $P_0(x_0, y_0, z_0)$, which is parallel to the vector $\mathbf{u}$.
   (a) $P_0 = (3, 4, -2)$, $\mathbf{u} = (4, -5, 2)$
   (b) $P_0 = (3, 2, 4)$, $\mathbf{u} = (-2, 5, 1)$
   (c) $P_0 = (0, 0, 0)$, $\mathbf{u} = (2, 2, 2)$
   (d) $P_0 = (-2, -3, 1)$, $\mathbf{u} = (2, 3, 4)$

6. In each of the following, find the parametric equations of the line through the given points.
   (a) $(2, -3, 1)$, $(4, 2, 5)$ (b) $(-3, -2, -2)$, $(5, 5, 4)$
   (c) $(-2, 3, 4)$, $(2, -3, 5)$ (d) $(0, 0, 0)$, $(4, 5, 2)$

7. For each of the lines in Exercise 6, find the equations in symmetric form.

8. State which of the following points are on the plane $3(x - 2) + 2(y + 3) - 4(z - 4) = 0$.
   (a) $(0, -2, 3)$ (b) $(1, -2, 3)$
   (c) $(1, -1, 3)$ (d) $(0, 0, 4)$

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$\vec{u}_1 \cdot \vec{u}_2 = 0$

9. In each of the following, find an equation of the plane passing through the given point and perpendicular to the given vector $\mathbf{n}$.

(a) $(0, 2, -3)$, $\mathbf{n} = (3, -2, 4)$

(b) $(-1, 3, 2)$, $\mathbf{n} = (0, 1, -3)$

(c) $(-2, 3, 4)$, $\mathbf{n} = (0, 0, -4)$

(d) $(5, 2, 3)$, $\mathbf{n} = (-1, -2, 4)$

10. In each of the following, find an equation of the plane passing through the given three points.

(a) $(0, 1, 2)$, $(3, -2, 5)$, $(2, 3, 4)$

(b) $(2, 3, 4)$, $(-1, -2, 3)$, $(-5, -4, 2)$

(c) $(1, 2, 3)$, $(0, 0, 0)$, $(-2, 3, 4)$

(d) $(1, 1, 1)$, $(2, 3, 4)$, $(-5, 3, 2)$

11. In each of the following, find parametric equations of the line of intersection of the given planes.

(a) $2x + 3y - 4z + 5 = 0$ and $-3x + 2y + 5z + 6 = 0$

(b) $3x - 2y - 5z + 4 = 0$ and $2x + 3y + 4z + 8 = 0$

(c) $-x + 2y + z = 0$ and $2x - y + 2z + 8 = 0$

12. In each of the following, find a pair of planes whose intersection is the given line.

(a) $x = 2 - 3t$
$y = 3 + t$
$z = 2 - 4t$

(b) $\dfrac{x-2}{-2} = \dfrac{y-3}{4} = \dfrac{z+4}{3}$

(c) $x = 4t$
$y = 1 + 5t$
$z = 2 - t$

13. Are the points $(2, 3, -2)$, $(4, -2, -3)$, and $(0, 8, -1)$ on the same line?

14. Are the points $(-2, 4, 2)$, $(3, 5, 1)$, and $(4, 2, -1)$ on the same line?

15. Find the point of intersection of the lines

$$x = 2 - 3s \quad y = 3 + 2s \quad z = 4 + 2s \qquad \text{and} \qquad x = 5 + 2t \quad y = 1 - 3t \quad z = 2 + t.$$

16. Which of the following pairs of lines are perpendicular?

(a) $x = 2 + 2t$, $y = -3 - 3t$, $z = 4 + 4t$ and $x = 2 + t$, $y = 4 - t$, $z = 5 - t$

(b) $x = 3 - t$, $y = 4 + t$, $z = 2 + 2t$ and $x = 2t$, $y = 3 - 2t$, $z = 4 + 2t$

17. Show that the following parametric equations define the same line.

$\vec{u}_1 \parallel \vec{u}_2$ and P_1 satisfy L_2
P_2 " " L_1

$x = 2 + 3t$, $y = 3 - 2t$, $z = -1 + 4t$ and $x = -1 - 9t$, $y = 5 + 6t$, $z = -5 - 12t$

18. Find parametric equations of the line passing through the point $(3, -1, -3)$ and perpendicular to the line passing through the points $(3, -2, 4)$ and $(0, 3, 5)$.

19. Find an equation of the plane passing through the point $(-2, 3, 4)$ and perpendicular to the line passing through the points $(4, -2, 5)$ and $(0, 2, 4)$.

20. Find the point of intersection of the line

P_1P_0

$$x = 2 - 3t \quad y = 4 + 2t \quad z = 3 - 5t$$

$n_0 \cdot \vec{P_1P_0} = 0$

and the plane $2x + 3y + 4z + 8 = 0$.

21. Find a plane containing the lines

$$x = 3 + 2t \quad y = 4 - 3t \quad z = 5 + 4t \qquad \text{and} \qquad x = 1 - 2t \quad y = 7 + 4t \quad z = 1 - 3t.$$

22. Find a plane that passes through the point $(2, 4, -3)$ and is parallel to the plane $-2x + 4y - 5z + 6 = 0$.

23. Find a line that passes through the point $(-2, 5, -3)$ and is perpendicular to the plane $2x - 3y + 4z + 7 = 0$.

Theoretical Exercises

T.1. Show that the graph of the equation $ax + by + cz + d = 0$, where a, b, c, and d are constants with a, b, and c not all zero, is a plane with normal $\mathbf{n} = (a, b, c)$.

T.2. Let the lines L_1 and L_2 be given parametrically by

$$L_1 : \mathbf{x} = \mathbf{w}_0 + s\mathbf{u} \quad \text{and} \quad L_2 : \mathbf{x} = \mathbf{w}_1 + t\mathbf{v}.$$

Show that

(a) L_1 and L_2 are parallel if and only if $\mathbf{u} = k\mathbf{v}$ for some scalar k.

(b) L_1 and L_2 are identical if and only if $\mathbf{w}_1 - \mathbf{w}_0$ and $\mathbf{u}$ are both parallel to $\mathbf{v}$.

(c) L_1 and L_2 are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

(d) L_1 and L_2 intersect if and only if $\mathbf{w}_1 - \mathbf{w}_0$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

T.3. The lines L_1 and L_2 in R^3 are said to be skew if they are not parallel and do not intersect. Give an example of skew lines L_1 and L_2.

Exercise 5.2

Q1 — Q2 : Similar to EX-1 Page-7 (Book Page-265)

Q3 — Q4 : state which of the following points are on the line

Q.3: $x = 3+2t$
$y = -2+3t \quad -\infty < t < \infty$
$z = 4-3t$

a) (1,1,1) (S) In symmetric form, the given line is as

$$\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-4}{-3} \quad \text{— (1)}$$

For (1,1,1), (1) $\Rightarrow \frac{1-3}{2} = \frac{1+2}{3} = \frac{1-4}{-3} \Rightarrow -1 = 1 = 1$, which is invalid

Hence the given point is not on the given line.

Q5: similar to EX-2 Page-8 (Book Page-266)

Q6: similar to EX-3 Page-8 (Book Page-266)

Q7: similar to EX-4 Page-8 (Book Page-266)

Q8: state which of the following points are on the plane

$3(x-2)+2(y+2)-4(z-4) = 0$ — (1)

a) (0,-2,3), (1) $\Rightarrow 3(0-2)+2(-2+2)-4(3-4) = 0$

$\Rightarrow -6+0+4 = 0 \Rightarrow -2 = 0$ invalid.

Hence given point is not on the given plane.

Q9: Similar to EX-5 Page-9 (Book Page-267)

Q10: similar to EX-6 Page-9 (Book Page-267)

Q11: a) $2x+3y-4z+5=0$ — (1) and $-3x+2y+5z+6=0$ — (2)

$2\times$eq.(1) $- 3\times$eq.(2) $\Rightarrow 13x-23z-8=0 \Rightarrow x = \frac{8+23z}{13}$ — (3)

$3\times$eq.(1) $+ 2\times$eq.(2) $\Rightarrow 13y-2z+27=0 \Rightarrow y = \frac{-27+2z}{13}$ — (4)

Let $z = t$, — (5), where $t \in \mathbb{R}$. Then from (3), (4) and (5), we have

$x = \frac{8}{13} + \frac{23}{13}t$
$y = -\frac{27}{13} + \frac{2}{13}t$, $-\infty < t < \infty$ are the required parametric equations of the line of intersection of the given planes.
$z = t$

Exercise 5.2

Q12: Similar to Ex-9 Page-10 (Book Page-268).

Q13 – Q14 ⇒ Are the Points (2, 3, −2), (4, −2, −3) and (0, 8, −1) on the same line?

③ Let $P_1(2,3,-2)$, $P_2(4,-2,-3)$ and $P_3(0,8,-1)$ be the given Points, then

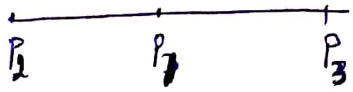
$$\overrightarrow{P_1P_2} = (4-2)\underline{i} + (-2-3)\underline{j} + (-3+2)\underline{k} \Rightarrow \overrightarrow{P_1P_2} = 2\underline{i} - 5\underline{j} - \underline{k}$$

$$\overrightarrow{P_2P_3} = -4\underline{i} + 10\underline{j} + 2\underline{k} \quad \text{and} \quad \overrightarrow{P_1P_3} = -2\underline{i} + 5\underline{j} + \underline{k}$$

Now $\|\overrightarrow{P_1P_2}\| = \sqrt{(2)^2 + (-5)^2 + (-1)^2} = \sqrt{4+25+1} = \sqrt{30} =$

$$\|\overrightarrow{P_2P_3}\| = \sqrt{16+100+4} = \sqrt{120} = 2\sqrt{30}$$

$$\|\overrightarrow{P_1P_3}\| = \sqrt{4+25+1} = \sqrt{30}$$



since $\|\overrightarrow{P_1P_2}\| + \|\overrightarrow{P_1P_3}\| = \|\overrightarrow{P_2P_3}\| \Rightarrow$ The given Points are on the same line

Q15, Q17, Q19, Q21 and Q23 ⟶ Do yourself.

Q16: which of the following pairs of lines are perpendiculars?

(a)
$$\left.\begin{aligned} x &= 2+2t \\ y &= -3-3t \\ z &= 4+4t \end{aligned}\right\} \text{—①} \quad \text{and} \quad \left.\begin{aligned} x &= 2+t \\ y &= 4-t \\ z &= 5-t \end{aligned}\right\} \leftarrow ②$$

vectors parallel to lines ① and ② are respectively as

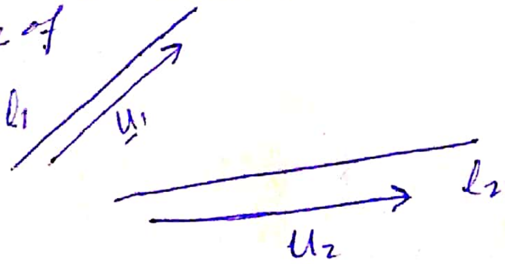
$$\underline{u}_1 = 2\underline{i} - 3\underline{j} + 4\underline{k} \quad \text{and} \quad \underline{u}_2 = \underline{i} - \underline{j} - \underline{k}$$

since $\underline{u}_1 \cdot \underline{u}_2 = (2)(1) + (-3)(-1) + 4(-1) = 2+3-4 = 1$

i.e. $\underline{u}_1 \cdot \underline{u}_2 \neq 0 \Rightarrow \underline{u}_1$ is not $\perp$ to $\underline{u}_2$

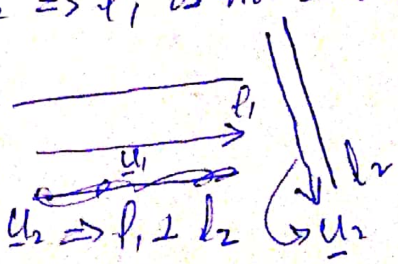
Hence lines are not perpendicular to each other

in case of not ⊥



$\underline{u}_1$ is not $\perp$ $\underline{u}_2 \Rightarrow l_1$ is not $\perp$ to l_2

In case of ⊥



$\Rightarrow \underline{u}_1 \perp \underline{u}_2 \Rightarrow l_1 \perp l_2$

Exercise 5.2

Q19: vector Parallel to the given line passing through the Points (3,-2,4) and (0,3,5) is given as

$$\underline{u} = [0-3 \quad 3-(-2), \quad 5-4] = [-3 \quad 5 \quad 1] = -3\underline{i}+5\underline{j}+\underline{k}$$

Let $\underline{v} = [a \ b \ c] = a\underline{i}+b\underline{j}+c\underline{k}$ be vector Parallel to the desired line passing through the Point (3,-1,-3). As the desired line is perpendicular to the given line, hence

$$\underline{u}\cdot\underline{v} = 0 \Rightarrow (-3)(a)+(5)(b)+(1)(c)=0$$

$$\Rightarrow -3a+5b+c = 0 \quad (*) \Rightarrow (-3)(2)+5(3)-9=0 \text{ for}$$

$a=2$, $b=3$ and $c=-9$ $\Rightarrow \underline{v} = 2\underline{i}+3\underline{j}-9\underline{k}$ (many other values for a, b and c can be chosen here satisfying eq. (*))

Thus equation of the line passing through the Point (3,-1,-3) and Parallel to $\underline{v} = 2\underline{i}+3\underline{j}-9\underline{k}$ is given as

$$\begin{aligned} x &= 3+2t \\ y &= -1+3t \\ z &= -3-9t \end{aligned}, \quad -\infty < t < \infty \quad \text{(Parametric equations)}$$

Q20 Equations of the given line are as

$$\left.\begin{aligned} x &= 2-3t \\ y &= 4+2t \\ z &= 3-5t \end{aligned}\right\} \longrightarrow (1)$$

Equation of the given Plane is as

$$2x+3y+4z+8=0 \quad - (2)$$

Let P(x,y,z) be the Point of intersection of (1) and (2), then

(1) in (2) $\Rightarrow 2(2-3t)+3(4+2t)+4(3-5t)+8=0 \Rightarrow$

$$4-\cancel{6t}+12+\cancel{6t}+12-20t+8=0 \Rightarrow 20t=36 \Rightarrow t=9/5$$

$$\text{Hence (1)} \Rightarrow \left.\begin{aligned} x &= 2-\tfrac{27}{5} \Rightarrow x = -17/5 \\ y &= 4+18/5 \Rightarrow y = 38/5 \\ z &= 3-\tfrac{45}{5} \Rightarrow z = -6 \end{aligned}\right\} \Rightarrow P(-17/5, 38/5, -6) \text{ is the desired point of intersection}$$

$$\boxed{\begin{aligned} &(2) \Rightarrow 2(-\tfrac{17}{5})+3(38/5)+4(-6)+8=0 \\ &\Rightarrow \frac{-34+114}{5}-16=0 \\ &\frac{-34+114-80}{5}=0 \Rightarrow \frac{-114+114}{5}=0 \\ &\Rightarrow 0=0 \end{aligned}}$$

Exercise 5.2

Q_{22} Given plane is $-2x+4y-5z+6=0$ — ①

General plane equation is $ax+by+cz+d=0$ — ②, where

$\underline{u}=[a\ \ b\ \ c]$ be the vector normal to ②

~~Since~~ compare ① and ②, we have

$\underline{u}=[-2\ \ 4\ \ -5]$ is normal vector to ①

since the desired plane passing through the point $(2,4,-3)$ is parallel to the given plane ①, so $\underline{u}=[-2, 4\ \ -5]$ is also normal to the desired plane. Thus equation of the desired plane passing through the point $(2,4,-3)$ and normal to vector $\underline{u}=[-2\ \ 4\ \ -5]$ is as

$-2(x-2)+4(y-4)-5(z+3)=0 \Rightarrow -2x+4+4y-16-5z-15=0$

$\Rightarrow -2x+4y-5z-27=0$ is the equation of the desired plane.