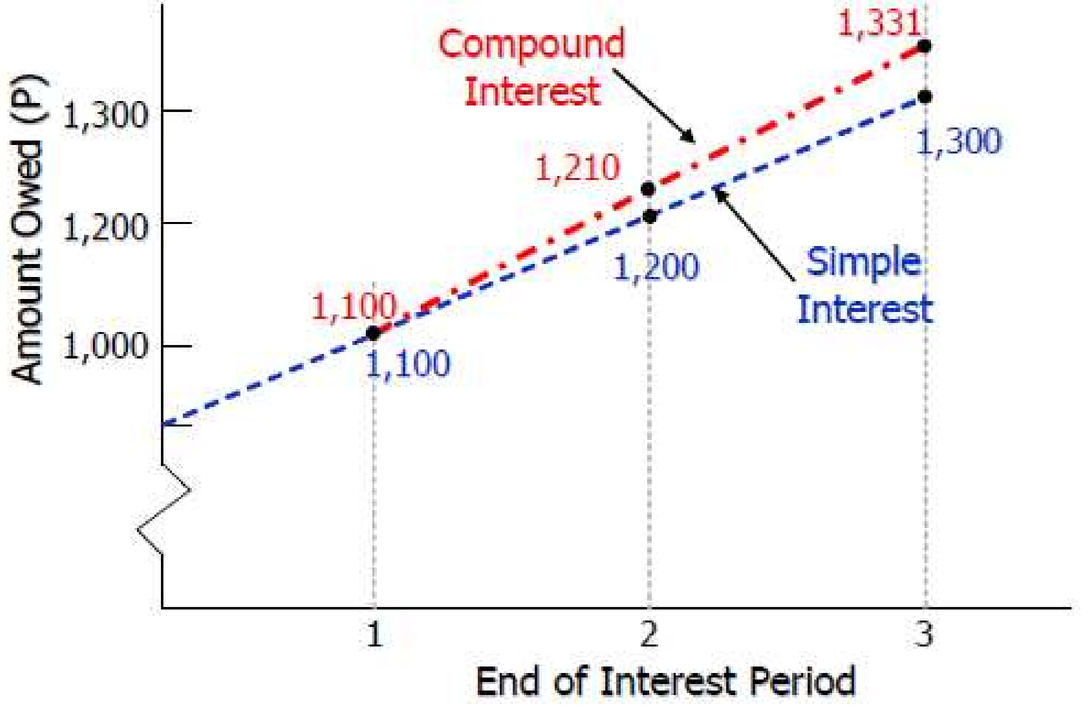
Money Time Relationship and Equivalence

- Intersection of Simple and Compound Interest
- Concept of Equivalence
- Notation and Cash Flow Diagrams
- Interest Formula
- Multiple Interest Formula
- Nominal Interest Rates
- Effective Interest Rates

Intersection of Simple and Compound Interest

Example:

Loan of ₱1,000 with 10% interest over 3 years.



Concept of Equivalence

- Economic equivalence is established when there is indifference between a future payment, or a series of future payments, and a present sum of money.

- It includes the comparison of alternative options, or proposals, by reducing them to an equivalent basis, depending on:
  - interest rate
  - amounts of money involved
  - timing of the affected monetary receipts and/or expenditures
  - manner in which the interest , or profit on invested capital is paid and the initial capital is recovered

Concept of Equivalence

Illustrative Example

Consider a situation in which we borrow ₱8,000 and agree to repay it in four years at an interest rate of 10% per year.

Year	Amount Owed at Beginning of Year	Interest Accrued for Year	Total Money Owed at End of Year	Principal Payment	End-of-Year Payment (Cash Flow)
Plan 1: At end of each year pay 2,000 principal plus interest due					
1	8,000	800	8,800	2,000	2,800
2	6,000	600	6,600	2,000	2,600
3	4,000	400	4,400	2,000	2,400
4	2,000	200	2,200	2,000	2,200
	20,000	**2,000**		**8,000**	**10,000**
		(total interest)			(total amount repaid)
Plan 2: Pay interest due at end of each year and principal at end of four years					
1	8,000	800	8,800	0	800
2	8,000	800	8,800	0	800
3	8,000	800	8,800	0	800
4	8,000	800	8,800	8,000	8,800
	32,000	**3,200**		**8,000**	**11,200**
		(total interest)			(total amount repaid)
Plan 3: Pay in four equal end-of-year payments					
1	8,000	800	8,800	1,724	2,524
2	6,276	628	6,904	1,896	2,524
3	4,380	438	4,818	2,086	2,524
4	2,294	230	2,524	2,294	2,524
	20,950	**2,096**		**8,000**	**10,096**
		(total interest)			(total amount repaid)
Plan 4: Pay principal and interest in one payment at end of four years					
1	8,000	800	8,800	0	0
2	8,800	880	9,680	0	0
3	9,680	968	10,648	0	0
4	10,648	1,065	11,713	8,000	11,713
	37,128	**3,713**		**8,000**	**11,713**
		(total interest)			(total amount repaid)

Notation and Cash Flow Diagrams

i = effective interest rate per interest period

N = number of compounding periods (e.g., years)

P = present sum of money; the equivalent value of one or more cash flows at the present time reference point

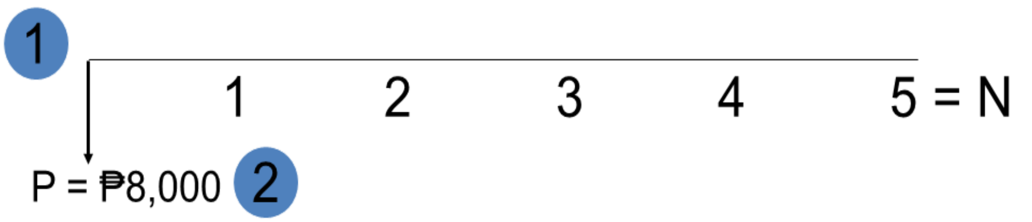
F = future sum of money; the equivalent value of one or more cash flows at a future time reference point

A = end-of-period cash flows (or equivalent end-of-period values) in a uniform series continuing for a specified number of periods, starting at the end of the first period and continuing through the last period

G = uniform gradient amounts – used if cash flows increase by a constant amount in each period

Notation and Cash Flow Diagrams

Cash Flow Diagram Notation

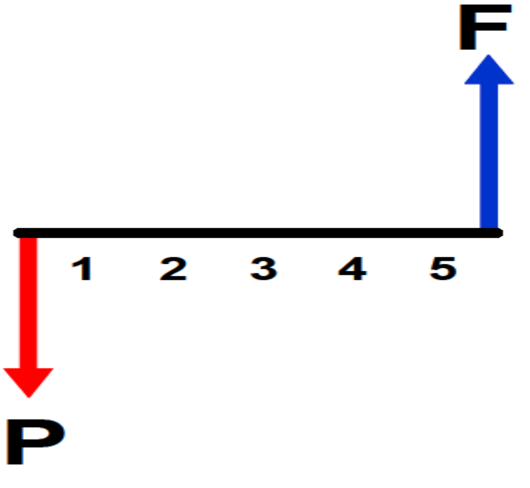


1. Time scale with progression of time moving from left to right; the numbers represent time periods (e.g. years, months, or quarters) and may be presented within a time interval or at the end of a time interval.
2. Present expense (cash outflow) of ₱8,000 for lender

Notation and Cash Flow Diagrams

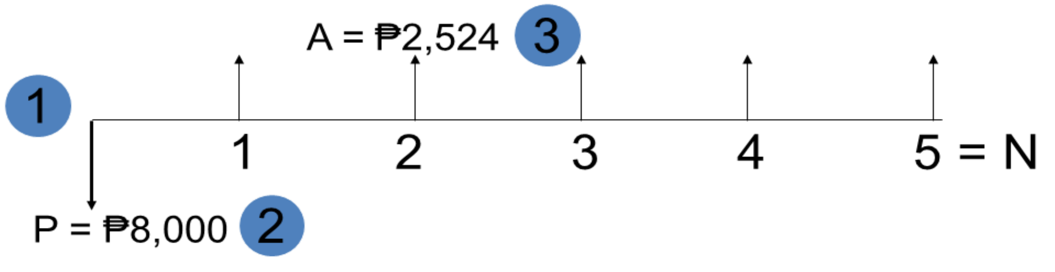
Cash Flow Diagrams Direction

- Upward arrow represents positive cash flow
  - Revenues from the sale of goods or services
  - Savings or cost reductions resulting from the success of your project
- Downward arrow represents expense or negative cash flows
  - Money invested in the project
  - Ongoing costs of doing the project



Notation and Cash Flow Diagrams

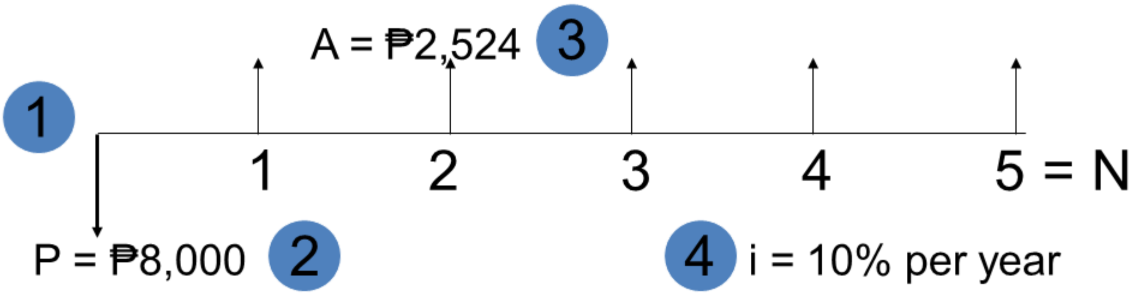
Cash Flow Diagram Notation



1. Time scale with progression of time moving from left to right; the numbers represent time periods (e.g., years, months, or quarters) and may be presented within a time interval or at the end of a time interval.
2. Present expense (cash outflow) of ₱8,000 for lender.
3. Annual income (cash inflow) of ₱ 2,524 for lender.

Notation and Cash Flow Diagrams

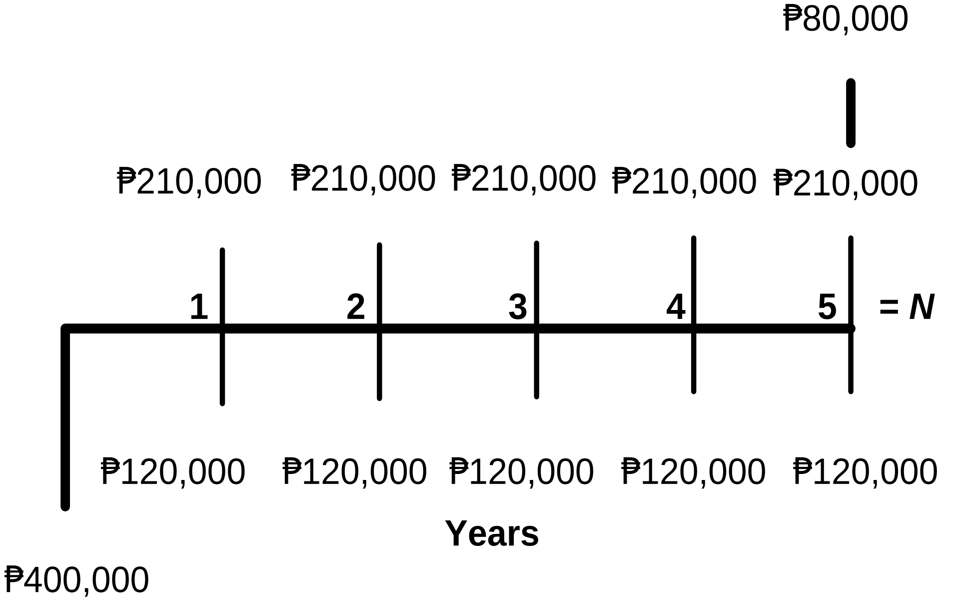
Cash Flow Diagram Notation



1. Time scale with progression of time moving from left to right; the numbers represent time periods (e.g., years, months, or quarters) and may be presented within a time interval or at the end of a time interval.
2. Present expense (cash outflow) of ₱8,000 for lender.
3. Annual income (cash inflow) of ₱ 2,524 for lender.
4. Interest rate of loan.

Example of Cash Flow Diagramming

Before evaluating the economic merits of a proposed investment, the XYZ Corporation insists that its engineers develop a cash flow diagram of the proposal. An investment of ₱400,000 can be made that will produce uniform annual revenue of ₱210,000 for five years and then have a market (recovery) value of ₱80,000 at the end of year five. Annual expenses will be ₱120,000 at the end of each year for operating and maintaining the project. Draw the cash flow diagram for the five-year life of the project. Use the corporation's viewpoint.



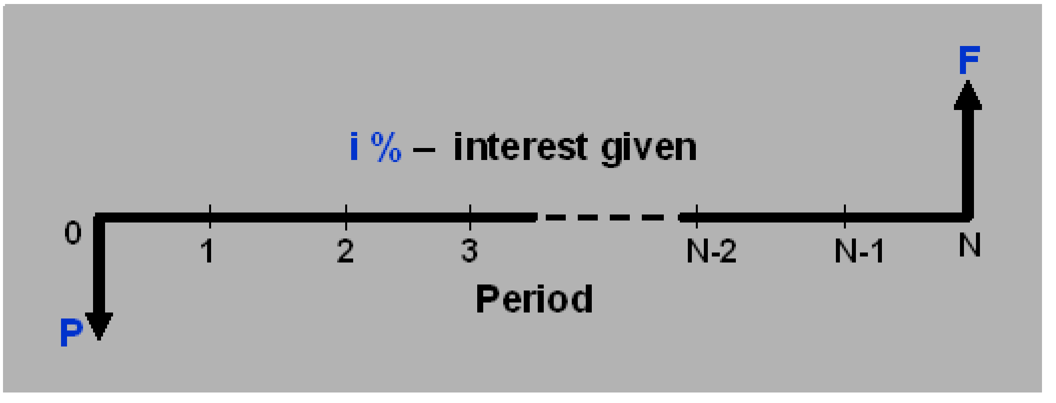
Interest Formula

- relating present and future values of single cash flows
- relating a uniform series (annuity) to present and future equivalent values
  - for discrete compounding and discrete cash flows
  - for deferred annuities (uniform series)
- equivalence calculations involving multiple interest
- relating a uniform gradient of cash flows to annual and present equivalents
- relating a geometric sequence of cash flows to present and annual equivalents

Interest Formula

- relating nominal and effective interest rates
- relating to compounding more frequently than once a year
- relating to cash flows occurring less often than compounding periods
- for continuous compounding and discrete cash flows
- for continuous compounding and continuous cash flows

Interest Formula – Future Worth



$$F = P(1+i)^N$$

where : F = future worth of money

N = number of interest period

$(1+i)^N$ **=** called the *single payment compound amount factor;* symbolized by (F/P, i%, N)

Hence

$$F = P(F/P, i\%, N)$$

Interest Formula – Future Worth

Illustrative Examples of Future Worth

1. Suppose you borrow ₱8,000 now, promising to repay the loan principal plus accumulated interest in four years at i = 10% / year. How much would you repay at the end of four years?

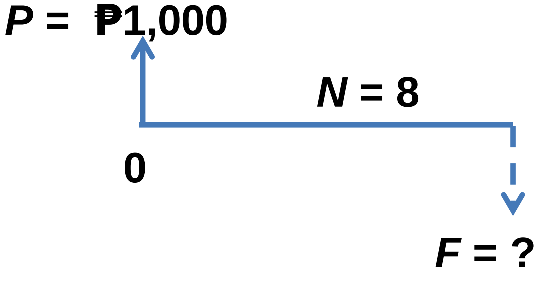
2. A firm borrows P1,000 at an interest of 10% per year for eight years. How much must it repay in a lump sum at the end of the eighth year?

Interest Formula – Future Worth

Solution to Example 1

Year	Amount Owed at the Beginning of Year	Interest Owed for Each Year	Amount Owed at End of Year	Total End-of-Year Payment
1	P = ₱ 8,000	iP = ₱ 800	$P(1+i)$ = ₱ 8,800	0
2	$P(1+i)$ = ₱ 8,800	$iP(1+i)$ = ₱ 880	$P(1+i)^2$ = ₱ 9,680	0
3	$P(1+i)^2$ = ₱ 9,680	$iP(1+i)^2$ = ₱ 968	$P(1+i)^3$ = ₱ 10,648	0
4	$P(1+i)^3$ = ₱ 10,648	$iP(1+i)^3$ = ₱ 1,065	$P(1+i)^4$ = ₱ 11,713	**F = ₱ 11,713**

Solution to Example 2



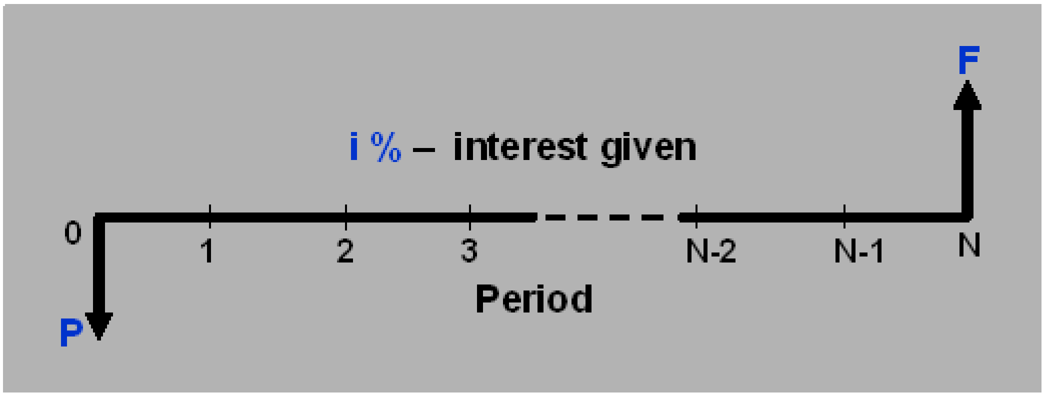
F = P (F/P, 10%, 8)

= ₱ 1,000 $(1+0.10)^8$

= ₱ 1,000 (2.1436)

= **₱ 2,143.60**

Interest Formula – Present Worth



$$P = F(1+i)^{-N}$$

where : P = present worth of money

N = number of interest period

$(1+i)^{-N}$ **=** called the *single payment present worth factor;* symbolized by $(P/F, i\%, N)$

Hence

$$P = F(P/F, i\%, N)$$

Interest Formula – Present Worth

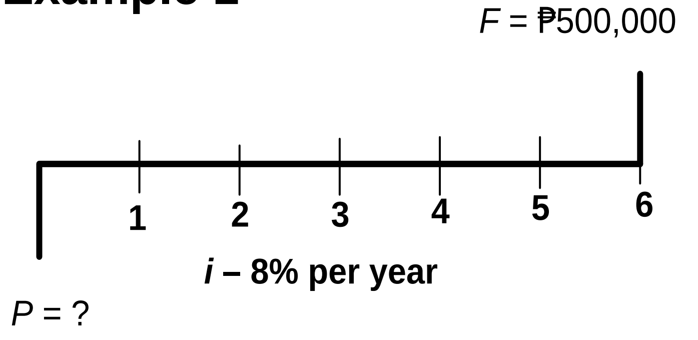
Illustrative Examples of Present Worth

1. Leo is thinking of purchasing a tract of land that will be worth ₱500,000 in six years. If the value of the land increases at 8% each year, how much should Leo be willing to pay now for this property?

2. A firm wishes to have ₱2,143.60 eight years from now. What amount should be deposited now to provide for it? Assume an interest rate of 10% per year.

Interest Formula – Present Worth

Solution to Example 1

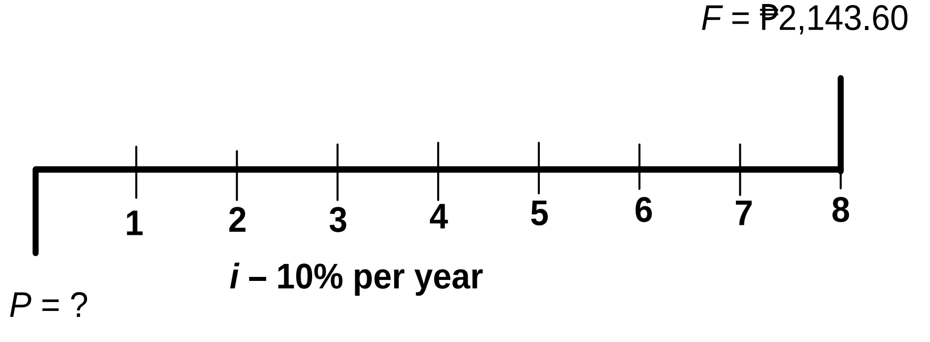


$$P = F(1+i)^{-N}$$

$$P = ₱500{,}000(1+0.08)^{-6}$$

$$\mathbf{P = ₱315{,}085}$$

Solution to Example 2



$$P = F(P/F, 10\%, 8)$$

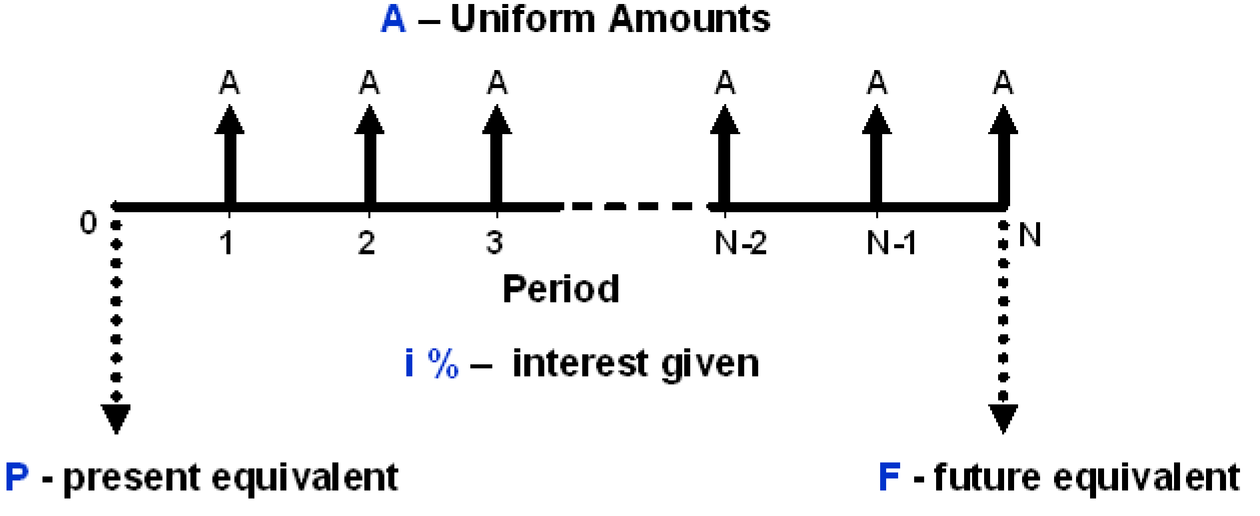
$$P = ₱2{,}143.60(1+0.10)^{-8}$$

$$\mathbf{P = ₱1{,}000.00}$$

Interest Formula - Annuity

Annuity

- a series of uniform (equal) receipts, each of amount *A*, occurring at the end of each period for *N* periods with interest at *i*% per period



Interest Formula - Annuity

❖ Finding future equivalent income (inflow) value given a series of uniform equal payments

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Uniform series compound amount factor, functionally expressed as $F = A(F/A, i\%, N)$

❖ Finding present equivalent income (inflow) value given a series of uniform equal payments

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

Uniform series present worth factor, functionally expressed as $P = A(P/A, i\%, N)$

Interest Formula - Annuity

Illustrative Example #1

Suppose you make 15 equal annual deposits of ₱1,000 each into a bank account having 5% interest per year. The first deposit will be made one year from this day. How much money can be withdrawn from this bank account immediately after the 15th deposit?

Solution:

$$F = A(F/A, 5\%, 15)$$

$$= A\left[\frac{(1+i)^N - 1}{i}\right]$$

$$= 1000\left[\frac{(1+0.05)^{15} - 1}{0.05}\right]$$

$$\boldsymbol{F} = \mathbf{21{,}578.60}$$

Interest Formula - Annuity

Illustrative Example #2

If you are 20 years of age and started to save ₱1.00 each day for the rest of your life, will you become a millionaire? Assume that you can live up to the age of 60 and save it in a bank that gives an annual interest rate of 10%.

Solution:

$$F = A(F/A, 10\%, 60)$$

$$= A\left[\frac{(1+i)^N - 1}{i}\right]$$

$$= 365\left[\frac{(1+0.10)^{60} - 1}{0.10}\right]$$

$$\boldsymbol{F} = \mathbf{1{,}107{,}707}$$

Interest Formula - Annuity

Illustrative Example #3

If a certain machine undergoes a major overhaul now, its output can be increased by 20%. This means additional cash flow of ₱20,000 at the end of each year for five years. If *i*=15%, how much can we afford to invest to overhaul this machine?

Solution:

$$P = A(P/A, 15\%, 5)$$

$$= A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

$$= 20{,}000\left[\frac{(1+0.15)^5 - 1}{0.15(1+0.15)^5}\right]$$

$$\boldsymbol{P} = \mathbf{67{,}043}$$

Interest Formula - Annuity

Illustrative Example #4

Mark is retiring and wishes to distribute his ₱1,000,000 to his sons at the rate of ₱100,000/yr. If the ₱1,000,000 is deposited in a bank account that earns 6% interest/yr, how many years will it take to completely deplete the account?

Solution:

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

Use trial and error.

$$\frac{(0.06 \times 1{,}000{,}000)}{100{,}000} = \left[\frac{(1+0.06)^N - 1}{(1+0.06)^N}\right]$$

Assume a starting value for *N*.

Interest Formula - Annuity

Assume that N =10.

$$\left[\frac{(1+0.06)^N-1}{(1+0.06)^N}\right]=0.44$$

Increase the value of N. Try N =16 years.

$$\left[\frac{(1+0.06)^N-1}{(1+0.06)^N}\right]=0.61$$

Decrease the value of N. Try N = 15 years.

$$\left[\frac{(1+0.06)^N-1}{(1+0.06)^N}\right]=0.58$$

Interest Formula - Annuity

Thus, the value of *N* is between 15 to 16 years. Using interpolation, we get

N	Result
15	0.58
?	0.6
16	0.61

$$\frac{N-15}{0.60-0.58}=\frac{16-15}{0.61-0.58}$$

$$\frac{N-15}{0.02}=\frac{1}{0.03}$$

$$N=\left(\frac{1\times 0.02}{0.03}\right)+15$$

$$N=15.67\ \text{years}$$

Multiple Interest Formula

- Considered where a series of cash outflows occur over a number of years
- Considered when the value of the outflows is unique for each of a number (i.e. first three years)
- Considered when the value of outflows is the same for the last four years
- Used to find the following:
  - Present equivalent expenditure
  - Future equivalent expenditure
  - Annual equivalent expenditure

Multiple Interest Formula

Illustrative Example:

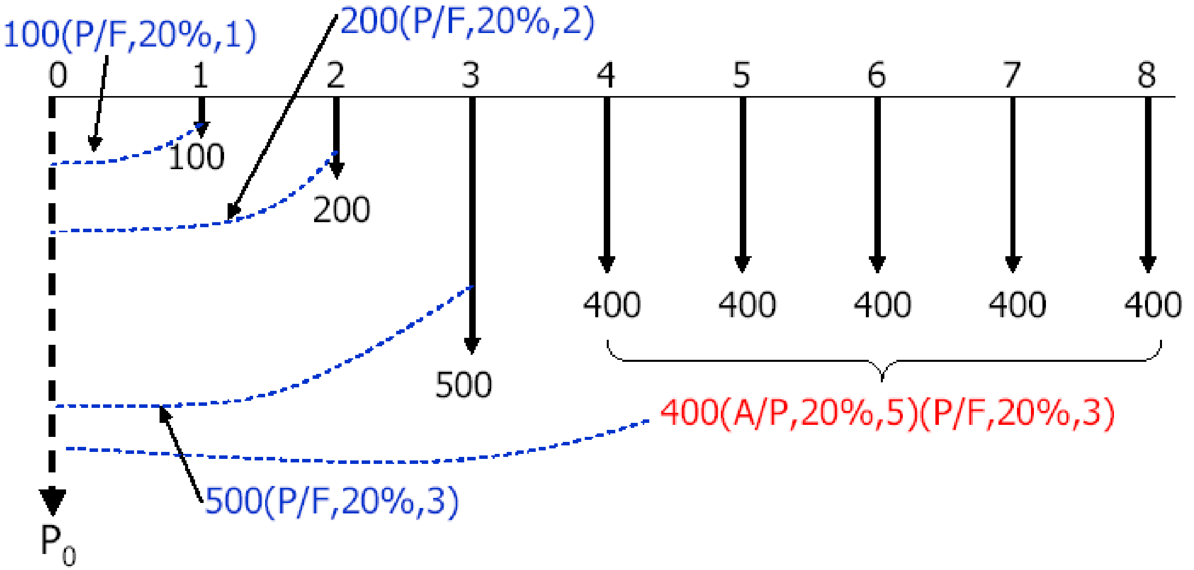
The expected maintenance expenditures for a certain equipment over eight years amounts as follows: ₱100 for the first year, ₱200 for the 2^{nd}, ₱500 for the 3^{rd}, and ₱400 for each year onwards. Make a cash flow diagram representing the expenditures. If *i*=20% per year, find the following:

a.) present equivalent expenditure (P_0)

b.) future equivalent expenditure (F_8)

c.) annual equivalent expenditure (A)

Multiple Interest Formula



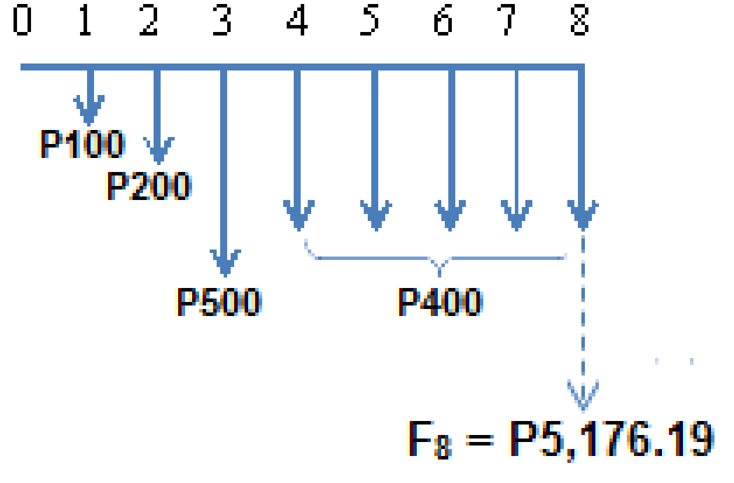
a. Find P_0.

$$P_0 = F_1(P/F,20\%,1) + F_2(P/F,20\%,2) + F_3(P/F,20\%,3)$$
$$+ A(P/A,20\%,5)(P/F,20\%,3)$$
$$= 100(0.8333) + 200(0.6944) + 500(0.5787)$$
$$+ 400(2.9900)(0.5787)$$
$$= 83.33 + 138.88 + 289.35 + 692.26$$
$$\boldsymbol{P_0 = 1{,}203.82}$$

Multiple Interest Formula

b. Find F_8.

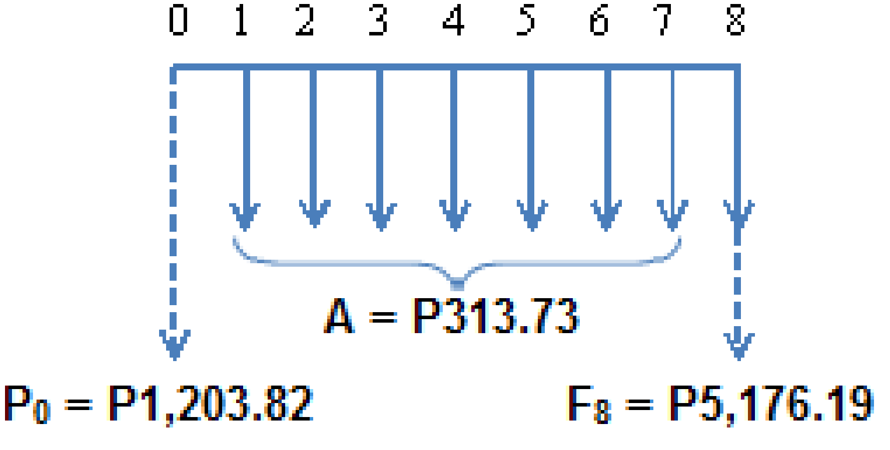
$$F_8 = P_0(F/P, 20\%, 8)$$
$$= 1{,}203.82\left[\frac{(1+0.20)^8}{0.20}\right]$$
$$= 1{,}203.82(4.2998)$$
$$\boldsymbol{F_8 = 5{,}176.19}$$



Multiple Interest Formula

c. Find A using P_0.

$$A = P_0(A/P, 20\%, 8)$$
$$= \frac{1{,}203.82}{\left[\frac{(1+0.20)^8 - 1}{0.20(1+0.20)^8}\right]}$$
$$= \frac{1{,}203.82}{3.8371}$$
$$\boldsymbol{A = 313.73}$$



Nominal Interest Rates

- If the compounding period isn't a year:
  - interest is quoted as an annual rate (r) called the *nominal interest rate*
  - to solve any problem, you must use a per period rate

$$(r/m) = i$$

where m is the number of compounding periods in a year

So: $F = P(1+i)^n$

Becomes: $F = P(1+r/m)^n$

where n is the <u>number of periods</u> not the number of years.

Effective Interest Rates

- Effective interest rate (i_{eff}) is the actual or exact rate of interest earned on the principal during one year.

- This is usually expressed on annual basis.

- Converting nominal interest to effective interest rate:

$$i = (1 + r/N)^N - 1$$

where : i_{eff} = effective interest rate

r = nominal rate of interest

N = number of compounding period/yr

- Effective interest rate is only equal to nominal rate of interest when compounding is on annual basis.

When $N>1$, $i>r$.

Effective Interest Rates

Illustrative Problem

A credit company charges at a rate of 1.375% per month on the unpaid balance of all accounts. The annual interest rate is 16.5%. What is the effective rate of interest per year being charged by the company?

Solution:

$$i_{eff} = \left(1 + \frac{r}{N}\right)^N - 1$$

$$= \left(1 + \frac{0.165}{12}\right)^{12} - 1$$

$$= 0.1781$$

$$\boldsymbol{i_{eff}} = \mathbf{17.81\%/year}$$