Adding Eqs. (3.20a) and (3.20b), and using the above formula, we obtain

$$\cos \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \tag{3.21}$$

The spectrum of cos $\omega_0 t$ consists of two impulses at $\omega_0$ and $-\omega_0$, as shown in Fig. 3.13. The result also follows from qualitative reasoning. An everlasting sinusoid cos $\omega_0 t$ can be synthesized by two everlasting exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$. Therefore, the Fourier spectrum consists of only two components of frequencies $\omega_0$ and $-\omega_0$.

**EXAMPLE 3.7** Find the Fourier transform of the sign function sgn $t$ (pronounced signum $t$), shown in Fig. 3.14. Its value is $+1$ or $-1$, depending on whether $t$ is positive or negative:

$$\text{sgn } t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{3.22}$$

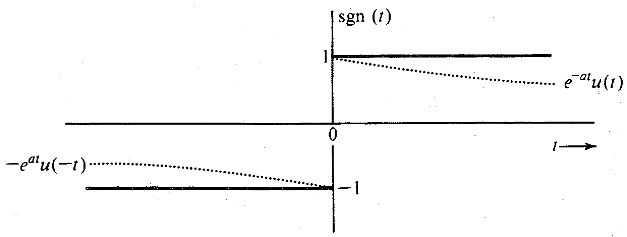


**Figure 3.14** Sign function.

The transform of sgn $t$ can be obtained by considering sgn $t$ as a sum of two exponentials, as shown in Fig. 3.14, in the limit as $a \to 0$:

$$\text{sgn } t = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

Therefore,

$$\mathcal{F}[\text{sgn } t] = \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\}$$

$$= \lim_{a\to 0}\left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right) \quad \text{(see pairs 1 and 2 in Table 3.1)}$$

$$= \lim_{a\to 0}\left(\frac{-2j\omega}{a^2+\omega^2}\right) = \frac{2}{j\omega} \tag{3.23}$$

## 3.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

We now study some of the important properties of the Fourier transform and their implications as well as their applications. Before embarking on this study, it is important to point out a pervasive aspect of the Fourier transform—the **time-frequency duality**.



**Table 3.1**

**Short Table of Fourier Transforms**

| | $g(t)$ | $G(\omega)$ | |
|---|---|---|---|
| 1 | $e^{-at}u(t)$ | $\frac{1}{a+j\omega}$ | $a > 0$ |
| 2 | $e^{at}u(-t)$ | $\frac{1}{a-j\omega}$ | $a > 0$ |
| 3 | $e^{-a\|t\|}$ | $\frac{2a}{a^2+\omega^2}$ | $a > 0$ |
| 4 | $te^{-at}u(t)$ | $\frac{1}{(a+j\omega)^2}$ | $a > 0$ |
| 5 | $t^n e^{-at}u(t)$ | $\frac{n!}{(a+j\omega)^{n+1}}$ | $a > 0$ |
| 6 | $\delta(t)$ | $1$ | |
| 7 | $1$ | $2\pi\delta(\omega)$ | |
| 8 | $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ | |
| 9 | $\cos\omega_0 t$ | $\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ | |
| 10 | $\sin\omega_0 t$ | $j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$ | |
| 11 | $u(t)$ | $\pi\delta(\omega)+\frac{1}{j\omega}$ | |
| 12 | $\text{sgn } t$ | $\frac{2}{j\omega}$ | |
| 13 | $\cos\omega_0 t\, u(t)$ | $\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$ | |
| 14 | $\sin\omega_0 t\, u(t)$ | $\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$ | |
| 15 | $e^{-at}\sin\omega_0 t\, u(t)$ | $\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$ | $a > 0$ |
| 16 | $e^{-at}\cos\omega_0 t\, u(t)$ | $\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$ | $a > 0$ |
| 17 | $\text{rect}\left(\frac{t}{\tau}\right)$ | $\tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$ | |
| 18 | $\frac{W}{\pi}\,\text{sinc}(Wt)$ | $\text{rect}\left(\frac{\omega}{2W}\right)$ | |
| 19 | $\Delta\left(\frac{t}{\tau}\right)$ | $\frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$ | |
| 20 | $\frac{W}{2\pi}\,\text{sinc}^2\left(\frac{Wt}{2}\right)$ | $\Delta\left(\frac{\omega}{2W}\right)$ | |
| 21 | $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ | $\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$ | $\omega_0 = \frac{2\pi}{T}$ |
| 22 | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$ | |