Control Systems - 7th Semester

Lecture 3

Model of Systems

Last week, we studied about transfer function models

Last week, we also studied how to obtain poles, zeros, and analyze stability of transfer function model

Last week, we also studied a new language of modelling which is called as state-space modelling

Last week, we also studied about obtaining state-space models from differential equations

This week, we will study the conversion techniques from state-space models to transfer function models (and vise versa)

Converting State Space to Transfer Function

The general form or template of ss model is as follows:

$$\dot{x} = Ax + Bu(t)$$

$$y = Cx + Du(t)$$

Let $G(s)$ denote the transfer function after converting to transfer function domain. The formula is:

$$G(s) = D + C[(sI - A)^{-1}B]$$

Example of conversion from ss to tf

Convert the following state-space model to transfer function

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Let us first obtain $(sI - A)^{-1}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

Example of conversion from ss to tf

$$sI - A = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

Now let us find $(sI - A)^{-1}$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)}$$

$$\text{adjoint}(sI - A) = \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

$$\begin{aligned} \det(sI - A) &= (s-1)(s-4) - (-2)(-3) \\ &= (s^2 - 5s + 4) - (6) \\ &= s^2 - 5s + 4 - 6 \\ &= s^2 - 5s - 2 \end{aligned}$$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)} = \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

Example of conversion from ss to tf

Next, we post-multiply with matrix $\boldsymbol{B}$ as follows:

$$(s\boldsymbol{I} - \boldsymbol{A})^{-1} \times B = \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} \big((s-4) \times 5\big) + \big(2 \times 6\big) \\ \big(3 \times 5\big) + \big((s-1) \times 6\big) \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} 5s - 20 + 12 \\ 15 + 6s - 6 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2} \begin{bmatrix} 5s - 8 \\ 6s + 9 \end{bmatrix}$$

Example of conversion from ss to tf

Now, let us pre-multiply with matrix $\boldsymbol{C}$ as follows:

$$C(sI - A)^{-1}B = \frac{1}{s^2 - 5s - 2}\begin{bmatrix} 1 & 2 \end{bmatrix} \times \begin{bmatrix} 5s - 8 \\ 6s + 9 \end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2}\left[1 \times (5s - 8) + 2 \times (6s + 9)\right]$$

$$= \frac{1}{s^2 - 5s - 2}\left[5s - 8 + 12s + 18\right]$$

$$= \frac{1}{s^2 - 5s - 2}\left[17s + 10\right]$$

$$= \frac{17s + 10}{s^2 - 5s - 2}$$

Example of conversion from ss to tf

Convert the following state-space model to transfer function

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & -\mathbf{2} & -\mathbf{3} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \mathbf{10} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{y} = [\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]\boldsymbol{x}$$

Let us first obtain $(\boldsymbol{sI} \ - \boldsymbol{A})$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

Example of conversion from ss to tf

Now let us find $(\boldsymbol{sI} - \boldsymbol{A})^{-1}$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}=\frac{\begin{bmatrix}(s^2+3s+2) & s+3 & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}}{s^3+3s^2+2s+1}$$

Next, we post-multiply with matrix $\boldsymbol{B}$ as follows:

$$(\boldsymbol{sI} - \boldsymbol{A})^{-1}\times \boldsymbol{B} = \frac{1}{s^3+3s^2+2s+1}\begin{bmatrix}(s^2+3s+2) & s+3 & 1\\ -1 & s(s+3) & s\\ -s & -(2s+1) & s^2\end{bmatrix}\times\begin{bmatrix}10\\0\\0\end{bmatrix}$$

Example of conversion from ss to tf

$$(sI - A)^{-1} \times B = \frac{1}{s^3 + 3s^2 + 2s + 1} \begin{bmatrix} 10s^2 + 30s + 20 \\ -10 \\ -10s \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s^3 + 3s^2 + 2s + 1} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10s^2 + 30s + 20 \\ -10 \\ -10s \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s^3 + 3s^2 + 2s + 1} [1 \times (10s^2 + 30s + 20) + 0 \times (-10) + 0 \times (-10s)]$$

$$G(s) = C(sI - A)^{-1}B = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Example of conversion from ss to tf

MATLAB code for conversion of ss to tf

```
A = [0 1 0; 0 0 1;  -1 -2 -3];

B = [10 ; 0; 0];

C = [1 0 0];

D = [0];

[num,den] = ss2tf(A,B,C,D);

g = tf(num,den)
```

```
Transfer function:
 10 s^2 + 30 s + 20
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s^3 + 3 s^2 + 2 s + 1
```

Conversion from tf to ss

Converting from tf to state-space is not unique process

There are various techniques to convert form transfer function domain to state-space domain

We call each technique as **canonical form**. Let us study the first canonical form

Conversion from tf to ss - Canonical Form 1

For nth order transfer function:

$$\boldsymbol{G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots b_1 s^1 + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots a_1 s + a_0}}$$

We write the following state-space model (using Canonical Form 1):

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_{n-2} & -a_{n-1} \end{bmatrix}} \qquad \boldsymbol{B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}}$$

$$\boldsymbol{C = [1 \quad 0 \quad 0 \quad \ldots \quad 0]}$$

Conversion from tf to ss - Canonical Form 1

For a 2nd order transfer function:

$$\boldsymbol{G(s) = \frac{b_1 s^1 + b_0}{s^2 + a_1 s + a_0}}$$

We write the following state-space model (using Canonical Form 1):

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}} \qquad \boldsymbol{B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}}$$

$$\boldsymbol{C = \begin{bmatrix} 1 & 0 \end{bmatrix}}$$

Conversion from tf to ss - Canonical Form 1

For a 3rd order transfer function:

$$G(s) = \frac{b_2 s^2 + b_1 s^1 + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \qquad B = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Conversion from tf to ss - Canonical Form 1

For a 4th order transfer function:

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s^1 + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \qquad B = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Conversion from tf to ss - Canonical Form 1

Example: Convert the following transfer function to state-space domain

$$\boldsymbol{G(s) = \frac{24}{s^3 + 9s^2 + 26s + 24}}$$

Solution: In this example, $a_0 = 24, a_1 = 26$, $a_2 = 9$, and $b_0 = 24$, we can obtain the following:

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}} \qquad \boldsymbol{B = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}}$$

$$\boldsymbol{C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}$$

Conversion from tf to ss - Canonical Form 1

Example: Convert the following transfer function to state-space domain

$$\boldsymbol{G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}}$$

Solution: In this example, $a_0 = 24, a_1 = 26,\ a_2 = 9,$ and $b_0 = 2,\ b_1 = 7,$ and $b_2 = 1,$ we can obtain the following:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{-24} & \mathbf{-26} & \mathbf{-9} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} \\ \mathbf{7} \\ \mathbf{2} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$