Control Systems - 7th Semester

Lecture 6

Step Response of First Order System

A first order system without zeros can be written as follows:

$$\boldsymbol{G(s) = \frac{b}{s+a}}$$

The inverse of $\boldsymbol{a}$ is called time constant i.e.,

$$\boldsymbol{\tau = \frac{1}{a}}$$

The gain ***K*** is also called as dc-gain or steady-state gain of a system

$$\boldsymbol{K = \frac{b}{a}}$$

Step Response of First Order System

Rise Time: $\boldsymbol{T_r}$, time taken to reach **90%** or **0.9** of final value from **10%** or **0.1**

Mathematically:

$$\boldsymbol{T_r = \frac{2.2}{a}}$$

Settling Time: $\boldsymbol{T_s}$, time taken to stay within **2%** of its final value (or reach **98%** of final value).

Mathematically:

$$\boldsymbol{T_s = \frac{4}{a}}$$

First Order Systems Summary

In first order system, we only have 2 parameters: dc gain and time-constant

Varying these two parameters only change the speed or amplitude of step response

Which parameter changes the speed of first order transfer function?

Which parameter changes the amplitude of first order transfer function?

Poles Location of Second Order System

A second order system has 2 poles. So, the following possibilities can occur:

- Both poles are real and equal
- Both poles are real and unequal
- Both poles are complex conjugate

Wait, one more possibility is also there

- Both poles are complex conjugate with real part equal to zero

General Second Order System

A general second order system can be written as follows:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ω_n is called the natural frequency of second order system and ζ is called damping ratio

ω_n is pronounced as omega n

ζ is pronounced as zeta

General Second Order System

Analyze this second order transfer function and determine $\boldsymbol{\omega_n}$ and $\boldsymbol{\zeta}$

$$\boldsymbol{G(s) = \frac{4}{s^2 + 2s + 4}}$$

Let us compare it with general form of second order systems

$$\boldsymbol{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

$\boldsymbol{\omega_n}$ = 2 and $\boldsymbol{\zeta = 0.5}$

General Poles of Second Order System

You can apply quadratic formula and compute the poles of transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles of the transfer function are

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Response Types of Second Order System

Now, we can have the following four possibilities:

- Overdamped response: The system has two real poles which are unequal - in this case $\zeta > 1$
- Critically damped response: The system has two real poles which are equal - in this case $\zeta = 1$
- Underdamped response: The system has two complex conjugate poles with some real part - in this case $0 < \zeta < 1$
- Undamped response: The system has two imaginary poles with zero real part - in this case $\zeta = 0$

Over Damped Second Order System

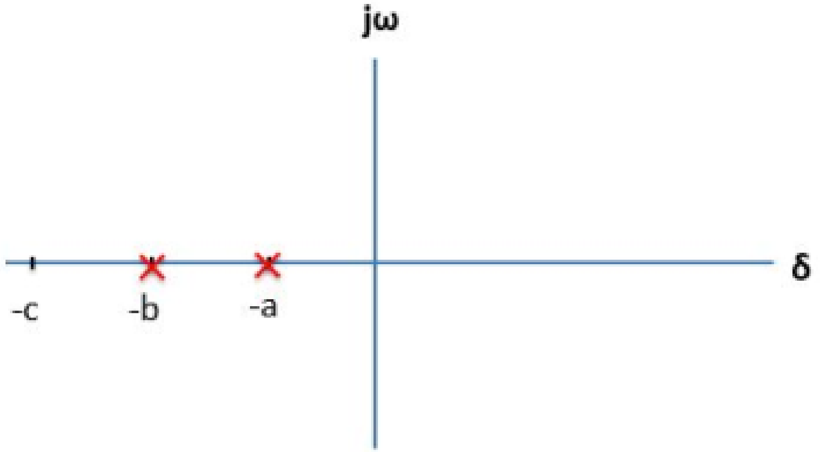

Figure: Poles Location of Over Damped System

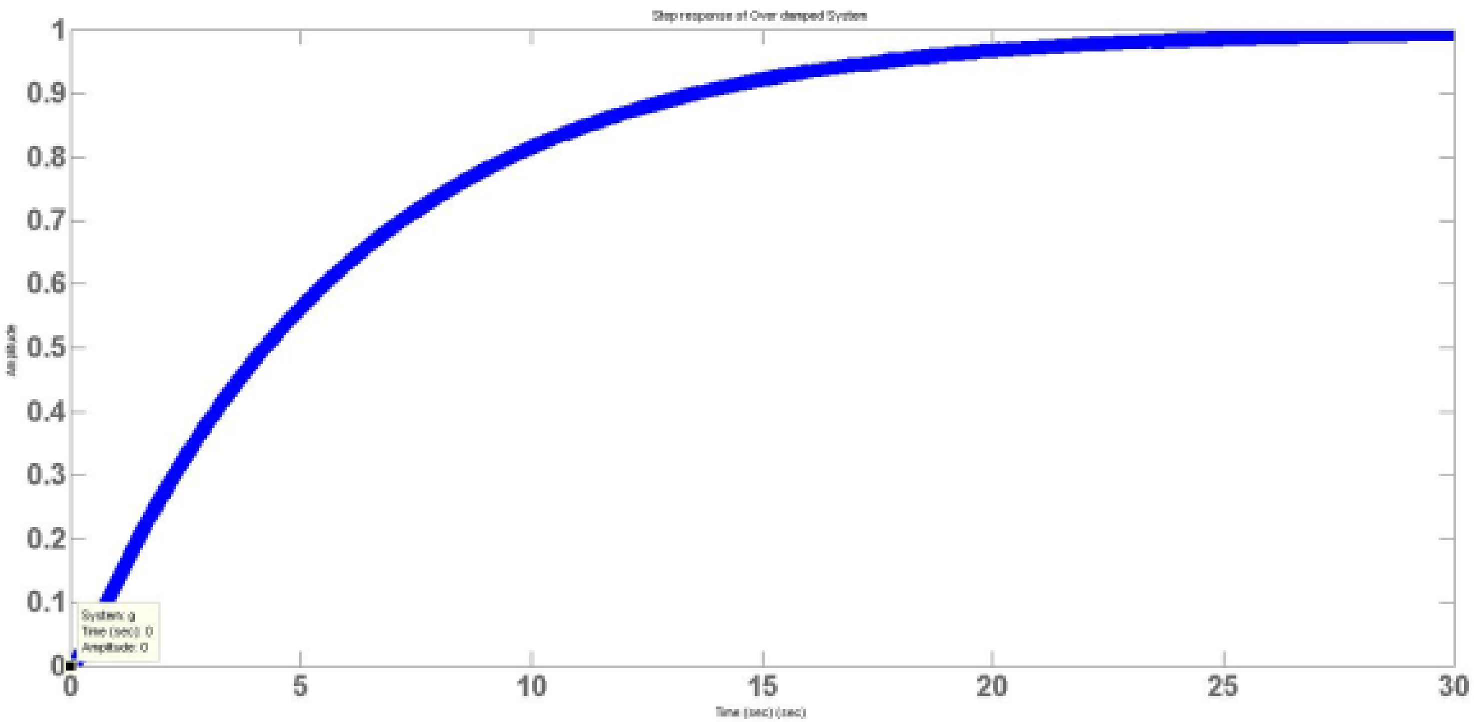

Figure: Step Response of Over Damped System

Critically Damped Second Order System

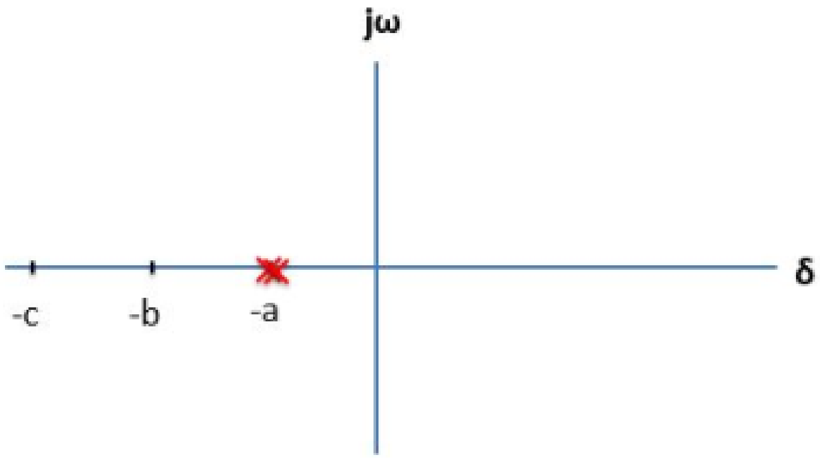

Figure: Poles Location of Critically Damped System

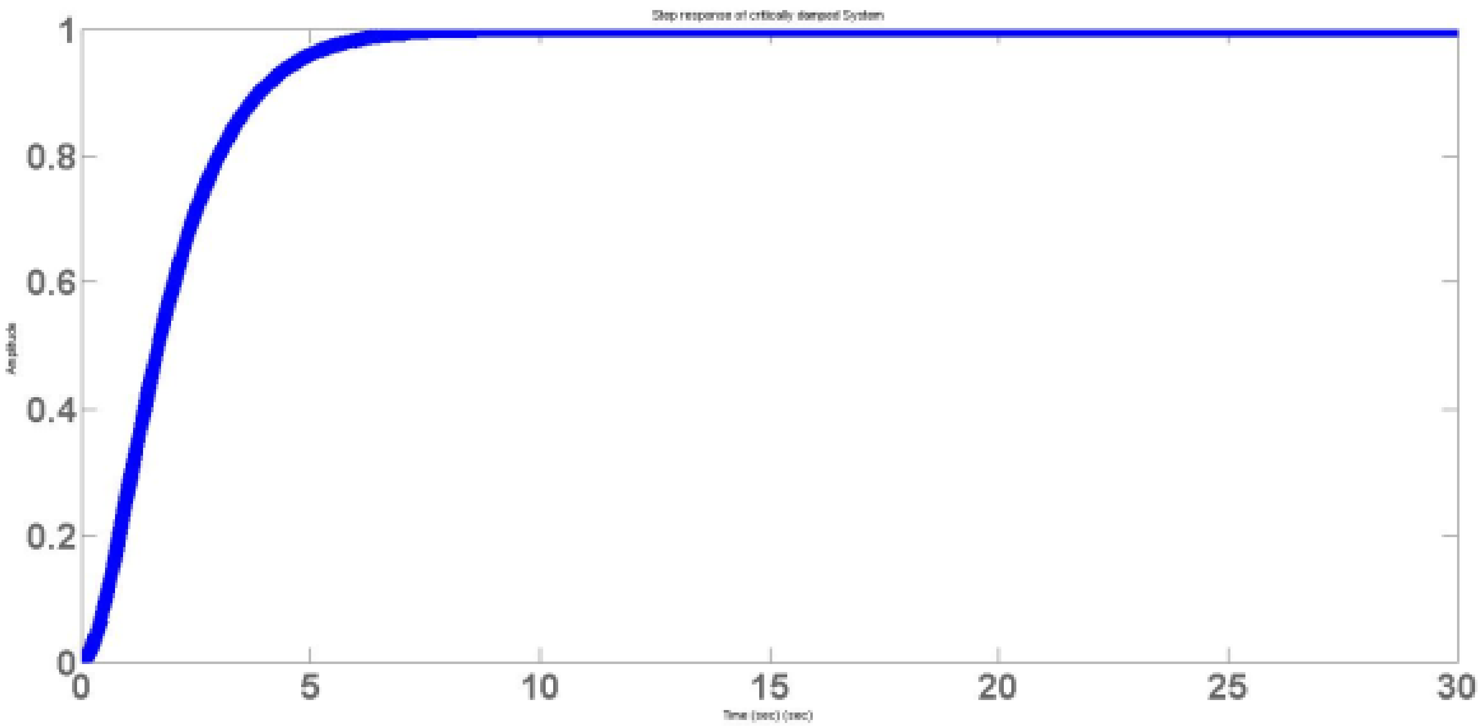

Figure: Step Response of Critically Damped System

Under Damped Second Order System

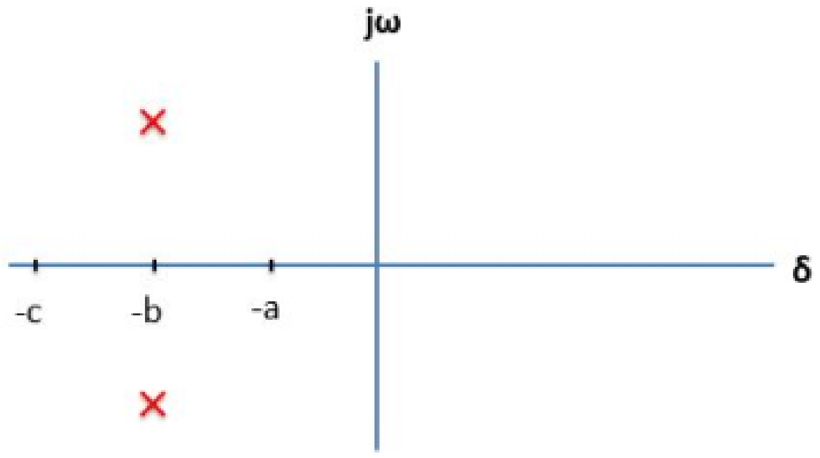

Figure: Poles Location of Under Damped System

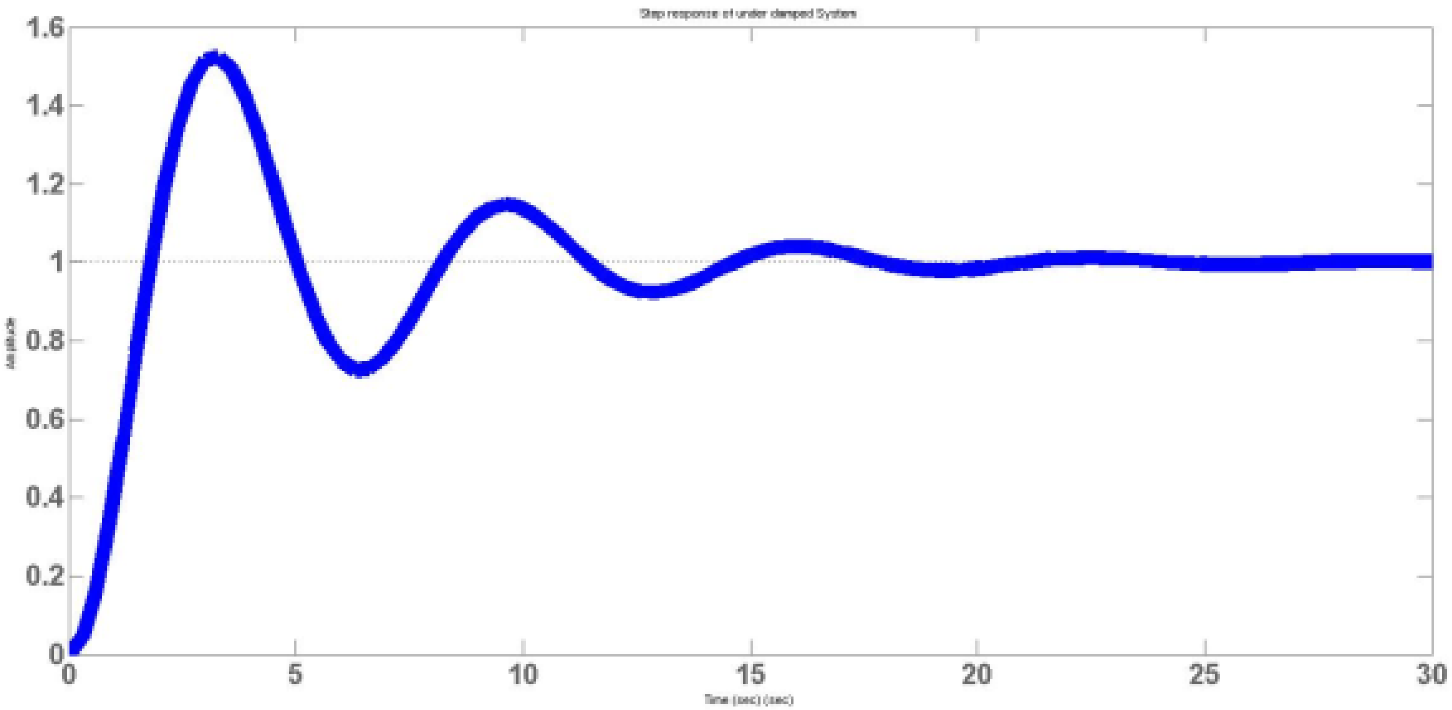

Figure: Step Response of Under Damped System

Un Damped Second Order System

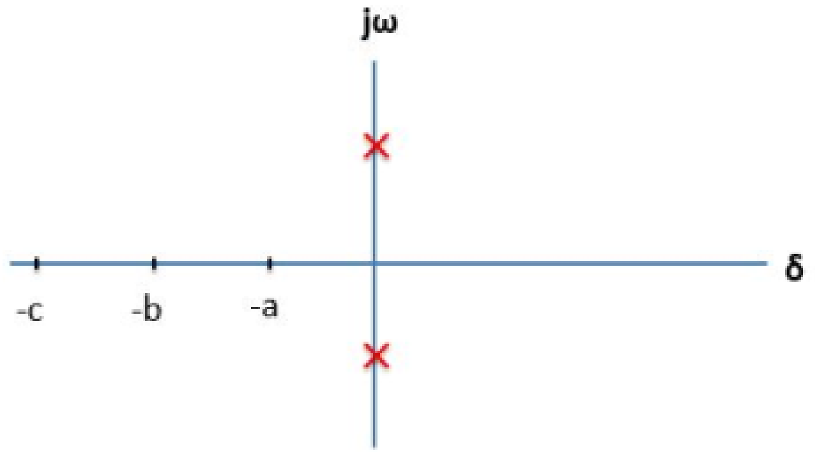

Figure: Poles Location of Un Damped System

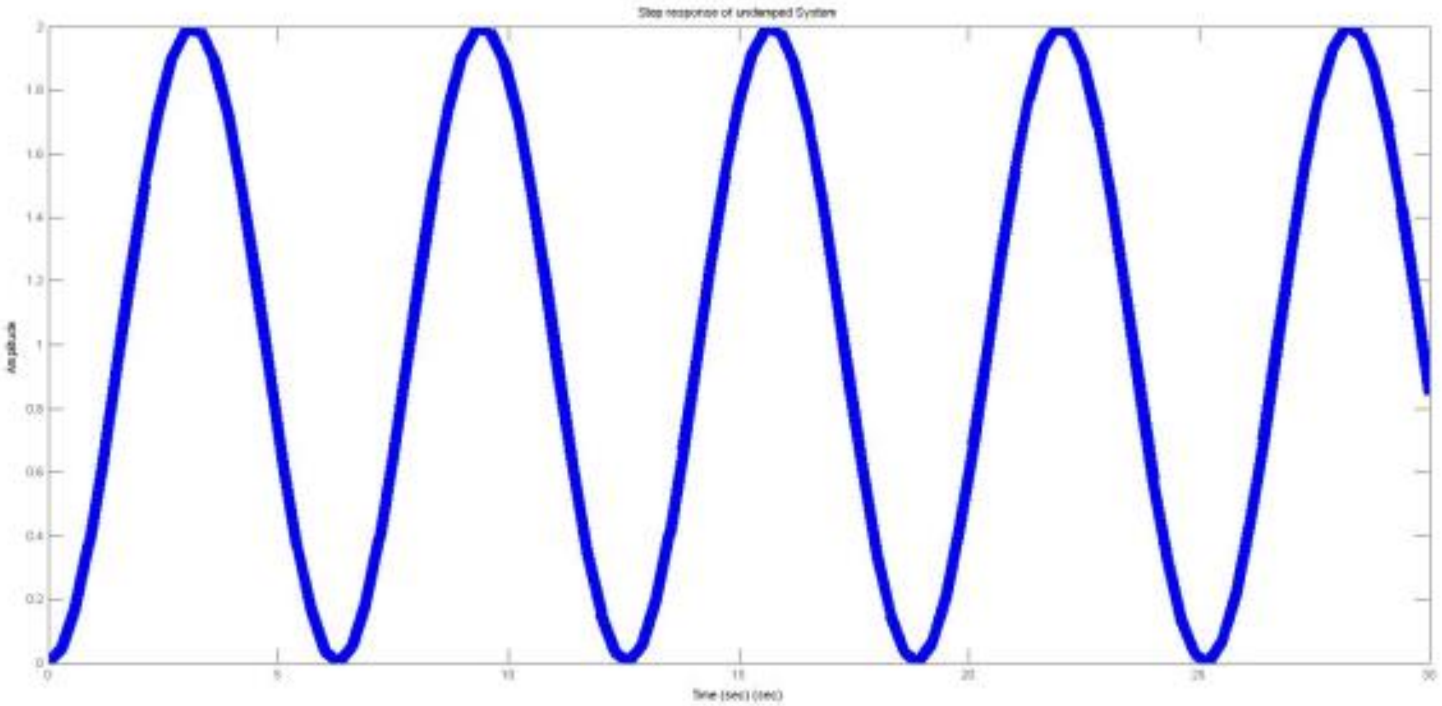

Figure: Step Response of Un Damped System

Role of ω_n

The natural frequency ω_n tells us about the distance from origin till the poles in s plane

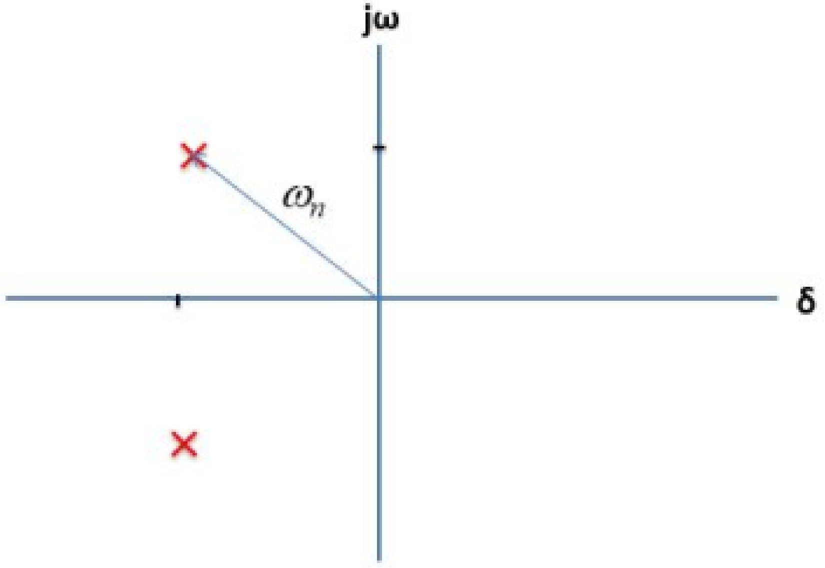

Figure: Role of ω_n in computation of poles

Role of ω_n

Let us draw a circle of radius 3 in the s plane

If a pole is located anywhere on the circumference of this circle, then $\boldsymbol{\omega_n}$ **= 3** rad/sec

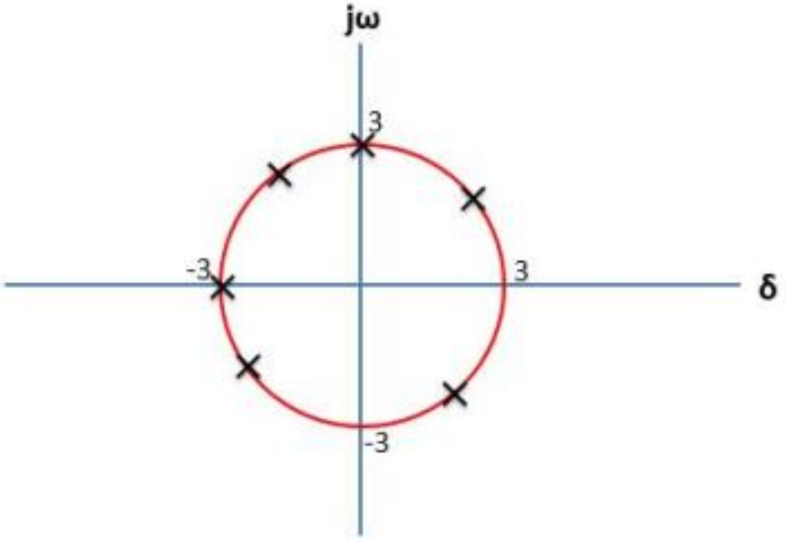

Figure: Example to demonstrate role of ω_n

Role of ζ

Now that we know that the role of ω_n is in the distance from origin till pole. What is the role of damping ratio, ζ, then?

ζ is the cosine of angle from –ve real axis to the vector connecting origin and pole

$$\zeta = \cos(\theta)$$

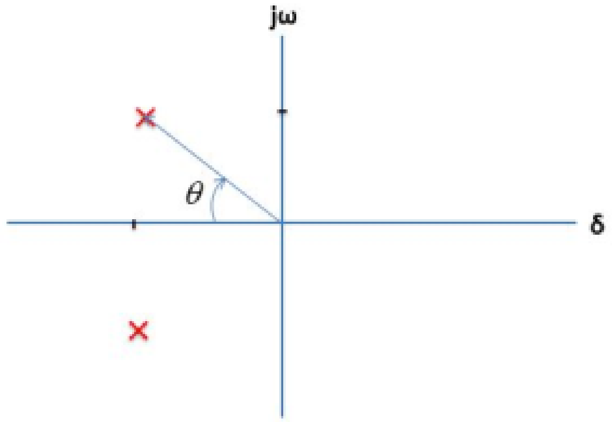

Figure: Role of ζ in determining the poles

Role of ζ - An example

If $\boldsymbol{\zeta = 0.707}$ meaning $\mathbf{cos}(\boldsymbol{\theta}) = \mathbf{0.707} \Rightarrow \boldsymbol{\theta = 45}$, can you trace the location of poles

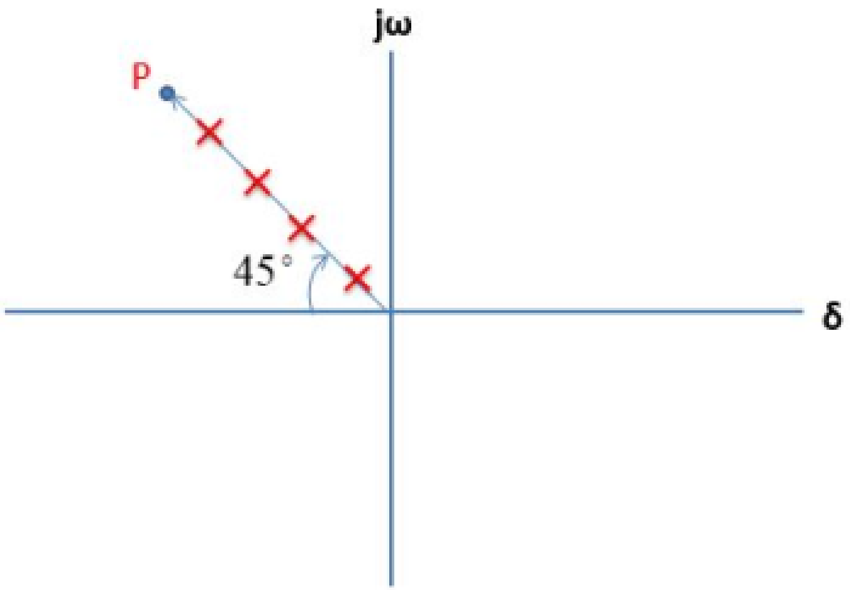

Figure: Role of ζ in determining the poles

Role of ζ and ω_n

To summarize again: $\boldsymbol{\omega_n}$ is used to compute the distance from origin till the poles

$\boldsymbol{\zeta}$ is the cosine of angle of the vector connecting origin and pole

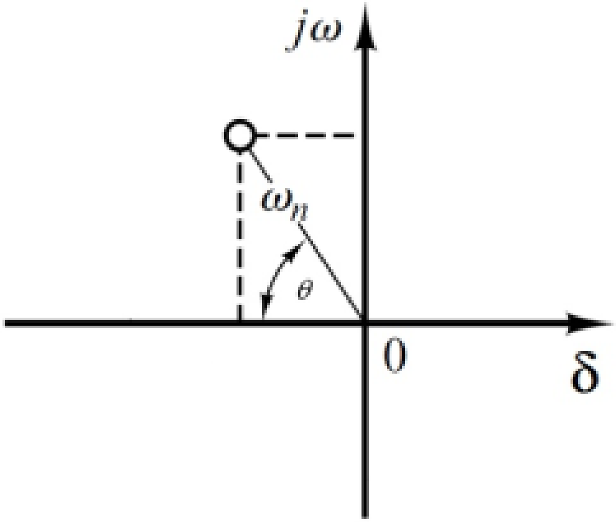

Figure: Role of $\boldsymbol{\zeta}$ and $\boldsymbol{\omega_n}$ in exactly determining the poles

Second Order System Analysis

You should know (for examination purposes):

- The four types of step responses of second order systems
- Being able to identify from graph, the type of response
- Know the location of poles, ζ and ω from plots

Second Order System Analysis - Example

Compute ω_n and ζ for the following transfer function

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Let us compare it with general form of second order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second Order System Analysis – Example

Transfer function

$$\boldsymbol{G(s) = \frac{2}{ms^2 + bs + k}}$$

(assume $\boldsymbol{m = 3, k = 2}$ and $\boldsymbol{b = 8}$)

$$\boldsymbol{G(s) = \frac{2}{3s^2 + 8s + 2}}$$

Now what we do, what is $\boldsymbol{\omega_n}$ and what is $\boldsymbol{\zeta}$? Let us compare it with more general form

$$\boldsymbol{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

Second Order System Analysis – Example

Now we have the following:

$$\boldsymbol{G(s) = \frac{2}{3s^2 + 8s + 2}}$$

Let us first eliminate the term 3 from this transfer function

$$\boldsymbol{G(s) = \frac{2/3}{s^2 + 8/3s + 2/3}}$$

Comparing it with standard form, we obtain $\boldsymbol{\omega_n^2} = \boldsymbol{2/3}$, which gives us $\boldsymbol{\omega = 0.8165}$

Let us determine $\boldsymbol{\zeta}$ now, which can be computed as follows: **(2)** $\boldsymbol{(\zeta)}$ $\boldsymbol{(\omega_n)}$ **=** $\boldsymbol{\frac{8}{3}}$

$$\mathbf{(2)\ (\zeta)\ (0.8165) = \frac{8}{3}}$$

Second Order System Analysis – Example

Which gives us **ζ = 1.6330.** Now based on **ζ,** what would be the type of step response (underdamped or overdamped or undamped or critically damped)

So, for this system, the response type will be over damped and the poles would be real and unequal

Let us use MATLAB to verify the same

$$-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Second Order System Analysis – Example

MATLAB code for obtaining step response

```
num = [2] ;
den = [3 8 2] ;
step(num , den)
```

MATLAB code for analyzing step response

```
zeta = 1.6330;
omegan = 0.8165;
-zeta*omegan + (omegan*sqrt(zeta*zeta-1))
-zeta*omegan - (omegan*sqrt(zeta*zeta-1))
```