Control Systems - 7th Semester

Lecture 2

Model

A model is representation or abstraction of reality/system

Who invent model? We, human beings, invent model based on our knowledge

This means the more knowledge a person has, the better he/she can write a model

What is mathematical model?

- ❑ A set of equations (linear or differential) that describes the relationship between input and output of a system

Types of Model and System

In mathematics, we broadly classify systems into 2 types, namely stochastic (random, probabilistic, uncertain) and deterministic (fixed relation between input and output)

To write model for a deterministic system, there are three techniques

- Black Box
- Grey Box
- White Box

Black Box Model

It is used when only input and output data are available

The internal dynamics are either too complex or totally unknown (sometimes for cyber security purposes, we do not want to label/show the hardware)

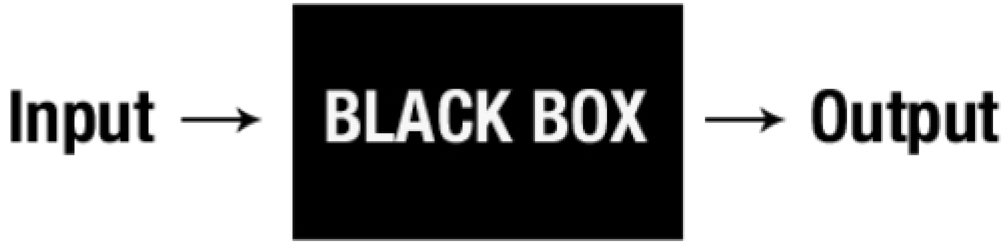


Figure: Black Box Model of a System

It is very hard to analyze or conclude something based on I/O data without having knowledge about the system

Grey Box Model

It is used when input and output data is known, plus some information (information means knowledge) about internal dynamics of the system are known

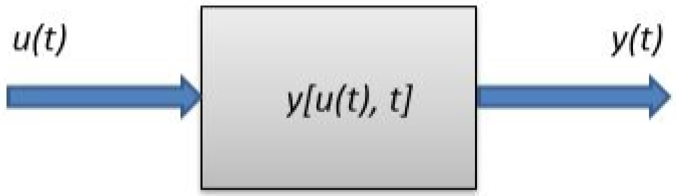


Figure: Grey Box Model of a System

In complex systems, we use grey box modeling to identify or estimate the system model

White Box Model

It is used when the input, output and internal dynamics of the system are known

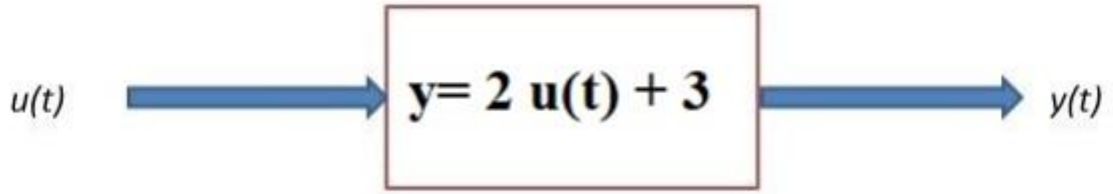


Figure: Grey Box Model of a System

White box models are very easy to predict any future values

Obtaining white box models requires us to know exact mathematical formulas and equations

Summary of Model Types

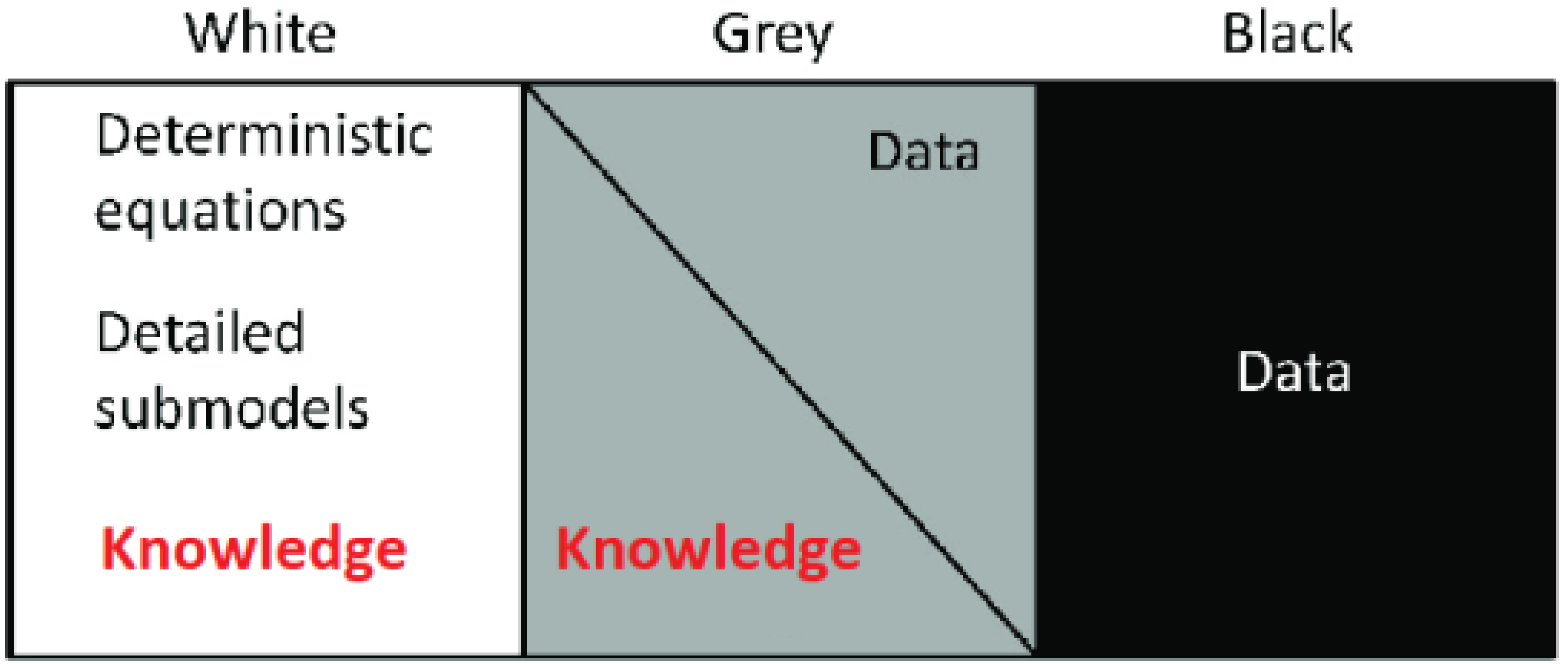


Figure: Techniques for obtaining models of a system

Equation Writing

In mathematics, we can write static equations as follows:

$$\boldsymbol{y = 2u + 3}$$

If the equations are a function of time (means time-varying), then we equations as follows:

$$\boldsymbol{y(t) = 2u(t) + 3}$$

Equation Example

For example: if ***u(t)*** is given as follows:

Time	Value
1	1
2	3
3	5
4	8

Table: Example of *u(t)*

MATLAB code for plotting the above signal ***u(t)***

```
clear;
clc;
u=[1 3 5 8];
stem(u)
```

Equation Plot

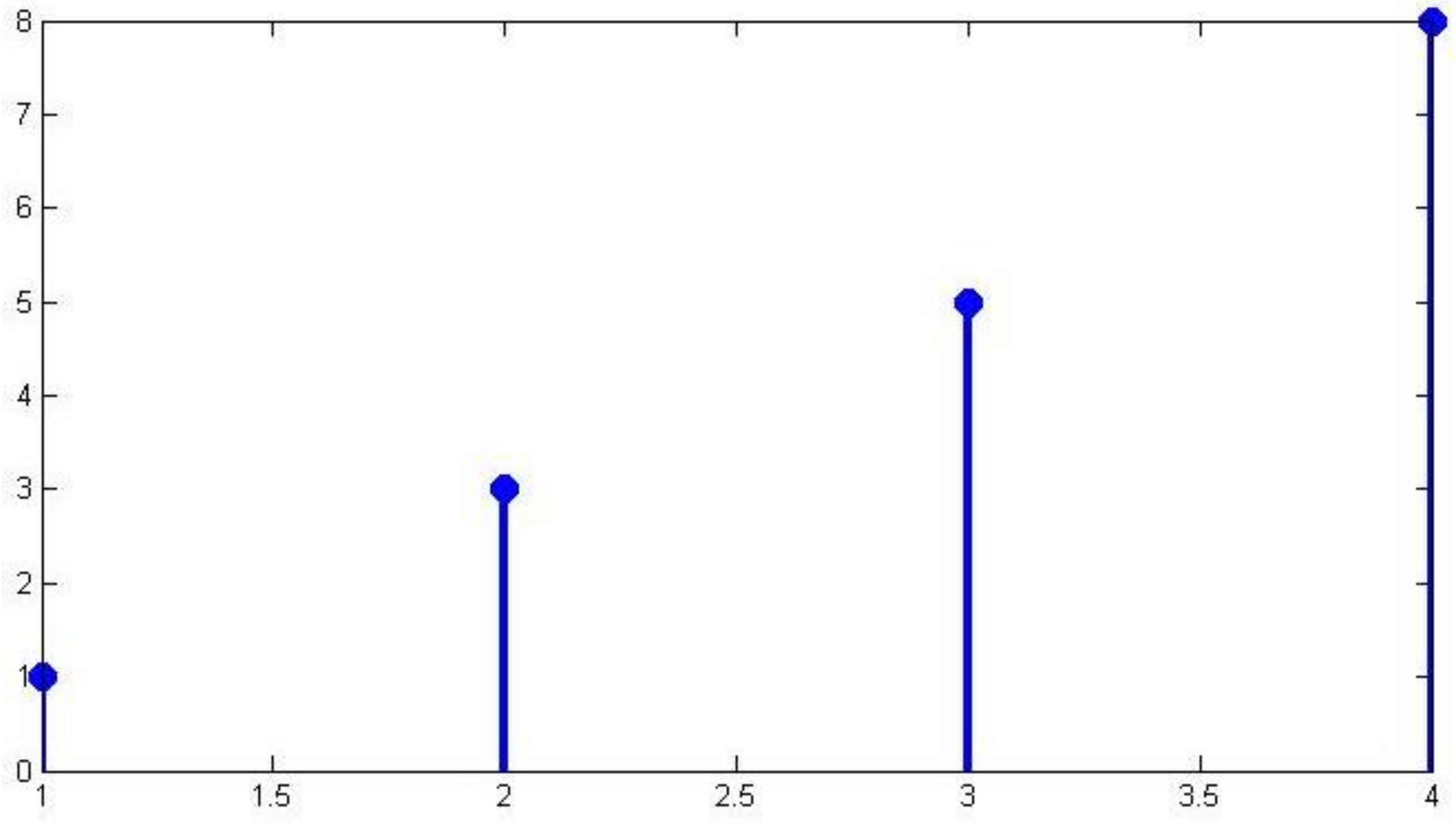


Figure: Plot of *u(t)*

Equation Plot

Let us put axis function in MATLAB code as follows:

```
clear;

clc;

u=[1 3 5 8];

stem(u)

axis([0 6 0 9])
```

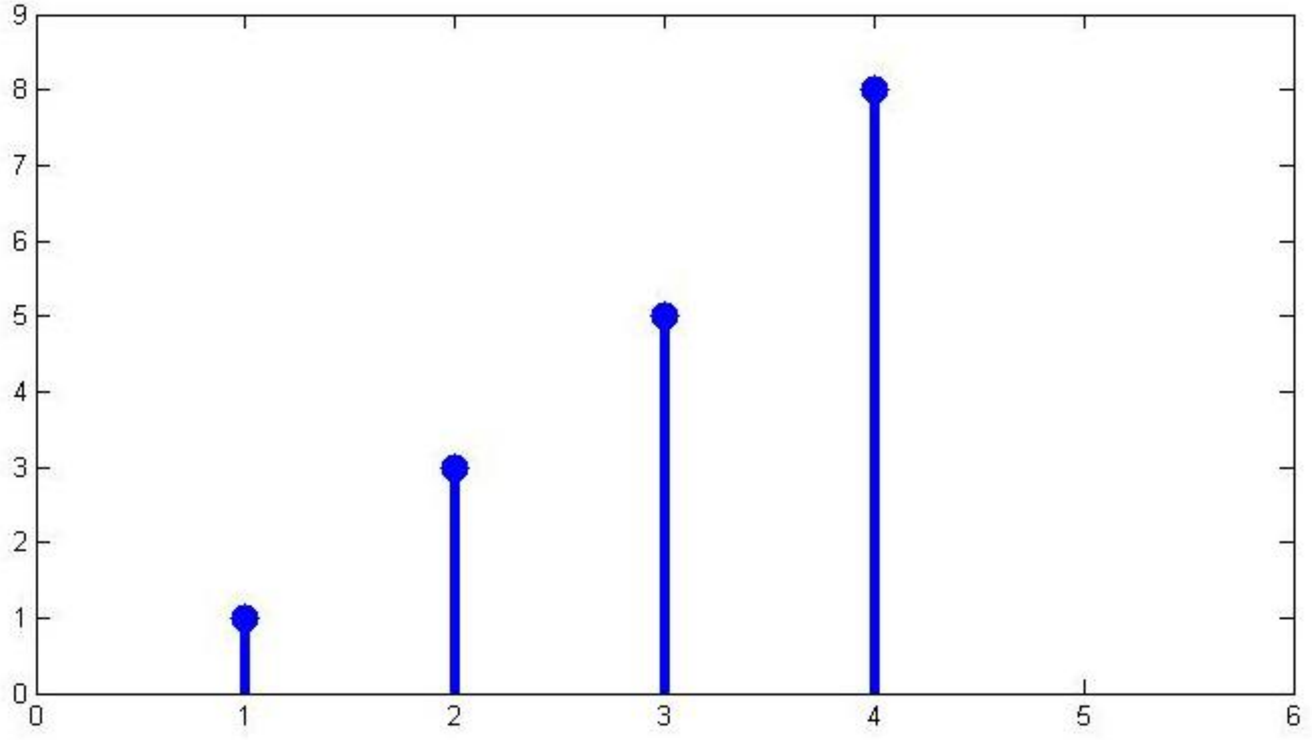


Figure: Plot of *u(t)* with extended axis

Still missing something: the labels for x-axis and y-axis

Equation Plot

Let us put xlabel and ylabel function in MATLAB code as follows:

```
clear;

clc;

u=[1 3 5 8];

stem(u)

axis([0 6 0 9])

xlabel('Time (sec)')

ylabel('Amplitude')
```

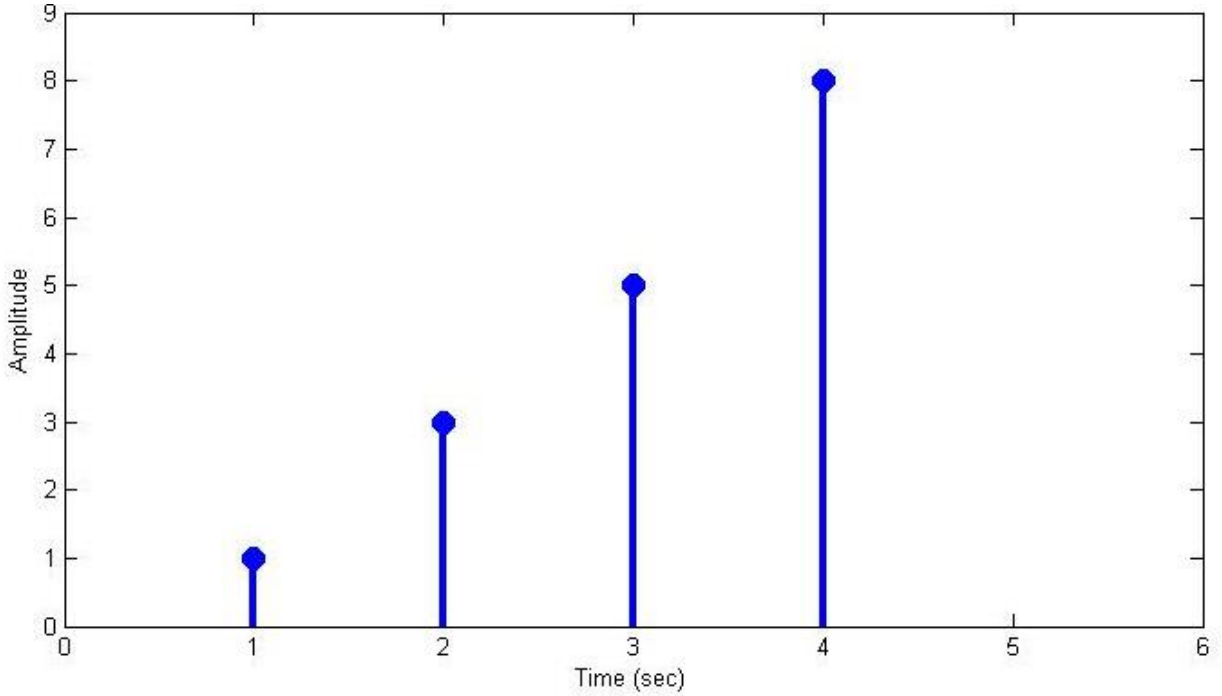


Figure: Plot of *u(t)* with correct labels of axes

Equation Plot

Now, we have the following equation:

$$y(t) = 2u(t) + 3$$

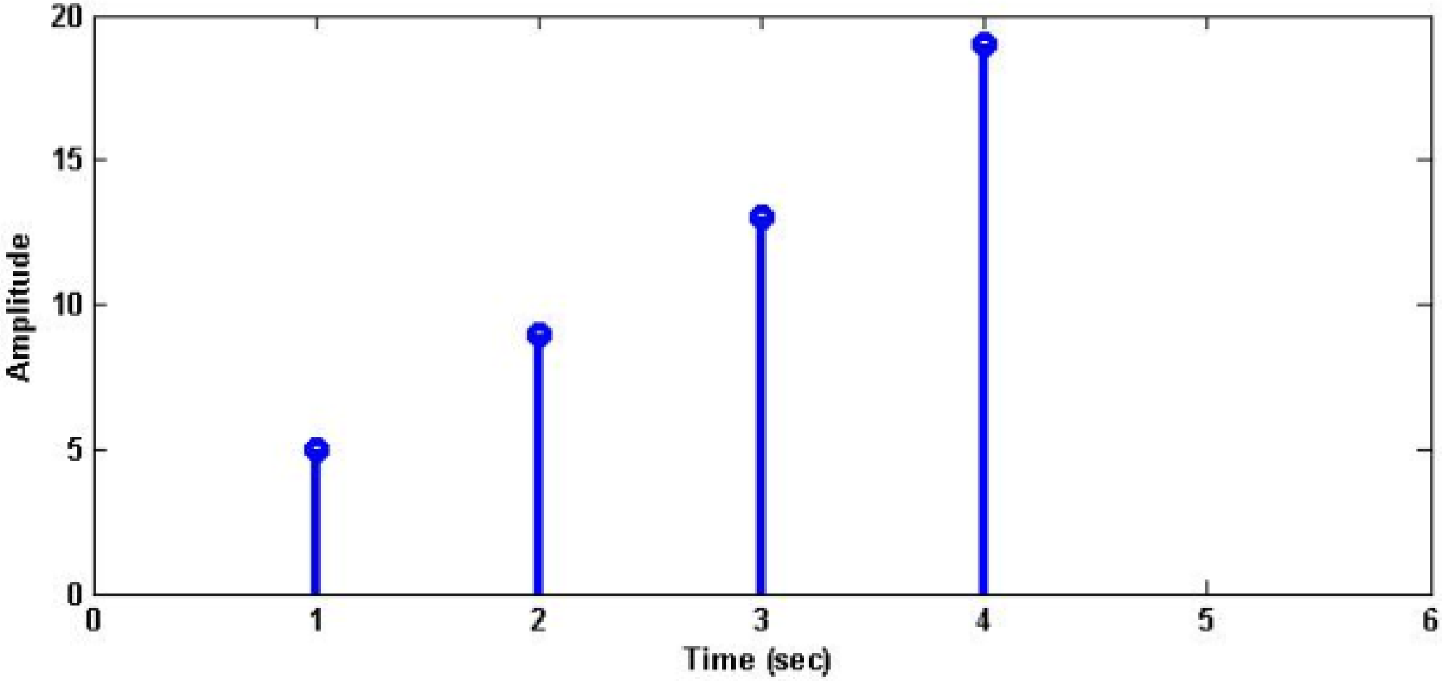


Figure: Plot of *y(t) = 2u(t) + 3*

Equation Plots

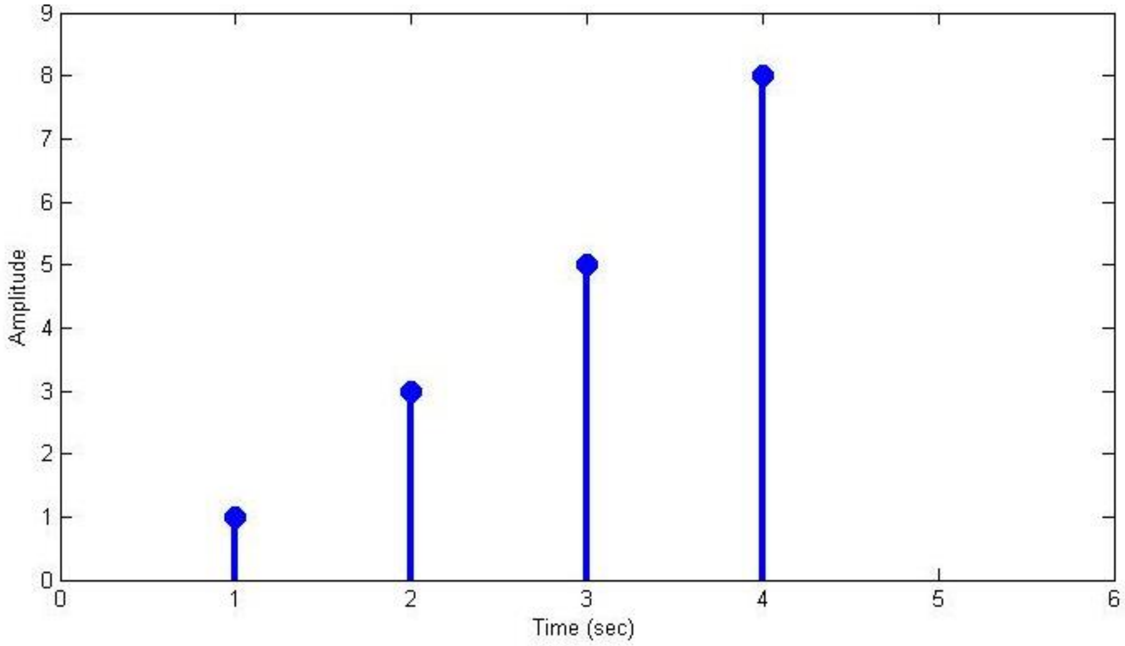


Figure: Plot of *u(t)*

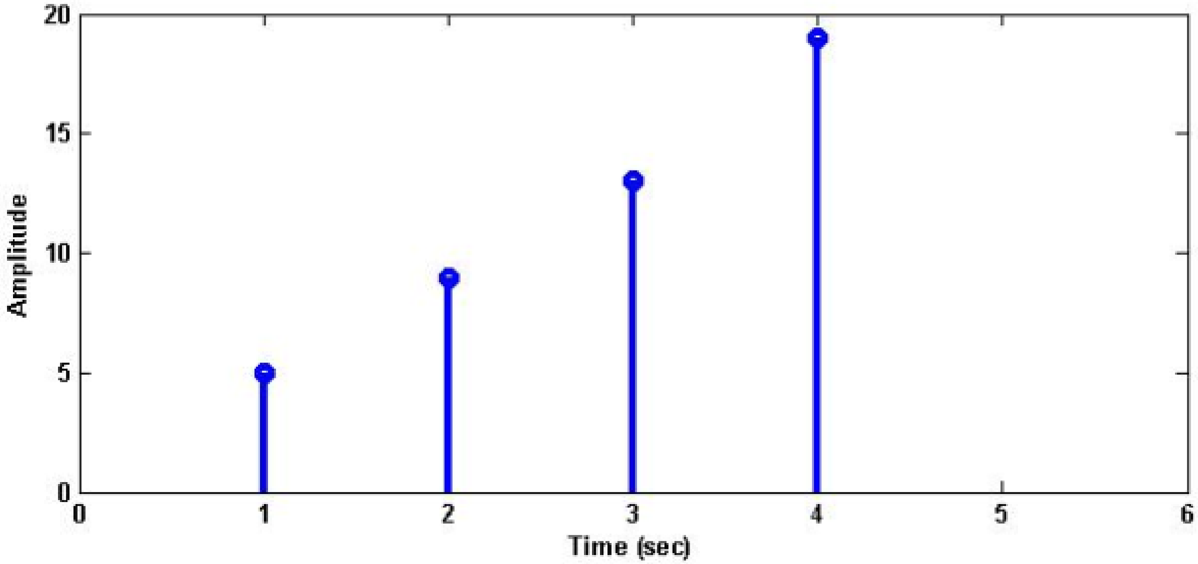


Figure: Plot of *y(t) = 2u(t) + 3*

Transfer Function

Transfer function: Mathematical model (or relationship) between input and output of a system

Transfer function G(s) = $\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}}$

Transfer Function Symbol

Ohm Law: $\boldsymbol{V = IR}$

Is $\boldsymbol{A = BC}$ Ohm Law?

The answer is yes, but if ***A*** denotes voltage, ***B*** denotes current and ***C*** denotes resistance

V is popular symbol to denote voltage and similarly ***I*** for current and ***R*** for resistance

A transfer function can be denoted by any alphabet from ***A*** to ***Z***, but popular symbols are ***G(s), P (s), H(s)*** and ***T (s)***

Transfer Function Symbol

Example 1: Find the transfer function of the system given by:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2\,y(t) = \ x(t)$$

Where x(t) is input and y(t) is output

Example 2: Find the transfer function of the system given by:

$$A\frac{d^2x(t)}{dt^2} = C\frac{dy(t)}{dt} - B\frac{dx(t)}{dt}$$

Where y(t) is input and x(t) is output

Transfer Function Analysis

Zeros: Roots of numerator of a transfer function

Poles: Roots of denominator of a transfer function

What information do poles and zeros convey (you have already learnt this in DSP)?

A continuous-time system is stable if all poles are negative

A discrete-time system is stable if all poles are within unit circle

Stability Definition

Absence of input: If the output goes towards zero (or an equilibrium point),then the system is stable

Presence of bounded input: If the output remains bounded, then the system is stable

Sometimes, we call it BIBO stability. Going back towards our discussion:

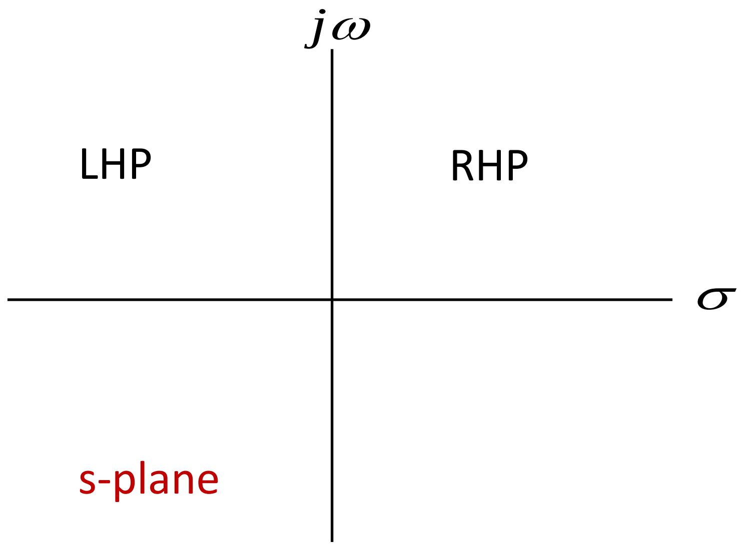
If all poles are negative, then the system is stable

Stability of Control Systems

The poles and zeros of the system are plotted in s-plane to check the stability of the system

- ❑ If all the poles of the system lie in left half plane the system is said to be **Stable**
- ❑ If any of the poles lie in right half plane the system is said to be **unstable**
- ❑ If pole(s) lie on imaginary axis the system is said to be **marginally stable**

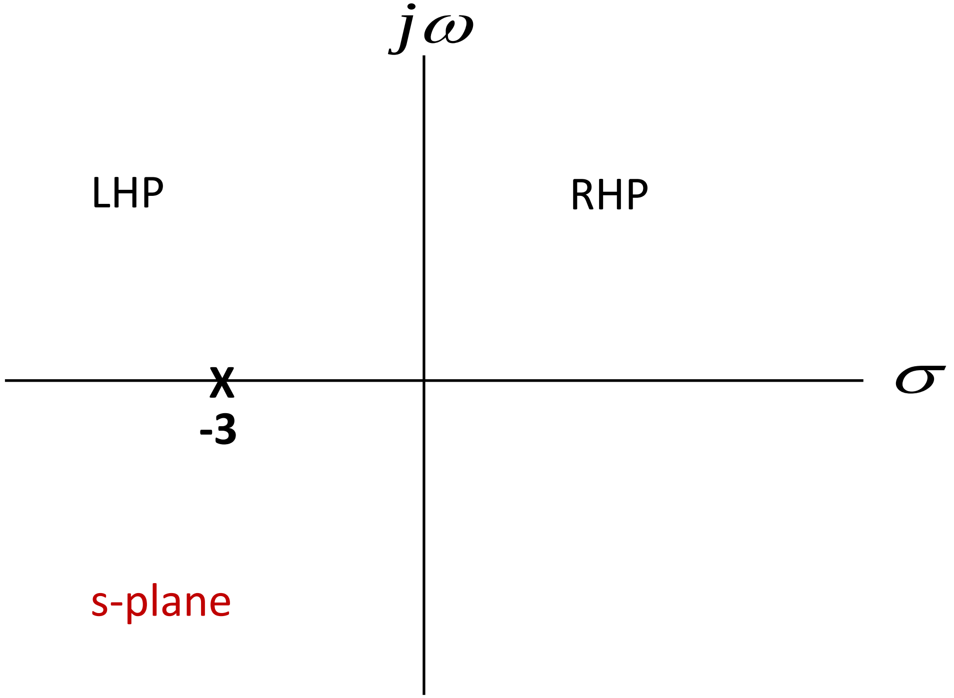
Poles of the system are represented by '**x**' and zeros of the system are represented by '**o**'.



Stability of Control Systems

For Example: $G(s) = \frac{C}{As+B}$ *if A = 1, B = 3 and C = 10*

Pole = -3



Stability Analysis

Consider the following transfer functions

- ❑ Calculate poles and zeros of system
- ❑ Draw the pole-zero map
- ❑ Determine stability of system

$$G_1(s) = \frac{(s-3)}{(s+5)}$$

$$G_2(s) = \frac{(s-3)}{(s-5)}$$

$$G_3(s) = \frac{(s-3)(s+2)}{(s+5)(s-10)}$$

$$G_4(s) = \frac{s(s+2)}{(s+5)(s-10)}$$

$$G_5(s) = \frac{3s}{(s+5)(s-10)}$$

$$G_6(s) = \frac{3s}{2s(s+5)(s-10)}$$

System Modeling

To recap, in control systems literature, a system has

- input
- output
- variables
- constants

In order to obtain white-box models, we introduce a famous (very popular) technique of modeling which is called as state-space model

In state-space modeling of a system, we classify system elements (or parameters) as either constants or variables

System Modeling

To summarize again, we are studying state-space models of deterministic systems

Inside a system, we have either constant or variable parameters

Among the variables present in a system, we choose some variables as state-space variable (based on certain criteria which we will study later on), and call them state-space variables

State-space variables: Those variables which completely describe the behavior of a system

State-space variables are abbreviated as ss variables (or sometimes state variable)

State-space Model

State-space variables are used to obtain mathematical model of a system

In a system, we can have one or two or many state-space variables

If there is one (1) state-space variable, then we denote it by x

In case of more than one (1) state-space variables, then we stack them in a vector and denote it by $\boldsymbol{x}$

State-space Model

In control systems literature, we use the symbol $\boldsymbol{u}(\boldsymbol{t})$ to denote input and $\boldsymbol{y}(\boldsymbol{t})$ to denote output

Before showing you mathematics, I summarize again the main points:

- input denoted by $\boldsymbol{u}(\boldsymbol{t})$
- output denoted by $\boldsymbol{y}(\boldsymbol{t})$
- variables
- constants
- state-space variables denoted by $\boldsymbol{x}(\boldsymbol{t})$

If any parameter is just constant (not a function of time $\boldsymbol{t}$), then we do not write the term $(\boldsymbol{t})$

State-space Model

The standard state-space model (or template) is as follows:

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t})}$$ State Equation

$$\boxed{\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t})}$$ Output Equation

where

- $\boldsymbol{x}(\boldsymbol{t})$ denotes the vector having state-space variables
- $\dot{\boldsymbol{x}}$ or $\frac{\boldsymbol{dx}}{\boldsymbol{dt}}$ represent the derivative of state-space variables
- $\boldsymbol{u}(\boldsymbol{t})$ denotes the input to a system
- $\boldsymbol{y}(\boldsymbol{t})$ denotes the output of a system
- A denotes system matrix and B denotes input matrix
- C denotes output matrix and D denotes feedforward matrix

State-space Model

A system is composed of variables, constants, inputs and outputs

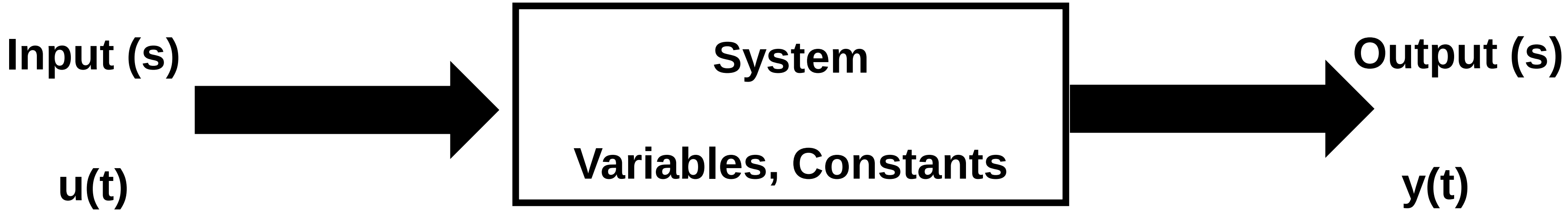


Figure: Graphical sketch of a system

State-space Model

A system is composed of variables, constants, inputs and outputs

Input (s)

u(t)

System

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Output (s)

y(t)

Figure: Graphical sketch of a system

Differential Equations to State Space

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots\ldots + a_1\frac{dy}{dt} + a_o\, y = b_0 u$$

Choosing the state variables x_i

$$x_1 = y \qquad \dot{x}_1 = \frac{dy}{dt} \qquad \dot{x}_1 = x_2$$

$$x_2 = \frac{dy}{dt} \qquad \dot{x}_2 = \frac{d^2y}{dt^2} \qquad \dot{x}_2 = x_3$$

$$x_3 = \frac{d^2y}{dt^2} \qquad \dot{x}_3 = \frac{d^3y}{dt^3} \qquad \vdots \qquad \dot{x}_n = -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0u$$

$$\vdots \qquad \vdots$$

$$x_n = \frac{d^{n-1}y}{dt^{n-1}} \qquad \dot{x}_n = \frac{d^n y}{dt^n} \qquad \dot{x}_{n-1} = x_n$$

Differential Equations to State Space

State Equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

Output Equation

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Differential Equations to State Space

Example: Write the state space representation for the system described by

$$\ddot{y} + 4\dot{y} + 3y = 2u$$

Solution:

Converting a Transfer Function to State Space

Find the state-space representation for the transfer function

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\mathbf{24}}{\boldsymbol{s^3 + 9s^2 + 26s + 24}}$$

Step 1: Cross-multiplying

Step 2: Inverse Laplace Transform

Step 3: Select state variables