Control Systems - 7th Semester

Lecture 9

Controller Design Techniques

Recalling again, we know that there are 3 types of techniques to design controllers which are:

- Full-state feedback controller or state feedback controller
- Observer-based state feedback controller
- PID Controller

Last week, we studied (and then simulated) the design of full-state feedback controller and its pre-requisites.

Today, we will study the design and pre-requisites of observer-based state feedback controller.

Controller Design Techniques

What is the difference between state feedback and observer-based state feedback controller?

It depends on matrix C whether it is identity matrix or not.

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$

$$y = Cx(t) + Du(t)$$

Controller Design Techniques

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$

$$y = Cx(t) + Du(t)$$

For example:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If such a system is unstable, how can we stabilize it using controller? Observer-based state feedback controller may be the possible solution in such a scenario

Observer based state feedback controller

There are 3 pre-requisites to full-fill before we can proceed to design of observer-based state feedback controller.

- Matrix $\boldsymbol{C}$ must NOT be equal to identity and matrix $\boldsymbol{D}$ must be equal to zero (or absent)
- The system must pass controllability test.
- The system must pass observability test.

The first 2 pre-requisites seem easy or familiar but what is observability test. Let us study observability test.

Pre-req 3: Observability Test

A system is observable or it passes observability test if the following criteria is satisfied:

- First, determine the order of the system and call it n.
- Second, using n, construct matrix Q follows:

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{1}$$

- Third, compute rank of matrix Q
- Finally, check if rank of matrix Q is equal to n or not.

If $rank(Q) = n$, then the system is observable and we can proceed to design of controller, otherwise STOP. No controller can be designed.

Example

Check whether do we need to design a controller for the following system:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If we need a controller, identify which controller to design, and then design it and place the eigenvalues at $(-3, -5)$. If you need observer, then place observer eigen values at $(-10, -20)$.

Checking Stability to know whether we require controller

First, we check stability of this system. The eigenvalues of this system can be obtained from $det(\lambda I - A) = 0$

$$
\begin{aligned}
det(\lambda I - A) &= det \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \\
&= det \begin{bmatrix} \lambda - 2 & -3 \\ 0 & \lambda - 5 \end{bmatrix} \\
&= (\lambda - 2)(\lambda - 5) - (0)(-3) \\
&= (\lambda - 2)(\lambda - 5) - (0) \\
&= (\lambda - 2)(\lambda - 5)
\end{aligned}
$$

The eigenvalues of matrix A are at 2 and 5, which indicates it is an unstable system.

Deciding controller type

Now, which controller to choose?

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As matrix C is NOT equal to identity matrix, we proceed to design of observer-based state feedback controller.

Prerequisite 2- Controllability Test

Let us compute now pre-requisite number $\mathbf{2}$ which is the controllability test.

In this case $\boldsymbol{n} = \mathbf{2}$, we matrix $\boldsymbol{P}$ would have the following shape:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix}$$

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{8} \\ \mathbf{2} & \mathbf{10} \end{bmatrix}$$

$$\boldsymbol{det}(\boldsymbol{P}) = -\mathbf{6}$$

As determinant $\boldsymbol{P}$ is non-zero, so $\boldsymbol{rank}(\boldsymbol{P}) = \mathbf{2}$, and it passes controllability test.

Let us proceed to Observability Test.

Prerequisite 3 - Observability Test

Let us compute now pre-requisite number $\mathbf{3}$ which is the observability test.

In this case $\boldsymbol{n = 2}$, we matrix $\boldsymbol{Q}$ would have the following shape:

$$\boldsymbol{Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \tag{2}$$

$$\boldsymbol{det(Q) = 3}$$

As determinant $\boldsymbol{Q}$ is non-zero, so $\boldsymbol{rank(Q) = 2}$, and it passes observability test.

Let us proceed to design of controller now.

Design Steps - Observer Design

To design controller, first we need to design observer and then state feedback controller as follows:

Observer:

- Construct matrix $\boldsymbol{L}$ whose size is transpose the size of $\boldsymbol{C}$
- Populate matrix $\boldsymbol{L}$ with elements starting from $\boldsymbol{l_1}$, $\boldsymbol{l_2}$ and so on
- Post-multiply $\boldsymbol{C}$ with $\boldsymbol{L}$ to obtain $\boldsymbol{LC}$, and then compute $\boldsymbol{det\big(sI-(A-LC)\big)}$
- Obtain the desired characteristic equation for observer and compare coefficients to obtain the values of $\boldsymbol{l_1}$, $\boldsymbol{l_2}$, and so on

Design Steps - Controller Design

State feedback Controller:

- Construct matrix $\boldsymbol{K}$ whose size is transpose the size of $\boldsymbol{B}$
- Populate matrix $\boldsymbol{K}$ with elements starting from $\boldsymbol{k_1}$, $\boldsymbol{k_2}$ and so on
- Pre-multiply $\boldsymbol{B}$ with $\boldsymbol{K}$ to obtain $\boldsymbol{BK}$, and then compute $\boldsymbol{det\big(sI - (A - BK)\big)}$
- Obtain the desired characteristic equation and compare coefficients to obtain the values of $\boldsymbol{k_1}$, $\boldsymbol{k_2}$, $\boldsymbol{k_3}$ and so on

Solution - Observer Design

$$\boldsymbol{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

$$\boldsymbol{LC} = \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

$$\boldsymbol{A - LC} = \begin{bmatrix} 2 - l_1 & 3 \\ -l_2 & 5 \end{bmatrix}$$

$$\boldsymbol{sI - (A - LC)} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 - l_1 & 3 \\ -l_2 & 5 \end{bmatrix}$$

$$\boldsymbol{sI - (A - LC)} = \begin{bmatrix} l_1 + s - 2 & -3 \\ l_2 & s - 5 \end{bmatrix}$$

Solution - Observer Design

$$sI - (A - LC) = \begin{bmatrix} s - 2 + k_1 & -3 + k_2 \\ 2k_1 & 2k_2 + s - 5 \end{bmatrix}$$

$$det(sI - (A - LC)) = s^2 + (l_1 - 7)s + (3l_2 - 5l_1 + 10)$$

Now lets compare it with desired characteristic equation:

$$(s + 10)(s + 20) = s^2 + 30s + 20$$

Compare coefficients to obtain values of l_1 and l_2.

Solution - State Feedback Controller

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$BK = \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 2 - k_1 & 3 - k_2 \\ 0 - 2k_1 & 5 - 2k_2 \end{bmatrix}$$

$$sI - (A - BK) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 - k_1 & 3 - k_2 \\ 0 - 2k_1 & 5 - 2k_2 \end{bmatrix}$$

$$sI - (A - BK) = \begin{bmatrix} s - 2 + k_1 & -3 + k_2 \\ 2k_1 & 2k_2 + s - 5 \end{bmatrix}$$

Solution - State Feedback Controller

$$sI - (A - BK) = \begin{bmatrix} s - 2 + k_1 & -3 + k_2 \\ 2k_1 & 2k_2 + s - 5 \end{bmatrix}$$

$$det(sI - (A - BK)) = s^2 + (k_1 + 2k_2 - 7)s + (-4k_2 + 10)$$

Now lets compare it with desired characteristic equation:

$$(s + 3)(s + 5) = s^2 + 8s + 15$$

Compare coefficients to obtain values of k_1 and k_2.