Stat202C Project No. 3 (15 points)

Due date: May 28, Monday 11:55pm on CCLE

Cluster sampling of the Ising/Potts model

This project is a continuum of Project 2. For simplicity, we consider the Ising model in an n x n lattice (n is between 64 and 256) with 4-nearest neighbor. X is the image (or state) defined on the lattice and the variable X_s at each site s takes value in {0, +1}. The model is

$$\pi(X) = \frac{1}{Z}\exp\{\beta \sum_{<s,t>} 1(x_s = x_t)\} \text{ or } \frac{1}{Z}\exp\{-\beta \sum_{<s,t>} 1(x_s \neq x_t)\}$$

When n x n is large enough, we know from physics that the probability mass of $\pi(X)$ is focused on the following set uniformly, with zero probability outside the set:

$$\Omega(h) = \{X: \ H(X) = h\}, \qquad H(X) = \frac{1}{2n^2}\sum_{<s,t>} 1(x_s \neq x_t)$$

H(X) is the "sufficient statistics" of X. Intuitively it measures the length of the total boundaries (cracks) in X and is normalized by the total number of edges. Two images X_1 and X_2 have the same probability if $H(X_1) = H(X_2)$. $\Omega(h)$ is an *equivalence class* of images. Theoretically, in the absence of phase transition, there is a one-to-one correspondence between β and h, $h=h(\beta)$. So, empirically, we can diagnose the convergence by monitoring whether $H(X)$ has converged to a constant value h over time.

We choose three β values: $\beta_1 = 0.65$, $\beta_2 = 0.75$, $\beta_3 = 0.85$. We have the three images X_1, X_2, X_3 at the coalescence time t_1, t_2, t_3 (sweeps) from Project 2. From these images (in project 2) we compute their sufficient statistics h_1^*, h_2^*, h_3^* respectively.

For each $\beta_i, i = 1,2,3,$ we run two Markov chains using Cluster sampling.
MC1 starts from a constant image (black or white) --- $h=0$ is the smallest (no crack); and
MC2 starts from a checkerboard image --- $h=1$ is the largest (cracks everywhere).
When they meet at $\Omega(h_i^*)$, i.e. a set not a state (much easier), we say they have converged.

1, Plot the sufficient statistics $H(X)$ of the current state $X(t)$ over time t and stop when h is within an epsilon distance from h_i^* , $i = 1,2,3$.

2, Mark the convergence time t_1, t_2, t_3 (sweeps) in the plots for comparison between the three parameters and against the Gibbs sampler convergence in project 2.

3, Plot the average sizes of the CPs (number of pixels that are flipped together at each step (sweep) and compare them between the three $\beta_i, i = 1,2,3$ settings.

4. Run the two MCs for $\beta_4 = 1.0$ to see how fast they converge (i.e. meeting in an $H(X)$).

Compare the experiments using two versions of the Cluster sampling algorithm:
Version 1: Form CPs over the entire image and flip them all randomly. So each step is a sweep.
Version 2: Randomly pick up a pixel, and grow a CP from it, flip this CP only. Accumulate the number of pixels that you have flipped and divide the number by n^2 to get the sweep number.

[The total number of tests is: 4 temperatures x 2 initial states x 2 SW versions = 16 trials.]