UCLA School of Education & Information Studies Department of Education

EDUC 231D
Advanced Quantitative Methods: Multilevel Analysis
Winter 2025

Cross-Level Interactions

Lecture 6 Presentation Slides

January 23, 2025

Today's Topics

- Intercept and slopes as outcomes
- R walkthrough #2

Random intercept and slope multilevel model (from the last lecture)

Let's remember this as Model #1

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\,, \qquad r_{ij} \sim N(0, \sigma^2)$$

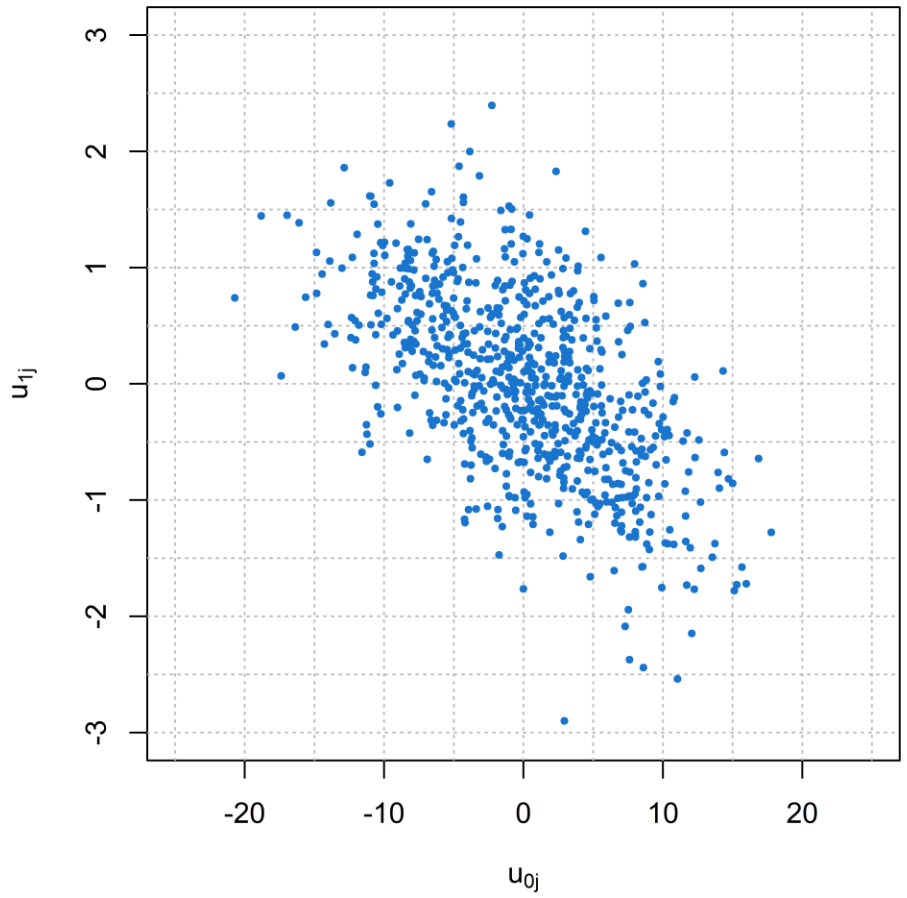
- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$
$$\beta_{1j} = \gamma_{10} + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

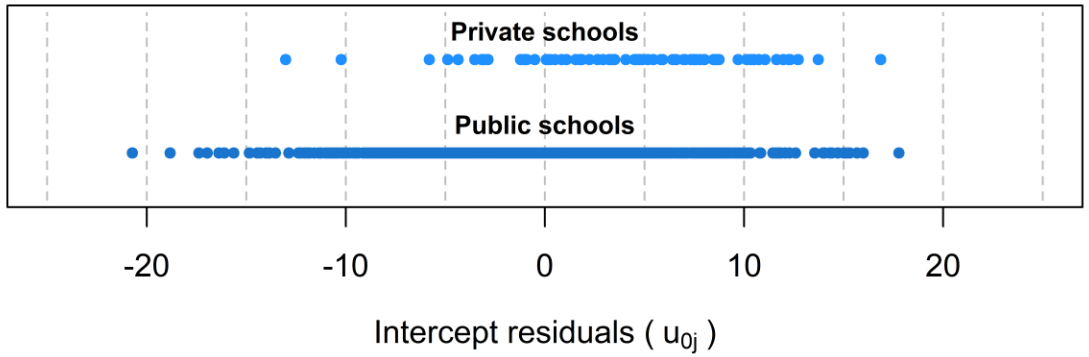
Between-group variation in the intercept and slope

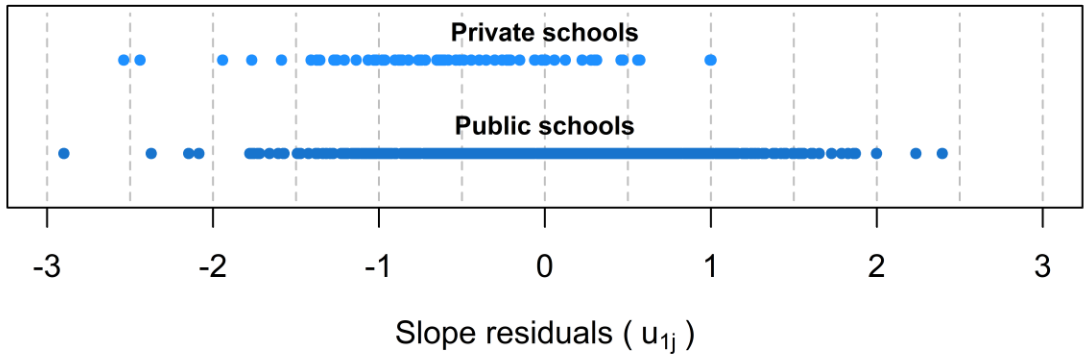
- From last lecture, we saw variability in the u_{0j}'s and u_{1j}'s
- Can we explain some of that variability with some school-level characteristics?



Between-group variation in the intercept and slope

- How are the intercept and slope residuals related to school sector (public vs. private)?





	Public schools (N=672)	**Private schools** (N=70)
Intercept residual (u_{0j})		
Mean (SD)	-0.482 (6.40)	4.62 (5.83)
Median [Min, Max]	-0.225 [-20.7, 17.8]	4.84 [-13.0, 16.9]
Slope residual (u_{1j})		
Mean (SD)	0.0628 (0.772)	-0.603 (0.711)
Median [Min, Max]	0.0686 [-2.90, 2.40]	-0.634 [-2.54, 1.00]

Intercept and Slope as Outcomes

Model intercept and slope as function of level-2 variable(s)

Let's remember this as Model #2

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\,, \qquad r_{ij} \sim N(0, \sigma^2)$$

- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} sector_j + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} sector_j + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

For our example, let's start by testing whether the intercept and slope are related to school sector

Note: *sector* = 1 for private schools and 0 for public schools

Model intercept and slope as function of level-2 variable(s)

- γ_{00} : expected school mean achievement score for a public school
- γ_{01} : expected difference in school mean achievement between private and public schools
- u_{0j} : deviation of the true mean achievement score for school *j* from an expected value based on the school's sector
- τ_{00} : variance in true school mean achievement, conditional on sector; or remaining parameter variance in achievement means after taking sector into account

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\,, r_{ij} \sim N(0, \sigma^2)$$

- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} sector_j + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} sector_j + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

Model intercept and slope as function of level-2 variable(s)

- γ_{10} : expected SES-Achievement slope for a public school
- γ_{11} : expected difference in SES-Achievement slope between private and public schools
- u_{1j} : deviation of the true SES-Achievement slope for school j from an expected value based on the school's sector
- τ_{11} : variance in true SES-Achievement slope, conditional on sector; or remaining parameter variance in slopes after taking sector into account
- $\tau_{01} = \tau_{10}$: covariance between true achievement means and true SES-Achievement slopes, conditional on sector

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\,, r_{ij} \sim N(0, \sigma^2)$$

- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} sector_j + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$

$$\beta_{1j} = \gamma_{10} + \gamma_{11} sector_j + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

Model intercept and slope as function of level-2 variable(s)

- Looking at the combined model clarifies why we are estimating a "cross-level interaction" model

- Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{01} sector_j + \gamma_{10}\left(SES_{ij} - \overline{SES}_{.j}\right) + \gamma_{11}\left(sector_j\right)\left(SES_{ij} - \overline{SES}_{.j}\right) + u_{0j} + u_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}$$

Cross-level interaction: interaction between a level-2 variable and a level-1 variable

Estimate the hierarchical model in R

- Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{01} sector_j + \gamma_{10}\left(SES_{ij} - \overline{SES}_{.j}\right) + \gamma_{11}\left(sector_j\right)\left(SES_{ij} - \overline{SES}_{.j}\right) + u_{0j} + u_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}$$

```
m2 <- lmer(g1rscore ~ 1 + famsesc + sector + sector*famsesc
                + (1 + famsesc | schid), data = ex1)

summary(m2)
```

Estimate the hierarchical model in R

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: g1rscore ~ 1 + famsesc + sector + sector * famsesc + (1 + famsesc |      schid)
   Data: ex1

REML criterion at convergence: 93100.6

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.1289 -0.6694  0.0275  0.6948  3.5955

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schid    (Intercept)  51.32    7.164
          famsesc       3.86    1.965   -0.13
 Residual             233.83   15.292
Number of obs: 11091, groups:  schid, 742

Fixed effects:
               Estimate Std. Error       df t value Pr(>|t|)
(Intercept)     94.6495     0.3169 733.2777 298.691  < 2e-16 ***
famsesc          7.7031     0.2615 584.7418  29.459  < 2e-16 ***
sector           6.5148     1.0304 729.5902   6.322 4.49e-10 ***
famsesc:sector  -3.8923     0.8832 728.1022  -4.407 1.21e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Estimate the hierarchical model in R

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---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

γ_{00} : Mean achievement score for public schools

γ_{01} : mean difference in achievement for private vs. public schools

Estimate the hierarchical model in R

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: g1rscore ~ 1 + famsesc + sector + sector * famsesc + (1 + famsesc |      schid)
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REML criterion at convergence: 93100.6

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famsesc:sector  -3.8923     0.8832 728.1022  -4.407 1.21e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

γ_{10} : mean SES-Ach. slope for public schools

γ_{11} : mean difference in SES-Ach. slope for private vs. public schools

Estimate the hierarchical model in R

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: g1rscore ~ 1 + famsesc + sector + sector * famsesc + (1 + famsesc |      schid)
   Data: ex1

REML criterion at convergence: 93100.6

Scaled residuals:
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---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

τ_{00} : variance in true school mean achievement, conditional on sector

τ_{11} : variance in true SES-Achievement slope, conditional on sector

Level 2 residual variance

- To what extent does school sector account for the between-school variation in mean achievement and the SES-achievement slope?
- Useful to compare the level 2 residual variance estimates from a model without sector to the model with sector

$$\frac{(\tau_{00}^{M1} - \tau_{00}^{M2})}{\tau_{00}^{M1}} \times 100$$

	Model 1 Unconditional model	**Model 2** Control for sector	**Percentage of variance explained**
Intercept variance (τ_{00})	54.89	51.32	6.5%
Slope variance (τ_{11})	4.67	3.86	17.3%

Expanding the intercept and slope models

- We know the private and public schools serve students with different family resources, on average
- Could differences in school mean SES help explain differences in school mean achievement and the SES-achievement slope?
- We can add a measure of school mean SES to the model to investigate this question

Expanding the intercept and slope models

Let's remember this as Model #3

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\,, \qquad r_{ij} \sim N(0, \sigma^2)$$

- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} sector_j + \gamma_{02}\left(schses_j - \overline{schses}_{..}\right) + u_{0j}$$
$$\beta_{1j} = \gamma_{10} + \gamma_{11} sector_j + \gamma_{12}\left(schses_j - \overline{schses}_{..}\right) + u_{1j}$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

The mean SES value for each school is centered on the grand mean SES value, which lets us interpret the other parameter estimates as the value for "a school with average mean SES"

Expanding the intercept and slope models

- Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{01} sector_j + \gamma_{02}\left(schses_j - \overline{schses}_{..}\right)$$
$$+ \gamma_{10}\left(SES_{ij} - \overline{SES}_{.j}\right) + \gamma_{11}\left(sector_j\right)\left(SES_{ij} - \overline{SES}_{.j}\right)$$
$$+ \gamma_{12}\left(schses_j - \overline{schses}_{..}\right)\left(SES_{ij} - \overline{SES}_{.j}\right)$$
$$+ u_{0j} + u_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}$$

Now we have two cross-level interactions

Expanding the intercept and slope models

- What does it mean to have family SES at level 1 and school mean SES at level 2?
- Modeling the within-school relationship at level 1
- Modeling the between-school relationship at level 2

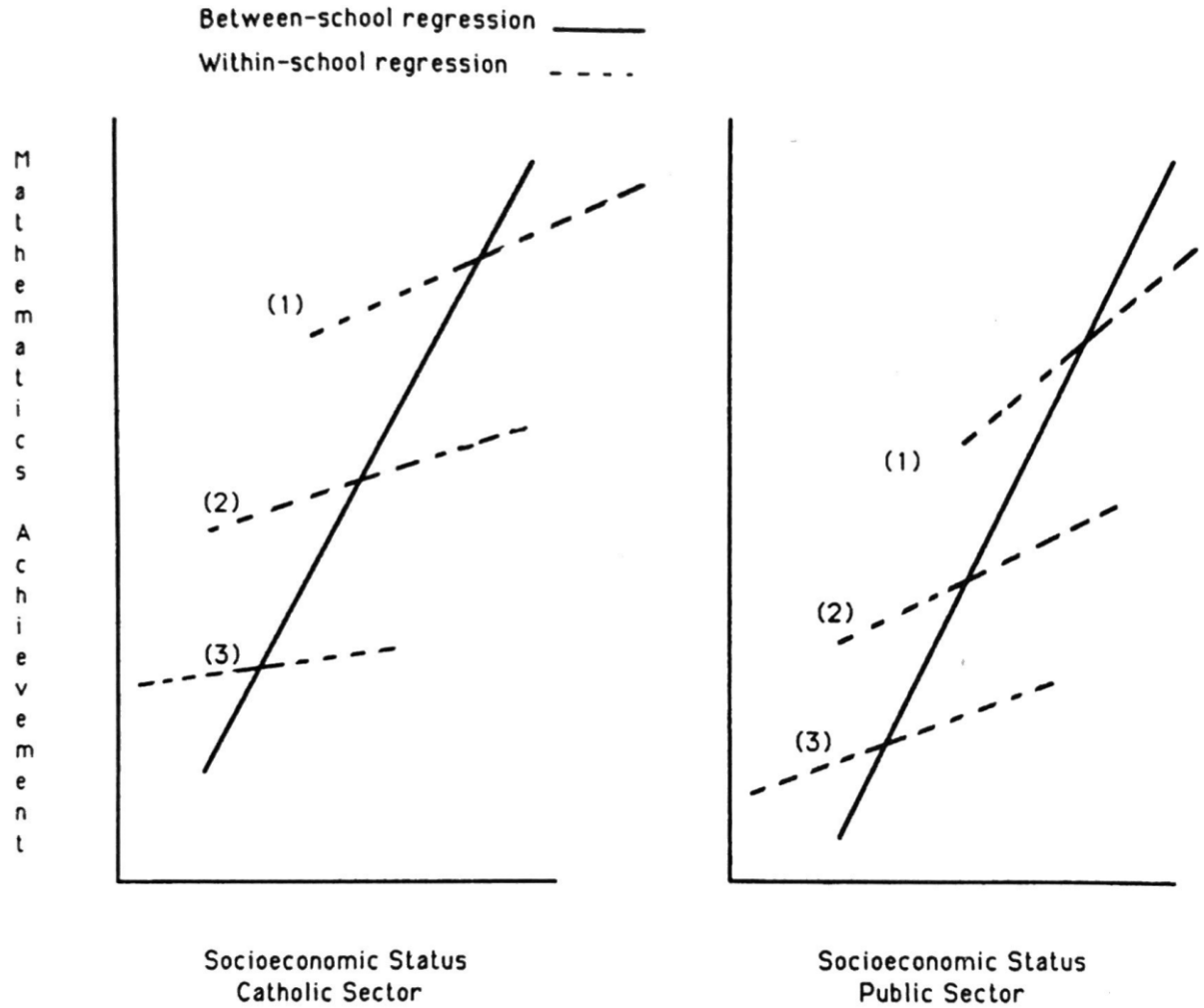


Image source: Raudenbush & Bryk. 2002. *Hierarchical Linear Models: Applications and Data Analysis Methods*. p. 83

Small group discussion, part 1

- In groups of 3-4, take 10 minutes to define the following parameters for Model 3:

- γ_{00} :
- γ_{01} :
- γ_{02} :
- u_{0j} :
- τ_{00} :

- γ_{10} :
- γ_{11} :
- γ_{12} :
- u_{1j} :
- τ_{11} :

Small group discussion, part 1

- γ_{00} : expected school mean achievement score for a public school with average school-mean SES (or controlling for school-mean SES)
- γ_{01} : expected difference in school mean achievement between private and public schools, controlling for school-mean SES
- γ_{02} : expected change in school mean achievement as school-mean SES changes, controlling for sector; or between-school SES-Achievement slope, controlling for sector
- u_{0j} : deviation of the true mean achievement score for school *j* from an expected value based on the school's sector and school-mean SES
- τ_{00} : variance in true school mean achievement, conditional on sector and school-mean SES; or remaining parameter variance in achievement means after taking sector and school-mean SES into account
- γ_{10} : expected within-school SES-Achievement slope for a public school with average school-mean SES (or controlling for school-mean SES)
- γ_{11} : expected difference in the within-school SES-Achievement slope between private and public schools, controlling for school-mean SES
- γ_{12} : expected change in the within-school SES-Achievement slope as school-mean SES changes, controlling for sector
- u_{1j} : deviation of the true within-school SES-Achievement slope for school *j* from an expected value based on the school's sector and school-mean SES
- τ_{11} : variance in true within-school SES-Achievement slope, conditional on sector and school-mean SES; or remaining parameter variance in slopes after taking sector and school-mean SES into account

Estimate the hierarchical model in R

- Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{01} sector_j + \gamma_{02}\left(schses_j - \overline{schses}_{..}\right)$$
$$+ \gamma_{10}\left(SES_{ij} - \overline{SES}_{.j}\right) + \gamma_{11}\left(sector_j\right)\left(SES_{ij} - \overline{SES}_{.j}\right)$$
$$+ \gamma_{12}\left(schses_j - \overline{schses}_{..}\right)\left(SES_{ij} - \overline{SES}_{.j}\right)$$
$$+ u_{0j} + u_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}$$

```
m3 <- lmer(g1rscore ~ 1 + sector + schsesgdm
           + famsesc + sector*famsesc + schsesgdm*famsesc
           + (1 + famsesc | schid), data = ex1)
summary(m2)
```

Estimate the hierarchical model in R

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: g1rscore ~ 1 + sector + schsesgdm + famsesc + sector * famsesc +      schsesgdm * famsesc + (1 + famsesc | schid)
   Data: ex1

REML criterion at convergence: 92568.2

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.1558 -0.6723  0.0367  0.7014  3.7314

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schid    (Intercept)  16.816   4.101
          famsesc       3.869   1.967   -0.03
 Residual             233.840  15.292
Number of obs: 11091, groups:  schid, 742

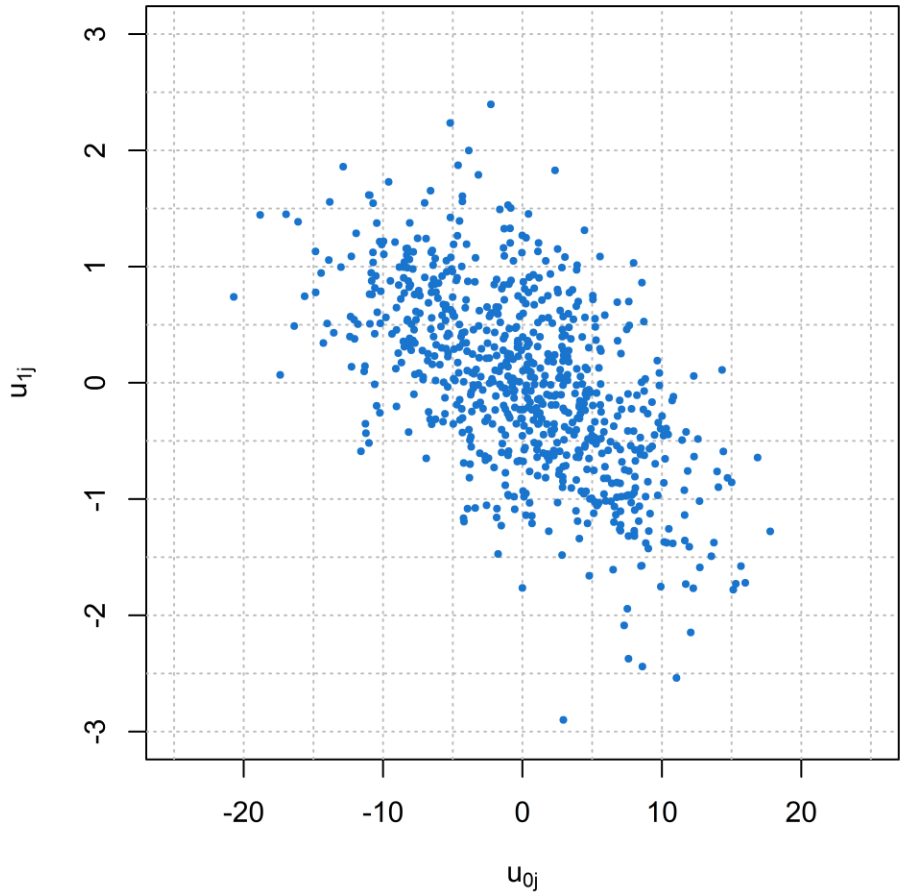
Fixed effects:
                 Estimate Std. Error       df t value Pr(>|t|)
(Intercept)       95.1707     0.2216 720.5701 429.552  < 2e-16 ***
sector             0.3517     0.7509 717.7296   0.468    0.640
schsesgdm         11.5450     0.4145 729.4274  27.856  < 2e-16 ***
famsesc            7.6947     0.2622 602.6232  29.343  < 2e-16 ***
sector:famsesc    -3.6007     0.9185 708.0561  -3.920  9.7e-05 ***
schsesgdm:famsesc -0.5633     0.5237 729.7443  -1.076    0.282
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Small group discussion, part 2

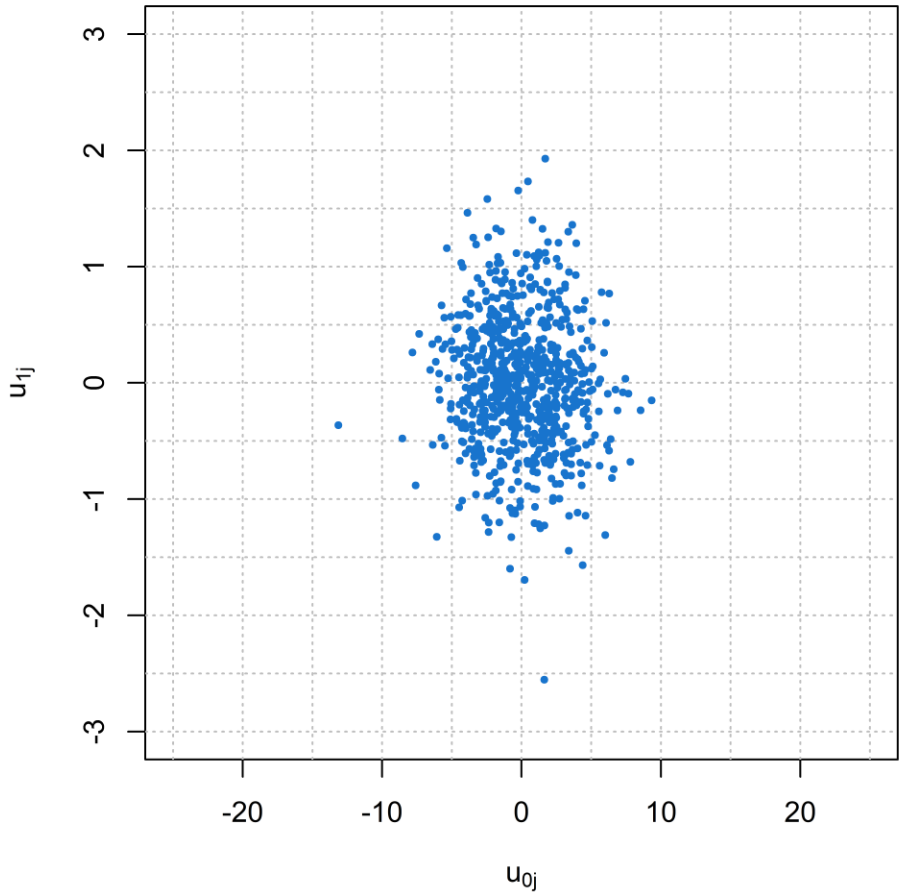
- In groups of 3-4, take 15 minutes to answer the following questions based on the model output on the previous slide:
  - What's the expected mean reading score for a public school with average mean SES?
  - What's the expected mean reading score for a private school with average mean SES?
  - Is the mean reading score for private vs. public schools significantly different after controlling for school mean SES?
  - What's the expected SES-Ach. slope for a public school with average mean SES?
  - What's the expected SES-Ach. slope for a private school with average mean SES?
  - Is the SES-Ach. slope for private vs. public schools significantly different after controlling for school mean SES? How would you describe what this difference means to somebody with not statistics background?
  - How is school mean SES related to school mean reading score (controlling for sector)?
  - How is school mean SES related to the school SES-Ach. slope (controlling for sector)?
  - To what extent does school sector AND school mean SES account for the between-school variation in mean achievement and the SES-achievement slope?

Plot residuals

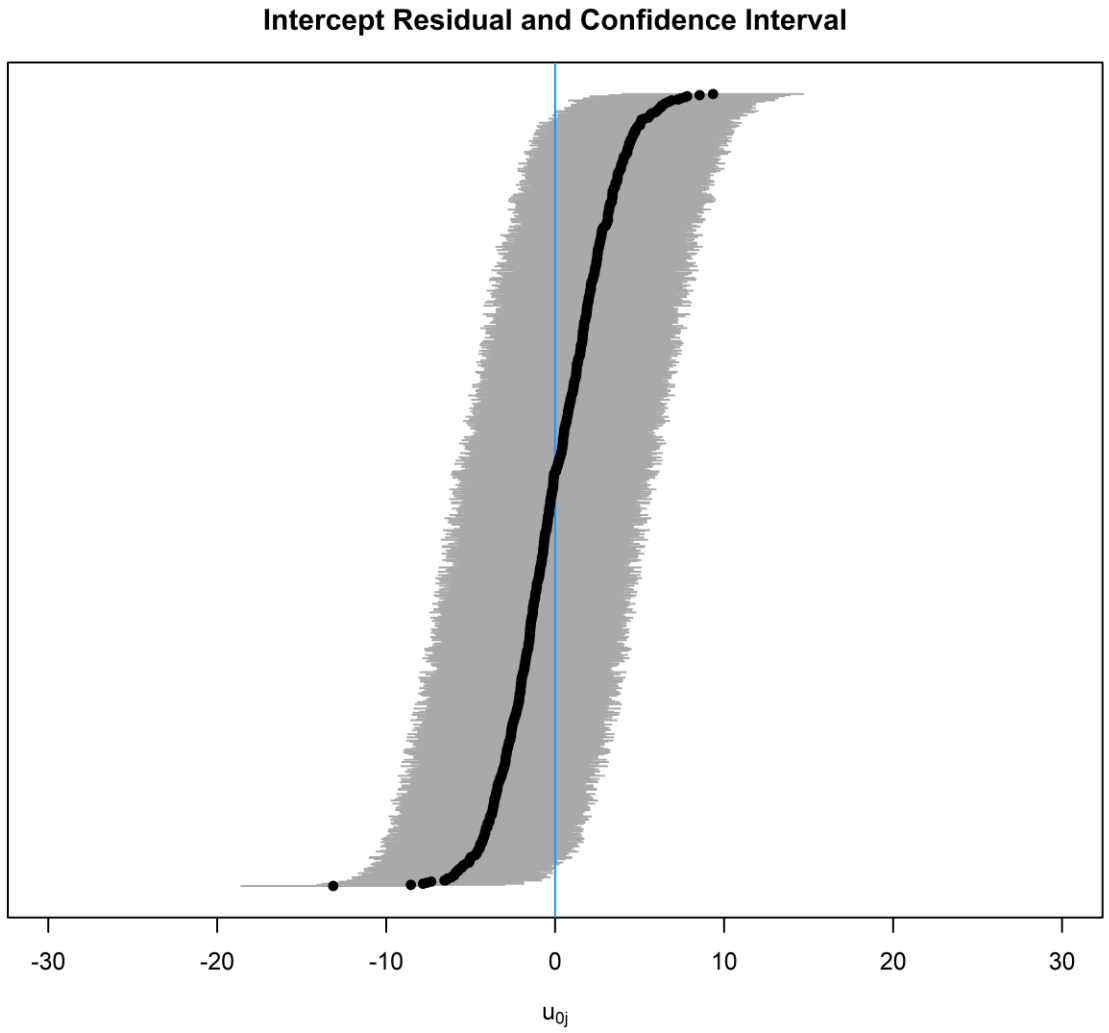
Residual plot from unconditional model (Model 1)

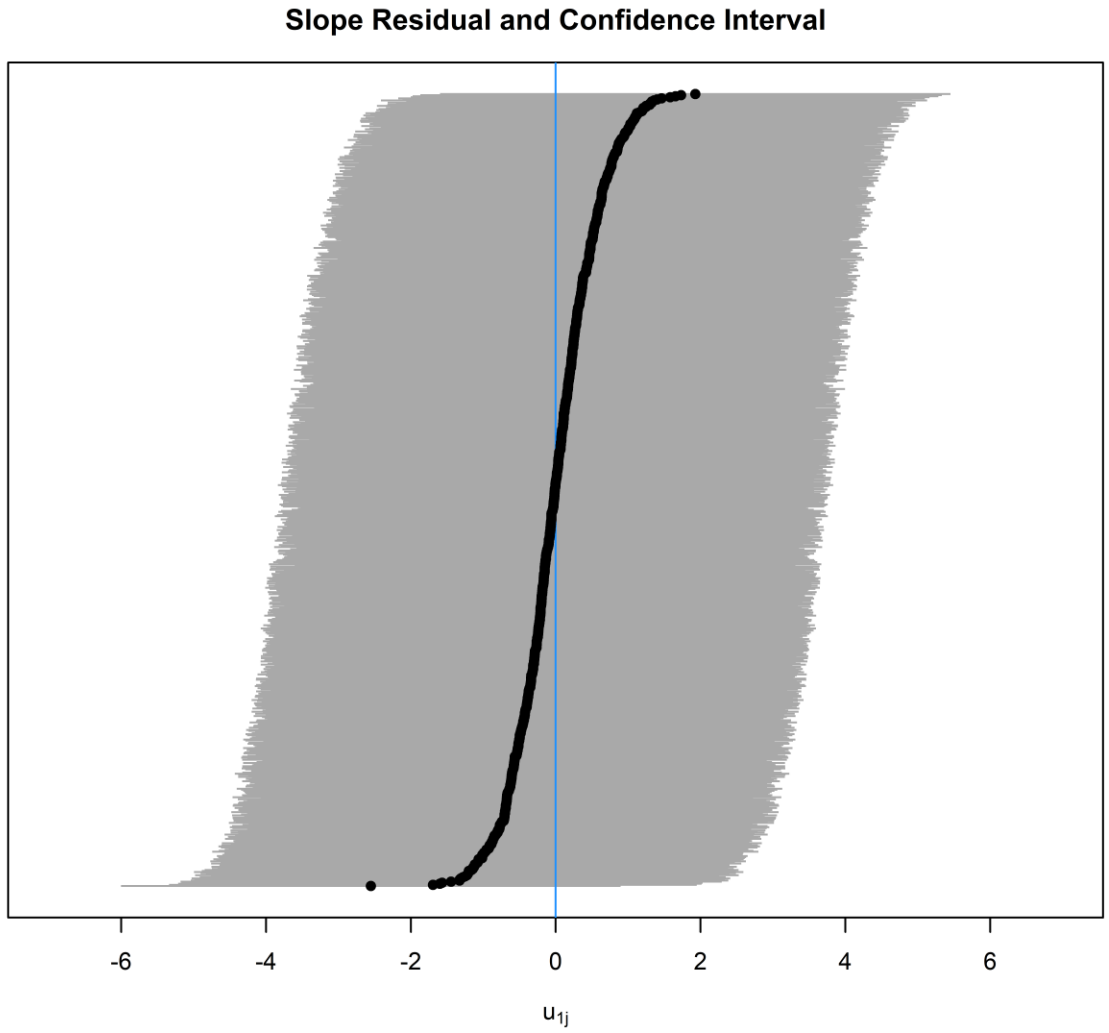


Residual plot from conditional model (Model 3)



Residual caterpillar plots: Model 3





Is there significant between-school residual variance in the slope?

- Model 1: unconditional

npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
4	93,174.34	93,203.60	-46,583.17	93,166.34			
6	93,170.49	93,214.37	-46,579.24	93,158.49	7.86	2	0.02

- Model 3: conditional on sector and school mean SES

npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
8	92,588.6	92,647.1	-46,286.3	92,572.6			
10	92,589.3	92,662.4	-46,284.7	92,569.3	3.3	2	0.19

Using R to Estimate Random Intercept and Random Slope Models

See R Walkthrough #2 Handout