UCLA School of Education & Information Studies Department of Education

EDUC 231D
Advanced Quantitative Methods: Multilevel Analysis
Winter 2025

Two-Level Models with Random Intercept and Random Slopes

Lecture 5 Presentation Slides

January 21, 2025

Today’s Topics

- Level 1: within-school model
- Level 2: between-school model
- Random intercept and slope model
- Level 2 residuals

What does random intercept and random slope mean?

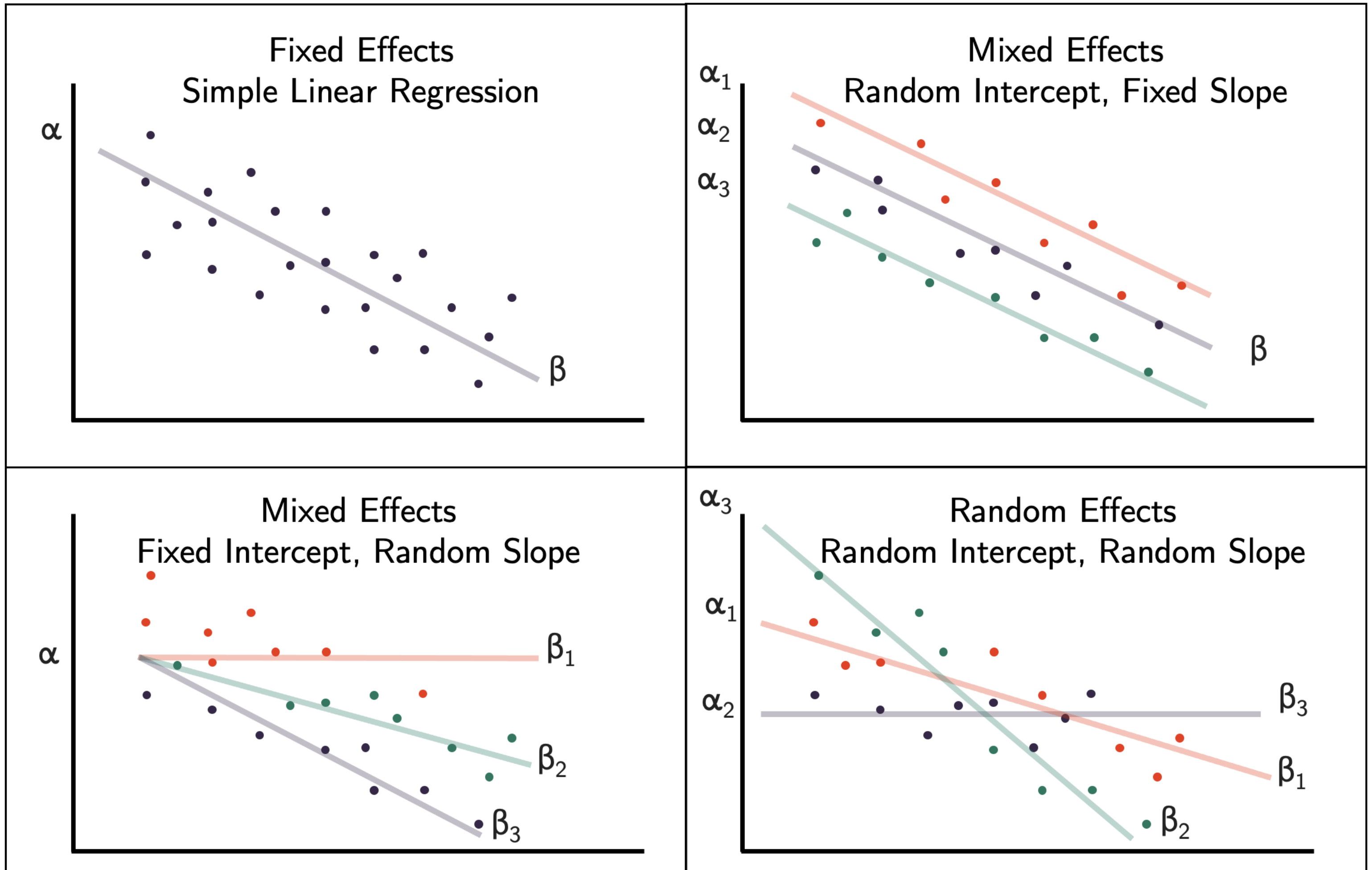
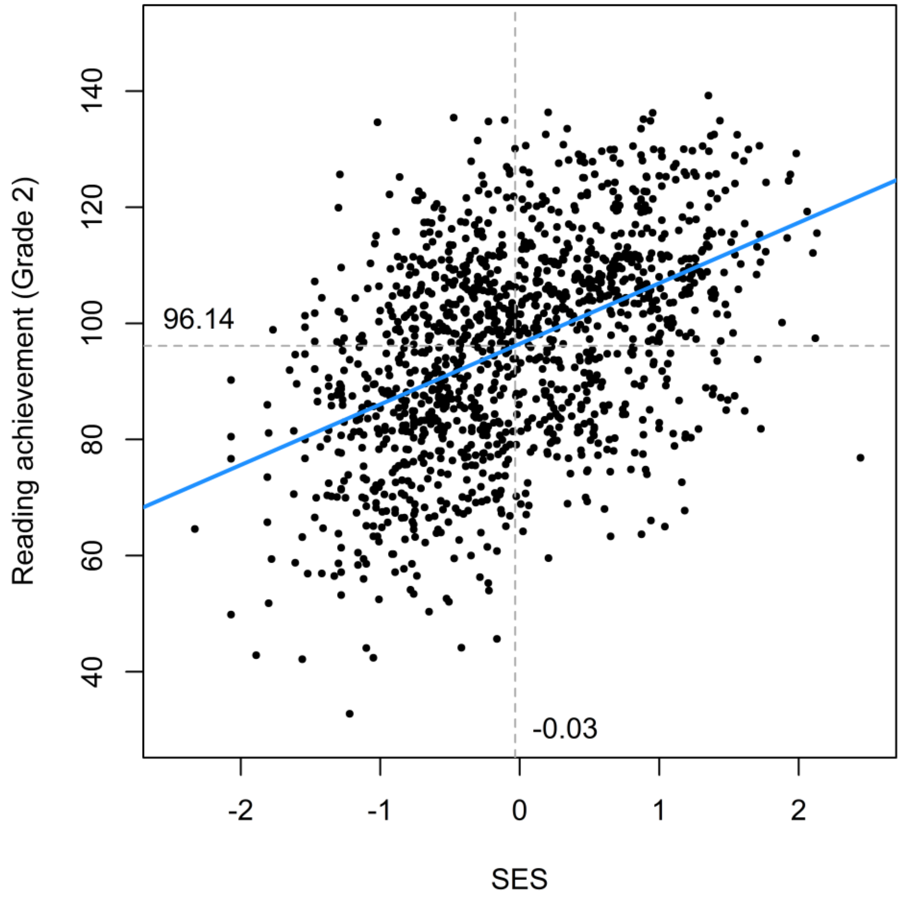


Image source: Midway, S. 2022. *Data Analysis in R*.
https://bookdown.org/steve_midway/DAR/

Motivating example

- Does the relationship between SES and student achievement systematically differ across schools?
- ECLS-K:2011 example from first lecture
  - The data include 11,091 first grade students, 742 schools
  - Number of students per school ranges from 10 to 25 students



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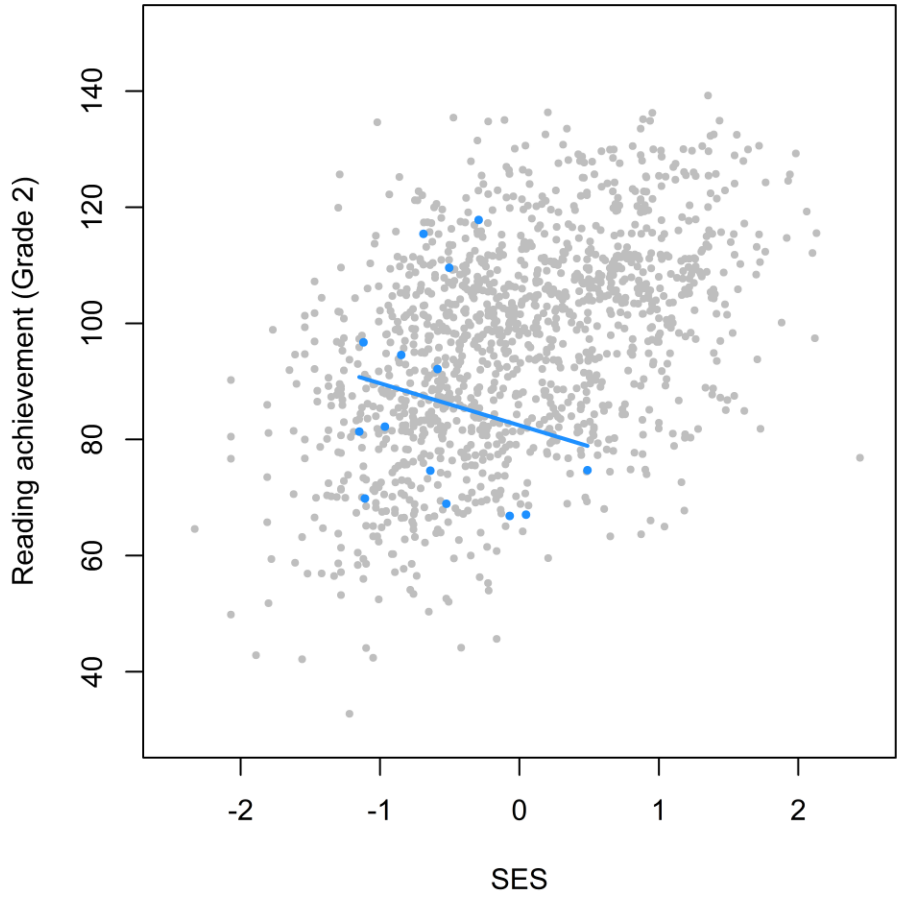

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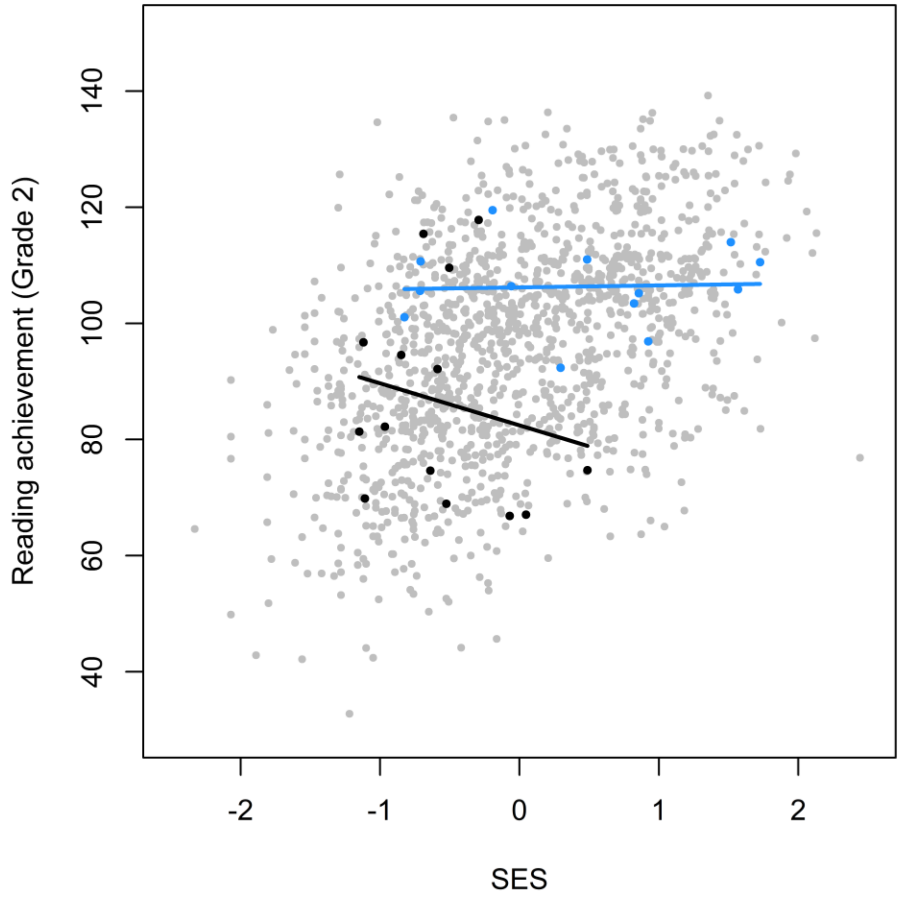

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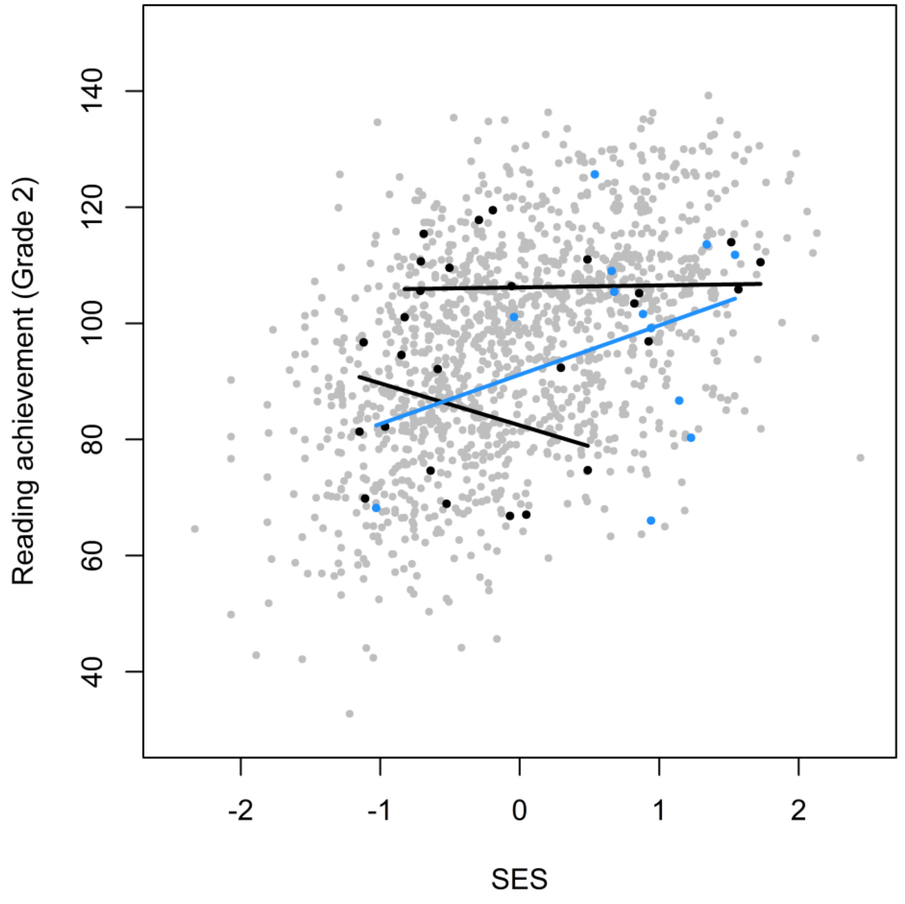

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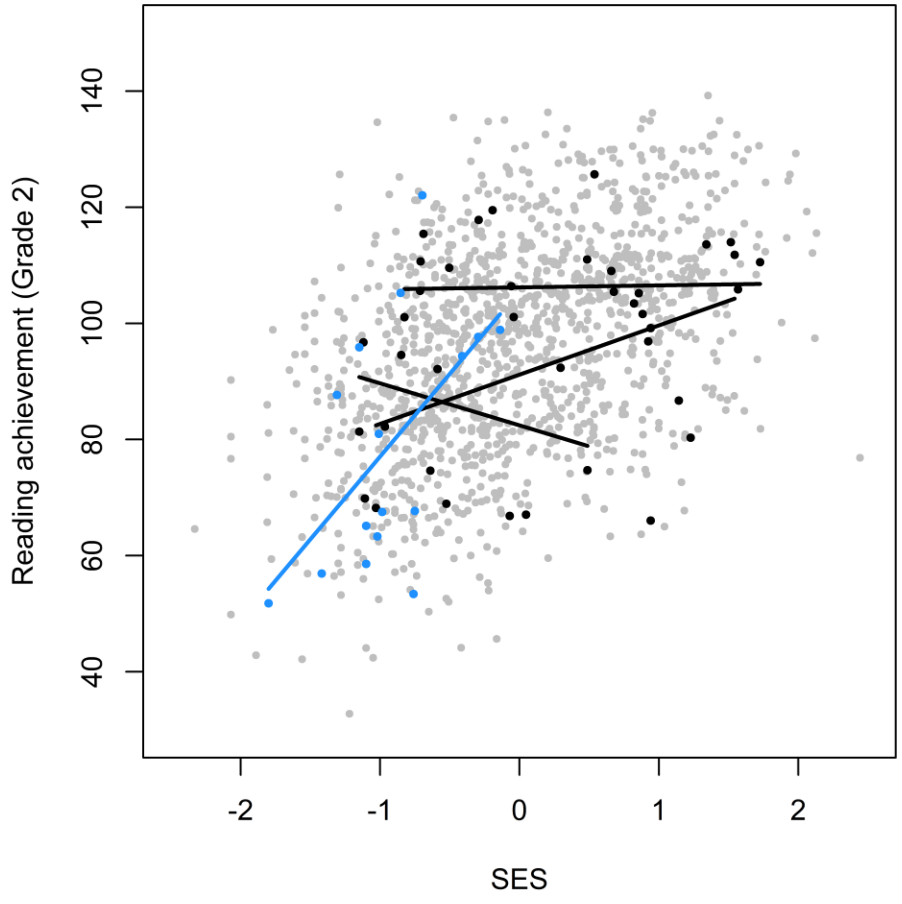

Level 1: Within-School Model

Within-school model

- Model level-1 intercepts and slopes as varying across level-2 units
- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}, \qquad r_{ij} \sim N(0, \sigma^2)$$

True mean reading score
for school *j*
(because SES is centered on school
mean SES)

The true SES-Y slope
for school *j*

Mean SES value
for the students in school *j*

Deviation of student *i*'s reading score
(Y_{ij}) from the expected score based
on the student's SES value and
regression model for school *j*

Pooled within school
residual variance

OLS estimates for each school

- The within-school model provides parameter estimates for each school
- Can use these estimates to address many interesting questions
- Questions about average trends across the school population:
  - What's the overall estimate of school mean achievement for the population?
  - What's the overall estimate of the relationship between SES and reading achievement?

schid	B0_hat	B1_hat
1002	102.76	21.65
1003	107.85	13.63
1006	101.86	5.10
1014	101.81	11.51
1015	96.65	3.94
1016	92.62	-7.03
1017	103.61	1.04
1019	105.85	10.13
1020	103.36	6.42
1021	93.69	6.02
1022	104.71	-1.76
1023	99.26	14.96
1025	105.31	23.46
1031	104.02	-0.62
1032	105.79	6.70
1033	98.65	2.47
1034	107.88	13.38
1035	103.89	-1.42
1036	102.76	11.95
1039	92.56	6.97

OLS estimates for each school

- Questions about variation across the school population (parameter variance):
  - How much does school mean achievement vary across schools?
  - How much does the SES-Achievement relationship vary across schools?
  - How much of the variation across schools in B0 and B1 is attributable to error variance vs. actual true differences?

schid	B0_hat	B1_hat
1002	102.76	21.65
1003	107.85	13.63
1006	101.86	5.10
1014	101.81	11.51
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OLS estimates for each school

- Questions about organizational and context effects:
  - How do differences in various school policies and practices relate to differences in school mean reading achievement? What about differences in the SES-Achievement relationship?
  - How do differences in various schooling conditions relate to differences in school mean reading achievement? What about differences in the SES-Achievement relationship?

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1002	102.76	21.65
1003	107.85	13.63
1006	101.86	5.10
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Level 2: Between-School Model

Between-school model

- Model level-2 intercepts and slopes as a function of fixed population means
- Level-2 (between-school) model:

Deviation of the true mean for school *j* from the grand mean

$$\beta_{0j} = \gamma_{00} + u_{0j}, \qquad u_{0j} \sim N(0, \tau_{00})$$

Variance of the true school means around the grand mean

$$\beta_{1j} = \gamma_{10} + u_{1j}, \qquad u_{1j} \sim N(0, \tau_{11})$$

Grand mean achievement score for our population of schools

Between-school model

- Model level-2 intercepts and slopes as a function of fixed population means
- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$

$$\beta_{1j} = \gamma_{10} + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

Grand mean SES-Achievement slope for our population of schools

Deviation of the true SES-Achievement slope for school *j* from the grand mean slope

Variance of the true school SES-Achievement slopes around the grand mean slope

Between-school model

- Model level-2 intercepts and slopes as a function of fixed population means
- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + u_{0j}\,, \qquad u_{0j} \sim N(0, \tau_{00})$$

$$\beta_{1j} = \gamma_{10} + u_{1j}\,, \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

The covariance between u_{1j} and u_{0j}; the covariance between true mean reading scores and the SES-Achievement slopes for schools in our population. (For example, do schools with higher mean reading achievement scores tend to have flatter SES-Achievement slopes?)

Random intercept and slope multilevel model

- Level-1 (within-school) model:

$$Y_{ij} = \beta_{0j} + \beta_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}\ , \qquad r_{ij} \sim N(0, \sigma^2)$$

- Level-2 (between-school) model:

$$\beta_{0j} = \gamma_{00} + u_{0j}\ , \qquad u_{0j} \sim N(0, \tau_{00})$$
$$\beta_{1j} = \gamma_{10} + u_{1j}\ , \qquad u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

Random intercept and slope multilevel model

- Combined model:

$$Y_{ij} = \gamma_{00} + \gamma_{10}\left(SES_{ij} - \overline{SES}_{.j}\right) + u_{0j} + u_{1j}\left(SES_{ij} - \overline{SES}_{.j}\right) + r_{ij}$$

$$r_{ij} \sim N(0, \sigma^2)$$
$$u_{0j} \sim N(0, \tau_{00})$$
$$u_{1j} \sim N(0, \tau_{11})$$

$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

Random intercept and slope multilevel model

- To estimate γ_{00}, the multilevel model uses a weighted average of the $\hat{\beta}_{0j}$'s, with the weights based on:

$$\frac{1}{\hat{\tau}_{00} + \frac{\hat{\sigma}^2}{n_j}}$$

- To estimate γ_{10}, the multilevel model uses a weighted average of the $\hat{\beta}_{1j}$'s, with the weights based on:

$$\frac{1}{\hat{\tau}_{11} + \frac{\hat{\sigma}^2}{\sum_{j=1}^{J}\left(X_{ij} - \bar{X}_{.j}\right)^2}}$$

Random Intercept and Slope Multilevel Model

Estimate the hierarchical model in R

- Combined model:

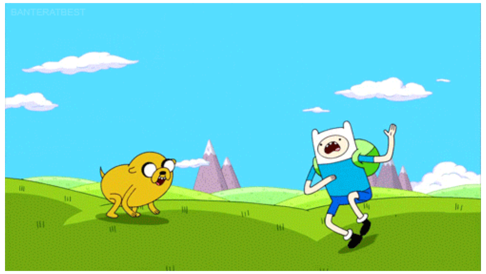
$$Y_{ij} = \gamma_{00} + \gamma_{10}(SES_{ij} - \overline{SES}_{.j}) + u_{0j} + u_{1j}(SES_{ij} - \overline{SES}_{.j}) + r_{ij}$$

```
m1 <- lmer(g1rscore ~ 1 + famsesc + (1 + famsesc | schid),
           data = ex1)

summary(m1)
```

Estimate the hierarchical model in R

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: g1rscore ~ 1 + famsesc + (1 + famsesc | schid)
   Data: ex1

REML criterion at convergence: 93159.9

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.0917 -0.6706  0.0348  0.6974  3.5802

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 schid    (Intercept)  54.893   7.409
          famsesc       4.656   2.158   -0.24
 Residual             233.954  15.296
Number of obs: 11091, groups:  schid, 742

Fixed effects:
            Estimate Std. Error       df t value Pr(>|t|)
(Intercept)  95.2620     0.3094 734.9216  307.86   <2e-16 ***
famsesc       7.3826     0.2523 591.6774   29.26   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Small group discussion



- In groups of 3-4, take 10 minutes to answer the following questions based on the model output on the previous slide:
  - What's the grand-mean reading score ($\hat{\gamma}_{00}$)?
  - What's the variance of the school mean reading scores ($\hat{\tau}_{00}$)?
  - What's the grand-mean SES-Achievement slope ($\hat{\gamma}_{10}$)?
  - What's the variance of the school SES-Achievement slopes ($\hat{\tau}_{11}$)?
  - What's the expected SES-Achievement slope for a school with a slope 1 standard deviation below the grand mean slope? What about for a school with a slope 1 standard deviation above the grand mean slope?
  - Does the SES-Achievement relationship tend to be stronger or weaker for schools with higher vs. lower mean reading achievement? What in the model output helped you answer this question?

Level 2 Residuals

Why examine the level 2 residuals

- Level-2 residuals represent the deviation of the true parameter value(s) for school *j* from the model-predicted value
- Also referred to as the level-2 random effects

$$\beta_{0j} = \gamma_{00} + u_{0j}\ ,$$
$$u_{0j} \sim N(0, \tau_{00})$$

$$\beta_{1j} = \gamma_{10} + u_{1j}\ ,$$
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$$\begin{pmatrix} u_{0j} \\ u_{1j} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}),$$
$$\mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

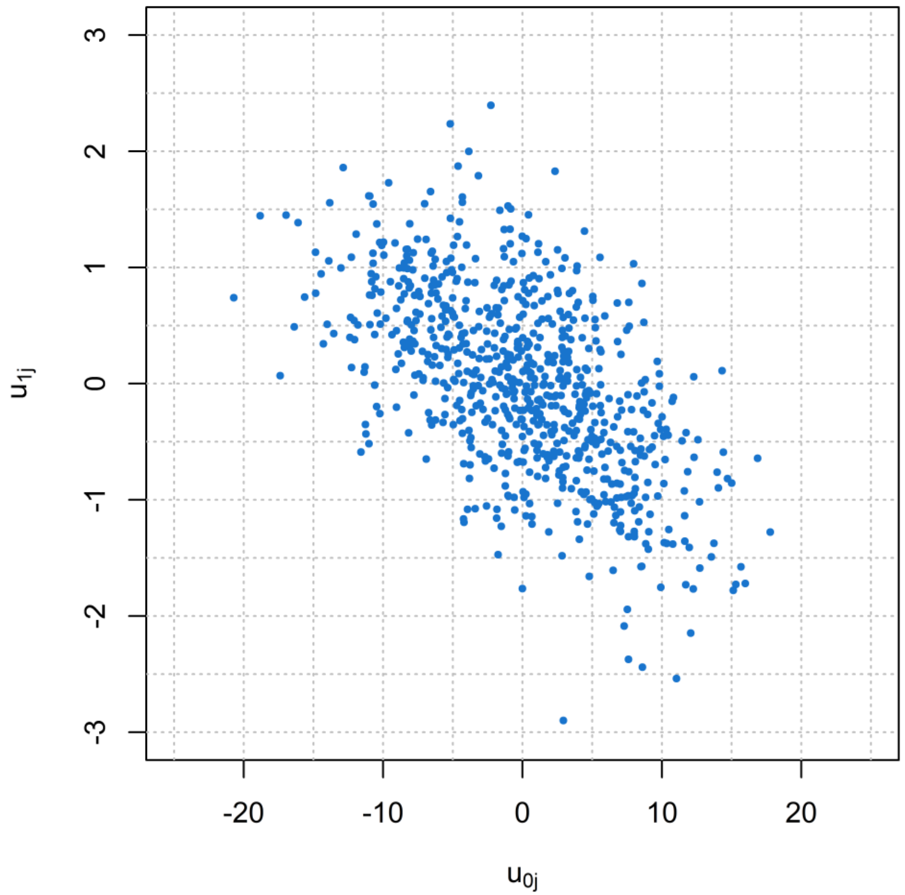
Why examine the level 2 residuals

- At least four reasons to examine the residuals:
  - Check modeling assumptions or possible data anomalies
  - Help understand magnitude of between-group variation
  - Help identify groups that significantly deviate from the norm (could be in a “good” way or in a “bad” way)
  - Explore potential “explanations” for between-group variation

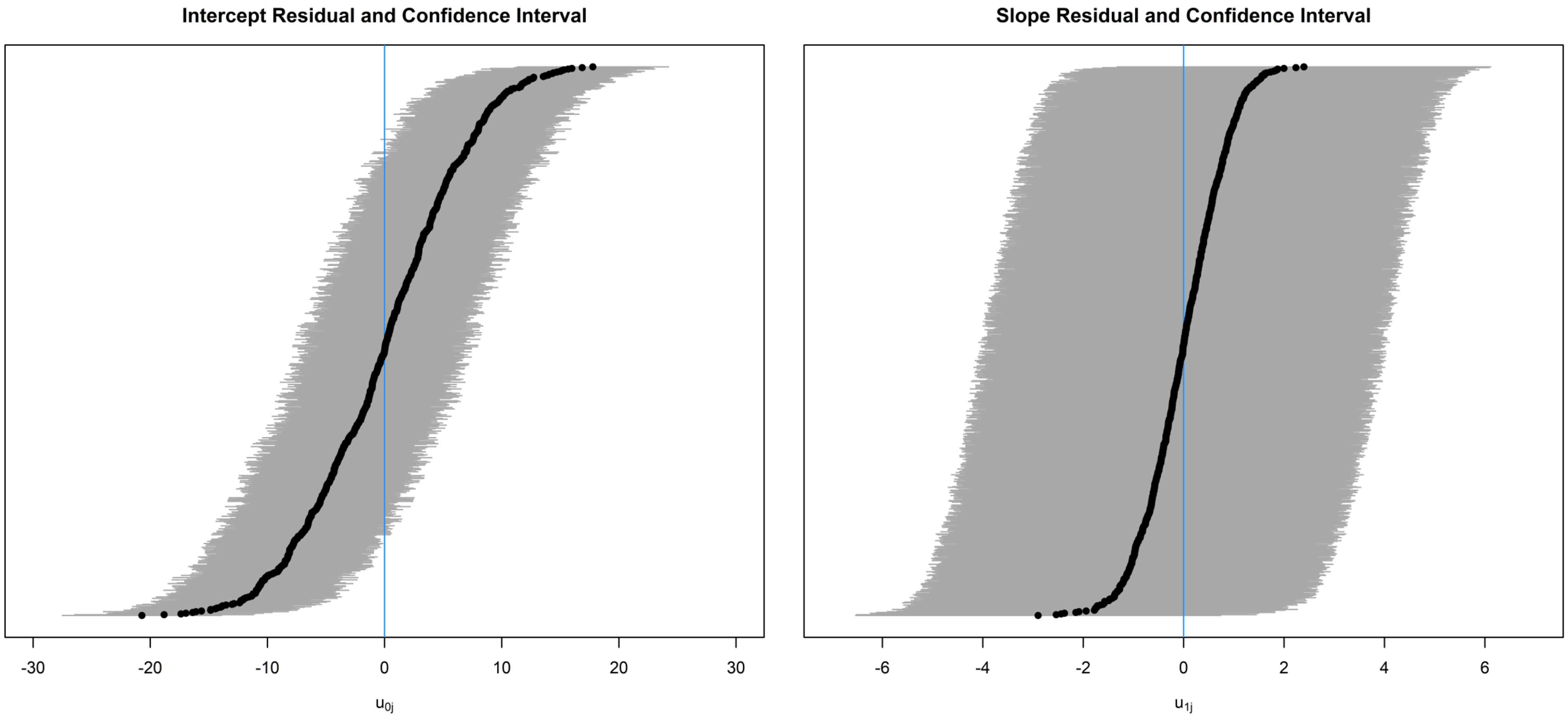
variable	mean	sd	p0	p25	p50	p75	p100	hist
u0j	0	6.51	-20.72	-4.57	0.17	4.51	17.78	▁▂▇▆▁
u1j	0	0.79	-2.90	-0.54	0.01	0.56	2.40	▁▂▇▆▁

Plot residuals

- What, if anything, stands out in the plot?
- Do you see any points that are of particular interest or concern?



Residual caterpillar plots: understanding uncertainty



Is there significant between-school variability in the slope?

- Can conduct a likelihood ratio test (Chi-square test) to compare the model fit between models with and without the random slope

```
# model with random slope
m1 <- lmer(g1rscore ~ 1 + famsesc + (1 + famsesc | schid), data = ex1)

# model with no random slope
m2 <- lmer(g1rscore ~ 1 + famsesc + (1 | schid), data = ex1)

# test model fit
anova(m2, m1, test = "LRT")
```

npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
4	93,174.34	93,203.60	-46,583.17	93,166.34			
6	93,170.49	93,214.37	-46,579.24	93,158.49	7.86	2	0.02