UCLA School of Education & Information Studies
Department of Education

EDUC 231D
Advanced Quantitative Methods: Multilevel Analysis
Winter 2025

Use of Multilevel Models for Longitudinal Analysis

Lecture 12 Presentation Slides

February 18, 2025

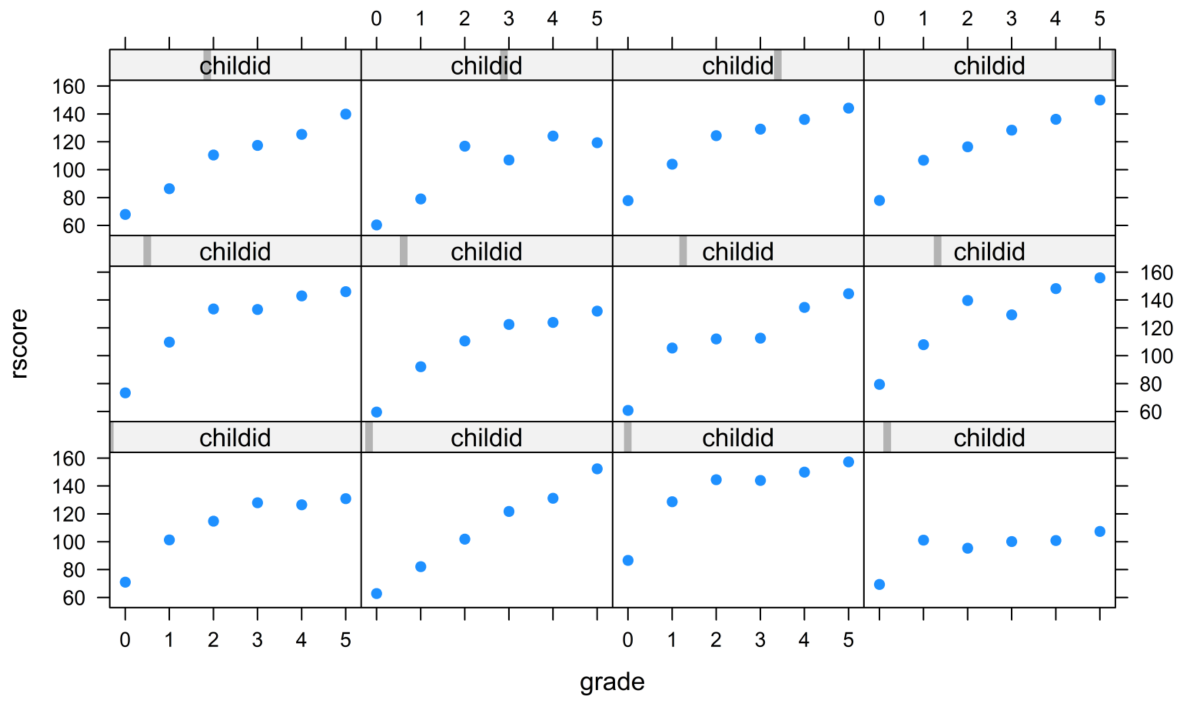
Today’s Topics

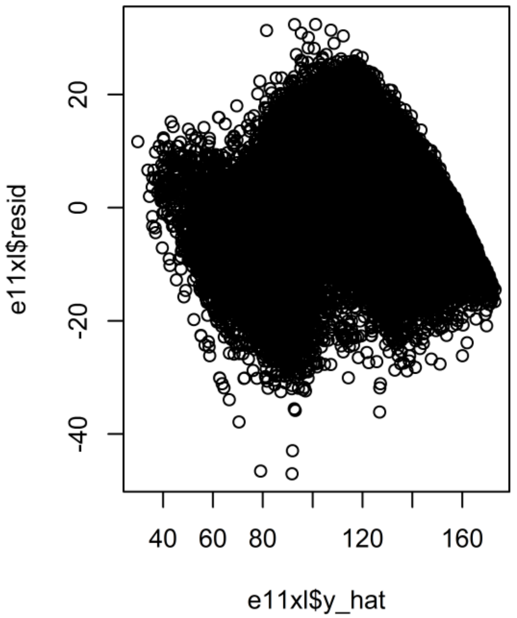
- Piecewise linear growth models
- Quadratic growth models
- Centering time

Piecewise Linear Growth Models

A more flexible approach to time

- In the ECLS-K data, we saw last class that the reading growth trajectory might not be linear



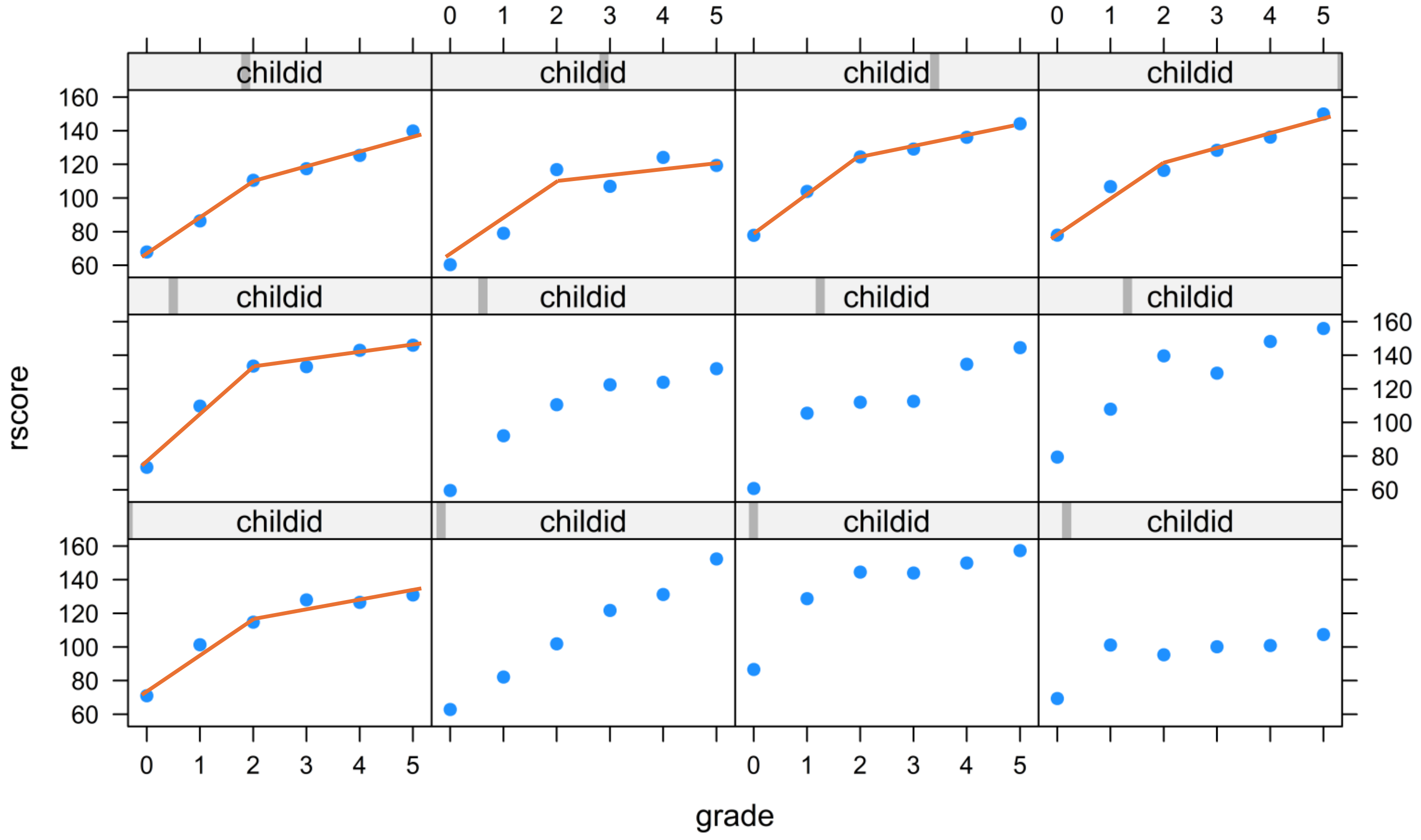


Modeling change during two different periods

- One approach to relaxing the linear growth constraint of the model is to let the slope differ during different periods of time
- This "piecewise" approach makes most sense when there are substantive or theoretical reasons to believe the rate of change depends on the time period covered, for example:
  - Prior research suggests learning growth is more rapid in the earlier grades than the later grades
  - In intervention studies, some time points might occur during the active treatment period (when change could be more pronounced) while later time points occur after the treatment period (when the treatment effect might fade out or flatten out)

Modeling change during two different periods

- What the piecewise linear growth model might look like with the ECLS-K data



Recoding time for a piecewise linear growth model

- Need to create two variables to represent time, one variable for each time period
- Two main options for coding the two variables
  - Coding for a “two-rate” model, where each variable captures the slope for that time period
  - Coding for an “increment/decrement” model, where the first variable represents the base rate of change and the second variable represents how much the rate increases or decreases in the second time period

Recoding time for a piecewise linear growth model

- Easier to see the coding options with real data
- For the ECLS-K data, let's code for the K – 2 period and the grade 3 – 5 period

Coding for the two-rate model

childid	grade	gradek2	grade35
10003426	0.00	0.00	0.00
10003426	1.00	1.00	0.00
10003426	2.00	2.00	0.00
10003426	3.00	2.00	1.00
10003426	4.00	2.00	2.00
10003426	5.00	2.00	3.00
10002116	0.00	0.00	0.00
10002116	1.00	1.00	0.00
10002116	2.00	2.00	0.00
10002116	3.00	2.00	1.00
10002116	4.00	2.00	2.00
10002116	5.00	2.00	3.00
10009310	0.00	0.00	0.00
10009310	1.00	1.00	0.00
10009310	2.00	2.00	0.00
10009310	3.00	2.00	1.00
10009310	4.00	2.00	2.00
10009310	5.00	2.00	3.00

Coding for the increment model

childid	grade	gradek2	grade35
10003426	0.00	0.00	0.00
10003426	1.00	1.00	0.00
10003426	2.00	2.00	0.00
10003426	3.00	3.00	1.00
10003426	4.00	4.00	2.00
10003426	5.00	5.00	3.00
10002116	0.00	0.00	0.00
10002116	1.00	1.00	0.00
10002116	2.00	2.00	0.00
10002116	3.00	3.00	1.00
10002116	4.00	4.00	2.00
10002116	5.00	5.00	3.00
10009310	0.00	0.00	0.00
10009310	1.00	1.00	0.00
10009310	2.00	2.00	0.00
10009310	3.00	3.00	1.00
10009310	4.00	4.00	2.00
10009310	5.00	5.00	3.00

The piecewise linear growth model

- (Level 1) Within-person model

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Time_{1ti}) + \pi_{2i}(Time_{2ti}) + e_{ti} \ ,$$

π_{0i} : true initial status for person *i*

π_{1i} : true rate of change for person *i* during first time period

π_{2i} : true rate of change for person *i* during second time period

e_{ti} : deviation of observed score at time *t* for person *i* from the expected growth trajectory for person *i*

The piecewise linear growth model

- (Level 2) Between-person model

$$\pi_{0i} = \beta_{00} + r_{0i}$$

$$\pi_{1i} = \beta_{10} + r_{1i}$$

$$\pi_{2i} = \beta_{20} + r_{2i}$$

$$\begin{pmatrix} r_{0i} \\ r_{1i} \\ r_{2i} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} & \tau_{02} \\ \tau_{10} & \tau_{11} & \tau_{12} \\ \tau_{20} & \tau_{21} & \tau_{22} \end{pmatrix}$$

Piecewise linear growth model: ECLS-K example

- Model estimation in R, using coding for the two-rate model

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Gradek2_{ti}) + \pi_{2i}(Grade35_{ti}) + e_{ti}$$

$$\pi_{0i} = \beta_{00} + r_{0i}$$
$$\pi_{1i} = \beta_{10} + r_{1i}$$
$$\pi_{2i} = \beta_{20} + r_{2i}$$

grade	gradek2	grade35
0.00	0.00	0.00
1.00	1.00	0.00
2.00	2.00	0.00
3.00	2.00	1.00
4.00	2.00	2.00
5.00	2.00	3.00

```
m1 <- lmer(rscore ~ 1 + gradek2 + grade35
          + (1 + gradek2 + grade35 | childid), data = e11xl,
          control = lmerControl(optimizer ="Nelder_Mead"))
```

Piecewise linear growth model: ECLS-K example

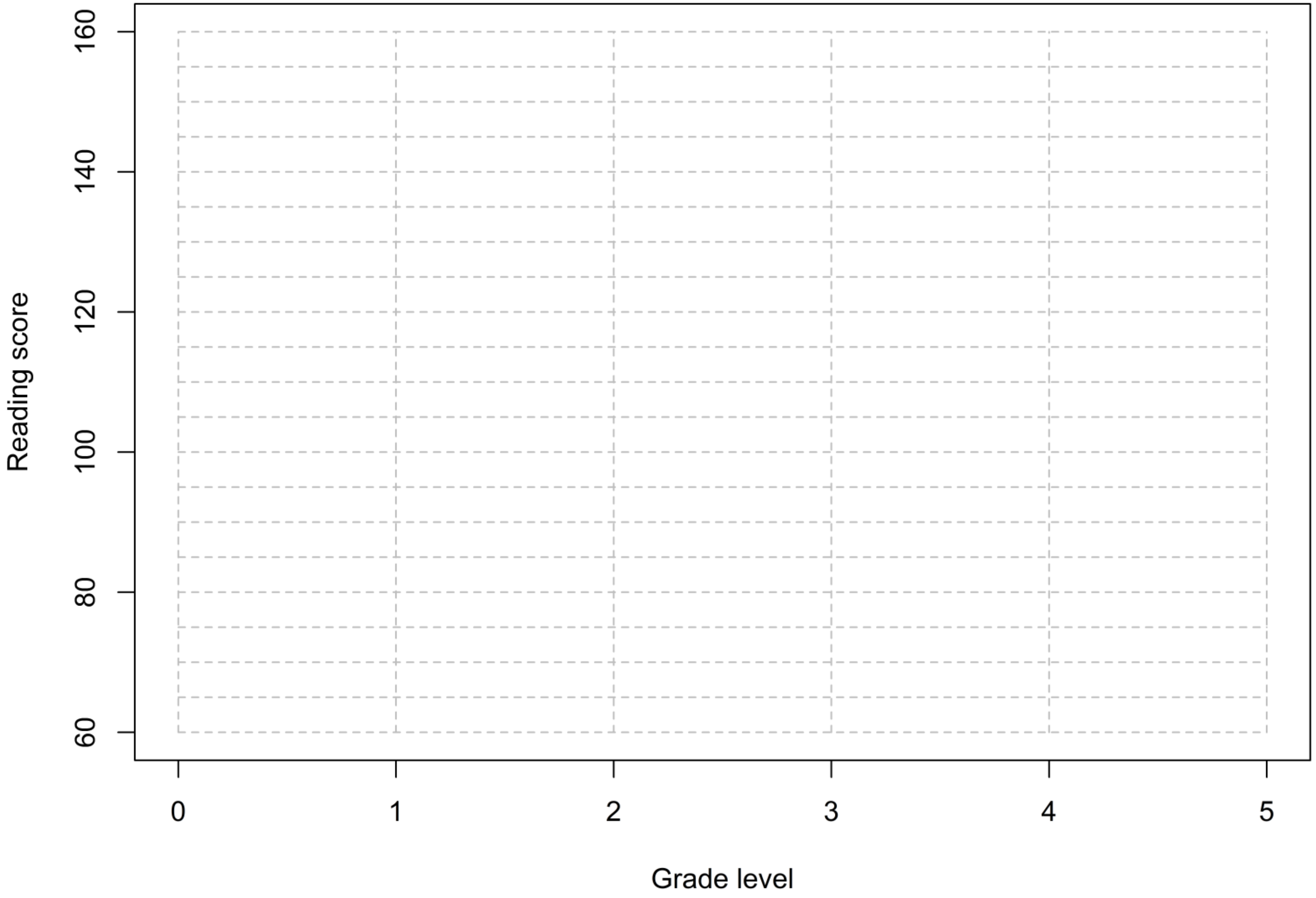
```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: rscore ~ 1 + gradek2 + grade35 + (1 + gradek2 + grade35 | childid)
   Data: e11xl
Control: lmerControl(optimizer = "Nelder_Mead")

REML criterion at convergence: 340081.5

Scaled residuals:
    Min      1Q  Median      3Q     Max 
-4.7107 -0.5574 -0.0201  0.5340  4.3537 

Random effects:
 Groups   Name        Variance Std.Dev. Corr       
 childid  (Intercept) 199.075  14.109              
          gradek2      19.319   4.395   -0.09      
          grade35       3.413   1.847   -0.68 -0.01
 Residual              43.461   6.592              
Number of obs: 46170, groups:  childid, 7695

Fixed effects:
             Estimate Std. Error        df t value Pr(>|t|)    
(Intercept) 7.115e+01  1.747e-01 7.694e+03   407.2   <2e-16 ***
gradek2     2.162e+01  6.992e-02 7.694e+03   309.2   <2e-16 ***
grade35     7.589e+00  3.852e-02 7.694e+03   197.0   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Fixed Effect	Estimate	SE	p-value	
(Intercept)	71.146	0.175	0.0000	***
gradek2	21.618	0.070	0.0000	***
grade35	7.589	0.039	0.0000	***

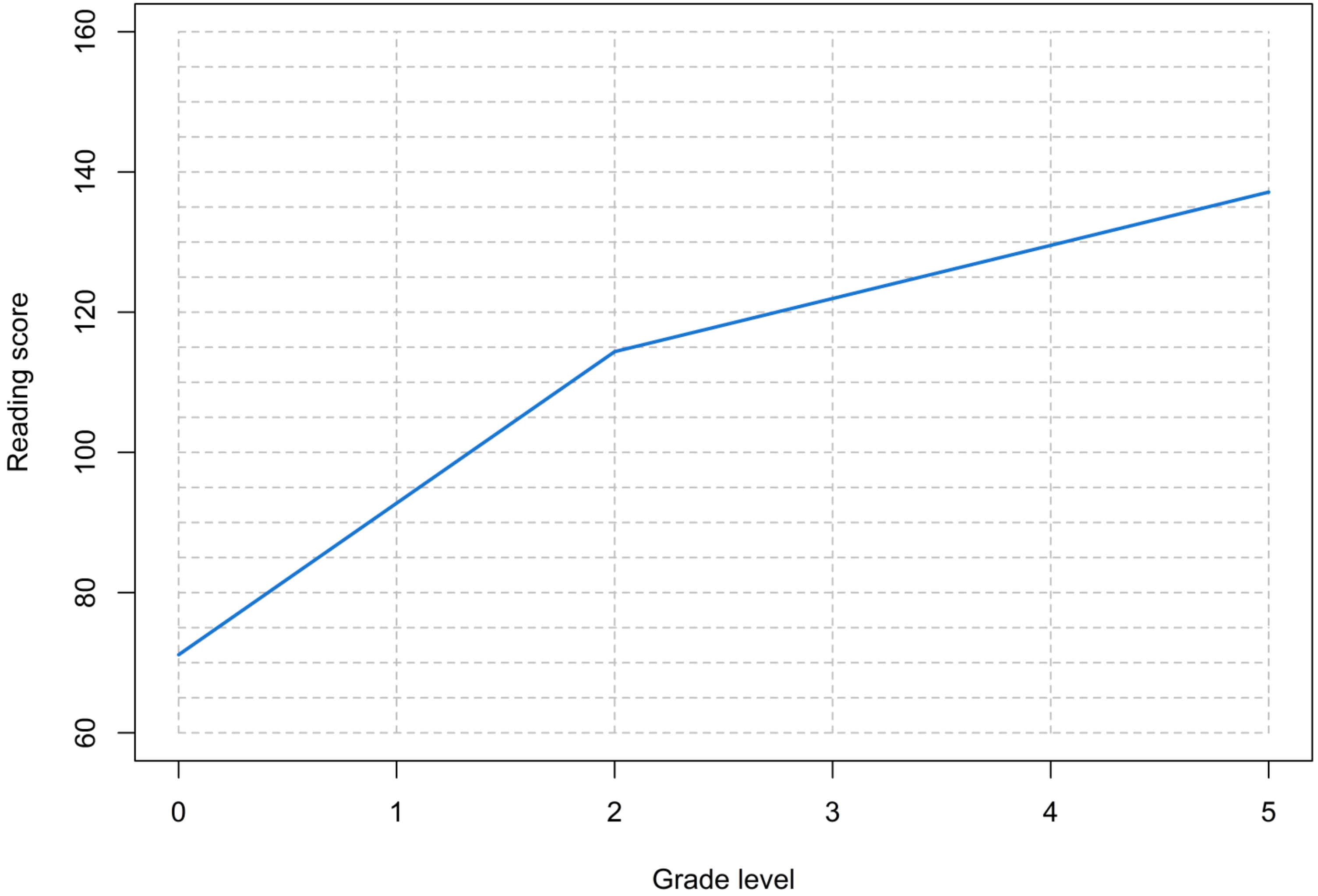
Small group discussion

- In groups of 3-4, take 10 minutes to answer the following questions based on the model output on the previous slide:
  - What's the estimated grand-mean reading score when students are in kindergarten?
  - What's the estimated grand-mean rate of reading growth from kindergarten through grade 2?
  - What's the estimated grand-mean rate of reading growth from grade 3 through grade 5?
  - To what extent do the rates of reading growth vary between students?
  - What's the expected reading score in spring of 2nd grade for a student with average reading achievement in kindergarten and an average rate of growth?
  - What's the expected reading score in spring of 5th grade for a student with average reading achievement in kindergarten and an average rate of growth?
  - Sketch out the estimated linear growth trajectory for the average student. (You can use the empty graph on the next slide.)
  - What do these results suggest about the typical rate of reading growth through elementary school?

Piecewise linear growth model: ECLS-K example



Piecewise linear growth model: ECLS-K example



Piecewise linear growth model: ECLS-K example

- Can conduct a post-hoc test to see whether the two slopes are statistically significantly different:

$$H_0: \beta_{10} = \beta_{20}$$

```
library("car")
linearHypothesis(m1, "gradek2=grade35")
```

Piecewise linear growth model: ECLS-K example

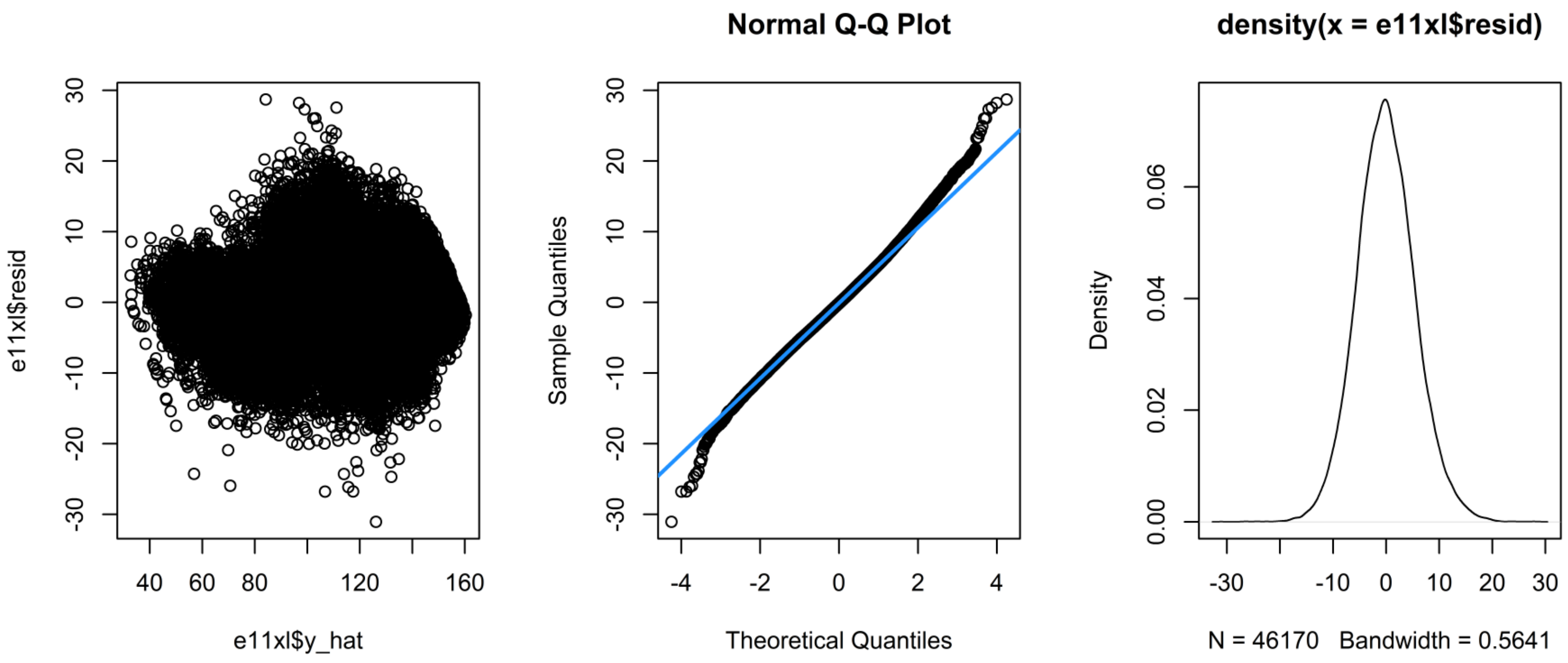
- Can conduct a post-hoc test to see whether the two slopes are statistically significantly different:

```
Linear hypothesis test

Hypothesis:
gradek2 - grade35 = 0

Model 1: restricted model
Model 2: rscore ~ 1 + gradek2 + grade35 + (1 + gradek2 + grade35 | childid)

  Df Chisq Pr(>Chisq)
1
2  1 24051  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Chis-sq test of equal coefficients suggests that the two slopes are significantly different, which provides some confidence that the piecewise linear growth model fits the data better than a simple linear growth model

Piecewise linear growth model: ECLS-K example

- And let's check the level-1 residuals to see if the heteroscedasticity we saw last time still exists with the piecewise model



Quadratic Growth Models

Quadratic growth curve model

- Another way to relax the linear growth constraint of the model is to let the slope increase/decrease over time by adding a quadratic term to the model:
- (Level 1) Within-person model

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Time_{ti}) + \pi_{2i}(Time_{ti}^{2}) + e_{ti} \ ,$$

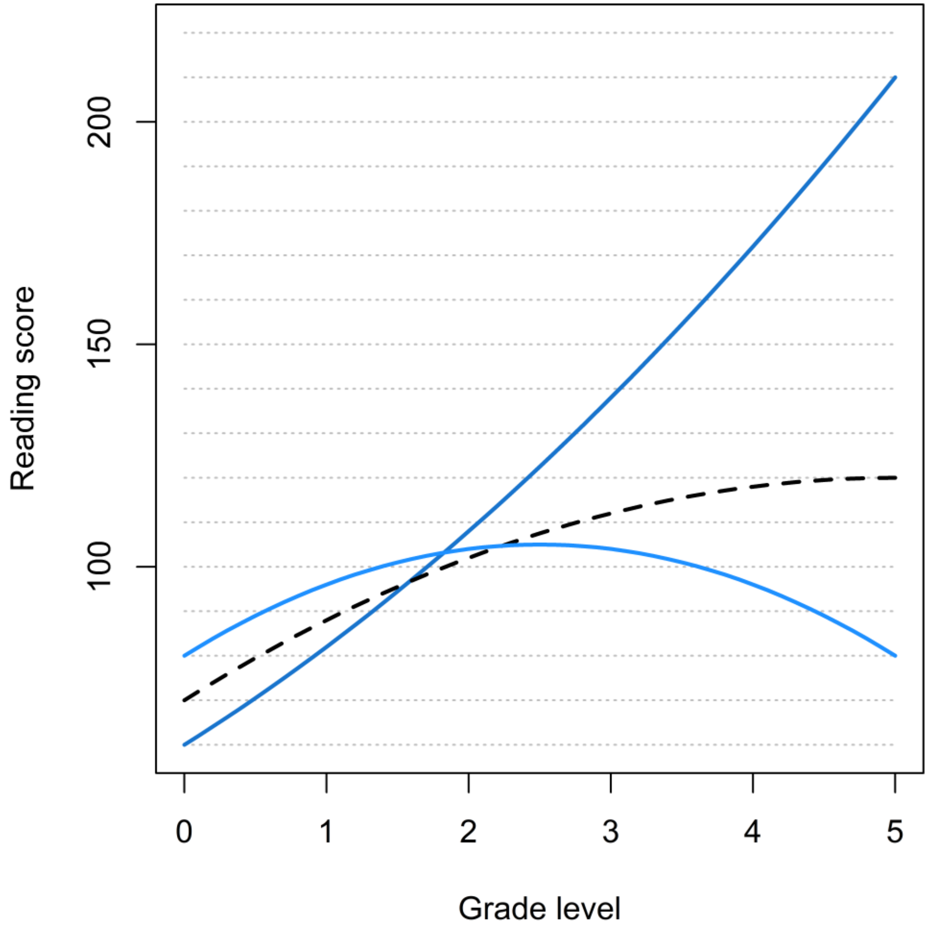
π_{0i} : true initial status for person *i*

π_{1i} : true rate of change for person *i* at the initial time point

π_{2i} : true acceleration/deceleration of change for person *i*

Quadratic growth curve model

- Hypothetical examples of quadratic growth curves



$$Y_{ti} = 60 + 20(Time_{ti}) + 2(Time_{ti}^2)$$

$$Y_{ti} = 70 + 20(Time_{ti}) - 2(Time_{ti}^2)$$

$$Y_{ti} = 80 + 20(Time_{ti}) - 4(Time_{ti}^2)$$

Quadratic growth curve model

- (Level 2) Between-person model

$$\pi_{0i} = \beta_{00} + r_{0i}$$

$$\pi_{1i} = \beta_{10} + r_{1i}$$

$$\pi_{2i} = \beta_{20} + r_{2i}$$

$$\begin{pmatrix} r_{0i} \\ r_{1i} \\ r_{2i} \end{pmatrix} \sim MVN(\mathbf{0}, \mathbf{T}), \qquad \mathbf{T} = \begin{pmatrix} \tau_{00} & \tau_{01} & \tau_{02} \\ \tau_{10} & \tau_{11} & \tau_{12} \\ \tau_{20} & \tau_{21} & \tau_{22} \end{pmatrix}$$

Quadratic growth curve model: ECLS-K example

- Model estimation in R

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Grade_{ti}) + \pi_{2i}(Grade_{ti}^2) + e_{ti}$$

$$\pi_{0i} = \beta_{00} + r_{0i}$$
$$\pi_{1i} = \beta_{10} + r_{1i}$$
$$\pi_{2i} = \beta_{20} + r_{2i}$$

grade	grade_sq
0.00	0.00
1.00	1.00
2.00	4.00
3.00	9.00
4.00	16.00
5.00	25.00

```
m1 <- lmer(rscore ~ 1 + grade + grade_sq
          + (1 + grade + grade_sq | childid), data = e11xl,
          control = lmerControl(optimizer ="Nelder_Mead"))
```

Quadratic growth curve model: ECLS-K example

- Model with random effect for π_{2i} does not converge!

```
Warning messages:
1: In checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv,  :
  Model failed to converge with max|grad| = 0.00767597 (tol = 0.002, component 1)
2: In checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv,  :
  Model is nearly unidentifiable: very large eigenvalue
 - Rescale variables?
```

- Likely an indication that there's very little between-person variation for π_{2i} : difficult to estimate τ_{22}
- Respecify the model so that there's no random effect for the quadratic term: $\pi_{2i} = \beta_{20}$

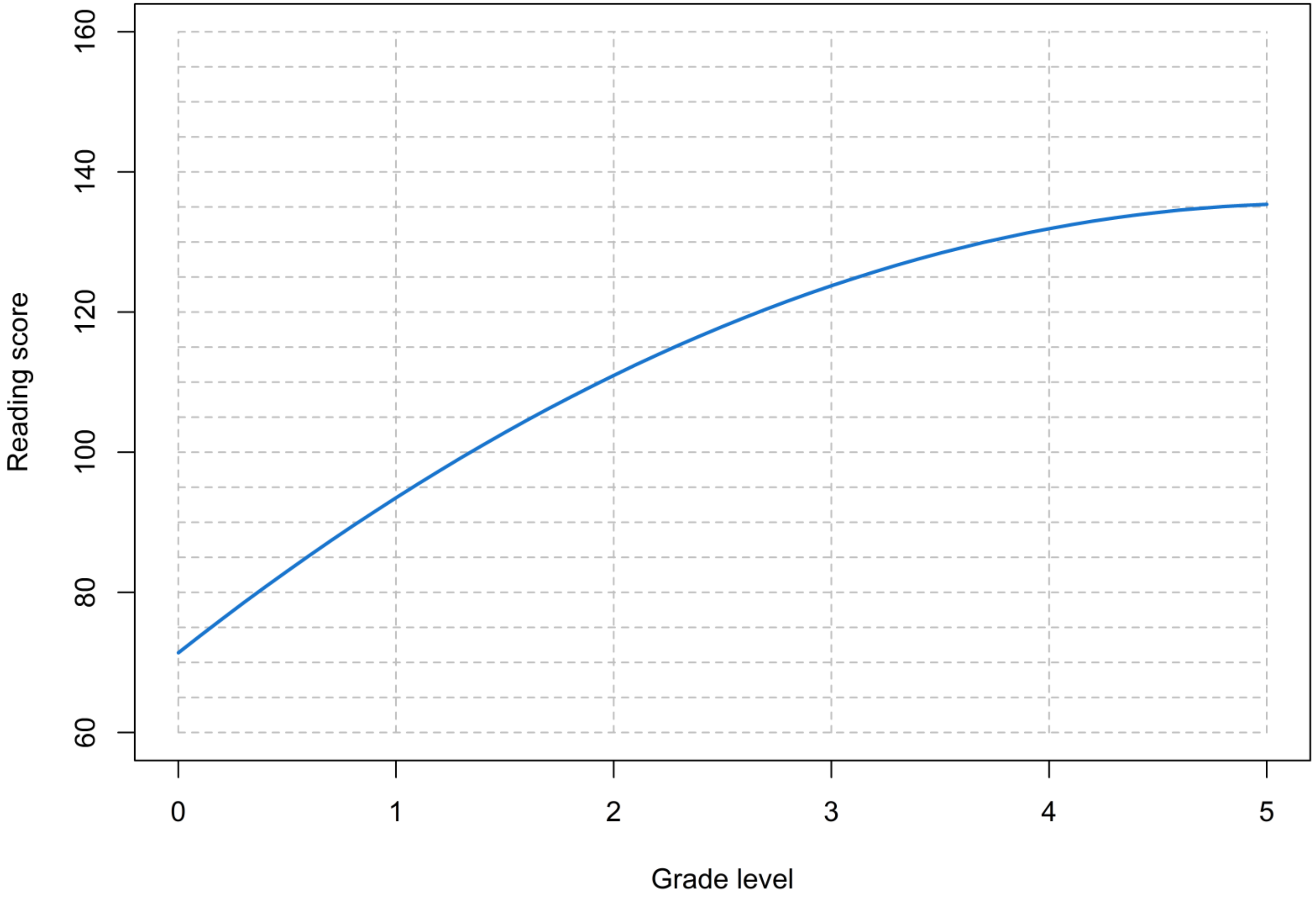
Quadratic growth curve model: ECLS-K example

```
m2 <- lmer(rscore ~ 1 + grade + grade_sq
     + (1 + grade | childid), data = e11xl)
```

```
Linear mixed model fit by REML. t-tests use Satterthwaite's method ['lmerModLmerTest']
Formula: rscore ~ 1 + grade + grade_sq + (1 + grade | childid)
   Data: e11xl

REML criterion at convergence: 343874.8

Scaled residuals:
    Min      1Q  Median      3Q     Max
-4.8891 -0.5794  0.0001  0.5710  4.3090

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 childid  (Intercept) 215.799  14.690
          grade         3.454   1.858   -0.33
 Residual              52.438   7.241
Number of obs: 46170, groups:  childid, 7695

Fixed effects:
              Estimate Std. Error         df t value Pr(>|t|)
(Intercept)  7.140e+01  1.834e-01  8.704e+03   389.3   <2e-16 ***
grade        2.445e+01  7.350e-02  3.800e+04   332.7   <2e-16 ***
grade_sq    -2.330e+00  1.351e-02  3.078e+04  -172.4   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

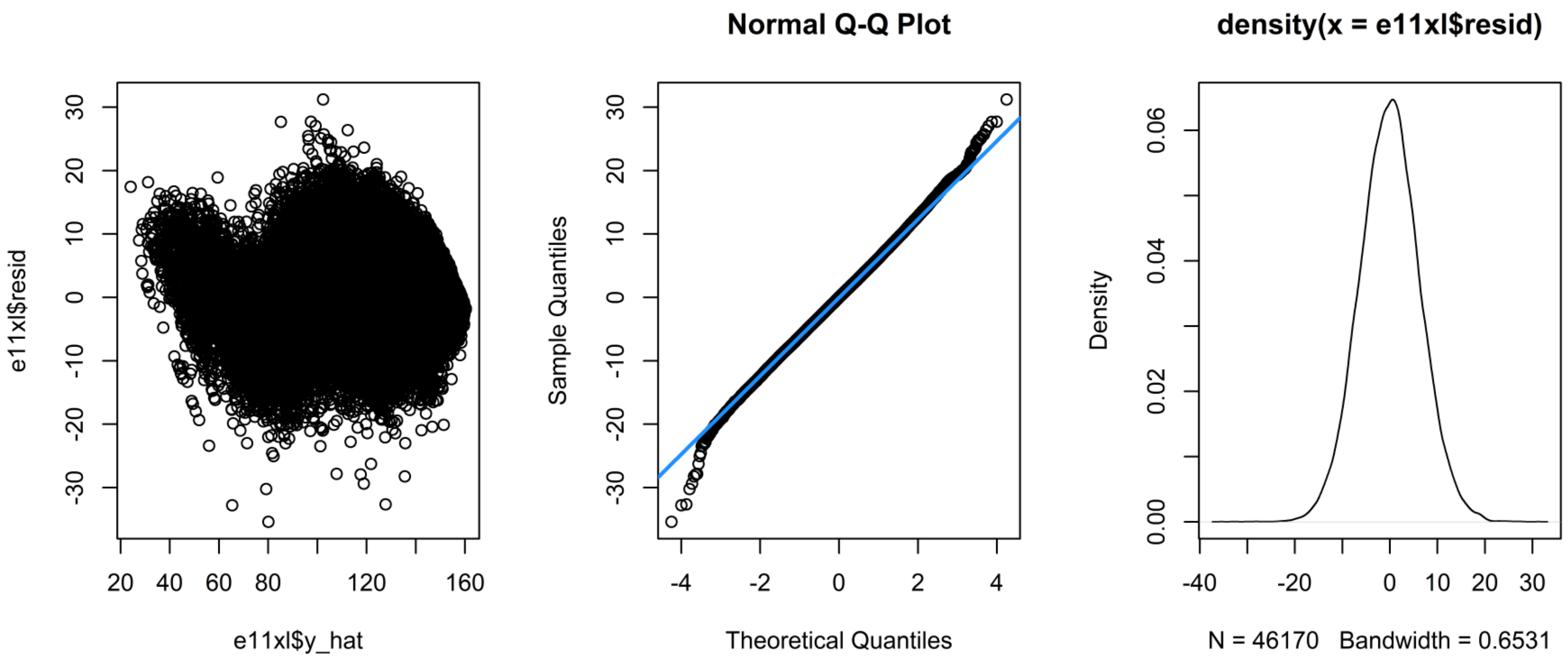
Fixed Effect	Estimate	SE	p-value	
(Intercept)	71.396	0.183	0.0000	***
grade	24.449	0.073	0.0000	***
grade_sq	-2.330	0.014	0.0000	***

Quadratic growth curve model: ECLS-K example



Quadratic growth curve model: ECLS-K example

- And let's check the level-1 residuals



Centering Time

Interpretation of growth models depends on how time is coded

- For interpretation (and sometimes estimation), it is important to consider the most useful way to center the time variable so that a value of zero has meaning
- For the linear model, interpretation of the intercept (π_{0i}) and random effect variance (τ_{00}) is based on when *Time* = 0

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Time_{ti}) + e_{ti}$$

- How should we code time if we were studying grades 7 to 12?
- What if we were most interested in a student's status in 9th grade?

Interpretation of growth models depends on how time is coded

- For a quadratic model, interpretation of the intercept (π_{0i}), growth rate, random effect variances (τ_{00}) differ depend on how time is centered

$$Y_{ti} = \pi_{0i} + \pi_{1i}(Time_{ti}) + \pi_{2i}\left(Time_{ti}^{2}\right) + e_{ti}$$

- The growth rate at any particular time point is the first derivative of the growth model:

$$Growth\ rate\ at\ time\ t = \pi_{1i} + 2\pi_{2i}(Time_{ti})$$

- Useful to center time at a meaningful time point for interpretation

Centering time: ECLS-K example

- Test centering grade at different grade levels (K, 2, and 5)

Centered at Grade 2
$(grade_{ti} - 2)$

Centered at kindergarten
$(grade_{ti} - 0)$

grade	grade.c2	grade.c5
$\boxed{0.00}$	-2.00	-5.00
1.00	-1.00	-4.00
2.00	$\boxed{0.00}$	-3.00
3.00	1.00	-2.00
4.00	2.00	-1.00
5.00	3.00	$\boxed{0.00}$

Centered at Grade 5
$(grade_{ti} - 5)$

Centering time: ECLS-K example

- Model fixed-effect estimates under different centering options
  - Linear growth model

With $(grade_{ti} - 0)$		
Fixed Effect	Estimate	SE
(Intercept)	79.162	0.178
grade	12.800	0.029
With $(grade_{ti} - 2)$		
Fixed Effect	Estimate	SE
(Intercept)	104.761	0.162
grade	12.800	0.029
With $(grade_{ti} - 5)$		
Fixed Effect	Estimate	SE
(Intercept)	143.161	0.176
grade	12.800	0.029

  - Quadratic growth model

With $(grade_{ti} - 0)$		
Fixed Effect	Estimate	SE
(Intercept)	71.396	0.183
grade	24.449	0.073
grade_sq	-2.330	0.014
With $(grade_{ti} - 2)$		
Fixed Effect	Estimate	SE
(Intercept)	110.974	0.166
grade	15.130	0.032
grade_sq	-2.330	0.014
With $(grade_{ti} - 5)$		
Fixed Effect	Estimate	SE
(Intercept)	135.394	0.182
grade	1.150	0.073
grade_sq	-2.330	0.014