1

(a) Notation $f_n = f(t_n, x_n)$ is used for convenience.

$$x_{n+1} = c_1 x_n + c_0 x_{n-1} + h(\alpha_2 f(t_{n+1}, x_{n+1}) + \alpha_1 f(t_n, x_n) + \alpha_0 f(t_{n-1}, x_{n-1})) \tag{1}$$

Using Taylor's expansion for Eq.(1) and $x'_n = f(t_n, x_n)$ we have

$$\begin{aligned}
x_{n+1} &= c_1 x_n + c_0 x_{n-1} + h(\alpha_2 f(t_{n+1}, x_{n+1}) + \alpha_1 f(t_n, x_n) + \alpha_0 f(t_{n-1}, x_{n-1})) \\
&= c_1 x_n + c_0 x_{n-1} + (\alpha_2 h x'_{n+1} + \alpha_1 h x'_n + \alpha_0 h x'_{n-1}) \\
&= c_1 x_n \\
&+ c_0 \left[x_n - h x'_n + \frac{h^2}{2} x''_n - \frac{h^3}{6} x'''_n + \frac{h^4}{24} x_n^{(4)} - \cdots \right] \\
&+ \alpha_2 h \left[x'_n + h x''_n + \frac{h^2}{2} x'''_n + \frac{h^3}{6} x_n^{(4)} + \cdots \right] \\
&+ \alpha_1 h x'_n \\
&+ \alpha_0 h \left[x'_n - h x''_n + \frac{h^2}{2} x_n''' - \frac{h^3}{6} x_n^{(4)} + \cdots \right] \\
&= c_1 x_n \\
&+ c_0 \left[x_n - h x'_n + \frac{h^2}{2} x''_n - \frac{h^3}{6} x'''_n + \frac{h^4}{24} x_n^{(4)} - \cdots \right] \\
&+ \alpha_2 \left[h x'_n + h^2 x''_n + \frac{h^3}{2} x'''_n + \frac{h^4}{6} x_n^{(4)} + \cdots \right] \\
&+ \alpha_1 h x'_n \\
&+ \alpha_0 \left[h x'_n - h^2 x''_n + \frac{h^3}{2} x'''_n - \frac{h^4}{6} x_n^{(4)} + \cdots \right]
\end{aligned} \tag{2}$$

Combining like terms in Eq.(2), we have

$$\begin{aligned}
x_{n+1} =& (c_0 + c_1) x_n + (\alpha_0 + \alpha_1 + \alpha_2 - c_0) h x'_n + (c_0 - 2\alpha_0 + 2\alpha_2) \frac{h^2}{2} x''_n \\
&+ (3\alpha_0 + 3\alpha_2 - c_0) \frac{h^3}{6} x'''_n + (4\alpha_2 - 4\alpha_0 + c_0) \frac{h^4}{24} x_n^{(4)} + O(h)
\end{aligned} \tag{3}$$

Show that this method is consistent if and only if the coefficient of x_n equals to 1 and x'_n equals to 1. That is

$$c_0 + c_1 = 1, \alpha_0 + \alpha_1 + \alpha_2 - c_0 = 1 \tag{4}$$

(b) The characteristic polynomial of Eq.(1), using the fact that $c_0 = 1 - c_1$

$$\begin{aligned}
P(x) &= x^2 - c_1 x - c_0 \\
&= x^2 - c_1 x - 1 + c_1 \\
&= (x - 1)(x - c_1 + 1)
\end{aligned} \tag{5}$$

whose roots are 1 and $1 - c_1$. If $|1 - c_1| > 1$, the method in (a) is consistent, but unstable. (c) In order to obtain a 3 order scheme, we have the following equations,

$$\begin{cases}
c_0 + c_1 = 1 \\
\alpha_0 + \alpha_1 + \alpha_2 - c_0 = 1 \\
c_0 - 2\alpha_0 + 2\alpha_2 = 1 \\
3\alpha_0 + 3\alpha_2 - c_0 = 1
\end{cases} \tag{6}$$

Solve Eq.(3), we have

$$\begin{cases} \alpha_0 = 2 + \alpha_2 \\ \alpha_1 = 4 - 2\alpha_2 \\ \alpha_2 = \alpha_2 \\ c_0 = 5 \\ c_1 = -4 \end{cases} \tag{7}$$

If we want a 4 order scheme, it is needed that $4\alpha_2 - 4\alpha_0 + c_0 = 1$, but

$$4\alpha_2 - 4\alpha_0 + c_0 = 4\alpha_2 - 4(2 + \alpha_2) + 5 = -3 \neq 1 \tag{8}$$

Hence we can say that the method has order 3, and that it does not have order 4.

2

(c) Linearized $\ddot{\theta} + \gamma\dot{\theta} + \sin(\theta) = 0$ is $\ddot{\theta} + \gamma\dot{\theta} + \theta = 0$, which can be transformed into matrix form

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -\gamma \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \tag{9}$$

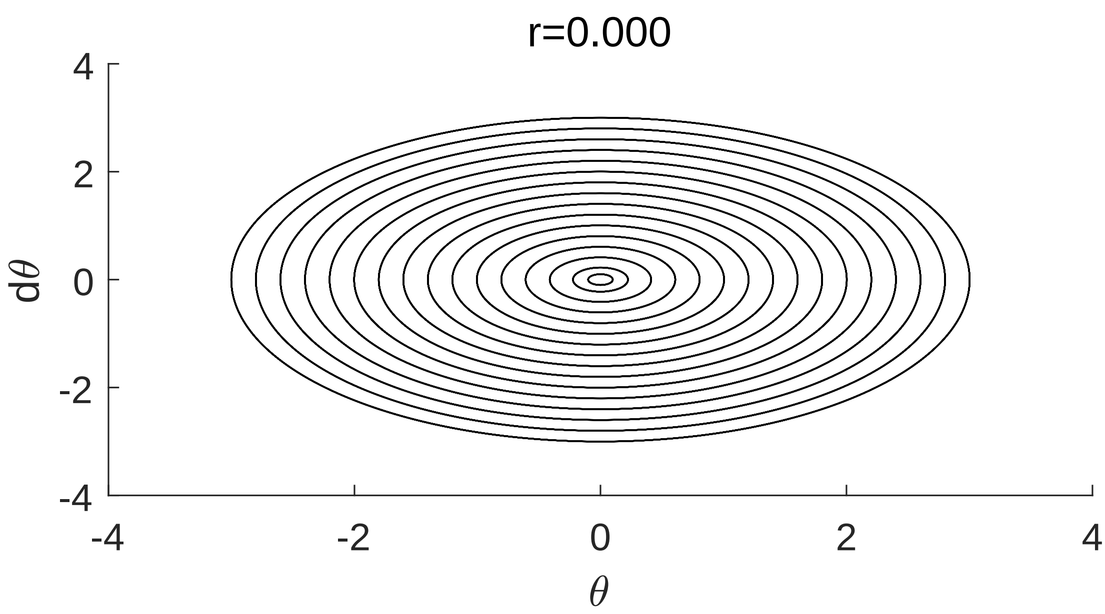

Figure 1: r=0

3

(a) Consider $f(x, y) = [\dot{x}, \dot{y}] = (y, g(x))$, we have

$$\begin{cases} y = \dot{x} \\ g(x) = \dot{y} \end{cases} \tag{10}$$

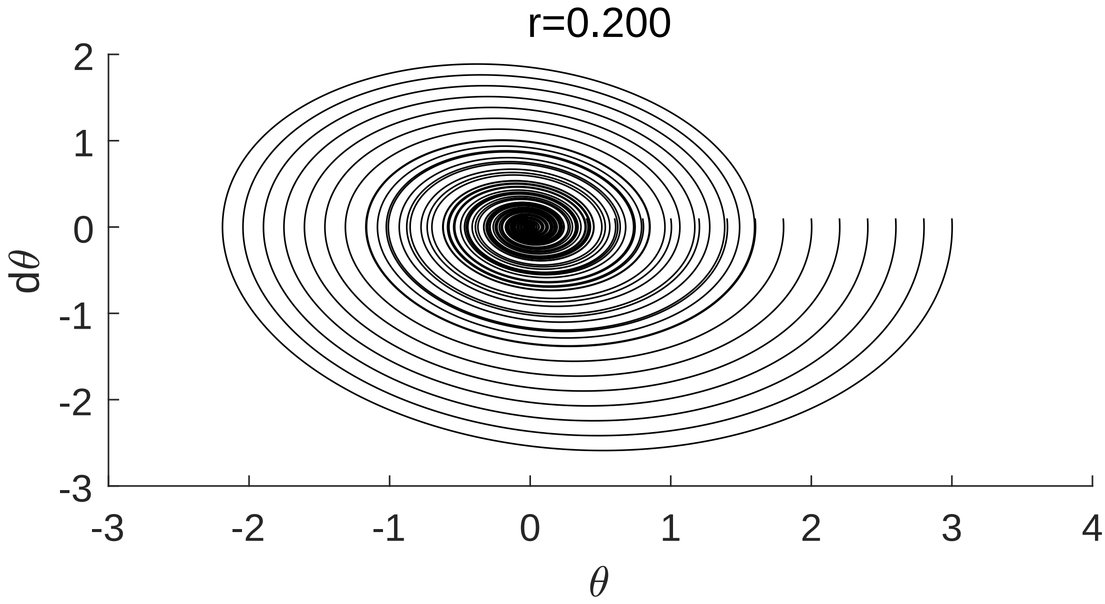

Figure 2: r=0.2

$$
\begin{aligned}
H(x,y) &= \frac{1}{2}y^2 - \int_{x_0}^{x} g(s)ds \\
&= \frac{1}{2}\dot{x}^2 - \int_{x_0}^{x} \dot{y}ds \\
&= \frac{1}{2}\dot{x}^2 - \int_{x_0}^{x} \ddot{s}ds \\
&= \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{x}^2 + \frac{1}{2}x_0^2 \\
&= \frac{1}{2}x_0^2
\end{aligned} \tag{11}
$$

From Eq.(11), it can be seen that the function $H(x,y)$ is constant. For the undamped pendulum equations $f(\theta,\dot{\theta}) = (\dot{\theta}, -\sin(\theta))$, we have the expression for level $H(\theta,\dot{\theta})$ as follows, Figure 6 and Figure 7 shows the level versus time.

$$
\begin{aligned}
H(\theta,\dot{\theta}) &= \frac{1}{2}\dot{\theta}^2 - \int_{\theta_0}^{\theta} -\sin(s)ds \\
&= \frac{1}{2}\dot{\theta}^2 - \cos(\theta) + \cos(\theta_0)
\end{aligned} \tag{12}
$$

(b) From Figure 8 and Figure 9, the trajectory become flat when θ arrives 2π, and this problem cannot be fixed by decreasing time step.
(c) $g(x) = \gamma_x$, for $x = 0.1$, using the program, we obtain the $r_{max} = 3.2e^{-4}$(Figure 10); and for $x = 0.3$, $r_{max} = 0.0054$(Figure 11).
(d) Lineaization at $\dot{\theta} = 0.5$,$\theta = [-\pi, 0.2, 0.8, \pi, 2\pi, 3\pi, 4\pi, 5\pi]$ (See Figure 12 to Figure 19)

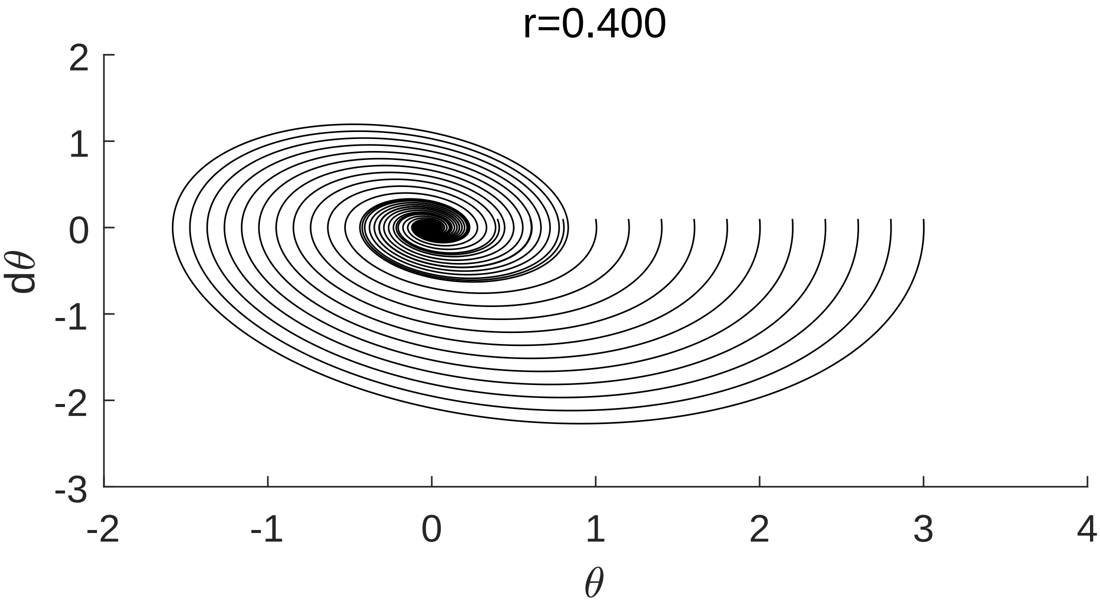

Figure 3: r=0.4

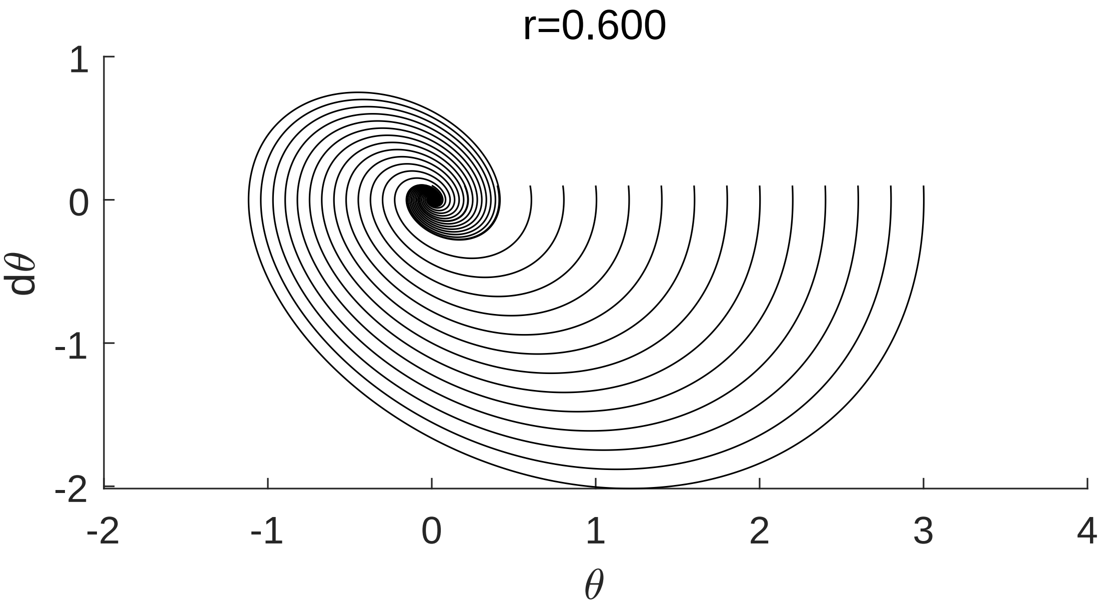

Figure 4: r=0.6

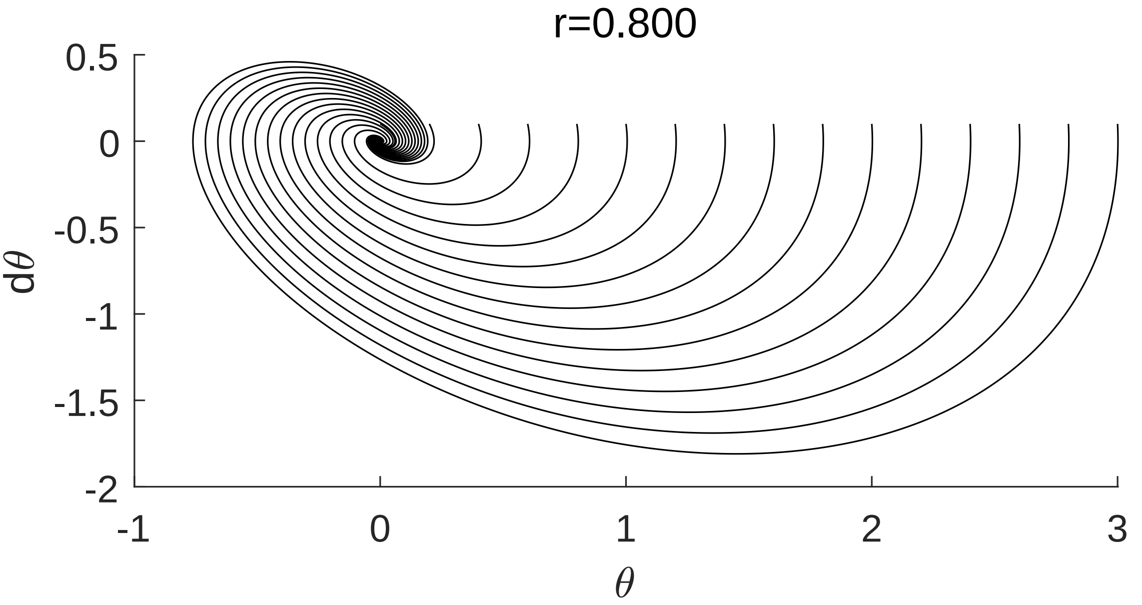

Figure 5: r=0.8

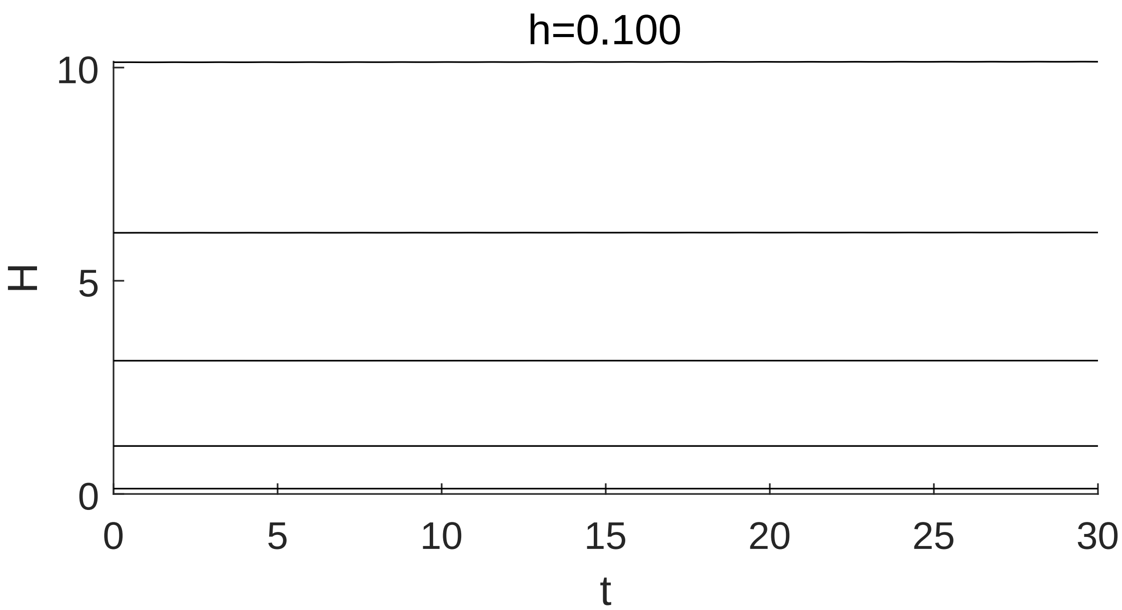

Figure 6: h=0.1

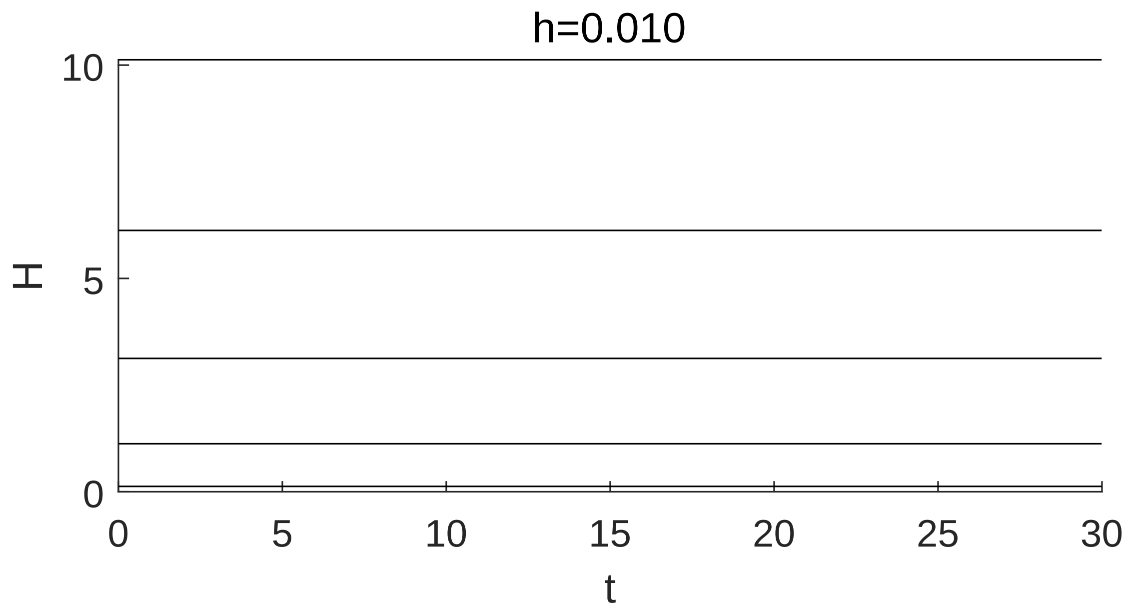

Figure 7: h=0.01

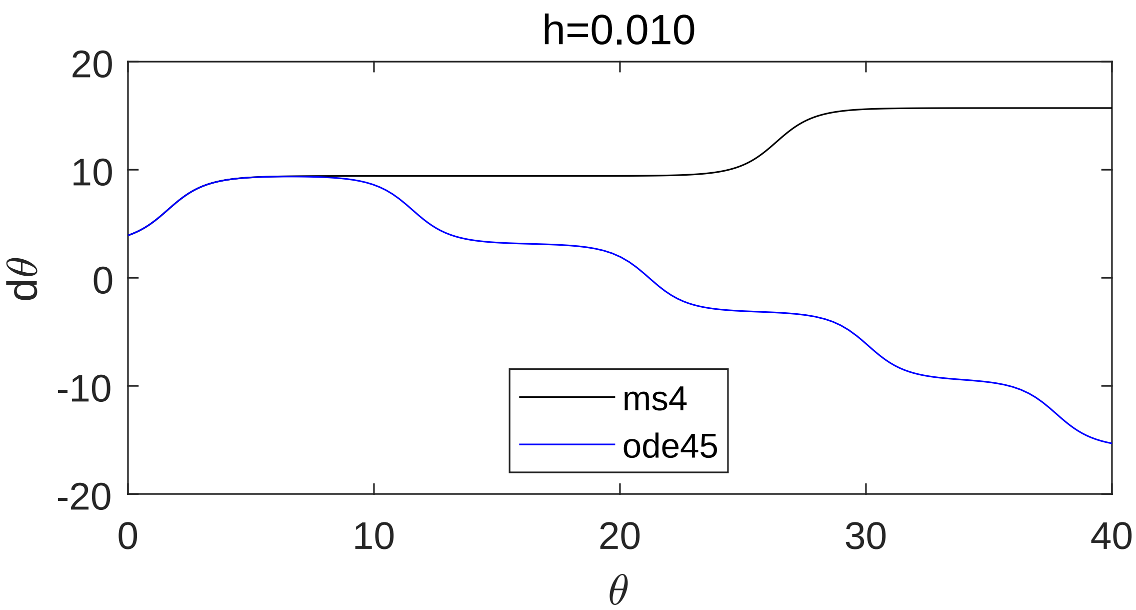

Figure 8: h=0.01

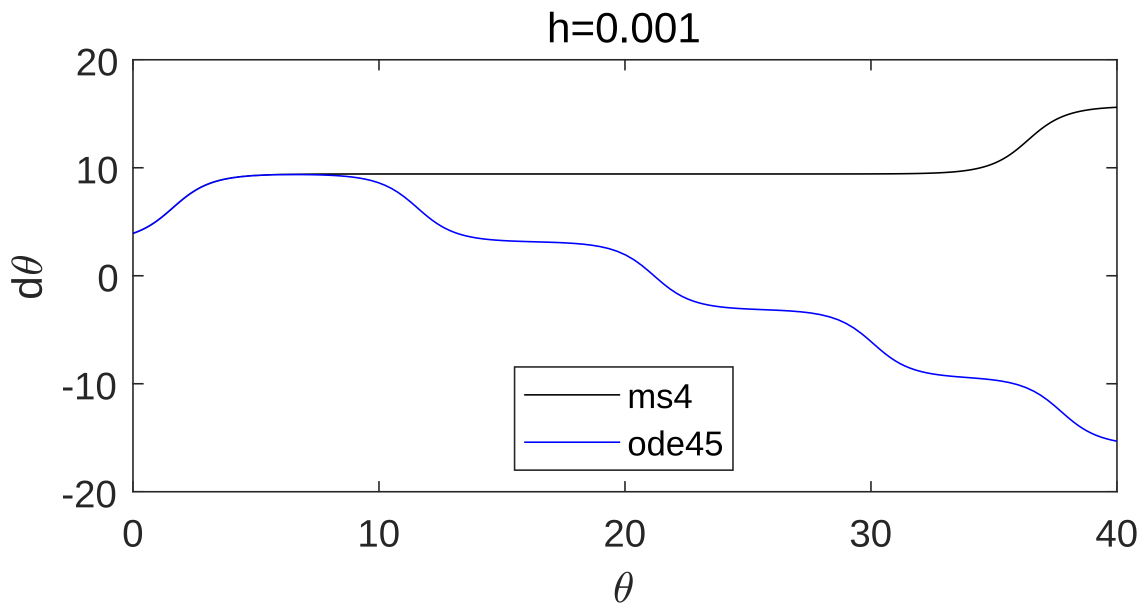

Figure 9: h=0.001

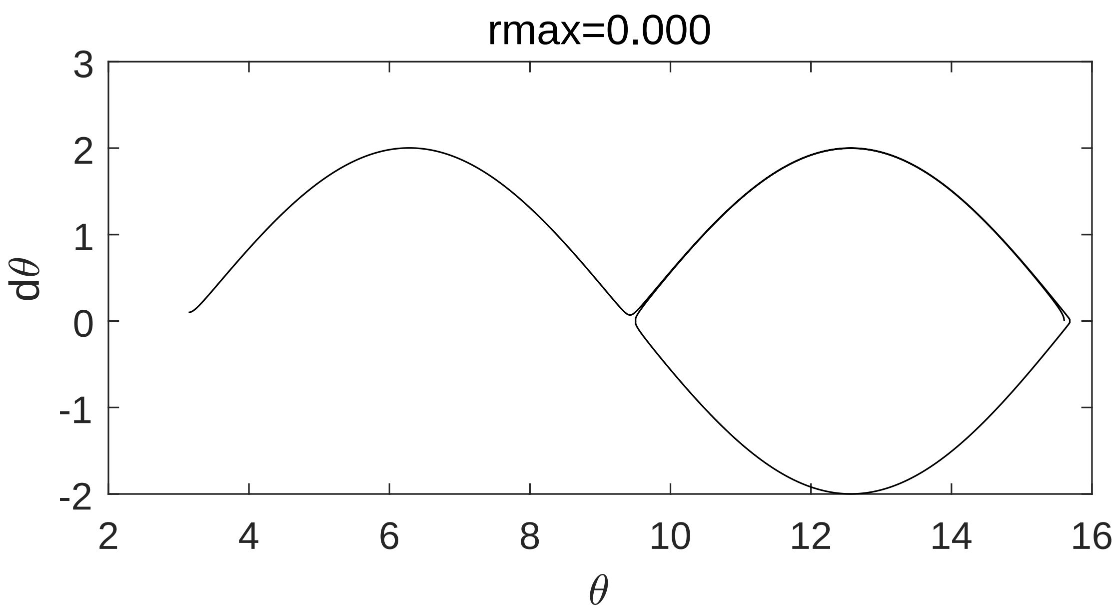

Figure 10: h=0.01

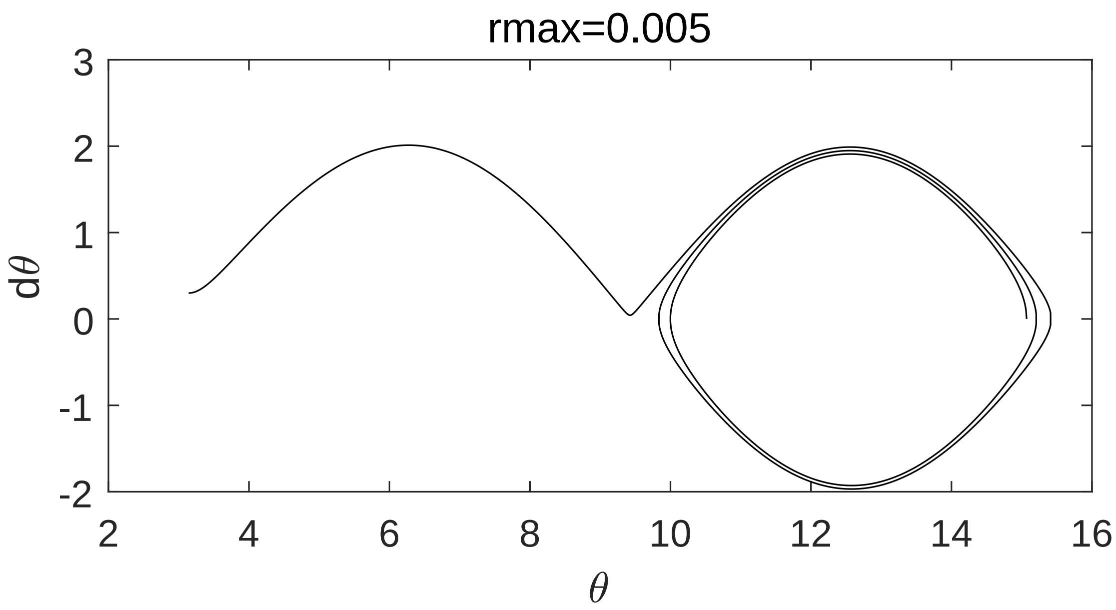

Figure 11: h=0.001

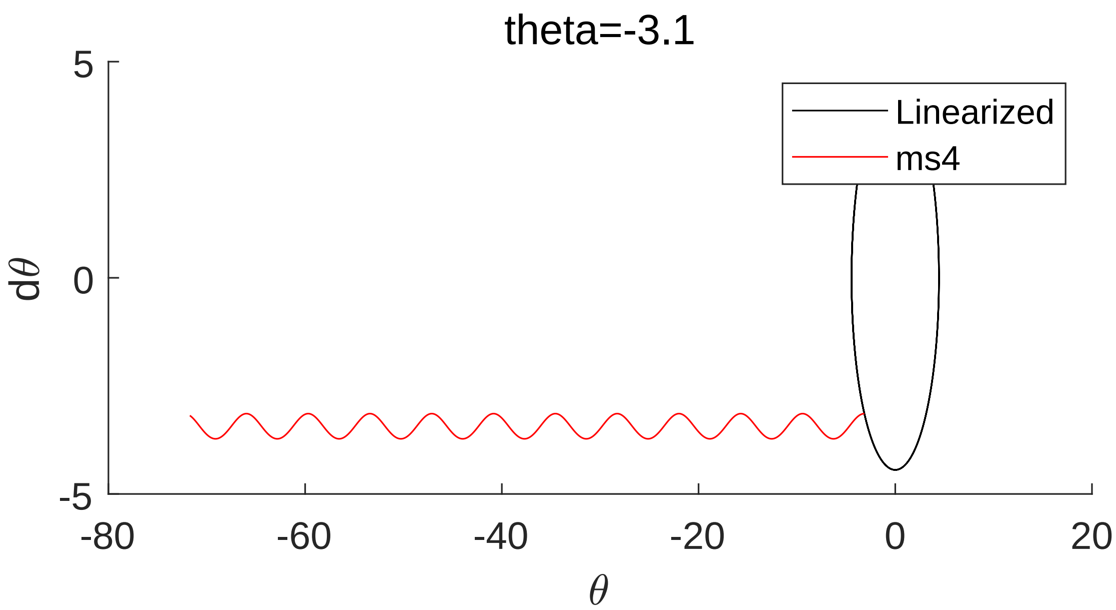

Figure 12: $\theta = -\pi, \dot{\theta} = 0.5$

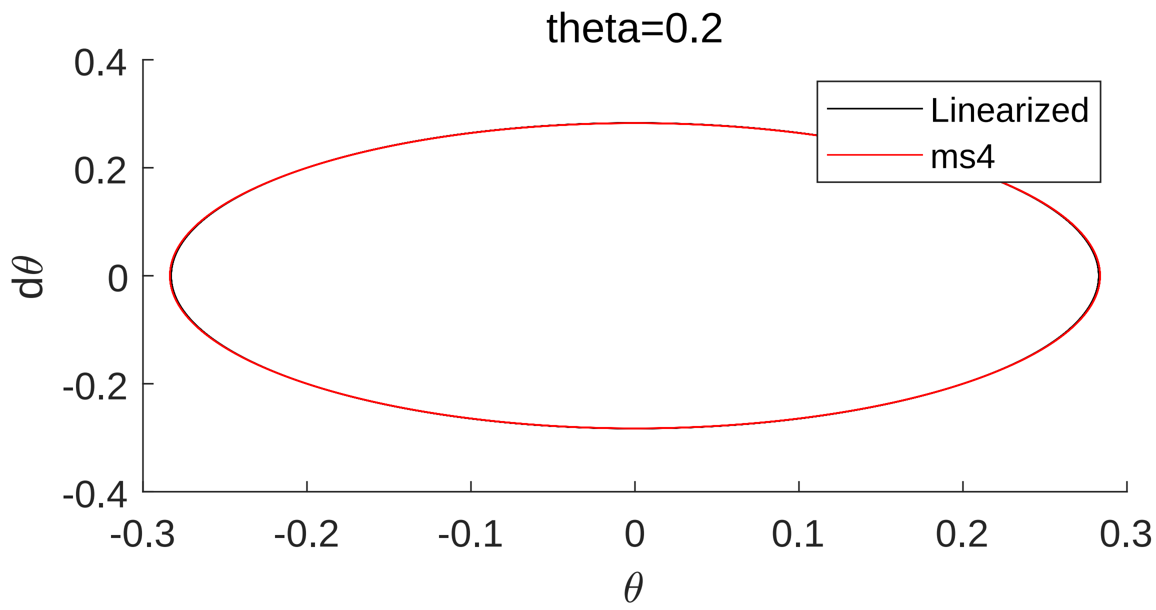

Figure 13: $\theta = 0.2, \dot{\theta} = 0.5$

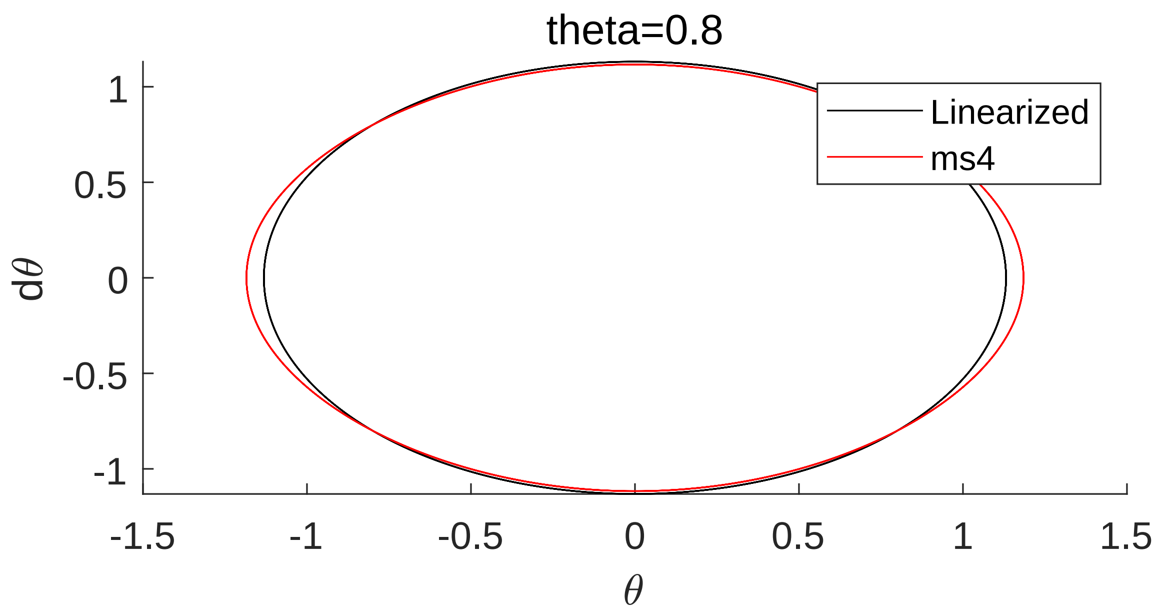

Figure 14: $\theta = 0.8, \dot{\theta} = 0.5$

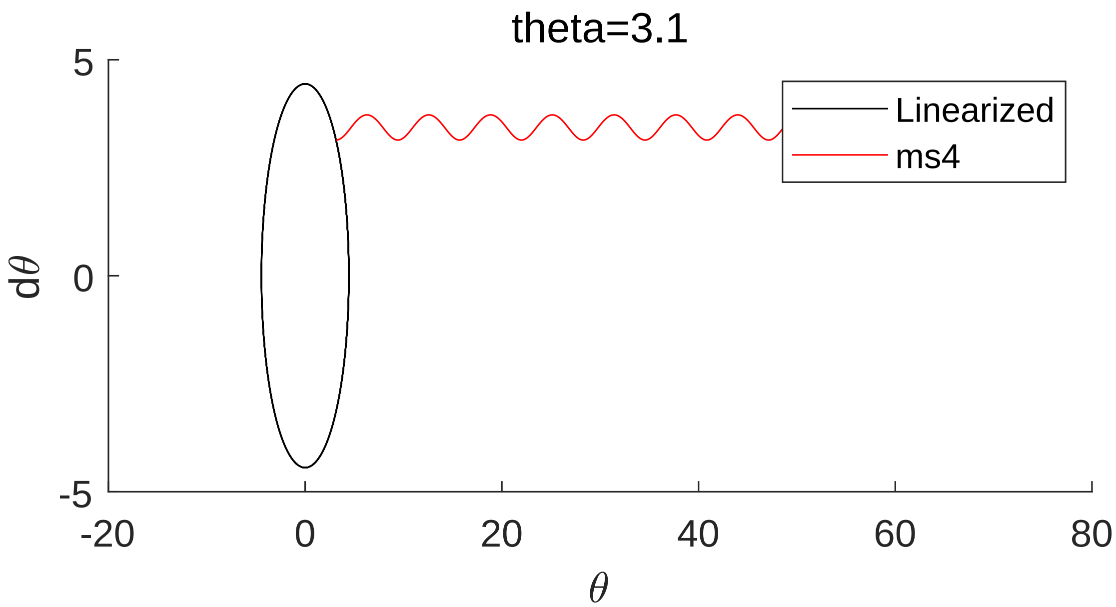

Figure 15: $\theta = \pi, \dot{\theta} = 0.5$

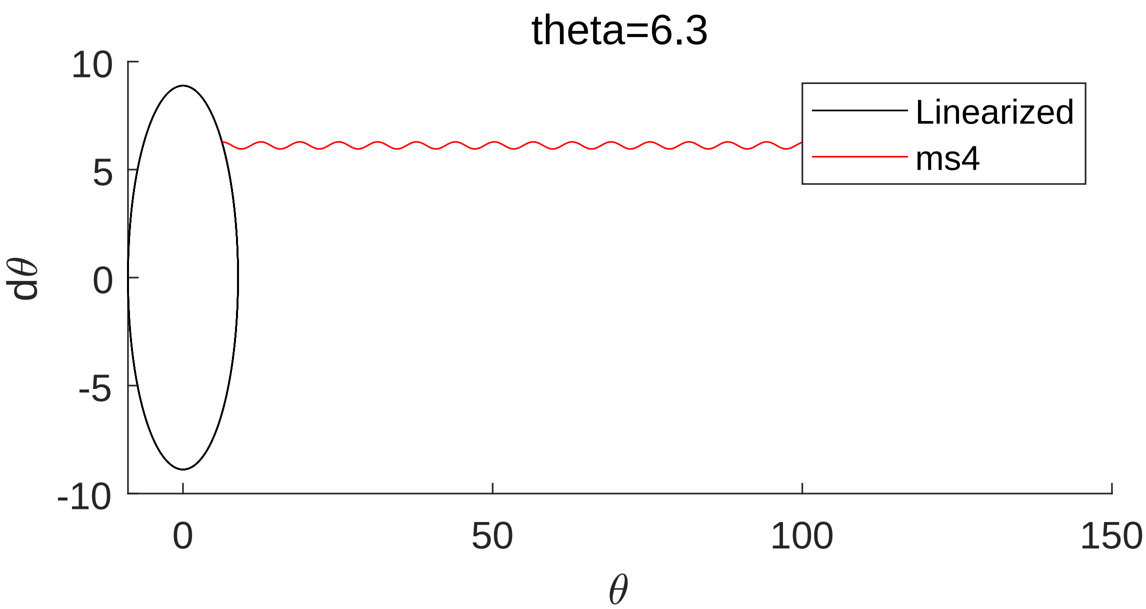

Figure 16: $\theta = 2\pi, \dot{\theta} = 0.5$

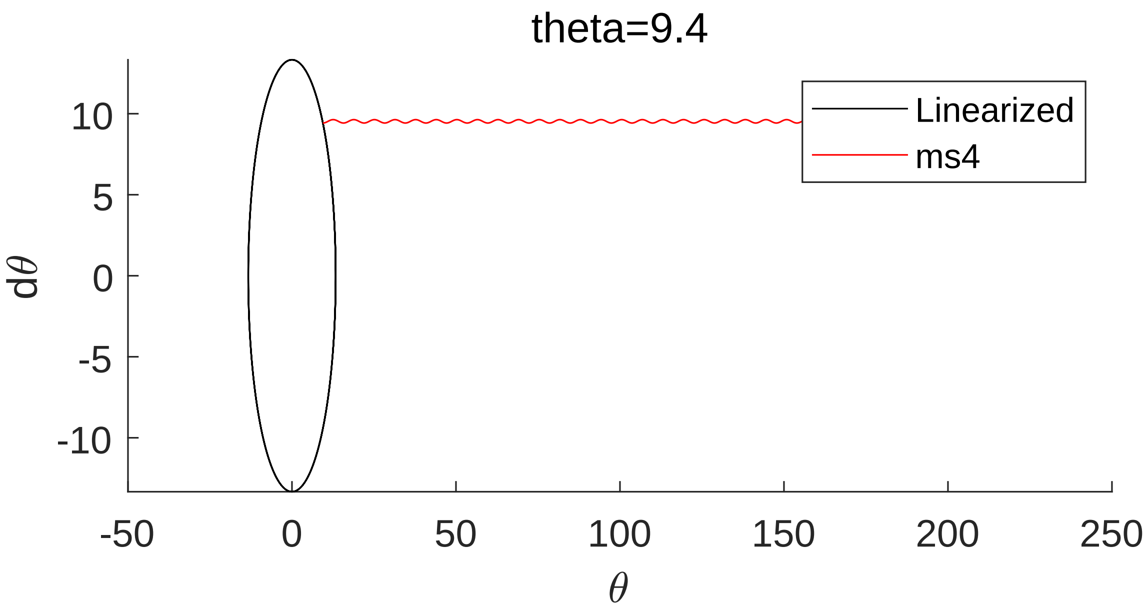

Figure 17: $\theta = 3\pi, \dot{\theta} = 0.5$

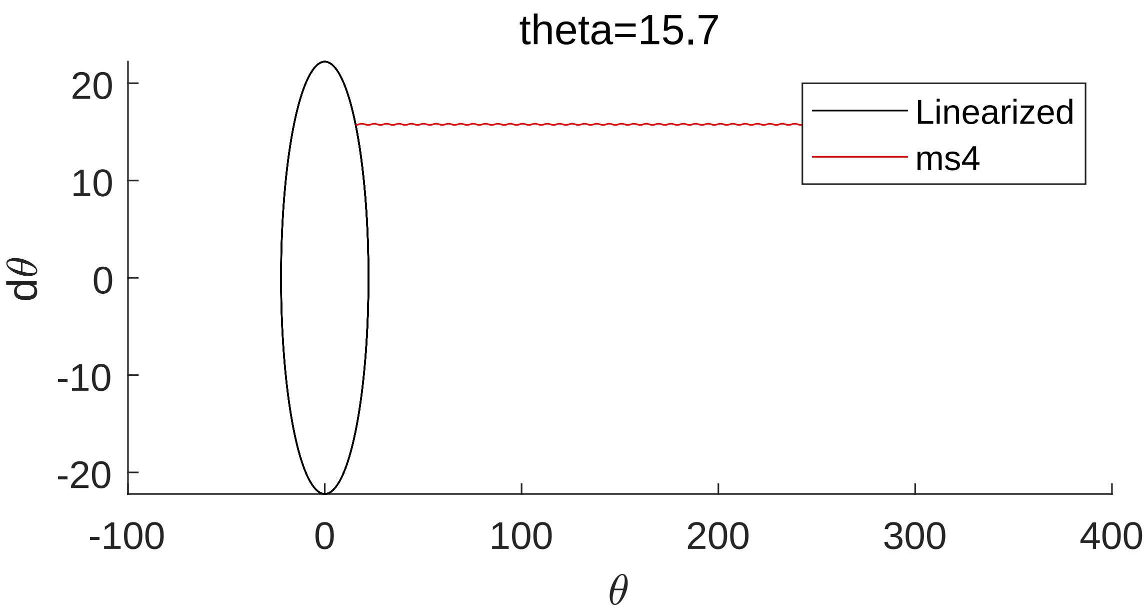

Figure 18: $\theta = 3\pi, \dot{\theta} = 0.5$

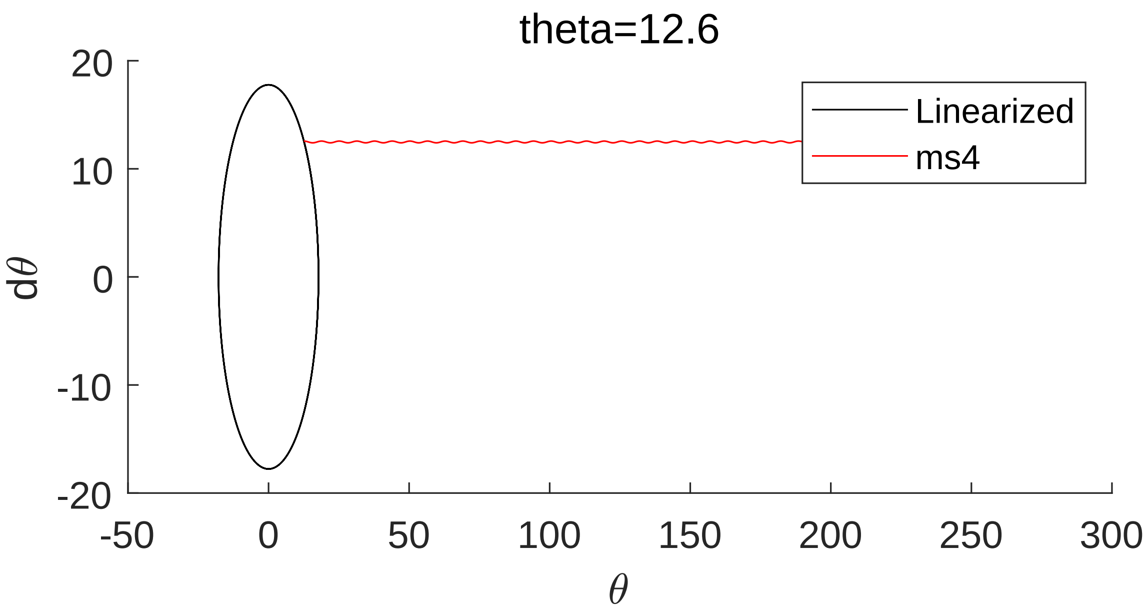

Figure 19: $\theta = 4\pi, \dot{\theta} = 0.5$