Homework 3

Alice Pagano
1236916

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Second-order Feynman diagrams for the Green's function

1. write the analytic expressions associated with diagrams (**a**), (**b**), and (**c**) in coordinate space;
2. show that (**c**) and (**d**) are distinct diagrams;
3. write the analytic expressions associated with diagram (**c**) in momentum space.

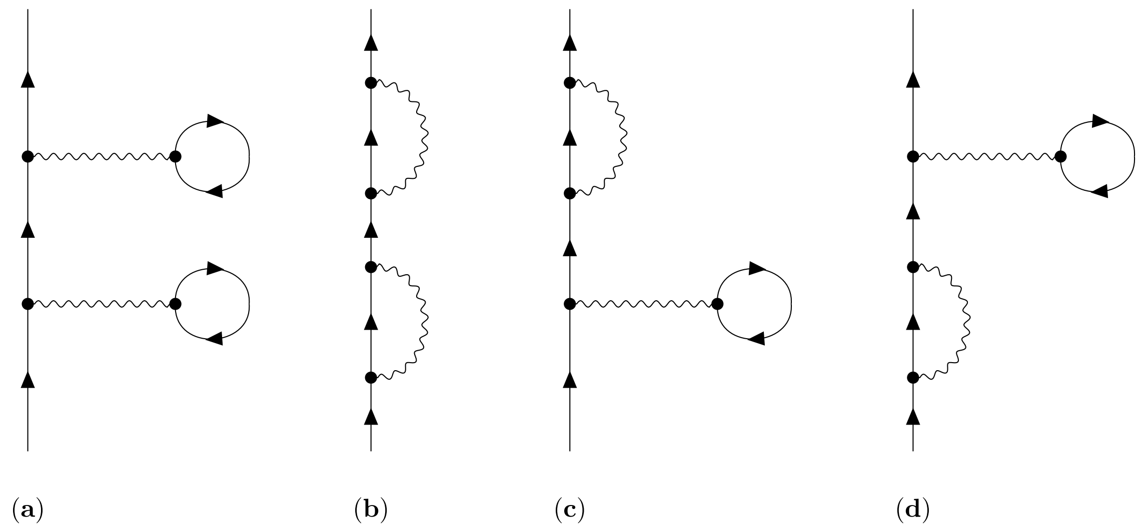


Figure 1: Four of the 10 topologically distinct connected diagrams at 2-nd order perturbative expansion for the Green's function $G_{\alpha\beta}(x, y)$.

Solution

1

Let us write the analytic expression associated with diagrams (**a**), (**b**), and (**c**) in coordinate space, using the following **Feynman rules**; to find the n-th order contribution to the single-particle Green's function $G_{\alpha\beta}(x, y)$ (for a system of identical particles interacting trough a 2-body potential):

1. Draw all topological distinct connected diagrams with n interaction lines U and $(2n+1)$ fermion lines which correspond to the non-interacting Green's functions G^0.
2. Label each vertex with a four-dimensional space-time point x_i.
3. Each solid line represents a Green's function $G^0_{\alpha\beta}(x, y)$ running from y to x.

4. Each wavy line represents an interaction

$$U(x, x') = V(\vec{\mathbf{x}}, \vec{\mathbf{x}}')\delta(t_x - t_{x'})$$

   which is an instantaneous 2-body potential.

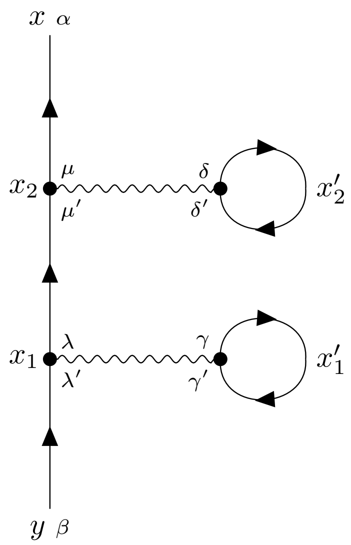
5. Integrate over all (space,time) variables that are internal and sum over the corresponding spin variables.

6. Affix a sign factor $(-1)^F$ to each term, where F is the number of closed fermion loops in the diagram.

7. To compute $G_{\alpha\beta}(x, y)$ assign a factor $(i/\hbar)^n$ to each n-th order term.

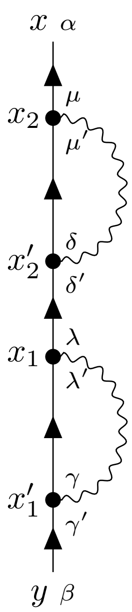
The diagrams in Fig.1 are four of the 10 topologically distinct connected diagrams at the 2-th order perturbative expansion for the Green's function $G_{\alpha\beta}(x, y)$. Let us write the analityc expression of each of them:

(a) For the first diagram, we obtain:



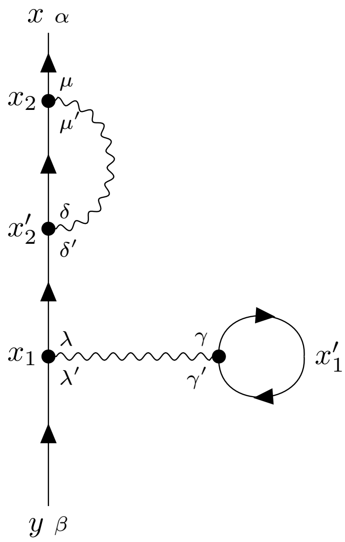
$$G^{(2a)}_{\alpha\beta}(x, y) = (-1)^2 \left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x'_1 \int \mathrm{d}^4x_2 \int \mathrm{d}^4x'_2 \times$$
$$\times G^0_{\alpha\mu}(x, x_2)U(x_2, x'_2)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\lambda}(x_2, x_1)G^0_{\delta'\delta}(x'_2, x'_2)\times$$
$$\times U(x_1, x'_1)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\beta}(x_1, y)G^0_{\gamma'\gamma}(x'_1, x'_1) \tag{1}$$

(b) For the second diagram, we obtain:



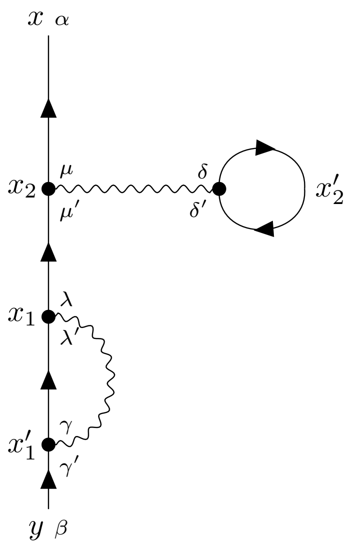
$$G^{(2b)}_{\alpha\beta}(x, y) = \left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x'_1 \int \mathrm{d}^4x_2 \int \mathrm{d}^4x'_2 \times$$
$$\times G^0_{\alpha\mu}(x, x_2)U(x_2, x'_2)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\delta}(x_2, x'_2)G^0_{\delta'\lambda}(x'_2, x_1)\times$$
$$\times U(x_1, x'_1)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\gamma}(x_1, x'_1)G^0_{\gamma'\beta}(x'_1, y) \tag{2}$$

(c) For the third diagram, we obtain:



$$
\begin{aligned}
G^{(2c)}_{\alpha\beta}(x,y) = (-1)^1 \left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x'_1 \int \mathrm{d}^4x_2 \int \mathrm{d}^4x'_2 \times \\
\times G^0_{\alpha\mu}(x,x_2) U(x_2,x'_2)_{\substack{\mu\mu'\\ \delta\delta'}} G^0_{\mu'\delta}(x_2,x'_2) G^0_{\delta'\lambda}(x'_2,x_1) \times \\
\times U(x_1,x'_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}} G^0_{\lambda'\beta}(x_1,y) G^0_{\gamma'\gamma}(x'_1,x'_1)
\end{aligned}
\tag{3}
$$

(d) For the fourth diagram, we obtain:



$$
\begin{aligned}
G^{(2d)}_{\alpha\beta}(x,y) = (-1)^1 \left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x'_1 \int \mathrm{d}^4x_2 \int \mathrm{d}^4x'_2 \times \\
\times G^0_{\alpha\mu}(x,x_2) U(x_2,x'_2)_{\substack{\mu\mu'\\ \delta\delta'}} G^0_{\mu'\lambda}(x_2,x_1) G^0_{\delta'\delta}(x'_2,x'_2) \times \\
\times U(x_1,x'_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}} G^0_{\lambda'\gamma}(x_1,x'_1) G^0_{\gamma'\beta}(x'_1,y)
\end{aligned}
\tag{4}
$$

2

Let us note that with the Feynman method, the arrows designate the direction of momentum flow. It is because of the arrows that the diagrams (**c**) and (**d**) are topologically distinct. In particular, graphically two diagrams are topologically distinct in the Feynman sense if they are visualized as made of rubber bands and one cannot be deformed into the other.

Now, in order to see analitically that diagram (**c**) and (**d**) are distinct, let us rewrite the analityc formula for the two diagrams by highliting their differences. For the (**c**) diagram:

$$
\begin{aligned}
G^{(2c)}_{\alpha\beta}(x,y) = (-1)^1 \left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x'_1 \int \mathrm{d}^4x_2 \int \mathrm{d}^4x'_2 \times G^0_{\alpha\mu}(x,x_2) U(x_2,x'_2)_{\substack{\mu\mu'\\ \delta\delta'}} \times \\
\times\, G^0_{\mu'\delta}(x_2,x'_2) G^0_{\delta'\lambda}(x'_2,x_1)\, U(x_1,x'_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}}\, G^0_{\lambda'\beta}(x_1,y) G^0_{\gamma'\gamma}(x'_1,x'_1)
\end{aligned}
$$

while for the (**d**) diagram:

$$G^{(2d)}_{\alpha\beta}(x,y) = (-1)^1\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_1' \int \mathrm{d}^4x_2 \int \mathrm{d}^4x_2'\, G^0_{\alpha\mu}(x,x_2)U(x_2,x_2')_{\substack{\mu\mu'\\ \delta\delta'}}\times$$
$$\times\; G^0_{\mu'\lambda}(x_2,x_1)G^0_{\delta'\delta}(x_2',x_2')\, U(x_1,x_1')_{\substack{\lambda\lambda'\\ \gamma\gamma'}}\, G^0_{\lambda'\gamma}(x_1,x_1')G^0_{\gamma'\beta}(x_1',y)$$

Let us see if with an exchange of the dummy variables, the two analityc formula are the same. For the internal variables we change in the following way:

$$x_1 \leftrightarrow x_2, \quad x_1' \leftrightarrow x_2'$$

while for the spin indices

$$\lambda \leftrightarrow \mu,\ \lambda' \leftrightarrow \mu',\ \gamma \leftrightarrow \delta, \gamma' \leftrightarrow \delta'$$

Now, we make these substitutions on the second equation (the analityc expression for the (**d**) diagram) obtaining:

$$G^{(2d)}_{\alpha\beta}(x,y) = (-1)^1\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_2 \int \mathrm{d}^4x_2' \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_1'\, G^0_{\alpha\mu}(x,x_1)U(x_1,x_1')_{\substack{\mu\mu'\\ \delta\delta'}}\times$$
$$\times\, G^0_{\mu'\lambda}(x_1,x_2)G^0_{\delta'\delta}(x_1',x_1')U(x_2,x_2')_{\substack{\lambda\lambda'\\ \gamma\gamma'}}G^0_{\lambda'\gamma}(x_2,x_2')G^0_{\gamma'\beta}(x_2',y)$$

we see that the result are still different. We cannot find any change of variables suitable.

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Finally, we just focus on (**c**) diagram and we write its analytic expression in momentum space either just taking the Fourier transform of its contribution in coordinate space and using the Feynman rules in momentum space.

Fourier antitransformation

Let us consider the analitycal formula of the (**c**) diagram in coordinate space (Eq.(3)):

$$G^{(2c)}_{\alpha\beta}(x,y) = (-1)^1\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_1' \int \mathrm{d}^4x_2 \int \mathrm{d}^4x_2'\, G^0_{\alpha\mu}(x,x_2)U(x_2,x_2')_{\substack{\mu\mu'\\ \delta\delta'}}\times$$
$$\times\, G^0_{\mu'\delta}(x_2,x_2')G^0_{\delta'\lambda}(x_2',x_1)U(x_1,x_1')_{\substack{\lambda\lambda'\\ \gamma\gamma'}}G^0_{\lambda'\beta}(x_1,y)G^0_{\gamma'\gamma}(x_1',x_1')$$

If we suppose spatial and temporal invariance (**uniform system**) then we can write a full Fourier representation. First of all, let us consider the Fourier transform of each term separately in the above expression:

$$
\begin{aligned}
U(x_2,x_2')_{\substack{\mu\mu'\\ \delta\delta'}} &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k_2\, e^{ik_2(x_2-x_2')}U(k_2)_{\substack{\mu\mu'\\ \delta\delta'}}\\
U(x_1,x_1')_{\substack{\lambda\lambda'\\ \gamma\gamma'}} &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k_1\, e^{ik_1(x_1-x_1')}U(k_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}}\\
G^0_{\alpha\mu}(x,x_2) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{ik(x-x_2)}G^0_{\alpha\mu}(k)\\
G^0_{\mu'\delta}(x_2,x_2') &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4q_1\, e^{iq_1(x_2-x_2')}G^0_{\mu'\delta}(q_1)\\
G^0_{\delta'\lambda}(x_2',x_1) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4p_1\, e^{ip_1(x_2'-x_1)}G^0_{\delta'\lambda}(p_1)\\
G^0_{\lambda'\beta}(x_1,y) &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4p_2\, e^{ip_2(x_1-y)}G^0_{\lambda'\beta}(p_2)\\
G^0_{\gamma'\gamma}(x_1',x_1') &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4q_2\, e^{iq_2(x_1'-x_1')}G^0_{\gamma'\gamma}(q_2)
\end{aligned}
$$

By substituting the last expressions in the upper equation we obtain:

$$
\begin{aligned}
G^{(2c)}_{\alpha\beta}(x,y) = \frac{(-1)^1}{(2\pi)^{28}}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_1' \int \mathrm{d}^4x_2 \int \mathrm{d}^4x_2' \int \mathrm{d}^4k\, \mathrm{d}^4k_2\, \mathrm{d}^4q_1\, \mathrm{d}^4p_1\, \mathrm{d}^4k_1\, \mathrm{d}^4p_2\, \mathrm{d}^4q_2 \times\\
\times G^0_{\alpha\mu}(k)U(k_2)_{\substack{\mu\mu'\\ \delta\delta'}}G^0_{\mu'\delta}(q_1)G^0_{\delta'\lambda}(p_1)U(k_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}}G^0_{\lambda'\beta}(p_2)G^0_{\gamma'\gamma}(q_2)\times\\
\times e^{ik(x-x_2)}e^{ik_2(x_2-x_2')}e^{iq_1(x_2-x_2')}e^{ip_1(x_2'-x_1)}e^{ik_1(x_1-x_1')}e^{ip_2(x_1-y)}\cancel{e^{iq_2(x_1'-x_1')}}
\end{aligned}
$$

If we consider the property:

$$
\delta^{(4)}(t) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4x\, e^{itx}
$$

we can integrate over the internal variables x_1, x_1', x_2, x_2' obtaining delta functions. Hence:

$$
\begin{aligned}
\int \mathrm{d}^4x_2\, e^{i(k_2+q_1-k)} &= (2\pi)^4\delta^{(4)}(k_2+q_1-k)\\
\int \mathrm{d}^4x_2'\, e^{i(p_1-q_1-k_2)} &= (2\pi)^4\delta^{(4)}(p_1-q_1-k_2)\\
\int \mathrm{d}^4x_1\, e^{i(k_1+p_2-p_1)} &= (2\pi)^4\delta^{(4)}(k_1+p_2-p_1)\\
\int \mathrm{d}^4x_1'\, e^{i(-k_1)} &= (2\pi)^4\delta^{(4)}(-k_1)
\end{aligned}
$$

Thus the analitycal Fourier transformed expression for diagram (**c**) becomes:

$$
\begin{aligned}
G^{(2c)}_{\alpha\beta}(x,y) = \frac{(-1)^1}{(2\pi)^{12}}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu'\\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4k\, \mathrm{d}^4k_2\, \mathrm{d}^4q_1\, \mathrm{d}^4p_1\, \mathrm{d}^4k_1\, \mathrm{d}^4p_2\, \mathrm{d}^4q_2 \times\\
\times G^0_{\alpha\mu}(k)U(k_2)_{\substack{\mu\mu'\\ \delta\delta'}}G^0_{\mu'\delta}(q_1)G^0_{\delta'\lambda}(p_1)U(k_1)_{\substack{\lambda\lambda'\\ \gamma\gamma'}}G^0_{\lambda'\beta}(p_2)G^0_{\gamma'\gamma}(q_2)\times\\
\times e^{ikx}e^{-ip_2y}\delta^{(4)}(k_2+q_1-k)\delta^{(4)}(p_1-q_1-k_2)\delta^{(4)}(k_1+p_2-p_1)\delta^{(4)}(-k_1)
\end{aligned}
$$

By exploiting the delta functions:

$$
\begin{cases}
k = k_2 + q_1 & \Rightarrow p_1 = k\\
p_1 = q_1 + k_2 & \Rightarrow k_2 = k - q_1\\
p_1 = k_1 + p_2 & \Rightarrow p_2 = k\\
-k_1 = 0 & \Rightarrow k_1 = 0
\end{cases}
$$

So, we can write:

$$G^{(2c)}_{\alpha\beta}(x,y) = \frac{(-1)^1}{(2\pi)^{12}}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4k \int \mathrm{d}^4q_1\,\mathrm{d}^4q_2\, e^{ikx}e^{-iky} \times$$
$$\times\, G^0_{\alpha\mu}(k) U(k-q_1)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\delta}(q_1) G^0_{\delta'\lambda}(k) U(0)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\beta}(k) G^0_{\gamma'\gamma}(q_2)$$

More explicitly:

$$G^{(2c)}_{\alpha\beta}(x,y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, e^{ik(x-y)}\Big[\frac{(-1)^1}{(2\pi)^8}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4q_1\,\mathrm{d}^4q_2 \times$$
$$\times\, G^0_{\alpha\mu}(k) U(k-q_1)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\delta}(q_1) G^0_{\delta'\lambda}(k) U(0)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\beta}(k) G^0_{\gamma'\gamma}(q_2)\Big]$$

Note that the quanity in the brackets is identified as $G_{\alpha\beta}(k) \equiv G_{\alpha\beta}(\vec{\mathbf{k}},\omega)$:

$$G^{(2c)}_{\alpha\beta}(k) = \frac{(-1)^1}{(2\pi)^8}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4q_1\,\mathrm{d}^4q_2 \times$$
$$\times\, G^0_{\alpha\mu}(k) U(k-q_1)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\delta}(q_1) G^0_{\delta'\lambda}(k) U(0)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\beta}(k) G^0_{\gamma'\gamma}(q_2) \tag{7}$$

Feynman rules in momentum space

Now, let us derive the analytic expression of (**c**) diagram in momentum space by using the Feynman rules. The Feynman rules for the n-th order contribution to $G_{\alpha\beta}(\vec{\mathbf{k}},\omega) \equiv G_{\alpha\beta}(k)$ are:

1. Draw all topological distinct connected diagrams with n interaction lines U and $(2n+1)$ fermion lines which correspond to the non-interacting Green's functions G^0.

2. Assign a (conventional) direction to each interaction line.

3. Associate a directed four-momentum with each line and conserve four-momentum at each vertex.

4. Each Green's function corresponds to a factor

$$G^0_{\alpha\beta}(\vec{\mathbf{k}},\omega) = \delta_{\alpha\beta}\left[\frac{\Theta(|\vec{k}|-k_F)}{\omega-\omega_k+i\eta} + \frac{\Theta(k_F-|\vec{k}|)}{\omega-\omega_k-i\eta}\right]$$

5. Each interaction corresponds to a factor $U(q)_{\substack{\lambda\lambda' \\ \mu\mu'}} = V(\vec{\mathbf{q}})_{\substack{\lambda\lambda' \\ \mu\mu'}}$.

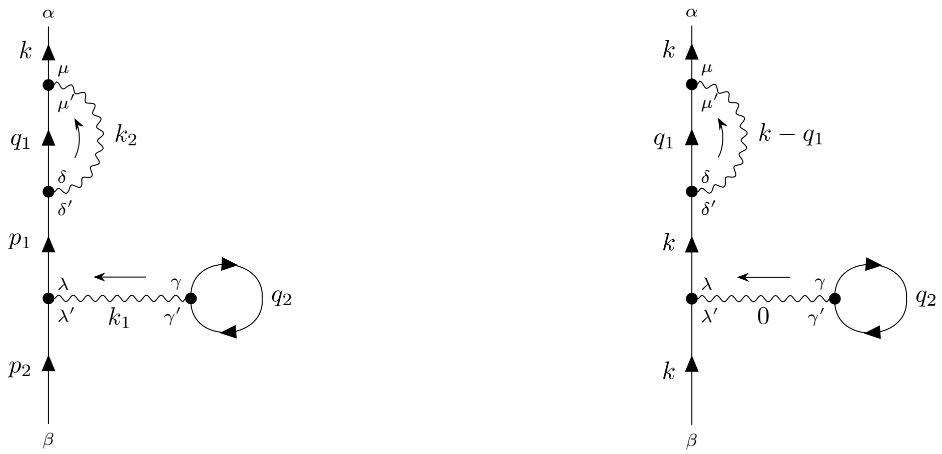
6. Perform a spin summation along each continuous particle line including the potential at each vertex.

7. Integrate over the n internal four-momenta.

8. Affix a factor $(i/\hbar)^n(2\pi)^{-4n}(-1)^F$ where F is the number of closed fermion loops.

9. Any single-particle line that forms a closed loop or that is linked by the same interaction line is interpreted as $e^{i\omega\eta}G_{\alpha\beta}(\vec{\mathbf{k}},\omega)$, where $\eta \to 0^+$ at the end of the calculation.

Using the Feynman rules, we obtain the two diagrams in Fig.2 before and after the conservation of momentum at each vertex (dots in figure).



(a) Before. **(b)** After.

Figure 2: Feynman diagram in momentum space of (**c**) before and after conservation of momenta.

Altough the topological structure is identical with the corresponding diagrams in coordinate space, the labeling and interpretation are naturally quite different. In particular, the analitycal expression of (**c**) diagram in momentum space results:

$$G^{(2c)}_{\alpha\beta}(k) = \frac{(-1)^1}{(2\pi)^8}\left(\frac{i}{\hbar}\right)^2 \sum_{\substack{\lambda\lambda',\mu\mu' \\ \gamma\gamma',\delta\delta'}} \int \mathrm{d}^4 q_1 \, \mathrm{d}^4 q_2 \, \times$$

$$\times \, G^0_{\alpha\mu}(k) U(k-q_1)_{\substack{\mu\mu' \\ \delta\delta'}} G^0_{\mu'\delta} e^{i\omega_1\eta}(q_1) G^0_{\delta'\lambda}(k) U(0)_{\substack{\lambda\lambda' \\ \gamma\gamma'}} G^0_{\lambda'\beta}(k) G^0_{\gamma'\gamma}(q_2) e^{i\omega_2\eta}$$