Spatial and temporal coherence

Spatial and temporal coherence are independent concepts: a wave can have perfect spatial coherence but partial temporal coherence, and viceversa.

Spatial coherence is associated with the wave's properties **transverse** to the direction of propagation.

Spatial coherence is associated with a **distribution of propagation vectors** $\vec{k}$ associated with the wave, i.e., with a departure of the wave from the ideal plane wave.

Temporal coherence is associated with the wave's properties **along** the direction of propagation.

It is associated with the **frequency distribution** of the source.

$$\tau_c \Delta\nu \approx 1 \quad \Longrightarrow \quad \tau_c \approx \frac{1}{\Delta\nu} = \frac{\lambda^2}{c\Delta\lambda}$$

First-order auto-correlation function

$$\boxed{g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}}$$

Degree of first-order **longitudinal** (**temporal**) coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

Let's consider a **quasi-monochromatic** em wave, with central frequency ω_0 and bandwidth $\Delta\omega$:

$$E(t) = E_0 e^{-i(\omega_0 t+\phi(t))}$$

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle} = \frac{\left\langle E_0 e^{i(\omega_0 t+\phi(t))} E_0 e^{-i(\omega_0 (t+\tau)+\phi(t+\tau))}\right\rangle}{E_0^2}$$

$$= e^{-i\omega_0\tau}\left\langle e^{-i[\phi(t+\tau)-\phi(t)]}\right\rangle$$

The real part of $g^{(1)}(\tau)$ is an oscillating function with period $\frac{2\pi}{\omega_0}$

First-order auto-correlation function

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

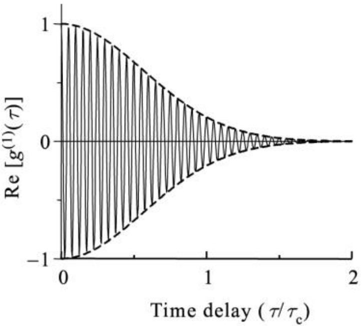
Degree of first-order **longitudinal (temporal)** coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

The variation of the modulus of $g^{(1)}(\tau)$ contains information on the coherence of the light:

$$\left|g^{(1)}(\tau)\right| = \left|\left\langle e^{-i[\phi(t+\tau)-\phi(t)]}\right\rangle\right|$$

$$\left|g^{(1)}(0)\right| = 1 \quad \text{always}$$



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First-order auto-correlation function

$$\boxed{g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}}$$

Degree of first-order **longitudinal** (**temporal**) coherence

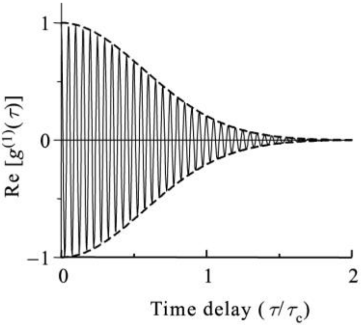
$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

$$0 < \tau \ll \tau_c \quad \Longrightarrow \quad \phi(t+\tau) \approx \phi(t)$$

$$\Longrightarrow \quad \left|g^{(1)}(\tau)\right| \cong 1$$

If τ increases, $\left|g^{(1)}(\tau)\right|$ decreases since it increases the probability of the phase relation to become random.

$\tau \gg \tau_c \quad \Longrightarrow \quad \phi(t+\tau), \phi(t)$ become totally uncorrelated



$$\Longrightarrow \quad \left|g^{(1)}(\tau)\right| = \left|\left\langle e^{-i[\phi(t+\tau)-\phi(t)]}\right\rangle\right| = 0 \qquad \text{on average}$$

First-order auto-correlation function

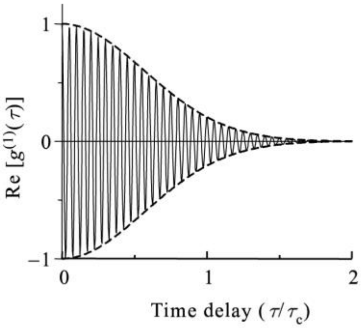
$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

Degree of first-order **longitudinal** (**temporal**) coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

$\Longrightarrow$ $\left|g^{(1)}(\tau)\right|$ varies from $1 \longrightarrow 0$ on a time scale of the order of τ_c

The functional shape of $g^{(1)}(\tau)$ for partially coherent light depends on the spectral broadening



Lorentzian lineshape

(homogeneous broadening)

$$g^{(1)}(\tau) = e^{-i\omega_0\tau}e^{-|\tau|/\tau_c} \qquad \Delta\omega_0 = 2\pi\Delta\nu_0 \qquad \tau_c = \frac{2\pi}{\Delta\omega_0} = \frac{1}{\Delta\nu_0}$$

First-order auto-correlation function

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

Degree of first-order **longitudinal** (**temporal**) coherence

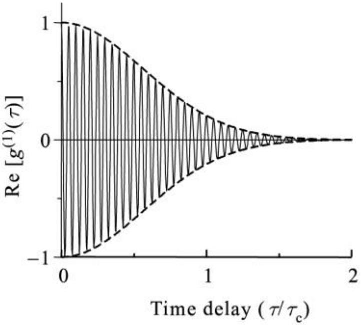
$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

Gaussian lineshape

(inhomogeneous broadening)

$$g^{(1)}(\tau) = e^{-i\omega_0\tau}e^{-(\pi/2)(\tau/\tau_c)^2}$$

$$\tau_c = \frac{2\sqrt{2\pi\ln 2}}{\Delta\omega_0}$$



Lorentzian lineshape

(homogeneous broadening)

$$g^{(1)}(\tau) = e^{-i\omega_0\tau}e^{-|\tau|/\tau_c} \qquad \Delta\omega_0 = 2\pi\Delta\nu_0 \qquad \tau_c = \frac{2\pi}{\Delta\omega_0} = \frac{1}{\Delta\nu_0}$$

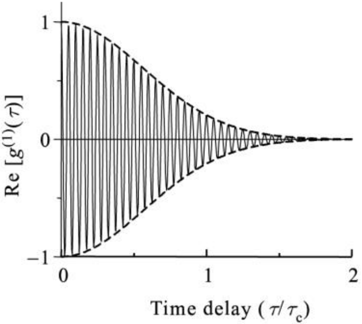
First-order auto-correlation function

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

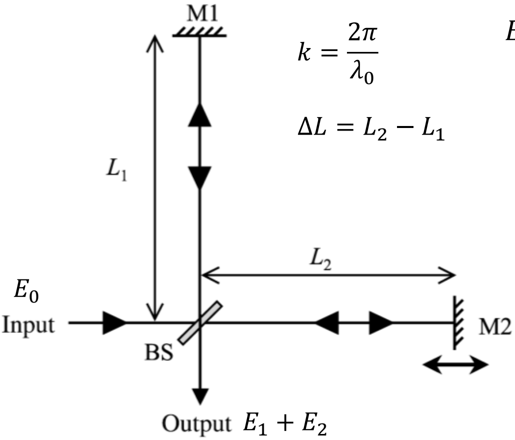
Degree of first-order **longitudinal (temporal)** coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

From the experimental point of view the first-order auto-correlation function $g^{(1)}(\tau)$ can be measured by using a **Michelson's interferometer**



Michelson's interferometer

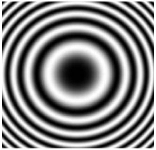


$$k = \frac{2\pi}{\lambda_0}$$

$$\Delta L = L_2 - L_1$$

$$
\begin{aligned}
E_{out} &= E_1 + E_2 \\
&= \frac{E_0}{2} e^{i2kL_1} + \frac{E_0}{2} e^{i2kL_2} e^{i\Delta\phi} \\
&= \frac{E_0}{2} e^{i2kL_1}\left(1 + e^{i2k\Delta L} e^{i\Delta\phi}\right) \\
&= \frac{E_0}{2} e^{i2kL_1}\left(1 + e^{i\delta}\right)
\end{aligned}
$$

$$\delta = \frac{4\pi}{\lambda_0}\Delta L + \Delta\phi$$

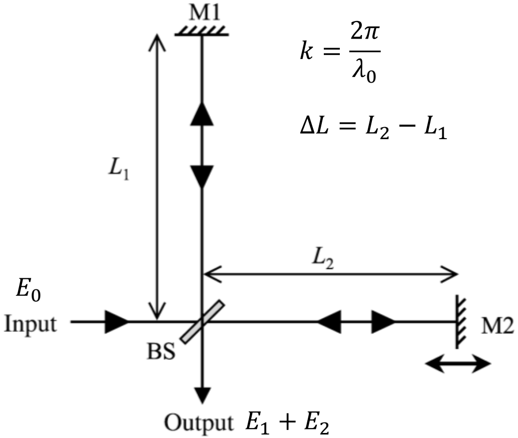


$$I_{out} \propto |E_{out}|^2$$

$$I_{out} = I_0^2 \cos^2\left(\frac{\delta}{2}\right)$$

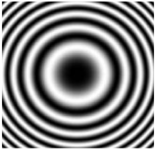
$$cos\delta = 1 - 2\left(sin\frac{\delta}{2}\right)^2$$

Michelson's interferometer



$$k = \frac{2\pi}{\lambda_0}$$

$$\Delta L = L_2 - L_1$$



$$I_{out} = I_0^2 \cos^2\left(\frac{\delta}{2}\right)$$

Interference **maxima**

$$\delta = \frac{4\pi}{\lambda_0}\Delta L + \Delta\phi = 2m\pi$$

Interference **minima**

$$\delta = \frac{4\pi}{\lambda_0}\Delta L + \Delta\phi = (2m+1)\pi$$

Fringe visibility

$$\boxed{V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}}$$

First-order auto-correlation function

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

Degree of first-order **longitudinal** (**temporal**) coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

...using a Michelson's interferometer

$$E_{out}(t) = E(t) + E(t+\tau)$$

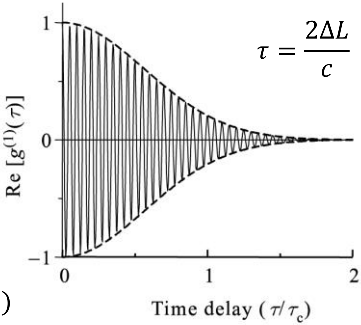
$$\mathrm{I}(\tau) \propto \langle E^*_{out}(t)E_{out}(t)\rangle =$$

$$= \langle E^*(t)E(t)\rangle + \langle E^*(t+\tau)E(t+\tau)\rangle +$$

$$+\langle E^*(t)E(t+\tau)\rangle + \langle E(t)E^*(t+\tau)\rangle$$

$$= 2\langle E^*(t)E(t)\rangle + 2Re(\langle E^*(t+\tau)E(t+\tau)\rangle)$$

$$= 2\langle E^*(t)E(t)\rangle\left[1 + Re\left(g^{(1)}(\tau)\right)\right]$$



$$\tau = \frac{2\Delta L}{c}$$

First-order auto-correlation function

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle |E(t)|^2\rangle}$$

Degree of first-order **longitudinal (temporal)** coherence

$$\langle E^*(t)E(t+\tau)\rangle = \frac{1}{T}\int_T E^*(t)E(t+\tau)dt$$

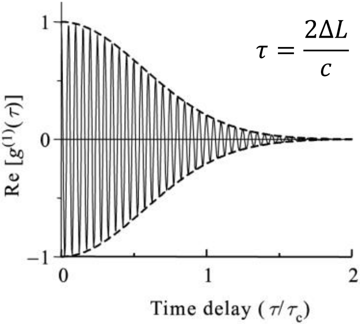
...using a Michelson's interferometer

$$E_{out}(t) = E(t) + E(t+\tau)$$

$$\mathrm{I}(\tau) = I_0\left[1 + \mathcal{Re}\left(g^{(1)}(\tau)\right)\right]$$

$$I_{\max,\min} = I_0\left(1 \pm \left|g^{(1)}(\tau)\right|\right)$$

$$\boxed{V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \left|g^{(1)}(\tau)\right|}$$



First-order auto-correlation function

$$g_{12}^{(1)}(\tau) = \frac{\langle E_1(t)^* E_2(t+\tau)\rangle}{\sqrt{\langle |E_1(t)|^2\rangle \langle |E_2(t)|^2\rangle}}$$

Degree of first-order **longitudinal** (**temporal**) coherence

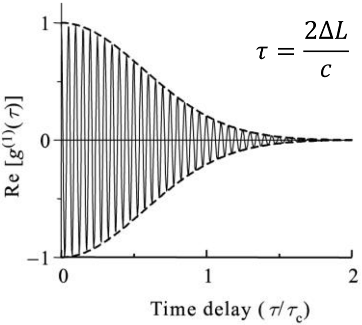
If the amplitudes in the two arms are different:

$$E_{out}(t) = E_1(t) + E_2(t+\tau)$$

$$\mathrm{I}(\tau) = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathcal{R}e\left(g_{12}^{(1)}(\tau)\right)$$

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\left|g_{12}^{(1)}(\tau)\right|$$

$$\tau = \frac{2\Delta L}{c}$$



Temporal coherence of light

Perfectly **monochromatic** light

- $\Delta\omega = 0$
- Perfect temporal coherence
- Coherence time: $\tau_c = \infty$
- $|g^{(1)}(\tau)| = 1$

Chaotic light

- $\Delta\omega \neq 0$ (finite bandwidth)
- Partial temporal coherence
- Coherence time: $\tau_c \propto \frac{1}{\Delta\omega}$
- $0 < |g^{(1)}(\tau)| < 1$

Incoherent light

- $\Delta\omega = \infty$
- No temporal coherence
- Coherence time: $\tau_c = 0$
- $|g^{(1)}(\tau)| = 0$

Temporal coherence of light

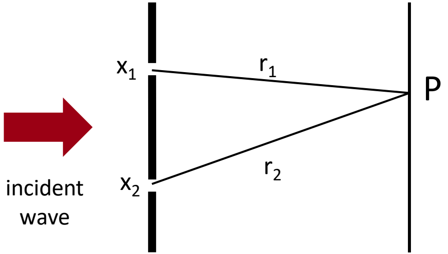
Source	Wavelength (nm)	Frequency (10^{14} Hz)	Bandwidth ($\Delta\nu$)	Coherence time, τ_c (s)	Coherence length, L_c
white light	400-600	5-7.5	2.5×10^{14} Hz	4×10^{-15}	1.2 μm (a few λ)
Hg lamp with single isotope	546.1	5.49	300 MHz	3×10^{-9}	1 m
He-Ne laser	632.8	4.74	1 GHz	10^{-9}	0.3 m
Stabilized He-Ne laser	632.8	4.74	10 kHz	10^{-4}	30 km

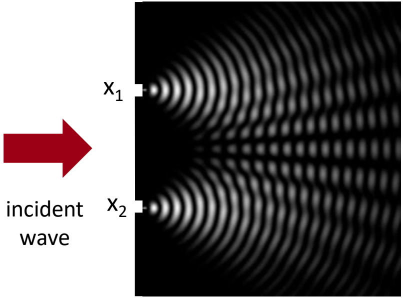
First-order cross-correlation function

Degree of first-order **transverse (spatial)** coherence

$$g_{12}^{(1)}(\tau) \equiv g_{12}^{(1)}(x_1, x_2, \tau) = \frac{\langle E(x_1,t)^* E(x_2, t+\tau)\rangle}{\sqrt{\langle |E(x_1,t)|^2\rangle \langle |E(x_2,t)|^2\rangle}}$$

$$\langle E(x_1,t)^* E(x_2,t+\tau)\rangle = \frac{1}{T}\int_T E(x_1,t)^* E(x_2,t+\tau)dt$$



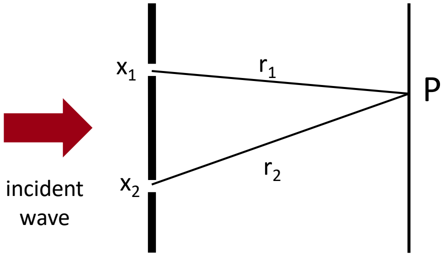


First-order cross-correlation function

Degree of first-order **transverse (spatial)** coherence

$$g_{12}^{(1)}(\tau) \equiv g_{12}^{(1)}(x_1, x_2, \tau) = \frac{\langle E(x_1, t)^* E(x_2, t+\tau)\rangle}{\sqrt{\langle |E(x_1, t)|^2\rangle \langle |E(x_2, t)|^2\rangle}}$$

$$\langle E(x_1, t)^* E(x_2, t+\tau)\rangle = \frac{1}{T}\int_T E(x_1, t)^* E(x_2, t+\tau) dt$$



$\left|g_{12}^{(1)}(\tau = 0)\right| = 1$ **Full coherence**

$0 < \left|g_{12}^{(1)}(\tau = 0)\right| < 1$ **Partial coherence**

$\left|g_{12}^{(1)}(\tau = 0)\right| = 0$ **Incoherence**

between the wavetrains passing in x_1 and x_2

Second-order correlation function

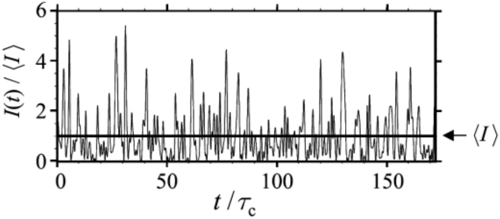
Degree of second-order (**temporal**) coherence

$$g^{(2)}(\tau) = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle} = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}$$

Let's assume a source with constant average intensity: $\langle I(t)\rangle = \langle I(t+\tau)\rangle = \langle I\rangle$

$I(t) = \langle I\rangle + \Delta I(t)$ with $\langle \Delta I(t)\rangle = 0$

$\boxed{\tau \gg \tau_c}$



$\langle I(t)I(t+\tau)\rangle =$

$= \langle(\langle I\rangle + \Delta I(t))(\langle I\rangle + \Delta I(t+\tau))\rangle =$

$$= \langle I\rangle^2 + \langle I\rangle\underbrace{\cancel{\langle \Delta I(t)\rangle}}_{=0} + \langle I\rangle\underbrace{\cancel{\langle \Delta I(t+\tau)\rangle}}_{=0} + \underbrace{\cancel{\langle \Delta I(t)\Delta I(t+\tau)\rangle}}_{=0\ \ \tau\gg\tau_c} = \langle I\rangle^2$$

Degree of second-order (**temporal**) coherence

$$\boxed{g^{(2)}(\tau) = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle} = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}}$$

Let's assume a source with constant average intensity: $\langle I(t)\rangle = \langle I(t+\tau)\rangle = \langle I\rangle$

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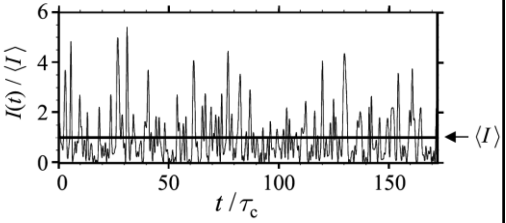
$$\boxed{\tau \gg \tau_c}$$



$$\langle I(t)I(t+\tau)\rangle_{\tau\gg\tau_c} = \langle I\rangle^2$$

$$g^{(2)}(\tau \gg \tau_c) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = 1$$

Second-order correlation function

Degree of second-order (**temporal**) coherence

$$g^{(2)}(\tau) = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle} = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}$$

Let's assume a source with constant average intensity: $\langle I(t)\rangle = \langle I(t+\tau)\rangle = \langle I\rangle$

$I(t) = \langle I\rangle + \Delta I(t)$ with $\langle \Delta I(t)\rangle = 0$

$\boxed{\tau \gg \tau_c}$

$$\langle I(t)I(t+\tau)\rangle_{\tau\gg\tau_c} = \langle I\rangle^2$$

$$g^{(2)}(\tau \gg \tau_c) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = 1$$

$\boxed{\tau \ll \tau_c}$

The intensity fluctuations at time t and $t+\tau$ might be correlated

$\rightarrow$ $\tau = 0$

$$g^{(2)}(\tau = 0) = \frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2} \geq 1$$

Second-order correlation function

Degree of second-order (**temporal**) coherence

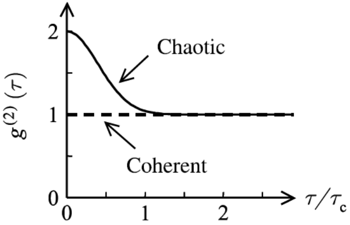
$\forall\ I(t)$ $$\boxed{g^{(2)}(0) \geq 1 \quad \text{and} \quad g^{(2)}(0) \geq g^{(2)}(\tau)}$$

Perfectly coherent light: $\boxed{g^{(2)}(\tau) = 1}$ $\forall\, \tau$

Partially coherent (**chaotic**) light:

Gaussian lineshape: $g^{(2)}(\tau) = 1 + e^{-\pi(\tau/\tau_c)^2}$

Lorentzian lineshape: $g^{(2)}(\tau) = 1 + e^{-2|\tau|/\tau_c}$



$$\boxed{g^{(2)}(\tau) = 1 + \left|g^{(1)}(\tau)\right|^2}$$

For every **classical** source of light!

Second-order correlation function

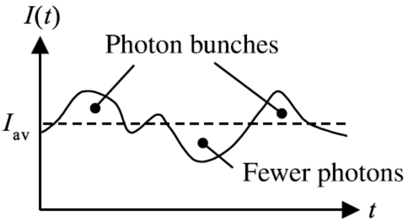
COHERENT light: $g^{(2)}(0)=1$ Perfectly coherent light

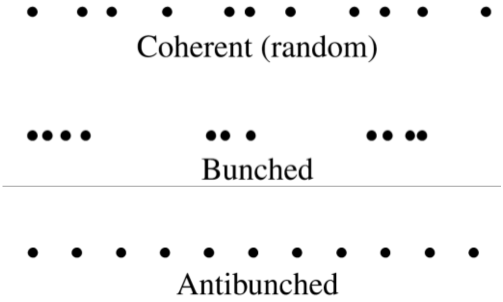
BUNCHED light: $g^{(2)}(0)>1$ Partially coherent light

$\Big\}$ **classical**

ANTIBUNCHED light: $g^{(2)}(0)<1$ **Quantum** nature of light!

Bunched light



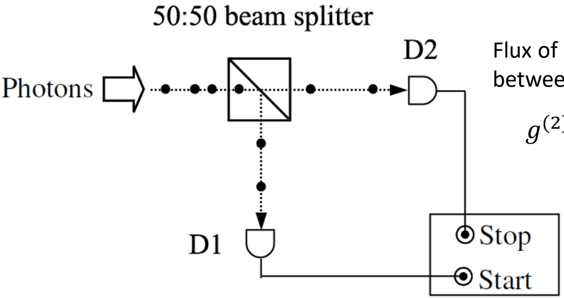


Second-order correlation function

...in terms of photons: $$g^{(2)}(\tau) = \frac{\langle n_1(t) n_2(t+\tau)\rangle}{\sqrt{\langle n_1(t)\rangle\langle n_2(t+\tau)\rangle}}$$

$g^{(2)}(\tau)$ is proportional to the **conditional probability** of detecting a second photon at time $t=\tau$, given that we detected one at $t=0$.

Hanbury-Brown & Twiss (HBT) configuration



Antibunched light

Flux of photons with long temporal intervals between two consecutive photons

$g^{(2)}(0) \cong 0 \qquad g^{(2)}(0) \leq g^{(2)}(\tau)$

in contrast with the results for classical light:

$g^{(2)}(0) \geq 1 \qquad g^{(2)}(0) \geq g^{(2)}(\tau)$

Second-order correlation function

...in terms of photons: $$g^{(2)}(\tau) = \frac{\langle n_1(t) n_2(t+\tau)\rangle}{\sqrt{\langle n_1(t)\rangle\langle n_2(t+\tau)\rangle}}$$

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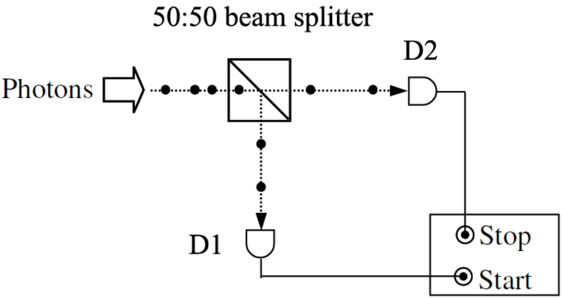
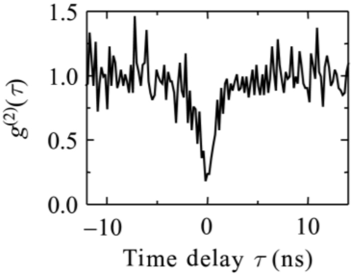


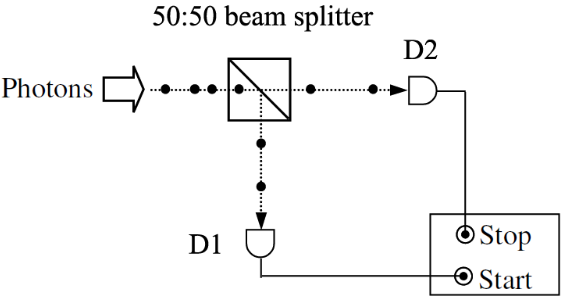
Antibunched light



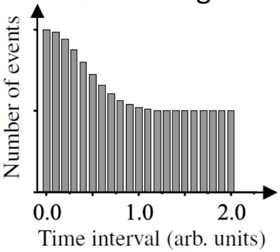
…in terms of photons:
$$g^{(2)}(\tau) = \frac{\langle n_1(t) n_2(t+\tau)\rangle}{\sqrt{\langle n_1(t)\rangle\langle n_2(t+\tau)\rangle}}$$

$g^{(2)}(\tau)$ is proportional to the **conditional probability** of detecting a second photon at time $t = \tau$, given that we detected one at $t = 0$.

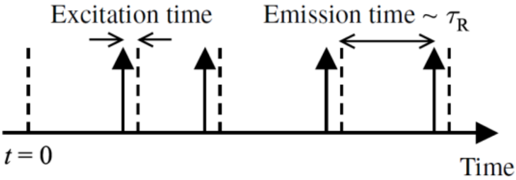
Hanbury-Brown & Twiss (HBT) configuration



Bunched light



Photon antibunching

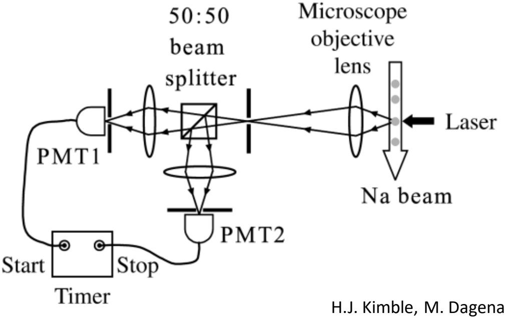


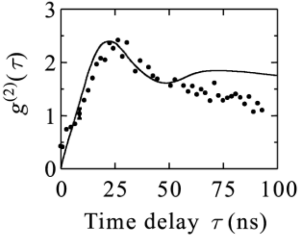
Single atom:

the probability of emission of two photons with temporal separation $\tau \ll \tau_R$ is very low

$$\Longrightarrow \quad g^{(2)}(0) \cong 0$$

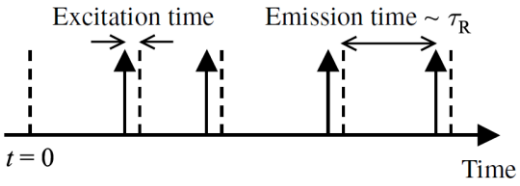
Hanbury-Brown & Twiss (HBT) configuration





H.J. Kimble, M. Dagenais and L. Mandel, *Phys. Rev. Lett.*, **39**, 691 (1977)

Photon antibunching



Single atom:

the probability of emission of two photons with temporal separation $\tau \ll \tau_R$ is very low

$$\Longrightarrow \quad g^{(2)}(0) \cong 0$$

Triggered **Single-Photon Sources**