The general case: multipolar expansion

$$\begin{cases} \bar{\nabla}\cdot(\varepsilon\bar{E}_c)=0 \\ \bar{\nabla}\times\bar{E}_c=i\omega\mu\bar{H}_c \\ \nabla\cdot\mu\bar{H}_c=0 \\ \bar{\nabla}\times\bar{H}_c=-i\omega\varepsilon\bar{E}_c \end{cases}$$

$$\begin{cases} \nabla^2\bar{E}_c+k^2\bar{E}_c=0 \\ \nabla^2\bar{H}_c+k^2\bar{H}_c=0 \end{cases}$$ Helmholtz

$$k^2=\omega^2\varepsilon\mu$$

The scalar function Ψ and two vectorial functions $\bar{M}$ ed $\bar{N}$:

$$\begin{cases} \bar{M}\equiv\bar{\nabla}\times(\bar{r}\Psi) \\ \bar{N}\equiv\dfrac{1}{k}\bar{\nabla}\times\bar{M} \end{cases}$$

$$\bar{\nabla}\cdot\bar{M}\equiv 0$$ Divergence of a Curl

$$\nabla^2\bar{M}+k^2\bar{M}=\bar{\nabla}\times[\bar{r}(\nabla^2\Psi+k^2\Psi)]$$

If: $$\nabla^2\Psi+k^2\Psi=0$$

$$\begin{cases} \nabla^2\bar{M}+k^2\bar{M}=0 \\ \nabla^2\bar{N}+k^2\bar{N}=0 \end{cases}$$

$$\begin{cases} \Psi_{o,l,m}(r,\theta,\varphi)=j_l(kr)\sin(m\varphi)P_l^m(\cos\theta) \\ \Psi_{e,l,m}(r,\theta,\varphi)=j_l(kr)\cos(m\varphi)P_l^m(\cos\theta) \end{cases}$$

$j_l(kr)$ Spherical Bessel Func.

$P_l^m(\cos\theta)$ Associated Legendre Func.

The general case: multipolar expansion

$Y_l^m(\theta,\varphi)$

Spherical Harmonics

$$Y_\ell^m(\theta,\varphi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y_\ell^m\, Y_{\ell'}^{m'\,*}\, d\Omega = \delta_{\ell\ell'}\,\delta_{mm'}$$

$$Y_{\ell m} = \begin{cases} (-1)^m\sqrt{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, P_\ell^{|m|}(\cos\theta)\;\sin(|m|\varphi) & \text{if } m<0 \\ \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell^m(\cos\theta) & \text{if } m=0 \\ (-1)^m\sqrt{2}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\;\cos(m\varphi) & \text{if } m>0 \end{cases}$$

The general case: multipolar expansion

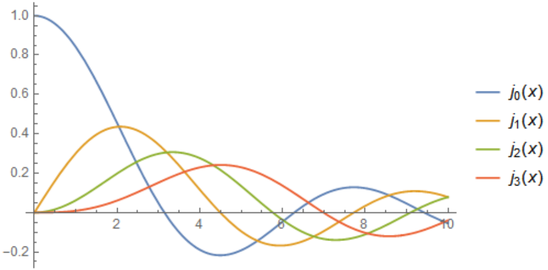
$j_n(x)$

Spherical Bessel Functions

$$j_0(x) = \frac{\sin(x)}{x}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2}$$



$$j_n(x) = Re\left[h_n^{(1)}(x)\right]$$

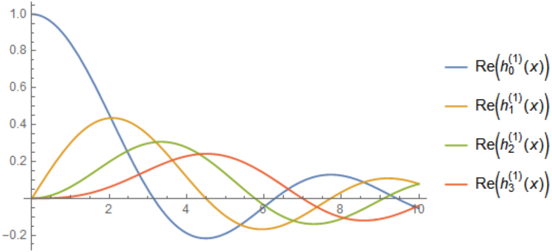
$h_n^{(1)}(x)$

Spherical Hankel functions

The general case: multipolar expansion

$$h_n^{(1)}(x)$$

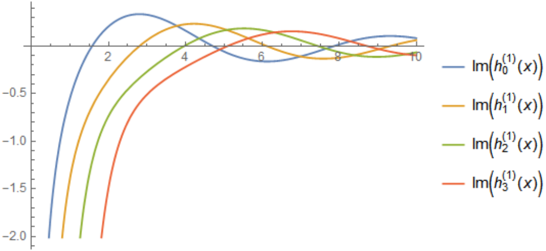
Spherical Hankel functions



$$h_0^{(1)}(x) = -i\, e^{ix}\,\frac{1}{x}$$

$$h_1^{(1)}(x) = -e^{ix}\,\frac{x+i}{x^2}$$

$$h_2^{(1)}(x) = i e^{ix}\,\frac{x^2+3ix-3}{x^3}$$



The general case: multipolar expansion

1. Multipolar expansion of the plane wave

$$\bar{E}_i = E_0 e^{ikz} e^{-i\omega t} \hat{u}_x = E_0 e^{-i\omega t} \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} \left(\bar{M}_{ol1} - i\bar{N}_{el1}\right)$$

2. Multipolar expansion of the scattered and transmitted fields:

$$\bar{E}_s = E_0 e^{-i\omega t} \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} \left(i a_l \bar{N}_{el1} - b_l \bar{M}_{ol1}\right)$$

$$\bar{E}_t = E_0 e^{-i\omega t} \sum_{l=1}^{\infty} i^l \frac{2l+1}{l(l+1)} \left(c_l \bar{M}_{ol1} - i d_l \bar{N}_{el1}\right)$$

$a_l\,,\, b_l\,,\, c_l\,,\, d_l \in \mathbb{C}$

3. Continuity at R of the tangential components of the fields:

$$\hat{u}_r \times \left(\vec{E}_i + \vec{E}_s\right) = \hat{u}_r \times \vec{E}_t$$

$$\hat{u}_r \times \left(\vec{H}_i + \vec{H}_s\right) = \hat{u}_r \times \vec{H}_t$$

The general case: multipolar expansion

$a_l\,, b_l\,, c_l\,, d_l \in \mathbb{C}$

$x = kR$ *Size parameter*

$m = \dfrac{n}{n_m}$ *Relative refractive index*

μ, μ_1 *Magnetic permeability*

$j_n(x)$ *Spherical Bessel functions*

$h_n^{(1)}(x)$ *Spherical Hankel functions*

$$a_n = \frac{\mu m^2 j_n(mx)[xj_n(x)]' - \mu_1 j_n(x)[mxj_n(mx)]'}{\mu m^2 j_n(mx)[xh_n^{(1)}(x)]' - \mu_1 h_n^{(1)}(x)[mxj_n(mx)]'}$$

$$b_n = \frac{\mu_1 j_n(mx)[xj_n(x)]' - \mu j_n(x)[mxj_n(mx)]'}{\mu_1 j_n(mx)[xh_n^{(1)}(x)]' - \mu h_n^{(1)}(x)[mxj_n(mx)]'},$$

$$c_n = \frac{\mu_1 j_n(x)[xh_n^{(1)}(x)]' - \mu_1 h_n^{(1)}(x)[xj_n(x)]'}{\mu_1 j_n(mx)[xh_n^{(1)}(x)]' - \mu h_n^{(1)}(x)[mxj_n(mx)]'},$$

$$d_n = \frac{\mu_1 m j_n(x)[xh_n^{(1)}(x)]' - \mu_1 m h_n^{(1)}(x)[xj_n(x)]'}{\mu m^2 j_n(mx)[xh_n^{(1)}(x)]' - \mu_1 h_n^{(1)}(x)[mxj_n(mx)]'}$$

The general case: multipolar expansion

$a_l\,,b_l\,,c_l\,,d_l \in \mathbb{C}$

$x = kR$ *Size parameter*

$m = \dfrac{n}{n_m}$ *Relative refractive index*

Defining the Riccati-Bessel functions $\psi_n(\rho) = \rho j_n(\rho), \quad \xi_n(\rho) = \rho h_n^{(1)}(\rho)$

Considering only the case: $\mu_1 = \mu$

It results:

$$a_n = \frac{m\psi_n(mx)\psi_n'(x) - \psi_n(x)\psi_n'(mx)}{m\psi_n(mx)\xi_n'(x) - \xi_n(x)\psi_n'(mx)}$$

$$b_n = \frac{\psi_n(mx)\psi_n'(x) - m\psi_n(x)\psi_n'(mx)}{\psi_n(mx)\xi_n'(x) - m\xi_n(x)\psi_n'(mx)}$$

$$a_1 = -i\frac{2}{3}(kR)^3\frac{\varepsilon-\varepsilon_m}{\varepsilon+2\varepsilon_m}$$

$$b_1 = -i\frac{1}{45}(kR)^5\frac{\varepsilon-\varepsilon_m}{\varepsilon_m}$$

1. Cross-sections

$$\left\{\begin{aligned} \sigma_{sca} &= \frac{2\pi}{k^2}\sum_{l=1}^{\infty}(2l+1)\left(|a_l|^2+|b_l|^2\right) \\ \sigma_{ext} &= \frac{2\pi}{k^2}\sum_{l=1}^{\infty}(2l+1)\,\Re e(a_l+b_l) \\ \sigma_{abs} &= \sigma_{ext}-\sigma_{sca} \end{aligned}\right.$$

$$k^2 \equiv \omega^2\varepsilon\mu = \left(\frac{\omega}{c}\right)^2 n^2 = \left(\frac{2\pi}{\lambda_0}\right)^2 n^2$$

l = multipolarity
l = 1 dipole
l = 2 quadrupole
l = 3 octupole
…

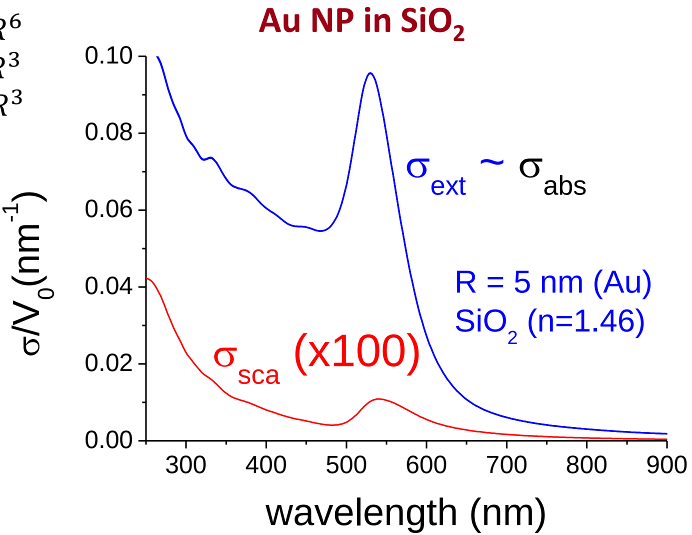
$$a_l \approx (kR)^{2l+1}$$

$$b_l \approx (kR)^{2l+3}$$

$$\sigma_{ext} \approx (kR)^{2l+1} \approx \sigma_{abs}$$

$$\sigma_{sca} \approx (kR)^{2(2l+1)}$$

$\sigma_{sca} \sim R^6$
$\sigma_{ext} \sim R^3$
$\sigma_{abs} \sim R^3$



Mie theory

$\sigma_{sca} \sim R^6$

$\sigma_{ext} \sim R^3$

$\sigma_{abs} \sim R^3$