Metamaterials

Negative Index Materials

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THE ELECTRODYNAMICS OF SUBSTANCES WITH SIMULTANEOUSLY NEGATIVE VALUES OF ε AND μ

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Usp. Fiz. Nauk 92, 517–526 (July, 1964)

Victor Veselago
(Ucraina, Russia 1929)

$$k^2 = \frac{\omega^2}{c^2} n^2 = k_0^2 n^2 \qquad \begin{cases} n^2 = \varepsilon \\ \mu = 1 \end{cases}$$ non-magnetic systems

$$n^2 = \varepsilon\mu$$

For non dissipative materials $\varepsilon, \mu \in \mathbb{R}$

- $\varepsilon > 0\,,\ \mu > 0$
- $\varepsilon < 0\,,\ \mu < 0$

Three possibilities:

1. Properties unaffected by simultaneous change of sign
2. $\varepsilon < 0\,,\ \mu < 0$ may contradict some natural law
3. Substances with $\varepsilon < 0\,,\ \mu < 0$ may exist but with different properties

negative μ and ε occur in nature, but not simultaneously: metamaterials

Maxwell's Equations (ME)

Macroscopic ME

$$\nabla \cdot \mathbf{D} = \rho_{\text{ext}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{ext}} + \frac{\partial \mathbf{D}}{\partial t}$$

Constitutive Eqs.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mu \mathbf{H}$$

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

«... should look for such substances. Since in our opinion the electrodynamics of substances with $\epsilon < 0$ and $\mu < 0$ is undoubtedly of interest, independently of our now having such substances available, we shall at first consider the matter purely formally. There- ...»*

M.E. vs Rightness

$$\begin{cases} \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{H} = \dfrac{\partial \boldsymbol{D}}{\partial t} \\ \boldsymbol{B} = \mu_0 \mu \, \boldsymbol{H} \\ \boldsymbol{D} = \varepsilon_0 \varepsilon \, \boldsymbol{E} \end{cases}$$

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} e^{-i\omega t}$$

$$\begin{cases} \nabla \leftrightarrow i\boldsymbol{k} \\ \dfrac{\partial}{\partial t} \leftrightarrow -i\omega \end{cases}$$

$$\begin{cases} \boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \mu \, \boldsymbol{H} \\ \boldsymbol{k} \times \boldsymbol{H} = -\omega \varepsilon_0 \varepsilon \, \boldsymbol{E} \end{cases}$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

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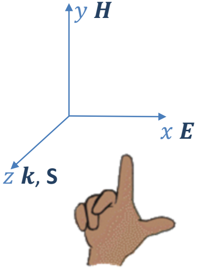
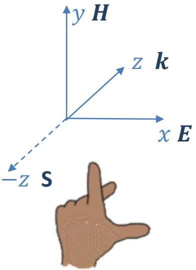


right-hand

$$\begin{cases} \boldsymbol{k} \times \boldsymbol{E} = \omega\mu_0\mu\, \boldsymbol{H} \\ \boldsymbol{k} \times \boldsymbol{H} = -\omega\varepsilon_0\varepsilon\, \boldsymbol{E} \end{cases}$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$



left-hand

$\varepsilon > 0\,, \quad \mu > 0$	$\varepsilon < 0\,, \quad \mu < 0$
E, H, k right-handed triplet	***E, H, k*** left-handed triplet
E, H, S right-handed triplet	***E, H, S*** right-handed triplet
$\boldsymbol{S} \cdot \boldsymbol{k} > 0$	$\boldsymbol{S} \cdot \boldsymbol{k} < 0$

Phase velocity

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}}$$

Group velocity (negative for LH)

$$\boldsymbol{v}_g = \bar{\nabla}_{\boldsymbol{k}}\omega$$

Defining the *direction cosines* for $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{k}$: $\alpha_j, \beta_j, \gamma_j$

$$\boldsymbol{G} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

$$A_{i,j} = cof_{i,j}(G) = (-1)^{\,i+j}\det(\tilde{G}_{i,j})$$

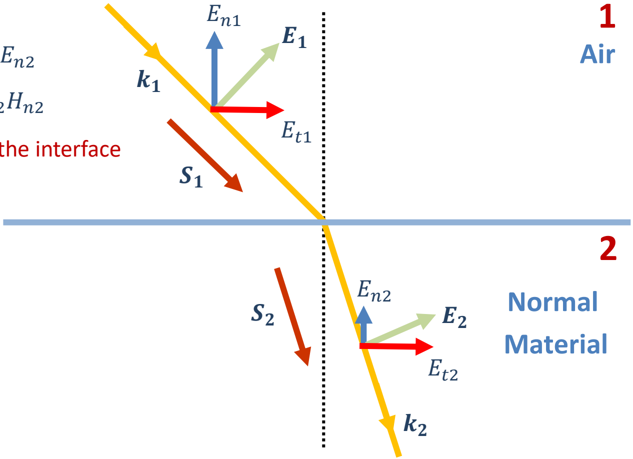
$$G_{i,j} = p\ A_{i,j}$$

$$\det(\boldsymbol{G}) = p = \begin{cases} 1\ RH \\ -1\ LH \end{cases}$$

Refraction (Snell law)

$$\begin{cases} E_{t1} = E_{t2} \\ H_{t1} = H_{t2} \\ \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2} \\ \mu_1 H_{n1} = \mu_2 H_{n2} \end{cases}$$

Continuity at the interface



Refraction (Snell law)

$$E_{t1} = E_{t2}$$

$$H_{t1} = H_{t2}$$

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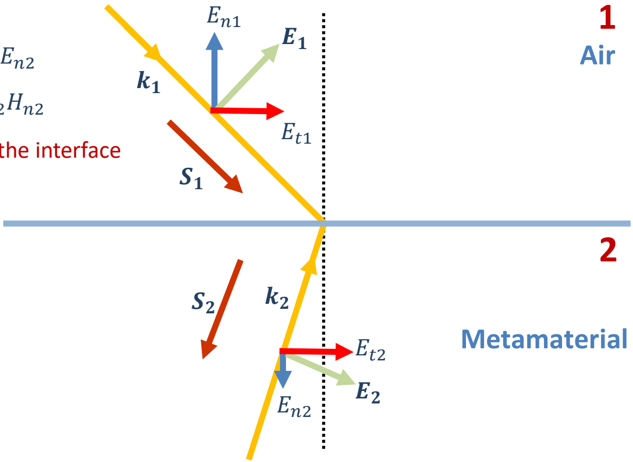
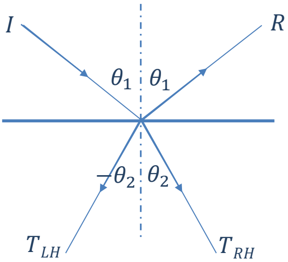
Continuity at the interface



Refraction (Generalized Snell law)

$$\begin{cases} E_{t1} = E_{t2} \\ H_{t1} = H_{t2} \\ \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2} \\ \mu_1 H_{n1} = \mu_2 H_{n2} \end{cases}$$

Continuity at the interface



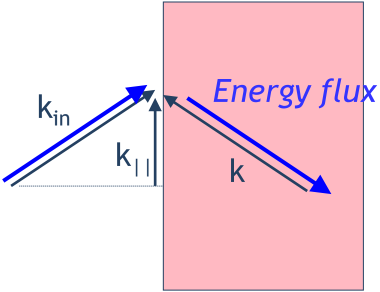
I incident ray
R reflected ray
T_{RH} refracted for RH
T_{LH} refracted for LH

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} = \frac{p_2}{p_1}\left|\sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}}\right|$$

$$n_{RH} = \sqrt{\mu\varepsilon} = (ab)^{1/2}$$

$$n_{LH} = \sqrt{\mu\varepsilon} = \left((ae^{i\pi})(be^{i\pi})\right)^{1/2} = \sqrt{ab}\ e^{i\pi} = -n_{RH}$$

Refraction (Snell law)



$$\sin\theta_{in} = n\sin\theta_{out}$$

Negative refraction

Negative refraction:

(1) $k_{||}$ is conserved

(2) Causality:
carry energy away from the interface

Phase fronts (***k***) travel opposite to energy if *n<0*

Perfect lensing

RH-material LH-material RH-material

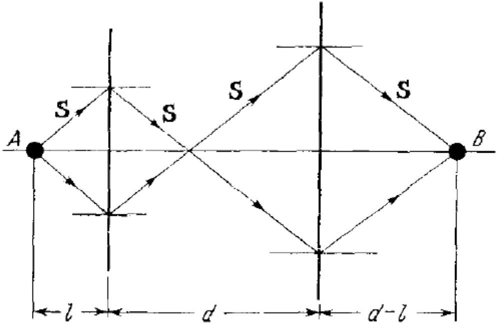


FIG. 4. Passage of rays of light through a plate of thickness d made of a left-handed substance. A – source of radiation; B – detector of radiation.

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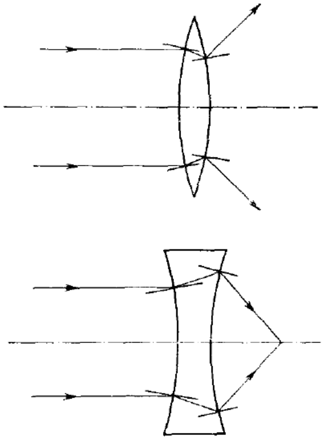

FIG. 5. Paths of rays through lenses made of left-handed substances, situated in vacuum.

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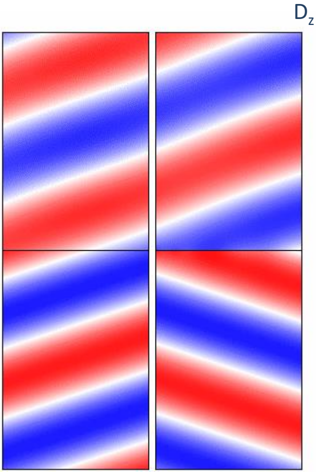
Refraction (Snell law)

Negative refraction

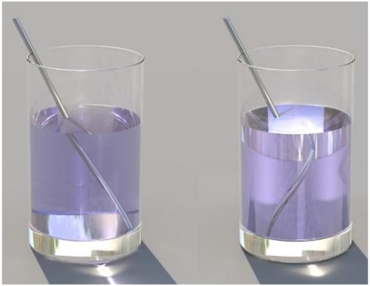
Air
$\varepsilon = 1.0$

RH material
$\varepsilon = 1.2$
$\mu = 1$



Air
$\varepsilon = 1.0$

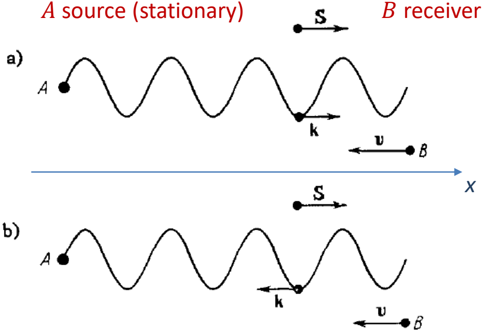
LH material
$\varepsilon = -1.2$
$\mu = -1$



Phase fronts (***k***) travel opposite to energy if *n<0*

Reversed Doppler Effect

A source (stationary) *B* receiver



$$\boldsymbol{v}_f = \frac{\omega}{k}\hat{\boldsymbol{k}}$$

RH Material ($p = +1$)

LH Material ($p = -1$)

$$\omega = \omega_0 \left(\frac{u_S - p v_B}{u_S + v_A} \right)$$

$$\omega = \omega_0 \left(1 - p \frac{v_B}{u_S} \right)$$

v_A = 0

v_B = velocity of B (< 0 if B is moving toward A)

u_S= velocity of the energy flux (e.g., c) > 0

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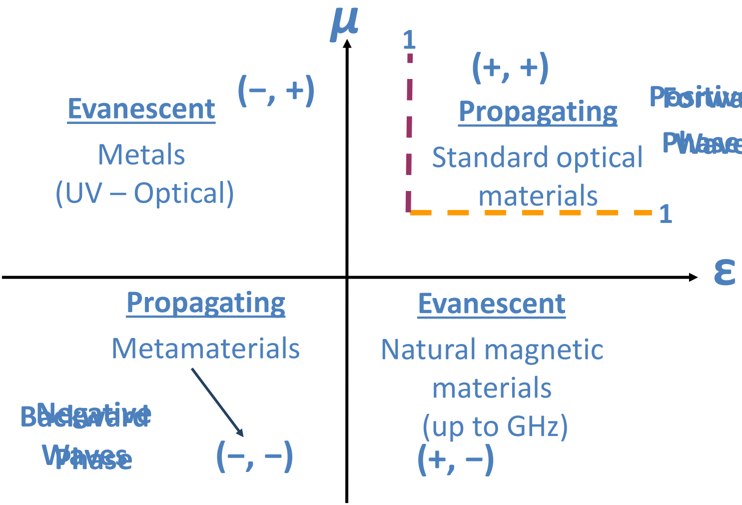
Materials Classification

$$n = \pm\sqrt{\varepsilon\mu} \quad \textbf{OR} \quad n = \pm\sqrt{(-\varepsilon)(-\mu)}$$



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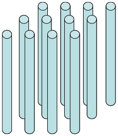
μ (vertical axis), ε (horizontal axis)



ABSORPTION

(ENG) ε < 0, μ > 0

Plasma

POSITIVE REFRACTION

(DPS) Dielectrics

(DNG) ε < 0, μ < 0

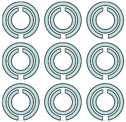
Not found in nature

NEGATIVE REFRACTION

(MNG) ε > 0, μ < 0

Gyrotropic

ABSORPTION



Metamaterial challenges

$\varepsilon < 0$ *easy (any metal)*
$\mu < 0$ *metamaterials*:

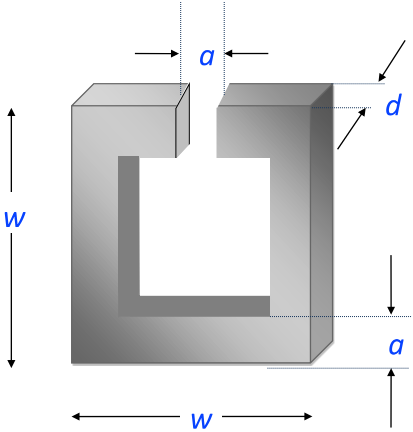
- ***$\lambda/10$* sized artificial units** with a magnetic response
- **nonmagnetic materials**

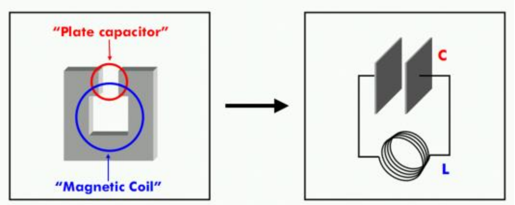
Obtained as:

- **Localized currents** induced by incident radiation to *circulate in loops*
- **Resonances** to get the strongest magnetic response

Resonant Metamaterials

Split ring resonators (SRR)





Split ring resonators (SRR) have a resonance at

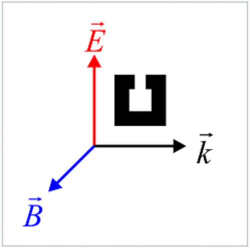
$$C = \varepsilon_0 \varepsilon \frac{S}{a} = \varepsilon_0 \varepsilon \frac{ad}{a} = \varepsilon_0 \varepsilon\, d$$

$$L = \mu_0 \frac{w^2}{d}$$

$$\omega_{LC} = \frac{1}{\sqrt{L\,C}} = \frac{c}{\sqrt{\varepsilon}\;w} \propto \frac{1}{size}$$

Faraday: flux change sets up a voltage over a loop

$$V = -\frac{d\Phi(B)}{dt} = -\oint \frac{dB}{dt} dA = i\omega\mu_0 H e^{-i\omega t} A_{loop}$$



Ohm's law: current depending on impedance

$$I = \frac{V}{Z} \qquad Z = i\omega L + \frac{1}{i\omega C} + R$$

Resonance when $|Z|$ is minimum (or 0)

Circulating current *I* has a magnetic dipole moment

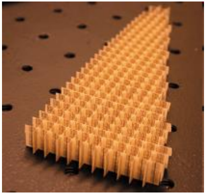
$$m = I A_{loop}$$ (pointing out of the loop)

$$\lambda_{LC} = \frac{2\pi c}{\omega_{LC}} = 2\pi w \sqrt{\varepsilon}$$

$\varepsilon = 1$
$w = 100$ nm
$\lambda_{LC} = 600$ nm

Negative refraction proof (microwaves)

Fig. 1. Photograph of the left-handed metamaterial (LHM) sample. The LHM sample consists of square copper split ring resonators and copper wire strips on fiber glass circuit board material. The rings and wires are on opposite sides of the boards, and the boards have been cut and assembled into an interlocking lattice.



$$\frac{\mu(\omega)}{\mu_0} = 1 - \frac{\omega_{mp}^2 - \omega_{mo}^2}{\omega^2 - \omega_{mo}^2 + i\gamma\omega}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_{ep}^2 - \omega_{eo}^2}{\omega^2 - \omega_{eo}^2 + i\gamma\omega}$$

$$\omega_{mp} = 2\pi \times 10.95\ GHz$$
$$\omega_{mo} = 2\pi \times 10.05\ GHz$$
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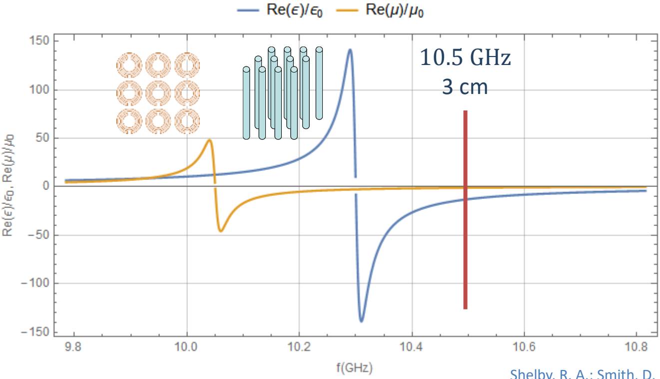
$$\omega_{eo} = 2\pi \times 10.30\ GHz$$
$$\gamma = 2\pi \times 0.02\ GHz$$



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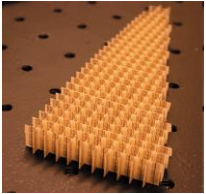

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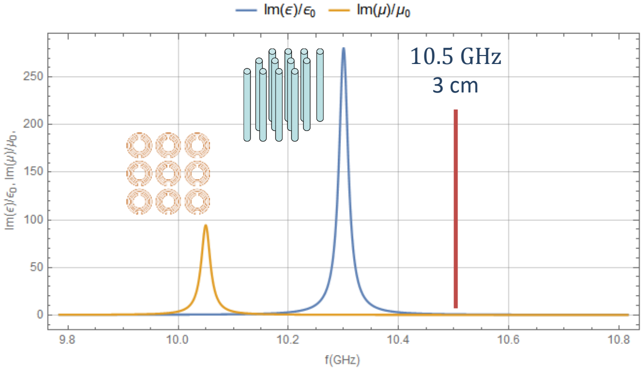

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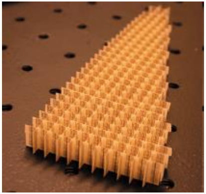

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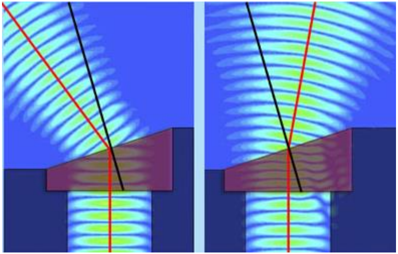
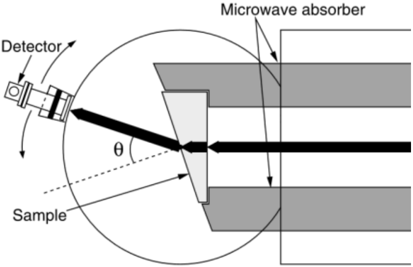
LHM RHM





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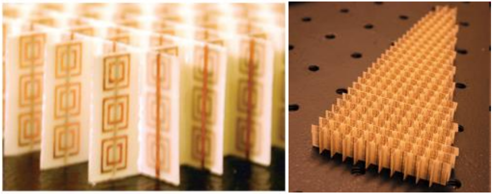

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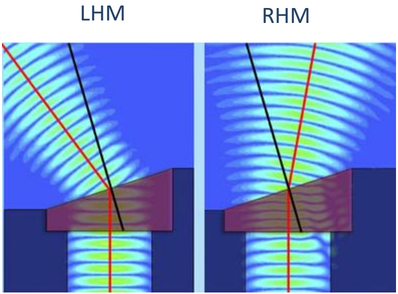

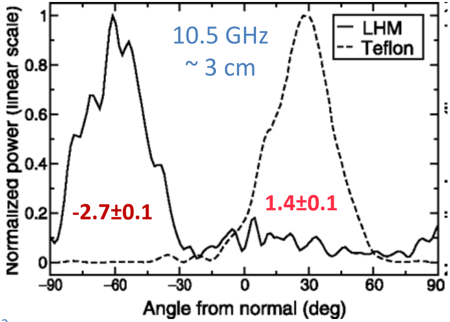

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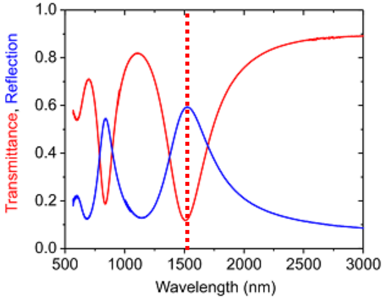
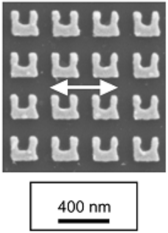
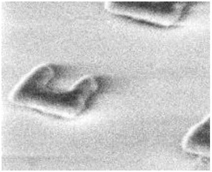
From microwaves to visible



Karlsruhe (2005)



AMOLF (2008)



200 nm sized SRR's,
Gold on glass
λ=1500 nm

Can we make smaller split rings for λ ~ 500 nm wavelength?

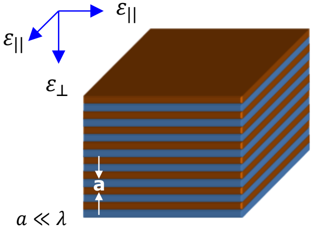
No: at visible ω metals have a plasmonic response

Metamaterials with periodic nanostructures

Multilayered metal-dielectric metamaterial

- Two isotropic layers with bulk permittivities ε_1 and ε_2
- Filling fractions f for ε_1, $1-f$ for ε_2
- 2 ordinary and one extra-ordinary axes (uniaxial)
- 2 effective permittivities (with respect to the anisotropy axis) $\varepsilon_\perp$ and $\varepsilon_{||}$

Note: parallel=ordinary



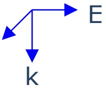
- For isotropic layers $D_i = \varepsilon_i E_i$
- effective fields

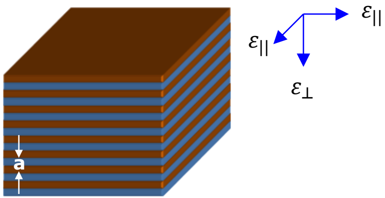
$$\begin{cases} E_{eff} = E_{ave} = fE_1 + (1-f)E_2 \\ D_{eff} = D_{ave} = fD_1 + (1-f)D_2 \end{cases}$$

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_\perp \end{pmatrix}$$

Multilayered metal-dielectric metamaterial

Parallel polarization





$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$

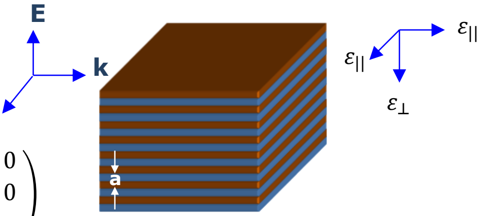
Boundary conditions $\quad E_1 = E_2 \equiv E$

$$\begin{cases} E_{eff} = E_{ave} = fE + (1-f)E = E \\ D_{eff} = D_{ave} = fD_1 + (1-f)D_2 = f\varepsilon_1 E + (1-f)\varepsilon_2 E = \varepsilon_{eff} E \end{cases}$$

$$\varepsilon_{||} = f\varepsilon_1 + (1-f)\varepsilon_2$$

Multilayered metal-dielectric metamaterial

Normal polarization



$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$

$D_1 = D_2 \equiv D$ Boundary conditions

$$\begin{cases} E_{eff} = E_{ave} = fE_1 + (1-f)E_2 \\ D_{eff} = D_{ave} = fD + (1-f)D = D \end{cases}$$

$$E_{eff} = f\frac{D}{\varepsilon_1} + (1-f)\frac{D}{\varepsilon_2} = \frac{D}{\varepsilon_{eff}}$$

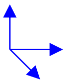
$$\frac{1}{\varepsilon_{\perp}} = \frac{f}{\varepsilon_1} + \frac{(1-f)}{\varepsilon_2}$$

Nanowire metal-dielectric metamaterial

- Two isotropic constituents with bulk permittivities ε_1, ε_2
- Filling fractions f for ε_1, 1-f for ε_2
- 2 ordinary and one extra-ordinary axes (uniaxial)
- 2 effective permittivities, $\varepsilon_\perp$ and $\varepsilon_{||}$

$a << \lambda$

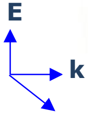
Note: parallel=extraordinary

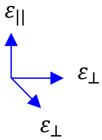


$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{||} \end{pmatrix}$$

Nanowire metal-dielectric metamaterial

Parallel polarization





$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_{\perp} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{||} \end{pmatrix}$$

$$E_1 = E_2 \equiv E$$ Boundary conditions

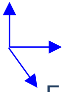
$$\begin{cases} E_{eff} = E_{ave} = fE + (1-f)E = E \\ D_{eff} = D_{ave} = fD_1 + (1-f)D_2 = f\varepsilon_1 E + (1-f)\varepsilon_2 E = \varepsilon_{eff} E \end{cases}$$

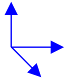
$$\varepsilon_{||} = f\varepsilon_1 + (1-f)\varepsilon_2$$

Nanowire metal-dielectric metamaterial

Normal polarization

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} \varepsilon_{\perp} & 0 & 0 \\ 0 & \varepsilon_{\perp} & 0 \\ 0 & 0 & \varepsilon_{||} \end{pmatrix}$$





- More complicated derivation
  - Homogenization (not simple averaging)
  - Assume small inclusions (<20%)
  - Maxwell-Garnett Theory (MGT)

(metal nanowires in dielectric host)

$$\varepsilon_{\perp}(\varepsilon_{xx} = \varepsilon_{yy}) = \varepsilon_d \frac{(1+f)\varepsilon_m + (1-f)\varepsilon_d}{(1-f)\varepsilon_m + (1+f)\varepsilon_d}$$

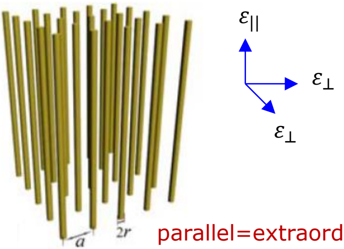
Shekhar, P., Atkinson, J. & Jacob, Z. Hyperbolic metamaterials: fundamentals and applications. *Nano Convergence* **1**, 14 (2014).

Strongly anisotropic dielectric Metamaterial

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{||} \end{pmatrix}$$

$$\varepsilon_{||} = f\varepsilon_m + (1-f)\varepsilon_d$$

$$\varepsilon_\perp(\varepsilon_{xx} = \varepsilon_{yy}) = \varepsilon_d \frac{(1+f)\varepsilon_m + (1-f)}{(1-f)\varepsilon_m + (1+f)}$$

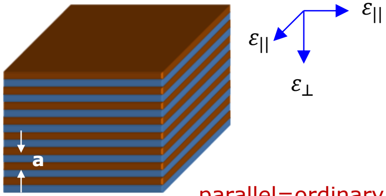


parallel=extraordinary

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_\perp \end{pmatrix}$$

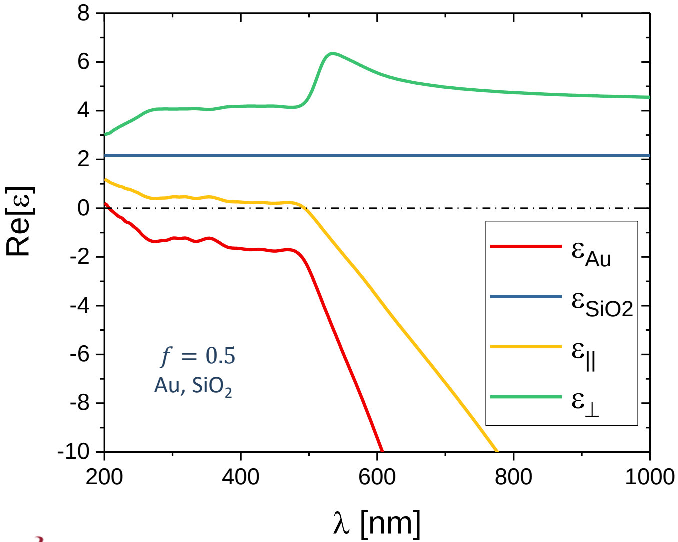
$$\varepsilon_{||} = f\varepsilon_m + (1-f)\varepsilon_d$$

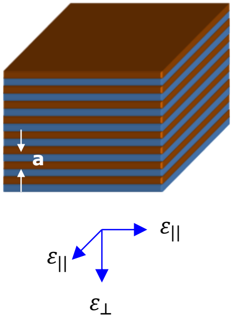
$$\frac{1}{\varepsilon_\perp} = \frac{f}{\varepsilon_m} + \frac{(1-f)}{\varepsilon_d}$$



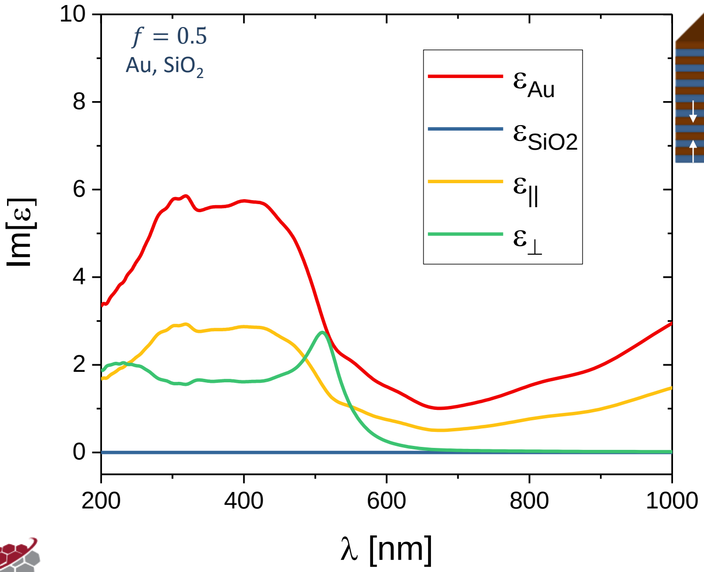
parallel=ordinary

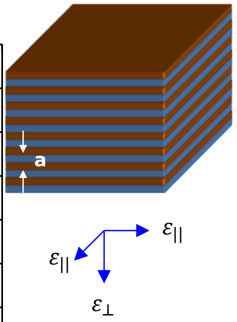
For most visible and IR wavelengths $|\varepsilon_m| >> \varepsilon_d \quad \rightarrow \varepsilon_{||} < 0, \varepsilon_\perp > 0$





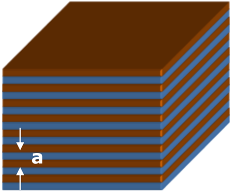
$\varepsilon_{||} < 0, \varepsilon_{\perp} > 0$

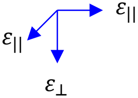

$f = 0.5$
Au, SiO_2
ε_{Au}
ε_{SiO2}
$\varepsilon_{||}$
$\varepsilon_{\perp}$
Im[ε]
λ [nm]
0
2
4
6
8
10
200
400
600
800
1000


a
$\varepsilon_{||}$
$\varepsilon_{||}$
$\varepsilon_{\perp}$

Hyperbolic Metamaterials
(HMM)

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$





Maxwell equations for time-harmonic waves

$$\boldsymbol{k} \times \boldsymbol{H} = -\omega \boldsymbol{D}$$
$$\boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \boldsymbol{H}$$

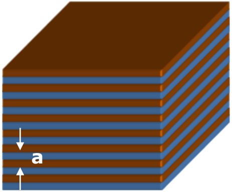
$$\boldsymbol{D} = \overleftrightarrow{\varepsilon}\boldsymbol{E} = \varepsilon_0(\varepsilon_{xx}E_x\hat{\boldsymbol{x}} + \varepsilon_{yy}E_y\hat{\boldsymbol{y}} + \varepsilon_{zz}E_z\hat{\boldsymbol{z}})$$

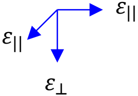
$$\boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E} = -\omega\mu_0\boldsymbol{k} \times \boldsymbol{H} = -k_0^2(\varepsilon_{xx}E_x\hat{\boldsymbol{x}} + \varepsilon_{yy}E_y\hat{\boldsymbol{y}} + \varepsilon_{zz}E_z\hat{\boldsymbol{z}})$$

$$\boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E} \equiv \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}) - k^2\boldsymbol{E} = -k_0^2(\varepsilon_{xx}E_x\hat{\boldsymbol{x}} + \varepsilon_{yy}E_y\hat{\boldsymbol{y}} + \varepsilon_{zz}E_z\hat{\boldsymbol{z}})$$

$$k_x(k_xE_x + k_yE_y + k_zE_z) = (k^2 - k_0^2\varepsilon_{xx})E_x$$
$$k_y(k_xE_x + k_yE_y + k_zE_z) = (k^2 - k_0^2\varepsilon_{yy})E_y$$
$$k_z(k_xE_x + k_yE_y + k_zE_z) = (k^2 - k_0^2\varepsilon_{zz})E_z$$

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$





Maxwell equations for time-harmonic waves

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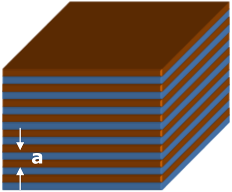
$$\boldsymbol{D} = \overleftrightarrow{\varepsilon}\,\boldsymbol{E} = \varepsilon_0(\varepsilon_{xx} E_x \hat{\boldsymbol{x}} + \varepsilon_{yy} E_y \hat{\boldsymbol{y}} + \varepsilon_{zz} E_z \hat{\boldsymbol{z}})$$

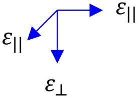
$$\boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E} = -\omega \mu_0 \boldsymbol{k} \times \boldsymbol{H} = -k_0^2(\varepsilon_{xx} E_x \hat{\boldsymbol{x}} + \varepsilon_{yy} E_y \hat{\boldsymbol{y}} + \varepsilon_{zz} E_z \hat{\boldsymbol{z}})$$

$$\boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E} \equiv \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{E}) - k^2 \boldsymbol{E} = -k_0^2(\varepsilon_{xx} E_x \hat{\boldsymbol{x}} + \varepsilon_{yy} E_y \hat{\boldsymbol{y}} + \varepsilon_{zz} E_z \hat{\boldsymbol{z}})$$

$$[k_x^2 - (k^2 - k_0^2 \varepsilon_{xx})] E_x + k_x k_y E_y + k_x k_z E_z = 0$$
$$k_y k_x E_x + \left[k_y^2 - \left(k^2 - k_0^2 \varepsilon_{yy}\right)\right] E_y + k_y k_z E_z = 0$$
$$k_z k_x E_x + k_z k_y E_y + [k_z^2 - (k^2 - k_0^2 \varepsilon_{zz})] E_z = 0$$

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix} \qquad \begin{matrix} \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{||} \\ \varepsilon_{zz} = \varepsilon_{\perp} \end{matrix}$$





Maxwell equations for time-harmonic waves

$$\boldsymbol{k} \times \boldsymbol{H} = -\omega \boldsymbol{D}$$
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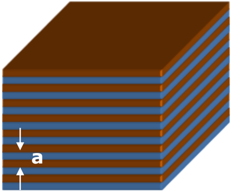
$$\boldsymbol{D} = \overleftrightarrow{\varepsilon}\,\boldsymbol{E} = \varepsilon_0 (\varepsilon_{xx} E_x \hat{\boldsymbol{x}} + \varepsilon_{yy} E_y \hat{\boldsymbol{y}} + \varepsilon_{zz} E_z \hat{\boldsymbol{z}})$$

$$\bar{\bar{M}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} k_0^2 - (k_y^2 + k_z^2) & k_x k_y & k_x k_z \\ k_x k_y & \varepsilon_{yy} k_0^2 - (k_x^2 + k_z^2) & k_y k_z \\ k_x k_z & k_y k_z & \varepsilon_{zz} k_0^2 - (k_x^2 + k_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$Det\ \bar{\bar{M}} = 0 \qquad \left(k^2 - \varepsilon_{||} k_0^2\right)\left[\varepsilon_{||}\varepsilon_{\perp} k_0^4 - \varepsilon_{||} k_0^2 \left(k_x^2 + k_y^2\right) - \varepsilon_{\perp} k_0^2 k_z^2\right] = 0$$

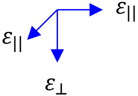
$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{||}$$
$$\varepsilon_{zz} = \varepsilon_{\perp}$$



1. $k^2 - \varepsilon_{||} k_0^2 = 0$ **Ordinary waves (TE)**

- $E \parallel x\ (y)$
- Dielectric function: $\varepsilon_{TE} = \varepsilon_{xx} = \varepsilon_{||}$



2. $\varepsilon_{||}\varepsilon_{\perp} k_0^4 - \varepsilon_{||} k_0^2 (k_x^2 + k_y^2) - \varepsilon_{\perp} k_0^2 k_z^2 = 0$

$$\frac{k_x^2 + k_y^2}{\varepsilon_{\perp}} + \frac{k_z^2}{\varepsilon_{||}} = k_0^2$$

Extraordinary waves (TM)

- $E \in xz\ (yz)\ plane$
- $\boldsymbol{E} \cdot \hat{\boldsymbol{z}} = E \cos\theta$
- Dielectric function: $\varepsilon_{TM}(\theta) = \dfrac{\varepsilon_{||}\,\varepsilon_{\perp}}{\varepsilon_{\perp}\sin^2\theta + \varepsilon_{||}\cos^2\theta}$

HMM Isofrequency surfaces

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{||} & 0 & 0 \\ 0 & \varepsilon_{||} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix} \qquad \overleftrightarrow{\mu} = \mu_0 \begin{pmatrix} \mu_{||} & 0 & 0 \\ 0 & \mu_{||} & 0 \\ 0 & 0 & \mu_{\perp} \end{pmatrix}$$

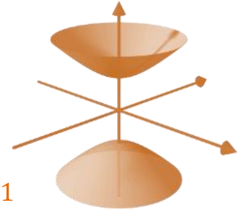
$$\varepsilon_{||}\,\varepsilon_{\perp} < 0 \qquad \mu_{||} = \mu_{\perp} > 0$$

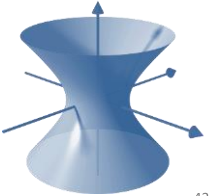
$$\frac{k_x^2 + k_y^2}{\varepsilon_{\perp}} + \frac{k_z^2}{\varepsilon_{||}} = k_0^2$$

TM modes

1. $\varepsilon_{||} > 0, \varepsilon_{\perp} < 0$ HMM type 1
2. $\varepsilon_{||} < 0, \varepsilon_{\perp} > 0$ HMM type 2

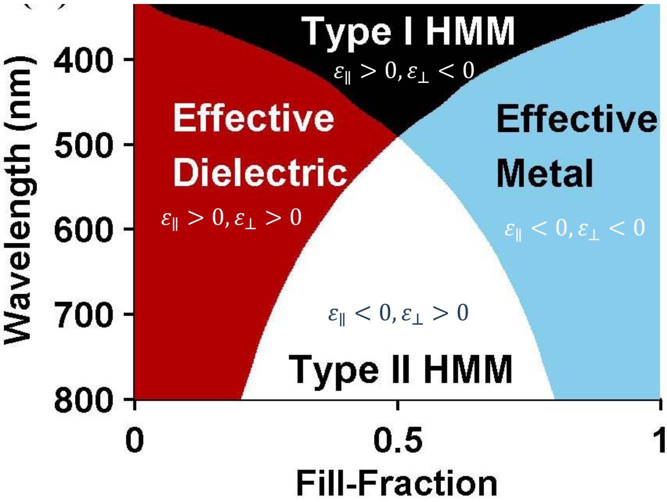
$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = +1 \end{cases}$$





Poddubny, A.; Iorsh, I.; Belov, P.; Kivshar, Y. Hyperbolic Metamaterials. *Nat Photon* **2013**, *7*, 948–957.

Ag/TiO$_2$ multilayer HMM

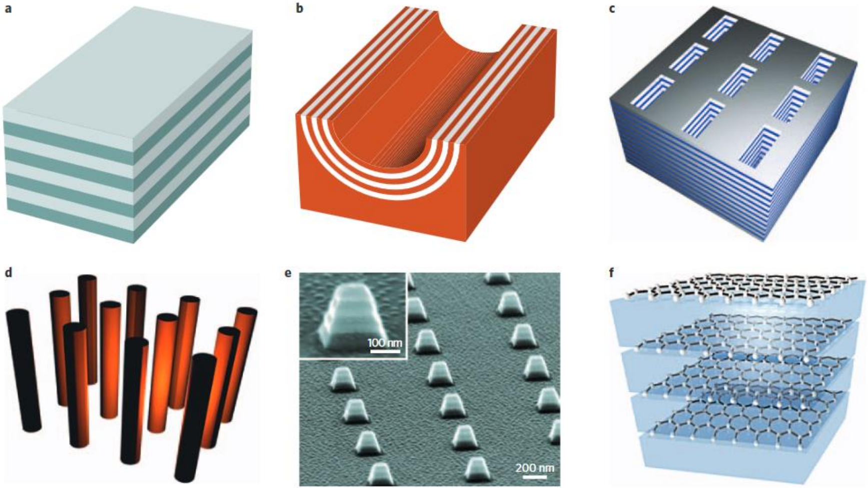


$$\frac{{k_x}^2}{\varepsilon_\parallel} + \frac{{k_z}^2}{\varepsilon_\perp} = k_0^2$$

$$\varepsilon_\parallel = f\varepsilon_m + (1-f)\varepsilon_d$$

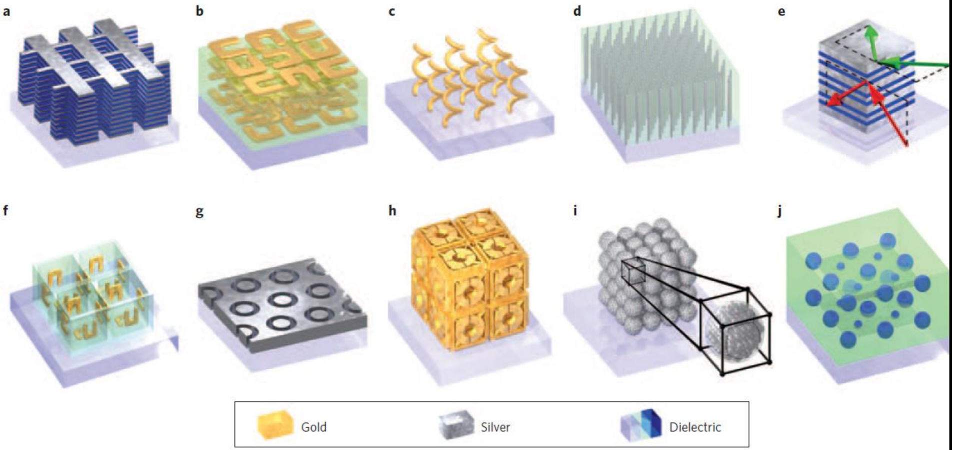
$$\frac{1}{\varepsilon_\perp} = \frac{f}{\varepsilon_m} + \frac{(1-f)}{\varepsilon_d}$$

Cortes, C. L., et al. Quantum nanophotonics using hyperbolic metamaterials. *J. Opt.* **14**, 063001 (2012).



Poddubny, A.; Iorsh, I.; Belov, P.; Kivshar, Y. Hyperbolic Metamaterials. *Nat Photon* **2013**, *7*, 948–957.

Metamaterials



C. M. Soukoulis and M. Wegener; ***Nature Photonics*** (5) 523–530 (2011)

Hyperbolic metamaterials

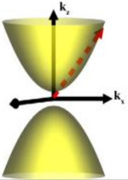
Photonic density of states

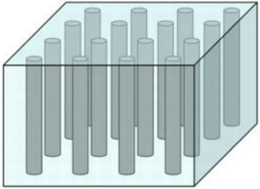
$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon_{\parallel} & 0 & 0 \\ 0 & \varepsilon_{\parallel} & 0 \\ 0 & 0 & \varepsilon_{\perp} \end{pmatrix}$$

$$\boxed{\boldsymbol{\varepsilon_{\parallel} \cdot \varepsilon_{\perp} < 0}}$$

$$(\varepsilon_x = \varepsilon_y = \varepsilon_{\parallel},\ \varepsilon_z = \varepsilon_{\perp})$$

- **Type I**
  $\varepsilon_{\parallel} > 0, \varepsilon_{\perp} < 0$

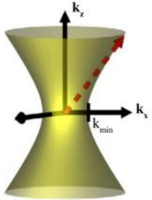


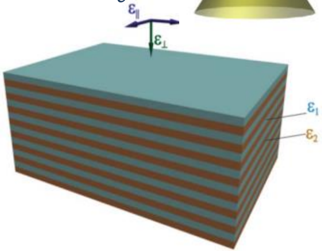


- **Type II**
  $\varepsilon_{\parallel} < 0, \varepsilon_{\perp} > 0$

  with:

  $$k_{\parallel}^{crit} = \sqrt{\varepsilon_{\perp}}\frac{\omega}{c}$$





$$g_{if} = \frac{2p}{\hbar}\left|M_{if}\right|^2 r_f \longrightarrow \rho_f(\omega) \sim k_{max}^3$$ (PDOS diverges)

Z. Jacob, J. Y. Kim, G. V. Naik, A. Boltasseva, E. E. Narimanov, and V. M. Shalaev, **Appl. Phys. B** 100(1), 215–218 (2010)

Hyperbolic metamaterials

Effective medium approximation

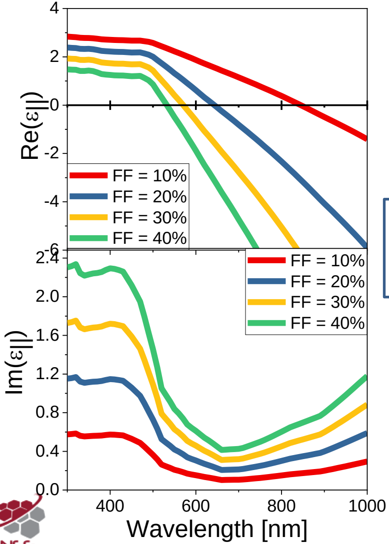
$$\varepsilon_{\parallel} = ff_m \varepsilon_m + (1 - ff_m)\varepsilon_d$$

Metal filling fraction:

$$ff_m = \frac{t_m}{t_m + t_d}$$

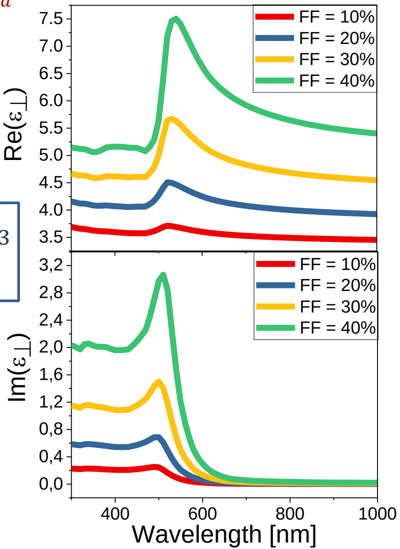
$$\frac{1}{\varepsilon_{\perp}} = \frac{ff_m}{\varepsilon_m} + \frac{1 - ff_m}{\varepsilon_d}$$

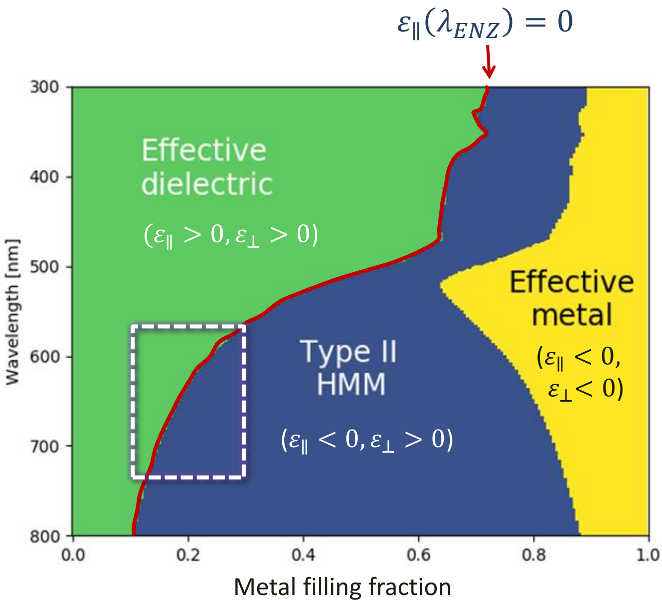
with $t_{m,d} << \lambda$



$$\boxed{\begin{aligned}\varepsilon_d &= Al_2O_3 \\ \varepsilon_m &= Au\end{aligned}}$$

Type II




$\varepsilon_{\parallel}(\lambda_{ENZ}) = 0$
Effective dielectric
$(\varepsilon_{\parallel} > 0, \varepsilon_{\perp} > 0)$
Type II HMM
$(\varepsilon_{\parallel} < 0, \varepsilon_{\perp} > 0)$
Effective metal
$(\varepsilon_{\parallel} < 0,$
$\varepsilon_{\perp} < 0)$
Wavelength [nm]
300
400
500
600
700
800
0.0
0.2
0.4
0.6
0.8
1.0
Metal filling fraction

ENZ materials

Refractive properties:

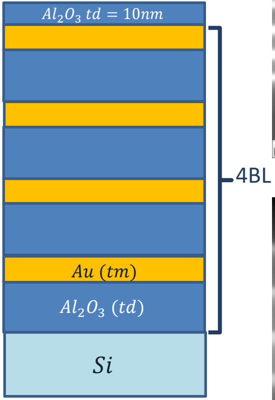
- The refractive index n is also nearly 0
- Phase velocity (c/n) and wavelength (λ/n) are infinite
- Photons oscillate in time coherently

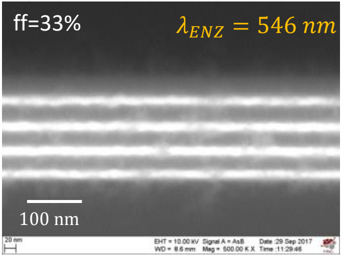
Radiative processes are modified by ENZ materials

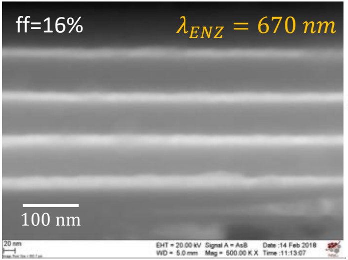
- Spontaneous emission: Einstein $A = nA_{vac}$ coefficient and the spontaneous emission lifetime ($1/A$)
- Stimulated emission: Einstein $B = B_{vac}/n^2$ coefficient (large optical gain)

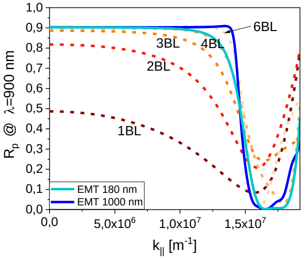
Hyperbolic metamaterials

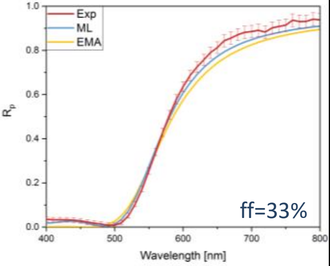
Multilayer synthesis







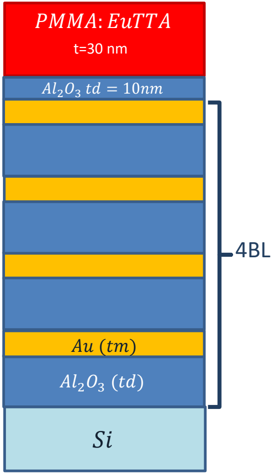




t_m=15 nm=cost.

Eu^{3+} - HMM

PL Intensity



t_m=15 nm=cost.