

Chapter 1

Spontaneous symmetry breaking

1.1 Spontaneous symmetry breaking

When we talk about a broken symmetry, we often refer to a situation as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

where \mathcal{H}_0 is invariant under the group \mathcal{G} and \mathcal{H}_1 is invariant under a subgroup $\mathcal{G}' \subset \mathcal{G}$.

Example 1: Ising with magnetic field

Let us consider the Hamiltonian for the Ising model with a magnetic field $H \neq 0$:

$$\mathcal{H} = J \sum_{\langle ij \rangle} S_i S_j + \sum_i H_i S_i$$

The second term, $\sum_i H_i S_i$, breaks the \mathbb{Z}^2 symmetry satisfied by the first alone.

Example 2: Hydrogen atom with an external field

An example, in quantum mechanics, is the hydrogen atom in presence of an electric field $\vec{\mathbf{E}}$ (Stark effect) or a magnetic one, $\vec{\mathbf{B}}$, (Zeeman effect). If \mathcal{H}_1 is small, the original symmetry is weakly violated and perturbative approaches are often used.

In all the above examples, one says that the symmetry is broken explicitly.

Definition 1: Spontaneous symmetry breaking

The Hamiltonian maintains the original symmetry but the variables used to describe the system become asymmetric.

At this point it is convenient to distinguish between

- **Discrete symmetries:** for instance \mathbb{Z}^2 , \mathbb{Z}_q .
- **Continuous symmetries:** for instance XY , $O(n)$.

Let us consider first the discrete ones by focusing on the \mathbb{Z}^2 symmetry (Ising). As previously said, if $H = 0$, the Hamiltonian of the Ising model, $\mathcal{H}_{\text{Ising}}$, is invariant with respect to the change $S_i \rightarrow -S_i$, hence the discrete group is

$$\mathcal{G} = \mathbb{Z}^2$$

A Ginzburg-Landau theory of the Ising model is given by

$$\beta\mathcal{H}(\Phi) = \int d^d\vec{x} \left[\frac{1}{2} (\vec{\nabla}\Phi)^2 + \frac{r_0}{2} \Phi^2 + \frac{u_0}{4} \Phi^4 - h\Phi \right] \quad (1.1)$$

Lecture n.
Tuesday 0th
January, 2020.
Compiled:
Wednesday 5th
February, 2020.

Hence, the partition function can be computed as:

$$Z(r_0, u_0, h) = \int \mathcal{D}[\Phi] e^{-\beta \mathcal{H}(\Phi)} \quad (1.2)$$

If we have $h = 0$, the symmetry is $\Phi \rightarrow -\Phi$. The equation of state obtained with saddle point approximation is

$$h = -\vec{\nabla}^2 \Phi + r_0 \Phi + u_0 \Phi^3$$

If h does not depend on \vec{x} , i.e. $h(\vec{x}) = h$, the last equation reduces to the equation of state of the Landau theory of uniform system:

$$h = r_0 \Phi + u_0 \Phi^3$$

Let us remind that the saddle point approximation consists in approximating the functional integral of Eq.(1.2) with its dominant term, i.e. with the one for which the exponent (Eq.(1.1)) is minimum. Therefore, in the uniform case (namely $\vec{\nabla} \Phi = 0$), it is equivalent to find the uniform value Φ_0 that is the extrema of the potential:

$$V(\Phi) = \frac{1}{2} r_0 \Phi^2 + \frac{u_0}{4} \Phi^4 - h \Phi$$

Hence, if $h = 0$, the extrema of the potential can be computed as

$$V' = (r_0 + u_0 \Phi^2) \Phi = 0$$

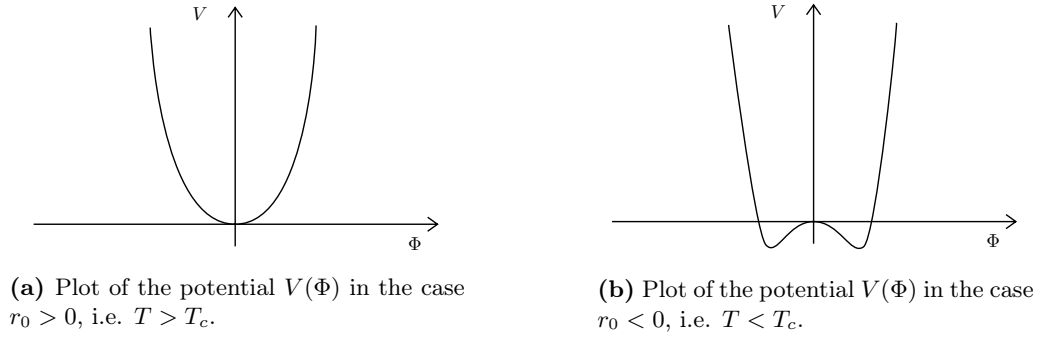


Figure 1.1

Let us remember that $r_0 \propto (T - T_c)$. In order to find the extrema of the potential $V(\Phi)$, we should distinguish two cases:

1. Case $r_0 > 0$ ($T > T_c$): there is only one solution $\Phi_0 = 0$, as we can see in Figure 1.1a.
2. Case $r_0 < 0$ ($T < T_c$): there are two solutions $\Phi_0 = \pm \sqrt{-\frac{r_0}{u_0}}$, as illustrated in Figure 1.1b.

We note that the two solution $\pm \Phi_0$ are related by the \mathbb{Z}^2 transformation, namely $\Phi \rightarrow -\Phi$. Moreover, in this case with $T < T_c$, the two states (phases) $\pm \Phi_0$ have a lower symmetry than the state $\Phi_0 = 0$.

If the thermal fluctuations $\delta \Phi$ are sufficiently strong to allow passages between the two states $\pm \Phi_0$ at $T < T_c$, we have $\langle \Phi \rangle = 0$ (preserves states).

However, for $T < T_c$ and $N \rightarrow +\infty$, transition between the two states will be less and less probable and the system will be trapped into one of the two states ($\pm \Phi_0$). In other words, the system choose spontaneously one of the two less symmetric state. Therefore, its physics is not any more described by Φ but by the fluctuations $\delta \Phi$ around the chosen minimum Φ_0 . There is a spontaneous symmetry breaking. It means that the variable Φ is not any more symmetric and one has to look at $\Phi \rightarrow \Phi_0 + \delta \Phi$, where $\delta \Phi$ is a new variable!

1.2 Spontaneous breaking of continuous symmetries and the onset of Goldstone particles

Let us start with a simple model in which the order parameter is a scalar complex variable

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}$$

and with an Hamiltonian \mathcal{H} that is invariant with respect to a global continuous transformation. For instance, the simplest model in statistical mechanics that is invariant with respect to a continuous symmetry is the XY model with $O(2)$ symmetry, or a Ginzburg-Landau model for a superfluid or a superconductor (with no magnetic field). Hence, we suppose that the Hamiltonian has the following form:

$$\beta\mathcal{H}_{eff} = \int d^d\vec{x} \left[\frac{1}{2} \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi^* + \frac{r_0}{2} \Phi^* \Phi + \frac{u_0}{4} (\Phi^* \Phi)^2 \right]$$

where

$$\Phi(\vec{x}) = \frac{1}{\sqrt{2}} [\Phi_1(\vec{x}) + i\Phi_2(\vec{x})], \quad \text{or} \quad \Phi(\vec{x}) = \psi(\vec{x}) e^{i\alpha(\vec{x})}$$

The physical meaning of Φ depends on the case considered. If we have:

- Superfluid: Φ is the macroscopic wave function of the Bose condensate (density of superfluid $n = |\Phi|^2$).
- Superconductor: Φ is the single particle wave function describing the position of the centre of mass of the Cooper pair.

1.2.1 Quantum relativistic case (field theory)

In quantum mechanics the analog of the Hamiltonian \mathcal{H} is the action

$$S(\Phi) = \int d^4\vec{x} \mathcal{L}(\Phi)$$

where

$$\mathcal{L}(\Phi) = -\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi^* - \frac{r_0}{2} \Phi \Phi^* - \frac{u_0}{4} (\Phi \Phi^*)^2$$

The Lagrangian $\mathcal{L}(\Phi)$ describes a scalar complex (i.e. charged) *muonic field* with mass $m \equiv \sqrt{r_0}$; we note that, if $\mathcal{L}(\Phi)$ describes a muonic field, we should have $r_0 > 0$ to have the mass m well defined. Moreover, the term $(\Phi \Phi^*)^2$ means self-interaction with strength $\lambda \equiv u_0$.

In all cases ($r_0 > 0$ or $r_0 < 0$), the original symmetry is $U(1)$; it means that both the Hamiltonian \mathcal{H} and the Lagrangian \mathcal{L} are invariant with respect to the transformation

$$\Phi \rightarrow e^{i\theta} \Phi, \quad \Phi^* \rightarrow e^{-i\theta} \Phi^*$$

where the phase θ does not depend on \vec{x} (global symmetry). In components the transformation becomes

$$\begin{cases} \Phi_1 \rightarrow \Phi_1 \cos \theta - \Phi_2 \sin \theta \\ \Phi_2 \rightarrow \Phi_2 \cos \theta + \Phi_1 \sin \theta \end{cases} \Rightarrow (\Phi'_1, \Phi'_2) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

Now, let us focus first on the statistical mechanics model and to the most interesting case of $r_0 < 0$. In components, \mathcal{H} can be expressed as

$$\beta\mathcal{H} = \int d^d\vec{x} \left[(\vec{\nabla}\Phi_1)^2 + (\vec{\nabla}\Phi_2)^2 \right] + \int d^d\vec{x} V(\Phi_1, \Phi_2) \quad (1.3)$$

where the potential is

$$V(\Phi_1, \Phi_2) = \frac{r_0}{2}(\Phi_1^2 + \Phi_2^2) + \frac{u_0}{4}(\Phi_1^2 + \Phi_2^2)^2 \quad (1.4)$$

In the case $r_0 < 0$, it is called *mexican hat potential* and it is shown in Figure 1.2.

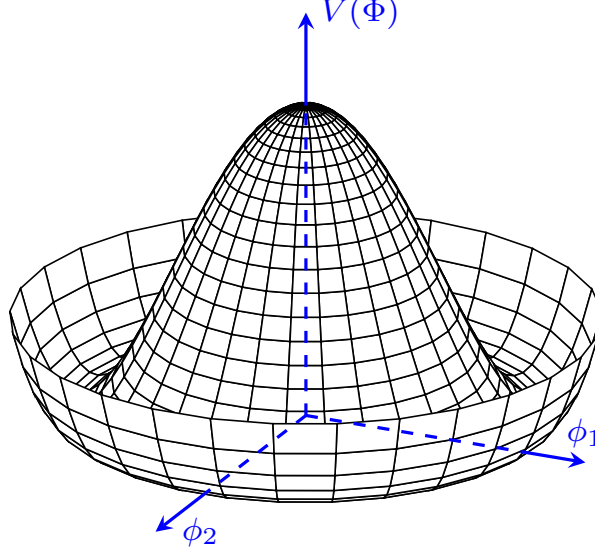


Figure 1.2: Case $r_0 < 0$. The potential $V(\Phi)$ is a mexican hat potential.

In the uniform case we have $(\nabla\Phi_1 = \nabla\Phi_2 = 0)$. Let us define $S = \sqrt{\Phi_1^2 + \Phi_2^2}$, the potential in Eq.(1.4) can be rewritten as:

$$V(S) = \frac{r_0}{2}S^2 + \frac{u_0}{4}S^4$$

In the uniform case, the solution is given by the minima of the potential $V(S)$; hence, in order to find the extrema points, we derive the potential with respect to S and we impose the condition $V' = 0$:

$$\frac{dV(S)}{dS} = r_0S + u_0S^3 = 0$$

We have a maximum at $S = 0$ and a minimum at $S^2 \equiv v^2 = -r_0/u_0$. Hence, for $r_0 < 0$, the Hamiltonian \mathcal{H} displays a minimum when

$$\Phi_1^2 + \Phi_2^2 \equiv v^2 = -\frac{r_0}{u_0}$$

It could be represented in the $2d$ plane (Φ_1, Φ_2) , where the minimum lies on a circle of radius

$$v = \sqrt{-\frac{r_0}{u_0}}$$

as show in Figure 1.3. The spontaneous symmetry breaking occurs when the system "chooses" one of the infinite available minima. In our example, let us suppose that the chosen minimum is

$$\Phi_1 = v = \sqrt{-\frac{r_0}{u_0}}, \quad \Phi_2 = 0 \quad (1.5)$$

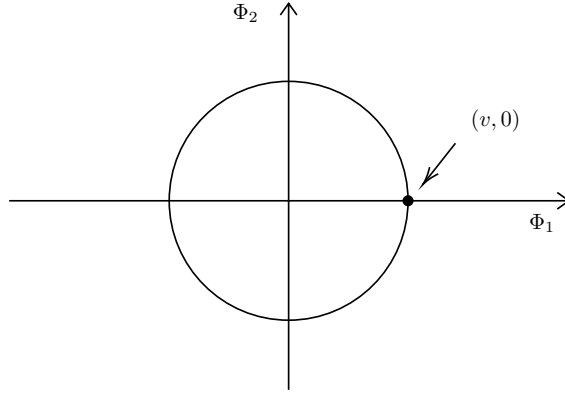


Figure 1.3: Plane (Φ_1, Φ_2) . The minimum lies on a circle of radius $v = \sqrt{-\frac{r_0}{u_0}}$.

Interpretation in relativistic quantum mechanics

Now, let us give a physical interpretation of the results previously obtained and of the considerations we have done in order to obtain them. In particular:

1. Choosing $r_0 < 0$ corresponds to an *imaginary mass*. This is because moving away from $\Phi = 0$, the system experiences a *negative resistance* in both directions, being $\Phi = 0$ a relative local maximum.
2. The minimum has the lowest energy and therefore it must correspond to the *empty state*. In this case, however, there is an infinite number of empty states!

In summary, the starting Hamiltonian \mathcal{H} (or Lagrangian \mathcal{L}) is invariant with respect to $U(1)$, but the one that describes the fluctuation dynamics around one of the chosen minimum state is not invariant with respect to $U(1)$. Let us see in more details why the Hamiltonian, or Lagrangian, it is not invariant anymore.

First of all, let us write the Lagrangian with respect to the fluctuations of Φ_1 and Φ_2 around the chosen state $(v, 0)$ (Eq.(1.5)), we obtain:

$$\begin{cases} \Phi_1 = v + \delta\Phi_1 \\ \Phi_2 = 0 + \delta\Phi_2 \end{cases} \Rightarrow \Phi = \Phi_1 + i\Phi_2 = v + (\delta\Phi_1 + i\delta\Phi_2)$$

where we omit the factor $1/\sqrt{2}$ for simplicity. Let us note that

$$\begin{cases} \delta\Phi_1 = \Phi_1 - v \\ \delta\Phi_2 = \Phi_2 \end{cases} \Rightarrow \langle \delta\Phi_1 \rangle_{\Phi_0} = \langle \delta\Phi_2 \rangle_{\Phi_0} = 0$$

indeed, as expected, the expectation of the empty state is back to be zero.

For the quantum relativistic Lagrangian, let us write

$$r_0 \rightarrow m^2, \quad u_0 \rightarrow \lambda, \quad v^2 = -\frac{m^2}{\lambda} \quad (1.6)$$

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu(\delta\Phi_1 + i\delta\Phi_2)\partial_\mu(\delta\Phi_1 - i\delta\Phi_2) \\
&\quad -\frac{m^2}{2}(v + \delta\Phi_1 + i\delta\Phi_2)(v + \delta\Phi_1 - i\delta\Phi_2) \\
&\quad -\frac{\lambda}{4}[(v + \delta\Phi_1 + i\delta\Phi_2)(v + \delta\Phi_1 - i\delta\Phi_2)]^2 \\
&= -\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1) - \frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2) \\
&\quad -\frac{m^2}{2}(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2) \\
&\quad -\frac{\lambda}{4}(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2)^2
\end{aligned} \tag{1.7}$$

Since $m^2 = -v^2\lambda$,

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1) - \frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2) \\
&\quad + \frac{\lambda v^2}{2}(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2) \\
&\quad - \frac{\lambda}{4}(v^4 + 4v^2\delta\Phi_1^2 + (\delta\Phi_1^2 + \delta\Phi_2^2)^2 + 4v^3\delta\Phi_1 + 2v^2(\delta\Phi_1^2 + \delta\Phi_2^2) + 4v\delta\Phi_1(\delta\Phi_1^2 + \delta\Phi_2^2))
\end{aligned} \tag{1.8}$$

Neglecting the constant terms in v

$$\begin{aligned}
\mathcal{L}(\delta\Phi_1, \delta\Phi_2) &= -\frac{1}{2}(\partial_\mu\delta\Phi_1)^2 - \frac{1}{2}(\partial_\mu\delta\Phi_2)^2 \\
&\quad -\lambda v^2\delta\Phi_1^2 - v\lambda\delta\Phi_1((\delta\Phi_1)^2 + (\delta\Phi_2)^2) \\
&\quad -\frac{\lambda}{4}((\delta\Phi_1)^2 + (\delta\Phi_2)^2)^2
\end{aligned} \tag{1.9}$$

Remark. The term $-\lambda v^2\delta\Phi_1^2$ indicates that the field $\delta\Phi_1$ (related to the transversal fluctuations) has a null empty state ($\langle\delta\Phi_1\rangle = 0$) and a mass M such that:

$$M^2 = 2\lambda v^2 = -2r_0 \tag{1.10}$$

Therefore, it represents a real, massive, mesonic scalar field that is physically acceptable.

However, \mathcal{L} is not any more invariant under the transformation $\delta\Phi_1 \rightarrow -\delta\Phi_1$.

Remark. The field $\delta\Phi_2$ has no mass! It describes the fluctuations along the circle where the potential V is in its minimum which implies no dynamical inertia, that implies no mass!

So, starting with one complex scalar field $\Phi(\vec{x})$ having mass m , when $m^2 < 0$ one gets a real scalar field $\delta\Phi_1$ with mass $M = \sqrt{-2m^2}$ and a second scalar field $\delta\Phi_2$ that is massless. This is called the *Goldstone boson*.

Theorem 1

If a continuous symmetry is spontaneously broken and there are no long range interactions, exists an elementary excitation with zero momentum or particle of zero mass called Goldstone boson.

More generally, let \mathcal{P} be a subgroup of \mathcal{G} . If \mathcal{G} has N independent generators and \mathcal{P} has M independent generators, if \mathcal{P} is the new (lower) symmetry, therefore $N - M$ Goldstone bosons exist.

In the previous case $\mathcal{G} = U(1) \Rightarrow N = 1$ whereas $M = 0$ (we have chosen a specific minimum).

Example 3

XY model in statistical mechanics:

- $\delta\Phi_1$: fluctuation of the modulus of m .
- $\delta\Phi_2$: fluctuations of the spin directions \Rightarrow spin waves.

Remark. In particle physics the presence of Goldstone bosons brings a serious problem in field theory since the corresponding particles are not observed!

Higgs-Englert-Brout (1964)

Higgs mechanism gives back the mass to the Goldstone particles. The basic idea is that the Goldstone theorem that works for a continuous global symmetry it can fail for local gauge theories!

1.3 Spontaneous symmetry breaking in Gauge symmetries

Statistical mechanics, G1 model for superconductors in presence of a magnetic field (*Meissner effect*, i.e. the magnetic induction $\vec{\mathbf{B}} = 0$ inside the superconductor).

$$\mathcal{H}(\Phi) = \int d^D \vec{x} \left[\frac{1}{2} B^2 + \left| \left(\vec{\nabla} - 2i\vec{\mathbf{A}} \right) \Phi \right|^2 \right] + \frac{r_0}{2} \Phi^* \Phi + \frac{u_0}{4} (\Phi^* \Phi)^2 - \vec{\mathbf{B}} \cdot \vec{\mathbf{H}} \quad (1.11)$$

where $\frac{B^2}{2}$ is the energy of the magnetic field $\vec{\mathbf{B}}$ and $\vec{\nabla} \rightarrow [\vec{\nabla} + iq\vec{\mathbf{A}}]$ is the minimal coupling. Consider $\vec{\mathbf{H}}$ the external magnetic field

$$\vec{\mathbf{B}} = \vec{\mathbf{H}} + \vec{\mathbf{M}} \quad (1.12)$$

is the induction field.

Normal conductor corresponding to $\Phi_0 = 0$, that implies $\vec{\mathbf{B}} = \vec{\mathbf{H}}$. For a superconductor we have $\Phi \neq 0$, a spontaneous symmetry breaking.

1.4 Fiedl theory analog

Scalar charged mesonic fields selfinteracting and in presence of an electromagnetic field with potential quadrivector $A_\mu(\vec{x})$.

$$\partial_\mu \rightarrow D_\mu = [\partial_\mu + iqA_\mu] \quad (1.13)$$

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \Phi^* = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2) \quad (1.14)$$

Remark. Because of the presence of $A_\mu(\vec{x})$, we should consider a theory that satisfies symmetry $U(1)$ locally! The transformations are

$$\begin{cases} \Phi(\vec{x}) \rightarrow e^{i\alpha(\vec{x})} \Phi(\vec{x}) \\ \Phi^*(\vec{x}) \rightarrow e^{-i\alpha(\vec{x})} \Phi^*(\vec{x}) \end{cases} \quad (1.15)$$

hence, exists $A_\mu(\vec{x})$ interacting with $\Phi(\vec{x})$.

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(\vec{x}) F^{\mu\nu}(\vec{x}) + (D_\mu \Phi(\vec{x}))^* (D_\mu \Phi(\vec{x})) - V(\Phi, \Phi^*) \quad (1.16)$$

where

$$D_\mu \Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{Gauge-covariant derivative} \quad (1.17a)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{field strength tensor} \quad (1.17b)$$

$$V(\Phi, \Phi^*) = \frac{m^2}{2} \Phi \Phi^* + \frac{\lambda}{4} (\Phi \Phi^*)^2 \quad (1.18)$$

- Case $m^2 > 0$: the minimum is in $\Phi = 0$.
- Case $m^2 < 0$: the minimum is in $\Phi = \sqrt{-\frac{m^2}{\lambda}} \equiv v$ (circle of radius $|\Phi| = v$).

Let us choose the state

$$\bar{\Phi}_1 = v, \quad \bar{\Phi}_2 = 0 \quad (1.19)$$

and consider

$$\Phi(x) = (v + \delta\Phi_1) + i\delta\Phi_2 \quad (1.20)$$

Inserting in the Lagrangian and keeping in mind $-m^2 = v^2\lambda$

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\delta\Phi_1)^2 + \frac{1}{2}(\partial_\mu\delta\Phi_2)^2 \\ & -\lambda v^2\delta\Phi_1^2 + q^2v^2A_\mu A^\mu - qvA^\mu\partial_\mu\delta\Phi_2 + \text{higher order terms} \end{aligned} \quad (1.21)$$

1. Term $\lambda v^2\delta\Phi_1^2$: the field $\delta\Phi_1$ is massive with mass $m = v\sqrt{2}$ (Higgs boson).
2. Term $q^2v^2A_\mu A^\mu$: this terms means that the Gauge boson A_μ , the photon, has got a mass

$$M_A = 2qv \quad (1.22)$$

Remark. Since now A_μ is massive, it has three independent polarization states.

3. Term $qvA^\mu\partial_\mu\delta\Phi_2$: the field $\delta\Phi_2$ is not massive (no term $\propto \delta\Phi_2^2$) and is mixed with A_μ . Dynamically this means that a propagating photon can transform itself into a field $\delta\Phi_2$ (photon r Goldstone boson).

Since $\delta\Phi_2$ does not seem to be a physical field it should be eliminated by a Gauge transformation.

Indeed a gauge transformation is also characterized by the transformation

$$A_\mu(\vec{x}) \rightarrow A_\mu(\vec{x}) + \frac{1}{qv}\partial_\mu\delta\Phi_2(x) \quad (1.23)$$

Inserting (1.23) in the Lagrangian, we eliminate the mixed term $qvA^\mu\partial_\mu\delta\Phi_2$ and $\frac{1}{2}(\partial_\mu\delta\Phi_2)^2$.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\delta\Phi_1)^2 - \lambda v^2\delta\Phi_1^2 + q^2v^2A_\mu A^\mu + \text{higher order terms} \quad (1.24)$$

Remark. Among the higher order terms there are

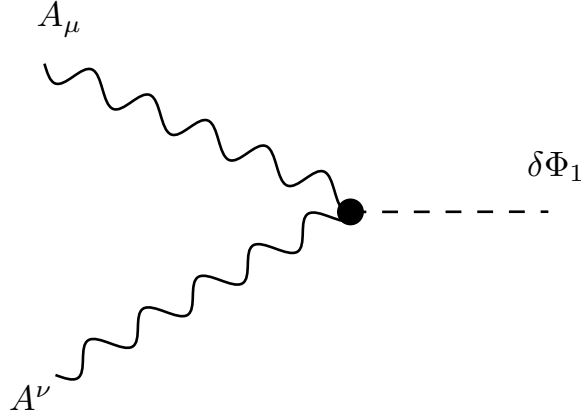
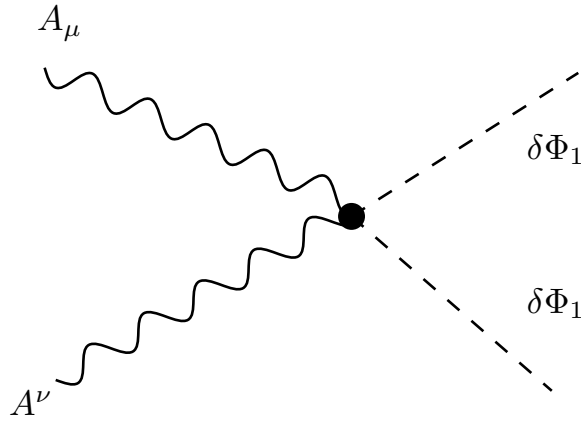
- $\propto \delta\Phi_1 A_\mu A^\mu$.
- $\propto \delta\Phi_1^2 A_\mu A^\mu$.

The new Lagrangian contains two fields: one is a massive photon with spin 1 and the second field $\delta\Phi_1$ is massive too, but has spin 0 (scalar).

The Goldstone boson has been "laten" by the Gauge boson that now is massive!

The mechanics trough which the gauge boson becomes massive is the so called *Higgs mechanism*.

In summary: (according to the degrees of freedom)

**Figure 1.4:** Description.**Figure 1.5:** Description.

- Global $U(1) \Rightarrow 2$ scalar fields with mass $1+1 \xrightarrow{\text{symmetry breaking}} 1$ scalar field massive and 1 scalar field not massive $1+1$.
- Local gauge $U(1) \Rightarrow 2$ massive scalar fields + 1 photon $(2+2)$ two polarizations $\xrightarrow{\text{symmetry breaking}} 1$ massive scalar field, 1 massive photon $1 + 3$ polarization.

Remark. The presence of the massive photon $m_A^2 = q^2 v^2$, $q = 2l$ in superconductivity, gives rise to the experimental drop

$$B(x) = B(0) \exp\left(-\frac{x}{l}\right) \quad (1.25)$$

inside the system.

- We cannot introduce by hand a massive photon i.e. a term like $\frac{1}{2}m_A^2 A_\mu A^\mu$ in the Lagrangian because we would violate explicitly the gauge symmetry!
- The Lagrangian is gauge invariant.
- Symmetry breaking occurs at the level of the vacuum state.
- A gauge symmetry that is explicitly broken is not renormalizable.

1.4.1 Non abelian gauge theories

Example 4

Electro-weak interactions theory (Glashow-Weinberg-Salam) (theory of leptons).

Lagrangian has $\underbrace{SU(2)}_{\text{weak interactions}} \times \underbrace{U(1)}_{\text{electromagnetian}}$

Example 5: Quantum chromodynamic (quarks+gluons)

In this case, one has a term that is $SU(3)$ invariant + the GWS lagrangian with symmetry $SU(2) \times U(1)$, implies

$$SU(3) \times SU(2) \times U(1) \quad (1.26)$$

Because of the groups $SU(2)$ and $SU(3)$ the symmetries above are not abelian. (For example in $SU(2)$ two matrices $U(\alpha)$ and $U(\beta)$ do not commute in general).

1.5 Extension of Higgs mechanism to non abelian theories

1.5.1 GWS model

Complex field $SU(2)$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi_3 + i\Phi_4 \end{pmatrix} = \begin{pmatrix} \Phi_a(\vec{x}) \\ \Phi_b(\vec{x}) \end{pmatrix} \quad (1.27)$$

where Φ_a, Φ_b are complex fields.

Gauge transformation $SU(2) \times U(1)$:

$$\begin{pmatrix} \Phi_a(\vec{x}) \\ \Phi_b(\vec{x}) \end{pmatrix} \rightarrow e^{\frac{i}{2}\alpha_0(\vec{x})} e^{\frac{i}{2}\vec{\tau} \cdot \vec{\alpha}(\vec{x})} \begin{pmatrix} \Phi_a(\vec{x}) \\ \Phi_b(\vec{x}) \end{pmatrix} \quad (1.28)$$

where $\vec{\tau}$ are Pauli matrices, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are four real functions (4 vectorial mesons).

$$\vec{\alpha}(\vec{x}) \rightarrow W_\mu^a(\vec{x}) = \left(W_\mu^{(1)}(\vec{x}), W_\mu^{(2)}(\vec{x}), W_\mu^{(3)}(\vec{x}) \right) \quad (1.29)$$

The scalar gauge field is

$$\alpha_0(\vec{x}) \rightarrow B_\mu(\vec{x}) \quad (1.30)$$

with B_μ is a linear combination of A_μ and $W_\mu^{(3)}$.

Lagrangian:

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 - \frac{1}{4} b^{\mu\nu} b_{\mu\nu} - \frac{1}{4} f_a^{\mu\nu} f_{\mu\nu}^a \quad (1.31)$$

$$D_\mu \rightarrow \partial_\mu - \frac{1}{2} i g \tau^a W_\mu^a - \frac{i}{2} g' B_\mu \quad (1.32)$$

$$f_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g \varepsilon^{abc} W_\mu^b W_\nu^c \quad (1.33)$$

$$b_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (1.34)$$

$$W_\mu^a \rightarrow W_\mu^a - \varepsilon^{abc} \alpha_b(\vec{x}) W_\mu^c(\vec{x}) + \frac{1}{g} \partial_\mu \alpha^a(\vec{x}) \quad (1.35a)$$

$$B_\mu \rightarrow B_\mu + \frac{1}{g'} \frac{\partial \alpha_0}{\partial x_\mu} \quad (1.35b)$$

$$\nu \sim \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 = v^2 \quad (1.36)$$

Choosing the direction on the sphere in \mathbb{R}^4 , 3 symmetries are broken a 3 Goldstone bosons.

1.5.2 Higgs mechanism

Higgs scalar field

$$\delta\Phi = \begin{pmatrix} \Phi^+ \\ \Phi_0 \end{pmatrix} \quad (1.37)$$

such that

$$\langle 0 | \Phi | 0 \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \quad (1.38)$$

$$\Rightarrow \mathcal{L}_{Higgs} = \frac{1}{2}(g\nu)^2 W_\mu^+ W^{-\mu} + \frac{1}{2}v^2 \left(gW_\mu^{(3)} - g'B_\mu \right)^2 \quad (1.39)$$

where

$$W_\mu^{(1)} = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-) \quad (1.40a)$$

$$W_\mu^{(2)} = \frac{1}{\sqrt{2}}(W_\mu^+ - W_\mu^-) \quad (1.40b)$$

Mass of the W^+ particle and its antiparticle

$$M_W^2 = \frac{1}{2}(gv)^2 \quad (1.41)$$

The 2^{nd} term is a linear combination of W_μ^3 and B_μ which corresponds to Z^0 , the field for a third weak gauge boson.

To make Z_μ^0 and A_μ orthogonal we should consider

$$A_\mu = (\cos \theta_W) B_\mu + (\sin \theta_W) W_\mu^3 \quad (1.42a)$$

$$Z_\mu^0 = (-\sin \theta_W) B_\mu + (\cos \theta_W) W_\mu^3 \quad (1.42b)$$

where θ_W is the Weiberg angle:

$$\tan \theta_W = \frac{g'}{g} \quad (1.43)$$

$$M_{Z^0}^2 = \frac{1}{2} \left(\frac{vg}{\cos \theta_W} \right)^2 = \frac{Mw^2}{\cos^2 \theta_W} \quad (1.44)$$