The partition function is

$$Z_N(T, J, H) = \sum_{\{S\}} \exp\left[ \frac{\beta J}{2N} \sum_{ij} S_i S_j + \beta H \sum_i S_i \right] \tag{1}$$

Since there are no restrictions on the double sum we can write

$$\sum_{ij} S_i S_j = \left( \sum_i S_i \right)\left( \sum_j S_j \right) = \left( \sum_i S_i \right)^2 \tag{2}$$

Rewriting the partition function:

$$Z_N(T, J, H) = \sum_{\{S\}} \exp\left[ \frac{K}{2N} \left( \sum_i S_i \right)^2 + h \sum_i S_i \right] \tag{3}$$

In order to transform the quadratic term into al inear one we make use of the integral identity known as the *Hubbard-stratonovitch transformation* (we can do it in any dimension). The idea is to rewrite something as a square.

Let us define:

$$x \equiv \sum_i S_i \tag{4}$$

The following identity holds

$$e^{\frac{Kx^2}{2N}} = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{NK}{2}y^2 + Kxy} \, dy, \quad \text{with} \quad \text{Re}(K) > 0 \tag{5}$$

where $y$ is a random field that follows a random distribution. To show the identity above it is sufficient to complete the square

*Proof.* (of (5))
$$-\frac{NK}{2}y^2 + Kxy = -\frac{NK}{2}\left( y - \frac{x}{N} \right)^2 + \frac{Kx^2}{2N} \tag{6}$$

Then we integrate:

$$e^{\frac{Kx^2}{2N}} \int_{-\infty}^{+\infty} e^{-\frac{NK}{2}\left( y - \frac{x}{N} \right)^2} \, dy \stackrel{(a)}{=} e^{\frac{Kx^2}{2N}} \sqrt{\frac{2\pi}{NK}} \tag{7}$$

where in $(a)$ we have considered $z \equiv \left( y - \frac{x}{N} \right)$ , $dz = dy$:

$$\rightarrow \int_{-\infty}^{+\infty} e^{-\alpha z^2} \, dz = \sqrt{\frac{\pi}{\alpha}} \tag{8}$$

with $\alpha \equiv \frac{NK}{2}$. $\qquad\qquad\square$

By using (5) we have

$$Z_N(K, h) = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} dy \, e^{-\frac{NK}{2}y^2} \underbrace{\left[ \sum_{\{S\}} e^{(h+Ky)\sum_i S_i} \right]}_{Q_y} \tag{9}$$

Sometimes $y$ is called *auxiliary field*. If $y$ is a fluctuating external field with Gaussian distribution

$$Q_y = \sum_{\{S\}} e^{(h+Ky)\sum_i S_i} = \prod_i^N \left( \sum_{S_i=\pm 1} \exp[(h + Ky)S_i] \right)$$
$$= (2\cosh(h + Ky))^N \tag{10}$$

The partition function becomes

$$Z_N(K,h) = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}y \, e^{-\frac{NK}{2}y^2} (2\cosh(h+Ky))^N = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}y \, e^{N\alpha(K,h,y)} \tag{11}$$

where

$$\alpha(K,h,y) = \ln\left[2\cosh(h+Ky)\right] - \frac{K}{2}y^2 \tag{12}$$

*Remark.* In the limit $N \to \infty$ the integral can be computed exactly by the *saddle point method*. We can replace the medium of the integral with the maximum of the integrand, we say that all the information is coming only from a bit of information. Replace the all integral with the integrand computed where it is maximum. That is an approximation and we are loosing information, it depends on the form of the function. For example, for a delta function it works better. In general:

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}y \to f(\bar{x}) \tag{13}$$

where $\bar{x} = \max_x f(x)$.

Indeed as $N \to \infty$, since the integrand is $\exp(N\alpha(K,h,y))$, the integral is dominated by the global maximum in $y$ of the function $\alpha(K,h,y)$.

$$Z_N(K,h) \overset{N\gg 1}{\approx} \sqrt{\frac{NK}{2\pi}} \max_y \left[ e^{N\alpha(K,h,y)} \right] \tag{14}$$

Let $y_s$ be the value of $y$ at which

$$\alpha(K,h,y_s) = \max_y \alpha(K,h,y) \tag{15}$$

therefore

$$Z_N(K,h) \overset{N\gg 1}{\approx} \sqrt{\frac{NK}{2\pi}} e^{N\alpha(K,h,y_s)} \tag{16}$$

When we are ample to compute the $y_s$ we can do this approximation and

$$f_b(K,h) = \lim_{N\to\infty} \frac{1}{N}(-k_B T \log Z_N) = -k_B T \alpha(K,h,y_s) \tag{17}$$

Looking for $y_s$ we consider the condition $\frac{\partial \alpha}{\partial y} = 0$:

$$\frac{\partial \alpha}{\partial y} = \frac{\sinh(h+Ky)K}{\cosh(h+Ky)} - Ky = 0 \quad \Rightarrow y_s = \tanh(h + Ky_s) \tag{18}$$

The last one is an implicit equation that can be solved graphically as a function of $K$ and $h$.

The magnetization in the $N \to \infty$ limit is given by

$$m = -\left(\frac{\partial f}{\partial H}\right)_T = \lim_{N\to\infty} \frac{1}{\beta N} \frac{\partial \ln Z_N(K,h)}{\partial H}$$
$$= \frac{\partial \alpha(K,h,y_s)}{\partial h} + \frac{O(\log N)}{N} = \frac{2\sinh(Ky_s+h)}{2\cosh(ky_s+h)} \tag{19}$$
$$= \tanh(Ky_s + h)$$

At the end $m \equiv y_s$

$$m = \tanh(h + Km) \tag{20}$$

that is a self consistent equation. We have solved analitically this problem.

# Chapter 1

# Mean field theories of phase transitions adn variational mean field

## 1.1 Mean field theories

Increasing the dimension of the systems the effort to solve analitically the problems increase , in fact we have seen that

- In $D = 1$: many (simple) models can be solved exactly using techniques such as the transfer matrix method.

- In $D = 2$: few models can still be solved exactly (often with a lot of effort).

- In $D = 3$: almost no model can be exactly solved.

so, approximations are need. The most important and most used one is the *mean field approximation*. The idea is to try to simplify the problem by neglectic the correlation between the fluctuations of the order parameter. It is equivalent to a statistical indipendence of the microscopic degrees of freedom.

The mean field approximation has different names depending on the system considered:

- Magnetic: Weiss theory.

- Fluids: Van der Walls.

- Polymers: Flory's theory.

### 1.1.1 Mean field for Ising model

$$Z_N(T, H, \{J_{ij}\}) = \sum_{\{S\}} e^{\beta\left[\frac{1}{2}\sum_{\langle ij \rangle} J_{ij} S_i S_j + \sum_i H_i S_i\right]} = \exp(-\beta F_N(T, H, \{J_{ij}\})) \quad (1.1)$$

The magnetization per spin is $m \equiv \langle S_i \rangle$ and let us consider $H_i = H$ uniform. Consider the idenity

$$\begin{aligned}
S_i S_j &= (S_i - m + m)(S_j - m + m) \\
&= (S_i - m)(S_j - m) + m^2 + m(S_j - m) + m(S_i - m)
\end{aligned} \quad (1.2)$$

*Remark.* The term

$$(S_i - m)(S_j - m) = (S_i - \langle S_i \rangle)(S_j - \langle S_j \rangle) \tag{1.3}$$

measures correlation between fluctuations.

The mean field approximation consists in neglecting this term obtaining

$$S_i S_j \approx m^2 + m(S_i - m) + m(S_j - m) \tag{1.4}$$

We have:

$$\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j \overset{MF}{\approx} \frac{1}{2} \sum_{i,j} J_{ij} \left[ -m^2 + m(S_i + S_j) \right] \tag{1.5}$$

The term

$$\frac{1}{2} \sum_{i,j} J_{ij} m(S_i + S_j) = \frac{2}{2} \sum_{i,j} J_{ij} m S_i \tag{1.6}$$

and assuming ($n.n.$ = neirest neighbour)

$$\sum_{j\, n.n.\, of\, i} J_{ij} = z \hat{J}_i \simeq z \hat{J} \tag{1.7}$$

where $z$ is the coordination number of the underlying lattice (hypercubic lattice $z = 2D$). Therefore, (1.6) became

$$\frac{2}{2} \sum_{i,j} J_{ij} m S_i = m \hat{J} z \sum_i S_i \tag{1.8}$$

$$-\mathcal{H} = \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j + H \sum_i S_i \simeq -\frac{1}{2} N \hat{J} z m^2 + (\hat{J} z m + H) \sum_{i=1}^{N} S_i \tag{1.9}$$

Let us do the same story over j for the sum over i:

$$
\begin{aligned}
Z_N(T, H, \hat{J}) &= e^{-\frac{\beta m^2 N}{2} z \hat{J}} \sum_{\{S\}} e^{\beta(\hat{J} z m + H) \sum_i S_i} \\
&= e^{-\frac{\beta m^2 N}{2} z \hat{J}} \left( \sum_{S = \pm 1} \exp\left( \beta(\hat{J} z m + H) S \right) \right)^N \\
&= e^{-\frac{\beta m^2 N}{2} z \hat{J}} \left( 2 \cosh[\beta(\hat{J} z m + H)] \right)^N
\end{aligned}
\tag{1.10}
$$

The free energy per spin is

$$
\begin{aligned}
f_b(T, H, \hat{J}) &= -\frac{1}{N} k_B T \log Z_N(T, H, \hat{J}) \\
&= \frac{1}{2} \hat{J} z m^2 - k_B T \log\left[ \cosh\left( \beta(\hat{J} z m + H) \right) \right] - k_B T \ln 2
\end{aligned}
\tag{1.11}
$$

*Remark.* We are replacing the interaction of the J with a field close to the $S_i$. We called $\hat{J} z m = H_{eff}$, the mean field!

In order to be a self-consistent equation has to satisfy the thermodynamic relation:

$$m = -\left( \frac{\partial f}{\partial H} \right)_T \quad \Rightarrow m = \tanh\left( \beta(\hat{J} z m + H) \right) \tag{1.12}$$

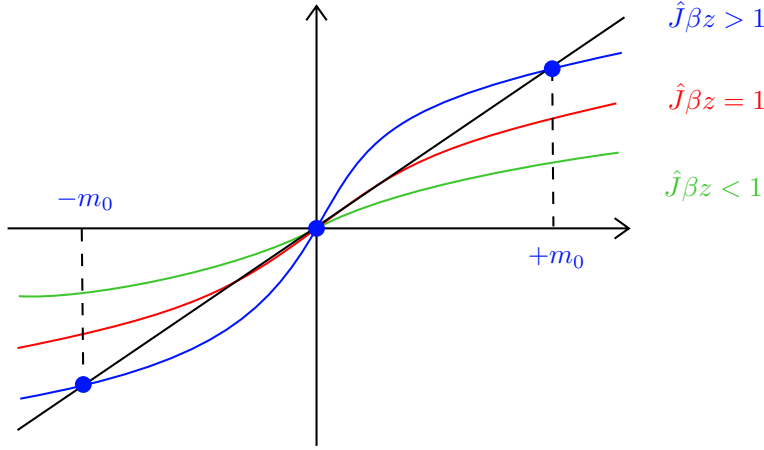*Remark.* The results it is similar to the Ising with infinite range ($\hat{J} z \leftrightarrow J$).

**Figure 1.1**

If $H = 0$, we have

$$m = \tanh\left(\beta(\hat{J}zm)\right) \tag{1.13}$$

and the graphical solution is shown in Figure 1.1 (hyperbolic function).

In particular:

- case $\beta\hat{J}z > 1$: there are three solutsions, one at $m = 0$ and two symmetric at $m = \pm m_0$. Magnetization is $\neq 0 (= |m_0|)$ for $H = 0$ (*ordered phase*). The two solution are symmetric because they are related by the $\mathbb{Z}^2$ symmetry;

- case $\beta\hat{J}z < 1$: single solution at $m = 0$ (*disordered or paramagnetic phase*);

- case $\beta\hat{J}z = 1$: the three solutions coincide at $m = 0$.

The condition $\beta_c\hat{J}z = 1$, define the critical point. The critical temperature $T_c$ is given by

$$\frac{z\hat{J}}{k_B T_c} = 1 \Rightarrow T_c = \frac{z\hat{J}}{k_B} \neq 0! \tag{1.14}$$

*Remark.* $T_c$ depends on $z$ and hence on $D$.

### 1.1.2 Free-energy expansion for $m \sim 0$

The critical point is characterized by the order parameter that is zero. Now we want to expand the free energy around the critical point. Let us put $H = 0$.

$$f(m, 0, T, \hat{J}) = \frac{1}{2}\hat{J}zm^2 - k_B T \ln\left[\cosh\left(\beta\hat{J}zm\right)\right] \tag{1.15}$$

Define $x \equiv \beta\hat{J}zm \simeq 0$, so

$$\cosh(x) = 1 + \underbrace{\frac{x^2}{2} + \frac{x^4}{4!}}_{t \simeq 0} + \dots \tag{1.16}$$

$$\log(1 + t) \simeq t - \frac{1}{2}t^2 \tag{1.17}$$

$$\log(\cosh x) \simeq \frac{x^2}{2} + \frac{x^4}{4!} - \frac{1}{2}\frac{x^4}{4} + O(x^6) = \frac{x^2}{2} - \frac{x^4}{12} + O(x^6) \tag{1.18}$$

This gives

$$f(m, 0, T, \hat{J}) = const + \frac{A}{2}m^2 + \frac{B}{4}m^4 + O(m^6) \tag{1.19}$$

with

$$A = \hat{J}z\left(1 - \beta\hat{J}z\right) \tag{1.20a}$$

$$B = \beta^3 \frac{(\hat{J}z)^4}{3} > 0 \tag{1.20b}$$

- case $\beta\hat{J}z > 1 \Rightarrow A < 0$: two stable symmetric minima at $m = \pm m_0$. Coexistence between the two ordered phases.
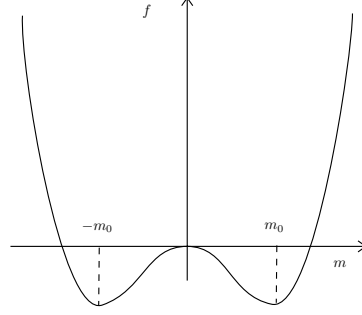


**Figure 1.2**

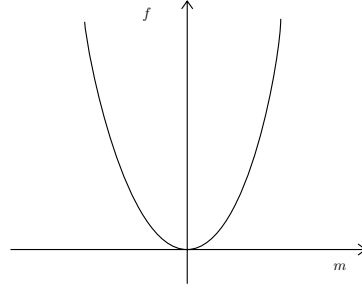- case $\beta\hat{J}z < 1 \Rightarrow A > 0$: one minimum at $m = 0$.



**Figure 1.3**

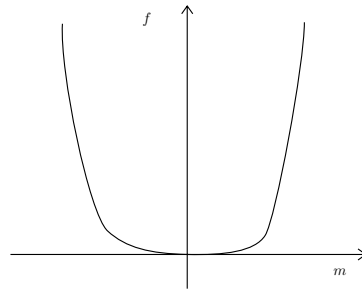- case $\beta\hat{J}z = 1 \Rightarrow A = 0$: 3 minima coincide at $m = 0$.



**Figure 1.4**