Chapter 1

Spontaneous symmetry breaking

1.1 Spontaneous symmetry breaking

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When we talk about a broken symmetry, we oftene refer to a situation as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{1.1}$$

where $\mathcal{H}_0$ is invariant under the group $\mathcal{G}$ and $\mathcal{H}_1$ is invariant under a subgroup $\mathcal{G}' \subset \mathcal{G}$.

Example 1 (Ising with magnetic field)**.**

$$\mathcal{H} = J \sum_{\langle ij \rangle} S_i S_j + \sum_i H_i S_i \tag{1.2}$$

The second term, $\mathcal{H}_1$, breaks the $\mathbb{Z}^2$ symmetry satisfied by the 1^{st} alone.

Example 2. In quantum mechanics: hydrogen atom in presence of an electric field $\vec{\mathbf{E}}$ (Stark effect) or a magnetic one, $\vec{\mathbf{B}}$, (Zeeman effect). If $\mathcal{H}_1$ is small, the original symmetry is weakly violated and perturbativ approaches are often used.

In all the above examples, one says that the symmetry is broken explicitly.

Definition 1 (Spontaneous symmetry breaking)**.** The Hamiltonian maintains the original symmetry but the variables used to describe the system become asymmetric.

At this point it is convenient to distinguish between

- Discrete symmetries: examples are $\mathbb{Z}^2$, $\mathbb{Z}_q$.
- Continuous symmetries: examples are xy, $O(n)$.

Let us consider first the discrete ones by focusing on $\mathbb{Z}^2$ (Ising).

If $H = 0$, $\mathcal{H}_{Ising}$ is invariant with respect to the change $S_i \to -S_i$, hence the discrete group is

$$\mathcal{G} = \mathbb{Z}^2 \tag{1.3}$$

A Ginzburg-Landau theory of the Ising is given by

$$-\beta\mathcal{H}(\Phi) = \int \mathrm{d}^D\vec{\mathbf{x}} \left[\frac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + \frac{r_0}{2}\Phi^2 + \frac{u_0}{4}\Phi^4 - h\Phi\right] \tag{1.4}$$

and

$$Z(r_0, u_0, h) = \int \mathrm{D}\Phi e^{-\beta\mathcal{H}(\Phi)} \tag{1.5}$$

The symmetry is $\Phi \to -\Phi$ if $h = 0$. Consider the saddle point equation of state

$$-\boldsymbol{\nabla}^2\Phi + r_0\Phi + u_0\Phi^3 = h \tag{1.6}$$

If h does not depend on $\vec{\mathbf{x}}$, uniform solution ($\boldsymbol{\nabla}\Phi = 0$).

The saddle point is equivalent to find the uniform value Φ_0 that is the extrema of the potential

$$V(\Phi) = \frac{1}{2} r_0 \Phi^2 + \frac{u_0}{4} \Phi^4 - h\Phi \tag{1.7}$$

For $h = 0$, $V' = (r_0 + u_0 \Phi^2)\Phi = 0$. Remembering that $r_0 \propto (T - T_c)$, we have two cases

1. Case $T > T_c$ ($r_0 > 0$): there is only one solution $\Phi_0 = 0$.

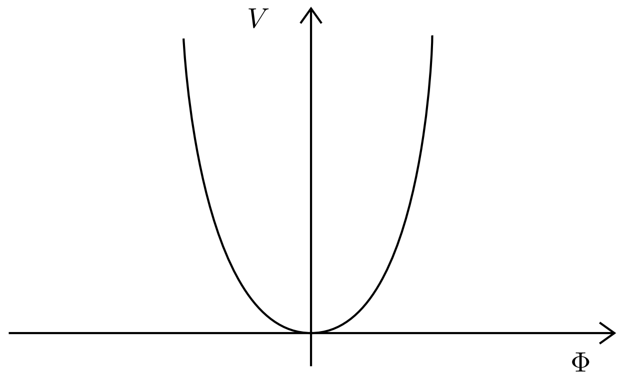

Figure 1.1: Description.

2. Case $T < T_c$ ($r_0 < 0$): there are two solutions $\Phi_0 = \pm\sqrt{-\frac{r_0}{u_0}}$.

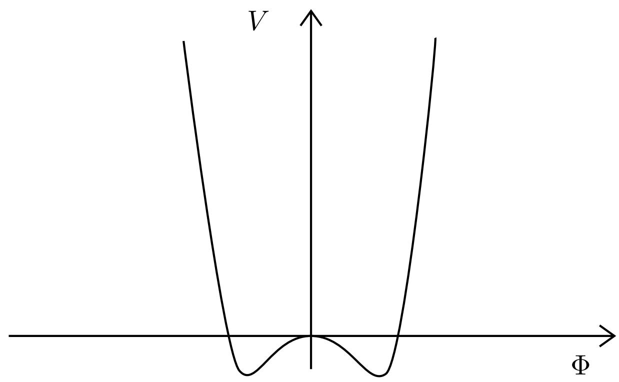

Figure 1.2: Description.

Remark. The two solution $\pm\Phi_0$ are related by the transformation $\in \mathbb{Z}^2$: $\Phi \to -\Phi$.

Remark. For $T < T_c$ the two states (phases) $\pm\Phi_0$ have a lower symmetry than the state $\Phi_0 = 0$.

Remark. If the thermal fluctuations $\delta\Phi$ are sufficiently strong to allow passages between the two states $\pm\Phi_0$ at $T < T_c$, we have $\langle\Phi\rangle = 0$ (preserves states).

However, for $T < T_c$ and $N \to +\infty$, transition between the two states will be less and less probable and the system will be trapped into one of the two states ($\pm\Phi_0$). The system choose spontaneously one of the two less symmetric state. Therefore, its physics is not any more described by Φ but the fluctuations $\delta\Phi$ around the chosen minimum Φ_0. There is spontaneous symmetry breaking.

The variable Φ is not any more symmetric and one has to look at $\Phi \to \Phi_0 + \delta\Phi$, where $\delta\Phi$ is a new variable!

1.2 Spontaneous breaking of continuous symmetries and the anset of Goldstone particles

Let us start with a simple model in which the order parameter is a scalar complex variable

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}} \tag{1.8}$$

and with an $\mathcal{H}$ that is invariant with respect to a global continuous transformation.

The simplest model in statistical mechanics is the XY model with $O(2)$ symmetry or a GL model for a superfluid or superconductor (no magnetic field)

$$\mathcal{H}_{eff} = \int \mathrm{d}^D\vec{\mathbf{x}} \left[\boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi^* + \frac{r_0}{2}\Phi^*\Phi + \frac{u_0}{4}(\Phi^*\Phi)^2\right] \tag{1.9}$$

where

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{\sqrt{2}}[\Phi_1(\vec{\mathbf{x}}) + i\Phi_2(\vec{\mathbf{x}})] \tag{1.10}$$

or

$$\Phi(\vec{\mathbf{x}}) = \psi(\vec{\mathbf{x}})e^{i\alpha(\vec{\mathbf{x}})} \tag{1.11}$$

- Superfluid: Φ macroscopic wave function of the Bose condensate (density of superfluid $n = \left|\Phi^2\right|$).
- Superconductor: Φ single particle wave function describing the position of the centre of mass of the Cooper pair.

1.2.1 Quantum relativistic case (field theory)

The analog of $\mathcal{H}$ is the action

$$S(\Phi) = \int \mathrm{d}^D\vec{\mathbf{x}}\, \mathcal{L}(\Phi) \tag{1.12}$$

where

$$\mathcal{L}(\Phi) = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - \frac{r_0}{2}\Phi\Phi^* - \frac{u_0}{4}(\Phi\Phi^*)^2 \tag{1.13}$$

It describes a scalar complex (i.e. charged) muonic field with mass m. Note that in this case $r_0 > 0$ and $m \equiv \sqrt{r_0}$. The term $(\Phi\Phi^*)^2$ means self-interaction with strenght $\lambda \equiv u_0$.

In all cases, the original symmetry is $U(1)$, i.e. both $\mathcal{H}$ and $\mathcal{L}$ are invariant with respect to the transformation

$$\Phi \to e^{i\theta}\Phi, \quad \Phi^* \to e^{-i\theta}\Phi^* \tag{1.14}$$

Remark. The phase θ does not depend on $\vec{\mathbf{x}}$ (global symmetry).

In components (1.14) become

$$\begin{cases} \Phi_1 \to \Phi_1\cos\theta - \Phi_2\sin\theta \\ \Phi_2 \to \Phi_2\cos\theta + \Phi_1\sin\theta \end{cases} \tag{1.15}$$

$$(\Phi_1', \Phi_2') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{1.16}$$

Let us focus first on the statistical mechanics model and to the most interesting case of $r_0 < 0$.

In components $\mathcal{H}$ becomes

$$\mathcal{H} = \int \mathrm{d}^D\vec{\mathbf{x}} \left[(\boldsymbol{\nabla}\Phi_1)^2 + (\boldsymbol{\nabla}\Phi_2)^2 \right] + \int \mathrm{d}^D\vec{\mathbf{x}}\, V(\Phi_1, \Phi_2) \tag{1.17}$$

where

$$V(\Phi_1, \Phi_2) = \frac{r_0}{2}\left(\Phi_1^2 + \Phi_2^2\right) + \frac{u_0}{4}\left(\Phi_1^2 + \Phi_2^2\right)^2 \tag{1.18}$$

It is the mexican hat potential, shown in Figure 1.3

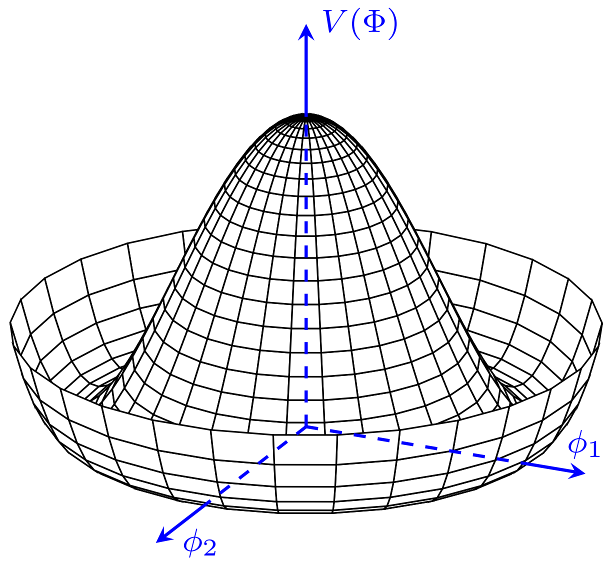

Figure 1.3: Case $r_0 < 0$.

For $r_0 < 0$, there is a uniform solution ($\boldsymbol{\nabla}\Phi_1 = \boldsymbol{\nabla}\Phi_2 = 0$). Let $S = \sqrt{\Phi_1^2 + \Phi_2^2}$,

$$V(S) = \frac{r_0}{2}S^2 + \frac{u_0}{4}S^4 \tag{1.19}$$

$$\frac{\mathrm{d}V(S)}{\mathrm{d}S} = r_0 S + u_0 S^3 = 0 \tag{1.20}$$

There is 1 maximum at $S = 0$ and minima for $S^2 = -\frac{r_0}{u_0}$

For $r_0 < 0$, $\mathcal{H}$ displays minima when

$$\Phi_1^2 + \Phi_2^2 \equiv v^2 = -\frac{r_0}{u_0} \tag{1.21}$$

On the $2D$ plane (Φ_1, Φ_2) the minima lie on the circle of radius

$$v = \sqrt{-\frac{r_0}{u_0}} \tag{1.22}$$

The spontaneous symmetry breaking occurs when the system "chooses" one of the infinite available minima.

In our example, suppose that the chosen minimum is

$$\Phi_1 = v = \sqrt{-\frac{r_0}{u_0}}, \quad \Phi_2 = 0 \tag{1.23}$$

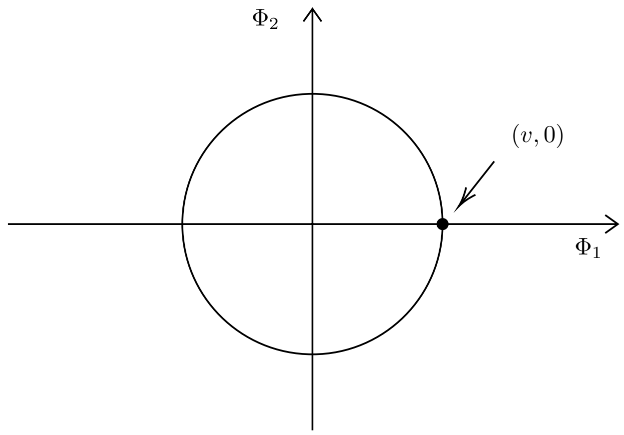

Figure 1.4: Description.

1.3 Interpretation in relativistic quantum mechanics

1. $r_0 < 0$ corresponds to an imaginary mass. This is because to move away from $\Phi = 0$, the system experiences a negative resistence in both directions, being $\Phi = 0$ a relative local minimum.

2. The minimum has the lowest energy and therefore it must correspond to the empty state. In this case, however, there is an infinite number of empty states!

Summarizing: the starting Hamiltonian (or Lagrangian) is invariant with respect to $U(1)$ but the one that describes the fluctuation dynamics around one of the chosen minimum state is not invariant with respect to $U(1)$.

Let us now write the Lagrangian with respect to the fluctuations of Φ_1 and Φ_2 around the chosen state

$$\begin{cases} \Phi_1 = v + \delta\Phi_1 \\ \Phi_2 = 0 + \delta\Phi_2 \end{cases} \tag{1.24}$$

or

$$\Phi = v + (\delta\Phi_1 + i\delta\Phi_2) \tag{1.25}$$

Note that, since

$$\begin{cases} \delta\Phi_1 = \Phi_1 - v \\ \delta\Phi_2 = \Phi_2 \end{cases} \tag{1.26}$$

we have

$$\langle\delta\Phi_1\rangle_{\Phi_0} = \langle\delta\Phi_2\rangle_{\Phi_0} = 0 \tag{1.27}$$

As expected the expectation of the empty state is back to be zero.

For the quantum relativistic Lagrangian, let us write

$$r_0 \to m^2, \quad u_0 \to \lambda, \quad v^2 = -\frac{m^2}{\lambda} \tag{1.28}$$

$$
\begin{aligned}
\mathcal{L} =& -\frac{1}{2}\partial_\mu(\delta\Phi_1 + i\delta\Phi_2)\partial_\mu(\delta\Phi_1 - i\delta\Phi_2) \\
&- \frac{m^2}{2}(v + \delta\Phi_1 + i\delta\Phi_2)(v + \delta\Phi_1 - i\delta\Phi_2) \\
&- \frac{\lambda}{4}[(v + \delta\Phi_1 + i\delta\Phi_2)(v + \delta\Phi_1 - i\delta\Phi_2)]^2 \\
=& -\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1) - \frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2) \\
&- \frac{m^2}{2}\left(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2\right) \\
&- \frac{\lambda}{4}\left(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2\right)^2
\end{aligned}
\tag{1.29}
$$

Since $m^2 = -v^2\lambda$,

$$
\begin{aligned}
\mathcal{L} =& -\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1) - \frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2) \\
&+ \frac{\lambda v^2}{2}\left(v^2 + 2v\delta\Phi_1 + \delta\Phi_1^2 + \delta\Phi_2^2\right) \\
&- \frac{\lambda}{4}\Big(v^4 + 4v^2\delta\Phi_1^2 + \left(\delta\Phi_1^2 + \delta\Phi_2^2\right)^2 4v^3\delta\Phi_1 + 2v^2\left(\delta\Phi_1^2 + \delta\Phi_2^2\right) + 4v\delta\Phi_1\left(\delta\Phi_1^2 + \delta\Phi_2^2\right)\Big)
\end{aligned}
\tag{1.30}
$$

Neglecting the constant terms in v

$$
\begin{aligned}
\mathcal{L}(\delta\Phi_1, \delta\Phi_2) =& -\frac{1}{2}(\partial_\mu\delta\Phi_1)^2 - \frac{1}{2}(\partial_\mu\delta\Phi_2)^2 \\
&- \lambda v^2\delta\Phi_1^2 - v\lambda\delta\Phi_1\Big((\delta\Phi_1)^2 + (\delta\Phi_2)^2\Big) \\
&- \frac{\lambda}{4}\Big((\delta\Phi_1)^2 + (\delta\Phi_2)^2\Big)^2
\end{aligned}
\tag{1.31}
$$

Remark. The term $-\lambda v^2\delta\Phi_1^2$ indicates that the field $\delta\Phi_1$ (related to the transversal fluctuations) has a null empty state ($\langle\delta\Phi_1\rangle = 0$) and a mass M such that:

$$
M^2 = 2\lambda v^2 = -2r_0 \tag{1.32}
$$

Therefore, it represents a real, massive, mesonic scalar field that is physically accettable.

However, $\mathcal{L}$ is not any more invariant under the transformation $\delta\Phi_1 \to -\delta\Phi_1$.

Remark. The field $\delta\Phi_2$ has no mass! It describes the fluctuations along the circle where the potential V is in its minimum which imples no dynamical inertia, that implies no mass!

So, starting with one complex scalar field $\Phi(\vec{\mathbf{x}})$ having mass m, when $m^2 < 0$one gets a real scalar field $\delta\Phi_1$ with mass $M = \sqrt{-2m^2}$ and a second scalar field $\delta\Phi_2$ that is massless. This is called the *Goldstone boson.*

Theorem 1.3.1. *If a continuous symmetry is spontaneously broken and there are no long range interactions, exists an elementary eccitation with zero momentum or particle of zero mass called Goldstone boson.*

More generally, let $\mathcal{P}$ be a subgroup of $\mathcal{G}$. If $\mathcal{G}$ has N indipendent generators and $\mathcal{P}$ has M indipendent generators, if $\mathcal{P}$ is the new (lower) symmetry, therefore exist $N - M$ Goldstone bosons.

In the previous case $\mathcal{G} = U(1) \Rightarrow N = 1$ whereas $M = 0$ (we have chosen a specific minimum).

Example 3. XY model in statistical mechanics:

- $\delta\Phi_1$: fluctuation of the modulus of m.
- $\delta\Phi_2$: fluctuations of the spin directions $\Rightarrow$ spin waves.

Remark. In particle physics the presence of Goldstone bosons brings a serious problem in field theory since the corresponding particles are not observed!

Higgs-Englert-Brout (1964)

Higgs mechanism gives back the mass to the Goldstone particles. The basic idea is that the Goldstone theorem that works for a continuous global symmetry it can fail for loacl gauge theories!

1.4 Spontaneous symmetry breaking in Gauge symmetries

Statistical mechanics, Gl model for superconductors in presence of a magnetic field (*Meissner effect*, i.e. the magnetic induction $\vec{\mathbf{B}} = 0$ inside the superconductor).

$$\mathcal{H}(\Phi) = \int \mathrm{d}^D\vec{\mathbf{x}} \left[\frac{1}{2}B^2 + \left|\left(\vec{\boldsymbol{\nabla}} - 2i\vec{\mathbf{A}}\right)\Phi\right|^2\right] + \frac{r_0}{2}\Phi^*\Phi + \frac{u_0}{4}(\Phi^*\Phi)^2 - \vec{\mathbf{B}}\cdot\vec{\mathbf{H}} \tag{1.33}$$

where $\frac{B^2}{2}$ is the energy of the magnetic field $\vec{\mathbf{B}}$ and $\vec{\boldsymbol{\nabla}} \to \left[\vec{\boldsymbol{\nabla}} + iq\vec{\mathbf{A}}\right]$ is the minimal coupling. Consider $\vec{\mathbf{H}}$ the external magnetic field

$$\vec{\mathbf{B}} = \vec{\mathbf{H}} + \vec{\mathbf{M}} \tag{1.34}$$

is the induction field.

Normal conductor corresponding to $\Phi_0 = 0$, that implies $\vec{\mathbf{B}} = \vec{\mathbf{H}}$. For a superconfuctor we have $\Phi \neq 0$, a spontaneous symmetry breaking.