Lecture 19.
Wednesday 18th December, 2019.
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0.1 Scaling theory

It is used whenever you have a collective behaviour. The length scale of the problem are a, L, ξ, but ξ is the only relevant length scale in the problem.

Which are the experimental data which gives us this ideas? What you can see from experiment is figure 1

Widom $\to$ static scaling theory $\to$ homogeneous functions.

0.1.1 Single variable r

$f(r)$ is homogeneous in r, if $\forall \lambda \in \mathbb{R}$ we have $f(\lambda r) = \lambda f(r)$. More general,

$$f(\lambda r) = g(\lambda) f(r) \tag{1}$$

Example 1.

$$f(r) = Br^2 \tag{2}$$

$$f(\lambda r) = B(\lambda r)^2 = \lambda^2 f(r) \quad \Rightarrow g(\lambda) = \lambda^2 \tag{3}$$

Sine it is valid for any λ, you can choose also a λ in that way

$$f(r) = f(\lambda r_0) = g(\lambda) f(r_0) \tag{4}$$

Theorem 0.1.1.

$$g(\lambda) = \lambda^p \tag{5}$$

where p is the degree of the homogeneity of the function.

We can make it for any variable, not only for a single one.

0.1.2 Generalized homogeneous functions

We are discussing $f(x, y)$, that is a generalized homogeneous function if $f(\lambda^a x, \lambda^b y) = \lambda f(x, y)$. In general any polynomial is a generalized homogeneous function. If we choose $\lambda^p \equiv s$, we have

$$f(s^{a/p} x, s^{b/p} y) = s f(x, y) \tag{6}$$

Consider $t \equiv \frac{T - T_c}{T_c}, h = \frac{H - H_c}{H_c}$

$$f(T, H) = f_{ANA}(T, H) + f_{SING}(t, h) \tag{7}$$

where f_{ANA} is an analytic term and f_{SING} diverges, has a singularity.

The singular part of the free energy

$$f_s(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda f_s(t, h) \tag{8}$$

where $\forall \lambda \in \mathbb{R}$.

Another important feature, choose λ as

$$\lambda = h^{-1/p_2} \quad \Rightarrow f_s(t, h) = h^{1/p_2} f_s(h^{-p_1/p_2} t, 1) \tag{9}$$

$$\Delta \equiv \frac{p_1}{p_2} \tag{10}$$

is called the *gap exponent*.

M is the first derivative of f with respect to H.

$$\lambda^{p_2} M_s(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda M_s(t, h) \tag{11}$$

so you have the same story for the magnetization. Consider $h = 0$ and $t \to 0^-$, we have

$$M_s(t) \sim (-t)^\beta \tag{12}$$

starting from this one try to figure out what is happening at this level.

$$M_s(t,0) = \lambda^{p_2-1} M_s(\lambda^{p_1} t, 0) \tag{13}$$

$$\lambda^{p_1} t = -1 \quad \Rightarrow \lambda = (t)^{-1/p_1} \tag{14}$$

so

$$M_s(t,0) = -(t)^{\frac{1-p_2}{p_1}} M_s(-1,0) \tag{15}$$

$$\beta = \frac{1-p_2}{p_1} \tag{16}$$

For δ, we have $T = T_c$ and $h \to 0^+$, so the magnetization goes like $M_s \sim h^{1/\delta}$.

$$M_s(0,h) = \lambda^{p_2-1} M_s(0, \lambda^{p_2} h) \tag{17}$$

Now we want

$$\lambda^{p_1} h = 1 \quad \Rightarrow \lambda = h^{-1/p_2} \tag{18}$$

$$M_s(0,h) = h^{\frac{1-p_2}{p_2}} M_s(0,+1) \tag{19}$$

$$\delta = \frac{p_2}{1-p_2} \tag{20}$$

From this you have a very simple relation

$$p_1 = \frac{1}{\beta(\delta+1)}, \quad p_2 = \frac{\delta}{\delta+1} \tag{21}$$