Lecture 8. Wednesday 6^{th} November, 2019. Compiled: Tuesday 10^{th} December, 2019.

Since $\sum_k \lambda_k^N = Z_N$ for $k = +, -, 1, \dots, n$:

$$\langle S_1 S_R \rangle_N = \frac{\sum_{ij} \langle t_j | \mathbb{S}_1 | t_i \rangle \lambda_i^{R-1} \langle t_i | \mathbb{S}_R | t_j \rangle \lambda_j^{N-R+1}}{\sum_{k=1}^n \lambda_k^N} \tag{1}$$

If we now multiply and divide by λ_+^N, we get

$$\langle S_1 S_R \rangle_N = \frac{\sum_{ij} \langle t_j | \mathbb{S}_1 | t_i \rangle (\lambda_i/\lambda_+)^{R-1} \langle t_i | \mathbb{S}_R | t_j \rangle (\lambda_j/\lambda_+)^{N-R+1}}{\sum_{k=1}^n (\lambda_k/\lambda_+)^N} \tag{2}$$

Remark. In the thermodynamic limit $N \to \infty$, only the terms with $j = +$ and $k = +$ survive in the sum. Remind that R is fixed.

$$\langle S_1 S_R \rangle_N = \lim_{N\to\infty} \langle S_1 S_R \rangle_N = \sum_{i=\pm,1\dots n} \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \tag{3}$$

Rembember that $\lambda_+ > \lambda_T \geq \lambda_1 \dots \lambda_n$:

$$\langle S_1 S_R \rangle_N = \langle t_+ | \mathbb{S}_1 | t_+ \rangle \langle t_+ | \mathbb{S}_R | t_+ \rangle + \sum_{i\neq+}^n \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \tag{4}$$

Since one can prove, by a method entirely analogous to that followed above, that

$$\lim_{N\to\infty} \langle S_R \rangle_N = \langle t_+ | \mathbb{S}_R | t_+ \rangle \tag{5}$$

we obtain

$$\langle S_1 S_R \rangle = \langle S_1 \rangle \langle S_R \rangle + \sum_{i\neq+} \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \tag{6}$$

The correlation function then follows immediately as

$$\Gamma_R = \langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle = \sum_{i\neq+}^n \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \tag{7}$$

Remark. Γ_R depends only on the eigenvalues and eigenvectors of the transfer matrix $\mathbb{T}$ and by the values of the spins S_1 and S_R.

A much simpler formula is obtained for the correlation length (**??**). Taking the limit $R \to \infty$ the ratio (λ_-/λ_+) dominates the sum and hence

$$\begin{aligned}
\xi^{-1} &= \lim_{R\to\infty} \left\{ -\frac{1}{R-1} \log |\langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle| \right\} \\
&= \lim_{R\to\infty} \left\{ -\frac{1}{R-1} \log \left[\left(\frac{\lambda_-}{\lambda_+}\right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_- \rangle \langle t_- | \mathbb{S}_R | t_+ \rangle \right] \right\} \\
&= -\log \left[\left(\frac{\lambda_-}{\lambda_+}\right) \right] - \lim_{R\to\infty} \frac{1}{R-1} \log \langle t_+ | \mathbb{S}_1 | t_- \rangle \langle t_+ | \mathbb{S}_R | t_+ \rangle \\
&= -\log \left(\frac{\lambda_-}{\lambda_+}\right)
\end{aligned} \tag{8}$$

The important result is

$$\boxed{\xi^{-1} = -\log \left(\frac{\lambda_-}{\lambda_+}\right)} \tag{9}$$

It means that the correlation length does depend only on the ratio between the two largest eigenvalues of the transfer matrix $\mathbb{T}$.

0.0.1 Results for the *1-dimensional* Ising model

The transfer matrix is given by

$$\mathbb{T} = \begin{pmatrix} \exp(K+h) & \exp(-K) \\ \exp(-K) & \exp(K-h) \end{pmatrix} \tag{10}$$

Calculate the eigenvalues:

$$|\mathbb{T} - \lambda \mathbb{1}| = (e^{K+h} - \lambda)(e^{K-h} - \lambda) - e^{-2K} = 0 \tag{11}$$

The solutions are

$$\lambda_\pm = e^K \cosh(h) \pm \sqrt{e^{2K}\sinh^2(h) + e^{-2K}} \tag{12}$$

The free energy

The free energy is

$$\begin{aligned} f_b &\equiv \lim_{N\to\infty} \frac{-k_B T}{N} \log Z_N(K,h) \\ &= -k_B T \lim_{N\to\infty} \frac{1}{N} \log \left[\lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right) \right] \\ &= -k_B T \log \lambda_+ \end{aligned} \tag{13}$$

and inserting the explicit expression of λ_+ for the Ising model, we get

$$\begin{aligned} f_b &= -k_B T \log \left(e^K \cosh h + \sqrt{e^{2K}\sinh^2(h) + e^{-2K}} \right) \\ &= -K k_B T - k_B T \log \left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}} \right) \end{aligned} \tag{14}$$

Remark. Rembember that $K \equiv \beta J, h \equiv \beta H$.

Exercise 1. Check that if $h = 0$ we get back the expression found previously with the iterative method (what is the important of boundary conditions?).

Let us now consider the limits $T \to 0$ and $T \to \infty$ by keeping H fixed and J fixed.

- Case: $T \to 0 \Rightarrow K \to \infty, h \to \infty$.

$$e^{-4K} \overset{K\to\infty}{\longrightarrow} 0 \tag{15a}$$

$$\sqrt{\sinh^2 h} \overset{h\to\infty}{\sim} \sinh(h) \tag{15b}$$

  This implies that

$$\cosh(h) + \sinh h \sim \frac{2e^h}{2} \simeq e^h \tag{16}$$

  and

$$f \overset{\substack{h\to\infty \\ K\to\infty}}{\sim} -K k_B T - k_B T \log e^h \sim -J - H \quad const \tag{17}$$

  Therefore, as $T \to 0^+$, f goes to a constant that depends on J and H.

- Case: $T \to \infty \Rightarrow K \to 0, h \to 0$. In this case we suppose also that H and J that are fixed, are also finite.

$$e^{-4K} \simeq 1 \tag{18a}$$

$$\sqrt{\sinh^2 h + e^{-4K}} \sim \sqrt{1} \tag{18b}$$

  Since $\cosh h \overset{h\to 0}{\sim} 1$:

$$f_B \sim -K k_B T - k_B T \log(1+1) \sim -J - k_B T \ln 2 \tag{19}$$

  Therefore, as $T \to \infty$, the free energy goes linearly to zero, as in figure 1.

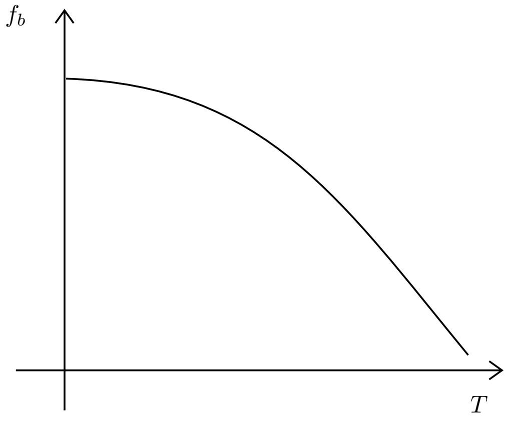

Figure 1: Plot of the free energy f_b in function of the temperature T. For $T \to 0$, the free energy becomes constant, while for $T \to \infty$ it goes linearly to zero.

The magnetization

This can be obtained by differentiating the negative of the free energy with respect to the magnetic field H:

$$m = -\frac{\partial f_b}{\partial H} = -\frac{1}{k_B T}\frac{\partial f_b}{\partial h} = \frac{\partial}{\partial h}\left[\log\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}\right)\right] \tag{20}$$

The result is

$$m = \frac{\sinh h + \frac{\sinh h \cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}}{\cosh h + \sqrt{\sinh^2 h + e^{-4K}}} \tag{21}$$

- Case: $T > 0$ fixed, $H \to 0 \leftrightarrow h \to 0$.

$$\sinh h \sim h \sim 0, \tag{22a}$$
$$\cosh h \sim 1 \tag{22b}$$

  In zero field $h \to 0$, we have $m \to 0$ for all $T > 0$. It means that there is no spontaneous magnetization!

The magnetic susceptibility

$$\chi_T \equiv \frac{\partial m}{\partial H} = \frac{1}{k_B T}\frac{\partial m}{\partial h} \tag{23}$$

If we consider the case $h \ll 1$, it is convenient first expand the (21) for $h \to 0$ and take the derivative to get χ_T.

Since $\sinh(h) \sim h + h^3$ and $\cosh(h) \sim 1 + h^2$, we have

$$m \overset{h \ll 1}{\approx} \frac{h(1+e^{2K})}{1+e^{-2K}} \tag{24}$$

If we now derive with respect to h

$$\chi_T = \frac{1}{k_B T}\frac{\partial m}{\partial h} \overset{h \ll 1}{\approx} \frac{1}{k_B T}\frac{(1+e^{2K})}{(1+e^{-2K})} \tag{25}$$

- Case: $T \to \infty \Leftrightarrow K \to 0$.

$$e^{2K} \simeq e^{-2K} \simeq 1 \tag{26}$$

  The *Curie's Law* for paramagnetic systems is:

$$\boxed{\chi_T \sim \frac{1}{k_B T}} \tag{27}$$

- Case: $T \to 0 \Leftrightarrow K \to \infty$.

$$e^{-2K} \simeq 0 \tag{28}$$

The *Curie's Law* for paramagnetic systems is:

$$\boxed{\chi_T \sim \frac{1}{k_B T} e^{2K} \sim \frac{1}{k_B T} e^{2J/k_B T}} \tag{29}$$

The correlation length

$$\xi^{-1} = -\log\left(\frac{\lambda_-}{\lambda_+}\right) = -\log\left[\frac{\cosh h - \sqrt{\sinh^2 h + e^{-4K}}}{\cosh h + \sqrt{\sinh^2 h + e^{-4K}}}\right] \tag{30}$$

For for $h = 0$, we have $\cosh h \to 1, \sinh h \to 0$:

$$\xi^{-1} = -\log\left[\frac{1 - e^{-2K}}{1 + e^{-2K}}\right] = -\log\left[\frac{1}{\coth K}\right] \tag{31}$$

Therefore:

$$\xi = \frac{1}{\log(\coth K)} \tag{32}$$

- Case: $T \to 0 \Leftrightarrow K \to \infty$.

$$\coth K = \frac{e^K + e^{-K}}{e^K - e^{-K}} \overset{K\to\infty}{\simeq} 1 + 2e^{-2K} + \ldots \quad \overset{K\to\infty}{\longrightarrow} 1 \tag{33}$$

It implies

$$\xi \overset{K\gg 1}{\sim} \frac{1}{\ln(1 + 2e^{-2K})} \sim \frac{e^{2K}}{2} \tag{34}$$

Hence

$$\xi \overset{T\to 0}{\sim} \frac{1}{2} e^{J/k_B T} \tag{35}$$

It diverges exponentially $\xi \to \infty$, as $T \to 0$.

- Case: $T \to \infty \Leftrightarrow K \to 0$.

$$\coth K = \frac{e^K + e^{-K}}{e^K - e^{-K}} \overset{K\to 0}{\simeq} \frac{1 + K + \frac{K^2}{2} + 1 - K + \frac{K^2}{2}}{1 + K + \frac{K^2}{2} - 1 + K - \frac{K^2}{2}} \sim \frac{2 + 2\frac{K^2}{2}}{2K} \sim \frac{1 + K^2}{K} \tag{36}$$

$$\xi^{-1} = \log(\coth K) \overset{K\to 0}{\sim} \ln\frac{1}{K} + \ln(1 + K^2) \sim +\infty \tag{37}$$

Therefore:

$$\xi \overset{K\to 0}{\longrightarrow} 0 \tag{38}$$

More precisely,

$$\xi \overset{K\to 0}{\sim} \frac{1}{\ln(1/K) + \ln(1 + K^2)} \overset{K\to 0}{\sim} -\frac{1}{\ln K} \tag{39}$$

0.1 Classical Heisenberg model for *d=1*

Suppose to study something different from the Ising model, we do not anymore assume spin that can assume values as -1 or $+1$, but spin that can assume a continuous value. This is the classical Heisenberg model.

Take a $d = 1$ dimensional lattice. In the classical Heisenberg model the spins are unit length vectors $\vec{\mathbf{S}}_i$, i.e. $\vec{\mathbf{S}}_i \in \mathbb{R}^3$, $\left|\vec{\mathbf{S}}_i\right|^2 = 1$ (continuous values on the unit sphere):

$$\vec{\mathbf{S}}_i = (S_i^x, S_i^y, S_i^z) \tag{40}$$

with periodic boundary condition: $\vec{\mathbf{S}}_{N+1} = \vec{\mathbf{S}}_1$.

Assuming $H = 0$, the model is defined through the following Hamiltonian::

$$-\beta\mathcal{H}(\{\vec{\mathbf{S}}\}) = K\sum_{i=1}^{N} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1} \quad (\longrightarrow \sum_i \vec{\mathbf{h}} \cdot \vec{\mathbf{S}}_i) \tag{41}$$

This model satisfies $O(3)$ symmetry. In the transfer matrix formalism:

$$Z_N(K) = \sum_{\{\vec{\mathbf{S}}\}} e^{-\beta\mathcal{H}} = \sum_{\{\vec{\mathbf{S}}\}} e^{K\sum_{i=1}^{N} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}} = \mathrm{Tr}\big(\mathbb{T}^N\big) \tag{42}$$

where $\left\langle \vec{\mathbf{S}}_i \middle| \mathbb{T} \middle| \vec{\mathbf{S}}_{i+1} \right\rangle = e^{K\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}}$.

Similarly to the Ising case:

$$\mathbb{T} = \sum_i |t_i\rangle\, \lambda_i\, \langle t_i| \tag{43}$$

and

$$\mathbb{T}_D = \mathbb{P}^{-1}\mathbb{T}\mathbb{P} \tag{44}$$

The problem is computing the eigenvalues λ_i of $\mathbb{T}$. Formally, we should find

$$\exp\Big[K\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2\Big] = \left\langle \vec{\mathbf{S}}_1 \middle| \mathbb{T} \middle| \vec{\mathbf{S}}_2 \right\rangle = \sum_{i \in \text{eigenvalues}} \lambda_i \left\langle \vec{\mathbf{S}}_1 \middle| t_i \right\rangle \left\langle t_i \middle| \vec{\mathbf{S}}_2 \right\rangle = \sum_i \lambda_i f_i(\vec{\mathbf{S}}_1) f^*(\vec{\mathbf{S}}_2) \tag{45}$$

Remark. We start by noticing that the term $e^{K\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2}$ is similar to the plane wave $e^{i\vec{\mathbf{q}} \cdot \vec{\mathbf{r}}}$, that in scattering problems is usually expanded in spherical coordinates. Plane wave can be expanded as a sum of spherical harmonics

$$e^{i\vec{\mathbf{q}} \cdot \vec{\mathbf{r}}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (i)^l j_l(qr) Y_{lm}^*(\hat{\mathbf{q}}) Y_{lm}(\hat{\mathbf{r}}) \tag{46}$$

where

$$j_l(qr) = -\frac{(i)^l}{2} \int_0^{\pi} \sin(\theta) e^{iqr\cos(\theta)} P_l(\cos(\theta))\, \mathrm{d}\theta \tag{47}$$

are the *spherical Bessel functions*, while the $P_l(\cos(\theta))$ are the *Legendre polynomial* of order l.

From a formal comparison we have

$$\vec{\mathbf{S}}_1 \leftrightarrow \hat{\mathbf{S}}_1, \qquad \begin{cases} i\vec{\mathbf{q}} \cdot \vec{\mathbf{r}} = iqr \\ K\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2 = K\left|\vec{\mathbf{S}}_1\right|\left|\vec{\mathbf{S}}_2\right| = K \end{cases} \tag{48}$$

multiplying by $(-i)$ we can write

$$qr = -iK\left|\vec{\mathbf{S}}_1\right|\left|\vec{\mathbf{S}}_2\right| = -iK \tag{49}$$

In our case we have $\hat{\mathbf{q}} = \vec{\mathbf{S}}_1, \hat{\mathbf{r}} = \vec{\mathbf{S}}_2$. Hence,

$$e^{K\vec{\mathbf{S}}_1\cdot\vec{\mathbf{S}}_2} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(i)^l j_l(-iK)Y_{lm}^*(\vec{\mathbf{S}}_1)Y_{lm}(\vec{\mathbf{S}}_2) = \sum_i \lambda_i f_i(\vec{\mathbf{S}}_1)f^*(\vec{\mathbf{S}}_2) \tag{50}$$

where

$$\lambda_i = \lambda_{lm}(K) = 4\pi(i)^l j_l(-iK) \tag{51}$$

Remark. Note that λ_i does not depend on m!

If $l = 0$, the largest eigenvalue is:

$$\lambda_+ = \lambda_0(K) = 4\pi j_0(-iK) = 4\pi\frac{\sin K}{K} \tag{52}$$

and

$$\lambda_- = \lambda_1(K) = 4\pi i j_1(-iK) = 4\pi\left[\frac{\cosh K}{K} - \frac{\sinh K}{K^2}\right] \tag{53}$$

Exercise 2. Given the largest eigenvalue λ_+:

$$\lambda_+ = 4\pi\frac{\sin(K)}{K} \tag{54}$$

find the bulk free energy density of the model and discuss its behaviour in the limits of low ($T \to 0$) and high ($T \to \infty$) temperatures.

How can we violate the hypothesis of the Perron-Frobenius theorem hoping to find a phase transition also in a $d = 1$ model? One of the hypothesis of the Perron-Frobenius theorem is the one in which $A_{ij} > 0$ for all i, j. Hence, one possibility is to build a model in which its transfer matrix has same A_{ij} that are equal to zero also for $T \neq 0$.