# Chapter 1

# Spontaneous symmetry breaking

## 1.1 Spontaneous symmetry breaking

***Lecture n.***
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When we talk about a broken symmetry, we often refer to a situation as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

where $\mathcal{H}_0$ is invariant under the group $\mathcal{G}$ and $\mathcal{H}_1$ is invariant under a subgroup $\mathcal{G}' \subset \mathcal{G}$.

**Example 1: Ising with magnetic field**

Let us consider the Hamiltonian for the Ising model with a magnetic field $H \neq 0$:

$$\mathcal{H} = J \sum_{\langle ij \rangle} S_i S_j + \sum_i H_i S_i$$

The second term, $\sum_i H_i S_i$, breaks the $\mathbb{Z}^2$ symmetry satisfied by the first alone.

**Example 2: Hydrogen atom with an external field**

An example, in quantum mechanics, is the hydrogen atom in presence of an electric field $\vec{\mathbf{E}}$ (Stark effect) or a magnetic one, $\vec{\mathbf{B}}$, (Zeeman effect). If $\mathcal{H}_1$ is small, the original symmetry is weakly violated and perturbative approaches are often used.

In all the above examples, one says that the symmetry is broken explicitly.

**Definition 1: Spontaneous symmetry breaking**

The Hamiltonian maintains the original symmetry but the variables used to describe the system become asymmetric.

At this point it is convenient to distinguish between

- **Discrete symmetries**: for instance $\mathbb{Z}^2$, $\mathbb{Z}_q$.
- **Continuous symmetries**: for instance $XY$, $O(n)$.

Let us consider first the discrete ones by focusing on the $\mathbb{Z}^2$ symmetry (Ising). As previously said, if $H = 0$, the Hamiltonian of the Ising model, $\mathcal{H}_{\text{Ising}}$, is invariant with respect to the change $S_i \to -S_i$, hence the discrete group is

$$\mathcal{G} = \mathbb{Z}^2$$

A Ginzburg-Landau theory of the Ising model is given by

$$\beta\mathcal{H}(\Phi) = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2}\left(\vec{\boldsymbol{\nabla}}\Phi\right)^2 + \frac{r_0}{2}\Phi^2 + \frac{u_0}{4}\Phi^4 - h\Phi\right] \tag{1.1}$$

Hence, the partition function can be computed as:

$$Z(r_0, u_0, h) = \int \mathcal{D}[\Phi] e^{-\beta \mathcal{H}(\Phi)} \tag{1.2}$$

If we have $h = 0$, the symmetry is $\Phi \to -\Phi$. The equation of state obtained with saddle point approximation is

$$h = -\vec{\nabla}^2 \Phi + r_0 \Phi + u_0 \Phi^3$$

If $h$ does not depend on $\vec{\mathbf{x}}$, i.e. $h(\vec{\mathbf{x}}) = h$, the last equation reduces to the equation of state of the Landau theory of uniform system:

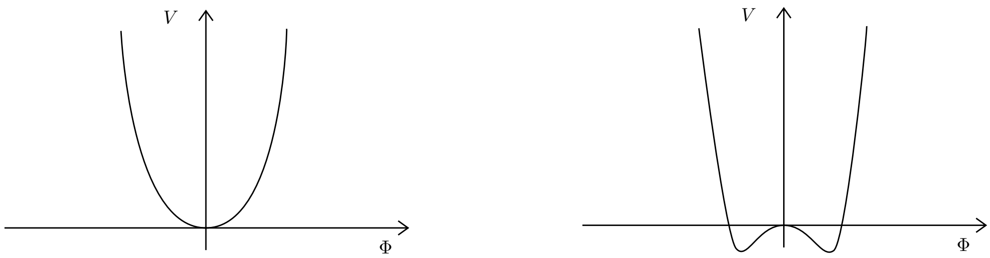
$$h = r_0 \Phi + u_0 \Phi^3$$

Let us remind that the saddle point appriximation constists in approximating the functional integral of Eq.(1.2) with its dominant term, i.e. with the one for which the exponent (Eq.(1.1)) is minimum. Therefore, in the uniform case (namely $\vec{\nabla}\Phi = 0$), it is equivalent to find the uniform value $\Phi_0$ that is the extrema of the potential:

$$V(\Phi) = \frac{1}{2} r_0 \Phi^2 + \frac{u_0}{4} \Phi^4 - h\Phi$$

Hence, if $h = 0$, the extrema of the potential can be computed as

$$V' = (r_0 + u_0 \Phi^2)\Phi = 0$$




**(a)** Plot of the potential $V(\Phi)$ in the case $r_0 > 0$, i.e. $T > T_c$.

**(b)** Plot of the potential $V(\Phi)$ in the case $r_0 < 0$, i.e. $T < T_c$.

**Figure 1.1**

Let us remember that $r_0 \propto (T - T_c)$. In order to find the extrema of the potential $V(\Phi)$, we should distinguish two cases:

1. Case $r_0 > 0$ ($T > T_c$): there is only one solution $\Phi_0 = 0$, as we can see in Figure 1.1a.

2. Case $r_0 < 0$ ($T < T_c$): there are two solutions $\Phi_0 = \pm\sqrt{-\frac{r_0}{u_0}}$, as illustrated in Figure 1.1b.

   We note that the two solution $\pm\Phi_0$ are related by the $\mathbb{Z}^2$ transformation, namely $\Phi \to -\Phi$. Moreover, in this case with $T < T_c$, the two states (phases) $\pm\Phi_0$ have a lower symmetry than the state $\Phi_0 = 0$.

   If the thermal fluctuations $\delta\Phi$ are sufficiently strong to allow passages between the two states $\pm\Phi_0$ at $T < T_c$, we have $\langle\Phi\rangle = 0$ (preserves states).

   However, for $T < T_c$ and $N \to +\infty$, transition between the two states will be less and less probable and the system will be trapped into one of the two states ($\pm\Phi_0$). In other words, the system choose spontaneously one of the two less symmetric state. Therefore, its physics is not any more described by $\Phi$ but by the fluctuations $\delta\Phi$ around the chosen minimum $\Phi_0$. There is a spontaneous symmetry breaking. It means that the variable $\Phi$ is not any more symmetric and one has to look at $\Phi \to \Phi_0 + \delta\Phi$, where $\delta\Phi$ is a new variable!

## 1.2 Spontaneous breaking of continuous symmetries and the onset of Goldstone particles

Let us start with a simple model in which the order parameter is a scalar complex variable

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}$$

and with an Hamiltonian $\mathcal{H}$ that is invariant with respect to a global continuous transformation. For instance, the simplest model in statistical mechanics that is invariant with respect to a continuous symmetry is the XY model with $O(2)$ symmetry, or a Ginzburg-Landau model for a superfluid or a superconductor (with no magnetic field). Hence, we suppose that the Hamiltonian as the following form:

$$\beta\mathcal{H}_{eff} = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2}\vec{\boldsymbol{\nabla}}\Phi \cdot \vec{\boldsymbol{\nabla}}\Phi^* + \frac{r_0}{2}\Phi\Phi^* + \frac{u_0}{4}(\Phi\Phi^*)^2\right]$$

where

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{\sqrt{2}}[\Phi_1(\vec{\mathbf{x}}) + i\Phi_2(\vec{\mathbf{x}})], \quad \text{or} \quad \Phi(\vec{\mathbf{x}}) = \psi(\vec{\mathbf{x}})e^{i\alpha(\vec{\mathbf{x}})}$$

The physical meaning of $\Phi$ depends on the case considered. If we have:

- Superfluid: $\Phi$ is the macroscopic wave function of the Bose condensate (density of superfluid $n = \left|\Phi^2\right|$).
- Superconductor: $\Phi$ is the single particle wave function describing the position of the centre of mass of the Cooper pair.

### 1.2.1 Quantum relativistic case (field theory)

In quantum mechanics the analog of the Hamiltonian $\mathcal{H}$ is the action

$$S(\Phi) = \int \mathrm{d}^4\vec{\mathbf{x}}\, \mathcal{L}(\Phi) \tag{1.3}$$

where

$$\mathcal{L}(\Phi) = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - \frac{r_0}{2}\Phi\Phi^* - \frac{u_0}{4}(\Phi\Phi^*)^2 \tag{1.4}$$

The Lagrangian $\mathcal{L}(\Phi)$ describes a scalar complex (i.e. charged) *muonic field* with mass $m \equiv \sqrt{r_0}$; we note that, if $\mathcal{L}(\Phi)$ describes a muonic field, we should have $r_0 > 0$ to have the mass $m$ well defined. Moreover, the term $(\Phi\Phi^*)^2$ means self-interaction with strenght $\lambda \equiv u_0$.

In all cases ($r_0 > 0$ or $r_0 < 0$), the original symmetry is $U(1)$; it means that both the Hamiltonian $\mathcal{H}$ and the Lagrangian $\mathcal{L}$ are invariant with respect to the transformation

$$\Phi \to e^{i\theta}\Phi, \quad \Phi^* \to e^{-i\theta}\Phi^* \tag{1.5}$$

where the phase $\theta$ does not depend on $\vec{\mathbf{x}}$ (global symmetry). In components the transformation becomes

$$\begin{cases} \Phi_1 \to \Phi_1\cos\theta - \Phi_2\sin\theta \\ \Phi_2 \to \Phi_2\cos\theta + \Phi_1\sin\theta \end{cases} \qquad \Rightarrow (\Phi_1', \Phi_2') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

Now, let us focus first on the statistical mechanics model and to the most interesting case of $r_0 < 0$. In components, $\mathcal{H}$ can be expressed as

$$\beta\mathcal{H} = \int \mathrm{d}^d\vec{\mathbf{x}} \left[(\boldsymbol{\nabla}\Phi_1)^2 + (\boldsymbol{\nabla}\Phi_2)^2\right] + \int \mathrm{d}^d\vec{\mathbf{x}}\, V(\Phi_1, \Phi_2) \tag{1.6}$$

where the potential is

$$V(\Phi_1, \Phi_2) = \frac{r_0}{2}\left(\Phi_1^2 + \Phi_2^2\right) + \frac{u_0}{4}\left(\Phi_1^2 + \Phi_2^2\right)^2 \tag{1.7}$$

In the case $r_0 < 0$, it is called *mexican hat potential* and it is shown in Figure 1.2.

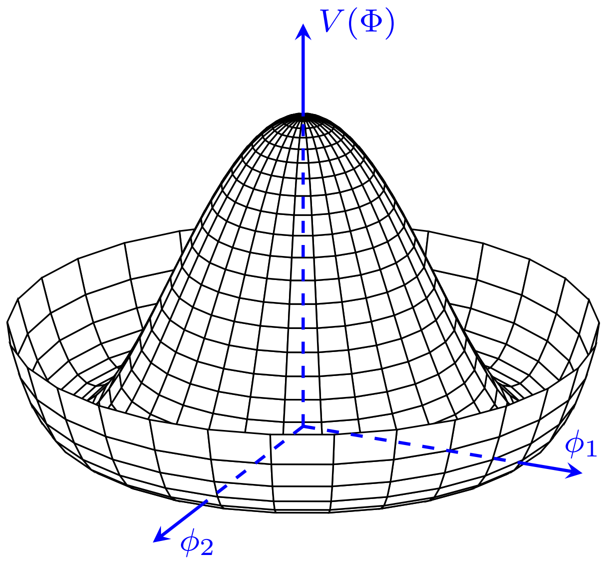



**Figure 1.2:** Case $r_0 < 0$. The potential $V(\Phi)$ is a mexican hat potential.

In the uniform case we have ($\boldsymbol{\nabla}\Phi_1 = \boldsymbol{\nabla}\Phi_2 = 0$). Let us define $S = \sqrt{\Phi_1^2 + \Phi_2^2}$, the potential in Eq.(1.7) can be rewritten as:

$$V(S) = \frac{r_0}{2}S^2 + \frac{u_0}{4}S^4$$

In the uniform case, the solution is given by the minima of the potential $V(S)$; hence, in order to find the extrema points, we derive the potential with respect to $S$ and we impose the condition $V' = 0$:

$$\frac{\mathrm{d}V(S)}{\mathrm{d}S} = r_0 S + u_0 S^3 = 0$$

We have a maximum at $S = 0$ and a minimum at $S^2 \equiv v^2 = -r_0/u_0$. Hence, for $r_0 < 0$, the Hamiltonian $\mathcal{H}$ displays a minimum when

$$\Phi_1^2 + \Phi_2^2 \equiv v^2 = -\frac{r_0}{u_0}$$

It could be represented in the $2d$ plane $(\Phi_1, \Phi_2)$, where the minimum lies on a circle of radius

$$v = \sqrt{-\frac{r_0}{u_0}}$$

as show in Figure 1.3. The spontaneous symmetry breaking occurs when the system "chooses" one of the infinite available minima. In our example, let us suppose that the chosen minimum is

$$\Phi_1 = v = \sqrt{-\frac{r_0}{u_0}}, \quad \Phi_2 = 0 \tag{1.8}$$

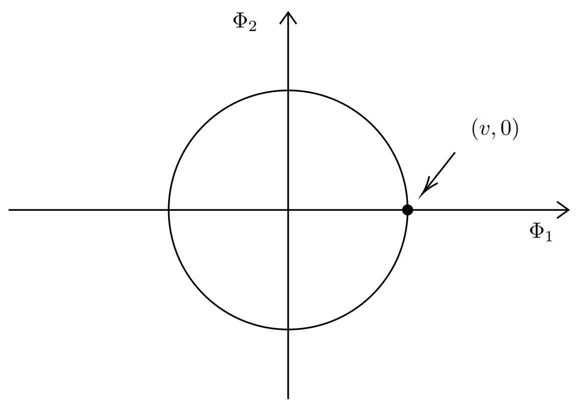



**Figure 1.3:** Plane $(\Phi_1, \Phi_2)$. The minimum lies on a circle of radius $v = \sqrt{-\frac{r_0}{u_0}}$.

### Interpretation in relativistic quantum mechanics

Now, let us give a physical interpretation of the results previously obtained and of the considerations we have done in order to obtain them. In particular:

1. Choosing $r_0 < 0$ corresponds to an *imaginary mass*. This is because moving away from $\Phi = 0$, the system experiences a *negative resistence* in both directions, being $\Phi = 0$ a relative local maximum.

2. The minimum has the lowest energy and therefore it must correspond to the *empty state*. In this case, however, there is an infinite number of empty states!

In summary, the starting Hamiltonian $\mathcal{H}$ (or Lagrangian $\mathcal{L}$) is invariant with respect to $U(1)$, but the one that describes the fluctuation dynamics around one of the chosen minimum state is not invariant with respect to $U(1)$. Let us see in more details why the Hamiltonian, or Lagrangian, it is not invariant anymore in the case $r_0 < 0$.

First of all, let us write the Lagrangian with respect to the fluctuations of $\Phi_1$ and $\Phi_2$ around the chosen state $(v, 0)$ (Eq.(1.8)), we obtain:

$$\begin{cases} \Phi_1 = v + \delta\Phi_1 \\ \Phi_2 = 0 + \delta\Phi_2 \end{cases} \qquad \Rightarrow \Phi = \Phi_1 + i\Phi_2 = v + (\delta\Phi_1 + i\delta\Phi_2)$$

where we omit the factor $1/\sqrt{2}$ for simplicity. Let us note that

$$\begin{cases} \delta\Phi_1 = \Phi_1 - v \\ \delta\Phi_2 = \Phi_2 \end{cases} \qquad \Rightarrow \langle\delta\Phi_1\rangle_{\Phi_0} = \langle\delta\Phi_2\rangle_{\Phi_0} = 0$$

indeed, as expected, the expectation value of the empty state is back to be zero.

Now, for the quantum relativistic Lagrangian $\mathcal{L}$, let us define

$$r_0 \to m^2, \quad u_0 \to \lambda, \quad v^2 = -\frac{m^2}{\lambda}$$

(recall that we are still in the case $r_0 < 0!$). Hence, the Lagrangian Eq.(1.4) becomes

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\partial_\mu(v+\delta\Phi_1+i\delta\Phi_2)\partial^\mu(v+\delta\Phi_1-i\delta\Phi_2)\\
&-\frac{m^2}{2}(v+\delta\Phi_1+i\delta\Phi_2)(v+\delta\Phi_1-i\delta\Phi_2)\\
&-\frac{\lambda}{4}[(v+\delta\Phi_1+i\delta\Phi_2)(v+\delta\Phi_1-i\delta\Phi_2)]^2\\
=&-\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1)-\frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2)\\
&-\frac{m^2}{2}\left(v^2+2v\delta\Phi_1+\delta\Phi_1^2+\delta\Phi_2^2\right)\\
&-\frac{\lambda}{4}\left(v^2+2v\delta\Phi_1+\delta\Phi_1^2+\delta\Phi_2^2\right)^2
\end{aligned}
$$

Since we have defined $m^2 = -v^2\lambda$, we can rewrite it as

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1)-\frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2)+\frac{\lambda v^2}{2}\left(v^2+\cancel{2v\delta\Phi_1}+\cancel{\delta\Phi_1^2+\delta\Phi_2^2}\right)\\
&-\frac{\lambda}{4}\left(v^4+4v^2\delta\Phi_1^2+\cancel{4v^3\delta\Phi_1}+\left(\delta\Phi_1^2+\delta\Phi_2^2\right)^2+\cancel{2v^2\left(\delta\Phi_1^2+\delta\Phi_2^2\right)}+4v\delta\Phi_1\left(\delta\Phi_1^2+\delta\Phi_2^2\right)\right)
\end{aligned}
$$

Neglecting the constant terms in $v$ (in green), finally we obtain

$$
\begin{aligned}
\mathcal{L}(\delta\Phi_1,\delta\Phi_2) = &-\frac{1}{2}(\partial_\mu\delta\Phi_1)^2-\frac{1}{2}(\partial_\mu\delta\Phi_2)^2\\
&-\frac{2}{2}\lambda v^2\delta\Phi_1^2-v\lambda\delta\Phi_1\left((\delta\Phi_1)^2+(\delta\Phi_2)^2\right)\\
&-\frac{\lambda}{4}\left((\delta\Phi_1)^2+(\delta\Phi_2)^2\right)^2
\end{aligned}
\tag{1.9}
$$

Comparing it with Eq.(1.4), we note that the yellow term $-\lambda v^2\delta\Phi_1^2$ indicates that the field $\delta\Phi_1$ (related to the transversal fluctuations) has a null empty state ($\langle\delta\Phi_1\rangle = 0$) and a mass $M$ such that (recall that $m^2 = -\lambda v^2 = r_0$):

$$M^2 = 2\lambda v^2 = -2r_0 = -2m^2$$

Therefore, it represents a real, massive, mesonic scalar field that is physically accettable (indeed we have $r_0 < 0$!). However, $\mathcal{L}$ is not any more invariant under the transformation $\delta\Phi_1 \to -\delta\Phi_1$, as we wanted to show! The symmetry is broken.

*Remark.* Note that the field $\delta\Phi_2$ has no mass (there is not a term $\propto \delta\Phi_2^2$)! Indeed, it describes the fluctuations along the circle where the potential $V$ is in its minimum which imples no dynamical inertia, that implies no mass!

In summary, starting with one complex scalar field $\Phi(\vec{\mathbf{x}})$ having mass $m$, when $m^2 < 0$ one gets a real scalar field $\delta\Phi_1$ with mass $M = \sqrt{-2m^2}$ and a second scalar field $\delta\Phi_2$ that is massless. This is called the **Goldstone boson**. Hence, we have the following theorem:

**Theorem 1: Goldstone's theorem**

*If a continuous symmetry is spontaneously broken and there are no long range interactions, exists an elementary excitation with zero momentum, or particle of zero mass, called Goldstone boson.*
*More generally, let $\mathcal{P}$ be a subgroup of $\mathcal{G}$ ($\mathcal{P} \subset \mathcal{G}$). If $\mathcal{G}$ has $N$ indipendent generators and $\mathcal{P}$ has $M$ indipendent generators, hence, if $\mathcal{P}$ is the new (lower) symmetry, therefore $N - M$ Goldstone bosons exist.*

In the previous case the symmetry group was $\mathcal{G} = U(1)$, hence $\mathcal{G}$ has $N = 1$ indipendent generators, whereas we have $M = 0$ (we have chosen a specific minimum). Therefore, we have only one Goldstone boson.

**Example 3: $XY$ model**

Let us consider the $XY$ model in statistical mechanics. We have that

- $\delta\Phi_1$ represents the fluctuation of the modulus of $m$.
- $\delta\Phi_2$ represents fluctuations of the spin directions, or spin waves.

*Remark.* In particle physics the presence of Goldstone bosons brings a serious problem in field theory since the corresponding particles are not observed! The resolution of this problem is given by Higgs-Englert-Brout (1964). In particular, the Higgs mechanism gives back the mass to the Goldstone particles, because the Goldstone theorem, that works well for a continuous global symmetry, can fail for local gauge theories!

## 1.3 Spontaneous symmetry breaking in Gauge symmetries

#### Statistical mechanics

In statistical mechanics, let us consider a Ginzburg-Landau model for superconductors in presence of a magnetic field (*Meissner effect*, i.e. the magnetic induction $\vec{\mathbf{B}} = 0$ inside the superconductor). The Hamiltonian of such a system is:

$$\beta\mathcal{H}(\Phi) = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2}B^2 + \left|\left(\vec{\boldsymbol{\nabla}} - 2i\vec{\mathbf{A}}\right)\Phi\right|^2 + \frac{r_0}{2}\Phi^*\Phi + \frac{u_0}{4}(\Phi^*\Phi)^2 - \vec{\mathbf{B}}\cdot\vec{\mathbf{H}}\right] \tag{1.10}$$

where $\frac{B^2}{2}$ is the energy of the magnetic field $\vec{\mathbf{B}}$ and $\vec{\boldsymbol{\nabla}} \to \left[\vec{\boldsymbol{\nabla}} + iq\vec{\mathbf{A}}\right]$ is the minimal coupling. If we have an external magnetic field $\vec{\mathbf{H}}$, the induction field is:

$$\vec{\mathbf{B}} = \vec{\mathbf{H}} + \vec{\mathbf{M}}$$

For normal conductors we have $\Phi_0 = 0$, which implies $\vec{\mathbf{B}} = \vec{\mathbf{H}}$, while for superconductors $\Phi \neq 0$ and we have a spontaneous symmetry breaking.

#### Field theory analog

Let us consider the analog of the previous system in field theory: a scalar charged mesonic fields selfinteracting and in presence of an electromagnetic field with potential quadrivector $A_\mu(\vec{\mathbf{x}})$. In particular, we have:

$$\partial_\mu \to D_\mu = [\partial_\mu + iqA_\mu]$$

The order parameter we are considering is defined again as a scalar complex variable:

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \Phi^* = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2)$$

In this case, because of the presence of $A_\mu(\vec{\mathbf{x}})$, we should consider a theory that satisfies symmetry $U(1)$ locally! Thus, the transformation is

$$\Phi(\vec{\mathbf{x}}) \to e^{i\alpha(\vec{\mathbf{x}})}\Phi(\vec{\mathbf{x}}), \quad \Phi^*(\vec{\mathbf{x}}) \to e^{-i\alpha(\vec{\mathbf{x}})}\Phi(\vec{\mathbf{x}}) \tag{1.11}$$

and exists $A_\mu(\vec{\mathbf{x}})$ interacting with $\Phi(\vec{\mathbf{x}})$. Hence, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(\vec{\mathbf{x}})F^{\mu\nu}(\vec{\mathbf{x}}) + (D_\mu\Phi(\vec{\mathbf{x}}))^*(D_\mu\Phi(\vec{\mathbf{x}})) - V(\Phi,\Phi^*) \tag{1.12}$$

where

$$D_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi \qquad \text{Gauge-covariant derivative}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{Field strength tensor}$$
$$V(\Phi,\Phi^*) = \frac{m^2}{2}\Phi\Phi^* + \frac{\lambda}{4}(\Phi\Phi^*)^2 \qquad \text{Potential}$$

Let us consider again two different cases:

- If $m^2 > 0$: there is a minimum at $\Phi = 0$.
- If $m^2 < 0$: there is a minimum at $\Phi = \sqrt{-\frac{m^2}{\lambda}} \equiv v$ (circle of radius $|\Phi| = v$).

Now, we consider the case $m^2 < 0$. Let us choose the state

$$\bar{\Phi}_1 = v, \quad \bar{\Phi}_2 = 0$$

Hence, we have:

$$\Phi(x) = (v + \delta\Phi_1) + i\delta\Phi_2$$

By inserting it in the Lagrangian and by keeping in mind that $m^2 = -v^2\lambda$, the Lagrangian transforms as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\delta\Phi_1)^2 + \frac{1}{2}(\partial_\mu\delta\Phi_2)^2 \\ &- \frac{2}{2}\lambda v^2\delta\Phi_1^2 + q^2v^2A_\mu A^\mu - qvA^\mu\partial_\mu\delta\Phi_2 + \text{higher order terms}\end{aligned} \tag{1.14}$$

Let us give a physical interpretation of the emphasized terms:

- $\lambda v^2\delta\Phi_1^2$ means that the field $\delta\Phi_1$ is massive with mass $M = v\sqrt{2\lambda}$ (Higgs boson).
- $q^2v^2A_\mu A^\mu$ means that the *Gauge boson* $A_\mu$, the photon, has got a mass

  $$M_A^2 = 2q^2v^2$$

  Note that since now $A_\mu$ is massive, it has three indipendent polarization states.
- $qvA^\mu\partial_\mu\delta\Phi_2$ means that the field $\delta\Phi_2$ is not massive (indeed there is no term $\propto \delta\Phi_2^2$) and that it is mixed with $A_\mu$. Dynamically, this means that a propagating photon can transform itself into a field $\delta\Phi_2$ (the photon becomes a Goldstone boson).

Since $\delta\Phi_2$ does not seem to be a physical field it should be eliminated by a Gauge transformation. Indeed a Gauge transformation is also characterized by the transformation

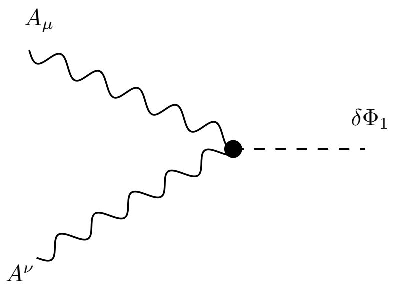
$$A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) - \frac{1}{q}\partial_\mu\alpha(\vec{\mathbf{x}})$$

We can choose $\alpha(\vec{\mathbf{x}}) = -\frac{1}{v}\delta\Phi_2(\vec{\mathbf{x}})$

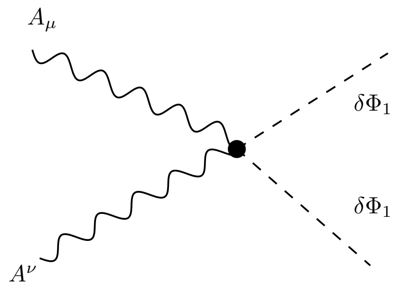
$$A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) + \frac{1}{qv}\partial_\mu\delta\Phi_2(\vec{\mathbf{x}})$$

By inserting it in the Lagrangian Eq.(1.14), we eliminate the mixed term $qvA^\mu\partial_\mu\delta\Phi_2$, in red, and the term $\frac{1}{2}(\partial_\mu\delta\Phi_2)^2$. Thus we obtain:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\delta\Phi_1)^2 - \lambda v^2\delta\Phi_1^2 + q^2v^2A_\mu A^\mu + \text{higher order terms} \tag{1.15}$$




**(a)** Feyman diagram of the term $\propto \delta\Phi_1 A_\mu A^\mu$.




**(b)** Feyman diagram of the term $\propto \delta\Phi_1^2 A_\mu A^\mu$.

**Figure 1.4**

*Remark.* Let us note that among the higher order terms we have neglected, there are

- A term proportional to $\propto \delta\Phi_1 A_\mu A^\mu$, shown in Figure 1.4a.
- A term proportional to $\propto \delta\Phi_1^2 A_\mu A^\mu$, shown in Figure 1.4b.

Now, let us consider the new Lagrangian Eq.(1.15). It contains two fields: one is a massive photon with spin 1 and the second field $\delta\Phi_1$ is massive too, but has spin 0 (scalar). Hence, the Goldstone boson has been "eaten" by the Gauge boson, the photon, that now is massive! The mechanics trough which the Gauge boson becomes massive is the so called **Higgs mechanism**.

In summary, according to the degrees of freedom, we have

- For a global $U(1)$ symmetry, we have 2 massive scalar fields, hence there are $1+1$ degrees of freedom.

  After symmetry breaking, we have 1 scalar field massive and 1 scalar field not massive. Hence, there are again $1+1$ degrees of freedom.

- For a local Gauge $U(1)$ symmetry, we have 2 massive scalar fields and one photon. Hence, there are $2+2$ degrees of freedom (the photon has 2 polarizations).

  After symmetry breaking, we have 1 massive scalar field and 1 massive photon. Hence, there are $1+3$ degrees of freedom (the massive photon has 3 polarizations).

*Remark.* The presence of the massive photon $m_A^2 = q^2 v^2$, $q = 2l$ in superconductivity, gives rise to the experimental drop

$$B(x) = B(0) \exp\Big(-\frac{x}{l}\Big) \tag{1.16}$$

inside the system.

- We cannot introduce by hand a massive photon i.e. a term like $\frac{1}{2} m_A^2 A_\mu A^\mu$ in the Lagrangian because we would violate explicitely the gauge symmetry!
- The Lagrangian is gauge invariant.
- Symmetry breaking occurs at the level of the vacuum state.
- A gauge symmetry that is explicitely broken is not renormalizable.

### 1.3.1 Non abelian gauge theories

**Example 4**

Electro-weak interactions theory (Glashow-Weinberg-Solam) (theory of leptons).
Lagrangian has $\underbrace{SU(2)}_{\text{weak interactions}} \times \underbrace{U(1)}_{\text{electromagnetian}}$

**Example 5: Quantum chromodynamic (quarks+gluons)**

In this case, one has a term that is $SU(3)$ invariant + the GWS lagrangian with symmetry $SU(2) \times U(1)$, implies

$$SU(3) \times SU(2) \times U(1) \tag{1.17}$$

Because of the groups $SU(2)$ and $SU(3)$ the symmetries above ar not abelian. (For example in $SU(2)$ two matrices $U(\alpha)$ and $U(\beta)$ do not commute in general).

## 1.4 Extension of Higgs mechanism to non abelian theories

### 1.4.1 GWS model

Complex field $SU(2)$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi_3 + i\Phi_4 \end{pmatrix} = \begin{pmatrix} \Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}}) \end{pmatrix} \tag{1.18}$$

where $\Phi_a, \Phi_b$ are complex fields.

Gauge transformation $SU(2) \times U(1)$:

$$\begin{pmatrix} \Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}}) \end{pmatrix} \to e^{\frac{i}{2}\alpha_0(\vec{\mathbf{x}})} e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(\vec{\mathbf{x}})} \begin{pmatrix} \Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}}) \end{pmatrix} \tag{1.19}$$

where $\vec{\tau}$ are Pauli matrices, $\alpha_0, \alpha_1, \alpha_2, \alpha 3$ are four real functions (4 vectorial mesons).

$$\vec{\alpha}(\vec{\mathbf{x}}) \to W^a_\mu(\vec{\mathbf{x}}) = \left( W^{(1)}_\mu(\vec{\mathbf{x}}), W^{(2)}_\mu(\vec{\mathbf{x}}), W^{(3)}_\mu(\vec{\mathbf{x}}) \right) \tag{1.20}$$

The scalar gauge field is

$$\alpha_0(\vec{\mathbf{x}}) \to B_\mu(\vec{\mathbf{x}}) \tag{1.21}$$

with $B_\mu$ is a linear combination of $A_\mu$ and $W^{(3)}_\mu$.

Lagrangian:

$$\mathcal{L} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 - \frac{1}{4} b^{\mu\nu} b_{\mu\nu} - \frac{1}{4} f^{\mu\nu}_a f^a_{\mu\nu} \tag{1.22}$$

$$D_\mu \to \partial_\mu \frac{1}{2} i g \tau^a W^a_\mu - \frac{i}{2} g' B_\mu \tag{1.23}$$

$$f^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g \varepsilon^{abc} W^b_\mu W^a_\nu \tag{1.24}$$

$$b_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.25}$$

$$W^a_\mu \to W^a_\mu - \varepsilon^{abc} \alpha_b(\vec{\mathbf{x}}) W^c_\mu(\vec{\mathbf{x}}) + \frac{1}{g} \partial_\mu \alpha^a(\vec{\mathbf{x}}) \tag{1.26a}$$

$$B_\mu \to B_\mu + \frac{1}{g'} \frac{\partial \alpha_0}{\partial x_\mu} \tag{1.26b}$$

$$\nu \sim \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 = v^2 \tag{1.27}$$

Choosing the direction on the sphere in $\mathbb{R}^4$, 3 symmetries are broken a 3 Goldstone bosons.

### 1.4.2 Higgs mechanism

Higgs scalar field

$$\delta\Phi = \begin{pmatrix} \Phi^+ \\ \Phi_0 \end{pmatrix} \tag{1.28}$$

such that

$$\langle 0| \, \Phi \, |0\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.29}$$

$$\Rightarrow \mathcal{L}_{Higgs} = \frac{1}{2}(g\nu)^2 W_\mu^+ W^{-\mu} + \frac{1}{2}v^2\left(gW_\mu^{(3)} - g'B_\mu\right)^2 \tag{1.30}$$

where

$$W_\mu^{(1)} = \frac{1}{\sqrt{2}}\left(W_\mu^+ + W_\mu^-\right) \tag{1.31a}$$

$$W_\mu^{(2)} = \frac{1}{\sqrt{2}}\left(W_\mu^+ - W_\mu^-\right) \tag{1.31b}$$

Mass of the $W^+$ particle and its antiparticle

$$M_W^2 = \frac{1}{2}(gv)^2 \tag{1.32}$$

The $2^{nd}$ term is a linear combination of $W_\mu^3$ and $B_\mu$ which corresponds to $Z^0$, the field for a third weak gauge boson.

To make $Z_\mu^0$ and $A_\mu$ orthogonal we should consider

$$A_\mu = (\cos\theta_W)B_\mu + (\sin\theta_W)W_\mu^3 \tag{1.33a}$$

$$Z_\mu^0 = (-\sin\theta_W)B_\mu + (\cos\theta_W)W_\mu^3 \tag{1.33b}$$

where $\theta_W$ is the Weiberg angle:

$$\tan\theta_W = \frac{g'}{g} \tag{1.34}$$

$$M_{Z^0}^2 = \frac{1}{2}\left(\frac{vg}{\cos\theta_W}\right)^2 = \frac{Mw^2}{\cos^2\theta_W} \tag{1.35}$$