*Lecture 13.* Wednesday 27th November, 2019. Compiled: Wednesday 27th November, 2019.

Last time: variational mean field for the Ising model.
Write the one point density distribution as (we assume $S_i = \pm 1$):

$$\rho_i = \rho(S_i) = a(1-\delta_{S_i,-1}) + b\delta_{S_i,-1} \tag{1}$$

The trace is

$$\begin{cases} \mathrm{Tr}(\rho_i) = 1 & \to a+b=1 \\ \mathrm{Tr}(\rho_i S_i) = m_i & \to a-b = m_i \end{cases} \tag{2}$$

$a, b$ are the functions of the order parameter. In that case we have not to write the functions for all the $i$. For $S_i = 1$ we have one value, for all the other values another one. The results or the previous equation is:

$$\begin{cases} a = \frac{1-m_i}{2} \\ b = \frac{1+m_i}{2} \end{cases} \tag{3}$$

So

$$\rho_i = \frac{1-m_i}{2}(1-\delta_{S_i,-1}) + \frac{1+m_i}{2}\delta_{S_i,-1} \tag{4}$$

Let us consider the Hamiltonian

$$\langle \mathcal{H} \rangle_{\rho_{MF}} = \left\langle -J\sum_{\langle ij \rangle} S_i S_j - \sum_i H_i S_i \right\rangle_{\rho_{MF}} = -J\sum_{\langle ij \rangle} \langle S_i \rangle \langle S_j \rangle - \sum_i H_i \langle S_i \rangle \tag{5}$$

with

$$\rho_{MF} = \prod_i \rho_i \tag{6}$$

the equation will trasform into

$$\langle \mathcal{H}_{MF} \rangle_{\rho_{MF}} = -J\sum_{\langle ij \rangle} m_i m_j - \sum_i H_i m_i \tag{7}$$

therefore

$$\langle \ln \rho \rangle_{\rho_{MF}} = \sum_i \mathrm{Tr}^{(i)}(\rho_i \ln \rho_i) = \sum_i \left[ \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \right] \tag{8}$$

The free energy mean field became

$$F_{\rho_{MF}} = \langle \mathcal{H}_{MF} \rangle + k_B T \langle \ln \rho \rangle_{MF} \tag{9}$$

so to obtain the minimal value

$$\left. \frac{\partial F_{\rho_{MF}}}{\partial m_i} \right|_{m_i = \bar{m}_i} = 0 \tag{10}$$

$$0 = -J\sum_{j \in n.n.\, i} \bar{m}_j - H_i + \frac{k_B T}{2} \ln \left[ \frac{1+\bar{m}_i}{1-\bar{m}_i} \right] \tag{11}$$

$$\tanh^{-1}(x) = \frac{1}{2} \ln \frac{1+x}{1-x} \tag{12}$$

so

$$k_B T \tanh^{-1}(\bar{m}_i) = J \sum_{j \in n.n.\, i} \bar{m}_j + H_i \tag{13}$$

$$\bar{m_i} = \tanh\left[(k_B T)^{-1}\left(J \sum_{j \in n.n.\, i} \bar{m_j} + H_i\right)\right] \tag{14}$$

now consider

$$z\bar{m_i} = \sum_{j \in n.n.\, i} \bar{m_j} \tag{15}$$

and put it into the last equation, what we obtain is

$$\bar{m_i} = \tanh\left[\beta(Jz\bar{m_i} + H_i)\right] \tag{16}$$

this is the as the Bragg William approximation.

## 0.1 second approach

We do not parameterize again the $\rho_i$ using simple parameter, but we calculate derivatives.

This model is called the *Blume-Emery-Griffith*. It is a spin model, is a deluter Ising model.

Let us consider Helium $He^4$. It is bose Einstein condensation. It goes from flud to superfluid transition. At $P = P_0$ (atmosferic pressure), we have $T_\lambda = 2,17K$. It is also called $\lambda$-transition because in the limit of $T_\lambda$ there is a critical point. Insert a figure 1 If we do the usual $(P, T)$ phase diagram we have Insert a figure 2 Now we start to enject $He^3$. This statisfies the fermion statistic.

Let us suppose for example $x$, the concentration of $He^3$. We have:

$$T_\lambda = T_\lambda(x) \tag{17}$$

The $\cos(x)$ decrease but the increase again.

Suppose a mixture like oil and water, during phase separation we saw small bubbles. At a given point there will be a phase separation. There is a phase rich of $He^4$ and another one rich of $He^3$.

The critical $x$ is:

$$x = x_t = \frac{n_3}{n_3 + n_4} \sim 0.67 \tag{18}$$

where $n_3$ is the number of $He^3$, etc.

So if $x > x_t$ we have the first order transition. Insert figure. The point $(x_t, T_t)$ is called *tricritical point*. There are different critical points that are characterize by different parameter. Can we describe by using a sort of spin model that? Ising model with vacancy. The idea is to diluted the typical Ising model. What happens if the concentration of the dilution will superate this given value?