Lecture 18. *Friday 13th December, 2019. Compiled: Friday 13th December, 2019.*

$$\left(-\boldsymbol{\nabla}^2 + \xi^{-2}(t)\right)G_c(\vec{\mathbf{r}} - \vec{\mathbf{r}}') = \frac{k_B T}{k}\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{1}$$

Let us try to do the Fourier transform. Let us define

$$\vec{\mathbf{x}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}' \tag{2}$$

e let us call $\widetilde{G}(q)$ the Fourier transform of the function G

$$\widetilde{G}(q) = \int_{-\infty}^{+\infty} \mathrm{d}|\vec{\mathbf{x}}|\, G_c(|\vec{\mathbf{x}}|)e^{-iq|\vec{\mathbf{x}}|} \tag{3}$$

we get

$$\widetilde{G}(q) = \frac{k_B T}{k}\frac{1}{q^2 + \xi^{-2}} \tag{4}$$

At $T = T_c$, we have $\xi \to \infty$ and $\widetilde{G}(q) \simeq \frac{1}{q^2}$. We have

$$G_c(|\vec{\mathbf{x}}|) = |\vec{\mathbf{x}}|^{2-D} \tag{5}$$

In this case we see immediately that $\eta = 0$. Go back and find why we have this.

$$G(\vec{\mathbf{x}}) = \int \mathrm{d}^D\vec{\mathbf{q}}\, \frac{1}{(2\pi)^D}\frac{1}{q^2 + \xi^{-2}}e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}} \tag{6}$$

Let us do it for $D = 3$:

$$\Rightarrow G(|x|) = \frac{4\pi}{(2\pi)^3}\int_0^{\infty} \mathrm{d}q\, \frac{q^2}{q^2 + \xi^{-2}}\int_{-1}^{+1} \mathrm{d}(\cos\theta)\, e^{iq|\vec{\mathbf{x}}|\cos\theta} \tag{7}$$

we get

$$= \frac{4\pi}{(2\pi)^3}|\vec{\mathbf{x}}|\int_0^{\infty} \mathrm{d}q\, \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}} \tag{8}$$

At the end

$$\Rightarrow G(|\vec{\mathbf{x}}|) = \frac{1}{2\pi}\frac{e^{-\frac{|\vec{\mathbf{x}}|}{\xi}}}{|\vec{\mathbf{x}}|} \tag{9}$$

Can we reach the simple level of fluctuation? The simple level is the one that follow gaussian distribution. Let us introduce fluctuations at the Gaussian level.

Idea: consider a function (an usual function)...

Consider $h = 0$: the saddle point solution is $m_0(r) = m_0$

$$\beta\mathcal{H}_{eff} = \int \mathrm{d}^D\vec{\mathbf{r}}\left(atm^2 + \frac{b}{2}m^4\frac{k}{2}(\boldsymbol{\nabla}m)^2\right) \tag{10}$$

$$m(\vec{\mathbf{r}}) = m_0 + \delta m(\vec{\mathbf{r}}) \tag{11}$$

we are assuming that the fluctuations $\delta m(\vec{\mathbf{r}})$ are small.

$$(\boldsymbol{\nabla}m)^2 = (\boldsymbol{\nabla}(m_0 + \delta m))^2 = (\boldsymbol{\nabla}(\delta m))^2 \tag{12}$$

$$m^2 = m_0^2 + 2m_0\delta m + (\delta m)^2 \tag{13}$$

$$\beta\mathcal{H}_{eff} = V\underbrace{\left(atm_0^2 + \frac{b}{2}m_0^4\right)}_{A_0} + \int \mathrm{d}^D\vec{\mathbf{r}}\left(\frac{k}{2}(\boldsymbol{\nabla}m)^2 + \left(at + 2bm_0^2\right)\delta m^2 + 2bm_0\delta m^3 + \frac{b}{2}\delta m^4\right) \tag{14}$$

$$\left(\underbrace{2atm_0}_{=0}+\frac{b}{2}4m_0^3\right)\delta m \tag{15}$$

the linear term in m_0 is equal to zero by definition.

At $T > T_c$ we know that $m_0 = 0$. In this cases

$$\beta\mathcal{H}^{>}_{eff} = \int \mathrm{d}^D\vec{\mathbf{r}}\left(\frac{k}{2}(\boldsymbol{\nabla}m)^2 + at(\delta m)^2 + \cancel{\frac{b}{2}(\delta m)^4}\right) \tag{16}$$

Remark. It is important to understand that these are fluctuations with respect to the solution.

We cannot do again the saddle point, otherwise we do not get too much information. We consider gaussian fluctuations: fluctuations that follow gaussian distribution. Therefore, the term in $(\delta m)^4$ is cancelled. What it is the difference in the exponent respect the mean field? Then we will to the same for $T < T_c$, the story is the same. Now, we are just taking the gaussian term

$$Z^G_{GL} = \int \mathrm{D}[\delta m]e^{-\int \mathrm{d}^D r\left(\frac{k}{2}(\boldsymbol{\nabla}\delta m)^2 + at(\delta m)^2\right)} \tag{17}$$

Consider a system in a box of volume $V = L^D$:

$$\delta m(\vec{\mathbf{r}}) = \frac{1}{V}\sum^{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\delta m_{\vec{\mathbf{k}}} \tag{18}$$

We have the integral

$$\delta m_{\vec{\mathbf{k}}} = \int_V \mathrm{d}^D\vec{\mathbf{r}}\,\delta m(\vec{\mathbf{r}})e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \tag{19}$$

with $\vec{\mathbf{k}} = k_1, \ldots, k_D = \frac{2\pi\bar{n}}{L}$ We have $\delta m_{\vec{\mathbf{k}}} \in \mathbb{C}$ but $\delta m(\vec{\mathbf{r}}) \in \mathbb{R}$, hence

$$\delta m_{\vec{\mathbf{k}}} = -\delta m_{-\vec{\mathbf{k}}} \tag{20}$$

$$\sum_{\vec{\mathbf{k}}} \to \frac{V}{(2\pi)^D}\int_{\mathbb{R}} \mathrm{d}\vec{\mathbf{k}} \tag{21}$$

$$\frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \to \frac{1}{V}\frac{V}{(2\pi)^2}\int_{\mathbb{R}} \mathrm{d}^D\vec{\mathbf{k}}\,e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} = \delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \tag{22}$$

$$\frac{1}{V}\int \mathrm{d}^D\vec{\mathbf{r}}\,e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}} = \delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'} \tag{23}$$

write immediately

$$V\delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'} \xrightarrow{V\to\infty} (2\pi)^D\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}') \tag{24}$$

$$a \Rightarrow \left|\vec{\mathbf{k}}\right| \leq \frac{\pi}{a} = \Lambda \tag{25}$$

that is the ultraviolet cut-off.

Change now the notation:

$$\delta m(\vec{\mathbf{r}}) \leftrightarrow \varphi(\vec{\mathbf{r}}), \quad k \leftrightarrow c \tag{26}$$

and obtain

$$\beta\mathcal{H}^{G,>}_{eff} = \int \mathrm{d}^D\vec{\mathbf{r}}\left[\frac{c}{2}(\boldsymbol{\nabla}\phi)^2 + at\phi^2\right] \tag{27}$$

$$\int \mathrm{d}^D r\,\frac{c}{2}(\boldsymbol{\nabla}\varphi)^2 = \frac{c}{2}\frac{1}{V^2}\int \mathrm{d}^D\vec{\mathbf{r}}\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}}\right)\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}'} e^{i\vec{\mathbf{k}}'\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}'}\right)$$
$$= \frac{c}{2}\sum_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}\left(-\vec{\mathbf{k}}\vec{\mathbf{k}}'\right)\varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\underbrace{\int \mathrm{d}^D\vec{\mathbf{r}}\,e^{i(\vec{\mathbf{k}}+\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}}_{(2\pi)^2\delta(\vec{\mathbf{k}}+\vec{\mathbf{k}}')} \tag{28}$$
$$= \frac{c}{2V}\sum_{\vec{\mathbf{k}}}\left|\vec{\mathbf{k}}\right|^2\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'}$$

$$\beta\mathcal{H}^{G,>}_{eff} \to \frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at+ck^2\right)\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'} \tag{29}$$

$$\int \mathrm{D}[\varphi(\vec{\mathbf{r}})] \to \int_{-\infty}^{+\infty}\prod_{|\vec{\mathbf{k}}|<\Lambda}\mathrm{d}(\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\})\,\mathrm{d}(\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}) \tag{30}$$

with $\varphi_{\vec{\mathbf{k}}} \in \mathbb{C}$.

$$\varphi^*_{\vec{\mathbf{k}}} = \varphi_{-\vec{\mathbf{k}}} \tag{31}$$

$$\mathrm{Re}\left\{\varphi_[\vec{\mathbf{k}}]\right\} = \mathrm{Re}\left\{\varphi_{-\vec{\mathbf{k}}}\right\},\quad \mathrm{Im}\left\{\varphi_[\vec{\mathbf{k}}]\right\} = -\,\mathrm{Im}\left\{\varphi_{-\vec{\mathbf{k}}}\right\} \tag{32}$$

$$\mathrm{Tr} = \int_{-\infty}^{+\infty}\prod_{\substack{|\vec{\mathbf{k}}|<\Lambda\\ k_D>0}}\mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\} \tag{33}$$

$$\widetilde{Z}^{G,>}_{GC} = \frac{1}{2}\int_{-\infty}^{+\infty}\prod_{\substack{\vec{\mathbf{k}}\\ |\vec{\mathbf{k}}|<\Lambda}}\mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}\,e^{-\beta\widetilde{\mathcal{H}}_{eff}[\varphi_{\vec{\mathbf{k}}}]} \tag{34}$$

$$x = \mathrm{Re}\,\varphi_{\vec{\mathbf{k}}},\quad y = \mathrm{Im}\,\varphi_{\vec{\mathbf{k}}} \tag{35}$$

$$\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{d}y\,e^{-A(x^2+y^2)} = \frac{\pi}{A} \tag{36}$$

where $A = \frac{1}{2V}$.

$$e^{-\beta\widetilde{F}^{>}_{GL}} = \left(\prod_{\substack{\vec{\mathbf{k}}\\ |\vec{\mathbf{k}}|<\Lambda\\ k_D>0}}\frac{2\pi V}{2at+c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{37}$$

$$\widetilde{F}^{G,>}_{GL} = -\frac{1}{2}k_BT\sum_{|\vec{\mathbf{k}}|>\Lambda}\log\left(\frac{2\pi V}{2at+c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{38}$$

$$c_V = -T\frac{\partial^2 F}{\partial T^2} = \frac{A}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda}\frac{1}{2at+c\left|\vec{\mathbf{k}}\right|^2} - \frac{B}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda}\frac{1}{\left(2at+c\left|\vec{\mathbf{k}}\right|^2\right)^2} \tag{39}$$

Question: what happens if I introduce guassian fluctuations.

It turns out that when we study the asimptotic behaviour of these integrals.

For the first term it turns out that

$$1^{st} \propto \begin{cases}\xi^{4-D} \sim t^{-\nu(4-D)} & D<4\\ <\infty & D>4\end{cases} \tag{40}$$

$$2^{nd} \propto \begin{cases} \xi^{2-D} \sim t^{-\nu(2-D)} & D < 2 \\ < \infty & D > 2 \end{cases} \tag{41}$$

At the end, the behaviour of the specific heat

$$c_V \sim \begin{cases} t^{-\nu(4-D)} & D < 4 \\ \infty & D > 4 \end{cases} \tag{42}$$

figure 1, figure 2.

$$\left\langle \varphi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'} \right\rangle_G = \frac{\int \prod \mathrm{d}\varphi_{\vec{\mathbf{k}}} \dots e^{-\beta \mathcal{H}_{eff}} \phi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'}}{Z_{GL}^{G,>}} = \delta_{\vec{\mathbf{k}},\vec{\mathbf{k}}'} \frac{V}{2at + ck^2} \tag{43}$$

$$\Rightarrow \begin{cases} \nu_G = \frac{1}{2} \\ \eta_G = 0 \end{cases} \tag{44}$$