Lecture 8. Wednesday 6th November, 2019. Compiled: Wednesday 6th November, 2019.

Last time we computed the lamnbda function, the fluctuation of the transfer matrix:

$$\langle s_1 s_R \rangle_N = \frac{\sum_{ij} \langle t_j | S_1 | t_i \rangle \lambda_i^{R-1} \langle t_i | S_R | T_j \rangle \lambda_j^{N-R+1}}{\sum_{k=1}^{n} \lambda_k^N} \tag{1}$$

$$\mathbb{s}_i = \sum_{s_i} |s_i\rangle S_i \langle s_i| \tag{2}$$

$$\mathbb{T} |t_i\rangle = \lambda_i |t_i\rangle \tag{3}$$

For the Ising model:

$$\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

$$\langle s_1 s_R \rangle_N = \frac{\sum_{ij} \langle t_j | S_1 | t_i \rangle \lambda_i^{R-1}/\lambda_+ \langle t_i | S_R | T_j \rangle \lambda_j^{N-R+1}/\lambda_T^{N-R+1}}{\sum_{k=1}^{n} \lambda_k^N/\lambda_+^N} \tag{5}$$

You keep the sum and the one that survives are $j = +, k = +$:

$$\lim_{N\to\infty} \langle s_1 s_R \rangle_N = \sum_{i=\pm}^{n} \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | s_1 | t_i \rangle \langle t_i | s_R | t_+ \rangle \tag{6}$$

rembember that $\lambda_+ > \lambda_T \geq \lambda_1 \dots \lambda_n$

$$\Rightarrow = \langle t_+ | \mathbb{s}_{\mathbb{1}} | t_+ \rangle \langle t_+ | s_R | t_+ \rangle + \sum_{i\neq +}^{n} \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | s_1 | t_i \rangle \langle t_i | s_R | t_+ \rangle \tag{7}$$

$$\Gamma_R = \langle s_1 s_R \rangle - \langle s_1 \rangle \langle s_R \rangle \tag{8}$$

$$\lim_{N\to\infty} \langle \mathbb{s}_i \rangle_N = \langle t_+ | \mathbb{s}_{\mathbb{1}} | t_+ \rangle \tag{9}$$

Because of (9):

$$\Gamma_R = \sum_{i\neq +}^{n} \left(\frac{\lambda_i}{\lambda_+}\right)^{R-1} \langle t_+ | s_1 | t_i \rangle \langle t_i | s_R | t_+ \rangle \tag{10}$$

$$\xi^{-1} = \lim_{N\to\infty} \left\{ -\frac{1}{R-1} \log |\langle s_1 s_R \rangle \langle s_1 \rangle \langle s_R \rangle| \right\} = -\log\left(\frac{\lambda_-}{\lambda_+}\right) - \lim_{R\to\infty} \frac{1}{R-1} \log \langle s_1 s_R \rangle - \langle s_1 \rangle \langle s_R \rangle \tag{11}$$

$$\xi^{-1} = -\log\left(\frac{\lambda_-}{\lambda_+}\right) \tag{12}$$

Let us try to apply this general result to the classical Ising model.

$$\mathbb{T} = \begin{pmatrix} \exp(k+h) & \exp(-k) \\ \exp(-k) & \exp(k-h) \end{pmatrix} \tag{13}$$

Calculate the eigenvalues:

$$|\mathbb{T} - \lambda \mathbb{1}| = 0 = (e^{k+h} - \lambda)(e^{k-h} - \lambda) - e^{-2k} = 0 \tag{14}$$

Calculate the two eigenvalues:

$$\lambda_\pm = e^k \cosh(h) \pm \sqrt{e^{2k} \sinh^2(h) + e^{-2k}} \tag{15}$$

$$f_b = -k_B T \log \lambda_+ = -k k_B T - k_B T \log\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4k}}\right) \tag{16}$$

rembember $k \equiv \beta J, h \equiv \beta H$.

Suppose we are looking for the $T \to 0$ limit, with H *fixed* and J *fixed*, in that case we have $k \to \infty, h \to \infty$.

$$e^{-4k} \overset{k\to\infty}{\longrightarrow} 0 \tag{17}$$

$\sqrt{\sinh^2 h} \approx \sinh h$ and $\cosh(h) + \sinh h \approx \frac{2e^h}{2} e^h$:

$$f \overset{\substack{h\to\infty \\ k\to\infty}}{\approx} -kk_BT - k_BT \log e^h \approx -J - H \quad const \tag{18}$$

So if $T \to \infty, h \to 0, k \to 0$:

$$f_B \approx -J - k_BT \ln 2 \tag{19}$$

The important is that it goes linearly as in Figure 1.

$$m = -\frac{\partial f_b}{\partial H} = -\frac{1}{k_BT}\frac{\partial f_b}{\partial h} = \frac{\sinh h + \frac{2\sinh h \cosh h}{\sqrt{\sinh^2 h + e^{-4k}}}}{\cosh h + \sqrt{\sinh^2 h + e^{-4k}}} \tag{20}$$

$$\chi = \frac{\partial m}{\partial H} = \frac{1}{k_BT}\frac{\partial m}{\partial h} \tag{21}$$

Just look for $h = 0$ ($\to \sinh h = 0$ and $\cosh h = 1$):

$$\xi^{-1} = -\log\left[\frac{1-e^{-2k}}{1+e^{-2k}}\right] = -\log\left[\frac{1}{\coth k}\right] \tag{22}$$

Suppose to study something different from the Ising model, we do not anymore assume spin that can assume values as -1 or +1, but spin that can assume a continuous value. This is the classical Heisenberg model.

0.1 Classica Heisenberg model

The spin are vector or modulus one, so they are versor. Figure 2 $\vec{\mathbf{s_i}} \in \mathbb{R}^3$ with the contraint $|\vec{\mathbf{s_i}}| = 1 \to \hat{\mathbf{s}}_\mathbf{i}$.

At $H = 0$:

$$-\beta\mathcal{H}(\{\vec{\mathbf{S}}_i\}) = k\sum_{i=1}^{N} \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1} (\longrightarrow \sum_i \vec{\mathbf{h}} \cdot \vec{\mathbf{S}}_i) \tag{23}$$

this is a $O(3)$ symmetric. Compute:

$$Z_N(k) = \sum_{\{\vec{\mathbf{S}}\}} e^{-\beta\mathcal{H}} = \mathrm{Tr}\big(\mathbb{T}^N\big) \tag{24}$$

$$\left\langle \vec{\mathbf{S}}_i \middle| \mathbb{T} \middle| \vec{\mathbf{S}}_{i+1} \right\rangle = e^{k\vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}} \tag{25}$$

$$\mathbb{T}_D = \mathbb{P}^{-1}\mathbb{T}\mathbb{P} \tag{26}$$

$$\mathbb{T}\left|t_i\right\rangle = \lambda_i \left|t_i\right\rangle \tag{27}$$

$$\mathbb{T} = \sum_i \left|t_i\right\rangle \lambda_i \left\langle t_i\right| \tag{28}$$

$$\exp\Big[k\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2\Big] = \left\langle \vec{\mathbf{S}}_1 \middle| \mathbb{T} \middle| \vec{\mathbf{S}}_2 \right\rangle = \sum_{i \in eigenval} \lambda_i \left\langle \vec{\mathbf{S}}_1 \middle| t_i \right\rangle \left\langle t_i \middle| \vec{\mathbf{S}}_2 \right\rangle = \sum_i \lambda_i f_i(\vec{\mathbf{S}}_1) f^*(\vec{\mathbf{S}}_2) \tag{29}$$

$$e^{k\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2} \iff e^{i\vec{\mathbf{q}} \cdot \vec{\mathbf{r}}} \tag{30}$$

$$e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{r}}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (i)^l j_l(qr) Y_{lm}(\hat{\mathbf{q}}) Y_{lm}^*(\hat{\mathbf{r}}) \tag{31}$$

$$j_l(qr) = -\frac{(i)^l}{2} \int_0^{\pi} \sin(\theta) e^{iqr\cos(\theta)} P_l(\cos(\theta))\,\mathrm{d}\theta \tag{32}$$

where $P_l(\cos(\theta))$ are the Legendre polynomial of order l.

$$qr = -ik\left|\vec{\mathbf{S}}_1\right|\left|\vec{\mathbf{S}}_2\right| = -ik \tag{33}$$

In our case we have $\hat{\mathbf{q}} = \vec{\mathbf{S}}_1, \hat{\mathbf{r}} = \vec{\mathbf{S}}_2$ Moreover,

$$\lambda_i = \lambda_{lm}(k) = 4\pi(i)^l j_l(-ik) \tag{34}$$

note that there is not m dependence If $l = 0$:

$$\lambda_+ = \lambda_0(k) = 4\pi j_0(-ik) = 4\pi\frac{\sin k}{k} \tag{35}$$

$$\lambda_- = \lambda_1(k) = 4\pi i j_1(-ik) = 4\pi\left[\frac{\cosh k}{k} - \frac{\sinh k}{k^2}\right] \tag{36}$$

0.2 Zipper model

Enea denaturation transition (?). You do not allow to have bubbles in the system as in figure 3, but we want as in figure 4. This is because it is called zipper.