Lecture 16. *Friday 6^{th} December, 2019. Compiled: Friday 6^{th} December, 2019.*

0.0.1 $O(n)$ model

The correct order parameter is a n-dimensional vector $\vec{\eta}$. We consider $\vec{\mathbf{h}} = \vec{\mathbf{0}}$:

$$\mathcal{L}(\vec{\eta}) = \frac{a}{2}\vec{\eta}\cdot\vec{\eta} + \frac{b}{4}(\vec{\eta}\cdot\vec{\eta})^2 + O\Big((\vec{\eta}\cdot\vec{\eta})^3\Big) \tag{1}$$

Generalize to include multicritical points, or phase transitions.

The phase transitions can be obtained by introducing a cubic term in the landau expansion. Remember that in the Ising model we have phase transition derived by symmetry breaking. Now, we have another type of phase transition.

The simplest Landau free energythat depends on a particular field is:

$$\mathcal{L} = at\eta^2 - w\eta^3 + \frac{b}{4}\eta^4 - h\eta \tag{2}$$

Remark. The minus sign w is not important, in this case $w > 0$.

with $t = \frac{T-T^*}{2}$. The equation of state is:

$$\frac{\partial\mathcal{L}}{\partial\eta} = 0 \quad \Rightarrow h = 2at\eta - 3w\eta^2 + b\eta^3 \tag{3}$$

$$h = 0 \quad \Rightarrow 0 = \eta(2at - 3w\eta + \beta\eta^2) \tag{4}$$

The two solutions are:

$$\bar{\eta} = 0, \quad \bar{\eta} = \frac{1}{4b}\Big(3w \pm \sqrt{qw^2 - 16abt}\Big) \tag{5}$$

$$\bar{\eta}_\pm = c \pm \sqrt{c^2 - \frac{at}{b}}, \quad c = \frac{3w}{4b} \tag{6}$$

$$\bar{\eta}_\pm \in \mathbb{R} \iff c^2 > \frac{at}{b} \quad \Rightarrow \frac{T-T^*}{2} < \frac{c^2 b}{a} \equiv t^{**} \equiv \frac{T^{**}-T^*}{2} \tag{7}$$

$$T^{**} = T^* + \frac{2c^2 b}{a} \tag{8}$$

$$T > T^{**} \Rightarrow \bar{\eta}_\pm \notin \mathbb{R} \Rightarrow \bar{\eta} = 0 \tag{9}$$

$$T < T^{**} \Rightarrow \bar{\eta}_\pm = c \pm \sqrt{c^2 - \frac{at}{b}} \in \mathbb{R} \tag{10}$$

$$T = T^{**} \Rightarrow \bar{\eta}_+ = \bar{\eta}_- \tag{11}$$

(Insert figure 1)

$$\mathcal{L}(\vec{\eta} = 0) = \mathcal{L}(\eta = \bar{\eta}_+) \tag{12}$$

In the last plot (the four) we see that there are two minima in the same line. That is a first order transition. Going under the $T < T_t$ the second minima decrease.

Another possibility is studying multicritical point.

$$\mathcal{L}_h(T,\Delta,\eta) = \frac{a(t,\Delta)}{2}\eta^2 + \frac{b(t,\Delta)}{4}\eta^4 + c\eta^6 - h\eta \tag{13}$$

Consider Δ (Δ_c is a critical value):

- $\Delta < \Delta_c$: as T decreases, it will reach a value, we have a that decreases.

$$T = T_c(\Delta) \Rightarrow \begin{cases} a(T_c,\Delta) = 0 \\ b(T_c,\Delta) > 0 \end{cases} \tag{14}$$

- $\Delta > \Delta_c$: as T decreases, we will have a and b that will decrease too. At that point:
$$b(\bar{T}, \Delta) = 0 \tag{15}$$
The free energy now is
$$\mathcal{L} = \frac{a}{2}\eta^2 + c\eta^6 \tag{16}$$
The plot of the free energy is in figure 2. We have the coexistence of three lines.

Tricritical point:
$$\Delta = \Delta_t, \quad T = T_t \tag{17}$$
$$a(\Delta_t, T_t) = b(\Delta_t, T_t) = 0 \tag{18}$$
$$\mathcal{L}_t = c\eta^6 - h\eta \tag{19}$$

As we approach the critical point $T \to T_c$, the correlation length $\xi \sim |T - T_c|^{-\nu}$ diverges.

Maybe mean field is not a very good approximation in proximity of the critical point. Question: how bad is the mean field approximation in proximity of the critical point?
$$\langle S_i S_j \rangle \stackrel{MF}{\to} \langle S_i \rangle \langle S_j \rangle \tag{20}$$
Calculate the error:
$$E_{ij} = \frac{|\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle|}{\langle S_i \rangle \langle S_j \rangle} \tag{21}$$
Define
$$G_c(i,j) \equiv \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle = \langle (S_i - \langle S_i \rangle)(S_j - \langle S_j \rangle) \rangle \tag{22}$$
If we want to compute the error in the mean field, is always zero. So, if we want calculate the average with respect to fluctuations it does not work. We can either look at the variation in which the field is the internal one, or we can somehow try to make a variation not because of thermal fluctuations but because we control it. We do this by using an external field. This is the response theory with a variation of the field.

In order to do that, instead using an H we use an H_i.
$$Z = \mathrm{Tr}_{\{S\}} \left(e^{-\beta\left(-J\sum_{\langle ij \rangle} S_i S_j - \sum_i H_i S_i\right)} \right) \tag{23}$$
The definition of the thermal average is
$$\langle S_i \rangle = \frac{\mathrm{Tr}_{\{S\}} \left(S_i e^{-\beta\left(-J\sum_{\langle ij \rangle} S_i S_j - \sum_i H_i S_i\right)} \right)}{Z} = \beta^{-1}\frac{\partial \ln Z}{\partial H_i} = -\frac{\partial F}{\partial H_i} \tag{24}$$
$$\langle S_i S_j \rangle = \frac{\beta^{-1}}{Z} \frac{\partial^2 Z}{\partial H_i \partial H_j} \tag{25}$$
$$G_c(i,j) = \beta^{-1} \frac{\partial^2 \ln Z}{\partial H_i \partial H_j} = -\frac{\partial^2 F(\{H_i\})}{\partial H_i \partial H_j} \tag{26}$$
$$\frac{\partial}{\partial H_j} \langle S_i \rangle = \frac{\partial}{\partial H_j}\left[-\frac{\partial F}{\partial H_i}\right] = G_c(i,j) \tag{27}$$
$$M = \sum_i \langle S_i \rangle \tag{28}$$
$$\frac{\partial M}{\partial H_j} = \sum_i \frac{\partial \langle S_i \rangle}{\partial H_j} = \sum_i G_c(i,j) \tag{29}$$

$$H_j = H_j(H) \tag{30}$$

$$\frac{\partial M}{\partial H} = \sum_j \frac{\partial M}{\partial H_j}\frac{\partial H_j}{\partial H} = \beta \sum_{ij} G_c(i,j) \tag{31}$$

the last one is the susceptibility χ_T. Therefore,

$$\chi_T = \beta \sum_{i,j} G_c(i,j) \tag{32}$$

$$G_c(i,j) \to G_c(|i-j|) \sim G(|\vec{\mathbf{r}}|) \tag{33}$$

More or less, we want to compute the total relative error

$$E_{TT} = \frac{\int_{V_\xi} \mathrm{d}^D\vec{\mathbf{r}}\, G_c(r)}{\int_{V_\xi} \mathrm{d}^D\vec{\mathbf{r}}\, \eta^2} \ll 1 \tag{34}$$

This quantity is related to χ_T. Because of the fluctutations we can say that the quantity above is

$$\sim \frac{\beta^{-1}\chi_T}{\int_{V_\xi} \mathrm{d}^D\vec{\mathbf{r}}\, \eta^2} \tag{35}$$

where $\chi_T \sim t^{-\gamma}$ and the denominator it is $\sim t^{2\beta}\xi^D$.

$$\frac{t^{-\gamma}}{t^{2\beta}t^{-\nu D}} \tag{36}$$

$$E \overset{t\to 0}{\sim}{}^{-\gamma-2\beta+\nu D} \ll 1 \tag{37}$$

$$-\gamma - 2\beta + \nu D \geq 0 \Rightarrow D > \frac{\gamma+2\beta}{\nu} \tag{38}$$

this is called the *Ginzburg criterium.*

In the mean field we have: $\gamma = 1$, $\beta = 1/2$. We obtain:

$$D > \frac{2}{\nu} \tag{39}$$

We have $\nu_{MF} = 1/2$. THerefore the dimension is $D > 4$ for the mean field. The

$$D_c = \frac{\gamma+2\beta}{2} \tag{40}$$

is called *upper critical dimension.*

Now we have a lower critical dimension (remember the last lessons!!) and an upper one.