The Hamiltonian is:

$$- \mathcal{H}(\{S\}) = \frac{J}{2N} \sum_{ij} S_i S_j + H \sum_i S_i \tag{1}$$

where the second term is associated to an external magnetic field. We have a function:

$$Z_N(T, J, H) = \sum_{\{S\}} \exp\left[\frac{\beta J}{2N} \sum_{ij} S_i S_j + \beta H \sum_i S_i \right] \tag{2}$$

where the term $\sum_{ij} S_i S_j = \left(\sum_i S_i \right)^2$. Rerite the partition function:

$$Z_N(T, J, H) = \sum_{\{S\}} \exp\left[\frac{K}{2N} \left(\sum_i S_i \right)^2 + h \sum_i S_i \right] \tag{3}$$

This is the Hubbard-stratonovitch transformation (you can do it in any dimension). The idea is to rewrite something as a square. Let us define:

$$x \equiv \sum_i S_i \tag{4}$$

The identity is the following

$$e^{\frac{K x^2}{2N}} = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{NK}{2} y^2 + Kxy} \, dy \tag{5}$$

The real part of K $\mathrm{Re}(K) > 0$. y is a random field that follows a random distribution. We want to do:

Proof.

$$- \frac{NK}{2} y^2 + Kxy = -\frac{NK}{2} \left(y - \frac{x}{N} \right)^2 + \frac{K x^2}{2N} \tag{6}$$

Then we integrate:

$$e^{\frac{K x^2}{2N}} \int_{-\infty}^{+\infty} e^{-\frac{NK}{2} \left(y - \frac{x}{N} \right)^2} \, dy \tag{7}$$

if $z \equiv y - \frac{x}{N}$, $dz = dy$:

$$\rightarrow \int_{-\infty}^{+\infty} e^{-\alpha z^2} \, dz = \sqrt{\frac{\pi}{\alpha}} \tag{8}$$

with $\alpha \equiv \frac{NK}{2}$, therefore

$$= e^{\frac{K x^2}{2N}} \sqrt{\frac{2\pi}{NK}} \tag{9}$$

$\square$

We have:

$$Z_N = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} dy \, e^{-\frac{NK}{2} y^2} \underbrace{\left[\sum_{\{S\}} e^{(h+Ky) \sum_i S_i} \right]}_{Q_y} \tag{10}$$

Sometimes the y is called *auxiliary field*.

$$Q_y = \prod_i^N \left(\sum_{S_i = \pm 1} \exp[(h + Ky)S_i] \right) = (2\cosh(h + Ky))^N \tag{11}$$

it becomes:

$$= \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} dy\, e^{-\frac{NK}{2}y^2} (2\cosh(h+Ky))^N = \sqrt{\frac{NK}{2\pi}} \int_{-\infty}^{+\infty} dy\, e^{N\alpha(K,h,y)} \quad (12)$$

where

$$\alpha(K,h,y) = \ln\left[2\cosh(h+Ky)\right] - \frac{K}{2}y^2 \quad (13)$$

Maybe we can replace the medium of the integral with the maximum of the integrand, we say that all the information is coming only from a bit of information. Replace the all integral with the integrand computed where it is maximum. That is an approximation and we are loosing information, it depends on the form of the function. For example, for a delta function it works better. This is the Saddle point approximation. In general:

$$\int_{-\infty}^{+\infty} f(x)\, dy \to f(\bar{x}) \quad (14)$$

where $\bar{x} = \max_x f(x)$

$$Z_N \approx \sqrt{\frac{NK}{2\pi}} \max_y \left[\exp[N\alpha]\right] \quad (15)$$

We call y_s be the value of y at which $\alpha(K,h,y_s) = \max_y \alpha$. Therefore:

$$\Rightarrow Z_N^S(K,h) \sim \sqrt{\frac{NK}{2\pi}} e^{N\alpha(K,h,y_s)} \quad (16)$$

Therefore when we are ample to compute the y_s we can do this approximation.

$$f_b = \lim_{N\to\infty} \frac{1}{N}(-k_B T \log Z_N) = -k_B T \alpha(h,K,y_s) \quad (17)$$

Consider the condition $\frac{\partial \alpha}{\partial y} = 0$:

$$\frac{\partial \alpha}{\partial y} = \frac{\sinh(h+Ky)K}{\cosh(h+Ky)} - Ky = 0 \quad (18)$$

$$\Rightarrow y_s = \tanh(h+Ky_s) \quad (19)$$

The magnetization is

$$m = -\frac{\partial f}{\partial H} \underset{\text{to do!}}{=} \frac{\partial \alpha}{\partial h} + \frac{O(\log N)}{N} = \tanh(Ky_s + h) \quad (20)$$

So at the end

$$m = \tanh(h + Km) \quad (21)$$

this will give us the equilibrium value of m! We have solved analitically this one.

The most important approximation is the *mean field approximation*. Try to simplify the problem by neglectic the correlation between the fluctuations.

$$Z_N = \sum_{\{S\}} e^{\beta\left[\frac{1}{2}\sum_{\langle ij\rangle} J_{ij} S_i S_j + \sum_i H_i S_i\right]} \quad (22)$$

The magnetization per spin is $m \equiv \langle S_i \rangle$. Let us consider $H_i = H$ uniform. The simplest approximation is: we have our nasty term, let us rewrite it in this way

$$S_i S_j = (S_i - m + m)(S_j - m + m) = \underbrace{(S_i - m)(S_j - m)}_{S_i - \langle S_i \rangle} + m^2 + m(S_j - m) + m(S_i - m)$$

$$(23)$$

where $S_i - \langle S_i \rangle$ are the fluctuations, so the mean field neglects correlations of the fluctuations:

$$S_i S_j = (S_i - m + m)(S_j - m + m) = \underbrace{(S_i - m)(S_j - m)} + m^2 + m(S_j - m) + m(S_i - m) \tag{24}$$

$$\Rightarrow \approx m^2 + m(S_i - m) + m(S_j - m) \tag{25}$$

We have:

$$\frac{1}{2} \sum_{\langle ij \rangle} S_i S_j \overset{MF}{\approx} \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \big[-m^2 + m(S_i + S_j) \big] \tag{26}$$

$$\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} m(S_i + S_j) = \frac{2}{2} \sum_{\langle ij \rangle} J_{ij} m S_i \tag{27}$$

Let us write in this way: (n.n. = neirest neighbour)

$$\sum_{j \text{ n.n. of } i} J_{ij} = z \hat{J}_i \simeq z \hat{J} \tag{28}$$

Therefore (27) became

$$= m \hat{J} z \sum_i S_i \tag{29}$$

If you do the same story over j for the sum over i:

$$Z_N \approx e^{-\frac{\beta m^2 N}{2} z \hat{J}} \sum_{\{S\}} e^{\beta(\hat{J}zm + H) \sum_i S_i} \tag{30}$$

$$\approx e^{-\frac{\beta m^2 N}{2} z \hat{J}} \Big(2 \cosh[\beta(\hat{J}zm + H)] \Big)^N \tag{31}$$

The free energy mean field is:

$$f_b^{MF} = -\frac{1}{N} k_B T \log Z_N = \frac{1}{2} \hat{J} z m^2 - k_B T \log \big[2 \cosh\big(\beta(\hat{J}zm + H) \big) \big] - k_B T \ln 2 \tag{32}$$

We are replacing the interaction of the J with a field close to the S_i. We called $\hat{J}zm = H_{eff}$, the mean field!

Self consistency must satisfy the thermodynamic relation:

$$m = -\left(\frac{\partial f}{\partial H} \right)_T \tag{33}$$

this is the *self-consistent constraint*. The self consistent relation is therefore:

$$m = \tanh\Big(\beta(\hat{J}zm + H) \Big) \tag{34}$$

The iperbolic function (insert Figure in cellulare (copiare da qualcuno)). $H = 0$. For the value of the parameter $\beta \hat{J} z > 1$, we find solution $m_0 \neq 0$. Two solution are symmetric because they are related by the z symmetry.

If we look for the csase $\beta \hat{J} z < 1$, we have just one solution that is equal to zero $m_0 = 0$.

The condition $\beta_c \hat{J} z = 1$, define the critical point

$$\frac{z \hat{J}}{k_B T_c} = 1 \Rightarrow T_c = \frac{z \hat{J}}{k_B} \neq 0 \tag{35}$$

The critical point is characterized by the order parameter that is zero. Now we want to expand the free energy around the critical point. Let us put $H = 0$.

$$\cosh(x) = 1 + \underbrace{\frac{x^2}{2} + \frac{x^4}{4!}}_{t} + \dots \tag{36}$$

$$\log(1 + t) \tag{37}$$

$$f(m, T) = cost + \frac{A}{2}m^2 + \frac{B}{4}m^4 + O(m^6) \tag{38}$$

$$A = \hat{J}z\left(1 - \beta\hat{J}z\right) \tag{39}$$

$$B = \beta^2\frac{(\hat{J}z)^4}{3} > 0 \tag{40}$$

Figure in cellulare 2.