0.0.1 Solution of (??) by Fourier transform

Lecture 18. *Friday 13th December, 2019. Compiled: Friday 31st January, 2020.*

Let us do the Fourier transform of Eq.(??):

$$\left(-\boldsymbol{\nabla}^2 + \xi^{-2}(t)\right)G_c(\vec{\mathbf{r}} - \vec{\mathbf{r}}') = \frac{k_B T}{k}\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$$

If we define $\vec{\mathbf{x}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ and we use the following convention for the Fourier transform $\widetilde{G}(q)$ of G:

$$\widetilde{G}(q) = \int_{-\infty}^{+\infty} G_c(|\vec{\mathbf{x}}|)e^{-i\vec{\mathbf{q}}\cdot|\vec{\mathbf{x}}|}\,\mathrm{d}^d|\vec{\mathbf{x}}|$$

then transforming both sides of the equation we get:

$$\left(q^2 + \xi^{-2}\right)\widetilde{G}(q) = \frac{k_B T}{k} \quad \Rightarrow \widetilde{G}(q) = \frac{k_B T}{k}\frac{1}{q^2 + \xi^{-2}} \tag{1}$$

where $q = |\vec{\mathbf{q}}|$. From this last equation we can also see that when $T = T_c$, since $\xi \to \infty$ we have $\widetilde{G}(q) \simeq \frac{1}{q^2}$ and so performing the inverse Fourier transform one gets

$$G_c(|\vec{\mathbf{x}}|) = |\vec{\mathbf{x}}|^{2-d}$$

from which we have that the critical exponent η is null (we will see that explicitly once we have computed G). In fact, at $T = T_c$ we have previously defined

$$G(r) \sim |\vec{\mathbf{x}}|^{2-d-\eta}$$

hence, in this case we have $\eta = 0$. Therefore, reminding that $\vec{\mathbf{x}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ we can now determine $G(\vec{\mathbf{x}})$ with the Fourier antitransform:

$$G(\vec{\mathbf{x}}) = \int \frac{\mathrm{d}^d\vec{\mathbf{q}}}{(2\pi)^d}\frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}}}{q^2 + \xi^{-2}} \tag{2}$$

This integral is a bit tedious to compute, and in general its result depends strongly on the dimensionality d of the system; the general approach used to solve it is to shift to spherical coordinates in $\mathbb{R}^d$ and then complex integration for the remaining part, which involves $|\vec{\mathbf{q}}|$. In order to do some explicit computations, let us consider the case $d = 3$; we will then have:

$$\begin{aligned}
G(\vec{\mathbf{x}}) &= \int \frac{\mathrm{d}^3\vec{\mathbf{q}}}{(2\pi)^3}\frac{e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}}}{q^2 + \xi^{-2}} \overset{\substack{\text{spherical}\\\text{coordinates}}}{=} \frac{1}{(2\pi)^3}\int_0^\infty \frac{q^2}{q^2 + \xi^{-2}}\,\mathrm{d}q \int_{-1}^{+1} e^{iq|\vec{\mathbf{x}}|\cos\theta}\,\mathrm{d}(\cos\theta)\int_0^{2\pi}\mathrm{d}\varphi \\
&\overset{z\equiv\cos(\theta)}{=} \frac{2\pi}{(2\pi)^3}\int_0^\infty \frac{q^2}{q^2 + \xi^{-2}}\,\mathrm{d}q\left[\frac{e^{iq|\vec{\mathbf{x}}|z}}{iq|\vec{\mathbf{x}}|}\right]_{-1}^{1} = \frac{1}{(2\pi)^2|\vec{\mathbf{x}}|}\int_0^\infty \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q
\end{aligned}$$

This last integral can be computed, using the residue theorem, extending it to the complex plane:

$$I = \int_0^\infty \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q = \frac{1}{2}\,\mathrm{Im}\oint \frac{ze^{iz|\vec{\mathbf{x}}|}}{(z^2 + \xi^{-2})}\,\mathrm{d}z$$

There are two poles at $z_P = \pm i\xi^{-1}$; we choose as the contour of integration γ which contains only the pole at $+i\xi^{-1}$ (see Figure 1) and so using the residue theorem we will have:

$$I = \frac{1}{2}\,\mathrm{Im}\oint_\gamma \frac{ze^{iz|\vec{\mathbf{x}}|}}{(z + i\xi^{-1})(z - i\xi^{-1})}\,\mathrm{d}z \overset{\substack{\text{residue}\\\text{theorem}}}{=} \frac{1}{2}\,\mathrm{Im}\left[2\pi i\mathrm{Res}(i\xi^{-1})\right]$$

Since,

$$\mathrm{Res}(i\xi^{-1}) = \frac{i\xi^{-1}e^{-\xi^{-1}|\vec{\mathbf{x}}|}}{2i\xi^{-1}} = \frac{e^{-|\vec{\mathbf{x}}|/\xi}}{2}$$

we obtain

$$I = \frac{1}{2}\,\mathrm{Im}\left[2\pi i \mathrm{Res}(i\xi^{-1})\right] = \frac{\pi}{2}e^{-|\vec{\mathbf{x}}|/\xi} \tag{3}$$

Therefore, in the end we have:

$$G(|\vec{\mathbf{x}}|) = \frac{1}{8\pi}\frac{e^{-|\vec{\mathbf{x}}|/\xi}}{|\vec{\mathbf{x}}|} \tag{4}$$

We see now clearly that the correlation function has indeed an exponential behaviour (as we have stated also in long range correlations) and that ξ is really the correlation length; furthermore, $G(\vec{\mathbf{x}}) \sim 1/|\vec{\mathbf{x}}|$ and from the definition of the exponent η we have $G(\vec{\mathbf{x}}) \sim 1/|\vec{\mathbf{x}}|^{d-2+\eta}$, so since $d = 3$ we indeed have $\eta = 0$.

One can also solve the equation for $G(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$ by using the spherical coordinates and use the Bessel functions.

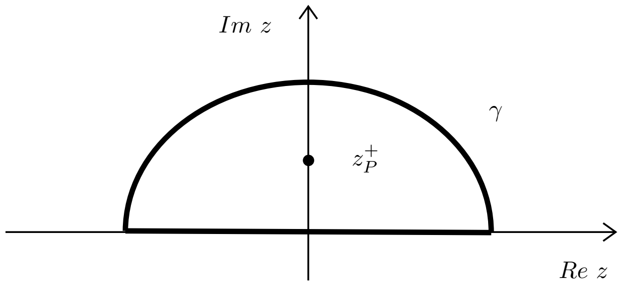


Figure 1: Positive integration contour γ in the complex plane for the integral I. It contains only the pole at $+i\xi^{-1}$.

Therefore, we have seen that for the Ising model $\nu = 1/2$. If we also consider the values of the other critical exponents we see that the upper critical dimension for this model is $d = 4$. In other words, mean field theories are actually good approximations for the Ising model if $d \geq 4$. We will later see some other confirmations of this fact.

0.1 Including fluctuations at the Gaussian level (non interacting fields)

Until now even if we have introduced Ginzburg-Landau theory we are still neglecting the effects of the fluctuations since we are regarding the mean field theory approximation for non-homogeneous systems as a saddle point approximation of a more general theory; in other words, since we are approximating

$$Z_{GL}[h] \overset{\substack{\text{saddle}\\\text{point}}}{\simeq} Z^0_{GL}[h] = e^{-L[m_0(\vec{\mathbf{r}})]}$$

we are still regarding the magnetization m as non fluctuating over the system. In order to include the fluctuations we must do more and go further the simple saddle point approximation. The simplest way we can include fluctuations in our description is expanding Z expressed as a functional integral around the stationary solution and keeping only quadratic terms; this means that we are considering fluctuations that follow a normal distribution around the stationary value. The important thing to note, however, is that in this approximation these fluctuations are independent, i.e.

they do not interact with each other. As we will see, with this assumption the values of some critical exponents will differ from the "usual" ones predicted by mean field theories.

Hence, let us introduce fluctuations at the Gaussian level. Consider consider $h = 0$ and $m_0(\vec{\mathbf{r}}) = m_0$ be the solution of the saddle point approximation. Let us expand the general expression

$$\beta\mathcal{H}_{eff}[m(\vec{\mathbf{r}})] = \int \left(atm^2 + \frac{b}{2}m^4 + \frac{k}{2}\left(\vec{\boldsymbol{\nabla}}m\right)^2 \right) \mathrm{d}^d\vec{\mathbf{r}}$$

by using

$$m(\vec{\mathbf{r}}) = m_0 + \delta m(\vec{\mathbf{r}})$$

If we assumed that the fluctuations $\delta m(\vec{\mathbf{r}})$ are small, we would obtain:

$$\begin{aligned}(\boldsymbol{\nabla} m)^2 &= (\boldsymbol{\nabla}(m_0 + \delta m))^2 = (\boldsymbol{\nabla}(\delta m))^2 \\ m^2 &= m_0^2 + 2m_0\delta m + (\delta m)^2 \\ m^4 &= m_0^4 + 4m_0^3\delta m + 6m_0^2(\delta m)^2 + 4m_0\delta m^3 + (\delta m)^4\end{aligned}$$

Hence, we have

$$\beta\mathcal{H}_{eff} = V \underbrace{\left(atm_0^2 + \frac{b}{2}m_0^4 \right)}_{A_0} + \int \left(\frac{k}{2}\left(\vec{\boldsymbol{\nabla}}(\delta m)\right)^2 + \left(at + 3bm_0^2\right)\delta m^2 + 2bm_0\delta m^3 + \frac{b}{2}\delta m^4 \right) \mathrm{d}^d\vec{\mathbf{r}} \tag{6}$$

where V is the volume of the system and the term proportional to δm, $\left(2atm_0 + 2bm_0^3\right)$, is zero since m_0 is the solution of the extremal condition (m_0 is the stationary solution)

$$\left.\frac{\delta\mathcal{H}_{eff}}{\delta m}\right|_{m=m_0} = 0$$

For simplicity let us first consider $T > T_c$; in this case, we know that $m_0 = 0$ and hence,

$$m(\vec{\mathbf{r}}) = m_0 + \delta m(\vec{\mathbf{r}}) = \delta m(\vec{\mathbf{r}})$$

We have also $A_0 = 0$, $3bm_0^2\delta m^2 = 0$ and $2bm_0\delta m^3 = 0$. Taking all of this into account, we obtain:

$$\beta\mathcal{H}_{eff}^{T>T_c}(\delta m) = \int \mathrm{d}^d\vec{\mathbf{r}} \left(\frac{k}{2}\left(\vec{\boldsymbol{\nabla}}\delta m\right)^2 + at(\delta m)^2 + \frac{b}{2}(\delta m)^4 \right)$$

The Gaussian approximation consists in neglecting the quartic term $(\delta m)^4$, hence we finally obtain:

$$\beta\mathcal{H}_{eff}^{G,T>T_c}(\delta m) \simeq \int \mathrm{d}^d\vec{\mathbf{r}} \left(\frac{k}{2}\left(\vec{\boldsymbol{\nabla}}\delta m\right)^2 + at(\delta m)^2 \right) \tag{7}$$

Remark. It is important to understand that these are fluctuations with respect to the solution m_0.

In order to compute this integral it is more convenient to shift to Fourier space.

0.1.1 Gaussian approximation for the Ising model in Ginzburg-Landau theory

For simplicity, consider the case $T > T_c$; now, let us compute the partition function

$$Z_G(\delta m) = \int \mathcal{D}[\delta m] e^{-\int \mathrm{d}^d r \left(\frac{k}{2}(\boldsymbol{\nabla}\delta m)^2 + at(\delta m)^2\right)} \tag{8}$$

in the Fourier space. Let us make some remarks on what happens when we apply Fourier transformations in this case. If our system is enclosed in a cubic box of volume $V = L^d$ (with periodic boundary conditions), we can define the Fourier components of the magnetization as:

$$\delta m_{\vec{\mathbf{k}}} = \int_V \delta m(\vec{\mathbf{r}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}} \tag{9}$$

where $\vec{\mathbf{k}} = k_1, \ldots, k_d = \frac{2\pi\vec{\mathbf{n}}}{L}$ with $k_\alpha = \frac{2\pi}{L} n_\alpha$ and $n_\alpha = 0, \pm 1, \ldots$. We can therefore expand the magnetization in a Fourier series:

$$\delta m(\vec{\mathbf{r}}) = \frac{1}{V} \sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} (\delta m_{\vec{\mathbf{k}}}) \tag{10}$$

Substituting this expression of m in $\delta m_{\vec{\mathbf{k}}}$ we obtain an integral representation for the Kronecker delta; in fact:

$$\delta m_{\vec{\mathbf{k}}} = \sum_{\vec{\mathbf{k}}'} \delta m_{\vec{\mathbf{k}}'} \left(\frac{1}{V} \int_V e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}} \right)$$

and this is true only if:

$$\frac{1}{V} \int_V e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}} = \delta_{\vec{\mathbf{k}},\vec{\mathbf{k}}'} \quad \Rightarrow \int_V e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}} = V\delta_{\vec{\mathbf{k}},\vec{\mathbf{k}}'}$$

Let us now make an observations; since $\delta m(\vec{\mathbf{r}}) \in \mathbb{R}$ (is real) we have that

$$\delta m^*_{\vec{\mathbf{k}}} = \delta m_{-\vec{\mathbf{k}}}$$

Useful relations

- Sometimes it is useful to convert the sum over $\vec{\mathbf{k}}$ by an integral by using the density of states in the $\vec{\mathbf{k}}$ space that is $V/(2\pi)^d$, hence one useful relation is

$$\boxed{\sum_{\vec{\mathbf{k}}} \to \frac{V}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d\vec{\mathbf{k}}} \tag{11}$$

- From the relation Eq.(11), we have:

$$\frac{1}{V} \sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \to \frac{1}{\cancel{V}} \frac{\cancel{V}}{(2\pi)^d} \int_{\mathbb{R}^D} \mathrm{d}^d\vec{\mathbf{k}}\, e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} = \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$$

  Hence, another useful relation is:

$$\boxed{\frac{1}{V} \sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \to \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')} \tag{12}$$

- As previously shown, by inserting $m(\vec{\mathbf{r}})$ into the expression for $m_{\vec{\mathbf{k}}}$

$$m(\vec{\mathbf{r}}) = \frac{1}{V} \sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}'\vec{\mathbf{r}}} m_{\vec{\mathbf{k}}}, \qquad m_{\vec{\mathbf{k}}} = \int_V m(\vec{\mathbf{r}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}}$$

  one gets

$$\boxed{\int_V e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\, \mathrm{d}^d\vec{\mathbf{r}} = V\delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}} \tag{13}$$

- Finally, since

$$\int_V e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\,\mathrm{d}^d\vec{\mathbf{r}} = V\delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'} \overset{V\to\infty}{\longrightarrow} (2\pi)^d\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}')$$

  We get the last useful relation:

$$V\delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'} \overset{V\to\infty}{\longrightarrow} (2\pi)^d\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}') \tag{14}$$

Remark. Our coarse graining procedure is based on the construction of blocks which have a linear dimension that cannot be smaller than a, the characteristic microscopic length of the system; this means that not all the $\vec{\mathbf{k}}$ are allowed, and in particular we must have

$$\left|\vec{\mathbf{k}}\right| \le \frac{\pi}{a} = \Lambda$$

It is the ultraviolet cut-off!

Gaussian Hamiltonian in Fourier space

We want to compute Eq.(7) in the Fourier space. For simplicity, let us change notation as follows

$$\delta m(\vec{\mathbf{r}}) \leftrightarrow \varphi(\vec{\mathbf{r}}), \qquad k \leftrightarrow c$$

Hence, Eq.(7) becomes

$$\beta\mathcal{H}_{eff}^{G,T>T_c}[\varphi] = \int \left[\frac{c}{2}(\boldsymbol{\nabla}\varphi)^2 + at\varphi^2\right]\mathrm{d}^d\vec{\mathbf{r}} \tag{15}$$

with

$$\varphi(\vec{\mathbf{r}}) = \frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}}, \quad \varphi_{\vec{\mathbf{k}}} \in \mathbb{C}$$

Let us consider the terms of expression (15) separately:

- Term $at\varphi^2$: the integral we are considering is

$$\int at\varphi^2(\vec{\mathbf{r}})\,\mathrm{d}^d\vec{\mathbf{r}} = \frac{at}{V^2}\sum_{\vec{\mathbf{k}},\vec{\mathbf{k}}'}\int_{\mathbb{R}^d} e^{i(\vec{\mathbf{k}}+\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\,\mathrm{d}^d\vec{\mathbf{r}} \overset{(13),(14)}{=} \frac{at}{V^2}\sum_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}\varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}(2\pi)^d\delta(\vec{\mathbf{k}}+\vec{\mathbf{k}}')$$

  On the other hand,

$$(2\pi)^d\delta(\vec{\mathbf{k}}+\vec{\mathbf{k}}') \overset{V\gg1}{\longrightarrow} V\delta_{\vec{\mathbf{k}},-\vec{\mathbf{k}}'}$$

  Hence, the term becomes

$$\int at\varphi^2(\vec{\mathbf{r}})\,\mathrm{d}^d\vec{\mathbf{r}} \overset{V\gg1}{\longrightarrow} \frac{1}{2V}\sum_{\vec{\mathbf{k}}} 2at\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'} \tag{16}$$

- Term $\frac{c}{2}(\boldsymbol{\nabla}\varphi)^2$: consider the integral

$$\int \frac{c}{2}(\boldsymbol{\nabla}\varphi)^2\,\mathrm{d}^d\vec{\mathbf{r}} = \frac{c}{2}\frac{1}{V^2}\int\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}}\right)\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}'} e^{i\vec{\mathbf{k}}'\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}'}\right)\mathrm{d}^d\vec{\mathbf{r}}$$
$$= \frac{c}{2V^2}\sum_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}\left(-\vec{\mathbf{k}}\cdot\vec{\mathbf{k}}'\right)\varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\underbrace{\int e^{i(\vec{\mathbf{k}}+\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}\,\mathrm{d}^d\vec{\mathbf{r}}}_{(2\pi)^d\delta(\vec{\mathbf{k}}+\vec{\mathbf{k}}')\to V\delta_{\vec{\mathbf{k}},-\vec{\mathbf{k}}'}} = \frac{c}{2V}\sum_{\vec{\mathbf{k}}}\left|\vec{\mathbf{k}}\right|^2\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'}$$

  Hence, the term becomes

$$\int \frac{c}{2}(\boldsymbol{\nabla}\varphi)^2\,\mathrm{d}^d\vec{\mathbf{r}} \overset{V\gg1}{\longrightarrow} \frac{c}{2V}\sum_{\vec{\mathbf{k}}}\left|\vec{\mathbf{k}}\right|^2\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'} \tag{17}$$

In conclusion, the Gaussian Hamiltonian in Eq.(7) in the Fourier space is the sum of the two terms in Eq.(16) and Eq.(17):

$$\boxed{\beta\mathcal{H}_{eff}^{G,T>T_c}[\varphi] \xrightarrow{V\gg 1} \frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at + c\left|\vec{\mathbf{k}}\right|^2\right)\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'}} \tag{18}$$

Now, thinking about the functional integral form of the partition function, what does the measure $\int \mathcal{D}[\varphi]$ become in Fourier space?

Since $\varphi(\vec{\mathbf{r}})$ is expressed in terms of the Fourier modes $\varphi_{\vec{\mathbf{k}}}$, which are in general complex,

$$\varphi(\vec{\mathbf{r}}) = \frac{1}{V}\sum_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}}e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}, \quad \varphi_{\vec{\mathbf{k}}} \in \mathbb{C}$$

the measure of the integral becomes:

$$\int \mathcal{D}[\varphi(\vec{\mathbf{r}})] \to \int_{-\infty}^{+\infty}\prod_{|\vec{\mathbf{k}}|<\Lambda} \mathrm{d}(\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\})\,\mathrm{d}(\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}) \tag{19}$$

However, since $\varphi(\vec{\mathbf{r}})$ is real (i.e. $\varphi^*_{\vec{\mathbf{k}}} = \varphi_{-\vec{\mathbf{k}}}$) the real and imaginary parts of the Fourier modes are not independent, because we have:

$$\begin{cases}\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\} = \mathrm{Re}\left\{\varphi_{-\vec{\mathbf{k}}}\right\} \\ \mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\} = -\,\mathrm{Im}\left\{\varphi_{-\vec{\mathbf{k}}}\right\}\end{cases}$$

This means that if we use the measure we have written above (Eq.(19)) we would integrate twice on the complex plane; we must therefore change the measure so as to avoid this double counting. We can for example simply divide everything by 2, or restrict the integration on the region where for example the last coordinate of $\vec{\mathbf{k}}$, let us call it k_z, is positive. Therefore:

$$\begin{aligned}\mathrm{Tr} \equiv \int \mathcal{D}[\varphi(\vec{\mathbf{r}})] &= \int_{-\infty}^{+\infty}\prod_{\substack{|\vec{\mathbf{k}}|<\Lambda \\ k_z>0}} \mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\} \\ &= \frac{1}{2}\int_{-\infty}^{+\infty}\prod_{|\vec{\mathbf{k}}|<\Lambda} \mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}\end{aligned} \tag{20}$$

For sake of brevity, we define:

$$\mathrm{Tr} \equiv \int_{-\infty}^{+\infty}{\prod_{\vec{\mathbf{k}}}}' \mathrm{d}\varphi_{\vec{\mathbf{k}}}\,, \qquad {\prod_{\vec{\mathbf{k}}}}' \mathrm{d}\varphi_{\vec{\mathbf{k}}} \equiv \prod_{\substack{|\vec{\mathbf{k}}|<\Lambda \\ k_z>0}} \mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\} \tag{21}$$

In the end, in the Fourier space we have:

$$\widetilde{Z}_G^{T>T_c} = \int \mathcal{D}[\varphi(\vec{\mathbf{r}})]e^{-\beta\widetilde{\mathcal{H}}_{eff}^{T>T_c}[\varphi_{\vec{\mathbf{k}}}]} = \int_{-\infty}^{+\infty}\left({\prod_{\vec{\mathbf{k}}}}' \mathrm{d}\varphi_{\vec{\mathbf{k}}}\right)e^{-\beta\widetilde{\mathcal{H}}_{eff}^{G,T>T_c}[\varphi_{\vec{\mathbf{k}}}]} \tag{22}$$

where

$$-\beta\widetilde{\mathcal{H}}_{eff}^{G,T>T_c}[\varphi_{\vec{\mathbf{k}}}] = -\frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at + c\left|\vec{\mathbf{k}}\right|^2\right)\left|\varphi_{\vec{\mathbf{k}}}\right|^2 \tag{23}$$

Free energy in Gaussian approximation

Let us consider again the case $T > T_c$ (for which we have $(m_0 = 0)$) and $h = 0$. In this case, the partition function of the system in the Fourier space is the one in Eq.(22):

$$\widetilde{Z}_G^{T>T_c} = \prod_{\substack{|\vec{\mathbf{k}}|<\Lambda \\ k_z>0}} \int_{-\infty}^{+\infty} \mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}\, e^{-\frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at+c\left|\vec{\mathbf{k}}\right|^2\right)\left|\varphi_{\vec{\mathbf{k}}}\right|^2}$$

Since $\left|\varphi_{\vec{\mathbf{k}}}\right|^2 = \mathrm{Re}^2\,\varphi_{\vec{\mathbf{k}}} + \mathrm{Im}^2\,\varphi_{\vec{\mathbf{k}}}$, changing variables to:

$$x \equiv \mathrm{Re}\,\varphi_{\vec{\mathbf{k}}}, \qquad y \equiv \mathrm{Im}\,\varphi_{\vec{\mathbf{k}}}$$

Thus, we have

$$\int_{-\infty}^{+\infty} \mathrm{d}x\,\mathrm{d}y\, e^{-A(x^2+y^2)} = \frac{\pi}{A}, \qquad A \equiv \frac{2at + c\left|\vec{\mathbf{k}}\right|^2}{2V}$$

Hence,

$$\widetilde{Z}_G^{T>T_c} = e^{-\beta \widetilde{F}_G^{T>T_c}} = \prod_{\substack{|\vec{\mathbf{k}}|<\Lambda \\ k_z>0}} \frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2} = \exp\left[\frac{1}{2}\sum_{|\vec{\mathbf{k}}|<\Lambda} \log\left(\frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\right)\right]$$

We therefore have that the free energy of the system is:

$$\widetilde{F}_G^{T>T_c} = -\frac{k_B T}{2}\sum_{|\vec{\mathbf{k}}|<\Lambda} \log\left(\frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{24}$$

Remark. For $T < T_c$ we have $m_0 = \pm(-at/b)^{1/2} \neq 0$. In addition, we have to redefine the quadratic term $(at + 3bm_0^2)$ (in Eq.(6)), that for $m_0^2 = -at/b$, becames $-2at$. Moreover, we have also the term $VA_0 = V(atm_0^2 + \frac{b}{2}m_0^4)$. Therefore, in the case $T < T_c$ the free energy of the system is

$$\widetilde{F}_G^{T<T_c} = VA_0 - \frac{k_B T}{2}\sum_{|\vec{\mathbf{k}}|<\Lambda} \log\left(\frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{25}$$

Specific heat in the Gaussian approximation

We can now compute the specific heat of the system, and so determine its critical exponent α. We therefore want to compute:

$$c_V^G = -T\frac{\partial^2}{\partial T^2}\frac{F_{GL}}{V}$$

The derivatives are straightforward, and in the end we get:

$$c_V^G = \underbrace{\frac{A}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda} \frac{1}{\left(2at + c\left|\vec{\mathbf{k}}\right|^2\right)^2}}_{1^{st}} - \underbrace{\frac{B}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda} \frac{1}{2at + c\left|\vec{\mathbf{k}}\right|^2}}_{2^{st}}$$

One can show that

$$1^{st} \propto \begin{cases} \xi^{4-d} \sim t^{-\nu(4-d)} & d < 4 \\ < \infty & d > 4 \end{cases}$$

and

$$2^{nd} \propto \begin{cases} \xi^{2-d} \sim t^{-\nu(2-d)} & d < 2 \\ < \infty & d > 2 \end{cases}$$

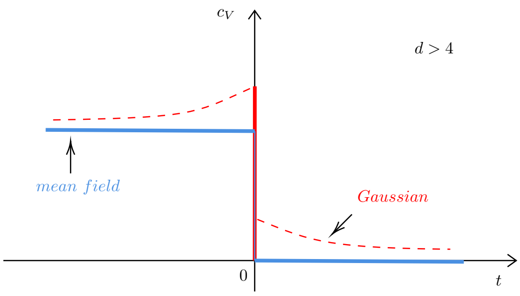
Therefore for $d < 2$ the 2^{nd} contribution to c_V^{GL} diverges, but in the same range of d the divergence of the first contribution is more relevant; on the other hand, for $2 \leq d < 4$ only the first contribution diverges. It is therefore the 1^{st} term that determines the divergence of the specific heat, and in particular for $d < 4$ we have $c_V^G \sim t^{-\nu(4-d)}$; in summary:

$$c_V^G \sim \begin{cases} t^{-\nu(4-d)} & d < 4 \\ < \infty & d > 4 \end{cases} \tag{26}$$

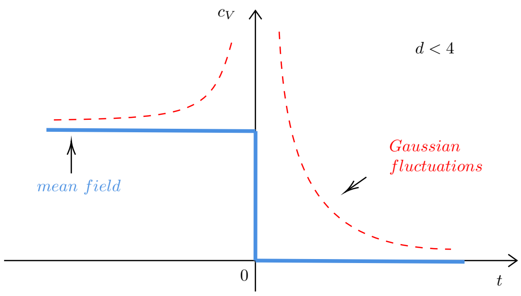
and so we see that in the Gaussian approximation the inclusion of the fluctuations has changed the behaviour of c_V at the transition point; in particular, has changed the value of the critical exponent α ($c_V \sim t^{-\alpha}$) to:

$$\alpha_G = \nu(4-d) \quad \text{for } d < 4 \tag{27}$$

In order to compute it, however, we still must determine ν so we now proceed to compute the two-point correlation function in order to determine both η and ν.



(a) Case $d > 4$.



(b) Case $d < 4$.

Figure 2: Behaviour of the specific heat c_V as a function of the rescaled temperature t. In blue it is represented its behaviour in the mean field theory, while in red the one with Gaussian approximations. We see that in the case $d < 4$ with Gaussian approximations the specific heat diverges near $t \sim 0$.

Two-point correlation function in the Gaussian approximation

We have to compute the 2-point correlation function for $\mathcal{H}_{eff}^G(\varphi)$. We know that the (simple) correlation function is defined as:

$$G(\vec{\mathbf{r}}, \vec{\mathbf{r}}') = \left\langle \varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{r}}') \right\rangle$$

so we first have to determine:

$$\varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{r}}') = \frac{1}{V^2} \sum_{\vec{\mathbf{k}}, \vec{\mathbf{k}}'} e^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} + \vec{\mathbf{k}}' \cdot \vec{\mathbf{r}}')} \varphi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'}$$

Shifting to Fourier space, we have (the subscript G stands for Gaussian):

$$\left\langle \varphi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'} \right\rangle_G = \frac{\int_{-\infty}^{+\infty} \mathrm{d}\varphi_{\vec{\mathbf{k}}_1} \dots \mathrm{d}\varphi_{\vec{\mathbf{k}}} \, \mathrm{d}\varphi_{\vec{\mathbf{k}}'} \, \varphi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'} e^{-\beta \mathcal{H}_{eff}}}{\int_{-\infty}^{+\infty} \mathrm{d}\varphi_{\vec{\mathbf{k}}_1} \dots \mathrm{d}\varphi_{\vec{\mathbf{k}}} \, \mathrm{d}\varphi_{\vec{\mathbf{k}}'} \, e^{-\beta \mathcal{H}_{eff}}}$$

where, as we said,

$$\beta \mathcal{H}_{eff}^{G} = VA_0 + \frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at + c\left|\vec{\mathbf{k}}\right|^2\right)\left|\varphi_{\vec{\mathbf{k}}}\right|^2$$

It is clear that in $\left\langle \varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\right\rangle_G$ all the integrals factorize since the Fourier modes are all independent (they are decoupled); therefore, all the integrals in the numerator that don't involve $\vec{\mathbf{k}}$ or $\vec{\mathbf{k}}'$ simplify with the same integrals in the denominator. Taking this into account, it is possible to show

$$\left\langle \varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\right\rangle_G = \frac{V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\delta_{\vec{\mathbf{k}},-\vec{\mathbf{k}}'}$$

that, in the limit $V \to \infty$ (Eq.(14)) becomes

$$\left\langle \varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\right\rangle_G \overset{V\to\infty}{\longrightarrow} \frac{(2\pi)^d}{2at + c\left|\vec{\mathbf{k}}\right|^2}\delta(\vec{\mathbf{k}} + \vec{\mathbf{k}}') \tag{28}$$

Going back to real space, by antitransforming, we have:

$$\begin{aligned}\left\langle \varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{r}}')\right\rangle_G &= \frac{1}{V^2}\sum_{\vec{\mathbf{k}},\vec{\mathbf{k}}'} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}+\vec{\mathbf{k}}'\cdot\vec{\mathbf{r}}')}\left\langle \varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\right\rangle_G = \frac{1}{V^2}\sum_{\vec{\mathbf{k}},\vec{\mathbf{k}}'} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}+\vec{\mathbf{k}}'\cdot\vec{\mathbf{r}}')}\frac{V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\delta_{\vec{\mathbf{k}},-\vec{\mathbf{k}}'} \\ &= \frac{1}{V}\sum_{\vec{\mathbf{k}}}\frac{e^{i\vec{\mathbf{k}}\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}')}}{2at + c\left|\vec{\mathbf{k}}\right|^2}\end{aligned}$$

We see that defining:

$$\xi(t) = \left(\frac{c}{2at}\right)^{1/2}$$

we get

$$\left\langle \varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{r}}')\right\rangle_G = \frac{1}{V}\sum_{\vec{\mathbf{k}}}\frac{1}{c}\frac{e^{i\vec{\mathbf{k}}\cdot(\vec{\mathbf{r}}-\vec{\mathbf{r}}')}}{\left|\vec{\mathbf{k}}\right|^2 + \xi^{-2}} = \frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\hat{G}(\vec{\mathbf{k}}) \tag{29}$$

where we have defined the correlation function

$$\hat{G}(\vec{\mathbf{k}}) = \frac{1}{c}\frac{1}{\left|\vec{\mathbf{k}}\right|^2 + \xi^{-2}} \tag{30}$$

this correlation function acquires the same form of the one computed in mean field theory. This means that the critical exponents ν and η now have the same values predicted by mean field theory (see Sec.(**??**) and Sec.(0.0.1)), namely:

$$\Rightarrow \begin{cases}\nu_G = \frac{1}{2} \\ \eta_G = 0\end{cases} \tag{31}$$

hence, there are no changes with Gaussian fluctuations! Interactions between $\varphi_{\vec{\mathbf{k}}}$ are needed!