Chapter 1

Widom's scaling theory. Block-spin Kadanoff's transformation

Lecture 19. *Wednesday 18th December, 2019. Compiled: Sunday 12th January, 2020.*

1.1 Static scaling hypothesis (Widom)

It is used whenever you have a collective behaviour. As $T \to T_c^{\pm}$ we know that $\xi \to L$. In this limit all the microscopic and intermediate scales are irrelevant. The length scale of the problem are a, L, ξ, but ξ is the only relevant length scale in the problem.

One should expect that, for $t \sim 0$, the free energy (better, its critical term) is invariant in form by a change of scale. This hypothesis is also suggested by experimental data such as the ones shown by Guggenheim for the gas phase diagrams and the ones shown for ferromagnetic materials at different temperatures.

Which are the experimental data which gives us this ideas? What you can see from experiment (Figure 1.1) is that data from different temperatures, if scaled properly, collapse into two (one for $t < 0$ and one for $t > 0$) unique curves.

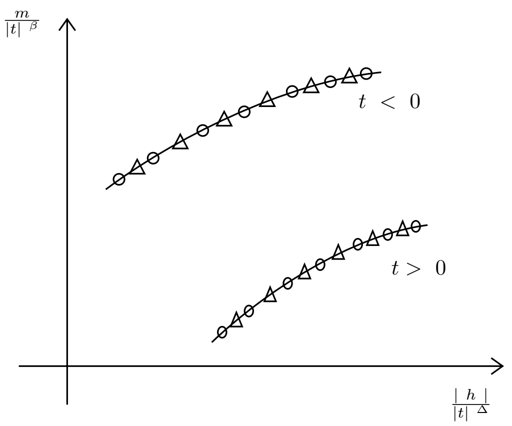

Figure 1.1: Description.

The Widom's static scaling theory is introduced also to explain the collapse of data shown above.

In order to properly define the scaling hypothesis, we should rely on the mathematical concept of homogeneous functions.

1.1.1 Homogeneous functions

Consider a single variable r.

Definition 1: Homogeneous function

$f(r)$ is homogeneous in r if $\forall\lambda \in \mathbb{R}$,

$$f(\lambda r) = g(\lambda) f(r) \tag{1.1}$$

where $g(\lambda)$ is, for the moment, arbitrary.

Example 1

Parabola:

$$f(r) = Br^2 \tag{1.2}$$

$$f(\lambda r) = B(\lambda r)^2 = \lambda^2 f(r) \quad \Rightarrow g(\lambda) = \lambda^2 \tag{1.3}$$

Remark. An interesting property of an homogeneous function is that, if $f(r_0)$ is known and $g(\lambda)$ is known, hence $f(r)$ is known $\forall r \in \mathbb{R}$. Indeed, we can always write $r = \lambda r_0$,

$$f(r) = f(\lambda r_0) = g(\lambda) f(r_0) \tag{1.4}$$

Theorem 1

The function $g(\lambda)$ is not arbitrary but it must be of the form

$$g(\lambda) = \lambda^p \tag{1.5}$$

where p is the degree of the homogeneity of the function.

Proof. Change of scale with respect to μ and λ.

$$f(\lambda(\mu r)) = g(\lambda) f(\mu r) = g(\lambda) g(\mu) f(r) \tag{1.6}$$

On the other hand,

$$f[(\lambda\mu) r] = g(\lambda\mu) f(r) \tag{1.7}$$

Hence,

$$g(\lambda\mu) = g(\lambda) g(\mu) \tag{1.8}$$

Let us assume that $g(\lambda)$ is differentiable (actually $g(\lambda)$ continuous is sufficient but proof more complicated)

$$\frac{\partial}{\partial\mu}[g(\lambda\mu)] = \frac{\partial}{\partial\mu}[g(\lambda)g(\lambda)] \tag{1.9}$$

$$\Rightarrow \lambda g'(\lambda\mu) = g(\lambda) g'(\mu) \tag{1.10}$$

Let $\mu = 1$ and let $g'(\mu = 1) = p$, equation (1.10) implies

$$\frac{g'(\lambda)}{g(\lambda)} = \frac{p}{\lambda} \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\ln g(\lambda)) = \frac{p}{\lambda} \quad \Rightarrow \ln g(\lambda) = p \ln \lambda + c \tag{1.12}$$

or

$$g(\lambda) = e^c \lambda^p \quad \Rightarrow g'(\lambda) = p e^c \lambda^{p-1} \tag{1.13}$$

and since $g'(1) = p$, we have $p = pe^c$ if and only if $c = 0$ which implies

$$g(\lambda) = \lambda^p \tag{1.14}$$

□

1.2 Generalized homogeneous functions

We can make it for any variable, not only for a single one. We are discussing $f(x, y)$, that is a generalized homogeneous function if as a most general form as

$$f(\lambda^a x, \lambda^b y) = \lambda f(x, y) \tag{1.15}$$

Indeed, if we consider instead

$$f(\lambda^a x, \lambda^b y) = \lambda^p f(x, y) \tag{1.16}$$

we can always choose $\lambda^p \equiv s$ such that

$$f(s^{a/p} x, s^{b/p} y) = s f(x, y) \tag{1.17}$$

and choosing $a' = a/p$ and $b' = b/p$ we are back to (1.15).
Remark. Since λ is arbitrary $\Rightarrow \lambda = y^{-1/b}$, we get

$$f(x, y) = y^{1/b} f\left(\frac{x}{y^{a/b}}, 1\right) \tag{1.18}$$

f depends on x and y only through the ratio $\frac{x}{y^{a/b}}$! Similarly, for x, one can choose $\lambda = x^{-1/a}$.

Example 2

Examples of non-homogeneous functions are

$$f(x) = e^{-x} \tag{1.19}$$

$$f(x) = \log x \tag{1.20}$$

1.3 Widom's theory of static scaling

Close to the critical point, the singular part of the free energy does not change form by a change of the scale.

More precisely, given $t \equiv \frac{T-T_c}{T_c}$ and $h = \frac{H-H_c}{H_c}$ and the free energy density

$$f(T, H) = f_{ana}(T, H) + f_{sing}(t, h) \tag{1.21}$$

where f_{ana} is an analytic term and f_{sing} diverges, has a singularity. Given that the scaling hypothesis is

$$f_s(\lambda^{p_1} t, \lambda^{p_2} h) = \lambda f_s(t, h), \quad \forall \lambda \in \mathbb{R} \tag{1.22}$$

where p_1 and p_2 are the degrees of the homogeneity of the singular part of the free energy.

Remark. The exponents p_1 and p_2 are not specified by the hypothesis but they will depend on the critical exponents.

Remark. Since f_s is a generalized homogeneous function, it is always possible to choose λ to eliminate one dependence.

For example, one can choose $\lambda = h^{-1/p_2}$ to obtain

$$f_s(t, h) = h^{1/p_2} f_s(h^{-p_1/p_2} t, 1) \tag{1.23}$$

where

$$\Delta \equiv \frac{p_1}{p_2} \tag{1.24}$$

is called the *gap exponent*.

Now, let us see how this simple hypothesis allow us, by simple differential calculus, to obtain relations between the thermodynamic critical exponents.

1.3.1 Scaling of the magnetization

Let us start from the scaling hypothesis

$$f_s(\lambda^{p_1}t,\lambda^{p_2}h)=\lambda f_s(t,h) \tag{1.25}$$

Since M is the 1^{st} derivative of f with respect to H, let us derive (1.25) with respect to h

$$\lambda^{p_2}\frac{\partial f_s}{\partial h}(\lambda^{p_1}t,\lambda^{p_2}h)=\lambda\frac{\partial f_s}{\partial h} \tag{1.26}$$

$$\Rightarrow \lambda^{p_2}M_s(\lambda^{p_1}t,\lambda^{p_2}h)=\lambda M_s(t,h) \tag{1.27}$$

On the other hand, we know that, for $h=0$ and $t\to 0^-$, $M_s(t)\sim(-t)^{\beta}$. Putting $h=0$ in (1.27)

$$M_s(t,0)=\lambda^{p_2-1}M_s(\lambda^{p_1}t,0) \tag{1.28}$$

Since λ is arbitrary, we can eliminate the dependence on t by choosing

$$\lambda^{p_1}t=-1 \quad \Rightarrow \lambda=(t)^{-1/p_1} \tag{1.29}$$

hence,

$$M_s(t,0)=-(t)^{\frac{1-p_2}{p_1}}M_s(-1,0) \tag{1.30}$$

Obtaining the 1^{st} relation:

$$\beta=\frac{1-p_2}{p_1} \tag{1.31}$$

Exponent δ

At $T=T_c$ ($t=0$) we have

$$M_s \overset{h\to 0^+}{\sim} h^{1/\delta} \tag{1.32}$$

$$\Rightarrow M_s(0,h)=\lambda^{p_2-1}M_s(0,\lambda^{p_2}h) \tag{1.33}$$

Now, we want

$$\lambda^{p_1}h=1 \quad \Rightarrow \lambda=h^{-1/p_2} \tag{1.34}$$

$$\Rightarrow M_s(0,h)=h^{\frac{1-p_2}{p_2}}M_s(0,1) \tag{1.35}$$

Since $M_s\sim h^{1/\delta}$, we get the 2^{nd} relation:

$$\delta=\frac{p_2}{1-p_2} \tag{1.36}$$

By getting p_1 and p_2 from the two relations we have

$$p_1=\frac{1}{\beta(\delta+1)},\quad p_2=\frac{\delta}{\delta+1} \tag{1.37}$$

and in particular the gap exponent is

$$\frac{p_2}{p_1}\equiv\Delta=\beta\delta \tag{1.38}$$

1.3.2 Equation of state

We can also predict the scaling form of the equation of state and explain the collapse of the experimental data. Let us start from

$$M_s(t,h) = \lambda^{p_2-1} M_s(\lambda^{p_1}t, \lambda^{p_2}h) \tag{1.39}$$

and choose $\lambda = |t|^{-1/p_1}$.

We have

$$M_s(t,h) = |t|^{\frac{1-p_2}{p_1}} M_s(\frac{t}{|t|}, \frac{h}{|t|^{\Delta}}) \tag{1.40}$$

Since $\beta = \frac{1-p_2}{p_1}$, $\delta = \frac{p_2}{p_1}$ we have

$$\frac{M_s(t,h)}{|t|^{\beta}} = M_s(\frac{t}{|t|}, \frac{h}{|t|^{\Delta}}) \tag{1.41}$$

We define the *scaled magnetization* as

$$\bar{m} \equiv |t|^{-\beta} M(t,h) \tag{1.42}$$

and the *scaled magnetic field* as

$$\bar{h} \equiv |t|^{-\Delta} h(t,M) \tag{1.43}$$

Hence, the equation (1.41) becomes

$$\bar{m} = F_{\pm}(\bar{h}) \tag{1.44}$$

universal curves!

1.3.3 Magnetic susceptibility

Compute $\frac{\partial^2}{\partial h^2}$ of the scaling hypothesis

$$f_s(\lambda^{p_1}t, \lambda^{p_2}h) = \lambda f_s(t,h) \tag{1.45}$$

From equation (1.41) we have

$$\lambda^{2p_2} \chi_T(\lambda^{p_1}t, \lambda^{p_2}h) = \lambda \chi_T(t,h) \tag{1.46}$$

Exponent γ

Consider $h=0$, $t \to 0^-$ (for example). Let $\lambda = -t^{-1/p_1}$, from equation (1.41)

$$\chi_T(t,0) = (-t)^{-\frac{2p_2-1}{p_1}} \chi_T(-1,0) \tag{1.47}$$

and since

$$\chi_T(t,0) \overset{t\to 0^-}{\sim} (-t)^{-\gamma_-} \tag{1.48}$$

we get

$$\gamma_- = \frac{2p_2-1}{p_1} = \beta(\delta - 1) \tag{1.49}$$

Remark. For $t \to 0^+$, we have $\lambda = t^{1/p_1}$ and we get

$$\gamma_- = \gamma_+ \equiv \gamma = \beta(\delta - 1) \tag{1.50}$$

relation between exponents!

1.3.4 Scaling of the specific heat

Starting again from

$$f_s(\lambda^{p_1}t, \lambda^{p_2}h) = \lambda f_s(t,h) \tag{1.51}$$

and compute $\frac{\partial^2}{\partial t^2}$ of both side, we get

$$\lambda^{2p_1}c_h(\lambda^{p_1}t, \lambda^{p_2}h) = \lambda c_H(t,h) \tag{1.52}$$

For $h = 0$ and $t \to 0^-$, we have $\lambda = (-t)^{-1/p_1}$ which implies

$$-t^{-2}c_H(-1,0) = -t^{-1/p_1}c_H(t,0) \tag{1.53}$$

and since

$$c_H(t,0) \overset{t\to 0^-}{\sim} t^{-\alpha_-} \tag{1.54}$$

we get

$$\alpha_- = 2 - \frac{1}{p_1} \tag{1.55}$$

Also in this case it is easy to show that

$$\alpha_- = \alpha_+ \tag{1.56}$$

Moreover, by inserting in (1.55) the relation $p_1 = \frac{1}{\beta(\delta+1)}$ one gets the *Griffith relation*

$$\alpha + \beta(\delta + 1) = 2 \tag{1.57}$$

Moreover, by combining Griffith with the relation $\gamma = \beta(\delta-1)$ one get the *Rushbrooke relation*

$$\alpha + 2\beta + \gamma = 2 \tag{1.58}$$

Remark. By using these hypothesis you can get equalities. All the inequality that you have in thermodynamics becames equalities. (lesson)

1.3.5 Second form of the static scaling hypothesis

$$f_s(\lambda^{p_1}t, \lambda^{p_2}h) = \lambda f_s(t,h) \tag{1.59}$$

The relation $\lambda = t^{-1/p_1}$ implies

$$f_s(1, t^{-p_2/p_1}h) = t^{-1/p_1}f_s(t,h) \tag{1.60}$$

Since $\Delta = \frac{p_2}{p_1}$ and $\alpha = 2 - \frac{1}{p_1}$, one gets

$$f_s(t,h) = t^{2-\alpha}f_s(1, \frac{h}{t^\Delta}) \tag{1.61}$$

that is the 2^{nd} scaling form of the free-energy.

1.4 Kadanoff's block spin and scaling of the correlation function

Widom's theory refers only to the free energy of the system. Moreover it do, not fully explain how the scaling hypothesis derives by ξ being the only relevant length of the system.

The first argument on this point was given 1966 by Kadonoff. The Kadonoff's intuition was: the divergence of ξ implies a relation between the coupling constants of a $\mathcal{H}_{eff}$ and the length scales over which m is defined.

We will that Kadanoff's intuition is correct. However, the relation found by Kadanoff is an approximation of the one proposed later by the theory of renormalization.

Remark. How can we relate the coupling costant of the two hamiltonian. This is the idea of the renormalization group. (lesson)

1.4.1 Kadanoff's argument

It is based on a coarse-grained operation on the system and two basic assumptions.

Coarse graining operation

Let us start from the Hamiltonian

$$-\beta\mathcal{H}_\Omega = k\sum_{\langle ij\rangle}\sigma_i\sigma_j + h\sum_i \sigma_i \tag{1.62}$$

with

$$\sigma_i = \pm 1, \quad k \equiv \frac{J}{k_B T}, \quad h \equiv \frac{H}{k_B T} \tag{1.63}$$

Given xi we know that, for distances $r < \xi$, the spins are correlated.

Let partition the system into blocks of size la (a is the lattice unit length while l is a adimensional scale) such that

$$r \ll \xi \quad \Rightarrow a \ll la \ll \xi(t) \tag{1.64}$$

The coarse-graining procedure consits in substituting the spins σ_i in a block of length la with a *superspin* or *block spin* S_I.

The transformation chosen by Kadanoff was the following

$$S_I \equiv \frac{1}{|m_l|}\frac{1}{l^D}\sum_{i\in I}\sigma_i \tag{1.65}$$

where m_l is the average magnetization of the I-esim block

$$m_l \equiv \frac{1}{l^D}\sum_{i\in I}\sigma_i \tag{1.66}$$

with the sum that is over all the sites with a given cell.

Remark. The division by $|m_l|$ in equation (1.65) is crucial because it rescales the new variables S_I to have the original values ± 1 (rescaling of the fields).

1^{st} crucial assumption

The frist assumption is that the Hamiltonian of the new system $\mathcal{H}_l$ is equal in form to $\mathcal{H}_\Omega$, the original one:

$$-\beta\mathcal{H}_l = k_l\sum_{\langle IJ\rangle}^{N_b} S_I S_J + h_l\sum_{I=1}^{N_b} S_I \tag{1.67}$$

Remark. This assumption is in general wrong!

Only the coupling constants k_l and h_l can change by the transformation. Note that the number of block spins N_l is given by $N_l = \frac{N}{l^D}$.

How you measure in the new system the lengths? the correlations lengths. Before we have changed the size of the system. We have increased the ruler, so the number is smaller now. It seems stupid but it is fundamental. An important point is: the new system has a lattice length equal to la, hence in the new system all the lengths will be measured in units of la. In particular

$$\xi_l = \frac{\xi}{l} \tag{1.68}$$

that is ξ_l has a numerical value smaller than the one measured in the original lattice.

What does it mean? Since $\xi_l < \xi$ the system described by $\mathcal{H}_l$ is more distant from the critical point than the original $\mathcal{H}(\sigma_i)$! We should expect $t_l > t$.

Similarly, h_l will be such that

$$h\sum_i \sigma_i = h\sum_I \sum_{i\in\sigma_i} \sigma_i = h\sum_I |m_l| l^D S_I = \underbrace{h|m_l|l^D}_{h_l}\sum_I S_I \equiv h_l \sum_I S_I \tag{1.69}$$

giving

$$h_l = |m_l| l^D h \tag{1.70}$$

Since $\mathcal{H}_l$ is equal in form to $\mathcal{H}$, so it is Z_l and hence the free energies per spins satisfy the equation

$$N_l f_s(t_l, h_l) = N f_s(t, h) \tag{1.71}$$

Hence,

$$\Rightarrow f_s(t_l, h_l) = l^D f_s(t, h) \tag{1.72}$$

Note that the homogeneity condition is recovered with $\lambda \equiv l^D$. We now should ask how t and h change under the block spin transformation.

2^{st} crucial assumption

The second assumption is

$$\begin{cases} t_l = t l^{Y_t} \\ h_l = h l^{Y_h} \end{cases} \tag{1.73}$$

where Y_t, Y_h are scaling exponents and are unknown! Inserting (1.73) in the free energy equation one obtains

$$f_s(t, h) = l^{-D} f_s(t l^{Y_t}, h l^{Y_h}) \tag{1.74}$$

As the parameter λ, in Widom also l can be chosen such that

$$l = |t|^{-1/Y_t} \tag{1.75}$$

so

$$f_s(t, h) = |t|^{D/Y_t} f_s(1, h|t|^{-Y_h/Y_t}) \tag{1.76}$$

Therefore,

$$\Delta = \frac{Y_h}{Y_t} \tag{1.77}$$

Remark. By comparing with the Widom's scaling

$$f_s(t, h) \sim t^{2-\alpha} f_s\left(1, \frac{h}{t^\Delta}\right) \tag{1.78}$$

we get the new relation

$$2 - \alpha = \frac{D}{Y_t} \tag{1.79}$$

1.5 Kadanoff's argument applied to the two-point correlation

Let us compute the two-point correlation function for the block spins

$$G_{IJ}(\vec{\mathbf{r}}_l, t_l) \equiv \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle \tag{1.80}$$

Since

$$h_l = h|m_l|l^D \quad \Rightarrow |m_l| = \frac{h_l l^{-D}}{h} \tag{1.81}$$

and using $h_l = hl^{Y_h}$, we obtain

$$|m_l| = l^{\frac{1}{h}-D} \tag{1.82}$$

Since

$$S_I = \frac{1}{|m_l|}\frac{1}{l^D}\sum_{i\in I}\sigma_i \tag{1.83}$$

we have

$$\begin{aligned} G_{IJ}(\vec{\mathbf{r}}_l, t_l) &= \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle \\ &= \frac{1}{l^{2(Y_h-D)}l^{2D}}\sum_{i\in I}\sum_{j\in J}\left[\underbrace{\langle \sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle}_{(a)}\right] \\ &= \frac{l^D l^D}{l^{2(Y_h-D)}l^{2D}}[\langle \sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle] \end{aligned} \tag{1.84}$$

in fact (a) we have made the assumption that since $la \ll \xi$, G_{ij} inside a block is fairly constant

Remark.

$$|\vec{\mathbf{r}}_l| = \frac{\vec{\mathbf{r}}}{l} \tag{1.85}$$

Hence,

$$G_{IJ}(\vec{\mathbf{r}}_l, t_l) = l^{2(D-Y_h)}G_{ij}(\vec{\mathbf{r}}, t) \tag{1.86}$$

If one includes also the h dependence:

$$G_{IJ}\left(\frac{\vec{\mathbf{r}}}{l}, tl^{Y_t}, hl^{Y_h}\right) = l^{2(D-Y_h)}G_{ij}(\vec{\mathbf{r}}, t, h) \tag{1.87}$$

By choosing $l = t^{-1/Y_t}$ one gets

$$G(\vec{\mathbf{r}}, t, h) = t^{\frac{2(D-Y_h)}{Y_t}}G(\vec{\mathbf{r}}t^{1/Y_t}, 1, ht^{-Y_h/Y_t}) \tag{1.88}$$

Since we are interested at large length scale in proximiti of $T = T_c$, we can choose $|\vec{\mathbf{r}}|$ such that

$$|\vec{\mathbf{r}}|t^{1/Y_t} = 1 \quad \Rightarrow t = |\vec{\mathbf{r}}|^{-Y_t} \tag{1.89}$$

Hence, inserting in (1.88)

$$G(\vec{\mathbf{r}}, t, h) = |\vec{\mathbf{r}}|^{-2(D-Y_h)}F_G\left(ht^{-Y_h/Y_t}\right) \tag{1.90}$$

where

$$F_G\left(ht^{-Y_h/Y_t}\right) \equiv G(1, ht^{-Y_h/Y_t}, 1) \tag{1.91}$$

Remembering the power law behaviour of G in proximity of the critical point, i.e. $G \sim |\vec{\mathbf{r}}|^{2-D-\eta}$ and $\xi \sim t^{-\nu}$. Finally we get

$$\nu = \frac{1}{Y_t}, \quad 2(D - Y_h) = D - 2 + \eta \tag{1.92}$$

Hence, we obtain the hyperscaling relation

$$\Rightarrow 2 - \alpha = \nu D \tag{1.93}$$