Lecture 7.
Wednesday 30th October, 2019.
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0.0.1 Transfer Matrix method

Given the Hamiltonian above we can write the corresponding partition function in the following symmetric form:

$$Z_N(k,h) = \sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1}\left[e^{kS_1S_2+\frac{h}{2}(S_1+S_2)}\right]\left[e^{kS_2S_3+\frac{h}{2}(S_2+S_3)}\right]\dots\left[e^{kS_NS_1+\frac{h}{2}(S_N+S_1)}\right] \tag{1}$$

Suppose you have a sort of $\sum_j M_{ij}P_{jk}$, what we have done is doing something like that. In the previous form Z_N can be written as a product of matrices

$$\begin{aligned} Z_N(h,k) &= \sum_{S_1=\pm1}\cdots\sum_{S_N=\pm1}\prod_{i=1}^{N}\exp\left[kS_iS_{i+1}+\frac{h}{2}(S_i+S_{i+1})\right] \\ &= \sum_{S_1=\pm1}\cdots\sum_{S_N=\pm1}\langle S_1|\,\mathbb{T}\,|S_2\rangle\langle S_2|\,\mathbb{T}\,|S_3\rangle\dots\langle S_N|\,\mathbb{T}\,|S_1\rangle \end{aligned} \tag{2}$$

where $\mathbb{T}$ is a 2×2 matrix defined as

$$\langle S|\,\mathbb{T}\,|S'\rangle = \exp\left[kSS'+\frac{h}{2}(S+S')\right] \tag{3}$$

Note that the labels of the matrix corresponds to the values of S_i. Hence its dimension depdens on the number of possible values a spin S_i can assume. It can also depends on how many spins are involved in the interacting terms that are present in the hamiltonian ($k_{LL}\sum S_iS_{i+1}S_{i+2}S_{i+3}$). For Ising $S_i=\pm1$ and neirest neighbour interaction implies that we have 2 values and that $\mathbb{T}$ is a 2×2 matrix whose components are

$$\langle +1|\,\mathbb{T}\,|+1\rangle = \exp[k+h] \tag{4a}$$
$$\langle +1|\,\mathbb{T}\,|-1\rangle = \langle -1|\,\mathbb{T}\,|+1\rangle = \exp[k-h] \tag{4b}$$
$$\langle -1|\,\mathbb{T}\,|-1\rangle = \exp[-k] \tag{4c}$$

The explicit representation is

$$\mathbb{T} = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix} \tag{5}$$

Let us now introduce some useful notations and relations using the bra-ket formalism: subequations

$$\left|S_i^{(+)}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}_i \qquad \left|S_i^{(-)}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}_i \tag{6}$$

$$\left\langle S_i^{(+)}\right| = (1^*,0)_i \qquad \left\langle S_i^{(-)}\right| = (0,1^*)_i \tag{7}$$

The identity relation is:

$$\sum_{S_i=\pm1}|S_i\rangle\langle S_i| = \left|S_i^{(+)}\right\rangle\!\left\langle S_i^{(+)}\right| + \left|S_i^{(-)}\right\rangle\!\left\langle S_i^{(-)}\right| = \mathbb{1} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{8}$$

By using the last property we can write

$$\begin{aligned} Z_N(K,h) &= \sum_{S_1=\pm1}\cdots\sum_{S_N=\pm1}\langle S_1|\,\mathbb{T}\,|S_2\rangle\langle S_2|\,\mathbb{T}\,|S_3\rangle\dots|S_i\rangle\langle S_i|\,\mathbb{T}\,|S_{i+1}\rangle\dots \\ &= \sum_{S_1=\pm1}\langle S_1|\,\mathbb{T}^N\,|S_1\rangle = \mathrm{Tr}\left[\mathbb{T}^N\right] \end{aligned} \tag{9}$$

this is exactly the trace of the matrix. Being $\mathbb{T}$ symmetric, we can diagonalize it by an unitary transformation

$$\mathbb{T}_D = \mathbb{P}^{-1}\mathbb{T}\mathbb{P} \tag{10}$$

with $\mathbb{P}\mathbb{P}^{-1} = \mathbb{1}$.

$$\begin{aligned}
\mathrm{Tr}\big[\mathbb{T}^N\big] &= \mathrm{Tr}\Bigg[\underbrace{\mathbb{T}\mathbb{T}\mathbb{T}\dots\mathbb{T}}_{N}\Bigg] = \mathrm{Tr}\big[\mathbb{P}\mathbb{P}^{-1}\mathbb{T}\mathbb{P}\mathbb{P}^{-1}\mathbb{T}\mathbb{P}\dots\mathbb{P}^{-1}\mathbb{T}\mathbb{P}\mathbb{P}^{-1}\big] \\
&= \mathrm{Tr}\big[\mathbb{P}\mathbb{T}_D^N\mathbb{P}^{-1}\big] \underset{\substack{\text{ciclyc property}\\ \text{of the trace}}}{=} \mathrm{Tr}\big[\mathbb{T}_D^N\mathbb{P}^{-1}\mathbb{P}\big] \\
&= \mathrm{Tr}\big[\mathbb{T}_D^N\big]
\end{aligned} \tag{11}$$

where

$$\mathbb{T}_D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \quad \mathbb{T}_D^N = \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix} \tag{12}$$

with $\lambda_\pm$ are the eigenvalues with $\lambda_+ > \lambda_-$.

Remark. $\mathbb{P}$ is the similitude matrix whose colums are given by the eigenvectors of $\lambda_\pm$.

We finally have:

$$Z_N(K,h) = \mathrm{Tr}\big[\mathbb{T}_{\mathbb{D}}^{\mathbb{N}}\big] = \lambda_+^N + \lambda_-^N \tag{13}$$

Remark. As mentioned previously the dimension of the transfer matrix $\mathbb{T}$ and hence the number of eigenvalues $\{\lambda\}$ depend both on the possible values of S_i and on the number of sites involved in terms of the Hamiltonian (range of interaction).

Example 1. For example consider the Ising ($S_i = \pm 1$) with neirest neighbour and next neirest neighbour interactions. The hamiltonian is:

$$\mathcal{H} = k_1 \sum_i S_i S_{i+1} + k_2 \sum_i S_i S_{i+1} S_{i+2} S_{i+3} \tag{14}$$

Because of the second term now there are $2^4 = 16$ possible configurations that can be described by using a 4×4 transfer matrix that we can write formally as

0.1 Lesson

We have $S_i = +1, 0, -1$, therefore it can assume three different values. This is a *deluted* ising model.

Let us suppose there are $(n+2)$ possible values:

$$\left\langle S_i^{(3)} \right| = (0, 0, 1^*, 0, \dots) \tag{15}$$

$$\left| S_i^{(3)} \right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{16}$$

$$\sum_{S_i} |S_i\rangle \langle S_i| = \mathbb{1}, \quad \mathbb{1} \in (n+2) \times (n+2) \tag{17}$$

$$\mathbb{S}_i = \sum_{S_i} |S_i\rangle \, S_i \, \langle S_i| \tag{18}$$

Now $\{\lambda_+, \lambda_-, \lambda_1, \dots, \lambda_n\}$, with $\lambda_+ > \lambda_- \geq \lambda_1 \geq \dots \geq \lambda_n$.

$$Z_N(\{k\}, h) = \lambda_+^N + \lambda_-^N + \sum_{i=1}^{n} \lambda_i^N \tag{19}$$

$$\mathbb{T} = \mathbb{P}\mathbb{T}_D\mathbb{P}^{-1} = \sum_i |t_i\rangle \lambda_i \langle t_i| \tag{20}$$

Now we are interested in the limit of the bulk free energy:

$$F_N() = -k_B T \log Z_N() \tag{21}$$

In general, looking at the thermodynamic limit:

$$f_b(\{k\}, h) = \lim_{N\to\infty} \frac{1}{N} F_N = \lim_{N\to\infty} \frac{1}{N}(-k_B T) \log \left[\lambda_+^N + \lambda_-^N + \sum_{i=1}^{n} \lambda_i^N\right] \tag{22}$$

$$\to = \lim_{N\to\infty} \frac{-k_B T}{N} \log \left[\lambda_+^N \left(1 + \frac{\lambda_-^N}{\lambda_+^N} + \sum_i \left(\frac{\lambda_i}{\lambda_+}\right)^N\right)\right] = -k_B T \log \lambda_+ \tag{23}$$

So we have obtained

$$f_b = -k_B T \log \lambda_+ \tag{24}$$

This is simply because λ_+ is the largest.

Theorem 0.1.1 (Perron-Frobenius)**.** *Let A be a $m \times m$ matrix. If* A *is finite ($m < \infty$) and $A_{ij} > 0, \forall i, j$, ($A_{ij} = A_{ij}(\vec{\mathbf{x}})$) therefore λ_+ has the following properties:*

1. $\lambda_+ \in \mathbb{R}^+$
2. $\lambda_+ \neq$ *from* $\{\lambda_i\}_{i=1,\dots,m-1}$
3. λ_+ *is a analytic function of its arguments*

Try to change $A_{ij} > 0$ or the hypothesis that A is *finite* and see what is obtained.

0.2 Correlation function

Now we calculate the two points correlation function. We want the fluctuation respect to the average:

$$\Gamma_R = \langle S_1 S_R\rangle - \langle S_1\rangle \langle S_R\rangle \tag{25}$$

we expect from physics that

$$\Gamma_R \underset{R\to\infty}{\sim} \exp[-R/\xi] \tag{26}$$

$$\xi^{-1} = \lim_{R\to\infty} \left[-\frac{1}{R} \log\left[\langle S_1 S_R\rangle - \langle S_1\rangle \langle S_R\rangle\right]\right] \tag{27}$$

$$\langle S_1 S_R\rangle_N = \frac{1}{Z_N} \sum_{\{S\}} S_1 S_R \exp[-\beta\mathcal{H}] \tag{28}$$

$$= \frac{1}{Z_N} \sum_{\{S\}} S_1 \langle S_1| \mathbb{T} |S_2\rangle \dots \langle S_{R-1}| \mathbb{T} |S_R\rangle S_R \langle S_R| \mathbb{T} |S_{R+1}\rangle \dots \langle S_N| \mathbb{T} |S_1\rangle \tag{29}$$

$$= \frac{1}{Z_N} \sum_{S_1, S_R} S_1 \langle S_1| \mathbb{T}^{R-1} |S_R\rangle S_R \langle S_R| \mathbb{T}^{N-R+1} |S_1\rangle \tag{30}$$

$$\mathbb{T}^{R-1} = \sum_{i=1}^{n+2} |t_i\rangle \, \lambda_i^{R-1} \, \langle t_i| \tag{31}$$

$$\mathbb{T}^{N-R+1} = \sum_{i=1}^{n+2} |t_i\rangle \, \lambda_i^{N-R+1} \, \langle t_i| \tag{32}$$

$$\langle S_1| \, \mathbb{T}^{R-1} \, |S_R\rangle = \sum_{i=1}^{n+2} \langle S_i|t_i\rangle \, \lambda^{R-1} \, \langle t_i|S_R\rangle \tag{33}$$

$$\sum_{\{S\}} S_1 S_R e^{-\beta\mathcal{H}} = \sum_{S_1 S_R} S_1 \sum_{i=1}^{n+2} \langle S_1|t_i\rangle \, \lambda_i^{R-1} \, \langle t_i|S_R\rangle \, S_R \sum_{j=1}^{n+2} \langle S_R|t_j\rangle \, \lambda_j^{N-R+1} \, \langle t_j|S_1\rangle \tag{34}$$

Define:

$$\mathbb{S}_1 = \sum_{S_1} |S_1\rangle \, S_1 \, \langle S_1| \tag{35}$$

$$\mathbb{S}_R = \sum_{S_R} |S_R\rangle \, S_R \, \langle S_R| \tag{36}$$

$$\rightarrow = \sum_{ij} \langle t_j| \, \mathbb{S}_1 \, |t_i\rangle \, \lambda_i^{R-1} \, \langle t_i| \, \mathbb{S}_R \, |t_j\rangle \, \lambda_j^{N-R+1} \tag{37}$$