Lecture 7.
Wednesday 30^{th} October, 2019.
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0.1 Transfer Matrix

Today we introduce a general tecnique used in many fields as graph theory: the transfer matrix. If you are able to diagonalize the transfer matrix, you can use trick as eigenvalues and eigenvectors.

Consider the Ising model in a circle, as in Figure 1. We are introducing the bulk

$$S_{N+1} = S_1 \tag{1}$$

$$\beta \mathcal{H} = k \sum_{i=1}^{N} S_i S_{i+1} + h \sum_{i=1}^{N} S_i \quad \text{with} \quad k \equiv \beta J, h \equiv H\beta \tag{2}$$

$$Z_N(k,h) = \sum_{S_1=\pm1} \sum_{S_2=\pm1} \cdots \sum_{S_N=\pm1} \left[e^{kS_1S_2+\frac{h}{2}(S_1+S_2)}\right] \ldots \left[e^{kS_NS_1+\frac{h}{2}(S_N+S_1)}\right] \tag{3}$$

Suppose you have a sort of $\sum_j M_{ij}P_{jk}$, what we have done is doing something like that. We can rewrite this formally:

$$\to Z_N = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \langle S_1| \mathbb{T} |S_2\rangle \langle S_2| \mathbb{T} |S_3\rangle \ldots \langle S_N| \mathbb{T} |S_1\rangle \tag{4}$$

where $\mathbb{T}$ is a 2x2 matrix and

$$\langle S| \mathbb{T} |S'\rangle = \exp\left[kSS' + \frac{h}{2}(S+S')\right] \tag{5}$$

For example:

$$\langle +1| \mathbb{T} |+1\rangle = \exp[k+h] \tag{6}$$

$$\langle +1| \mathbb{T} |-1\rangle = \exp[-k] \tag{7}$$

The matrix has the form:

$$\mathbb{T} = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix} \tag{8}$$

$$\left|S_i^{(+)}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

$$\left|S_i^{(-)}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

Therefore the bra is:

$$\left\langle S_i^{(+)}\right| = (1^*, 0) \tag{11}$$

$$\left\langle S_i^{(-)}\right| = (0, 1^*) \tag{12}$$

Now:

$$\sum_{S_i=\pm1} |S_i\rangle \langle S_i| = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

Therefore:

$$\to Z_N = \sum_{S_1=\pm1} \cdots \sum_{S_N=\pm1} \langle S_1| \mathbb{T} |S_2\rangle \langle S_2| \mathbb{T} |S_3\rangle \ldots |S_i\rangle \langle S_i| \mathbb{T} |S_{i+1}\rangle \ldots \tag{14}$$

$$\to Z_N(k,h) = \sum_{S_1=\pm1} \langle S_1| \mathbb{T}^N |S_1\rangle = \mathrm{Tr}\left[\mathbb{T}^N\right] \tag{15}$$

this is exactly the trace of the matrix. We can find a unitary transformation:

$$\mathbb{T}_D = \mathbb{P}\mathbb{T}\mathbb{P}^{-1} \tag{16}$$

with $\mathbb{P}\mathbb{P}^{-1} = \mathbb{1}$.

$$\rightarrow Z = \mathrm{Tr}\big[\mathbb{P}^{-1}\mathbb{P}\mathbb{T}\mathbb{P}^{-1}\mathbb{P}\mathbb{T}\ldots\mathbb{P}^{-1}\mathbb{P}\mathbb{T}\mathbb{P}^{-1}\mathbb{P}\big] \tag{17}$$

$$\rightarrow= \mathrm{Tr}\big[\mathbb{P}^{-1}\mathbb{T}_{\mathbb{D}}^{\mathbb{N}}\mathbb{P}\big] = \mathrm{Tr}\big[\mathbb{P}\mathbb{P}^{-1}\mathbb{T}_{\mathbb{D}}^{\mathbb{N}}\big] = \mathrm{Tr}\big[\mathbb{T}_{\mathbb{D}}^{\mathbb{N}}\big] \tag{18}$$

Now:

$$Z_N(k,h) = \mathrm{Tr}\big[\mathbb{T}_{\mathbb{D}}^{\mathbb{N}}\big] = \lambda_+^N + \lambda_-^N, \quad \lambda_+ \geq \lambda_- \tag{19}$$

We have $S_i = +1, 0, -1$, therefore it can assume three different values. This is a *deluted* ising model.

Let us suppose there are $(n+2)$ possible values:

$$\left\langle S_i^{(3)} \right| = (0, 0, 1^*, 0, \ldots) \tag{20}$$

$$\left| S_i^{(3)} \right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{21}$$

$$\sum_{S_i} |S_i\rangle \langle S_i| = \mathbb{1}, \quad \mathbb{1} \in (n+2) \times (n+2) \tag{22}$$

$$\mathbb{S}_i = \sum_{S_i} |S_i\rangle\, S_i \,\langle S_i| \tag{23}$$

Now $\{\lambda_+, \lambda_-, \lambda_1, \ldots, \lambda_n\}$, with $\lambda_+ > \lambda_- \geq \lambda_1 \geq \cdots \geq \lambda_n$.

$$Z_N(\{k\}, h) = \lambda_+^N + \lambda_-^N + \sum_{i=1}^n \lambda_i^N \tag{24}$$

$$\mathbb{T} = \mathbb{P}\mathbb{T}_D\mathbb{P}^{-1} = \sum_i |t_i\rangle\, \lambda_i \,\langle t_i| \tag{25}$$

Now we are interested in the limit of the bulk free energy:

$$F_N() = -k_B T \log Z_N() \tag{26}$$

In general, looking at the thermodynamic limit:

$$f_b(\{k\}, h) = \lim_{N\to\infty} \frac{1}{N} F_N = \lim_{N\to\infty} \frac{1}{N}(-k_B T) \log\left[\lambda_+^N + \lambda_-^N + \sum_{i=1}^n \lambda_i^N\right] \tag{27}$$

$$\rightarrow= \lim_{N\to\infty} \frac{-k_B T}{N} \log\left[\lambda_+^N\left(1 + \frac{\lambda_-^N}{\lambda_+^N} + \sum_i \left(\frac{\lambda_i}{\lambda_+}\right)^N\right)\right] = -k_B T \log \lambda_+ \tag{28}$$

So we have obtained

$$f_b = -k_B T \log \lambda_+ \tag{29}$$

This is simply because λ_+ is the largest.

Theorem 0.1.1 (Perron-Frobenius). *Let A be a $m \times m$ matrix. If* A *is finite ($m < \infty$) and $A_{ij} > 0, \forall i, j$, ($A_{ij} = A_{ij}(\vec{\mathbf{x}})$) therefore λ_+ has the following properties:*

1. $\lambda_+ \in \mathbb{R}^+$

2. $\lambda_+ \neq$ *from* $\{\lambda_i\}_{i=1,\dots,m-1}$

3. λ_+ *is a analytic function of its arguments*

Try to change $A_{ij} > 0$ or the hypothesis that A is *finite* and see what is obtained.

0.2 Correlation function

Now we calculate the two points correlation function. We want the fluctuation respect to the average:

$$\Gamma_R = \langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle \tag{30}$$

we expect from physics that

$$\Gamma_R \underset{R\to\infty}{\sim} \exp[-R/\xi] \tag{31}$$

$$\xi^{-1} = \lim_{R\to\infty} \left[-\frac{1}{R} \log \left[\langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle \right] \right] \tag{32}$$

$$\langle S_1 S_R \rangle_N = \frac{1}{Z_N} \sum_{\{S\}} S_1 S_R \exp[-\beta \mathcal{H}] \tag{33}$$

$$= \frac{1}{Z_N} \sum_{\{S\}} S_1 \langle S_1 | \mathbb{T} | S_2 \rangle \dots \langle S_{R-1} | \mathbb{T} | S_R \rangle S_R \langle S_R | \mathbb{T} | S_{R+1} \rangle \dots \langle S_N | \mathbb{T} | S_1 \rangle \tag{34}$$

$$= \frac{1}{Z_N} \sum_{S_1, S_R} S_1 \langle S_1 | \mathbb{T}^{R-1} | S_R \rangle S_R \langle S_R | \mathbb{T}^{N-R+1} | S_1 \rangle \tag{35}$$

$$\mathbb{T}^{R-1} = \sum_{i=1}^{n+2} | t_i \rangle \lambda_i^{R-1} \langle t_i | \tag{36}$$

$$\mathbb{T}^{N-R+1} = \sum_{i=1}^{n+2} | t_i \rangle \lambda_i^{N-R+1} \langle t_i | \tag{37}$$

$$\langle S_1 | \mathbb{T}^{R-1} | S_R \rangle = \sum_{i=1}^{n+2} \langle S_i | t_i \rangle \lambda^{R-1} \langle t_i | S_R \rangle \tag{38}$$

$$\sum_{\{S\}} S_1 S_R e^{-\beta \mathcal{H}} = \sum_{S_1 S_R} S_1 \sum_{i=1}^{n+2} \langle S_1 | t_i \rangle \lambda_i^{R-1} \langle t_i | S_R \rangle S_R \sum_{j=1}^{n+2} \langle S_R | t_j \rangle \lambda_j^{N-R+1} \langle t_j | S_1 \rangle \tag{39}$$

Define:

$$\mathbb{S}_1 = \sum_{S_1} | S_1 \rangle S_1 \langle S_1 | \tag{40}$$

$$\mathbb{S}_R = \sum_{S_R} | S_R \rangle S_R \langle S_R | \tag{41}$$

$$\to = \sum_{ij} \langle t_j | \mathbb{S}_1 | t_i \rangle \lambda_i^{R-1} \langle t_i | \mathbb{S}_R | t_j \rangle \lambda_j^{N-R+1} \tag{42}$$