Lecture 15. Wednesday 4th December, 2019. Compiled: Saturday 21st December, 2019.

This first approximation is reasonable if either

1. ρ is small enough. It implies that $|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j| \gg 1$ and hence $\Phi_{ij} \ll 1$.
2. Sufficiently high T such that $\Phi(|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|)/k_B T \ll 1$. What is important it is the ration between β and Φ_{ij}.

In either cases we have $\exp(-\beta\Phi_{ij}) \to 1$ and $f_{ij} \to 0$. By keeping only linear terms the partition function is

$$\begin{aligned} Q_N(V,T) &= \int_V \mathrm{d}\vec{\mathbf{r}}_1 \dots \mathrm{d}\vec{\mathbf{r}}_N \left(1 + \sum_{i,j>i} f_{ij} + \dots\right) \\ &= V^N + \sum_{i,j>i} \int_V \mathrm{d}\vec{\mathbf{r}}_1 \cdots \int_V \mathrm{d}\vec{\mathbf{r}}_N \, f_{ij} \\ &= V^N + V^{N-2} \sum_{i,j>i} \int_V \mathrm{d}\vec{\mathbf{r}}_i \, \mathrm{d}\vec{\mathbf{r}}_j \, f_{ij} + \dots \end{aligned} \tag{1}$$

We are summing up over all configurations ij. Let us try to compute the double integral:

$$\int_V \mathrm{d}\vec{\mathbf{r}}_i \, \mathrm{d}\vec{\mathbf{r}}_j \, f_{ij}(|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|) \underset{\substack{\text{translational} \\ \text{symmetry}}}{=} \int \mathrm{d}\vec{\mathbf{r}}_i \, \mathrm{d}\vec{\mathbf{r}}_j \, f(\vec{\mathbf{r}}) = V \int_V \mathrm{d}\vec{\mathbf{r}} \, f(|\vec{\mathbf{r}}|) \equiv -2B_2 V \tag{2}$$

where

$$B_2 \equiv -\frac{1}{2} \int_V \mathrm{d}\vec{\mathbf{r}} \, f(|\vec{\mathbf{r}}|) \tag{3}$$

so, what is important it is the relative distance. $\vec{\mathbf{r}}$ gives us the position from the center we have choosen. Rewriting again the partition function we obtain:

$$Q_N(V,T) = V^N - V^{N-1} N(N-1) B_2(T) \tag{4}$$

Remark. The factor $\frac{N(N-1)}{2}$ comes out becouse in the double sum are considered all the possible connections (bonds) between pairs of particles (i,j) with $j > i$.

$$Z_N(V,T) = \left(\frac{V^N}{N!\Lambda^{3N}}\right) \left(1 - \overbrace{\frac{N^2}{V}}^{N(N-1)\approx N^2} B_2(T) + \dots\right) \tag{5}$$

Remark. We do not care about the $(N-1)$ term, because N is big enough!

The free energy is:

$$F_N = F_N^{ideal} - k_B T \ln\left[1 - \frac{N^2}{V} B_2(T) + \dots\right] \tag{6}$$

Hence,

$$P_N = -\left(\frac{\partial F_N}{\partial V}\right)_{T,N} = \frac{N k_B T}{V} \left(1 + \frac{\frac{N}{V} B_2}{1 - \frac{N^2}{V} B_2}\right) = \frac{N k_B T}{V} \left(\frac{1 - \frac{N^2}{V} B_2 + \frac{N}{V} B_2}{1 - \frac{N^2}{V} B_2}\right) \tag{7}$$

Expanding the denominator for $\frac{N}{V} B_2 \ll 1$ $\rho \ll 1$ one gets

$$P_N \simeq \frac{N k_B T}{V} \left(1 + \frac{N}{V} B_2 + \dots\right) \tag{8}$$

here we see the ideal gas and the correction to the ideal gas.

Remark. The equation (8) gives an important relation between experimentally accessible observables as P_N and microscopic quantities such as $f(\vec{\mathbf{r}})$ (and hence $\Phi(\vec{\mathbf{r}})$) trough the estimate of B_2.

Therefore, it is important computing B_2, because one time we have this we have the expansion. Or if we wish, by doing the fit of data at different temperature we obtain B_2 from the experiment and we can see f_{ij}.

Remark. The virial expansion obtained in (8) is valid for small densities.

To consider higher order terms in the virial expansion we need to consider higher order products of the f_{ij}.

Before doing this, however, one can show that an higher order expansion can be obtained by using the following (rather rude) trick.

Since $(1-x)^{-1} = 1 + x + \dots$, ny going backward it is possible to write

$$\frac{PV}{Nk_BT} \approx 1 + \rho B_2 + \cdots \simeq \frac{1}{1 - B_2\rho} \tag{9}$$

this is the Clausius equation. On the other hand,

$$\frac{1}{1 - B_2\rho} \simeq 1 + B_2\rho + \underbrace{(B_2)^2\rho^2}_{B_3 \approx B_2^2} + \underbrace{(B_2)^3\rho^3}_{B_4 \approx B_2^3} + \dots \tag{10}$$

Identifying the coefficients for each power we get, for example

$$B_3 \approx (B_2)^2, \quad B_4 \approx (B_2)^3 \tag{11}$$

This is the approximation of higher order virial coefficients with powers of B_2.

Example 1. Exam: let us compute virial expansion of a gas in a potential.

0.0.1 Computation of B_2 for given Φ

Gas of hard spheres

The particles are interacting (it is not ideal!) and there is a size that is the range of the potential.

$$\Phi(r) = \begin{cases} \infty & r < \sigma \\ 0 & r \geq \sigma \end{cases} \tag{12}$$

Hence,

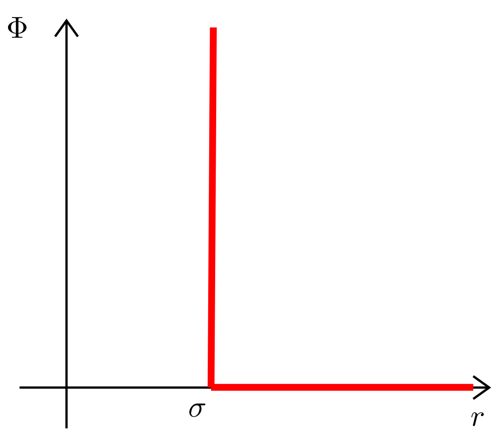

Figure 1: Plot of the potential $\Phi(r)$.

$$e^{-\beta\Phi(r)} = \begin{cases} 0 & r < \sigma \\ 1 & r \geq \sigma \end{cases} \tag{13}$$

This implies that

$$B_2(T) = -\frac{1}{2}\int_V \mathrm{d}\vec{\mathbf{r}}\, f(|\vec{\mathbf{r}}|) = -\frac{1}{2}4\pi\int_V \mathrm{d}r\, r^2\left[e^{-\beta\Phi(r)} - 1\right] = 2\pi\int_0^\sigma \mathrm{d}r\, r^2 = \frac{2}{3}\pi\sigma^3 \tag{14}$$

$$\Rightarrow B_2^{HS}(T) = \frac{2}{3}\pi\sigma^3 \tag{15}$$

this is the second virial coefficient for a hard sphere gas. There is no condensation in the gas spheres.

Remark. As expected B_2^{HS} does not depend on temperature (purely repulsive interaction).

For hard spheres we have:

$$PV = Nk_BT\left(1 + \frac{2}{3}\pi\sigma^3\frac{N}{V}\right) \tag{16}$$

Remark. The excluded volume interaction (hard sphere) increases the product PV with respect to the ideal gas.

Let us say, that the potential is not anymore zero but is $-\varepsilon$ when it is in the case $r \geq \sigma$. Or consider a case in which it is $-\varepsilon$ between $[\sigma, 2\sigma]$, then it goes to zero.

Gas with Lennard-Jones interaction

We can consider a Lennard-Jones potential. The 2-body Lennard-Jones potential energy is

$$\Phi = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{17}$$

where the first one is the repulsive term (Pauli excluded principle) and the second is the attractive term (fluctuation of the elecrtic dipole moment).

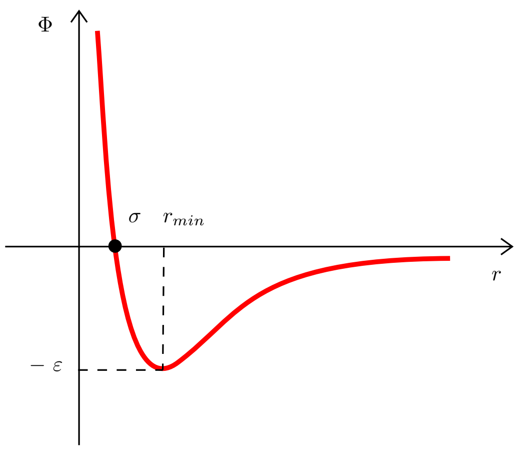

Figure 2: Plot of the Lennard-Jones potential.

The minimum is in $r_{min} = 2^{1/\sigma}$. You can play with the range of attraction by changing σ or by changing the ε. What it is important is that for the Lennard-Jones we have

$$B_2 \overset{LJ}{=} B_2(T) \tag{18}$$

$$B_2(T) = -2\pi\int_0^\infty r^2\left[e^{-\frac{4\varepsilon}{k_BT}\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]} - 1\right]\mathrm{d}r \tag{19}$$

It is not solvable exactly but by series expansion. Let us consider the following change of variables

$$x = \frac{r}{\sigma}, \quad \tau = \frac{k_B T}{\varepsilon} \tag{20}$$

Integrating by parts $\int f'g = fg - \int g'f$ where $f' = x^2 g = \exp[-()]$, we obtain

$$\begin{aligned} B_2(T^*) &= \frac{2}{3}\pi\sigma^3 \frac{4}{\tau} \int_0^\infty x^2 \left(\frac{12}{x^{12}} - \frac{6}{x^6}\right) e^{-\frac{4}{\tau}\left(\frac{1}{x^{12}} - \frac{1}{x^6}\right)} \,\mathrm{d}x \\ &= A \int_0^\infty \left(\frac{12}{x^{16}} - \frac{6}{x^4}\right) e^{-\frac{4}{\tau}\left(\frac{1}{x^{12}} - \frac{1}{x^6}\right)} \,\mathrm{d}x \end{aligned} \tag{21}$$

Expand the exponential and then integrate term by term. One gets a serie in inverse power of τ.

$$B_2(\tau) = -2A' \sum_{n=0}^{\infty} \frac{1}{4n!} \Gamma\left(\frac{2n-1}{4}\right) \left(\frac{1}{\tau}\right)^{\frac{2n+1}{4}} \tag{22}$$

where Γ is the Euler function.

Remark. Note that, because the Lennard-Jones potential has an attractive interaction term $\left[-\left(\frac{\sigma}{r}\right)^6\right]$, the second virial coefficient depends on temperature.

0.0.2 Higher order terms in the cluster expansion

Let us consider again the formal expansion

$$\prod_i \left(\prod_{j>i} (1 + f_{ij}) \right) = 1 + \sum_{i,j>i} f_{ij} + \sum_{\substack{i \\ j>i \\ l>k \\ k\geq i \\ (ij)\neq(kl)}} f_{ij} f_{kl} + \dots \tag{23}$$

The problem with this expansion is that it groups terms quite different from one another. Fro example the terms $f_{12}f_{23}$ and $f_{12}f_{34}$. Indeed the first term correspond to a diagram as in Figure 3a, while the second to two disconnected diagrams as in Figure 3b.

(a) Description

(b) Description

Figure 3

Another problem of the above expansion is that it does not recognize identical clusters formed by different particles. For example the terms $f_{12}f_{23}$ and $f_{12}f_{14}$ contribute in the same way to the partition function. It is then convenient to follow a diagrammatic approach similar to the Feymann approach in the reciprocal space.

For the linear term f_{ij} the only diagram is given by Figure 4. As we have seen this has multeplicity $\frac{N(N-1)}{2}$ and the integral is of the form

$$\int f_{12}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2 = V\int f(\vec{\mathbf{r}})\,\mathrm{d}\vec{\mathbf{r}} = -2VB_2 \tag{24}$$

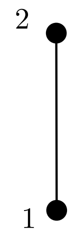

Figure 4

For the term $f_{ij}f_{kl}$ we can have the case as in Figure 5, that has molteplicity

$$\frac{N(N-1)}{2}\frac{(N-1)(N-3)}{2}\frac{1}{2} \tag{25}$$

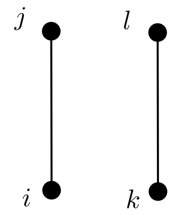

Figure 5

and the integral is of the form

$$\int f_{12}f_{34}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3\,\mathrm{d}\vec{\mathbf{r}}_4 \tag{26}$$

i.e. involving 4-particles

$$\begin{aligned}&\int f(|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|)f(|\vec{\mathbf{r}}_3-\vec{\mathbf{r}}_4|)\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3\,\mathrm{d}\vec{\mathbf{r}}_4 = \\ &= V^2\left(\int f(\vec{\mathbf{r}})\,\mathrm{d}\vec{\mathbf{r}}\right)^2 = 4V^2B_2^2\end{aligned} \tag{27}$$

The next case if for instance as in Figure 6. This involves 3 particles. The multiplicity of this diagram is

$$\frac{N(N-1)(N-2)}{3!}\times 3 \tag{28}$$

The integral is of the form

$$\begin{aligned}&\int f_{12}f_{23}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3 \simeq V\left(\int \mathrm{d}r\, f(r)\right)^2 = \\ &= \int f(|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|)f(|\vec{\mathbf{r}}_2-\vec{\mathbf{r}}_3|)\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3 = \\ &= V\left(\int f(\vec{\mathbf{r}})\,\mathrm{d}\vec{\mathbf{r}}\right)^2 = 4VB_2^2\end{aligned} \tag{29}$$

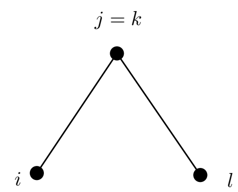

Figure 6

Another interesting diagram is the one in Figure 7. Its molteplicity is

$$\frac{N(N-1)(N-2)}{3!} \tag{30}$$

The associated integral involves 3 particles and it is of the form

$$\begin{aligned}&\int f_{12}f_{23}f_{31}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3 = \\ &= \int f(|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|)f(|\vec{\mathbf{r}}_2-\vec{\mathbf{r}}_3|)f(|\vec{\mathbf{r}}_3-\vec{\mathbf{r}}_1|)\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3 \\ &= \int f(|\vec{\mathbf{r}}_1-\vec{\mathbf{r}}_2|)f(|\vec{\mathbf{r}}_2-\vec{\mathbf{r}}_3|)f(|\vec{\mathbf{r}}_3-\vec{\mathbf{r}}_1|)\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_{21}\,\mathrm{d}\vec{\mathbf{r}}_{23}\end{aligned} \tag{31}$$

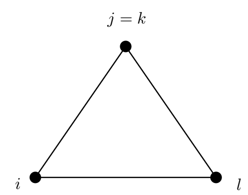

Figure 7

On the other hand $\vec{\mathbf{r}}_{13} = \vec{\mathbf{r}}_{23} - \vec{\mathbf{r}}_{21}$, which implies

$$f(|\vec{\mathbf{r}}_3 - \vec{\mathbf{r}}_1|) = f(|\vec{\mathbf{r}}_{23} - \vec{\mathbf{r}}_{21}|) \tag{32}$$

hence,

$$\int f(|\vec{\mathbf{r}}_{12}|)f(|\vec{\mathbf{r}}_{23}|)f(|\vec{\mathbf{r}}_{31}|)\,\mathrm{d}\vec{\mathbf{r}}_{21}\,\mathrm{d}\vec{\mathbf{r}}_{23}\,\mathrm{d}\vec{\mathbf{r}}_2 = \int f(|\vec{\mathbf{r}}_{12}|)f(|\vec{\mathbf{r}}_{23}|)f(|\vec{\mathbf{r}}_{23} - \vec{\mathbf{r}}_{21}|)\,\mathrm{d}\vec{\mathbf{r}}_{21}\,\mathrm{d}\vec{\mathbf{r}}_{23}\,\mathrm{d}\vec{\mathbf{r}}_2 \tag{33}$$

Let us call this integral

$$\int f_{12}f_{23}f_{31}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3 \equiv 3!V\left(B_3 - 2B_2^2\right) \tag{34}$$

The configurational partition function with these terms becomes

$$\begin{aligned} Q_N(V,T) =& V^N - V^N\frac{N(N-1)}{V}B_2 + V^N\frac{N(N-1)(N-2)(N-3)}{8V^2}(4B_2^2) \\ &+ V^N\frac{N(N-1)(N-2)}{2V^2}4B_2^2 \\ &= V^N\left(1 + \frac{N(N-1)}{V}B_2 + \frac{N(N-1)(N-2)(N-3)}{2V^2}B_2^2 + \frac{N(N-1)(N-3)}{V^2}B_3\right) \end{aligned} \tag{35}$$

Let us now face the problem in a slightly different ways. Let us remind that

$$Q_N(V,T) = \sum_{diagrams}\int\prod_{kl} f_{kl}\,\mathrm{d}^{3N}r \tag{36}$$

i.e. sum over all possible diagrams i.e. all possible ways in which a can draw edges between pairs of points (k,l). For each such diagrams I have to product between all edge and then integrate over the configurational space (N points).

Let us now consider only *connected* diagrams for i sites. In other words given i points (i particles) from a system of N points and I consider all the possible ways I can connect these i points (an example is shown in Figure 8).

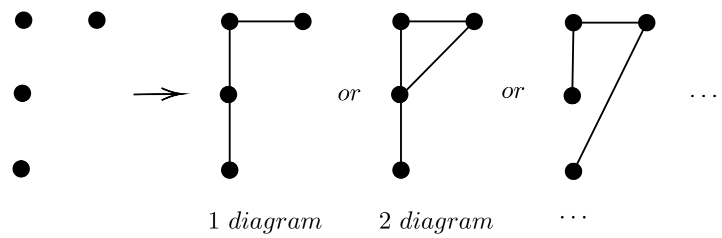

Figure 8: Example for $i = 4$.

For each diagram we take the product $\prod_{kl} f_{kl}$ and then integrate over the position of the i points (i particles). For a fixed diagram:

$$\int \prod_{lk\in\, diagram} f_{kl}\,\mathrm{d}\vec{\mathbf{r}}_1 \ldots \mathrm{d}\vec{\mathbf{r}}_i \tag{37}$$

Example 2 ($i = 4$)**.** For example the diagram 1 in Figure 8 gives the contribution

$$\int f_{12}f_{13}f_{34}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3\,\mathrm{d}\vec{\mathbf{r}}_4 \tag{38}$$

The diagram 2 gives

$$\int f_{12}f_{13}f_{23}f_{34}\,\mathrm{d}\vec{\mathbf{r}}_1\,\mathrm{d}\vec{\mathbf{r}}_2\,\mathrm{d}\vec{\mathbf{r}}_3\,\mathrm{d}\vec{\mathbf{r}}_4 \tag{39}$$

and so on.

Finally we sum over all these connected diagrams of i points.

$$\sum_{\substack{\text{connected}\\ \text{diagrams}}} \int \prod_{lk\in\, diagram} f_{kl}\, \mathrm{d}\vec{\mathbf{r}}_1 \ldots \mathrm{d}\vec{\mathbf{r}}_i \tag{40}$$

This is what we call $i!VB_i$ and defines B_i.

- Case $i = 1$: clearly $B_1 = 1$.
- Case $i = 2$: just one edge, one connected diagram

$$\int f_{12}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2 = -2VB_2 \tag{41}$$

- Case $i = 3$: the connected diagrams are shown in Figure 9.

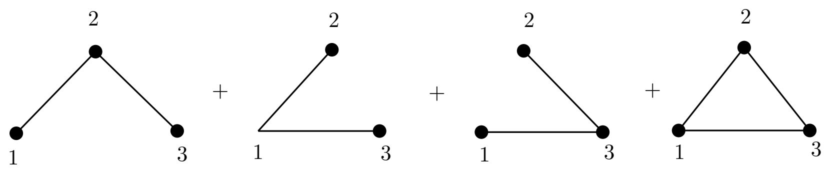

Figure 9: Description.

$$\begin{aligned} &= \underbrace{\int f_{12}f_{23}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2\, \mathrm{d}\vec{\mathbf{r}}_3 + \int f_{12}f_{13}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2\, \mathrm{d}\vec{\mathbf{r}}_3 + \int f_{13}f_{23}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2\, \mathrm{d}\vec{\mathbf{r}}_3}_{3V\left(\int f(\vec{\mathbf{r}})\mathrm{d}\vec{\mathbf{r}}\right)^2} \\ &+ \underbrace{\int f_{12}f_{23}f_{13}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2\, \mathrm{d}\vec{\mathbf{r}}_3}_{3!V(B_3-2B_2^2)} \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} \sum_{\substack{\text{connected diagrams}\\ \text{of } i=3 \text{ points}}} \int \prod_{kl\in\, diagram} f_{kl}\, \mathrm{d}\vec{\mathbf{r}}_1\, \mathrm{d}\vec{\mathbf{r}}_2\, \mathrm{d}\vec{\mathbf{r}}_3 &= 3V(-2B_2)^2 + 6V(B_3 - 2B_2) \\ &= 6VB_3 = 3!VB_3 \end{aligned} \tag{43}$$

For the partition function we have to sum over all possible clusters. Possible procedure:

1. Given the N points we can partition the into connected clusters. For all i points we can make m_i clusters of that size i.

$$\sum_i i m_i = N \tag{44}$$

   For each cluster of size i we have a term $(i!VB_i)$. If there are m_i of them we have a weight $(i!VB_i)^{m_i}$.

2. Now we have to count in how many ways we can make the partition of N in a set of $\{m_i\}$ clusters. Clearly if we permute the label of the N vertices we have possible different clusters. In principle this degenerancy is proportional to $N!$

On the other hand, if one changes the order of the labels within a cluster (in $i!$ ways) this does not change the cliuster and since there are m_i clusters of size i we have to divide by $(i!)^{m_i}$.

Moreover, since there are m_i clusters one can swap them (in $m_i!$ ways). THe degenerancy is $\frac{N!}{m_i!(i!)^{m_i}}$. Therefore,

$$Q_N(V,T) = \sum_{\{m_i\}} \prod_i \frac{N!}{m_i!(i!)^{m_i}} (i!VB_i)^{m_i} \tag{45}$$

Example 3 $(N = q)$**.** Consider the point in Figure 10.

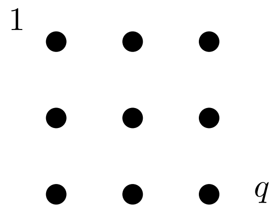

Figure 10: Description.

1. Parition these points into clusters.

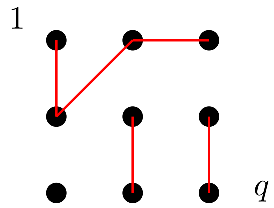

Figure 11: Description.

2. For this partition $\{m_i\}$ we have $m_4 = 1, m_2 = 2, m_1 = 1$. Now, the cluster of size 4 can be connected in a given different ways $(4!VB_4)^1$.

Exercise 1. TO DO: compute the degenerancy of this case.

Chapter 1

Landau theory of phase transition for homogeneous systems

1.1 Landau theory of phase transition (uniform systems)

It is a phenomenological mean field theory of phase transitions for uniform systems (no spatial variation of the order parameter). It is based on the following assumptions:

1. Existence of an unfirom order parameter η. Remember the definition of the order parameter:

$$\eta = \begin{cases} 0 & T \geq \bar{T} \text{ (disordered or symmetric phase)} \\ \neq 0 & T < \bar{T} \text{ (ordered symmetry is broken)} \end{cases} \tag{1.1}$$

   Well known examples are

$$\begin{cases} \eta \to m \\ \eta \to \rho_L - \rho_G \end{cases} \tag{1.2}$$

2. The free energy is an *analyc function* of the order parameter η. It is because you are doing the expansion close to...etc etc. Therefore, $\mathcal{L} = \mathcal{L}(\eta)$.

3. The form of $\mathcal{L}$ must satisfy the underlying symmetry of the system.

4. Equilibrium states correspond to the absolute minima of $\mathcal{L}$.

Remark. Since $\mathcal{L}$ os analytic it can be formally expanded in power of η, for $\eta \sim 0$.

$$\mathcal{L}(\eta) \approx a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \dots \tag{1.3}$$

1.2 Symmetries

To fix the ideas let us consider the theory for the Ising model. In this case η is a scalar (magnetization).

For $T > \bar{T}$ (critical point) we expect a paramagnetic phase. $\mathcal{L}$ has a minimum at $\eta = 0$, hence

$$\frac{\partial \mathcal{L}}{\partial \eta} = a_1 + 2a_2\eta + 3a_3\eta^2 + \dots = 0 \tag{1.4}$$

$\eta = 0$ is a solution if and only if $a_1 = 0$.

Remark. No linear term must be present!

Since Ising has $\mathbb{Z}^2$ symmetry, we should require

$$\mathcal{L}(-\eta) = \mathcal{L}(\eta) \tag{1.5}$$

which implies

$$a_k = 0 \quad \forall k \text{ odd} \tag{1.6}$$

Moreover, since $\mathcal{L}$ is analytic, terms proportional to $|\eta|$ are excluded. The minimal expression for $\mathcal{L}(\eta)$ that describes the equilibrium phase diagram of an Ising-like system is

$$\mathcal{L}(\eta) \simeq a_0(J,T) + a_2(J,T)\eta^2 + a_4(J,T)\eta^4 + O(\eta^6) \tag{1.7}$$

The coefficients of the expansion are functions of the physical parameters, J and T.

Since for $T > \bar{T}$, $\eta = \bar{\eta} = 0$ and

$$\mathcal{L}(\eta = 0) = a_0 \tag{1.8}$$

$a_0(T,J)$ value of $\mathcal{L}$ in the paramagnetic phase. Since what matters is the free-energy difference we can put $a_0 = 0$ identically.

Moreover, in order to have $\eta = \bar{\eta} \neq 0 < \infty$ for $T < \bar{T}$ (thermodynamic stability) we should impose that the coefficient of the highest power of η is always positive. In this case

$$a_4(J,T) > 0 \tag{1.9}$$

Indeed if this condition is violated $\mathcal{L}$ reaches it s absolute minimum for $|\eta| \to \infty$! Therefore,

$$\mathcal{L}(\eta) \simeq a_2\eta^2 + a_4\eta^4 \tag{1.10}$$

where he term a_4 it is positive and fixed.

If we now fix J and expand the coefficients a_2 and a_4 as a function of $t \equiv \frac{T-\bar{T}}{\bar{T}}$,

$$a_2 \sim a_2^0 + \frac{T-\bar{T}}{\bar{T}}\frac{a}{2} \tag{1.11a}$$

$$a_4 \sim \frac{b}{4} + \dots \tag{1.11b}$$

By choosing $a_2^0 = 0$ the sign of a_2 is determined by the one of

$$t \equiv \frac{T-\bar{T}}{\bar{T}} \tag{1.12}$$

In particular, at $T = \bar{T}$, one has $a_2 = 0$. Hence, for scalar, Ising like systems the minimal Landau free energy is given by

$$\mathcal{L} = \frac{a}{2}t\eta^2 + \frac{b}{4}\eta^4 + O(\eta^6) \tag{1.13}$$

Remark. Does not matter the coefficient in green in front, so in the next part of the course we will change it. If it is written in this way we have always $a > 0$. We have also $b > 0$.

1.3 Equilibrium phases

Now, the equilibrium states

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \quad \Rightarrow at\eta + b\eta^3 = 0 \tag{1.14}$$

Hence,

$$\bar{\eta} = \begin{cases} 0 & T > \bar{T} \\ \pm\sqrt{\frac{-at}{b}} & T < \bar{T} \end{cases} \tag{1.15}$$

At $T = \bar{T}$ the 3 solutions coincide!

- Case $T > \bar{T}$ ($t > 0$): only one solution $\bar{\eta} = 0$.

$$\frac{\partial^2 \mathcal{L}}{\partial \eta^2} = at + 3b\eta^2 \geq 0 \tag{1.16}$$

for $\bar{\eta} = 0$ and $t > 0$ implying that $\eta = \bar{\eta}$ is a global minimum, as in Figure 1.1.

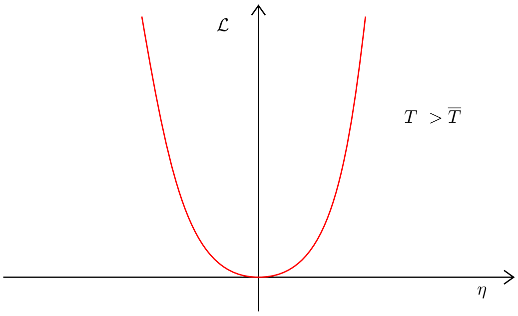

Figure 1.1: Description.

- Case $T < \bar{T}$ ($t < 0$): 3 solutions $\bar{\eta} = 0$ and $\bar{\eta} = \pm\sqrt{-\frac{at}{b}}$. Let us see wheter they are minima or local maxima.

$$\left.\frac{\partial^2 \mathcal{L}}{\partial \eta^2}\right|_{\bar{\eta}=0} = at < 0 \quad \Rightarrow \bar{\eta} = 0 \text{ local maximum (no equilibrium)} \tag{1.17}$$

$$\left.\frac{\partial^2 \mathcal{L}}{\partial \eta^2}\right|_{\bar{\eta}=\pm\sqrt{-\frac{at}{b}}} = at + 3b\left(-\frac{at}{b}\right) = -2at \tag{1.18}$$

since $t < 0$, $-2at > 0$ and hence $\bar{\eta} = \pm\sqrt{-\frac{at}{b}}$ are 2 minima!

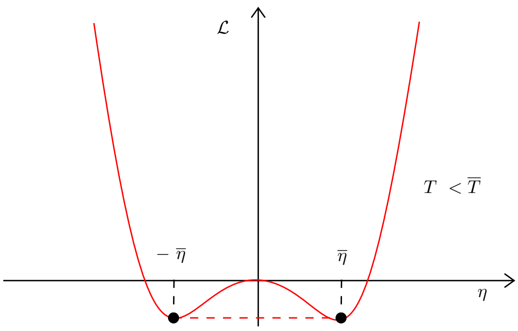

Figure 1.2: Description.

$$\mathcal{L}\left(\eta = \pm\sqrt{-\frac{at}{b}}\right) = -\frac{a^2t^2}{2b} + \frac{a^2t^2}{4b} = -\frac{a^2t^2}{4b} < 0 \tag{1.19}$$

The 2 minima are related by the group symmetry $\mathbb{Z}^2$ ($\bar{\eta} \to -\bar{\eta}$).

Remark. Note that, in presence of an external magnetic field h, one should consider the Legendre transform of $\mathcal{L}$ obtaining its Gibbs version:

$$\mathcal{L} = \frac{a}{2}t\eta^2 + \frac{b}{4}\eta^4 - h\eta \tag{1.20}$$

we have inserted a field coupled with the order parameter.

1.4 Critical exponents in Landau's theory

Consider $t \equiv \frac{T-\bar{T}}{\bar{T}}$.

Exponent β

We have $\eta \sim t^\beta$ for $h = 0$, $t \to 0^-$. Since $t < 0$, the minima of $\mathcal{L}$ are

$$\bar{\eta} = \pm\sqrt{-\frac{at}{b}} \quad \Rightarrow \beta = \frac{1}{2} \tag{1.21}$$

as expected.

Exponent α

We have $C \sim t^{-\alpha}$ for $h = 0$, $|t| \to 0$. Two cases:

- $t > 0$: $\bar{\eta} = 0$ and $\mathcal{L}(\bar{\eta}) = 0$.
- $t < 0$: $\mathcal{L}_{min} = \mathcal{L}\left(\bar{\eta} = \pm\sqrt{-\frac{at}{b}}\right) = -\frac{a^2t^2}{4b}$, that implies

$$\mathcal{L}_{min} = \begin{cases} 0 & t > 0 \\ -\frac{a^2t^2}{4b} & t < 0 \end{cases} \tag{1.22}$$

Hence,

$$c_V = -T\frac{\partial^2\mathcal{L}}{\partial T^2} = -T\frac{\partial^2}{\partial T^2}\left(-\frac{a^2}{4b\bar{T}^2}(T-\bar{T})^2\right) \tag{1.23}$$

We have

$$\frac{\partial}{\partial T}(\dots) = -\frac{a^2}{2b\bar{T}^2}(T-\bar{T}) \tag{1.24}$$

$$\frac{\partial^2}{\partial T^2} = \frac{\partial}{\partial T}\left[-\frac{a^2}{2b\bar{T}^2}(T-\bar{T})\right] = -\frac{a^2}{2b\bar{T}^2} \tag{1.25}$$

$$c_V \begin{cases} 0 & T > \bar{T} \\ \frac{a^2}{2b\bar{T}^2}T & T < \bar{T} \end{cases} \tag{1.26}$$

We have $t \to 0^-$ if and only if $T \to \bar{T}^-$, which implies $c_V \to \frac{a^2}{2b\bar{T}}$ that is constant.

In obth cases $\alpha = 0$.

Exponent δ

We have $h \sim \eta^\delta$ at $T = \bar{T}$. Let us start from the equation of state. This is obtained by computing the $\frac{\partial}{\partial \eta}$ of the Gibbs version.

$$\frac{\partial \mathcal{L}_G}{\partial \eta} = at\eta + b\eta^3 - h = 0 \tag{1.27}$$

that is the condition for equilibrium. The equation of state is

$$h = at\eta + b\eta^3 \tag{1.28}$$

Equation (1.28) tells us that, for fixed h, the extreme points of $\mathcal{L}$ are given by the values of η that satisfies (1.28).

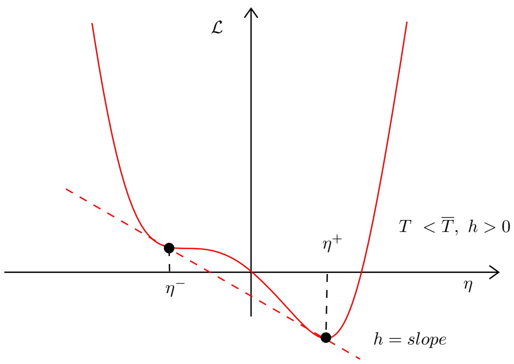

Figure 1.3: Description.

At $T = \bar{T}$ $(t = 0)$ we have $h \sim \eta^3$, hence $\delta = 3$.

Exponent γ

$\chi_T \sim t^{-\gamma}$ for $h = 0$, $|t| \to 0$. Let us derive the equation of state (1.28) with respect to h:

$$at\frac{\partial \eta}{\partial h} + 3b\eta^3\frac{\partial \eta}{\partial h} = 1 \tag{1.29}$$

and since

$$\chi = \frac{\partial \eta}{\partial h} \tag{1.30}$$

we have

$$\chi = \frac{1}{at + 3b\eta^2} \tag{1.31}$$

- Case $t > 0$, $\bar{\eta} = 0$: $\chi_T = -\frac{1}{at}$.
- Case $t < 0$, $\bar{\eta} = \pm\left(-\frac{at}{b}\right)^{1/2}$: $\chi_T = -\frac{1}{2at}$.

In both cases $\chi_T \sim 1/t$ and this gives

$$\gamma = \gamma' = 1 \tag{1.32}$$

Summary

In summary the Landau theory gives the following (mean field) values of the critical exponents

$$\beta = \frac{1}{2}, \quad \alpha = 0, \quad \delta = 3, \quad \gamma = 1 \tag{1.33}$$

Landau theory does not depend on the system dimension D (as expected since isa mean field theory) but only on its symmetries.

Remark. For a $O(n)$ (vector) model the order parameter η becomes a vector field $\vec{\eta}$ with n compnents and

$$\mathcal{L}_G(\vec{\eta}) = \frac{a}{2} t \vec{\eta} \cdot \vec{\eta} + \frac{b}{4} (\vec{\eta} \cdot \vec{\eta})^2 - \vec{\mathbf{h}} \cdot \vec{\eta} + O\Big((\vec{\eta} \cdot \vec{\eta})^3\Big) \tag{1.34}$$