Chapter 1

Spontaneous symmetry breaking

1.1 Spontaneous symmetry breaking

Lecture n.
Tuesday 0th January, 2020.
Compiled: Wednesday 5th February, 2020.

When we talk about a broken symmetry, we often refer to a situation as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

where $\mathcal{H}_0$ is invariant under the group $\mathcal{G}$ and $\mathcal{H}_1$ is invariant under a subgroup $\mathcal{G}' \subset \mathcal{G}$.

Example 1: Ising with magnetic field

Let us consider the Hamiltonian for the Ising model with a magnetic field $H \neq 0$:

$$\mathcal{H} = J \sum_{\langle ij \rangle} S_i S_j + \sum_i H_i S_i$$

The second term, $\sum_i H_i S_i$, breaks the $\mathbb{Z}^2$ symmetry satisfied by the first alone.

Example 2: Hydrogen atom with an external field

An example, in quantum mechanics, is the hydrogen atom in presence of an electric field $\vec{\mathbf{E}}$ (Stark effect) or a magnetic one, $\vec{\mathbf{B}}$, (Zeeman effect). If $\mathcal{H}_1$ is small, the original symmetry is weakly violated and perturbative approaches are often used.

In all the above examples, one says that the symmetry is broken explicitly.

Definition 1: Spontaneous symmetry breaking

The Hamiltonian maintains the original symmetry but the variables used to describe the system become asymmetric.

At this point it is convenient to distinguish between

- **Discrete symmetries**: for instance $\mathbb{Z}^2$, $\mathbb{Z}_q$.
- **Continuous symmetries**: for instance XY, $O(n)$.

Let us consider first the discrete ones by focusing on the $\mathbb{Z}^2$ symmetry (Ising). As previously said, if $H = 0$, the Hamiltonian of the Ising model, $\mathcal{H}_{\text{Ising}}$, is invariant with respect to the change $S_i \to -S_i$, hence the discrete group is

$$\mathcal{G} = \mathbb{Z}^2$$

A Ginzburg-Landau theory of the Ising model is given by

$$\beta\mathcal{H}(\Phi) = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2} \left(\vec{\boldsymbol{\nabla}} \Phi \right)^2 + \frac{r_0}{2} \Phi^2 + \frac{u_0}{4} \Phi^4 - h\Phi \right] \tag{1.1}$$

Hence, the partition function can be computed as:

$$Z(r_0, u_0, h) = \int \mathcal{D}[\Phi] e^{-\beta \mathcal{H}(\Phi)} \tag{1.2}$$

If we have $h = 0$, the symmetry is $\Phi \to -\Phi$. The equation of state obtained with saddle point approximation is

$$h = -\vec{\boldsymbol{\nabla}}^2 \Phi + r_0 \Phi + u_0 \Phi^3$$

If h does not depend on $\vec{\mathbf{x}}$, i.e. $h(\vec{\mathbf{x}}) = h$, the last equation reduces to the equation of state of the Landau theory of uniform system:

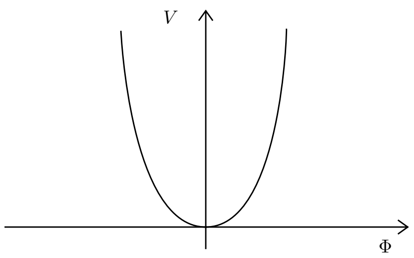
$$h = r_0 \Phi + u_0 \Phi^3$$

Let us remind that the saddle point appriximation constists in approximating the functional integral of Eq.(1.2) with its dominant term, i.e. with the one for which the exponent (Eq.(1.1)) is minimum. Therefore, in the uniform case (namely $\vec{\boldsymbol{\nabla}}\Phi = 0$), it is equivalent to find the uniform value Φ_0 that is the extrema of the potential:

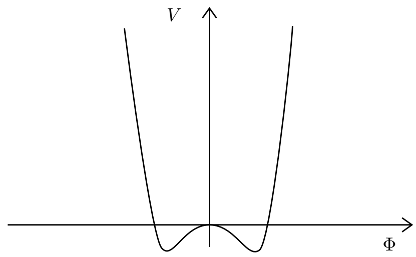
$$V(\Phi) = \frac{1}{2} r_0 \Phi^2 + \frac{u_0}{4} \Phi^4 - h\Phi$$

Hence, if $h = 0$, the extrema of the potential can be computed as

$$V' = (r_0 + u_0 \Phi^2)\Phi = 0$$



(a) Plot of the potential $V(\Phi)$ in the case $r_0 > 0$, i.e. $T > T_c$.



(b) Plot of the potential $V(\Phi)$ in the case $r_0 < 0$, i.e. $T < T_c$.

Figure 1.1

Let us remember that $r_0 \propto (T - T_c)$. In order to find the extrema of the potential $V(\Phi)$, we should distinguish two cases:

1. Case $r_0 > 0$ ($T > T_c$): there is only one solution $\Phi_0 = 0$, as we can see in Figure 1.1a.

2. Case $r_0 < 0$ ($T < T_c$): there are two solutions $\Phi_0 = \pm\sqrt{-\frac{r_0}{u_0}}$, as illustrated in Figure 1.1b.

   We note that the two solution $\pm\Phi_0$ are related by the $\mathbb{Z}^2$ transformation, namely $\Phi \to -\Phi$. Moreover, in this case with $T < T_c$, the two states (phases) $\pm\Phi_0$ have a lower symmetry than the state $\Phi_0 = 0$.

   If the thermal fluctuations $\delta\Phi$ are sufficiently strong to allow passages between the two states $\pm\Phi_0$ at $T < T_c$, we have $\langle\Phi\rangle = 0$ (preserves states).

   However, for $T < T_c$ and $N \to +\infty$, transition between the two states will be less and less probable and the system will be trapped into one of the two states ($\pm\Phi_0$). In other words, the system choose spontaneously one of the two less symmetric state. Therefore, its physics is not any more described by Φ but by the fluctuations $\delta\Phi$ around the chosen minimum Φ_0. There is a spontaneous symmetry breaking. It means that the variable Φ is not any more symmetric and one has to look at $\Phi \to \Phi_0 + \delta\Phi$, where $\delta\Phi$ is a new variable!

1.2 Spontaneous breaking of continuous symmetries and the onset of Goldstone particles

Let us start with a simple model in which the order parameter is a scalar complex variable

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}$$

and with an Hamiltonian $\mathcal{H}$ that is invariant with respect to a global continuous transformation. For instance, the simplest model in statistical mechanics that is invariant with respect to a continuous symmetry is the XY model with $O(2)$ symmetry, or a Ginzburg-Landau model for a superfluid or a superconductor (with no magnetic field). Hence, we suppose that the Hamiltonian as the following form:

$$\beta\mathcal{H}_{eff} = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2}\vec{\boldsymbol{\nabla}}\Phi \cdot \vec{\boldsymbol{\nabla}}\Phi^* + \frac{r_0}{2}\Phi\Phi^* + \frac{u_0}{4}(\Phi\Phi^*)^2\right]$$

where

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{\sqrt{2}}[\Phi_1(\vec{\mathbf{x}}) + i\Phi_2(\vec{\mathbf{x}})], \quad \text{or} \quad \Phi(\vec{\mathbf{x}}) = \psi(\vec{\mathbf{x}})e^{i\alpha(\vec{\mathbf{x}})}$$

The physical meaning of Φ depends on the case considered. If we have:

- Superfluid: Φ is the macroscopic wave function of the Bose condensate (density of superfluid $n = |\Phi^2|$).
- Superconductor: Φ is the single particle wave function describing the position of the centre of mass of the Cooper pair.

1.2.1 Quantum relativistic case (field theory)

In quantum mechanics the analog of the Hamiltonian $\mathcal{H}$ is the action

$$S(\Phi) = \int \mathrm{d}^4\vec{\mathbf{x}}\,\mathcal{L}(\Phi) \tag{1.3}$$

where

$$\mathcal{L}(\Phi) = -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi^* - \frac{r_0}{2}\Phi\Phi^* - \frac{u_0}{4}(\Phi\Phi^*)^2 \tag{1.4}$$

The Lagrangian $\mathcal{L}(\Phi)$ describes a scalar complex (i.e. charged) *muonic field* with mass $m \equiv \sqrt{r_0}$; we note that, if $\mathcal{L}(\Phi)$ describes a muonic field, we should have $r_0 > 0$ to have the mass m well defined. Moreover, the term $(\Phi\Phi^*)^2$ means self-interaction with strenght $\lambda \equiv u_0$.

In all cases ($r_0 > 0$ or $r_0 < 0$), the original symmetry is $U(1)$; it means that both the Hamiltonian $\mathcal{H}$ and the Lagrangian $\mathcal{L}$ are invariant with respect to the transformation

$$\Phi \to e^{i\theta}\Phi, \quad \Phi^* \to e^{-i\theta}\Phi^* \tag{1.5}$$

where the phase θ does not depend on $\vec{\mathbf{x}}$ (global symmetry). In components the transformation becomes

$$\begin{cases} \Phi_1 \to \Phi_1\cos\theta - \Phi_2\sin\theta \\ \Phi_2 \to \Phi_2\cos\theta + \Phi_1\sin\theta \end{cases} \Rightarrow (\Phi_1', \Phi_2') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

Now, let us focus first on the statistical mechanics model and to the most interesting case of $r_0 < 0$. In components, $\mathcal{H}$ can be expressed as

$$\beta\mathcal{H} = \int \mathrm{d}^d\vec{\mathbf{x}}\left[(\boldsymbol{\nabla}\Phi_1)^2 + (\boldsymbol{\nabla}\Phi_2)^2\right] + \int \mathrm{d}^d\vec{\mathbf{x}}\,V(\Phi_1, \Phi_2) \tag{1.6}$$

where the potential is

$$V(\Phi_1, \Phi_2) = \frac{r_0}{2}\left(\Phi_1^2 + \Phi_2^2\right) + \frac{u_0}{4}\left(\Phi_1^2 + \Phi_2^2\right)^2 \tag{1.7}$$

In the case $r_0 < 0$, it is called *mexican hat potential* and it is shown in Figure 1.2.

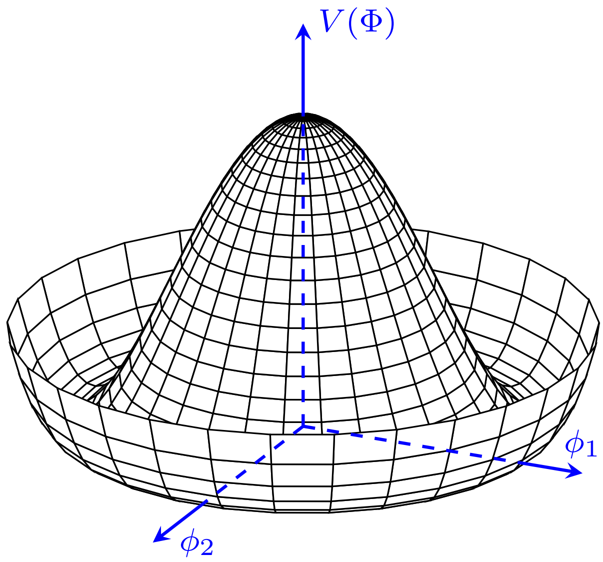


Figure 1.2: Case $r_0 < 0$. The potential $V(\Phi)$ is a mexican hat potential.

In the uniform case we have ($\boldsymbol{\nabla}\Phi_1 = \boldsymbol{\nabla}\Phi_2 = 0$). Let us define $S = \sqrt{\Phi_1^2 + \Phi_2^2}$, the potential in Eq.(1.7) can be rewritten as:

$$V(S) = \frac{r_0}{2}S^2 + \frac{u_0}{4}S^4$$

In the uniform case, the solution is given by the minima of the potential $V(S)$; hence, in order to find the extrema points, we derive the potential with respect to S and we impose the condition $V' = 0$:

$$\frac{\mathrm{d}V(S)}{\mathrm{d}S} = r_0 S + u_0 S^3 = 0$$

We have a maximum at $S = 0$ and a minimum at $S^2 \equiv v^2 = -r_0/u_0$. Hence, for $r_0 < 0$, the Hamiltonian $\mathcal{H}$ displays a minimum when

$$\Phi_1^2 + \Phi_2^2 \equiv v^2 = -\frac{r_0}{u_0}$$

It could be represented in the $2d$ plane (Φ_1, Φ_2), where the minimum lies on a circle of radius

$$v = \sqrt{-\frac{r_0}{u_0}}$$

as show in Figure 1.3. The spontaneous symmetry breaking occurs when the system "chooses" one of the infinite available minima. In our example, let us suppose that the chosen minimum is

$$\Phi_1 = v = \sqrt{-\frac{r_0}{u_0}}, \quad \Phi_2 = 0 \tag{1.8}$$

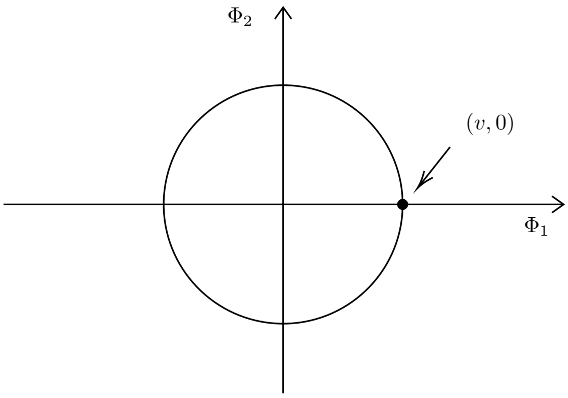


Figure 1.3: Plane (Φ_1, Φ_2). The minimum lies on a circle of radius $v = \sqrt{-\frac{r_0}{u_0}}$.

Interpretation in relativistic quantum mechanics

Now, let us give a physical interpretation of the results previously obtained and of the considerations we have done in order to obtain them. In particular:

1. Choosing $r_0 < 0$ corresponds to an *imaginary mass*. This is because moving away from $\Phi = 0$, the system experiences a *negative resistence* in both directions, being $\Phi = 0$ a relative local maximum.

2. The minimum has the lowest energy and therefore it must correspond to the *empty state*. In this case, however, there is an infinite number of empty states!

In summary, the starting Hamiltonian $\mathcal{H}$ (or Lagrangian $\mathcal{L}$) is invariant with respect to $U(1)$, but the one that describes the fluctuation dynamics around one of the chosen minimum state is not invariant with respect to $U(1)$. Let us see in more details why the Hamiltonian, or Lagrangian, it is not invariant anymore in the case $r_0 < 0$.

First of all, let us write the Lagrangian with respect to the fluctuations of Φ_1 and Φ_2 around the chosen state $(v, 0)$ (Eq.(1.8)), we obtain:

$$\begin{cases} \Phi_1 = v + \delta\Phi_1 \\ \Phi_2 = 0 + \delta\Phi_2 \end{cases} \Rightarrow \Phi = \Phi_1 + i\Phi_2 = v + (\delta\Phi_1 + i\delta\Phi_2)$$

where we omit the factor $1/\sqrt{2}$ for simplicity. Let us note that

$$\begin{cases} \delta\Phi_1 = \Phi_1 - v \\ \delta\Phi_2 = \Phi_2 \end{cases} \Rightarrow \langle\delta\Phi_1\rangle_{\Phi_0} = \langle\delta\Phi_2\rangle_{\Phi_0} = 0$$

indeed, as expected, the expectation value of the empty state is back to be zero.

Now, for the quantum relativistic Lagrangian $\mathcal{L}$, let us define

$$r_0 \to m^2, \quad u_0 \to \lambda, \quad v^2 = -\frac{m^2}{\lambda}$$

(recall that we are still in the case $r_0 < 0$!). Hence, the Lagrangian Eq.(1.4) becomes

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\partial_\mu(v+\delta\Phi_1+i\delta\Phi_2)\partial^\mu(v+\delta\Phi_1-i\delta\Phi_2)\\
&-\frac{m^2}{2}(v+\delta\Phi_1+i\delta\Phi_2)(v+\delta\Phi_1-i\delta\Phi_2)\\
&-\frac{\lambda}{4}[(v+\delta\Phi_1+i\delta\Phi_2)(v+\delta\Phi_1-i\delta\Phi_2)]^2\\
= &-\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1)-\frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2)\\
&-\frac{m^2}{2}\left(v^2+2v\delta\Phi_1+\delta\Phi_1^2+\delta\Phi_2^2\right)\\
&-\frac{\lambda}{4}\left(v^2+2v\delta\Phi_1+\delta\Phi_1^2+\delta\Phi_2^2\right)^2
\end{aligned}
$$

Since we have defined $m^2=-v^2\lambda$, we can rewrite it as

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(\partial_\mu\delta\Phi_1\partial^\mu\delta\Phi_1)-\frac{1}{2}(\partial_\mu\delta\Phi_2\partial^\mu\delta\Phi_2)+\frac{\lambda v^2}{2}\left(v^2+\cancel{2v\delta\Phi_1}+\cancel{\delta\Phi_1^2+\delta\Phi_2^2}\right)\\
&-\frac{\lambda}{4}\left(v^4+4v^2\delta\Phi_1^2+\cancel{4v^3\delta\Phi_1}+\left(\delta\Phi_1^2+\delta\Phi_2^2\right)^2+\cancel{2v^2\left(\delta\Phi_1^2+\delta\Phi_2^2\right)}+4v\delta\Phi_1\left(\delta\Phi_1^2+\delta\Phi_2^2\right)\right)
\end{aligned}
$$

Neglecting the constant terms in v (in green), finally we obtain

$$
\begin{aligned}
\mathcal{L}(\delta\Phi_1,\delta\Phi_2) = &-\frac{1}{2}(\partial_\mu\delta\Phi_1)^2-\frac{1}{2}(\partial_\mu\delta\Phi_2)^2\\
&-\frac{2}{2}\lambda v^2\delta\Phi_1^2-v\lambda\delta\Phi_1\left((\delta\Phi_1)^2+(\delta\Phi_2)^2\right)\\
&-\frac{\lambda}{4}\left((\delta\Phi_1)^2+(\delta\Phi_2)^2\right)^2
\end{aligned}
\tag{1.9}
$$

Comparing it with Eq.(1.4), we note that the yellow term $-\lambda v^2\delta\Phi_1^2$ indicates that the field $\delta\Phi_1$ (related to the transversal fluctuations) has a null empty state ($\langle\delta\Phi_1\rangle=0$) and a mass M such that (recall that $m^2=-\lambda v^2=r_0$):

$$
M^2=2\lambda v^2=-2r_0=-2m^2
$$

Therefore, it represents a real, massive, mesonic scalar field that is physically accettable (indeed we have $r_0<0$!). However, $\mathcal{L}$ is not any more invariant under the transformation $\delta\Phi_1\to-\delta\Phi_1$, as we wanted to show! The symmetry is broken.

Remark. Note that the field $\delta\Phi_2$ has no mass (there is not a term $\propto\delta\Phi_2^2$)! Indeed, it describes the fluctuations along the circle where the potential V is in its minimum which imples no dynamical inertia, that implies no mass!

In summary, starting with one complex scalar field $\Phi(\vec{\mathbf{x}})$ having mass m, when $m^2<0$ one gets a real scalar field $\delta\Phi_1$ with mass $M=\sqrt{-2m^2}$ and a second scalar field $\delta\Phi_2$ that is massless. This is called the **Goldstone boson**. Hence, we have the following theorem:

Theorem 1: Goldstone's theorem

If a continuous symmetry is spontaneously broken and there are no long range interactions, exists an elementary excitation with zero momentum, or particle of zero mass, called Goldstone boson.
More generally, let $\mathcal{P}$ be a subgroup of $\mathcal{G}$ ($\mathcal{P}\subset\mathcal{G}$). If $\mathcal{G}$ has N indipendent generators and $\mathcal{P}$ has M indipendent generators, hence, if $\mathcal{P}$ is the new (lower) symmetry, therefore $N-M$ Goldstone bosons exist.

In the previous case the symmetry group was $\mathcal{G} = U(1)$, hence $\mathcal{G}$ has $N = 1$ indipendent generators, whereas we have $M = 0$ (we have chosen a specific minimum). Therefore, we have only one Goldstone boson.

Example 3: XY model

Let us consider the XY model in statistical mechanics. We have that

- $\delta\Phi_1$ represents the fluctuation of the modulus of m.
- $\delta\Phi_2$ represents fluctuations of the spin directions, or spin waves.

Remark. In particle physics the presence of Goldstone bosons brings a serious problem in field theory since the corresponding particles are not observed! The resolution of this problem is given by Higgs-Englert-Brout (1964). In particular, the Higgs mechanism gives back the mass to the Goldstone particles, because the Goldstone theorem, that works well for a continuous global symmetry, can fail for local gauge theories!

1.3 Spontaneous symmetry breaking in gauge symmetries

1.3.1 Statistical mechanics

In statistical mechanics, let us consider a Ginzburg-Landau model for superconductors in presence of a magnetic field (*Meissner effect*, i.e. the magnetic induction $\vec{\mathbf{B}} = 0$ inside the superconductor). The Hamiltonian of such a system is:

$$\beta\mathcal{H}(\Phi) = \int \mathrm{d}^d\vec{\mathbf{x}} \left[\frac{1}{2}B^2 + \left|\left(\vec{\boldsymbol{\nabla}} - 2i\vec{\mathbf{A}}\right)\Phi\right|^2 + \frac{r_0}{2}\Phi^*\Phi + \frac{u_0}{4}(\Phi^*\Phi)^2 - \vec{\mathbf{B}}\cdot\vec{\mathbf{H}}\right] \tag{1.10}$$

where $\frac{B^2}{2}$ is the energy of the magnetic field $\vec{\mathbf{B}}$ and $\vec{\boldsymbol{\nabla}} \to \left[\vec{\boldsymbol{\nabla}} + iq\vec{\mathbf{A}}\right]$ is the minimal coupling. If we have an external magnetic field $\vec{\mathbf{H}}$, the induction field is:

$$\vec{\mathbf{B}} = \vec{\mathbf{H}} + \vec{\mathbf{M}}$$

For normal conductors we have $\Phi_0 = 0$, which implies $\vec{\mathbf{B}} = \vec{\mathbf{H}}$, while for superconductors $\Phi \neq 0$ and we have a spontaneous symmetry breaking.

1.3.2 Field theory analog: the Higgs mechanism for an abelian group

Let us consider the analog of the previous system in field theory: a scalar charged mesonic fields selfinteracting and in presence of an electromagnetic field with potential quadrivector $A_\mu(\vec{\mathbf{x}})$. We begin by applying the Higgs mechanism to an abelian[1], $U(1)$ gauge theory, to demonstrate how the mass of the corresponding gauge boson (the photon) comes about.

The $U(1)$ gauge invariant kinetic term of the photon is given by

$$\mathcal{L}_{photon} = -\frac{1}{4}F_{\mu\nu}(\vec{\mathbf{x}})F^{\mu\nu}(\vec{\mathbf{x}}), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

[1] In abstract algebra, an abelian group, also called a commutative group, is a group in which the result of applying the group operation to two group elements does not depend on the order in which they are written.

That is, $\mathcal{L}_{photon}$ is invariant under the transformation: $A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) - \partial_\nu \alpha(\vec{\mathbf{x}})$ for any α and x. If we naively add a mass term for the photon to the Lagrangian, we have

$$\mathcal{L}_{photon} = -\frac{1}{4} F_{\mu\nu}(\vec{\mathbf{x}}) F^{\mu\nu}(\vec{\mathbf{x}}) + \frac{1}{2} m_A^2 A_\mu A^\mu$$

the mass term violates the local gauge symmetry; hence, the $U(1)$ gauge symmetry requires the photon to be massless.

Now extend the model by introducing a complex scalar field, with charge q, that couples both to itself and to the photon. In this case, because of the presence of $A_\mu(\vec{\mathbf{x}})$, we should consider a theory that satisfies symmetry $U(1)$ locally! The Lagrangian of this model is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(\vec{\mathbf{x}}) F^{\mu\nu}(\vec{\mathbf{x}}) + (D_\mu \Phi(\vec{\mathbf{x}}))^* (D_\mu \Phi(\vec{\mathbf{x}})) - V(\Phi, \Phi^*) \tag{1.11}$$

where

$$\begin{aligned} D_\mu \Phi &= (\partial_\mu + iqA_\mu)\Phi && \text{gauge-covariant derivative} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu && \text{Field strength tensor} \\ V(\Phi, \Phi^*) &= \frac{m^2}{2} \Phi\Phi^* + \frac{\lambda}{4} (\Phi\Phi^*)^2 && \text{Potential} \end{aligned}$$

The complex scalar field we are considering is defined again as:

$$\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2), \quad \Phi^* = \frac{1}{\sqrt{2}}(\Phi_1 - i\Phi_2)$$

Hence, it is easily discerned that this Lagrangian is invariant under the gauge transformations

$$\begin{aligned} &\Phi(\vec{\mathbf{x}}) \to e^{i\alpha(\vec{\mathbf{x}})} \Phi(\vec{\mathbf{x}}), \quad \Phi^*(\vec{\mathbf{x}}) \to e^{-i\alpha(\vec{\mathbf{x}})} \Phi(\vec{\mathbf{x}}) \\ &A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) - \partial_\nu \alpha(\vec{\mathbf{x}}) \end{aligned}$$

Let us consider again two different cases:

- If $m^2 > 0$: the state of minimum energy will be that with $\Phi = 0$ and the potential will preserve the symmetries of the Lagrangian. Then the theory is simply QED with a massless photon and a charged scalar field Φ with mass m.
- If $m^2 < 0$: the field Φ will acquire a vacuum expectation value. The state of minimum energy will be that with $\Phi = \sqrt{-\frac{m^2}{\lambda}} \equiv v$ (circle of radius $|\Phi| = v$) and the global $U(1)$ gauge symmetry will be spontaneously broken.

Now, let us focus on the case $m^2 < 0$. In particular, it is convenient to parametrize Φ by choosing

$$\bar{\Phi}_1 = v, \quad \bar{\Phi}_2 = 0$$

Hence, we have:

$$\Phi(x) = (v + \delta\Phi_1) + i\delta\Phi_2$$

where Φ_1 is referred to as the Higgs boson and Φ_2 as the Goldstone boson. As said, they are real scalar fields which have no vacuum expecation value ($\langle \Phi_1 \rangle_{\Phi_0} = \langle \Phi_2 \rangle_{\Phi_0} = 0$). By inserting it in the Lagrangian and by keeping in mind that $m^2 = -v^2\lambda$, the Lagrangian transforms as

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \delta\Phi_1)^2 + \frac{1}{2} (\partial_\mu \delta\Phi_2)^2 \\ &- \frac{2}{2} \lambda v^2 \delta\Phi_1^2 + q^2 v^2 A_\mu A^\mu - qvA^\mu \partial_\mu \delta\Phi_2 + \text{higher order terms} \end{aligned} \tag{1.14}$$

Let us give a physical interpretation of the emphasized terms:

- $\lambda v^2 \delta\Phi_1^2$ means that the field $\delta\Phi_1$ is massive with mass $M = v\sqrt{2\lambda}$ (*Higgs boson*).
- $q^2 v^2 A_\mu A^\mu$ means that the *gauge boson* A_μ, the photon, has got a mass

$$M_A^2 = 2q^2v^2$$

  Note that since now A_μ is massive, it has three indipendent polarization states.

- $qvA^\mu \partial_\mu \delta\Phi_2$ means that the field $\delta\Phi_2$ is not massive (indeed there is no term $\propto \delta\Phi_2^2$) and that it is mixed with A_μ (*Goldstone boson*). Dynamically, this means that a propagating photon can transform itself into a field $\delta\Phi_2$ (the photon becomes a Goldstone boson).

Summarizing, this Lagrangian now describes a theory with a Higgs boson $\delta\Phi_1$ with mass $M = v\sqrt{2\lambda}$, a photon of mass $M_A = qv\sqrt{2}$ and a massless Goldstone $\delta\Phi_2$.

Moreover, since $\delta\Phi_2$ does not seem to be a physical field it should be eliminated by a gauge transformation. Hence, the strange $\delta\Phi_2 A^\mu$ mixing can be removed by making the following gauge transformation:

$$A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) - \frac{1}{q}\partial_\mu \alpha(\vec{\mathbf{x}})$$

We can choose $\alpha(\vec{\mathbf{x}}) = -\frac{1}{v}\delta\Phi_2(\vec{\mathbf{x}})$

$$A_\mu(\vec{\mathbf{x}}) \to A_\mu(\vec{\mathbf{x}}) + \frac{1}{qv}\partial_\mu \delta\Phi_2(\vec{\mathbf{x}})$$

and the gauge choice with the transformation above is called the unitary gauge. By inserting it in the Lagrangian Eq.(1.14), we eliminate the mixed term $qvA^\mu\partial_\mu\delta\Phi_2$, in red, and the term $\frac{1}{2}(\partial_\mu\delta\Phi_2)^2$. Thus we obtain:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\delta\Phi_1)^2 - \lambda v^2\delta\Phi_1^2 + q^2v^2A_\mu A^\mu + \text{higher order terms} \tag{1.15}$$

The Goldstone $\delta\Phi_2$ will then completely disappear from the theory and one says that the Goldstone has been "eaten" to give the photon mass. Therefore, the new Lagrangian in Eq.(1.15) contains two fields: one is a massive photon with spin 1 and the second field $\delta\Phi_1$ is massive too, but has spin 0 (scalar). The mechanics trough which the gauge boson becomes massive is the so called **Higgs mechanism**.

It is instructive to count the degrees of freedom (dof) before and after spontaneous symmetry broken has occurred. In particular, we have:

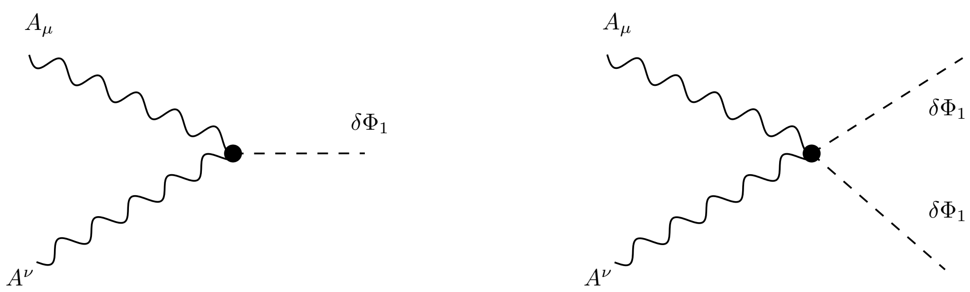
- For a global $U(1)$ symmetry, we have 2 massive scalar fields, hence there are $1+1$ degrees of freedom.

  After symmetry breaking, we have 1 scalar field massive and 1 scalar field not massive. Hence, there are again $1+1$ degrees of freedom.

- For a local gauge $U(1)$ symmetry, we have 2 massive scalar fields and one massless photon. Hence, there are $2+2$ degrees of freedom (the massless photon has 2 polarizations).

  After symmetry breaking, we have 1 massive scalar field and 1 massive photon. Hence, there are $1+3$ degrees of freedom (the massive photon has 3 polarizations).

Remark. Let us note that in Eq.(1.15) among the higher order terms we have neglected, there are



(a) Feyman diagram of the term $\propto \delta\Phi_1 A_\mu A^\mu$.

(b) Feyman diagram of the term $\propto \delta\Phi_1^2 A_\mu A^\mu$.

Figure 1.4

- A term proportional to $\propto \delta\Phi_1 A_\mu A^\mu$, shown in Figure 1.4a.
- A term proportional to $\propto \delta\Phi_1^2 A_\mu A^\mu$, shown in Figure 1.4b.

Remark. Note that the presence of the massive photon $M_A^2 = 2q^2v^2$, with $q = 2l$ in superconductivity, gives rise to the exponential drop

$$B(x) = B(0)e^{-x/l}$$

inside the system.

Now, let us do some considerations:

- As said, we cannot introduce by hand a massive photon i.e. a term like $\frac{1}{2}M_A^2 A_\mu A^\mu$ in the Lagrangian because we would violate explicitely the gauge symmetry!
- The Lagrangian is gauge invariant.
- Symmetry breaking occurs at the level of the vacuum state.
- A gauge symmetry that is explicitely broken is not renormalizable.

1.3.3 Non-abelian gauge theories

Let us illustrate some examples of non abelian gauge theories.

Example 4

An example of non abelian gauge theory is the gauge theory that contains the combined electromagnetic and weak interactions, which is generally referred to as the electroweak unification, or electroweak interactions theory (Glashow-Weinberg-Solam, or GWS). It is non abelian because the Lagrangian is invariant under the group $\underbrace{SU(2)}_{\text{weak interactions}} \times \underbrace{U(1)}_{\text{electromagnetism}}$, that is not an abelian group because of $SU(2)$.

Example 5: Quantum chromodynamic (quarks+gluons)

In the case of quantum chromodynamic, one has a term that is $SU(3)$ invariant and the GWS Lagrangian that has symmetry $SU(2) \times U(1)$. It implies that the Lagrangian of this theory is invariant under

$$SU(3) \times SU(2) \times U(1) \tag{1.16}$$

Because of the groups $SU(2)$ and $SU(3)$ the symmetries above ar not abelian (for example in $SU(2)$ two matrices $U(\alpha)$ and $U(\beta)$ do not commute in general).

1.3.4 Extension of Higgs mechanism to non-abelian theories

The abelian example of Higgs mechanism can now be generalized in a straightforward way to a non-abelian gauge theory. In particular, we discuss an electroweak interactions gauge theory. To spontaneously break the symmetry, consider a complex scalar field of $SU(2)$:

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_1 + i\Phi_2 \\ \Phi_3 + i\Phi_4\end{pmatrix} = \begin{pmatrix}\Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}})\end{pmatrix} \tag{1.17}$$

where Φ_a, Φ_b are complex fields. The Lagrangian of the system is invariant under the gauge transformation $SU(2) \times U(1)$:

$$\begin{pmatrix}\Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}})\end{pmatrix} \to e^{\frac{i}{2}\alpha_0(\vec{\mathbf{x}})} e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(\vec{\mathbf{x}})} \begin{pmatrix}\Phi_a(\vec{\mathbf{x}}) \\ \Phi_b(\vec{\mathbf{x}})\end{pmatrix}$$

where $\vec{\tau}$ are Pauli matrices, $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are four real functions (4 vectorial mesons). In particular,

$$\begin{aligned}\alpha_0(\vec{\mathbf{x}}) &\to B_\mu(\vec{\mathbf{x}}) \\ \vec{\alpha}(\vec{\mathbf{x}}) &\to W_\mu^a(\vec{\mathbf{x}}) = \left(W_\mu^{(1)}(\vec{\mathbf{x}}), W_\mu^{(2)}(\vec{\mathbf{x}}), W_\mu^{(3)}(\vec{\mathbf{x}})\right)\end{aligned}$$

where α_0 is the scalar gauge field and B_μ is a linear combination of A_μ and $W_\mu^{(3)}$. The Lagrangian is given by

$$\mathcal{L} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - \mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 - \frac{1}{4}b^{\mu\nu}b_{\mu\nu} - \frac{1}{4}f_a^{\mu\nu}f_{\mu\nu}^a \tag{1.19}$$

where

$$\begin{aligned}D_\mu &\to \partial_\mu \frac{1}{2} ig\tau^a W_\mu^a + \frac{i}{2} g' B_\mu \\ f_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g\varepsilon^{abc} W_\mu^b W_\nu^a \\ b_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ W_\mu^a &\to W_\mu^a - \varepsilon^{abc}\alpha_b(\vec{\mathbf{x}}) W_\mu^c(\vec{\mathbf{x}}) + \frac{1}{g}\partial_\mu \alpha^a(\vec{\mathbf{x}}) \\ B_\mu &\to B_\mu + \frac{1}{g'}\frac{\partial \alpha_0}{\partial x_\mu}\end{aligned}$$

Just as in the abelian example, the scalar field develops a nonzero vacuum expecation value for $\mu^2 < 0$, which spontaneously breaks the symmetry. There is an infinite number of degenerate states with minimum energy satisfying $\nu \equiv \Phi_1^2 + \Phi_2^2 + \Phi_3^2 + \Phi_4^2 = v^2$. After we have chosen the direction on the sphere in $\mathbb{R}^4$, we note that 3 symmetries are broken; hence, we have 3 Goldstone bosons.

Higgs mechanism

The W and Z gauge masses can now be generated in the same manner as the Higgs mechanism generated the photon mass in the abelian example. Let us consider a Higgs scalar field

$$\delta\Phi = \begin{pmatrix}\Phi^+ \\ \Phi_0\end{pmatrix}$$

such that

$$\langle 0 | \Phi | 0 \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\Rightarrow \mathcal{L}_{Higgs} = \frac{1}{2}(g\nu)^2 W_\mu^+ W^{-\mu} + \frac{1}{2} v^2 \Big(g W_\mu^{(3)} - g' B_\mu \Big)^2 \tag{1.21}$$

where

$$W_\mu^{(1)} = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-) \tag{1.22a}$$

$$W_\mu^{(2)} = \frac{1}{\sqrt{2}}(W_\mu^+ - W_\mu^-) \tag{1.22b}$$

Mass of the W^+ particle and its antiparticle

$$M_W^2 = \frac{1}{2}(gv)^2 \tag{1.23}$$

The 2^{nd} term is a linear combination of W_μ^3 and B_μ which corresponds to Z^0, the field for a third weak gauge boson.

To make Z_μ^0 and A_μ orthogonal we should consider

$$A_\mu = (\cos\theta_W) B_\mu + (\sin\theta_W) W_\mu^3 \tag{1.24a}$$

$$Z_\mu^0 = (-\sin\theta_W) B_\mu + (\cos\theta_W) W_\mu^3 \tag{1.24b}$$

where θ_W is the Weiberg angle:

$$\tan\theta_W = \frac{g'}{g} \tag{1.25}$$

$$M_{Z^0}^2 = \frac{1}{2}\left(\frac{vg}{\cos\theta_W}\right)^2 = \frac{Mw^2}{\cos^2\theta_W} \tag{1.26}$$