Lecture 18. Friday 13th December, 2019. Compiled: Sunday 22nd December, 2019.

0.0.1 Solution by Fourier transform

Let us try to do the Fourier transform and define

$$\vec{\mathbf{x}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}' \tag{1}$$

Taking the Fourier transform of

$$\big(-\boldsymbol{\nabla}^2 + \xi^{-2}(t)\big)G_c(\vec{\mathbf{r}} - \vec{\mathbf{r}}') = \frac{k_B T}{k}\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}') \tag{2}$$

and calling $\widetilde{G}(q)$ the Fourier transform of the function G, one gets

$$\widetilde{G}(q) = \int_{-\infty}^{+\infty} \mathrm{d}|\vec{\mathbf{x}}|\, G_c(|\vec{\mathbf{x}}|)e^{-iq|\vec{\mathbf{x}}|} \quad \Rightarrow \widetilde{G}(q) = \frac{k_B T}{k}\frac{1}{q^2 + \xi^{-2}} \tag{3}$$

Remark. At $T = T_c$, since $\xi \to \infty$ we have and $\widetilde{G}(q) \simeq \frac{1}{q^2}$ and performing the inverse Fourier transform one gets

$$G_c(|\vec{\mathbf{x}}|) = |\vec{\mathbf{x}}|^{2-D} \tag{4}$$

On the other hand, at $T = T_c$ we defined

$$G(r) \sim |\vec{\mathbf{x}}|^{2-D-\eta} \tag{5}$$

hence, we have $\eta = 0$.

Let us now obtain the full expression of $G(r)$ by computing the inverse Fourier transform

$$G(\vec{\mathbf{x}}) = \int \mathrm{d}^D\vec{\mathbf{q}}\, \frac{1}{(2\pi)^D}\frac{1}{q^2 + \xi^{-2}}e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}} \tag{6}$$

Let us do it for $D = 3$:

$$\begin{aligned}
G(\vec{\mathbf{x}}) &= \int \mathrm{d}^3\vec{\mathbf{q}}\, \frac{1}{(2\pi)^3}\frac{1}{q^2 + \xi^{-2}}e^{i\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}} \overset{\substack{\text{spherical}\\ \text{coordinates}}}{=} \\
&= \frac{4\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}q\, \frac{q^2}{q^2 + \xi^{-2}}\int_{-1}^{+1} \mathrm{d}(\cos\theta)\, e^{iq|\vec{\mathbf{x}}|\cos\theta} \\
&\overset{z\equiv\cos(\theta)}{=} \frac{2}{(2\pi)^2}\int_0^\infty \mathrm{d}q\, \frac{q^2}{q^2 + \xi^{-2}}\left[\frac{1}{iq|\vec{\mathbf{x}}|}e^{iq|\vec{\mathbf{x}}|z}\right]_{-1}^{1} \\
&= \frac{4}{(2\pi)^2|\vec{\mathbf{x}}|}\int_0^\infty \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q
\end{aligned} \tag{7}$$

Let us now consider the integral

$$I = \int_0^\infty \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{q\sin(q|\vec{\mathbf{x}}|)}{q^2 + \xi^{-2}}\,\mathrm{d}q \tag{8}$$

that can be computed in the complex plane, i.e. as

$$I = \frac{1}{2}\,\mathrm{Im}\oint \frac{ze^{iz|\vec{\mathbf{x}}|}}{(z^2 + \xi^{-2})}\,\mathrm{d}z \tag{9}$$

The integration contour is a semicirle in the positive half plane. There are two poles $z_P = \pm i\xi^{-1}$, but only one is enclosed by the semicirle (Figure 1).

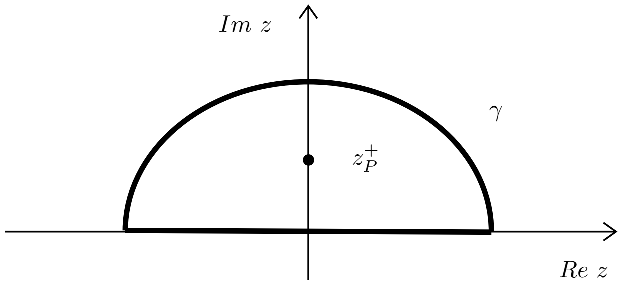

Figure 1: Description.

Using the Theorem of residues one has

$$\begin{aligned} I &= \frac{1}{2}\,\mathrm{Im}\oint_\gamma \frac{z e^{iz|\vec{\mathbf{x}}|}}{(z^2+\xi^{-2})}\,\mathrm{d}z = \frac{1}{2}\oint_\gamma \frac{z e^{iz|\vec{\mathbf{x}}|}}{(z+i\xi^{-1})(z-i\xi^{-1})}\,\mathrm{d}z \\ &\overset{th.\,of\,residues}{=} \frac{1}{2}\,\mathrm{Im}\left[2\pi i \mathrm{Res}(i\xi^{-1})\right] \end{aligned} \tag{10}$$

and since

$$\mathrm{Res}(i\xi^{-1}) = \frac{i\xi^{-1}e^{-\xi^{-1}|\vec{\mathbf{x}}|}}{2i\xi^{-1}} = \frac{e^{-|\vec{\mathbf{x}}|/\xi}}{2} \tag{11}$$

$$\Rightarrow I = \frac{1}{2}\,\mathrm{Im}\left[2\pi i \mathrm{Res}(i\xi^{-1})\right] = \frac{\pi}{2}e^{-|\vec{\mathbf{x}}|/\xi} \tag{12}$$

Hence, at the end

$$\Rightarrow G(|\vec{\mathbf{x}}|) = \frac{1}{2\pi}\frac{e^{-\frac{|\vec{\mathbf{x}}|}{\xi}}}{|\vec{\mathbf{x}}|} \tag{13}$$

where it is confirmed the exponential behaviour for $t \neq 0$ and $\eta = 0$.

One can also solve the equation for $G(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$ by using the spherical coordinates and use the Bessel functions.

0.1 Including fluctuations at the Gaussian level (non interacting fields)

Can we reach the simple level of fluctuation? The simple level is the one that follow gaussian distribution. Let us introduce fluctuations at the Gaussian level.

Let us consider $h = 0$ and let $m_0(\vec{\mathbf{r}}) = m_0$ be the solution of the saddle point approximation. Let us expand the general expression

$$\beta\mathcal{H}_{eff}[m] = \int \mathrm{d}^D\vec{\mathbf{r}}\left(atm^2 + \frac{b}{2}m^4 + \frac{k}{2}(\boldsymbol{\nabla} m)^2\right) \tag{14}$$

by using

$$m(\vec{\mathbf{r}}) = m_0 + \delta m(\vec{\mathbf{r}}) \tag{15}$$

We are assuming that the fluctuations $\delta m(\vec{\mathbf{r}})$ are small:

$$(\boldsymbol{\nabla} m)^2 = (\boldsymbol{\nabla}(m_0+\delta m))^2 = (\boldsymbol{\nabla}(\delta m))^2 \tag{16a}$$

$$m^2 = m_0^2 + 2m_0\delta m + (\delta m)^2 \tag{16b}$$

$$m^4 = m_0^4 + 4m_0^3\delta m + 6m_0^2(\delta m)^2 + 4m_0\delta m^3 + (\delta m)^4 \tag{16c}$$

Hence,

$$\beta\mathcal{H}_{eff} = V\underbrace{\left(atm_0^2 + \frac{b}{2}m_0^4\right)}_{A_0} + \int \mathrm{d}^D\vec{\mathbf{r}}\left(\frac{k}{2}(\boldsymbol{\nabla} m)^2 + \left(at + 3bm_0^2\right)\delta m^2 + 2bm_0\delta m^3 + \frac{b}{2}\delta m^4\right) \tag{17}$$

Remark. The term proportional to δm, $\left(2atm_0 + \frac{b}{2}4m_0^3\right)$ is zero since m_0 is the solution of the extremal condition

$$\left.\frac{\delta \mathcal{H}_{eff}}{\delta m}\right|_{m=m_0} = 0 \tag{18}$$

For simplicity let us first consider $T > T_c$, we know that $m_0 = 0$ and $m(\vec{\mathbf{r}}) = m_0 + \delta m(\vec{\mathbf{r}}) = \delta m(\vec{\mathbf{r}})$. We have $A_0 = 0$, $3bm_0^2\delta m^2 = 0$ and $2bm_0\delta m^3 = 0$, that implies

$$\beta\mathcal{H}_{eff}^{T>T_c}(\delta m) = \int \mathrm{d}^D\vec{\mathbf{r}} \left(\frac{k}{2}(\boldsymbol{\nabla}\delta m)^2 + at(\delta m)^2 + \cancel{\frac{b}{2}(\delta m)^4}\right) \tag{19}$$

Remark. It is important to understand that these are fluctuations with respect to the solution.

The Gaussian approximation consists in neglecting the quartic term $(\delta m)^4$

$$\Rightarrow \beta\mathcal{H}_{eff}^{T>T_c}(\delta m) \simeq \int \mathrm{d}^D\vec{\mathbf{r}} \left(\frac{k}{2}(\boldsymbol{\nabla}\delta m)^2 + at(\delta m)^2\right) \tag{20}$$

Remark. We cannot do again the saddle point, otherwise we do not get too much information. We consider gaussian fluctuations: fluctuations that follow gaussian distribution. Therefore, the term in $(\delta m)^4$ is cancelled. What it is the difference in the exponent respect the mean field? Then we will to the same for $T < T_c$, the story is the same. (lesson)

0.2 Ginzburg-Landau in the Gaussian approximation

Let us solve

$$Z_{GL}^G = \int \mathrm{D}[\delta m] e^{-\int \mathrm{d}^D r \left(\frac{k}{2}(\boldsymbol{\nabla}\delta m)^2 + at(\delta m)^2\right)} \tag{21}$$

in the Fourier space. Consider a system in a box of volume $V = L^D$ (periodic boundary conditions):

$$\delta m(\vec{\mathbf{r}}) = \frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}(\delta m_{\vec{\mathbf{k}}}) \tag{22}$$

where

$$\delta m_{\vec{\mathbf{k}}} = \int_V \delta m(\vec{\mathbf{r}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\,\mathrm{d}^D\vec{\mathbf{r}} \tag{23}$$

where $\vec{\mathbf{k}} = k_1, \dots, k_D = \frac{2\pi\vec{n}}{L}$ with $k_\alpha = \frac{2\pi}{L}n_\alpha$ ($n_\alpha = 0, \pm 1, \dots$).

Remark. We have $\delta m_{\vec{\mathbf{k}}} \in \mathbb{C}$ but $\delta m(\vec{\mathbf{r}}) \in \mathbb{R}$, hence

$$\delta m_{\vec{\mathbf{k}}} = -\delta m^*_{-\vec{\mathbf{k}}} \tag{24}$$

0.2.1 Useful relations

Sometimes it is useful to convert the sum over $\vec{\mathbf{k}}$ by an integral by using the density of states in the $\vec{\mathbf{k}}$ space that is $\frac{V}{(2\pi)^D}$,

$$\sum_{\vec{\mathbf{k}}} \to \frac{V}{(2\pi)^D}\int_{\mathbb{R}^D} \mathrm{d}\vec{\mathbf{k}} \tag{25}$$

Hence,

$$\frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \to \frac{1}{V}\frac{V}{(2\pi)^2}\int_{\mathbb{R}^D} \mathrm{d}^D\vec{\mathbf{k}}\, e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \\ = \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')}\,\mathrm{d}^D\vec{\mathbf{k}} = \delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}') \tag{26}$$

so

$$\boxed{\frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \to \delta(\vec{\mathbf{r}}-\vec{\mathbf{r}}')} \tag{27}$$

Inserting

$$m(\vec{\mathbf{r}}) = \frac{1}{V}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}'\vec{\mathbf{r}}} m_{\vec{\mathbf{k}}'} \tag{28}$$

In the expression

$$m_{\vec{\mathbf{k}}} = \int_V \mathrm{d}^D\vec{\mathbf{r}}\, e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} m(\vec{\mathbf{r}}) \tag{29}$$

one gets

$$\boxed{\frac{1}{V}\int \mathrm{d}^D\vec{\mathbf{r}}\, e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}} = \delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}} \tag{30}$$

Finally, since

$$\int \mathrm{d}^D\vec{\mathbf{r}}\, e^{i(\vec{\mathbf{k}}-\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}} \overset{V\to\infty}{\longrightarrow} (2\pi)^D\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}') \tag{31}$$

From (30) one gets,

$$\boxed{V\delta_{\vec{\mathbf{k}}\vec{\mathbf{k}}'} \overset{V\to\infty}{\longrightarrow} (2\pi)^D\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}')} \tag{32}$$

Remark. Since the minimal spatial length of the system is a,

$$a \Rightarrow \left|\vec{\mathbf{k}}\right| \le \frac{\pi}{a} = \Lambda \tag{33}$$

that is the ultraviolet cut-off.

0.3 Gaussian Hamiltonian in Fourier space

0.4 lesson

Change now the notation:

$$\delta m(\vec{\mathbf{r}}) \leftrightarrow \varphi(\vec{\mathbf{r}}), \quad k \leftrightarrow c \tag{34}$$

and obtain

$$\beta\mathcal{H}^{G,>}_{eff} = \int \mathrm{d}^D\vec{\mathbf{r}}\left[\frac{c}{2}(\boldsymbol{\nabla}\phi)^2 + at\phi^2\right] \tag{35}$$

$$\int \mathrm{d}^D r\,\frac{c}{2}(\boldsymbol{\nabla}\varphi)^2 = \frac{c}{2}\frac{1}{V^2}\int \mathrm{d}^D\vec{\mathbf{r}}\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}}\right)\left(\boldsymbol{\nabla}\sum_{\vec{\mathbf{k}}'} e^{i\vec{\mathbf{k}}'\cdot\vec{\mathbf{r}}}\varphi_{\vec{\mathbf{k}}'}\right) \\ = \frac{c}{2}\sum_{\vec{\mathbf{k}}\vec{\mathbf{k}}'}\left(-\vec{\mathbf{k}}\vec{\mathbf{k}}'\right)\varphi_{\vec{\mathbf{k}}}\varphi_{\vec{\mathbf{k}}'}\underbrace{\int \mathrm{d}^D\vec{\mathbf{r}}\, e^{i(\vec{\mathbf{k}}+\vec{\mathbf{k}}')\cdot\vec{\mathbf{r}}}}_{(2\pi)^2\delta(\vec{\mathbf{k}}+\vec{\mathbf{k}}')} \\ = \frac{c}{2V}\sum_{\vec{\mathbf{k}}}\left|\vec{\mathbf{k}}\right|^2\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'} \tag{36}$$

$$\beta\mathcal{H}^{G,>}_{eff} \to \frac{1}{2V}\sum_{\vec{\mathbf{k}}}\left(2at + ck^2\right)\varphi_{\vec{\mathbf{k}}}\varphi_{-\vec{\mathbf{k}}'} \tag{37}$$

$$\int \mathrm{D}[\varphi(\vec{\mathbf{r}})] \to \int_{-\infty}^{+\infty}\prod_{|\vec{\mathbf{k}}|<\Lambda}\mathrm{d}(\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\})\,\mathrm{d}(\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}) \tag{38}$$

with $\varphi_{\vec{\mathbf{k}}} \in \mathbb{C}$.

$$\varphi^*_{\vec{\mathbf{k}}} = \varphi_{-\vec{\mathbf{k}}} \tag{39}$$

$$\mathrm{Re}\Big\{\varphi[\vec{\mathbf{k}}]\Big\} = \mathrm{Re}\Big\{\varphi_{-\vec{\mathbf{k}}}\Big\}, \quad \mathrm{Im}\Big\{\varphi[\vec{\mathbf{k}}]\Big\} = -\,\mathrm{Im}\Big\{\varphi_{-\vec{\mathbf{k}}}\Big\} \tag{40}$$

$$\mathrm{Tr} = \int_{-\infty}^{+\infty}\prod_{\substack{|\vec{\mathbf{k}}|<\Lambda \\ k_D>0}}\mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\} \tag{41}$$

$$\widetilde{Z}^{G,>}_{GC} = \frac{1}{2}\int_{-\infty}^{+\infty}\prod_{\substack{\vec{\mathbf{k}} \\ |\vec{\mathbf{k}}|<\Lambda}}\mathrm{d}\,\mathrm{Re}\{\varphi_{\vec{\mathbf{k}}}\}\,\mathrm{d}\,\mathrm{Im}\{\varphi_{\vec{\mathbf{k}}}\}\,e^{-\beta\widetilde{\mathcal{H}}_{eff}[\varphi_{\vec{\mathbf{k}}}]} \tag{42}$$

$$x = \mathrm{Re}\,\varphi_{\vec{\mathbf{k}}}, \quad y = \mathrm{Im}\,\varphi_{\vec{\mathbf{k}}} \tag{43}$$

$$\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{d}y\,e^{-A(x^2+y^2)} = \frac{\pi}{A} \tag{44}$$

where $A = \frac{1}{2V}$.

$$e^{-\beta\widetilde{F}^{>}_{GL}} = \left(\prod_{\substack{\vec{\mathbf{k}} \\ |\vec{\mathbf{k}}|<\Lambda \\ k_D>0}}\frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{45}$$

$$\widetilde{F}^{G,>}_{GL} = -\frac{1}{2}k_B T\sum_{|\vec{\mathbf{k}}|>\Lambda}\log\left(\frac{2\pi V}{2at + c\left|\vec{\mathbf{k}}\right|^2}\right) \tag{46}$$

$$c_V = -T\frac{\partial^2 F}{\partial T^2} = \frac{A}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda}\frac{1}{2at + c\left|\vec{\mathbf{k}}\right|^2} - \frac{B}{V}\sum_{|\vec{\mathbf{k}}|<\Lambda}\frac{1}{\left(2at + c\left|\vec{\mathbf{k}}\right|^2\right)^2} \tag{47}$$

Question: what happens if I introduce guassian fluctuations.

It turns out that when we study the asimptotic behaviour of these integrals.

For the first term it turns out that

$$1^{st} \propto \begin{cases} \xi^{4-D} \sim t^{-\nu(4-D)} & D < 4 \\ < \infty & D > 4 \end{cases} \tag{48}$$

$$2^{nd} \propto \begin{cases} \xi^{2-D} \sim t^{-\nu(2-D)} & D < 2 \\ < \infty & D > 2 \end{cases} \tag{49}$$

At the end, the behaviour of the specific heat

$$c_V \sim \begin{cases} t^{-\nu(4-D)} & D < 4 \\ \infty & D > 4 \end{cases} \tag{50}$$

figure 1, figure 2.

$$\left\langle \varphi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'} \right\rangle_G = \frac{\int \prod \mathrm{d}\varphi_{\vec{\mathbf{k}}} \ldots e^{-\beta \mathcal{H}_{eff}} \phi_{\vec{\mathbf{k}}} \varphi_{\vec{\mathbf{k}}'}}{Z_{GL}^{G,>}} = \delta_{\vec{\mathbf{k}},\vec{\mathbf{k}}'} \frac{V}{2at + ck^2} \tag{51}$$

$$\Rightarrow \begin{cases} \nu_G = \frac{1}{2} \\ \eta_G = 0 \end{cases} \tag{52}$$