

Since  $\sum_k \lambda_k^N = Z_N$  for  $k = +, -, 1, \dots, n$ :

$$\langle S_1 S_R \rangle_N = \frac{\sum_{ij} \langle t_j | \mathbb{S}_1 | t_i \rangle \lambda_i^{R-1} \langle t_i | \mathbb{S}_R | t_j \rangle \lambda_j^{N-R+1}}{\sum_{k=1}^n \lambda_k^N} \quad (1)$$

If we now multiply and divide by  $\lambda_+^N$ , we get

$$\langle S_1 S_R \rangle_N = \frac{\sum_{ij} \langle t_j | \mathbb{S}_1 | t_i \rangle (\lambda_i / \lambda_+)^{R-1} \langle t_i | \mathbb{S}_R | t_j \rangle (\lambda_j / \lambda_+)^{N-R+1}}{\sum_{k=1}^n (\lambda_k / \lambda_+)^N} \quad (2)$$

*Remark.* In the thermodynamic limit  $N \rightarrow \infty$ , only the terms with  $j = +$  and  $k = +$  survive in the sum. Remind that  $R$  is fixed.

$$\langle S_1 S_R \rangle_N = \lim_{N \rightarrow \infty} \langle S_1 S_R \rangle_N = \sum_{i=\pm, 1, \dots, n} \left( \frac{\lambda_i}{\lambda_+} \right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \quad (3)$$

Remember that  $\lambda_+ > \lambda_T \geq \lambda_1 \dots \lambda_n$ :

$$\langle S_1 S_R \rangle_N = \langle t_+ | \mathbb{S}_1 | t_+ \rangle \langle t_+ | \mathbb{S}_R | t_+ \rangle + \sum_{i \neq +}^n \left( \frac{\lambda_i}{\lambda_+} \right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \quad (4)$$

Since one can prove, by a method entirely analogous to that followed above, that

$$\lim_{N \rightarrow \infty} \langle S_R \rangle_N = \langle t_+ | \mathbb{S}_R | t_+ \rangle \quad (5)$$

we obtain

$$\langle S_1 S_R \rangle = \langle S_1 \rangle \langle S_R \rangle + \sum_{i \neq +} \left( \frac{\lambda_i}{\lambda_+} \right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \quad (6)$$

The correlation function then follows immediately as

$$\Gamma_R = \langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle = \sum_{i \neq +}^n \left( \frac{\lambda_i}{\lambda_+} \right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_i \rangle \langle t_i | \mathbb{S}_R | t_+ \rangle \quad (7)$$

*Remark.*  $\Gamma_R$  depends only on the eigenvalues and eigenvectors of the transfer matrix  $\mathbb{T}$  and by the values of the spins  $S_1$  and  $S_R$ .

A much simpler formula is obtained for the correlation length (??). Taking the limit  $R \rightarrow \infty$  the ratio  $(\lambda_- / \lambda_+)$  dominates the sum and hence

$$\begin{aligned} \xi^{-1} &= \lim_{R \rightarrow \infty} \left\{ -\frac{1}{R-1} \log |\langle S_1 S_R \rangle - \langle S_1 \rangle \langle S_R \rangle| \right\} \\ &= \lim_{R \rightarrow \infty} \left\{ -\frac{1}{R-1} \log \left[ \left( \frac{\lambda_-}{\lambda_+} \right)^{R-1} \langle t_+ | \mathbb{S}_1 | t_- \rangle \langle t_- | \mathbb{S}_R | t_+ \rangle \right] \right\} \\ &= -\log \left[ \left( \frac{\lambda_-}{\lambda_+} \right) \right] - \lim_{R \rightarrow \infty} \frac{1}{R-1} \log \langle t_+ | \mathbb{S}_1 | t_- \rangle \langle t_+ | \mathbb{S}_R | t_+ \rangle \\ &= -\log \left( \frac{\lambda_-}{\lambda_+} \right) \end{aligned} \quad (8)$$

The important result is

$$\xi^{-1} = -\log \left( \frac{\lambda_-}{\lambda_+} \right) \quad (9)$$

It means that the correlation length does depend only on the ratio between the two largest eigenvalues of the transfer matrix  $\mathbb{T}$ .

**Lecture 8.**  
Wednesday 6<sup>th</sup>  
November, 2019.  
Compiled: Tuesday  
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2019.

### 0.0.1 Results for the 1-dimensional Ising model

The transfer matrix is given by

$$\mathbb{T} = \begin{pmatrix} \exp(K+h) & \exp(-K) \\ \exp(-K) & \exp(K-h) \end{pmatrix} \quad (10)$$

Calculate the eigenvalues:

$$|\mathbb{T} - \lambda \mathbb{1}| = (e^{K+h} - \lambda)(e^{K-h} - \lambda) - e^{-2K} = 0 \quad (11)$$

The solutions are

$$\lambda_{\pm} = e^K \cosh(h) \pm \sqrt{e^{2K} \sinh^2(h) + e^{-2K}} \quad (12)$$

#### The free energy

The free energy is

$$\begin{aligned} f_b &\equiv \lim_{N \rightarrow \infty} \frac{-k_B T}{N} \log Z_N(K, h) \\ &= -k_B T \lim_{N \rightarrow \infty} \frac{1}{N} \log \left[ \lambda_+^N \left( 1 + \left( \frac{\lambda_-}{\lambda_+} \right)^N \right) \right] \\ &= -k_B T \log \lambda_+ \end{aligned} \quad (13)$$

and inserting the explicit expression of  $\lambda_+$  for the Ising model, we get

$$\begin{aligned} f_b &= -k_B T \log \left( e^K \cosh h + \sqrt{e^{2K} \sinh^2(h) + e^{-2K}} \right) \\ &= -K k_B T - k_B T \log \left( \cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}} \right) \end{aligned} \quad (14)$$

*Remark.* Remember that  $K \equiv \beta J, h \equiv \beta H$ .

**Exercise 1.** Check that if  $h = 0$  we get back the expression found previously with the iterative method (what is the important of boundary conditions?).

Let us now consider the limits  $T \rightarrow 0$  and  $T \rightarrow \infty$  by keeping  $H$  fixed and  $J$  fixed.

- Case:  $T \rightarrow 0 \Rightarrow K \rightarrow \infty, h \rightarrow \infty$ .

$$e^{-4K} \xrightarrow{K \rightarrow \infty} 0 \quad (15a)$$

$$\sqrt{\sinh^2 h} \xrightarrow{h \rightarrow \infty} \sinh(h) \quad (15b)$$

This implies that

$$\cosh(h) + \sinh h \sim \frac{2e^h}{2} \simeq e^h \quad (16)$$

and

$$f \xrightarrow{K \rightarrow \infty} -K k_B T - k_B T \log e^h \sim -J - H \quad \text{const} \quad (17)$$

Therefore, as  $T \rightarrow 0^+$ ,  $f$  goes to a constant that depends on  $J$  and  $H$ .

- Case:  $T \rightarrow \infty \Rightarrow K \rightarrow 0, h \rightarrow 0$ . In this case we suppose also that  $H$  and  $J$  that are fixed, are also finite.

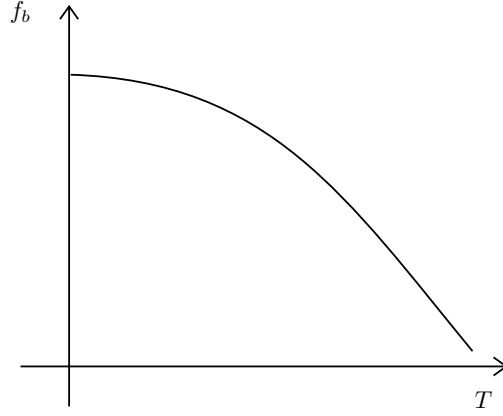
$$e^{-4K} \simeq 1 \quad (18a)$$

$$\sqrt{\sinh^2 h + e^{-4K}} \sim \sqrt{1} \quad (18b)$$

Since  $\cosh h \xrightarrow{h \rightarrow 0} 1$ :

$$f_B \sim -K k_B T - k_B T \log(1+1) \sim -J - k_B T \ln 2 \quad (19)$$

Therefore, as  $T \rightarrow \infty$ , the free energy goes linearly to zero, as in figure 1.



**Figure 1:** Plot of the free energy  $f_b$  in function of the temperature  $T$ . For  $T \rightarrow 0$ , the free energy becomes constant, while for  $T \rightarrow \infty$  it goes linearly to zero.

### The magnetization

This can be obtained by differentiating the negative of the free energy with respect to the magnetic field  $H$ :

$$m = -\frac{\partial f_b}{\partial H} = -\frac{1}{k_B T} \frac{\partial f_b}{\partial h} = \frac{\partial}{\partial h} \left[ \log \left( \cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}} \right) \right] \quad (20)$$

The result is

$$m = \frac{\sinh h + \frac{\sinh h \cosh h}{\sqrt{\sinh^2 h + e^{-4K}}}}{\cosh h + \sqrt{\sinh^2 h + e^{-4K}}} \quad (21)$$

- Case:  $T > 0$  fixed,  $H \rightarrow 0 \Leftrightarrow h \rightarrow 0$ .

$$\sinh h \sim h \sim 0, \quad (22a)$$

$$\cosh h \sim 1 \quad (22b)$$

In zero field  $h \rightarrow 0$ , we have  $m \rightarrow 0$  for all  $T > 0$ . It means that there is no spontaneous magnetization!

### The magnetic susceptibility

$$\chi_T \equiv \frac{\partial m}{\partial H} = \frac{1}{k_B T} \frac{\partial m}{\partial h} \quad (23)$$

If we consider the case  $h \ll 1$ , it is convenient first expand the (21) for  $h \rightarrow 0$  and take the derivative to get  $\chi_T$ .

Since  $\sinh(h) \sim h + h^3$  and  $\cosh(h) \sim 1 + h^2$ , we have

$$m \underset{h \ll 1}{\sim} \frac{h(1 + e^{2K})}{1 + e^{-2K}} \quad (24)$$

If we now derive with respect to  $h$

$$\chi_T = \frac{1}{k_B T} \frac{\partial m}{\partial h} \underset{h \ll 1}{\approx} \frac{1}{k_B T} \frac{(1 + e^{2K})}{(1 + e^{-2K})} \quad (25)$$

- Case:  $T \rightarrow \infty \Leftrightarrow K \rightarrow 0$ .

$$e^{2K} \simeq e^{-2K} \simeq 1 \quad (26)$$

The *Curie's Law* for paramagnetic systems is:

$$\chi_T \sim \frac{1}{k_B T} \quad (27)$$

- Case:  $T \rightarrow 0 \Leftrightarrow K \rightarrow \infty$ .

$$e^{-2K} \simeq 0 \quad (28)$$

The *Curie's Law* for paramagnetic systems is:

$$\chi_T \sim \frac{1}{k_B T} e^{2K} \sim \frac{1}{k_B T} e^{2J/k_B T} \quad (29)$$

### The correlation length

$$\xi^{-1} = -\log \left( \frac{\lambda_-}{\lambda_+} \right) = -\log \left[ \frac{\cosh h - \sqrt{\sinh^2 h + e^{-4K}}}{\cosh h + \sqrt{\sinh^2 h + e^{-4K}}} \right] \quad (30)$$

For  $h = 0$ , we have  $\cosh h \rightarrow 1, \sinh h \rightarrow 0$ :

$$\xi^{-1} = -\log \left[ \frac{1 - e^{-2K}}{1 + e^{-2K}} \right] = -\log \left[ \frac{1}{\coth K} \right] \quad (31)$$

Therefore:

$$\xi = \frac{1}{\log(\coth K)} \quad (32)$$

- Case:  $T \rightarrow 0 \Leftrightarrow K \rightarrow \infty$ .

$$\coth K = \frac{e^K + e^{-K}}{e^K - e^{-K}} \xrightarrow{K \rightarrow \infty} 1 + 2e^{-2K} + \dots \xrightarrow{K \rightarrow \infty} 1 \quad (33)$$

It implies

$$\xi \xrightarrow{K \gg 1} \frac{1}{\ln(1 + 2e^{-2K})} \sim \frac{e^{2K}}{2} \quad (34)$$

Hence

$$\xi \xrightarrow{T \rightarrow 0} \frac{1}{2} e^{J/k_B T} \quad (35)$$

It diverges exponentially  $\xi \rightarrow \infty$ , as  $T \rightarrow 0$ .

- Case:  $T \rightarrow \infty \Leftrightarrow K \rightarrow 0$ .

$$\coth K = \frac{e^K + e^{-K}}{e^K - e^{-K}} \xrightarrow{K \rightarrow 0} \frac{1 + K + \frac{K^2}{2} + 1 - K + \frac{K^2}{2}}{1 + K + \frac{K^2}{2} - 1 + K - \frac{K^2}{2}} \sim \frac{2 + 2\frac{K^2}{2}}{2K} \sim \frac{1 + K^2}{K} \quad (36)$$

$$\xi^{-1} = \log(\coth K) \xrightarrow{K \rightarrow 0} \ln \frac{1}{K} + \ln(1 + K^2) \sim +\infty \quad (37)$$

Therefore:

$$\xi \xrightarrow{K \rightarrow 0} 0 \quad (38)$$

More precisely,

$$\xi \xrightarrow{K \rightarrow 0} \frac{1}{\ln(1/K) + \ln(1 + K^2)} \xrightarrow{K \rightarrow 0} -\frac{1}{\ln K} \quad (39)$$

## 0.1 Classical Heisenberg model for $d=1$

Suppose to study something different from the Ising model, we do not anymore assume spin that can assume values as  $-1$  or  $+1$ , but spin that can assume a continuous value. This is the classical Heisenberg model.

Take a  $d = 1$  dimensional lattice. In the classical Heisenberg model the spins are unit length vectors  $\vec{\mathbf{S}}_i$ , i.e.  $\vec{\mathbf{S}}_i \in \mathbb{R}^3$ ,  $|\vec{\mathbf{S}}_i|^2 = 1$  (continuous values on the unit sphere):

$$\vec{\mathbf{S}}_i = (S_i^x, S_i^y, S_i^z) \quad (40)$$

with periodic boundary condition:  $\vec{\mathbf{S}}_{N+1} = \vec{\mathbf{S}}_1$ .

Assuming  $H = 0$ , the model is defined through the following Hamiltonian::

$$-\beta\mathcal{H}(\{\vec{\mathbf{S}}\}) = K \sum_{i=1}^N \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1} \quad (\longrightarrow \sum_i \vec{\mathbf{h}} \cdot \vec{\mathbf{S}}_i) \quad (41)$$

This model satisfies  $O(3)$  symmetry. In the transfer matrix formalism:

$$Z_N(K) = \sum_{\{\vec{\mathbf{S}}\}} e^{-\beta\mathcal{H}} = \sum_{\{\vec{\mathbf{S}}\}} e^{K \sum_{i=1}^N \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}} = \text{Tr}(\mathbb{T}^N) \quad (42)$$

where  $\langle \vec{\mathbf{S}}_i | \mathbb{T} | \vec{\mathbf{S}}_{i+1} \rangle = e^{K \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_{i+1}}$ .

Similarly to the Ising case:

$$\mathbb{T} = \sum_i |t_i\rangle \lambda_i \langle t_i| \quad (43)$$

and

$$\mathbb{T}_D = \mathbb{P}^{-1} \mathbb{T} \mathbb{P} \quad (44)$$

The problem is computing the eigenvalues  $\lambda_i$  of  $\mathbb{T}$ . Formally, we should find

$$\exp[K \vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2] = \langle \vec{\mathbf{S}}_1 | \mathbb{T} | \vec{\mathbf{S}}_2 \rangle = \sum_{i \in \text{eigenvalues}} \lambda_i \langle \vec{\mathbf{S}}_1 | t_i \rangle \langle t_i | \vec{\mathbf{S}}_2 \rangle = \sum_i \lambda_i f_i(\vec{\mathbf{S}}_1) f_i^*(\vec{\mathbf{S}}_2) \quad (45)$$

*Remark.* We start by noticing that the term  $e^{K \vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2}$  is similar to the plane wave  $e^{i \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}}$ , that in scattering problems is usually expanded in spherical coordinates. Plane wave can be expanded as a sum of spherical harmonics

$$e^{i \vec{\mathbf{q}} \cdot \vec{\mathbf{r}}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l (i)^l j_l(qr) Y_{lm}^*(\hat{\mathbf{q}}) Y_{lm}(\hat{\mathbf{r}}) \quad (46)$$

where

$$j_l(qr) = -\frac{(i)^l}{2} \int_0^\pi \sin(\theta) e^{iqr \cos(\theta)} P_l(\cos(\theta)) d\theta \quad (47)$$

are the *spherical Bessel functions*, while the  $P_l(\cos(\theta))$  are the *Legendre polynomial* of order  $l$ .

From a formal comparison we have

$$\vec{\mathbf{S}}_1 \leftrightarrow \hat{\mathbf{S}}_1, \quad \begin{cases} i \vec{\mathbf{q}} \cdot \vec{\mathbf{r}} = iqr \\ K \vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2 = K |\vec{\mathbf{S}}_1| |\vec{\mathbf{S}}_2| = K \end{cases} \quad (48)$$

multiplying by  $(-i)$  we can write

$$qr = -iK \left| \vec{\mathbf{S}}_1 \right| \left| \vec{\mathbf{S}}_2 \right| = -iK \quad (49)$$

In our case we have  $\hat{\mathbf{q}} = \vec{\mathbf{S}}_1, \hat{\mathbf{r}} = \vec{\mathbf{S}}_2$ . Hence,

$$e^{K\vec{\mathbf{S}}_1 \cdot \vec{\mathbf{S}}_2} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l (i)^l j_l(-iK) Y_{lm}^*(\vec{\mathbf{S}}_1) Y_{lm}(\vec{\mathbf{S}}_2) = \sum_i \lambda_i f_i(\vec{\mathbf{S}}_1) f_i^*(\vec{\mathbf{S}}_2) \quad (50)$$

where

$$\lambda_i = \lambda_{lm}(K) = 4\pi (i)^l j_l(-iK) \quad (51)$$

*Remark.* Note that  $\lambda_i$  does not depend on  $m$ !

If  $l = 0$ , the largest eigenvalue is:

$$\lambda_+ = \lambda_0(K) = 4\pi j_0(-iK) = 4\pi \frac{\sin K}{K} \quad (52)$$

and

$$\lambda_- = \lambda_1(K) = 4\pi i j_1(-iK) = 4\pi \left[ \frac{\cosh K}{K} - \frac{\sinh K}{K^2} \right] \quad (53)$$

**Exercise 2.** Given the largest eigenvalue  $\lambda_+$ :

$$\lambda_+ = 4\pi \frac{\sin(K)}{K} \quad (54)$$

find the bulk free energy density of the model and discuss its behaviour in the limits of low ( $T \rightarrow 0$ ) and high ( $T \rightarrow \infty$ ) temperatures.