Synchronization in networks of Rössler oscillators with long-range interactions

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***Abstract*—The interactions of units embedded in a complex network are modeled here taking into account not only the direct links among them, but also the indirect interaction between distant nodes. We propose a new framework combining the features of network topology with these *long-range interactions* by applying the concept of d-path Laplacian matrices, and study synchronization of a network of chaotic units. The main effect of considering this new type of interactions is an improvement in synchronization as it can be achieved even in networks originally not synchronizable.**

I. Introduction

Synchronization, the ability of multiple interacting dynamical systems to behave at unison, represents one of the most interesting collective behavior occurring in natural and man-made complex systems [1]. This phenomenon is made possible only thanks to the interactions among the agents, which thus constitute an essential feature of these systems. These interactions have been modeled by taking into account the different characteristics of the networks, such as webpages interacting through hyperlinks, or individuals in social groups interacting by means of their friendship or through other collaboration ties. In all of these cases it is important to model the entire system, the single units, and the way in which they evolve. A very simple and efficient way to represent is the use of the paradigm of complex networks [2], where these systems represent the agents and the links the connections among them. Several studies have elucidated the interplay between the unit dynamics and the network topology in the onset, stability and robustness of the synchronized state [2], [3], [4], [5], as well as the role of heterogeneity of links, delays in the signal interchange, and the time-dependent character of connections.

The main assumption underlying this network approach is that the nodes in the graph are directly connected to each other through the links. However, there are physical systems where the interactions are not direct but occur through a dynamic environment. Synchronization in these systems may be mediated either by passive [6] or active [7] media. Examples are the communication between cellular populations occurring thanks to small molecules diffused in the medium [8], chemical oscillators interacting through a stirred solution [9] or pedestrian crowd synchrony [6]. It turns out that agents, indirectly coupled by a medium that is distributed and dynamical, can also be modeled by networks if one resorts to use multilayer models [10].

A mechanism not incorporated in the previous models is the indirect peer pressure that agents receive from the others. In fact, an individual embedded in a network representing the direct interactions he has can also be influenced by others to which he is not directly connected, as it occurs for instance in social systems where an individual is not only influenced by peers directly connected to him, but also by those socially close to him but not necessarily connected to him. It is reasonable to assume that the indirect peer pressure is smaller than the direct one, but still significant to be considered. Recently, a new mathematical tool to account for both direct and indirect interactions has been introduced [11] and used to study consensus, leadership and innovations in social groups [12]. This tool is based on a generalization of the notion of the graph Laplacian to account for indirect interactions between agents separated by a distance measured by the length of the shortest path connecting them. The indirect influence that each agent receives from the others, or transmit to, decays as a function of this distance.

In this paper, we apply this tool to the study of synchronization in a network of Rössler oscillators in the presence of such long-range interactions (LRIs). After introducing, in Sec. II, the mathematical preliminaries on graphs and, in particular, on the generalized graph Laplacian, the mathematical model of the network of coupled Rössler oscillators is presented in Sec. III. The conditions for synchronization are derived and numerical simulations corroborating them are discussed in Sec. IV. Sec. V concludes the paper.

II. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, and adjacency matrix $A = [a_{ij}]_{N\times N}$, with $a_{ij} = a_{ji} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We assume that the graph under consideration is without self-loops, thus $a_{ii} = 0$. Let $\mathcal{L} = [l_{ij}]_{N\times N}$ be the Laplacian matrix of graph $\mathcal{G}$, defined as $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$. An important property of the Laplacian matrix $\mathcal{L}$ is that each row sum is zero, thus $\mathbf{1}_N = [1, 1, \ldots, 1]^T \in \mathcal{R}^N$ is an eigenvector of $\mathcal{L}$ associated with the null eigenvalue. The Laplacian can be also defined starting from the node-to-edges incidence matrix $\nabla \in \mathcal{R}^{N\times m}$, where m is the number of edges. This is a matrix whose entries are: $\nabla_{ij} = 1$, if node v_i is the head of the edge j; $\nabla_{ij} = -1$, if node v_i is the tail of edge j; and $\nabla_{ij} = 0$, otherwise. The Laplacian matrix is given by $\mathcal{L} = \nabla\nabla^T$.

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The set of neighbors of a node v_i is defined as $\mathcal{N}_i = \{v_j : (v_i, v_j) \in \mathcal{E}\}$, representing the set of all nodes with edges starting from v_i. A directed path between node v_i and node v_l is a sequence of edges (v_i, v_j), (v_j, v_k), ..., (v_k, v_l), starting from v_i and ending in v_l. If for any two nodes $v_i \in \mathcal{V}$ and $v_j \in \mathcal{V}$ there always exists a path between v_i and v_j, then, the graph $\mathcal{G}$ is said to be strongly connected. The shortest path is the path between v_i and v_j of minimum length.

Let $P_l\left(v_i, v_j\right)$ denote a shortest-path of length l between v_i and v_j. The irreducible set of shortest paths of length l in the graph is the set $P_l = \{P_l(v_i, v_j), P_l(v_i, v_r), ..., P_l(v_s, v_t)\}$ in which the endpoints of every shortest path are different. Every shortest-path in P_l is called an irreducible shortest-path.

We now introduce the concept of d-path Laplacian [11]. Let d_{max} denote the graph diameter, that is, the maximum shortest path distance in the graph. For any $d \leq d_{max}$, the entries of the d-path incidence matrix $\nabla_d \in \mathcal{R}^{N\times p}$ of a connected graph with N nodes and p irreducible shortest paths of length d, labeled as $p_1, p_2, \ldots, p_p$ are defined as: $\nabla_{d,ij} = 1$, if node v_i is the head of the irreducible shortest path p_j; $\nabla_{ij} = -1$, if node v_i is the tail of irreducible shortest path p_j; and $\nabla_{ij} = 0$, otherwise. For $d = 1$, one recovers the classical incidence matrix, i.e., $\nabla_1 = \nabla$.

The d-path Laplacian matrix, denoted by $\mathcal{L}_d \in \mathcal{R}^{N\times N}$, of a connected graph of N nodes is defined as:

$$\mathcal{L}_{d,ij} = \begin{cases} \delta_k\left(i\right) & \text{if } i = j \;, \\ -1 & \text{if } d_{ij} = d \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\delta_k\left(i\right)$ is the number of irreducible shortest paths of length d that start from node i. Similarly to the Laplacian matrix, it holds that $\mathcal{L}_d = \nabla_d \nabla_d^T$.

These tools are useful to generalize the interactions in a network. Let us now suppose that the nodes of a graph interact not only with their neighbors, but also with the other nodes at distance d (where the distance is measured as the length of the shortest path between them) with $d \leq d_{max}$. To account for interactions that become less important as the distance increases, let us assign different weights to them, according to two laws modeling the decay rate with d: the Mellin transform, where the decay rate is with d^{-s} $(s > 0)$, and the Laplace transform, where the decay rate is with $e^{-\lambda d}$ $(\lambda > 0)$. This yields to the following definition [11] of the Mellin and Laplace transformed Laplacian matrices of a graph $\mathcal{G}$:

$$\tilde{\mathcal{L}}_\tau = \begin{cases} \sum\limits_{d=1}^{d_{max}} d^{-s}\mathcal{L}_d, & \text{for } \tau = \text{Mell}, s > 0 \\ \mathcal{L} + \sum\limits_{d=2}^{d_{max}} e^{-\lambda d}\mathcal{L}_d, & \text{for } \tau = \text{Lapl}, \lambda > 0. \end{cases} \tag{2}$$

Here, s and λ are the positive constant parameters of the Mellin or Laplace transforms.

III. Network of Rössler oscillators with LRIs

Consider N dynamical units coupled with a coupling constant σ on a graph $\mathcal{G}$, described by:

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \sigma \sum_{(i,j)\in\mathcal{E}} \left(h(\mathbf{x}_j) - h(\mathbf{x}_i)\right), \tag{3}$$

where $\mathbf{x}_i \in \mathcal{R}^n$ is the state vector of oscillator i, $f(\mathbf{x}_i) : \mathcal{R}^n \to \mathcal{R}^n$ represents the dynamics of the oscillator when isolated from the rest of the network, and $h(\mathbf{x}_j) : \mathcal{R}^n \to \mathcal{R}^n$ is the coupling function.

Eqs. (3) can be rewritten in compact form as follows:

$$\dot{\mathbf{x}} = F(\mathbf{x}) - \sigma\mathcal{L} \otimes \mathrm{I}_n \cdot H(\mathbf{x}), \tag{4}$$

where $\mathbf{x} = \left[\mathbf{x}_1^T\ \mathbf{x}_2^T \ldots \mathbf{x}_N^T\right]^T$, $F(\mathbf{x}) = \left[f(\mathbf{x}_1)^T\ f(\mathbf{x}_2)^T \ldots f(\mathbf{x}_N)^T\right]^T$, I_n indicates the identity matrix of order n, and $H(\mathbf{x}) = \left[h(\mathbf{x}_1)^T\ h(\mathbf{x}_2)^T \ldots h(\mathbf{x}_N)^T\right]^T$.

Eqs. (4) represent a system of units coupled through the links of a network, according to the classical network approach. The use of the Mellin and Laplace transformed Laplacian makes straightforward the extension of this model to the inclusion of LRIs. Such system of N units coupled with LRIs may be described by the following equations:

$$\dot{\mathbf{x}}_{\text{Mell}} = F(\mathbf{x}) - \sigma\tilde{\mathcal{L}}_{\text{Mell}} \otimes \mathrm{I}_n \cdot H(\mathbf{x}), \tag{5}$$

$$\dot{\mathbf{x}}_{\text{Lapl}} = F(\mathbf{x}) - \sigma\tilde{\mathcal{L}}_{\text{Lapl}} \otimes \mathrm{I}_n \cdot H(\mathbf{x}), \tag{6}$$

In this paper as local units we have considered Rössler oscillators. For them, $n = 3$ and $\mathbf{x}_i = [x_{i,1}, x_{i,2}, x_{i,3}]^{\mathrm{T}}$. The uncoupled dynamics is given by:

$$f(\mathbf{x}_i) = \begin{bmatrix} -x_{i,2} - x_{i,3} \\ x_{i,1} + a_R x_{i,2} \\ b_R + x_{i,3}(x_{i,1} - c_R) \end{bmatrix} \tag{7}$$

where a_R, b_R and c_R are system parameters and the coupling function is

$$h(\mathbf{x}_i) = [x_{i,1}, 0, 0]^{\mathrm{T}} \tag{8}$$

In the following we study synchronization in this system, defined as $\lim\limits_{t\to+\infty} \|\mathbf{x}_i - \mathbf{x}_j\| = 0$ $\forall i$ and j. As measure for it, we consider the following synchronization error

$$e = \left\langle \frac{1}{N-1} \sum_{h=2}^{N} \sum_{j=1}^{n} \frac{|x_{h,j}(t) - x_{1,j}(t)|}{n} \right\rangle_T \tag{9}$$

where $\langle\rangle_T$ denotes average over a time window of length T.

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IV. CONDITIONS FOR SYNCHRONIZATION

The conditions for synchronization of the network of Rössler oscillators with long-range interactions are derived by applying the Master Stability Function (MSF) approach [3]. According to this approach the stability of the synchronized state is linked to the eigenvalues of the graph Laplacian.

In the absence of LRIs $\mathcal{L}$ has to be considered. Since it is symmetric and positive semi-definite, its eigenvalues can be ordered as follows: $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$. The Rössler oscillators with coupling defined as in Eq. (8) and parameters such that the uncoupled dynamics is chaotic ($a_R = b_R = 0.2$ and $c_R = 9$ [13]) belong to class III MSFs, and synchronization requires that

$$\sigma\lambda_2, \ldots, \sigma\lambda_N \in (\alpha_1, \alpha_2) \tag{10}$$

with $\alpha_1 = 0.186$ and $\alpha_2 = 4.614$ [14].

In the presence of LRIs, the transformed Laplacian matrix is taken into account and condition (10) is restated in terms of its eigenvalues. Since the matrix is positive semi-definite, its eigenvalues can be also ordered as: $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_N$. The condition for synchronization thus becomes:

$$\sigma\tilde{\lambda}_2, \ldots, \sigma\tilde{\lambda}_N \in (\alpha_1, \alpha_2) \tag{11}$$

where for brevity we have omitted the subscript indicating the Mellin or Laplace transform.

In [15] several interesting results on the eigenvalues of the transformed Laplacian matrix are presented. There, it is shown that $\tilde{\lambda}_2/N$ and $\tilde{\lambda}_N/N$ approaches 1 as $s \to 0$ (or $\lambda \to 0$), whereas they approach λ_2/N and λ_N/N (i.e., their corresponding values in the network without LRIs) as $s \to +\infty$ (or $\lambda \to +\infty$). In addition, it is shown that the behavior of $\tilde{\lambda}_2/N$ and $\tilde{\lambda}_N/N$ is non-increasing with s or λ.

Now, let us see with a concrete example how the inclusion of LRIs can favor synchronization. We consider a network of Rössler oscillators that is not synchronizable. In particular, we select a network of $N = 30$ units, generated by an Erdös-Rényi model with link probability $p = 0.2$ [16] and illustrated in Fig. 1. The eigenvalues of this network are $\lambda_2 = 0.4988$ and $\lambda_N = 12.0722$. We have then fixed $\sigma = 0.1$. For this value of the coupling, condition (10) is not satisfied, as $\sigma\lambda_2 = 0.0499 < \alpha_1 = 0.186$. At the same time all the eigenvalues satisfy the condition $\sigma\lambda_i > \alpha_2 \; \forall i = 2, \ldots, N$.

We examine now what happens in the presence of LRIs while we increase the strength of the interaction (in practice, decreasing s or λ). First, we note that $\tilde{\lambda}_N/N$ increases while decreasing s or λ; however, since its maximum values is one, then (since $\sigma < \alpha_2/N$) immediately follows that $\sigma\tilde{\lambda}_N < \alpha_2$. On the other hand, $\tilde{\lambda}_2/N$ also increases while decreasing s or λ, such that there exists a value of the transform parameters for which $\sigma\tilde{\lambda}_2$ becomes greater than α_1. This is shown in Fig. 2 which reports the evolution of $\tilde{\lambda}_2/N$ (red curve) and $\tilde{\lambda}_N/N$ (black curve) as function of the parameter of the transform for the Mellin (Fig. 2(a)) or the Laplace (Fig. 2(b)) transform. The dashed line indicates the value of $\alpha_1/(N\sigma)$. The condition is met for $s \leq 2.5$ for the Mellin transformed Laplacian eigenvalues and for $\lambda \leq 0.9$ for the Laplace transformed Laplacian eigenvalues.

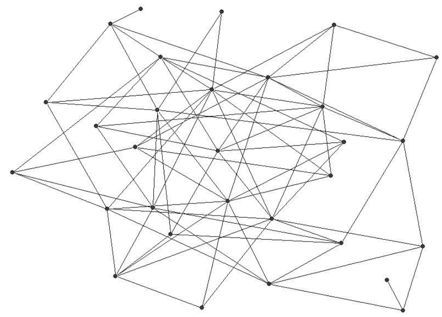


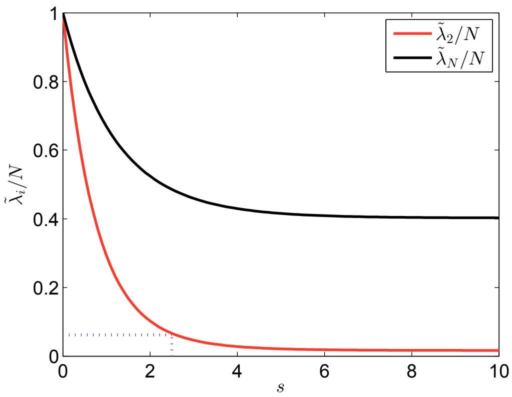
Fig. 1. A network of $N = 30$ Rössler oscillators used to illustrate synchronization in the absence and in the presence of LRIs.

We have then considered numerical simulations of this network starting from random initial conditions and systematically varying the transform parameter from 0 to 10 at steps of 0.1. In particular, 40 runs for each value of the transform parameters have been carried out. After that, the average synchronization error e has been calculated in a time window of $T = 1000$ arbitrary time units. The results are illustrated in Fig. 3. The effect of an increasing strength of LRIs (implemented by a decrease in s or λ) is to reduce the average synchronization error e, until it approaches zero. This occurs in correspondence of the values predicted by the above considerations. For comparison, the dashed line represents the average synchronization error e in the absence of LRIs, showing that for large values of the transform parameter the behavior of the original network is recovered. The comparison also shows that the presence of LRIs is always beneficial.

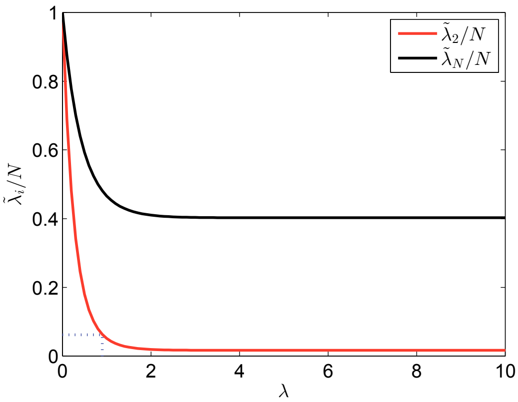
The above considerations clearly apply to any network for which the condition $\sigma < \alpha_2/N$ is satisfied. If this upper bound for the coupling strength holds, then LRIs can yield to synchronization for proper values of the transform parameter. In addition, note that type III MSF systems represent the more challenging case for synchronization. For different coupling functions (e.g., $h(\mathbf{x}_i) = [0, x_{i,2}, 0]^{\mathrm{T}}$), the Rössler units have a type II MSF, for which synchronization is attained if $\sigma\lambda_2 > \alpha_1$ without LRIs or $\sigma\tilde{\lambda}_2 > \alpha_1$ with LRIs. In this case, the positive effect of LRIs (increasing $\tilde{\lambda}_2$ and thus decreasing the minimum value of the coupling strength enabling synchronization) is apparent.

V. CONCLUSIONS

The integration of direct interactions, i.e., those represented in the classical Laplacian matrix, with the long-range (indirect) ones, modeled through the Mellin/Laplace transformed d-path Laplacian, shed light that all the connectivity paths existing in a complex network are important for synchronization. In fact, a network that is not synchronizable by only using direct links, may become synchronizable if LRIs of proper intensity are present. The key factor enabling the achievement of synchronization is the increase in the network algebraic connectivity deriving from the presence of LRIs. In the paper, we have shown that for type II MSF systems this yields synchronization without any restriction on the coupling coefficient, while for type III MSF systems an upper bound exists.

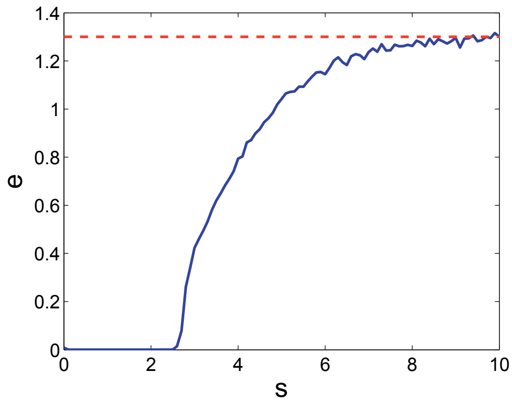


(a)

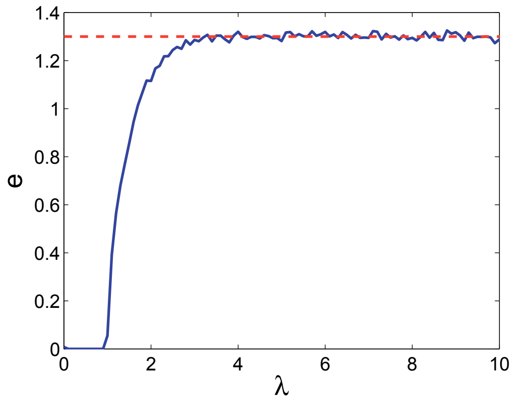


(b)

Fig. 2. Evolution of $\tilde{\lambda}_2/N$ (red curve) and $\tilde{\lambda}_N/N$ (black curve) vs. the parameter of the transform for: (a) Mellin transform; (b) Laplace transform. The dashed lines indicate for which value of the transform parameter $\tilde{\lambda}_2/N$ equals the predicted threshold for synchronization.



(a)



(b)

Fig. 3. Synchronization error e as a function of the parameter of the transform accounting for long-range interactions: (a) Mellin transformed Laplacian; (b) Laplace transformed Laplacian. The dashed line represents the average error for the network in the absence of long-range interactions.

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