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Boundary vertices in graphs

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Abstract

The distance $d(u,v)$ between two vertices u and v in a nontrivial connected graph G is the length of a shortest u–v path in G. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. A vertex v of G is a peripheral vertex if $e(v)$ is the diameter of G. The subgraph of G induced by its peripheral vertices is the periphery $\mathrm{Per}(G)$ of G. A vertex u of G is an eccentric vertex of a vertex v if $d(u,v)=e(v)$. A vertex x is an eccentric vertex of G if x is an eccentric vertex of some vertex of G. The subgraph of G induced by its eccentric vertices is the eccentric subgraph $\mathrm{Ecc}(G)$ of G. A vertex u of G is a boundary vertex of a vertex v if $d(w,v) \leqslant d(u,v)$ for all $w \in N(u)$. A vertex u is a boundary vertex of G if u is a boundary vertex of some vertex of G. The subgraph of G induced by its boundary vertices is the boundary $\partial(G)$ of G. A graph H is a boundary graph if $H = \partial(G)$ for some graph G. We study the relationship among the periphery, eccentric subgraph, and boundary of a connected graph and establish a characterization of all boundary graphs. It is shown that for each triple a,b,c of integers with $2 \leqslant a \leqslant b \leqslant c$, there is a connected graph G such that $\mathrm{Per}(G)$ has order a, $\mathrm{Ecc}(G)$ has order b, and $\partial(G)$ has order c. Moreover, for each triple r,s,t of rational numbers with $0 < r \leqslant s \leqslant t \leqslant 1$, there is a connected graph G of order n such that $|V(\mathrm{Per}(G))|/n = r$, $|V(\mathrm{Ecc}(G))|/n = s$, and $|V(\partial(G))|/n = t$.

MSC: 05C12

Keywords: Peripheral vertex; Eccentric vertex; Boundary vertex

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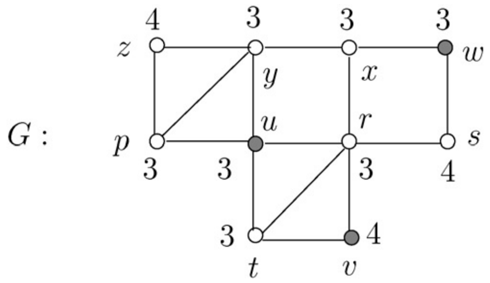

Fig. 1. Peripheral, eccentric, and boundary vertices in a graph.

1. Introduction

Let G be a nontrivial connected graph. The *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest u–v path in G. A u–v path of length $d(u,v)$ is called a u–v geodesic. If u and v are vertices in G and w is a neighbor of u, then $|d(u,v)-d(w,v)|\leqslant 1$. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $\operatorname{rad} G$ of G and the maximum eccentricity is its *diameter* $\operatorname{diam} G$. A vertex v in G is a *central vertex* if $e(v)=\operatorname{rad}(G)$, while the subgraph induced by the central vertices of G is the *center* $\operatorname{Cen}(G)$ of G. A vertex v in a connected graph G is a *peripheral vertex* if $e(v)=\operatorname{diam} G$. The subgraph of G induced by its peripheral vertices is the *periphery* $\operatorname{Per}(G)$.

Let u be a vertex in G. A vertex v is *an eccentric vertex of u* if $d(u,v)=e(u)$, that is, if every vertex at greatest distance from u is an eccentric vertex of u. A vertex x is an *eccentric vertex* of G if x is an eccentric vertex of some vertex of G (see [3]). Consequently, if v is an eccentric vertex of u and w is a neighbor of v, then $d(w,u)\leqslant d(u,v)$. A vertex v may have this property, however, without being an eccentric vertex of u. A vertex v is a *boundary vertex of u* if $d(w,u)\leqslant d(u,v)$ for all $w\in N(v)$. While the distance from a vertex u of G to an eccentric vertex v of u attains the global maximum $\max_{w\in V(G)}\{d(u,w)\}$, the distance from u to a boundary vertex v of u attains the local maximum $\max_{w\in N[v]}\{d(u,w)\}$. Equivalently, a vertex v is a boundary vertex of u if no u–v geodesic can be extended at v to a longer geodesic. Intuitively, then, beginning with a vertex u, a boundary vertex v of u is reached when, locally, we can move no further from u. Essentially, then, the distance from u and v may not attain an absolute maximum in terms of the distances from u to the other vertices of G, but is does attain a local maximum. A vertex v is *a boundary vertex of G* if v is a boundary vertex of some vertex of G. For the graph G of Fig. 1, the vertex v is a boundary vertex of u, but v is not an eccentric vertex of u. The vertex w is an eccentric vertex of u, but w is not a peripheral vertex of G.

There are certain vertices in a nontrivial connected graph that cannot be boundary vertices.

Proposition 1.1. *No cut-vertex of a graph is a boundary vertex.*

Proof. Assume, to the contrary, that there exists a graph G and a cut-vertex u of G such that u is a boundary vertex of some vertex v in G. Let G_1 and G_2 be two distinct components of $G-u$ such that $v \in V(G_1)$, and let w be a neighbor of u that belongs to G_2. Then $d(w,v)=d(u,v)+1$, contrary to hypothesis. □

In fact, determining which vertices in a graph G are boundary vertices can be restricted to the case where G is 2-connected.

Proposition 1.2. *Let v be a vertex in a connected graph G such that v belongs to a block B and v is not a cut-vertex of G. Then v is a boundary vertex of G if and only if v is a boundary vertex of B.*

Proof. Certainly, every boundary vertex of a block of some connected graph is a boundary vertex of the graph. It remains then only to verify the converse. Let G be a connected graph and let v be a boundary vertex of G. Thus v is a boundary vertex of some vertex w in G. Since v is not a cut-vertex, v belongs to a unique block B of G. If $w \in V(B)$, then the proof is complete. Thus, we may assume that $w \notin V(B)$. Let w belong to the block B', where then $B' \neq B$. For each $y \in V(B)$, every $w-y$ geodesic contains a unique cut-vertex x of G that belongs to B. Hence $d(w,v)=d(w,x)+d(x,v)$. Let $u \in N(v)$. Then $u \in V(B)$ and so $d(w,u)=d(w,x)+d(x,u)$. Because v is a boundary vertex of w, it follows that $d(w,u) \leqslant d(w,v)$. Therefore, $d(x,u) \leqslant d(x,v)$, which implies that v is a boundary vertex of x as well. □

A vertex in a graph is called *complete* (also *extreme* or *simplicial*) if the subgraph induced by its neighborhood is complete. In particular, every end-vertex is complete. Observe that if v is a complete vertex and u is a neighbor of v, then $d(w,u)=d(w,v)=1$ for every $w \in N(v)$. Thus v is a boundary vertex of u. Therefore, every complete vertex of a graph is a boundary vertex. In particular, every end-vertex of a graph is a boundary vertex. In fact, more can be said about the complete vertices of a graph.

Proposition 1.3. *Let G be a connected graph. A vertex v of G is a boundary vertex of every vertex distinct from v if and only if v is a complete vertex of G.*

Proof. First, let v be a complete vertex in G and let w be a vertex distinct from v. Let $w=v_0, v_1, \ldots, v_k=v$ be a w–v geodesic. Let u be a neighbor of v. If $u=v_{k-1}$, then $d(w,u)<d(w,v)$. So we may assume that $u \neq v_{k-1}$. Since v is complete, $uv_{k-1} \in E(G)$ and $w=v_0, v_1, \ldots, v_{k-1}, u$ is a $w-u$ path in G, implying that $d(w,u) \leqslant d(w,v)$. Hence v is a boundary vertex of w.

For the converse, let v be a vertex of G that is not a complete vertex. Then there exist distinct, nonadjacent vertices $u, w \in N(v)$. Since $d(u,w)>d(u,v)$, it follows that v is not a boundary vertex of u. □

We now present a result which is the reverse of Proposition 1.3.

Proposition 1.4. *Let G be nontrivial connected graph and let u be a vertex of G. Every vertex distinct from u is a boundary vertex of u if and only if* $e(u)=1$.

Proof. Assume first that $e(u)=1$ and let v be a vertex of G distinct from u. Let w be a neighbor of v. Then $d(u,w)\leqslant 1$ and $d(u,v)=1$. Hence v is a boundary vertex of u. For the converse, assume, to the contrary, that every vertex of G different from u is a boundary vertex of u but $e(u)\neq 1$. Then there exists a vertex x in G such that $d(x,u)=2$. Let x,y,u be a path in G. Then u is a neighbor of y and $d(x,u)=2$, while $d(y,u)=1$. Thus y is not a boundary vertex of u, which is a contradiction. □

2. Boundary graphs

The subgraph of G induced by its eccentric vertices is called the *eccentric subgraph* $\mathrm{Ecc}(G)$ of G (see [3]). We now define the subgraph of G induced by its boundary vertices to be the *boundary* $\partial(G)$ of G. We write $H\leqslant G$ to indicate that H is a subgraph of G. Thus for every connected graph G,

$$\mathrm{Per}(G)\leqslant \mathrm{Ecc}(G)\leqslant \partial(G)\leqslant G. \tag{1}$$

We define a graph H to be a *boundary graph* if $H=\partial(G)$ for some connected graph G. A connected graph G is a *self-boundary graph* if $G=\partial(G)$. Certainly, every self-boundary graph is a boundary graph. In [1] a characterization of all graphs that are the periphery of some connected graph was established; while in [2] a characterization of all graphs that are the eccentric subgraph of some connected graph was presented. We state these two results.

Theorem A. *A nontrivial graph F is the periphery of some connected graph if and only if every vertex of F has eccentricity* 1 *or no vertex of F has eccentricity* 1.

Theorem B. *A nontrivial graph F is the eccentric subgraph of some connected graph if and only if every vertex of F has eccentricity* 1 *or no vertex of F has eccentricity* 1.

According to Theorems A and B then, graphs that are the periphery of some graph are precisely those graphs that are the eccentric subgraph of a graph. For graphs that are the boundary of some graph, however, we have a different characterization. We begin with two lemmas.

Lemma 2.1. *Let G be connected graph of diameter* 2. *Then every vertex v is a boundary vertex of G unless v is the unique vertex of G having eccentricity* 1.

Proof. Let $v\in V(G)$. If $e(v)\neq 1$, then $e(v)=2$ since $\operatorname{diam} G=2$. Thus there is a vertex u such that $d(u,v)=2$. Since $d(w,u)\leqslant 2$ for all $w\in N(v)$, it follows that v is a boundary vertex of u and so v is a boundary vertex of G. If $e(v)=1$ and there is another vertex v' in G with $e(v')=1$, then v and v' are boundary vertices of each other. Finally, suppose that v is the unique vertex of G having eccentricity 1. We show that v is not a boundary vertex of G. Assume, to the contrary, that v is a boundary vertex of some vertex w. Since $e(w)\neq 1$, there exists a vertex u that is not adjacent to w. However, u is a neighbor of v, $d(u,w)=2$, and $d(w,v)=1$, which produces a contradiction. □

Lemma 2.2. *Let F be a nontrivial connected graph with no vertices of eccentricity 1 and let $G=F+K_k$, where $k\geqslant 1$. Then G is a self-boundary graph if and only if $k\geqslant 2$.*

Proof. Certainly, $e_G(v)=2$ if $v\in V(F)$ and $e_G(v)=1$ if $v\in V(K_k)$. Thus $\operatorname{diam} G=2$. If $k=1$, then $\partial(G)=F\neq G$ by Lemma 2.1 and so G is not a self-boundary graph.

For the converse, assume that $k\geqslant 2$. Let $v\in V(G)$. If v belongs to K_k, then v is a boundary vertex of each vertex in K_k that is distinct from v; while if v belongs to F, then v is a boundary vertex of each vertex in F that is distinct from v. Therefore, $\partial(G)=G$. □

Theorem 2.3. *A nontrivial graph H is the boundary of some connected graph if and only if H does not have exactly one vertex with eccentricity 1.*

Proof. Suppose first that H is the boundary of some connected graph G. Assume, to the contrary, that H has exactly one vertex, say v, with eccentricity 1. Then $\operatorname{diam} H=2$ and $H=F+K_1$, for some graph F, where $V(K_1)=\{v\}$ and no vertex in F has eccentricity 1 in F. By Lemma 2.2, H is not a self-boundary graph and so $H\neq G$. On the other hand, since H is the boundary of G, it follows that $\mathrm{Per}(G)\leqslant H$ and so $\operatorname{diam} G=2$ as well. Thus $e(v)=1$ or $e(v)=2$ for all $v\in V(G)$. Since $\mathrm{Per}(G)\leqslant H$, it follows that $e(v)=1$ for all $v\in V(G)-V(H)$. Let $k=|V(G)|-|V(H)|\geqslant 1$. Then $G=H+K_k=F+K_{k+1}$, where $k+1\geqslant 2$. Since no vertex in F has eccentricity 1 in F, it then follows from Lemma 2.2 that G is a self-boundary graph and so $G=\partial(G)$, which is impossible.

For the converse, suppose that H does not have exactly one vertex with eccentricity 1. We consider three cases.

Case 1: *All vertices of H have eccentricity* 1. Then $H=K_n$ for some n and $H=\partial(H)$.

Case 2: *No vertex of H has eccentricity* 1. Let $G=H+K_1$. It then follows from Lemma 2.2 that $H=\partial(G)$.

Case 3: *At least two vertices of H have eccentricity* 1 *and at least one vertex of H has eccentricity* 2 *or more*. Let $V(H)=V_1\cup V_2$ such that $e(v)=1$ if $v\in V_1$ and $e(v)\geqslant 2$ if $v\in V_2$. Let $|V_1|=n_1$ and $|V_2|=n_2$. By assumption, $n_1\geqslant 2$ and $n_2\geqslant 1$. However, since H contains at least two peripheral vertices, $n_2\geqslant 2$ as well. Furthermore, $H=K_{n_1}+F$, where $V(K_{n_1})=V_1$ and $V(F)=V_2$. Again, it follows by Lemma 2.2 that H is a self-boundary graph and so $H=\partial(H)$. □

Observation 2.4. *Every vertex-transitive graph is a self-boundary graph.*

The converse of Observation 2.4 is false. For example, consider the graphs G and H of Fig. 2, where each vertex of G and H is labeled by its eccentricity. Certainly, neither G nor H is vertex-transitive, but both are self-boundary graphs. Moreover, every vertex of G is an eccentric vertex; while only two vertices of G are periph-

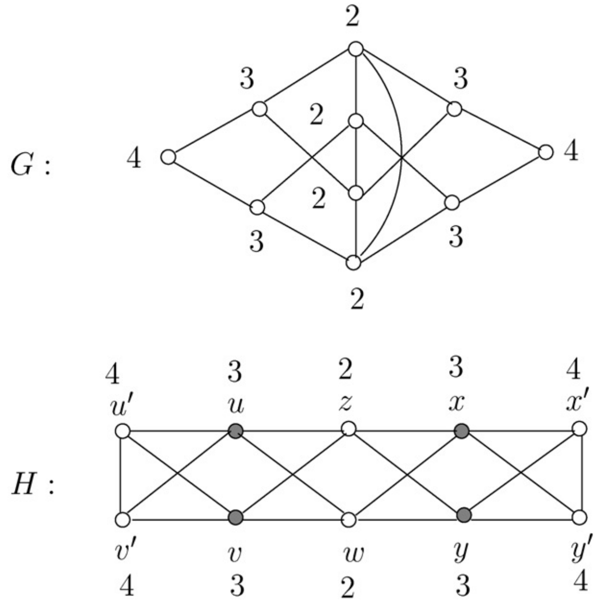

Fig. 2. Nontransitive self-boundary graphs.

eral vertices. On the other hand, the vertices u, v, x, y of H are not even eccentric vertices.

3. The boundary, eccentric subgraph, and periphery of a graph

For a connected graph G, let $\mathscr{B}(G) = V(\partial(G))$, $\mathscr{C}(G) = V(\mathrm{Cen}(G))$, $\mathscr{E}(G) = V(\mathrm{Ecc}(G))$, and $\mathscr{P}(G) = V(\mathrm{Per}(G))$. There are numerous connected graphs G such that $\mathscr{P}(G) = \mathscr{E}(G) = \mathscr{B}(G)$. For example, the wheel $W_n = C_n + K_1$, $n \geqslant 4$, has this property. Indeed, $\mathscr{P}(W_n) = \mathscr{E}(W_n) = \mathscr{B}(W_n) = V(C_n)$. The grid graph $P_m \times P_n$, where m and n are positive integers is another example of such graph, as we show next.

Proposition 3.1. *For every pair m, n of positive integers,*

$$\partial(P_m \times P_n) = \mathrm{Per}(P_m \times P_n).$$

Proof. By (1) we have that $V(\mathrm{Per}(P_m \times P_n)) \subseteq V(\partial(P_m \times P_n))$, so it suffices to show that no nonperipheral vertex of G is a boundary vertex. Let $P_m : u_1, u_2, \ldots, u_m$ and $P_n : v_1, v_2, \ldots, v_n$. The vertex set of $P_m \times P_n$ is the set $\{(u_i, v_j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. For brevity, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ we denote $w_{ij} = (u_i, v_j)$. The peripheral vertices of $P_m \times P_n$ are w_{11}, w_{1n}, w_{m1} and w_{mn} (and if $m = 1$ or $n = 1$ then at least two of these vertices are not distinct). For some i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, let w_{ij} be a nonperipheral vertex of $P_m \times P_n$ and suppose by way of contradiction that w_{ij} is a

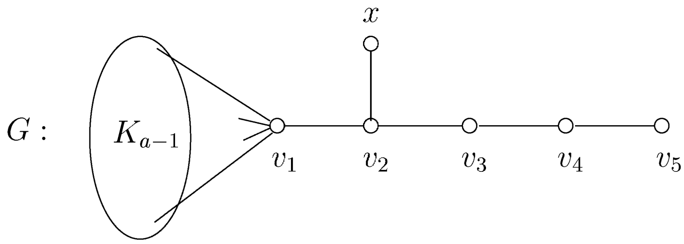

Fig. 3. The graph G in Case 2.

boundary vertex with respect to some vertex $w_{i'j'}$. Without loss of generality, we assume that $i' \leqslant i$ and $j' \leqslant j$. Since w_{ij} is not peripheral, either $i < m$ or $j < n$ (or both). We assume, again without any loss of generality, that $i < m$. Hence the vertex $w_{(i+1)j}$ is adjacent in $P_m \times P_n$ to w_{ij}. Consequently, $d(w_{i'j'}, w_{ij}) = i - i' + j - j'$, while $d(w_{i'j'}, w_{(i+1)j}) = i - i' + j - j' + 1$. Thus, $d(w_{i'j'}, w_{(i+1)j}) > d(w_{i'j'}, w_{ij})$ which contradicts our assumption that w_{ij} is a boundary vertex of $w_{i'j'}$. Hence no such vertex w_{ij} exists and the result follows. □

On the other hand, for some connected graphs G, it is possible that $\mathscr{P}(G)$ is a proper subset of $\mathscr{E}(G)$ or that $\mathscr{E}(G)$ is a proper subset of $\mathscr{B}(G)$, or both. Indeed, we have the following.

Theorem 3.2. *For each triple a, b, c of integers with $2 \leqslant a \leqslant b \leqslant c$, there is a connected graph G such that* $\mathrm{Per}(G)$ *has order a,* $\mathrm{Ecc}(G)$ *has order b, and $\partial(G)$ has order c.*

Proof. If $a = b = c$, then $G = K_a$ has the desired properties by Proposition 1.3. Thus, we consider the following three cases.

Case 1: $a < b = c$. Let $G = K_{b-a} + \bar{K}_a$. Then the order of G is b. Since $e(v) = 2$ if $v \in V(\bar{K}_a)$ and $e(v) = 1$ if $v \in V(K_{b-a})$, it follows that $\mathrm{Per}(G) = \bar{K}_a$ and $\mathrm{Ecc}(G) = \partial(G) = G$. Thus $\mathrm{Per}(G)$ has order a and $\mathrm{Ecc}(G)$ and $\partial(G)$ have order $b = c$.

Case 2: $a = b < c$. Suppose, first, that $c = b + 1$. Let G be the graph obtained from the graphs K_{a-1} and $P_5 : v_1, v_2, \ldots, v_5$ by adding a new vertex x, the edge xv_1, and joining v_1 to every vertex in $V(K_{a-1})$. The graph G is shown in Fig. 3. Since $e(v) = 5$ if $v \in V(K_{a-1}) \cup \{v_5\}$, $e(v) = 4$ if $v \in \{v_1, v_4, x\}$, and $e(v) = 3$ if $v \in \{v_2, v_3\}$, it follows that $\mathscr{P}(G) = V(K_{a-1}) \cup \{v_5\}$. Moreover, let v be an arbitrary vertex in G and let $u \in \{v_1, v_2, v_3, v_4, x\}$. Since $e(v) \neq d(v, u)$, it follows that $\mathscr{E}(G) = \mathscr{P}(G) = V(K_{a-1}) \cup \{v_5\}$. By Propositions 1.1 and 1.3, it follows that $\mathscr{B}(G) = \mathscr{P}(G) \cup \{x\}$.

For $c \geqslant b + 2$, let G' be the graph obtained by replacing the vertex x in the graph G in Fig. 3 by K_{c-b} and joining every vertex of K_{c-b} to v_2. The graph G' then has the desired properties.

Case 3: $a < b < c$. For $a = 2, b = 3$, and $c = 4$, let H be the graph shown in Fig. 4, where each vertex of G is also labeled with its eccentricity. It is routine to verify that $\mathscr{P}(H) = \{u, y\}$, $\mathscr{E}(H) = \{u, y, z\}$, and $\mathscr{B}(H) = \{u, w, y, z\}$, where z is an eccentric vertex of w, which is in turn a boundary vertex of s.

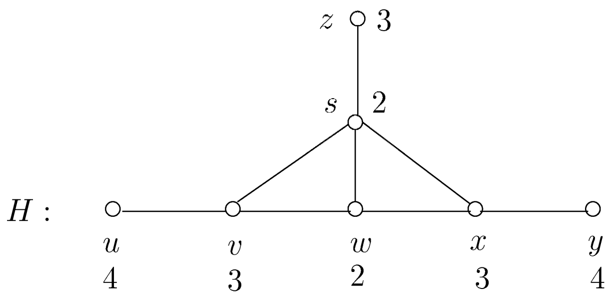

Fig. 4. The graph H in Case 3.

For each triple a,b,c of integers with $2 \leqslant a < b < c$, we construct a graph G from the graph H of Fig. 4 by (1) replacing u by K_{a-1} and joining every vertex of K_{a-1} to v, (2) replacing z by K_{b-a} and joining every vertex of K_{b-a} to s, and (3) replacing w by K_{c-b} and joining every vertex of K_{c-b} to every vertex in $\{s,v,x\}$. Then every vertex in K_{a-1} has eccentricity 4, every vertex in K_{b-a} has eccentricity 3, and every vertex in K_{c-b} has eccentricity 2. Then $\mathscr{P}(G) = V(K_{a-1}) \cup \{y\}$, $\mathscr{E}(G) = \mathscr{P}(G) \cup V(K_{b-a})$, and $\mathscr{B}(G) = \mathscr{E}(G) \cup V(K_{c-b})$. $\square$

The graph H of Fig. 4 shows that a graph exists for which exactly $\frac{2}{7}$ of its vertices are peripheral vertices, $\frac{3}{7}$ of its vertices are eccentric vertices, and $\frac{4}{7}$ of its vertices are boundary vertices. It is not difficult to show that there is no graph G of order n, where $4 \leqslant n \leqslant 6$, for which $\mathrm{Per}(G)$ has order 2, $\mathrm{Ecc}(G)$ has order 3, and $\partial(G)$ has order 4. So in particular, there is no graph of order 6 for which exactly $\frac{1}{3}$ of its vertices are peripheral vertices, $\frac{1}{2}$ of its vertices are eccentric vertices, and $\frac{2}{3}$ of its vertices are boundary vertices. This suggests the question of determining precisely which proportions of vertices of a graph can be peripheral vertices, eccentric vertices, and boundary vertices. This is answered in the following theorem.

Theorem 3.3. *For each triple r,s,t of rational numbers with $0 < r \leqslant s \leqslant t \leqslant 1$, there is a connected graph G of order n such that*

$$\frac{|\mathscr{P}(G)|}{n} = r, \quad \frac{|\mathscr{E}(G)|}{n} = s, \quad \textit{and} \quad \frac{|\mathscr{B}(G)|}{n} = t.$$

Proof. Let $r = a_1/b_1$, $s = a_2/b_2$, and $t = a_3/b_3$, where a_i, b_i are positive integers for $1 \leqslant i \leqslant 3$. We consider six cases.

Case 1: $0 < r < s < t < 1$. We use the graph H of order 7 in Fig. 4 as a basic graph. Let $n_1 = 7a_1b_2b_3$, $n_2 = 7a_2b_1b_3$, $n_3 = 7a_3b_1b_2$, and $n_4 = 7b_1b_2b_3$. We construct a graph G from H by (1) replacing u by K_{n_1-1} and joining every vertex of K_{n_1-1} to v, (2) replacing w by $K_{n_3-n_2}$ and joining every vertex of $K_{n_3-n_2}$ to every vertex in $\{s,v,x\}$, (3) replacing z by $K_{n_2-n_1}$ and joining every vertex of $K_{n_2-n_1}$ to s, and (4) replacing s by $K_{n_4-n_3-2}$ and joining every vertex of $K_{n_4-n_3-2}$ to every vertex in $\{v,x\} \cup V(K_{n_2-n_1}) \cup V(K_{n_3-n_2})$. Then every vertex in $V(K_{n_1-1}) \cup \{y\}$

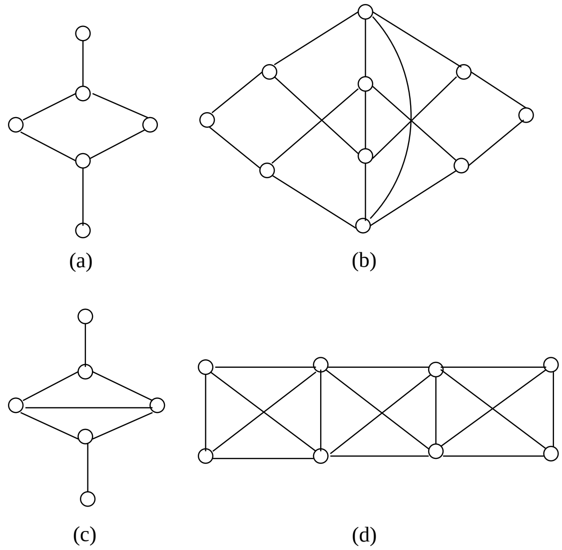

Fig. 5. The basic graphs in Cases 3–6.

has eccentricity 4, every vertex in $V(K_{n_2-n_1})\cup\{v,x\}$ has eccentricity 3, and every vertex in $V(K_{n_3-n_2})\cup V(K_{n_4-n_3-2})$ has eccentricity 2. Thus $\mathscr{P}(G)=V(K_{n_1-1})\cup\{y\}$, $\mathscr{E}(G)=\mathscr{P}(G)\cup V(K_{n_3-n_2})$ since every vertex of $K_{n_2-n_1}$ is an eccentric vertex of each vertex in $K_{n_3-n_2}$, and $\mathscr{B}(G)=\mathscr{E}(G)\cup V(K_{n_3-n_2})$ since every vertex of $K_{n_3-n_2}$ is a boundary vertex of $K_{n_4-n_3-2}$. Since the order of G is $n=n_4$,

$$\frac{|\mathscr{P}(G)|}{n}=\frac{n_1}{n_4}=r,\quad \frac{|\mathscr{E}(G)|}{n}=\frac{n_2}{n_4}=s,\quad\text{and}\quad \frac{|\mathscr{B}(G)|}{n}=\frac{n_3}{n_4}=t.$$

Case 2: $0<r<s<t=1$. We use the self-boundary graph H of order 10 in Fig. 2 as a basic graph. Assume that $a_3=b_3=1$. Let $n_1=10a_1b_2$, $n_2=10a_2b_1$, $n_3=10b_1b_2$. We construct a graph G from H by (1) replacing u' by K_{n_1-3} and joining every vertex of K_{n_1-3} to every neighbor of u', (2) replacing z by $K_{n_2-n_1-1}$ and joining every vertex of $K_{n_2-n_1-1}$ to every neighbor of z, and (3) replacing y by $K_{n_3-n_2-3}$ and joining every vertex of $K_{n_3-n_2-3}$ to every neighbor of y. Then every vertex in $V(K_{n_1-1})\cup\{u',x',y'\}$ has eccentricity 4, every vertex in $V(K_{n_3-n_2-3})\cup\{u,v,x\}$ has eccentricity 3, and every vertex in $V(K_{n_2-n_1-1})\cup\{w\}$ has eccentricity 2. Thus $\mathscr{P}(G)=V(K_{n_1-3})\cup\{u',x',y'\}$, $\mathscr{E}(G)=\mathscr{P}(G)\cup V(K_{n_2-n_1-1})\cup\{w\}$ since every vertex in $K_{n_2-n_1-1}$ is an eccentric vertex of w and w is an eccentric vertex of each vertex in $K_{n_2-n_1-1}$, and $\mathscr{B}(G)=V(G)$. Since the order of G is $n=n_3$,

$$\frac{|\mathscr{P}(G)|}{n}=\frac{n_1}{n_3}=r,\quad \frac{|\mathscr{E}(G)|}{n}=\frac{n_2}{n_3}=s,\quad\text{and}\quad \frac{|\mathscr{B}(G)|}{n}=t=1.$$

For the remaining cases, we construct four basic graphs shown in Fig. 5, which can be used to construct the desired graph G. Since the proof in each case is similar to the one used in Cases 1 and 2, we omit its proof.

Case 3: $0<r<s=t<1$. Let the basic graph be the graph in Fig. 5(a).

Case 4: $0<r<s=t=1$.

Let the basic graph be the graph in Fig. 5(b).

Case 5: $0<r=s<t<1$.

Let the basic graph be the graph in Fig. 5(c).

Case 6: $0<r=s<t=1$.

Let the basic graph be the graph in Fig. 5(d). □

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