Laplacian Energy of Directed Graphs

Kissani Perera
Graduate School of Mathematics,
Kyushu University.

Back Ground

- The concept of graph energy arose in chemistry where certain numerical quantities, such as the heat of formation of a hydrocarbon are related to total ∏-electron energy (sum of energies of all electrons) that can be calculated as the energy of an appropriate "molecular" graph.
- In 1978, Gutman defined it as $E(G)=\sum_{i=1}^{n}|\lambda_i|$,where λ_i are Eigen values of adjacency matrix of G.
- It is defined for all graphs (no matter whether these represent conjugated molecules or not).
- Later eigen values of other matrices have been studied, of which Laplacian matrix attracted the greatest attention.

- Then, at the very end of the 20th century, mathematicians suddenly became interested in graph energy and found energy as well as Laplacian energy of several kinds of graphs including circulant graphs, cayley trees ,random graphs and finally directed graphs.

Motivation

	Laplacian Energy LE(G)	
	$LE(G)=\sum_{i=1}^{n}\lambda_i{}^2$	$LE(G)=\sum_{i=1}^{n}\left\lvert\lambda_i-\frac{2m}{n}\right\rvert$
Undirected Graphs $L=D-A_{undir}$	Kragujevac(2006)	Gutman (2006)
Directed graphs $L=D-S$	Adiga (2009)	Adiga (2009)
$L=D^{out}-A_{dir}$	(2010) 	(2010) $LE(G)=\sum_{i=1}^{n}\left\lvert\lambda_i-\frac{m}{n}\right\rvert$

Introduction-Laplacian energy

- According to Kragujevac(2006), we define Laplacian energy for directed graphs as

$$LE(G) = \sum_{i=1}^{n} \lambda_i^{\,2}$$

- Where λ_i are Eigen values of Laplacian matrix

$$L = D^{out} - A$$

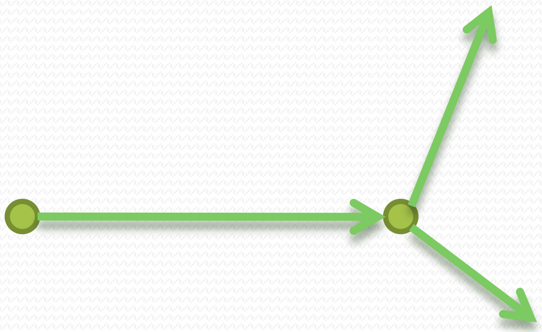
- D^{out} : Diagonal matrix with out degrees of vertices.
- A : Directed Adjacency Matrix

$$a_{ij} = \begin{cases} 1, & if \quad i \to j \\ 0, & otherwise \end{cases}$$

Theorem 1.1

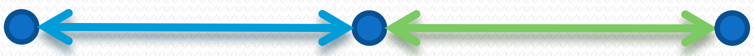
- Let G be a digraph with vertex degrees $d_1, d_2, \ldots, d_n$
- If G is a simple connected directed graph then

$$LE(G) = \begin{cases} \sum_{i=1}^{n} d_i^{\,2} & ;\ d_i \text{ total degree} \\ \sum_{i=1}^{n} \left(d_i^{\,out}\right)^2 & ;\ d_i \text{ out degree} \\ \sum_{i=1}^{n} \left(d_i^{\,in}\right)^2 & ;\ d_i \text{ in degree} \end{cases}$$



- If G is a complete directed graph then

$$LE(G)=\begin{cases}\displaystyle\sum_{i=1}^{n} d_i(d_i+1) & ;\text{if } d_i \text{ is in}-\text{degree or out-degree}\\ \displaystyle\frac{1}{2}\sum_{i=1}^{n} d_i(2d_i+1) & ;\text{ if } d_i \text{ is total degree}\end{cases}$$



Proof

- Suppose G is a **complete digraph**.

Case I: D contain out-degree (or in-degree)

Then each edge has bi directions. i.e., $a_{ij} = a_{ji}$

$$Trace(L) = \sum_i \lambda_i = \sum_{i=1}^{n} d_i^{out}$$

(Viète's law)

Sum of determinant of all 2×2 sub matrices are $\sum_{i<j} \lambda_i \lambda_j$

$$i.e., \sum_{i<j} \det\begin{pmatrix} d_i^{out} & -a_{ij} \\ -a_{ji} & d_j^{out} \end{pmatrix} = \sum_{i<j} d_i^{out} d_j^{out} - a_{ij}a_{ji} = \sum_{i<j} d_i^{out} d_j^{out} - a_{ij}^2$$

Since $a_{ij}^2 = a_{ij}$ for every $i < j$

$$\sum_{i \neq j} \lambda_i \lambda_j = 2\sum_{i<j} \lambda_i \lambda_j = \sum_{i \neq j} d_i^{out} d_j^{out} - a_{ij} = \sum_{i \neq j} d_i^{out} d_j^{out} - \sum_{i=1}^{n} d_i^{out} ------(1)$$

Therefore

$$LE(G) = \sum_i \lambda_i^2 = \left(\sum_i \lambda_i\right)^2 - \sum_{i \neq j} \lambda_i \lambda_j$$

$$= \left(\sum_i d_i^{out}\right)^2 - \left[\sum_{i \neq j} d_i^{out} d_j^{out} - \sum_{i=1}^{n} d_i^{out}\right]$$

$$= \sum_{i=1}^{n} \left(d_i^{out}\right)^2 + \sum_{i=1}^{n} d_i^{out}$$

$$= \sum_{i=1}^{n} d^{out}(d_i^{out} + 1)$$

Result similar to undirected graphs

Case II : D contain total degree of vertices

- Equation (1) changed as

$$\sum_{i\neq j}\lambda_i\lambda_j = 2\sum_{i<j}\lambda_i\lambda_j = \sum_{i\neq j} d_i d_j - a_{ij} = \sum_{i\neq j} d_i d_j - \frac{1}{2}\sum_{i=1}^{n} d_i$$

$$LE(G) = \sum_i \lambda_i^2 = \left(\sum_i \lambda_i\right)^2 - \sum_{i\neq j}\lambda_i\lambda_j$$

$$= \left(\sum_i d_i\right)^2 - \left[\sum_{i\neq j} d_i d_j - \frac{1}{2}\sum_{i=1}^{n} d_i\right]$$

$$= \frac{1}{2}\left(2\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n} d_i(2d_i + 1)$$

- Suppose G is a **Simple directed graph**.

$$a_{ij} = \begin{cases} 1, & i \to j \\ 0, & otherwise \end{cases}$$

$$i.e., \sum_{i<j} \det\begin{pmatrix} d_i^{out} & -a_{ij} \\ 0 & d_j^{out} \end{pmatrix} = \sum_{i<j} d_i^{out} d_j^{out}$$

$$\sum_{i\neq j} \lambda_i \lambda_j = 2\sum_{i<j} \lambda_i \lambda_j = \sum_{i\neq j} d_i^{out} d_j^{out}$$

$$LE(G) = \sum_i \lambda_i^2 = \left(\sum_i \lambda_i\right)^2 - \sum_{i\neq j} \lambda_i \lambda_j$$

$$= \left(\sum_i d_i^{out}\right)^2 - \left[\sum_{i\neq j} d_i^{out} d_j^{out}\right] = \sum_{i=1}^{n} \left(d_i^{out}\right)^2$$

Example:

Adiga(2009) **L=D-S** = $\sum_{i-1}^{n} d_i(d_i-1)$	**L=D^{out}-A** = $\sum_{i=1}^{n} \left(d_i^{out}\right)^2$	
S= $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	A= $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	
D= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	D^{out}= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	D= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
L= $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$	L = $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$	L= $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$
LE(G)=4	L(G)= 3	LE(G)=10

- (Corollary 1.1)
- Laplacian energy of directed path P_n is $(n-1)$.
- From theorem (1.1)

$$LE(P_n) = \sum_{i=1}^{n} \left(d_i^{out}\right)^2 = \sum_{i=1}^{n-1} 1^2 = (n-1)$$

- (Corollary 1.2)
- Laplacian energy of directed cycle C_n with $n \geq 3$ is n.

$$LE(C_n) = \sum_{i=1}^{n} \left(d_i^{out}\right)^2 = \sum_{i=1}^{n} 1^2 = n$$

Corollary (1.3)

- For any simple connected digraph with n>=2 vertices,

$$\boxed{n-1 \;\le\; LE(G) \;\le\; n^2(n-1)}$$

Moreover $LE(G) = n^2(n-1)$, iff G is a complete digraph

and LE(G)=(n-1) , iff G is directed path.

Proof (right-side)

Let G be any simple connected digraph with n(>=2) vertices. Since maximum degree of any vertex is less or equal to (n-1),

$$LE(G)=\sum_{i=1}^{n}\left(d_i^{out}\right)^2<\sum_{i=1}^{n}d_i^{out}(d_i^{out}+1)\leq n(n-1)(n)=n^2(n-1).$$

Since each complete digraph has maximum number of (n-1) degrees, maximum energy achieved for complete directed graphs.

Proof(left-side): *by induction* $LE(G) \geq n-1$

- *To form connected graph we need at least two nodes*

Only directed graphs with 2 nodes is a path.

Since eigenvalues of $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ *are 1 and 0 and* $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ *are 2*

and 0 result is true for n=2

- Suppose result is true for any connected digraph with (n-1) vertices. i.e., $LE(G) \geq n-2$

- We need to show that the result is true for any connected digraph.

➢ Let G be digraph with n vertices.

➢ Then there is an induced digraph H on (n-1) vertices.

$V(G) = V(H) \cup \{v_n\}$ and $LE(H) \geq n-2$

v_n is connected to at least one vertex in H and

$LE(G) \geq LE(H)+1$ and hence have $LE(G) \geq n-1$

- Further if G is a simple connected digraph with $LE(G) = n-1$ then G must be a di-path.
- Let n=2 then

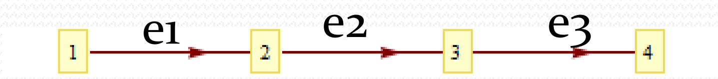
$$LE(G) = n-1 = 1$$

$$LE(G) = \sum_{i=1}^{2} \left(d_i^{out}\right)^2 = \left(d_1^{out}\right)^2 + \left(d_2^{out}\right)^2 = 1$$

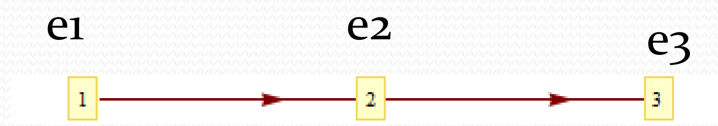
- This achieve for directed path.

Laplacian energy of Line digraphs LD(G)

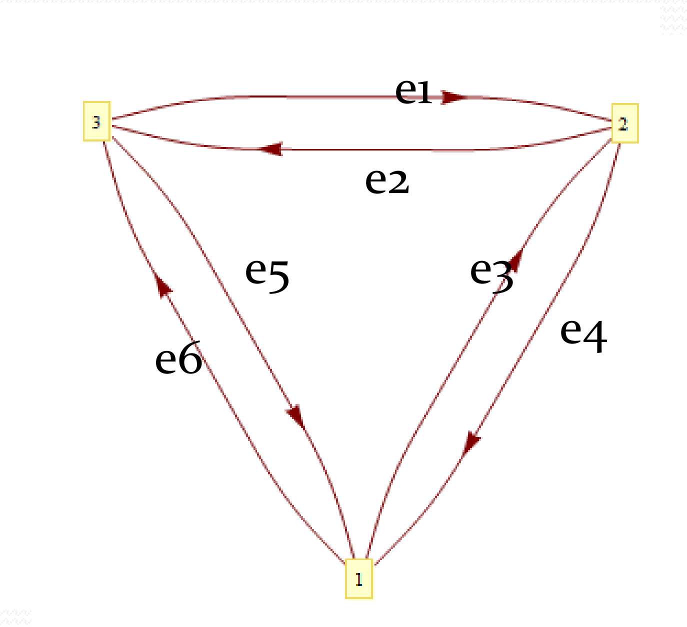
- (Definition):
- Line digraph LD(G) of a non empty graph G has vertex set $V(LD(G)) = \{e : e \in \Gamma(G)\}$
- $e_1 = \{x, y\}$ is adjacent to $e_2 = \{w, z\}$ in LD(G) iff (y=w) in G.



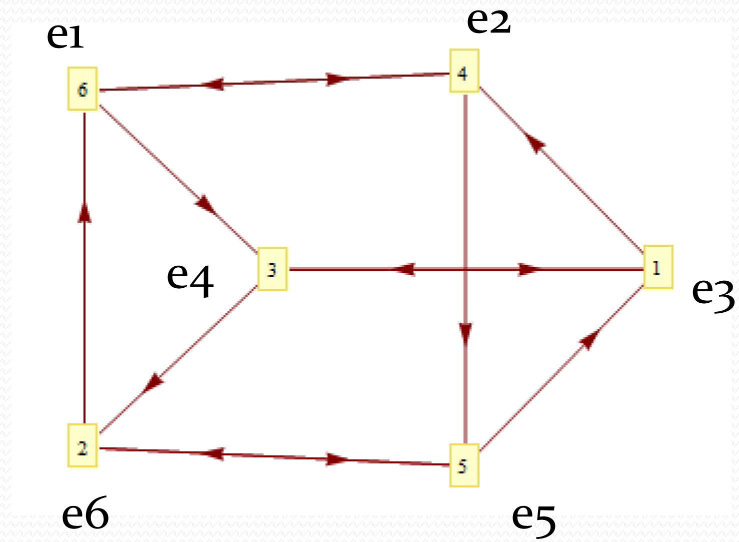
Path $-P_4$



Line dipath



K_3- Complete digraph



Complete Line digraph

Results

- Line graph of di-cycle has form a di-cycle with same number of vertices.

$$\boxed{LE(LD(C_n)) = LE(C_n) = n}$$

- Line graph of di-path with n vertices also form a di-path with (n-1) vertices.

$$\boxed{LE(LD(P_n)) = LE(P_{n-1}) = n-2}$$

Results

- Laplacian energy of Line digraph of complete digraph K_n with n≥3, is

$$\boxed{n(n-1)\left(n^2-2n+2\right)}$$

- Eigenvalues of LD

$$\begin{array}{l} \text{Eigenvalue} \\ \text{multiplicity} \end{array} \begin{pmatrix} n & n-1 & 0 \\ (n-1) & n(n-2) & 1 \\ & & \end{pmatrix}$$

Number of directed arcs in $K_n = n(n-1)$

Number of vertices in $LD(K_n) = n(n-1)$

By definition:

$$LE(G) = \sum_{i=1}^{n(n-1)} \lambda_i{}^2 = \left[n^2 * (n-1) + (n-1)^2 * n(n-2)\right]$$

$$=$$

$$= n(n-1)(n^2 - 2n + 2)$$

Nature of Line digraph

LD has one way arcs as well as bi directed edges.

First assume we have a simple directed graph .

Number of arcs	**Vertex degree**
n(n-1)/2	(n-1)
n(n-1)/2	(n-2)

Then remaining n(n-1)/2 arcs contributes to form bi directions.

For each arc add to form a bi direction increase by

$$S_n = 5 + 2(n-3), \qquad n \geq 3$$

Laplacian energy of LD

By structure of Line digraph

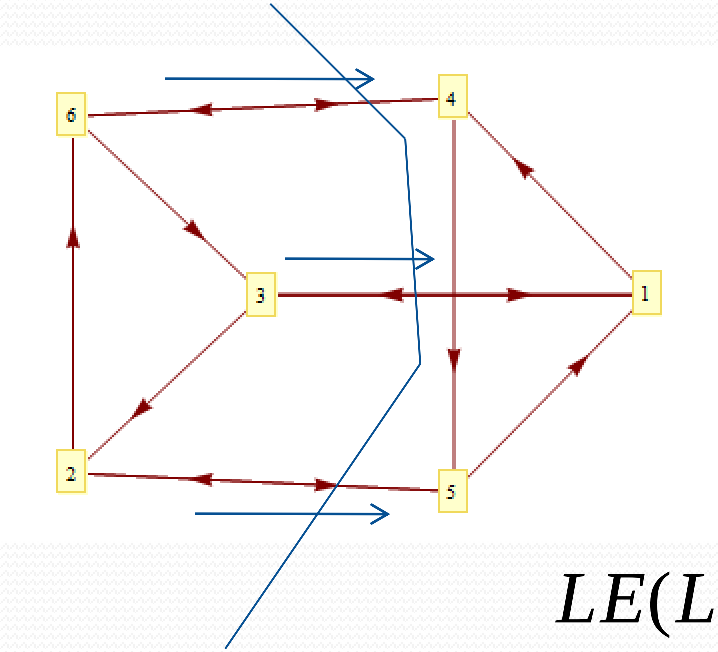
$$LE(G) = \underbrace{\left[\frac{n(n-1)}{2} * (n-1)^2 + \frac{n(n-1)}{2} * (n-2)^2\right]}_{\downarrow} + \left[\frac{n(n-1)}{2} * S_n\right]$$

$$= \sum_{i=1}^{n(n-1)} d_i^{\,2} + \left[\frac{n(n-1)}{2} * S_n\right]$$

$$= \frac{n(n-1)}{2}\left[(n-1)^2 + (n-2)^2 + 5 + 2(n-3)\right]$$

$$= \frac{n(n-1)}{2}(2n^2 - 4n + 4) = n(n-1)(n^2 - 2n + 2)$$

Example: Line digraph of K3



- 3 vertex with degree 1
- 3 vertex with degree 2
- 3 vertex contribute to form bi directions

$$LE(LD(K_3)) = 3*1^2 + 3*2^2 + 3*S_3$$

$$= 3 + 12 + 3*5$$

$$= 30$$

Examples

Line Digraph	Eigen values	LE(G)	$n(n-1)(n^2-2n+2)$
K3	{0,2,2,2,3,3}	30	=(3*2)(9-2*3+2) =30
K4	{0,3,3,3,3,3,3,3,3,4,4,4}	120	=(4*3)(16-2*4+2) =120
K5	{0,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,5,5,5,5 }	340	=(5*4)(25-2*5+2) =340
K6	{0,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5, 5,5,5,5,6,6,6,6,6}	780	=(6*5)(36-2*6+2) =780
K7	{0,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6, 6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,7, 7,7,7}	1554	=(7*6)(49-2*7+2) =1554

Minimum Laplacian energy change due to arc addition(observation)

- For all simple connected directed graphs with m=n, min .LE(G) = m. For m>n, minimum Laplacian energy is increase by 3 for each n arcs added by 5 for next n arcs by 7 for next n arcs until n(n-1)/2 number of arcs added.

$$\min.LE(G) = \min \sum_{i=1}^{n} d_i^2$$

$$m = n, \quad d_i = 1, \forall i$$

$$\therefore \min LE(G) = m$$

$$m = [n+1, 2n], \quad \min.LE(G) = \sum_{i=1}^{n-j} d_i^2 + \sum_{i=n-j+1}^{n} 2^2 = (n-j) + 4*(j) \quad for \quad j = 1,..,n$$

Table : LE(G) for arcs are increase

n	m	Min[LE[G]]	n	m	Min[LE[G]]	m	Min[LE[G]]
3	2	2	9	9	9	22	56
	3	3		10	12	23	61
4	3	3		11	15	24	66
	4	4		12	18	25	71 +5
	5	7 +3		13	21	26	76
	6	10		14	24 +3	27	81
5	4	4		15	27	28	88
	5	5		16	30	29	95
	6	8		17	33	30	102
	7	11		18	36	31	109
	8	14 +3		19	41	33	116 +7
	9	17		20	46 +5	34	123
	10	20		21	51	35	130

Enumerate the structure of digraphs with $LE(G) \le \alpha$

- **Theorem (1.2)**

- Lets consider the class $P(\alpha) = \{G \mid LE(G) \le \alpha\ \}$

For any $\alpha > 1$, the class $P(\alpha)$ of all non-isomorphic connected directed graphs with the property $LE(G) \le \alpha$ is finite.

(Proof)

Let G be simple connected directed graph such that $LE(G) \le \alpha$

Then $LE(G) = \sum_{i=1}^{n} \left(d_i^{out}\right)^2 \le \alpha$

But $n - 1 \le LE(G) \le \alpha$ corollary (1.3)

Hence we obtain $n - 1 \le \alpha$

Since n is finite class $P(\alpha)$ is finite.

(Corollary):

The class P(10) contains exactly 49 digraphs. More exactly 31 digraphs with n≤10, 8 directed cycles with n≤10, 10 directed paths with n≤ 11.

Every n-connected graph has at least (n-1) arcs.

Note that for n=12, $LE(G) \geq (n-1) = 11 > 10$

- For n=11, $LE(G) \geq 10$

Therefore all digraphs from class p(10) has at most 11 vertices.

- Since $LE(C_n) = n$ we have 8 digraphs with cycles.
- Since $LE(P_n) = n-1$ we have 10 dipath graphs .

- All other graphs are belong to simple connected digraphs with n≤10.
- Some of digraphs are listed here.

Conclusion

- Laplacian energy of simple directed graph is

$$LE(G)=\begin{cases}\sum_{i=1}^{n}\left(d_i^{out}\right)^2, & if\ \ D=diag(d_1^{out},d_2^{out},\dots,d_n^{out})\\ \sum_{i=1}^{n}\left(d_i^{in}\right)^2, & if\ \ D=diag(d_1^{in},d_2^{in},\dots,d_n^{in})\\ \sum_{i=1}^{n}\left(d_i^{tot}\right)^2, & if\ \ D=diag(d_1^{tot},d_2^{tot},\dots,d_n^{tot})\end{cases}$$

and complete digraph is

$$LE(G)=\begin{cases}\sum_{i=1}^{n}d_i^{in}(d_i^{in}+1)\ , & if\ \ d_i^{in}: indegree\ of\ i\\ \sum_{i=1}^{n}d_i^{out}(d_i^{out}+1)\ , & if\ \ d_i^{out}: outdegree\ of\ i\\ \frac{1}{2}\sum_{i=1}^{n}d_i^{tot}(2d_i^{tot}+1)\ , & if\ \ d_i^{tot}: total degree\ of\ i\end{cases}$$

Conclusion

- For any simple connected digraph with n>=2 vertices,

$$n-1 \quad \leq \quad LE(G) \quad \leq \quad n^2(n-1)$$

- Laplacian energy of Line digraph of complete graph K_n with n≥3, is

$$n(n-1)\left(n^2-2n+2\right)$$

- The class P(10) contains exactly 49 digraphs. More exactly 31 digraphs with n≤10, 8 directed cycles with n≤10, 10 directed paths with n≤ 11.