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Laplacian Energy of Directed Graphs and Minimizing Maximum Outdegree Algorithms

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Abstract

Energy has been studied in mathematical perspective as well as physical perspective for several years ago. In spectral graph theory, the eigenvalues of several kinds of matrices have been studied, of which Laplacian matrix attracted the greatest attention [5]. Recently, in 2009, Adiga considered Laplacian energy of directed graphs using skew Laplacian matrix, in which degree of vertex is considered as total of the out-degree and the in-degree. Since directed graphs play an important role in identifying the structure of web-graphs as well as communication graphs, we consider Laplacian energy of simple directed graphs and find some relations by using the general definition of Laplacian matrix. Unlike in [1], we derived two types of equations for simple directed graphs and symmetric directed graphs with $n \geq 2$ vertices by considering out-degree of vertex. Further we consider the class $P(\alpha)$ which consists of non isomorphic graphs with energy less than some α and find 47 non isomorphic directed graphs for class $P(10)$. Our objective extended to enumerate the structure of directed graphs using the Laplacian energy concept. Minimization maximum outdegree(MMO) algorithm defined in [3] can be used to find the directed graphs with minimum Laplacian energy.

Keywords: Laplacian energy, directed graph, MMO algorithms

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1. Introduction

The relation between Hückel theory and the theory of graph spectra was observed for a long time. The basic problem in Hückel theory is to deter-

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mine the eigenvalues and eigenvectors of the graph representing carbon atom connectivity of a given conjugated system. An interesting quantity is the sum of the energies of all the electrons in a molecule, called total π-electron energy [5, 6, 9, 4, 8]. Several criteria relate to energy such as energy change due to edge addition, maximal energy, equal energy has been considered in [2, 9, 4]. In spectral graph theory, the eigenvalues of several other matrices have been studied, of which Laplacian matrix attracted the greatest attention [5]. Therefore based on definitions of energy, in 2006, Gutman [8] defined Laplacian energy for undirected graph $G(n,m)$ as $LE_g(G) = \sum_{i=1}^{n} |\mu_i - 2m/n|$, and derive some lower bounds and upper bounds. He use axillary eigenvalues $\gamma_i, i = 1, ..., n$ defined by $\gamma_i = \mu_i - \frac{2m}{n}$ which satisfy $\sum_{i=1}^{n} \gamma_i = 0$ and $\sum_{i=1}^{n} \gamma_i^2 = 2M$ where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d(i) - \frac{2m}{n})^2$ and $d(i)$ is the degree of a vertex. The following bounds were obtained from the definition of $LE_g(G)$.

Theorem 1.1 (Gutman[8]).

$$LE_g(G) \leq \sqrt{2Mn} \tag{1}$$

Theorem 1.2 (Gutman[8]). *If G is a graph with one component then,*

$$LE_g(G) \leq \frac{2m}{n} + \sqrt{(n-1)(2M - (\frac{2m}{n})^2)} \tag{2}$$

Theorem 1.3 (Gutman[8]).

$$2\sqrt{M} \leq LE_g(G) \leq 2M \tag{3}$$

Further in [12], Total π-electron energy and Laplacian energy was compared. A similar problem for the usual Laplacian energy has been considered in [10] for undirected graphs using second spectral moment. According to [10] Laplacian energy was defined as $LE_k(G) = \sum_{i=1}^{n} \mu_i^2$ for eigenvalues μ_i of undirected Laplacian matrix $L = D - A$. Following results were obtained in [10].

Theorem 1.4 (Kragujevac[10]). *For any graph G on n vertices whose degree are $d(1), d(2), ..., d(n)$,*

$$LE_k(G) = \sum_{i=1}^{n} d(i)(d(i)+1) \tag{4}$$

Theorem 1.5 (Kragujevac[10]). *For any connected graph G on $n \geq 2$ vertices,*

$$LE_k(G) \geq 6n - 8. \tag{5}$$

Equality holds iff G is a path P_n on n vertices.

Theorem 1.6 (Kragujevac[10]). *For any $\alpha > 4$, the class $p(\alpha)$ of all non-isomorphic connected graphs with the property $LE_k(G) \leq \alpha$ is finite.*

By using eigenvalues of Laplacian matrix, Laplacian Estrada index was defined in [11] and derived some upper and lower boundaries. Further in 2009, Gutman found various relationships using incident energy [7]. In 2009, Adiga [1] introduce skew Laplacian energy for directed graphs as $SLE(G) = \sum_{i=1}^{n} \mu_i^2$, which is similar to [10]. Eigenvalues μ_i are the eigenvalues of skew Laplacian matrix $SL(G) = D - S(G)$ where $S(G)$ is the adjacency matrix with $s_{ij} = 1$ and $s_{ji} = -1$ whenever there is a arc from $i \to j$ and 0 otherwise. D is a diagonal matrix with $D(i,i) = d(i) = d^{out}(i) + d^{in}(i)$ where $d^{out}(i)$ is the outdegree and $d^{in}(i)$ is the indegree of vertex i. Later upper and lower bound for skew Laplacian energy $SLE(G) = \sum_{i} |\mu_i - \frac{2m}{n}|$ of simple directed graphs similar to equation (1), (2) and (3) are also derived as in the equation (7) in Theorem 1.7. Skew Laplacian eigenvalues of L satisfy the following relations.

$$\begin{aligned} \sum_{i=1}^{n} \mu_i &= 2m \\ \sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n} d(i)(d(i)-1) \end{aligned} \tag{6}$$

Theorem 1.7 (Adiga[1]).

$$\begin{aligned} SLE(G) &\leq \sqrt{2M_1 n} \\ SLE(G) &\leq k + \sqrt{(n-1)[2M_1 - k^2]} \\ 2\sqrt{M} \leq SLE(G) &\leq 2M_1 \end{aligned} \tag{7}$$

where $M_1 = M + 2m = m + \frac{1}{2}\sum_{i=1}^{n}(d(i) - \frac{2m}{n})^2.$

Most real world networks such as communication networks, web graphs etc. are directed graphs. Even though the energy concept originated in chemistry to find the energy of molecular structure, our objective is to investigate the applicability of the concept to find the structure of the web graphs. We enumerate graph structure in which Laplacian energy is less than some value. As an example, we consider the class $P(\alpha)$ and find the structure of directed graphs belong to the class $P(10)$. This paper is a small attempt to find the Laplacian energy and its behavior due to several criteria. We introduce notations and derive formulas for $LE(G)$, the Laplacian energy of a directed graph G by using Kirchoff matrix as in the Section 2. Then we find some relations between undirected and directed graphs of LE in Section 3. Finally we analyze the MMO algorithm [3] and discuss how it is useful to find the directed graphs with minimum Laplacian energy in Section 4.

2. Laplacian energy of directed graphs

A graph G, which has directed edge or arc is called a directed graph. Adjacency matrix A of G is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ whenever (v_i, v_j) is an directed edge and 0 otherwise. A directed graph having no multiple edges or self loops is called a *simple directed graph*. i.e., $a_{ij} \in \{0, 1\}$ and $a_{ij} = 1 \Rightarrow a_{ji} = 0$. A graph in which each edge is bidirected is called a *symmetric directed graph*. i.e., $a_{ij} = 1 \Rightarrow a_{ji} = 1$. Let $D = diag(d^{out}(1), d^{out}(2), d^{out}(3), ..., d^{out}(n))$ be diagonal matrix with outdegree of the vertices $v_1, v_2, ..., v_n$. Then we call $L(G) = D(G) - A(G)$, Laplacian matrix and its eigenvalues are denoted by $\{\mu_1, \mu_2, ..., \mu_n\}$. Since $L(G)$ is asymmetric matrix it does not give real eigenvalues always.

Definition 2.1. *Let $A(G)$ be the adjacency matrix of a directed graph G. Then Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{n} \mu_i^2$ where n is the order of G and $\mu_i, (i = 1, .., n)$ are the eigenvalues of the Laplacian matrix. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2)$ be two finite, directed graphs with disjoint sets of vertices $V(G1)$ and $V(G_2)$. Then the direct sum $G = G1 \oplus G2$ of these graphs is defined by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.*

Theorem 2.1. *If G is a disconnected directed graph with components $G_1, G_2, ..., G_n$,*

$$LE(G) = \sum_{i=1}^{n} LE(G_i). \tag{8}$$

Theorem 2.2. *Let G be a directed graph with vertex degrees $d^{out}(1), d^{out}(2), ..., d^{out}(n)$. Then the following relations are hold.*
If G is a simple directed graph then

$$LE(G) = \sum_{i=1}^{n} d^{out}(i)^2$$

If G is a symmetric directed graph then

$$LE(G) = \sum_{i=1}^{n} d^{out}(i)(d^{out}(i) + 1)$$

(Proof) Suppose G is a simple directed graph. Let D be a diagonal matrix with $D(i,i) = d^{out}(i)$for $i \in V$. If $i \to j$ is an arc then $a_{ij} = 1$ and $a_{ji} = 0$. From Viéte Rule, it is clear that $\sum_i \mu_i = Trace(L) = \sum_{i=1}^{n} d^{out}(i)$ and sum of the determinant of all the 2×2 principal sub matrices are $\sum_{i<j} \mu_i \mu_j$. i.e.,

$$\begin{aligned} \sum_{i<j} \mu_i \mu_j &= \sum_{i<j} det \begin{pmatrix} d^{out}(i) & -a_{ij} \\ 0 & d^{out}(j) \end{pmatrix} \\ &= \sum_{i<j} d^{out}(i) d^{out}(j) \end{aligned}$$

For every $i < j$,

$$\begin{aligned} \sum_{i \neq j} \mu_i \mu_j &= 2 \sum_{i<j} \mu_i \mu_j \\ &= \sum_{i \neq j} d^{out}(i) d^{out}(j) \\ &= \sum_{i \neq j} d^{out}(i) d^{out}(j) \end{aligned}$$

Therefore

$$\begin{aligned} LE(G) &= \sum_i \mu_i^2 = (\sum_i \mu_i)^2 - \sum_{i\neq j} \mu_i\mu_j \\ &= (\sum_i d^{out}(i))^2 - (\sum_{i\neq j} d^{out}(i)d^{out}(j)) \\ &= \sum_{i=1}^{n} (d^{out}(i))^2 \end{aligned} \tag{9}$$

If G is a symmetric directed graph then $d^{out}(i) = d^{in}(i) = d(i)$ for each node i. Hence Laplacian energy of symmetric directed graph is similar to the undirected graph as given by $LE(G_u) = \sum_{i=1}^{n} d(i)(d(i)+1)$ in [10]. We can replace $d(i)$ with $d^{out}(i)$ and obtained the result. □

Corollary 2.3. *For any directed graph G, its Laplacian energy $LE(G)$ is an integer.*

(Proof) Since degree of vertex is an integer we have integer values for $LE(G) = \sum_{i=1}^{n}(d^{out}(i))^2$ or $LE(G) = \sum_{i=1}^{n} d^{out}(i)(d^{out}(i)+1)$ for $i = 1, 2, ..., n$. □

Corollary 2.4. *The Laplacian energy of a simple directed path P_n with $n \geq 2$ is $(n-1)$.*

(Proof) Since every directed path P_n has exactly $(n-1)$ vertices with out-degree 1 and one vertex of degree 0, using theorem (2.2), we conclude that $LE(P_n) = (n-1)$. □

Corollary 2.5. *The Laplacian energy of a simple directed cycle C_n with $n \geq 3$ is n.*

(Proof) Since every vertex in C_n has out degree one, it follows from theorem (2.2) that $LE(G) = n$. □

Corollary 2.6. *For any connected directed graph on $n \geq 2$ vertices, we have*

$$n - 1 \leq LE(G) \leq n^2(n-1). \tag{10}$$

Moreover $LE(G) = n^2(n-1)$ if and only if G is a complete directed graph K_n and $LE(G) = n-1$ if and only if G is a directed path P_n on n vertices.

(Proof) Let G be a connected directed graph with $n(\geq 2)$ vertices. Maximum degree of any vertex is less than or equal to $(n-1)$. If G is a simple connected graph then $LE(G) = \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} (d^{out}(i))^2 < \sum_{i=1}^{n} d^{out}(i)(d^{out}(i) + 1)$.

If G is a symmetric directed graph then $LE(G) = \sum_{i=1}^{n} d^{out}(i)(d^{out}(i) + 1) \leq n^2(n-1)$. This implies that for any directed graph G, $LE(G) \leq n^2(n-1)$. Since each vertex of a complete directed graph has exactly $n-1$ degrees, it is clear that the maximum Laplacian energy of directed graphs with n vertices is achieved for the complete directed graph K_n.

We prove left side of inequality (10) by induction. As we know, to form a directed graph we need at least two nodes. Only connected graph which has two node is a simple or bi directed path. Since eigenvalues of $L(P_2) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ is 1 and 0 and $L(K_2) = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ is 2 and 0, the result is true for $n = 2$. Suppose the result is true for any connected directed graph with $n-1$ vertices. i.e., $LE(G) \geq n-2$. Then we need to prove the result for any arbitrary connected directed graph with n vertices. Let G be a connected directed graph with n vertices. Then, there is an induced subgraph $H \subset G$ on $n-1$ vertices which is also connected. Let $V(H) = \{v_1, v_2, ..., v_{n-1}\}$, $V(G) = V(H) \cup \{v_n\}$ and $LE(H) \geq n-2$. It is easy to show that $LE(G) \geq LE(H)+1$. So we have $LE(G) \geq n-1$. We can also prove that if G is a simple, connected directed graph with n vertices such that $LE(G) = n-1$, then G must be a directed path P_n. Suppose $LE(G) = n-1$. Let $n = 2$. Then $LE(G) = 1$. Since $LE(G) = \sum_{i=1}^{2} (d^{out}(i))^2$, we have $d^{out}(1) + d^{out}(2) = 1$. This happened when we have a one vertex with outdegree 1. That is there exists one directed edge between two nodes. Therefore G should be a directed path. $\square$

3. Relations between undirected graphs and directed graphs

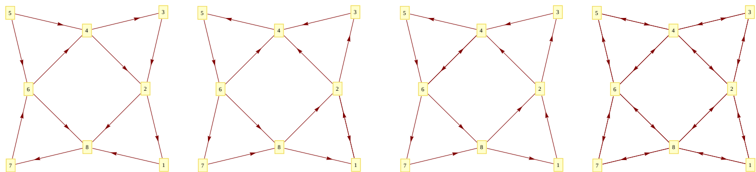
Every undirected graph can be converted to a directed graph by assigning directions. If each edge is replaced by two way directions then it is similar to the undirected graph.

Definition 3.1. *For a given directed graph $G_d = (V_d, E_d)$, we define an undirected graph $U(G_d) = (U(V_d), U(E_d))$ by $U(V_d) = V_u$ and $U(E_d) = \{\{v_1, v_2\} \mid (v_1, v_2) \in E_d$ or $(v_2, v_1) \in E_d\}$.*

Let $A(U(G)) = (a_{ij})$ be adjacency matrix of $U(G)$ and let $A'(G) = (a'_{ij})$ be adjacency matrix of G. Then

$$a_{ij} = \begin{cases} 1 & \text{if } a'_{ij} = 1 \text{ or } a'_{ji} = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 1. *Figure 1 shows four directed graphs $G_{d_1}, G_{d_2}, G_{d_3}, G_{d_4}$ with $U(G_{d_i}) = G_u$. The Laplacian energies $LE(G_{d_i})$ are different.*



(a) G_{d_1} (b) G_{d_2} (c) G_{d_3} (d) G_{d_4}

Figure 1: Representation of undirected to directed graphs

Theorem 3.1. *For any directed graph G_d, $LE(G_d) \leq LE_k(U(G_d))$*

(Proof) Let $G_d = (V_d, E_d)$ and $U(G_d) = (V_u, E_u)$. For any node $v \in V_d$, $d^{out}(v) \leq d(v)$.

$$\begin{aligned} LE_k(U(G_d)) &= \sum_{i=1}^{n} d(i)(d(i)+1) \\ &= \sum_{i=1}^{n} d(i)^2 + \sum_{i=1}^{n} d(i) \\ &\geq \sum_{i=1}^{n} (d^{out}(i))^2 + \sum_{i=1}^{n} d^{out}(i) \\ &\geq LE(G_d) \end{aligned}$$

Equality occur iff G_d is a symmetric directed graph. $\square$

Theorem 3.2. *Let G be a directed graph and $G' = G - e$ be a directed graph obtaining by deleting arc e. Then $LE(G') \leq LE(G)$.*

(Proof) Let $G = (V_d, E_d)$ be a directed graph with $|V_d| = n$. Let $H = (V_h, E_h)$ be edge induced sub graph with $|V_h| = n_1$ and $|E_h| = e$ edges. Define G' as $H \oplus (n - n_1)K_1$. Then Laplacian $L(G)$ of G is the $L(G - E_h) + L(G')$. $L(G')$ is a square matrix with maximum eigenvalue 1 and all other 0. Then $\sum_{i=1}^{n} \mu_i(G) - \sum_{i=1}^{n} \mu_i(G') = 1$. This implies that $\sum_{i=1}^{n} \mu_i(G) > \sum_{i=1}^{n} \mu_i(G')$ and exists at least one $\mu_i(G') < \mu_i(G)$. Hence prove the theorem. □

Proposition 3.1. *Let G_d be non-symmetric directed graph with $U(G_d) = G_u$. Then there exists a directed graph G'_d such that $U(G'_d) = G_u$ and $LE(G_d) < LE(G'_d)$.*

(Proof) Let G_d be non-symmetric directed graph. Then there exists $\{v_i, v_j\} \in E(G_u)$ and $\{v_i, v_j\} \notin E(G_d)$ or $\{v_j, v_i\} \notin E(G_d)$. Suppose $\{v_i, v_j\} \notin E(G_d)$. Let G'_d be a connected directed graph with $V(G'_d) = V(G_d)$, $E(G'_d) = E(G_d) \cup \{v_i, v_j\}$. Then $\sigma(G'_d) = G_u$ and by theorem 3.2, $LE(G'_d) > LE(G_d)$. By adding arcs for each node we can transform given non-symmetric graph to symmetric graph, which is identical to the undirected graph. □

Proposition 3.2. *Let G_d be a non-simple directed graph with $U(G_d) = G_u$. Then there exists a directed graph G'_d such that $U(G'_d) = G_u$ and $LE(G_d) > LE(G'_d)$.*

(Proof) Suppose G_d be a non-simple directed graph. Then there exists $\{v_i, v_j\} \in E(G_u)$ and $\{v_i, v_j\} \in E(G_d)$ and $\{v_j, v_i\} \in E(G_d)$. Suppose $\{v_i, v_j\} \in E(G_d)$. Let G'_d be a connected directed graph with $V(G'_d) = V(G_d)$ and $E(G'_d) = E(G_d) - \{v_i, v_j\}$. Then $U(G'_d) = G_u$ and by Theorem 3.2, $LE(G'_d) < LE(G_d)$. By deleting arcs from each node we can transform given non-simple graph to simple graph. □

Theorem 3.3. *Let $P(\alpha) = \{G \mid LE(G) \leq \alpha,\ G$ is a simple connected directed graph$\}$. For any $\alpha \geq 1$, the class $P(\alpha)$ of all non-isomorphic connected directed graphs with the property $LE(G) \leq \alpha$ is finite.*

(Proof) Let G be a directed graph with n vertices and m arcs such that $LE(G) \leq \alpha$. By Corollary 2.6, $n - 1 \leq LE(G) \leq \alpha$. Hence we obtain $n - 1 \leq \alpha$. Since n is finite, class $P(\alpha)$ is also finite. □

Corollary 3.4. *The class $P(10)$ contains exactly 47 directed graphs. More exactly 29 directed graphs with $n \leq 10$, 8 directed cycles with $n \leq 10$ and 10 directed paths with $n \leq 11$. Some of the graphs are listed in Figure 2.*

(Proof) Let $\alpha = 10$. Every simple connected directed graph with n vertices has at least $(n-1)$ arcs. Notice that for $n = 12$, $LE(G) \geq (n-1) = 11 > 10$. For $n = 11$, $LE(G) \geq 10$. Therefore all directed graphs from the class $P(10)$ have at most 11 vertices. Since $LE(P_n) = n - 1$ it has 10 directed path P_n with $n \leq 11$ and since $LE(C_n) = n$ it has 8 directed cycle with $n \leq 10$. $\square$

Theorem 3.5. *Let $G = (V_d, E_d)$ be a simple connected directed graph with $|V_d| = n$ and $|E_d| = m$. If $\triangle = \max\{d^{out}(v) \mid v \in V_d\}$ and $\delta = \min\{d^{out}(v) \mid v \in V_d\}$ then*

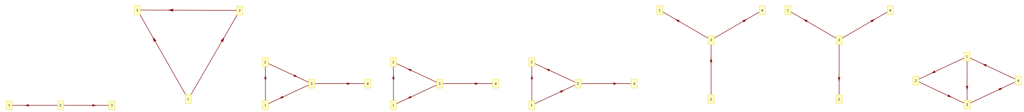
$$\frac{m^2}{n} \leq LE(G) \leq m(\triangle + \delta) - n\delta\triangle.$$

(Proof) By Cauchy Schwarz inequality, $LE(G) = \sum_{i=}^{n} \mu_i^2 = \sum_{i=1}^{n} (d^{out}(i))^2 \geq \frac{1}{n}(\sum_{i=1}^{n} d^{out}(i))^2 = \frac{m^2}{n}$. Lets consider the $(d^{out}(i) - \delta)(d^{out}(i) - \triangle)$. For all i, $(d^{out}(i) - \delta) \geq 0$ and $(d^{out}(i) - \triangle) \leq 0$. Therefore $(d^{out}(i) - \triangle)(d^{out}(i) - \delta) \leq 0, \forall i \in V_d$. Further $\sum_i (d^{out}(i) - \triangle)(d^{out}(i) - \delta) = \sum_i (d^{out}(i))^2 - (\triangle + \delta) \sum_i d^{out}(i) + n\delta\triangle \leq 0$. This shows that $\sum_i (d^{out}(i))^2 \leq m(\triangle + \delta) - n\delta\triangle \leq 0$. Hence $LE(G) \leq m(\triangle + \delta) - n\delta\triangle$. $\square$

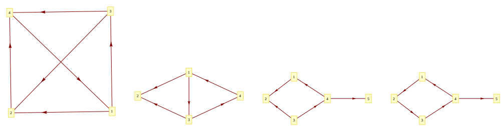
Remark: If $\triangle = \delta$ then $m = \sum_{i=1}^{n} d^{out}(i) = n\triangle$ and $m(\triangle + \delta) - n\delta\triangle$ $=2n \triangle^2 - n\triangle^2 = n\triangle^2 = \frac{m^2}{n}$.

4. Minimizing maximum outdegree algorithms

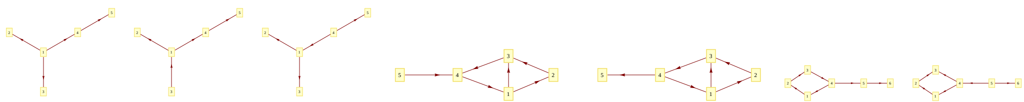
In this section we describe the relationship between minimum Laplacian energy of directed graphs and Minimizing maximum out degree algorithms called MMO algorithms [3].



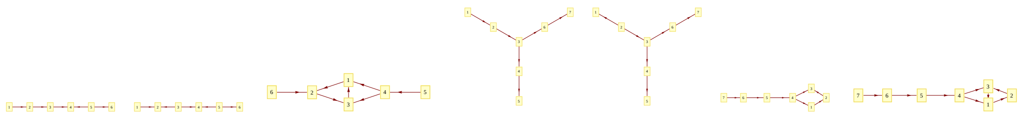
(a) $LE = 4$ (b) $LE = 5$ (c) $LE = 6$ (d) $LE = 10$ (e) $LE = 8$ (f) $LE = 5$ (g) $LE = 9$ (h) $LE = 7$



(i) $LE = 10$ (j) $LE = 9$ (k) $LE = 9$ (l) $LE = 7$



(m) $LE = 10$ (n) $LE = 6$ (o) $LE = 8$ (p) $LE = 8$ (q) $LE = 10$ (r) $LE = 8$ (s) $LE = 10$



(t) $LE = 7$ (u) $LE = 9$ (v) $LE = 9$ (w) $LE = 8$ (x) $LE = 10$ (y) $LE = 9$ (z) $LE = 10$

Figure 2: Directed graphs with nodes < 10

Definition 4.1. *Let G_u be an undirected graph. The optimal directed Laplacian energy $LE_{opt}(G_u)$ of G_u is defined by*
$LE_{opt}(G_u) = \min\{LE(G_d) \mid G_d \text{ is a directed graph and } U(G_d) = G_u\}$.

Definition 4.2. *For a directed graph $G_d = (V_d, E_d)$, the maximum out degree $\triangle(G_d)$ of G_d is defined by*
$\triangle(G_d) = \max\{d^{out}(v) \mid v \in V_d\}$.

Definition 4.3. *Let G_u be an undirected graph. The optimal maximum out-degree $\triangle_{opt}(G_u)$ of G_u is defined by*
$\triangle_{opt}(G_u) = \min\{\triangle(G_d) \mid G_d \text{ is a directed graph and } U(G_d) = G_u\}$.

Definition 4.4. *Let $G_d = (V_d, E_d)$ be a directed graph. We denote $v \to w$ if $(v, w) \in E_d$. We also denote $v \Rightarrow w$ if there exists $v_1, v_2, ..., v_k \in V_d, (k \geq 1)$ such that $v = v_1, v_1 \to v_2, .., v_{k-1} \to v_k$ and $v_k = w$.*

Proposition 4.1. *Let $G_u = (V_u, E_u)$ be an undirected graph and $G_d = (V_d, E_d)$ be a directed graph satisfying $U(G_d) = G_u$. If $\triangle(G_d) = p$ and $\{d^{out}(v) \mid v \in V_d\} = \{p, p-1\}$ then $LE(G_d) = LE_{opt}(G_u)$.*

(Proof) Let $k = |\{v \in V_d \mid d^{out}(v) = p\}|$ and $l = |\{v \in V_d \mid d^{out}(v) = p-1\}|$. Since $\sum\{d^{out}(v) \mid v \in V_d\} = |E_u|$, we have $pk + (p-1)l = |E_u|$. Since $k + l = |V_u|$, k and l is uniquely determined. $|E_u| = kp + l(p-1) = |V_u|p + (k - |V_u|)$ and $|V_u|p = |E_u| + |V_u| - k$. Hence $p = \frac{|E_u|}{|V_u|} + 1 - \frac{k}{|V_u|}$. Since $1 \leq k \leq |V_u|$ then $0 \leq 1 - \frac{k}{|V_u|} < 1$. So we have $p = \lceil \frac{|E_u|}{|V_u|} \rceil$.
Let $h(x_1, x_2, ..., x_n, \lambda) = \sum_{i=1}^{n} x_i^2 + 2\lambda(\sum_{i=1}^{n} x_i - |E_u|)$. Since $\frac{\partial h}{\partial x_i} = 2x_i + 2\lambda = 0$ and $\frac{\partial h}{\partial \lambda} = \sum_{i=1}^{n} x_i - |E_u| = 0$, the function h is minimum at $x_i = -\lambda, (i = 1, .., n)$ and $\sum_{i=1}^{n} x_i = |E_u|$. Since $\displaystyle\sum_{i=1}^{n} x_i = -|V_u|\lambda = |E_u|$, we have $\lambda = -\frac{|E_u|}{|V_u|}$ and $x_i = \frac{|E_u|}{|V_u|}, (i = 1, ..., n)$. If all x_i's are integer, the function h have minimum value with $x_i \in \{\lceil \frac{|E_u|}{|V_u|} \rceil, \lceil \frac{|E_u|}{|V_u|} - 1 \rceil\}$ *for* $(i = 1, ..., n)$. We consider $x_i = d^{out}(v_i)$. Then $h(x_1, x_2, ..., x_n, \lambda) = LE(G_d)$. So $LE(G_d) = kp^2 + l(p-1)^2$ gives the optimal solution. □

Finding the orientation of simple graph by minimizing maximum out degree of a node is studied in literature [13, 3] and defined as MMO (Minimizing maximum out degree) algorithms. In order to minimize the maximum out degree and find a optimal solution to MMO problems, [3] use simple algorithm called reverse algorithm as in Table 4.

Input	An undirected graph $G_u = (V_u, E_u)$
Output	Oriented graph $MMO(G_u) = G_d = (V_d, E_d)$
Step 1:	Set $E_d = \emptyset$
Step 2:	Find arbitrary orientation and update E_d
Step 3:	Compute out degree $d^{out}(v)$ for each $v \in V_d$. Let u be $\max\{d^{out}(v) \vert v \in V_d\}$
Step 4:	Find a directed path $P = u \rightarrow v_1 \rightarrow ... \rightarrow v_k$ of length $k(k \geq 1)$ which satisfy $d^{out}(v_i) \leq d^{out}(u), \forall 1 \leq i \leq k-1$ and $d^{out}(v_k) \leq d^{out}(u) - 2$ If such P exists then set $E_d = E_d \setminus \{P \cup \bar{P}\}$, where $\bar{P} = v_k \rightarrow ... \rightarrow v_1 \rightarrow u$ and goto Step 2. Otherwise halt.

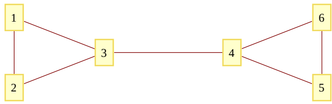
Table 1: MMO Algorithm

Reverse path cause to reduce the maximum outdegree by one and increase the outdegree of terminal vertex by one. It is proved in [3] that if $G_d = MMO(G_u)$ then $\triangle_{opt}(G_u) = \triangle(G_d)$.

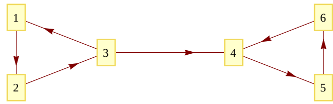
Example 2. *In Figure 3, we demonstrate two directed graphs G_{d_1} and G_{d_2} with $U(G_{d_1}) = U(G_{d_2}) = G_u$. The maximum outdegree of G_{d_1} and G_{d_2} are same. But $LE(G_{d_1}) \neq LE(G_{d_2})$. In Figure 4 we demonstrate four directed graphs $G_{d_1}, G_{d_2}, G_{d_3}, G_{d_4}$ with $U(G_{d_i}) = G_u$ and $LE(G_{d_i}) = LE_{opt}(G_u), (i = 1, 2, 3, 4)$. We can see $\{d^{out}(v) \mid v \in V_{d_i}\} = \{1, 2\}$ and $|\{v \in V_{d_i} \mid d^{out}(v) = 1\}| = |\{v \in V_{d_j} \mid d^{out}(v) = 1\}|$ and $|\{v \in V_{d_i} \mid d^{out}(v) = 2\}| = |\{v \in V_{d_j} \mid d^{out}(v) = 2\}|$ for $(i, j = 1, 2, 3, 4)$.*

Proposition 4.2. *Let $G_u = (V_u, E_u)$ be undirected graph and $G_d = (V_d, E_d) = MMO(G_u)$. Let $v_0 \in V_d$ be a vertex with $d^{out}(v_0) = \triangle(G_d)$. Define $V' = \{w \in V_d \mid v_0 \Rightarrow w\}$. $G'_u = (V', E'_u)$ be the induced undirected graph where $E'_u = \{\{u, v\} \in E_u \mid u, v \in V'\}$ and $G'_d = (V', E'_d)$ be the directed graph where $E'_d = \{(u, v) \in E_d \mid u, v \in V'\}$. Then the following relations are hold.*

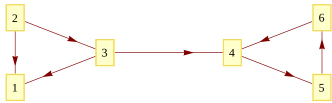
1. $d^{out}(v_0) \geq d^{out}(v') \geq d^{out}(v_0) - 1$ *for* $v' \in V'$.
2. $LE(G'_d) = LE_{opt}(G'_u)$



(a) G_u



(b) $LE(G_{d_1}) = 9$



(c) $LE(G_{d_2}) = 11$

Figure 3: Equal maximum degree and different Laplacian energy

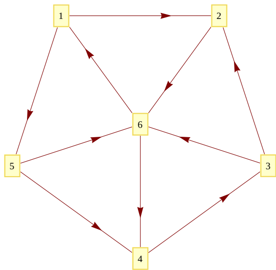
(Proof)

1. Let $v' \in V'$. Then we have a path $P = v_0 \to ... \to v'$ from v_0 *to* v'. Since $d^{out}(v_0) = \triangle(G_d)$, we have $d^{out}(v') \leq d^{out}(v_0)$. Further $G_d = MMO(G_u)$ imply that $d^{out}(v') > d^{out}(v_0) - 2$. Therefore $d^{out}(v') \geq d^{out}(v_0) - 1$.
2. From (1) we have $\{d^{out}(v) \mid v \in V'\} = \{d^{out}(v_0), d^{out}(v_0) - 1\}$.
   By Proposition 4.1, we have $LE(G'_d) = LE_{opt}(G'_u)$. □

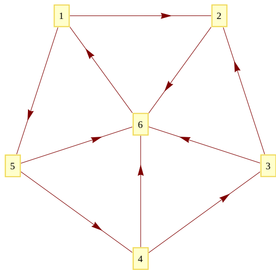
Theorem 4.1. *Let $G_u = (V_u, E_u)$ be an undirected graph and $G_d = (V_d, E_d) = MMO(G_u)$. Let $p = \triangle(G_d), V_p = \{v \in V_d \mid d^{out}(v) = p\}$ and $V_1 = \{w \in V_d \mid v_p \in V_p, v_p \Rightarrow w\}$. If $V_u = V_1$ then $LE(G_d) = LE_{opt}(G_u)$.*

(Proof) Since $G_d = MMO(G_u)$ then $\{d^{out}(v) \mid v \in V_1\} = \{p, p-1\}$. If $V_u = V_1$ then we have $LE(G_d) = LE_{opt}(G_u)$ by Proposition 4.1. □

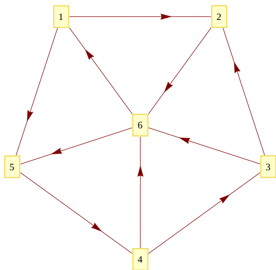
Example 3. *In figure 5(a) $MMO(G_u) = (V_d, E_d)$ and $\{d^{out}(v) \mid v \in V_d\} = \{3, 2, 1\}$. $p = \triangle(G_d) = 3, V_p = \{9, 11, 12\}, V_1 = \{9, 10, 11, 12, 13, 14\}$. So we cannot apply Theorem 4.1. In Figure 5(b) $MMO(G_u) = (V_d, E_d)$ and $\{d^{out}(v) \mid v \in V_d\} = \{2, 1\}$. We can see $p = \triangle(G_d) = 2$, $V_p = \{1, 2, 3, 4, 5, 6\}$ and $V_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then we have $LE(MMO(G_u)) = LE_{opt}(G_u)$ from the Theorem 4.1.*



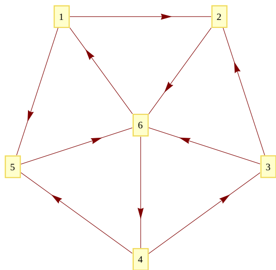
(a) G_{d_1}



(b) G_{d_2}



(c) G_{d_3}



(d) G_{d_4}

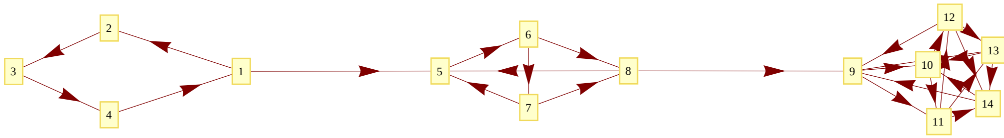
Figure 4: Oriented graphs with minimum Laplacian energy

5. Conclusion

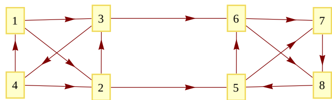
We build relations on Laplacian energy of directed graphs. Then we enumerated the stucture of the graphs whose Laplacian energy is less than some α value. Further we considered relationship between MMO algorithms and Laplacian energy. It is remained for the future to further analysis whether MMO algorithms always gives the optimal solution for minimum Laplacian enegy.

6. Acknowledgment

The authors thank Dr.Tetsuji Taniguchi for his valuable suggestions and discussions. This work has been partially supported by Kyushu University



(a) $MMO(G_u)$



(b) $MMO(G_u)$

Figure 5: Graphs consists of optimal Laplacian Energy

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