LAPLACIAN MATRIX AND APPLICATIONS

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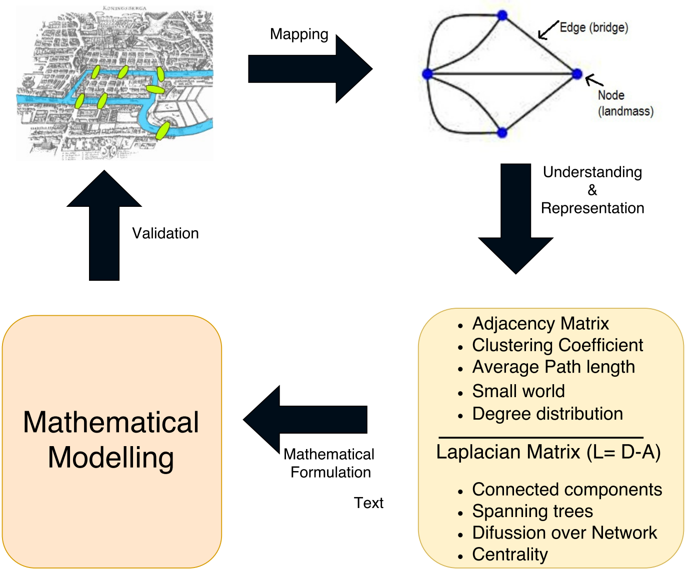
Complex Systems; Complex Network/Large graph Approach



Intuition of Networks

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Introduction to Networks

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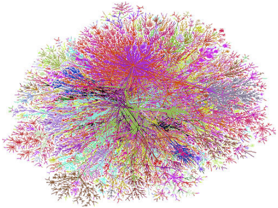
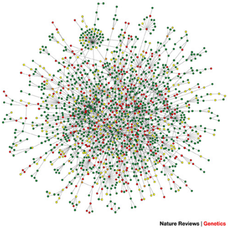
Formal Definition

A network, G, is a pair (V, E). Where V is the set of vertices (nodes) of G and E is the set of edges (links) of G. (Estrada & Knight). Categories of networks include simple networks, directed networks, undirected networks, weighted networks, etc (Estrada, 2015)

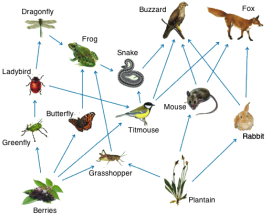
Real-world Networks



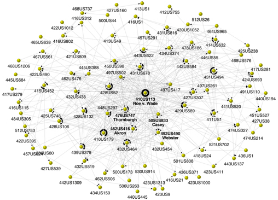
(a) Internet



(b) Protein-Protein



(c) Food web



(d) Citation network

Source: www.wikipedia.com

Laplacian Matrix

Definition

Consider a simple undirected network, the Laplacian matrix L is the difference between the Degree matrix D and Adjacency matrix A i.e $L = D - A$. The entries of L are given as

$$L_{i,j} = \begin{cases} k_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j \\ 0 & \text{otherwise,} \end{cases}$$

where k_i denotes the degree of node i (Estrada, 2011).

Spectrum of the Laplacian Matrix

Spectrum

Spectrum of a matrix is a set eigenvalues and their multiplicities. Let λ_i denote the eigenvalues of the Laplacian matrix. Considering the nondecreasing order: $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1 = 0$

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Insights from spectrum

- The multiplicity of 0 as an eigenvalue of $\mathbf{L}$ is equal to the number of connected components in the network.
- A network, G, is connected if its second smallest eigenvalue is nonzero. That is, $\lambda_2 > 0$ if and only if G is connected. The eigenvalue λ_2 is thus called the algebraic connectivity of a network, $a(G)$ (Estrada, 2011).

Applications of Laplacian Matrix

- Centrality measure
- Diffusion on network
- Consensus in multi-agent systems
- Synchronization

Centrality Measures

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- Subgraph centrality: Participation of a node in subgraphs in network
- Laplacian centrality : Impact of deactivation of a node from the network.

Laplacian Centrality

- Work presented is based on the paper: Laplacian centrality: A new centrality measure for weighted networks by Qi et al., 2012.

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- Standard centrality measures (degree, closeness, betweenness) have been extended to cater for weighted networks, however, these measures either capture the local or global characterisation of networks (T.Opsahl '2009, Newman '2001, A.Barrat '2004, U.Brandes '2001).

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- The Laplacian centrality is a measure between local and global (i.e intermediate) characterisation of the centrality of a node.

Laplacian Centrality Cont....

Laplacian Energy of a Network

The importance of a node is determined by the ability of the network to respond to the deactivation of the node from the network. The response is quantified by the relative drop in Laplacian energy (E_L) of the network (Qi et al., 2012).

$$E_L(G) = \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{i<j} w_{i,j}^2, \tag{1}$$

where $x_i^{\prime}s$ are vertex sums and w_{ij} are weights of edges between vertices i and j (Qi et al., 2012).

Mathematical Formulation of Laplacian Centrality

Mathematically, Laplacian centrality for a node i in network G is given by (Qi et al., 2012)

$$C_L(v_i, G) = \frac{(\Delta E)_i}{E_L(G)} = \frac{E_L(G) - E_L(G_i)}{E_L(G)}, \tag{2}$$

where

$E_L(G)$ - Laplacian energy of network G.

$E_L(G_i)$ - Laplacian energy of network G on removal of node i

Graph Theoretical Interpretation of Laplacian Centrality

Expressing Equation 2 in terms of 2-walks of the node i gives

$$(\Delta E)_i = 2 \cdot NW_2^M(v_i) + 2 \cdot NW_2^E(v_i) + 4 \cdot NW_2^C(v_i), \tag{3}$$

where $NW_2^C(v_i)$, $NW_2^E(v_i)$, and $NW_2^M(v_i)$ are closed 2-walks containing vertex v_i, non-closed 2-walks with vertex v_i as one of the end points and non-closed 2-walks with vertex v_i as the middle point (Qi et al., 2012).

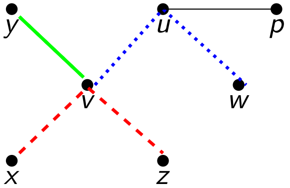

Figure: 2-walks at node v

Application to real-world networks

Zachary's Karate Network

- **The Zachary's Karate Network** was created from a dataset formed by observation of 34 members of a karate club over two years. Misunderstandings within the group led to a split into two groups, one led by the Administrator (1) and the other by the instructor (34).
- Nodes represent players in both groups while edges represent interactions outside karate activities.
- The weights on the edges correspond to different aspects of interactions between players
- Database Source: `http://nexus.igraph.org/api/dataset_info?id=1&format=html`

Zachary's Karate Network cont...

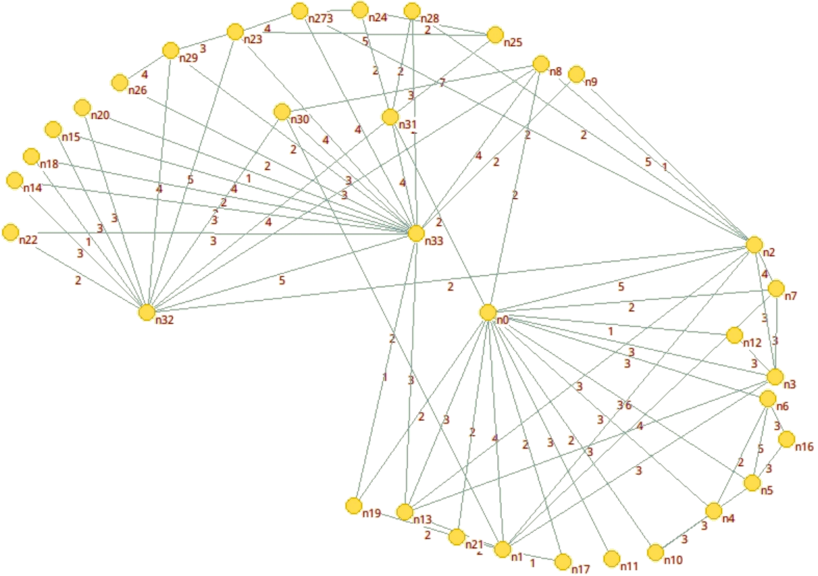

Figure: Zachary's Karate Network

Centrality rankings of the Zachary's Karate network

	Scores				Ranks			
Node	Degree	Betweenness	Closeness	Laplacian	Degree	Betweenness	Closeness	Laplacian
$\mathbf{n_0}$	**42**	**250.15**	**0.2538**	**0.2544**	**2**	**1**	**1**	**2**
n_1	29	33.80	0.2000	0.1725	5	8	8	5
n_2	33	36.65	0.1964	0.2166	4	6	11	4
n_3	18	1.33	0.1765	0.0965	8	18	17	10
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n_{18}	3	3.00	0.1875	0.0226	31	16	15	29
$\mathbf{n_{19}}$	**5**	**127.07**	**0.2481**	**0.0331**	**25**	**3**	**3**	**23**
n_{20}	4	0.00	0.2037	0.0280	28	24	6	26
n_{21}	4	0.00	0.1765	0.0246	28	24	17	27
n_{22}	5	0.00	0.1587	0.0382	25	24	24	19
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n_{31}	21	66.33	0.2089	0.1310	6	4	4	6
n_{32}	38	38.13	0.2000	0.2371	3	5	8	3
$\mathbf{n_{33}}$	**48**	**209.50**	**0.2519**	**0.3067**	**1**	**2**	**2**	**1**

Table: The scores and ranks based on four centrality measures for the Zachary's karate club network.

Interpretations of Results

- Code in Python to compute the Laplacian Centralities for nodes as shown in the table. `https://docs.google.com/a/aims.ac.za/viewer?a=v&pid=sites&srcid=YWltcy5hYy56YXxhcmNoaXZlfGd4OjcyYjZkNjJmN2ExNmI0YjQ`

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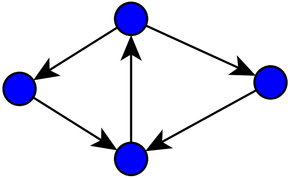
- For all the other centralities mentioned earlier and the laplacian centrality, both the administrator and Instructor scored highly.

- There is a good positive correlation between the degree and the laplacian centralities.

Possible Extension of Laplacian Centrality

- How the story will be with directed networks?



$$L = D_{out} - A$$

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

- To begin with, the laplacian matrix of the directed network is not symmetric. This perhaps brings in a twist in the whole story.

Diffusion on networks

Diffusion is a process by which information, epidermic, viruses, and any other behaviours spread over networks [?]. Take a simple undirected connected network. Consider a quantity of substance ϕ_i (heat) at each node i at time t. The diffusion of heat over the network is given by

$$\frac{d\phi_i}{dt} = C \sum_j A_{ij}(\phi_j - \phi_i) \tag{4}$$

In matrix notation,

$$\frac{d\boldsymbol{\phi}}{dt} + C\mathbf{L}\boldsymbol{\phi} = 0, \quad \boldsymbol{\phi}(0) = \boldsymbol{\phi}_0 \tag{5}$$

whose solution

$$\boldsymbol{\phi}(t) = \boldsymbol{\phi}_0 \; e^{-C\mathbf{L}t} \tag{6}$$

Equilibrium behaviour

As time t goes to infinity, the equilibrium state is completely determined by the **kernel of L**. The quantity of heat $\phi_j(t)$ at any node j at time t is given by

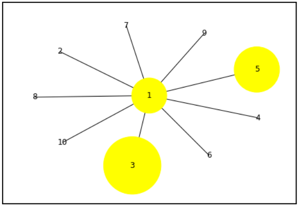
$$\lim_{t \to \infty} \phi_j(t) = \frac{1}{n} \sum_{i=1}^{n} \phi_i(0).$$

NOTE:

The structure of the network has no influence over the equilibrium value but plays a role in influencing the rate at which diffusion occurs.

Illustration of diffusion over a simple network

Suppose we assign to each node heat quantities given by $\phi_0 = [2, 0, 8, 0, 5, 0, 0, 0, 0, 0]$ in order node 1 to 10. Let $C = 1$.



(a) $t = 0$

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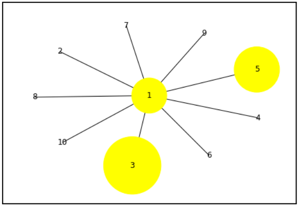

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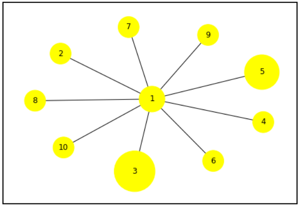

(b) $t = 1$

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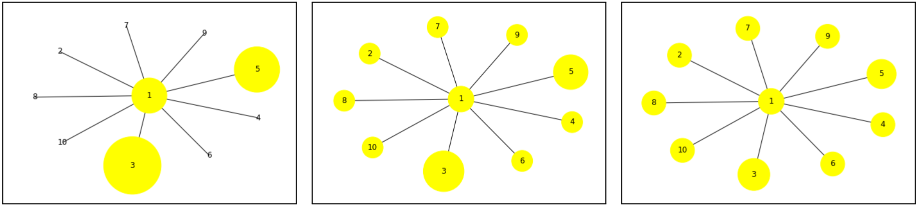

(a) $t = 0$ (b) $t = 1$ (c) $t = 2$

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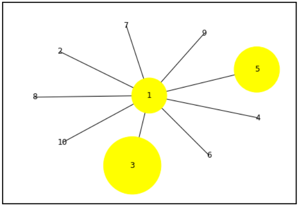

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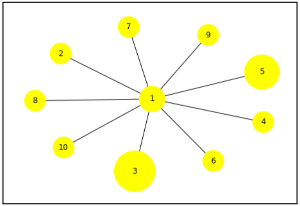
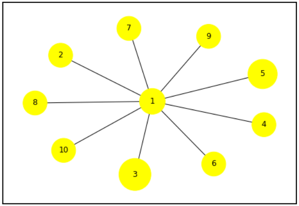
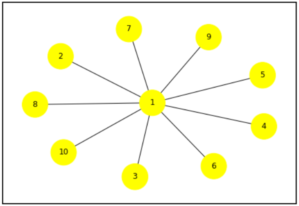


(b) $t = 1$



(c) $t = 2$



(d) $t = 5$

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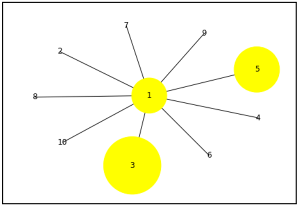

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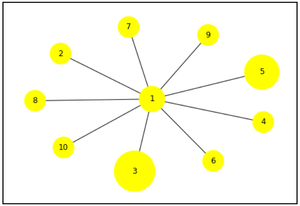
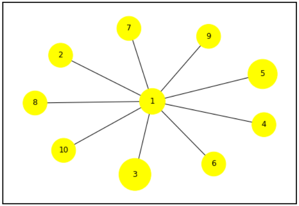
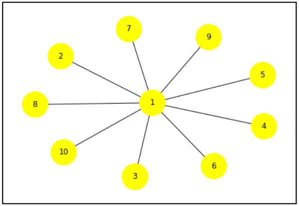


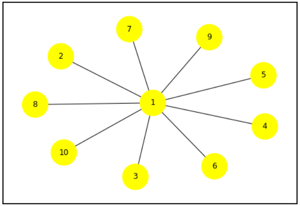
(b) $t = 1$



(c) $t = 2$



(d) $t = 5$



(e) $t = 7$

Illustration of diffusion over a simple network

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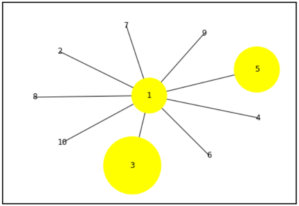
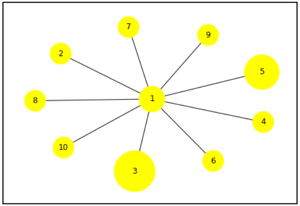
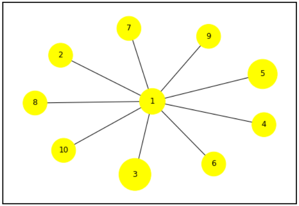


(a) $t = 0$



(b) $t = 1$



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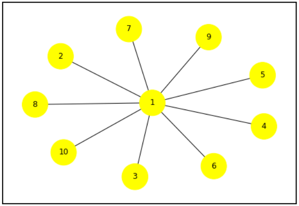

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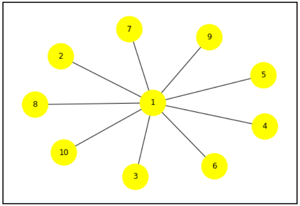
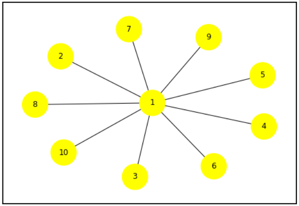
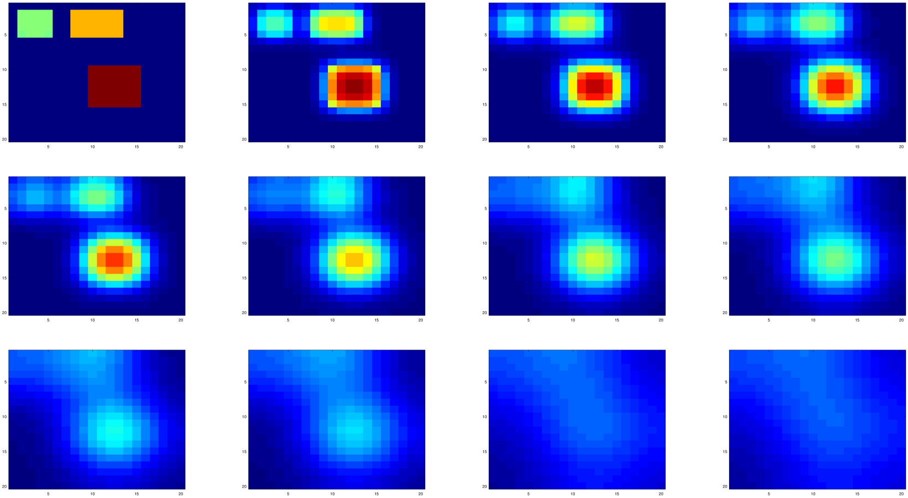


(e) $t = 7$



(f) $t = 9$

Diffusion on a Lattice



Animation: `www.wikipedia.com/laplacian_matrix`

Summary

- Overview of the Network theory approach to the study of complex systems
- Representation of Network by the Laplacian Matrix
- Application of the Laplacian Matrix
  - Laplacian centrality for directed weighted networks
  - Possible extension of laplacian centrality to directed networks
  - Diffusion over networks based on direct interactions between connected nodes
  - Consideration of non direct interactions in diffusion process on networks


DANKE
SCHÖN