3 Laplacian Centrality of weighted Networks

The centrality of a node is a measure of how important or central a node is with in a network. A variety of centrality measures have been introduced for undirected unweighted networks based on various definitions of importance of a node. These include: degree, closeness, betweenness, eigenvector and subgraph centralities. Standard centrality measures i.e degree, closeness, and betweenness were extended to weighted networks due to the fact that weighted networks provide more information about the network and therefore measures applied to these networks are of great importance. These standard centrality measures give information on either the local environment of a node (i.e degree centrality) or the global position of the node in the network (i.e closeness, betweenness and subgraph centralities). This implies that information about the intermediate (between local and global) environment of a node cannot be captured by any of the standard centralities, yet, such information is very useful in the study of real-world networks. For instance, quantifying the relative importance of a particular actor in a social network. It is for this reason that a new type of centrality known as the Laplacian centrality was introduced by Qui.,et al.

With Laplacian centrality measure, the importance of a node is determined by the ability of the network to respond to the deactivation of the node from the network. In other words, it is a measure of the relative drop of Laplacian energy in the network due to the removal (or deactivation) of the node from the network. The drop of Laplacian energy with respect to node v is determined by the number of 2-walks that v participates in the network.

3.1 Laplacian energy of a network

Let $G = (V, E, W)$ be a simple undirected weighted network with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, edge set E, where each edge $e = (v_i, v_j)$ is attached with a weight w_{ij}. If there is no edge between v_i and v_j, $w_{i,j} = 0$. In addition, $w_{i,i} = 0$ and $w_{i,j} = w_{j,i}$. We define

$$\mathbf{W}(\mathbf{G}) = \begin{pmatrix} 0 & w_{1,2} & \dots & w_{1,n} \\ w_{2,1} & 0 & \dots & w_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ w_{n,1} & w_{n,2} & \dots & 0 \end{pmatrix} \text{ and } \mathbf{X}(\mathbf{G}) = \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix},$$

where x_i is the sum-weight of vertex v_i given by $x_i = \sum_{j=1}^{n} w_{i,j} = \sum_{u \in N(v_i)} w_{v_i,u}$, where $N(v_i)$ is the neighborhood of v_i.

Definition 1 (Weighted Laplacian matrix) *The Laplacian matrix of a weighted network G is the matrix $\mathbf{L}(\mathbf{G}) = \mathbf{X}(\mathbf{G}) - \mathbf{W}(\mathbf{G})$.*

Definition 2 (Laplacian Energy of a network) *Let $G = (V, E, W)$ be a weighted network on n vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of its Laplacian matrix. The Laplacian energy of G is defined as*

$$E_L(G) = \sum_{i=1}^{n} \lambda_i^2.$$

As networks become larger, computing eigenvalues of the Laplacian matrix becomes very hard. We therefore, use the entries of the Laplacian matrix rather than its eigenvalues to compute the Laplacian energy of a network as given by Theorem 1.

Theorem 1 *For any network $G = (V, E, W)$ on n vertices whose vertex sum-weights are $x_1, x_2, \ldots, x_n$ respectively, we have*

$$E_L(G) = \sum_{i=1}^{n} x_i^2 + 2 \sum_{i<j} w_{i,j}^2. \tag{24}$$

3.2 Laplacian centrality of node

If $G = (V, E, W)$ is a network with n nodes $V = \{v_1, v_2, \ldots, v_n\}$. Let G_i be the network obtained by deleting v_i from G. The Laplacian centrality is given by

$$C_L(v_i, G) = \frac{(\Delta E)_i}{E_L(G)} = \frac{E_L(G) - E_L(G_i)}{E_L(G)} \tag{25}$$

For any vertex v, the denominator remains unchanged and from Corollary **??**, we can tell that $E_L(G) - E_L(G_i)$ is non-negative. We then focus on obtaining the expression for $(\Delta E)_i$. In order to obtain the graph theoretical descriptions of Laplacian centrality, we will study the k-walks (discussed in Chapter 2) for the weighted graph, specifically, for $k = 2$. For better understanding of the weighted network concept, we represent a weighted network as an unweighted multigraph network by replacing each edge $e = (v_i, v_j)$ with w_ij copies of multiedges as shown in Fig. **??**(**??**). For instance, for a 2-walk $v_1v_2v_3$ in a weighted network, the number of 2-walks in its corresponding unweighted network is $w_{v_1,v_2}w_{v_2,v_3}$.

Theorem 2 *Let $G = (V, E, W)$ be a weighted network of n vertices $v_1, v_2, \ldots, v_n$. Let G_i be the network obtained by deleting vertex v from G, then the drop of Laplacian energy with respect to v_i is*

$$(\Delta E)_i = E_L(G) - E_L(G_i) = 4 \cdot NW_2^C(v_i) + 2 \cdot NW_2^E(v_i) + 2 \cdot NW_2^M(v_i). \tag{26}$$

where $NW_2^C(v_i)$, $NW_2^E(v_i)$, and $NW_2^M(v_i)$ are closed 2-walks containing vertex v_i, non-closed 2-walks with vertex v_i as one of the end points and non-closed 2-walks with vertex v_i as the middle point (Qi et al., 2012).

3.3 Laplacian Centrality of an edge

The identification of the most central nodes in a network is an important concept that has applications in the control of spread of an epidermic, faster dissemination of information with in a social networks, etc. On the other hand, we believe that it is also vital to identify how central an edge is with in the network. this therefore prompts us to study the Laplacian centrality centrality of an edge and define a graph theoretical expression for the drop in energy when an edge is removed from the network.

Similar to laplacian centrality of a node, we define the Laplacian centrality of an edge as the drop in Laplacian energy when an edge is removed from a network. Let us consider an undirected weighted network $G = (V, E)$. The Laplacian energy of G is given by

$$E_L(G) = \sum_{i=1}^{n} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2, \tag{27}$$

where $x(v_i) = \sum_{j \in N(v_i)} w_{i,j}$.

On removing an arbitrary edge $e_{1,2}$, without loss of generality, assume $H = G - e_{1,2}$. Let $N(v_i)$ be the neighborhood of vertex v_i in G, $v_i \in e_{i,j}$ represent that edge $e_{i,j}$ is incident to vertex v_i in G, and $x'(v_i)$ be the corresponding sum-weight of the vertex v_i in H. We have:

$$x'(v_i) = \begin{cases} x(v_i) - w_{v_1,v_2} & \text{if } v_i \sim e_{1,2}, \\ x(v_i) & \text{otherwise.} \end{cases} \tag{28}$$

The Laplacian energy of the subgraph is given by

$$E_L(H) = \sum_{v_i \sim e_{1,2}} (x(v_i) - w_{v_1,v_2})^2 + \sum_{v_i \nsim e_{1,2}} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2 - 2 \cdot w_{v_1,v_2}^2 \tag{29}$$

Definition 3 (Laplacian centrality of an edge, $C_L(e)$) *The Laplacian centrality of an edge e, in Graph G is given by*

$$C_L(e) = \frac{E_L(G) - E_L(H)}{E_L(G)} = \frac{\Delta E_L}{E_L(G)} \tag{30}$$

From (30), we observe the denominators remain the same in computing the Laplacian centrality for edges. This then directs our attention to obtaining the graph theoretical descriptions of the drop in the Laplacian energy when a given node is removed from the graph.

Following a similar procedure in the proof for Theorem 2, the drop in Laplacian energy which the difference between (27) and (29) is given by

$$
\begin{aligned}
\Delta E &= \sum_{i=1}^{n} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2 - (\sum_{v_i \sim e_{1,2}} (x(v_i) - w_{v_1,v_2}^2)^2 + \sum_{v_i \nsim e_{1,2}} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2 - 2 \cdot w_{v_1,v_2}^2) \\
&= \sum_{i=1}^{n} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2 - \Big(\sum_{v_i \sim e_{1,2}} (x^2(v_i) - 2x(v_i) \cdot w_{v_1,v_2} + w_{v_1,v_2}^2) + \sum_{v_i \nsim e_{1,2}} x^2(v_i) + 2\sum_{i<j} w_{i,j}^2 - 2 \cdot w_{v_1,v_2}^2 \Big) \\
&= 2 \sum_{v_i \sim e_{ij}} x(v_i).w_{v_1,v_2} - \sum_{v_i \sim e_{ij}} w_{1,2}^2 + 2 \cdot w_{v_1,v_2}^2 \\
&= 2 \sum_{v_i \sim e_{i,j}} (x(v_i) \cdot w_{v_1,v_2}) - 2 \cdot w_{v_1,v_2}^2 + 2 \cdot w_{v_1,v_2}^2 \\
&= 2 \sum_{v_i \sim e_{i,j}} x(v_i) \cdot w_{v_1,v_2} \\
&= 2 \sum_{v_i \sim e_{i,j}} w_{v_1,v_2} \sum_{u \in N(v_i)} w_{v_i,u} \\
&= 2 \cdot \sum_{v_1} w_{v_2,v_1} \sum_{u \in N(v_1); u \neq v_2} w_{v_1,u} + 2 \cdot \sum_{v_2} w_{v_1,v_2} \sum_{u \in N(v_2); u \neq v_1} w_{v_2,u} + 2 \cdot w_{v_1,v_2} \cdot w_{v_1,v_2} + 2 \cdot w_{v_2,v_1} \cdot w_{v_2,v_1}
\end{aligned}
$$

$$\Delta E = 2 \cdot NW_2^U(v_1(E), v_2(M)) + 2 \cdot NW_2^U(v_1(E), v_2(M)) + 4 \cdot NW_2^C(v_1, v_2) \tag{31}$$

where

$NW_2^U(v_1(E), v_2(M))$ is the number of non-closed walks of length 2 with vertex v_2 as the middle point and vertex v_1 as an end point.

$NW_2^U(v_1(E), v_2(M))$ is the number of non-closed walks of length 2 with vertex v_1 as the middle point and vertex v_2 as an end point.

$NW_2^C(v_1, v_2)$ is the number of closed walks of length 2 between vertices v_1 and v_2.

From (31), we can easily tell the energy drop can be obtained by taking into account the immediate neighbourhood around the edge, that is, the nearest neighbours of the two nodes that are incident to the edge in question.

Example 1 *Let us consider the weighted graph in Fig. 1. We compute the drop in Laplacian energy by First, we compute the Laplacian energy $E(G)$ of the graph as follows:*

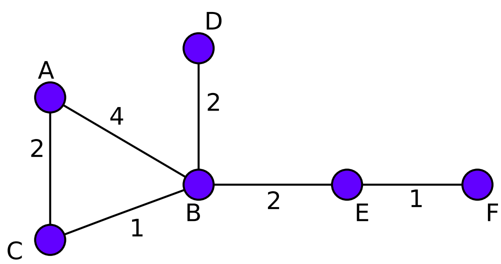


Figure 1

$$
\begin{aligned}
E_L(G) &= \sum_{i=1}^{n} x_i^2 + 2\sum_{i<j} w_{i,j}^2 \\
&= (6^2 + 3^2 + 9^2 + 3^2 + 1^2 + 2^2) + 2(4^2 + 2^2 + 1^2 + 2^2 + 2 + 1^2) \\
&= 200
\end{aligned}
$$

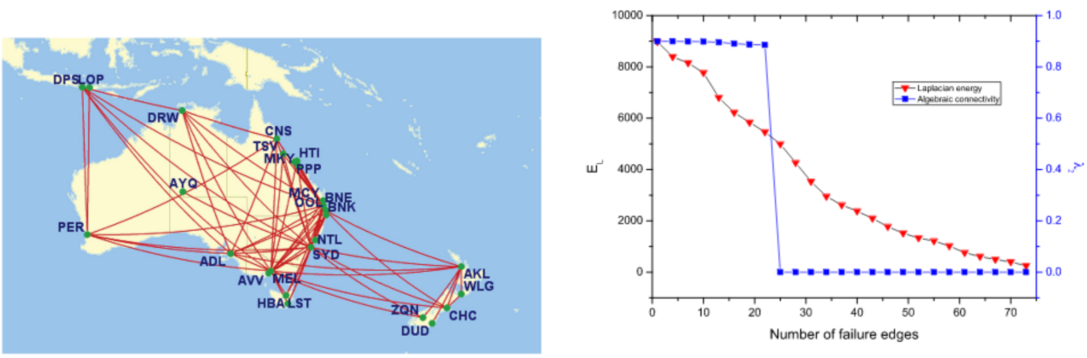
Table 1: Laplacian Centralities of edges

Edge(e)	$E(H); H = G - e$	$E(G) - E(H)$	$\Delta(E)$ by walks method in (31)
a,c	164	$200 - 164 = 36$	$2(2) + 2(8) + 4(4) = 36$
b,c	176	$200 - 176 = 24$	$2(2 + 2 + 4) + 2(2) + 4(1) = 24$
a,b	80	$200 - 80 = 120$	$2(8) + 2(4 + 8 + 8) + 2(16) = 120$
b,d	156	$200 - 156 = 44$	$2(0) + 2(4 + 8 + 2) + 4(4) = 44$
b,e	152	$200 - 158 = 88$	$2(2) + 2(4 + 8 + 2) + 4(4) = 48$
e,f	192	$200 - 192 = 8$	$2(0) + 2(2) + 4(1) = 8$

From Table 1, we observe the drop in energy computed by the difference between Laplacian energy of the graph G and that of the subgraph, H obtained on removing edge e is equal to that obtained using closed and non-closed walks in (31).

3.4 Laplacian Energy as a fair measure of robustness of network

With the fast increasing usage of air transport in most regions of the world, there is a pressing need to design robust air traffic networks that will ensure robustness when one or more flights are removed or added to the network. The second smallest eigenvalue of the Laplacian matrix of air traffic networks, λ_2, also known as the algebraic connectivity is a common measure of robustness in networks. Unfortunately, the algebraic connectivity captures only the global information about the connectivity of a network. Recently, the laplacian energy was introduced as another measure of robustness that is considered fair and effective compared to the algebraic connectivity. This is so because the Laplacian centrality captures the local information of the network (Yang,etal, 2015). In order to illustrate the relationship between Laplacian energy and air traffic network robustness, a real air traffic network of Jet-star Asia Airway among Indonesia, Australia, and New Zealand (Fig. 2a) was investigated.



(a) (b)

Figure 2: (a) is the air traffic network route map for Jetstar Asia Airway (Indonesia, Australia, and New Zealand) in 2015, (b) is a plot of Laplacian energy E_L and algebraic connectivity λ_2 against the number of randomly failed edges from the or Jetstar Asia Airway (Indonesia, Australia, and New Zealand).

From the Fig. (2b), we observe that values for both the laplacian energy and algebraic connectivity decreases with removal of edges from the network. However, on removal of 20 to 30 edges, the algebraic connectivity abruptly drops from 0.9 to close to 0 (that is in only one instance) which signifies a disconnection in the network that is, more than one connected component in the network. on the other hand though, the laplacian curve indicates gradual degradation of the network robustness on removal of 20 to 30 edges of the network. The ability of the Laplacian energy measure to capture the change from one connected component to more connected components in much more instances makes it an effective measure for network robustness over algebraic connectivity.