Generalized Algebraic Connectivity for Asymmetric Networks

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***Abstract*— The problem of connectivity assessment of an asymmetric network represented by a weighted directed graph is investigated in this paper. The notion of generalized algebraic connectivity is introduced for this type of network as an extension of conventional algebraic connectivity measure for symmetric networks. This new notion represents the expected asymptotic convergence rate of a cooperative algorithm used to control the network. The proposed connectivity measure is then described in terms of the eigenvalues of the Laplacian matrix of the graph representing the network. The effectiveness of this measure in describing the connectivity of asymmetric networks is demonstrated by some intuitive and counter-intuitive examples. A variation of the power iteration algorithm is then developed to compute the proposed connectivity measure. To this end, the Laplacian matrix of the network is properly transformed to a new matrix such that existing techniques can be used to find the eigenvalue representing network connectivity. The effectiveness of the proposed notion in describing network connectivity and also the efficiency of the developed algorithm are subsequently verified by simulations.**

I. Introduction

Ad-hoc wireless networks are composed of a number of fixed or mobile nodes that are capable of exchanging data through wireless channels without the support of a pre-existing infrastructure [1], [2]. The convergence time of different cooperative algorithms used for objectives such as consensus, swarming, target localization and data diffusion is highly dependent on the degree of connectivity of the network [3], [4]. This is due to the fact that a network with a higher degree of connectivity is capable of propagating information more effectively [5]. For the case of random networks, where the communication links are represented by random variables, the in-network information diffusion strictly depends upon the connectivity of the underlying expected communication graph [6].

The algebraic connectivity of a connected symmetric network is defined in the literature as the smallest nonzero eigenvalue of the Laplacian matrix of the network graph [7]. A survey on algebraic connectivity measure for different types of networks and their computation techniques can be found in [8]. A decentralized orthogonal iteration algorithm is introduced in [9] for computing the eigenvectors corresponding to the k dominant eigenvalues of a symmetric weighted network graph. However, the procedure requires centralized initialization and is not scalable to larger network sizes. A distributed algorithm for the estimation and control of the algebraic connectivity of ad-hoc networks with a random topology is developed in [10], which is only applicable to symmetric networks. A generalization of Fiedler's algebraic connectivity to directed graphs is introduced in [11], where several relationships between the algebraic connectivity and properties of the graph are investigated. Note that unlike symmetric networks, no standard definition exists for the rate of convergence of cooperative algorithms applied to asymmetric networks. Therefore, the notion of algebraic connectivity is not well developed for such networks. The notion of algebraic connectivity is extended to directed graphs in [12], where the magnitude of the second smallest eigenvalue of the Laplacian matrix is introduced as a measure of network connectivity. This notion, however, fails to capture important characteristics of the asymmetric network such as the convergence rate of cooperative algorithms running over the network. Moreover, the decentralized power iteration approach proposed in [12] requires the solution of a set of nonlinear equations with relatively high computational complexity, which limits the applicability of the algorithm to real scenarios.

There has been a growing interest in the application of asymmetric networks such as underwater acoustic sensor networks [13]. However, most of the papers cited in the previous paragraph are either concerned with the connectivity of symmetric networks or provide a simple extension of the Fiedler value to describe the connectivity of asymmetric networks. It is noted in [3] that the addition of new links in a directed graph representing an asymmetric network does not necessarily improve the rate of convergence to consensus. However, this observation is not quantitatively characterized in the context of connectivity. Furthermore, the measure given in [12] is a straightforward extension of the algebraic connectivity metric for symmetric networks, and does not reflect specific characteristics of the asymmetric networks in terms of the rate of convergence to consensus. The generalized algebraic connectivity is introduced in this work as a novel measure of connectivity of asymmetric networks represented by weighted directed graphs. The proposed connectivity measure is directly related to the expected convergence speed of cooperative algorithms (e.g., consensus algorithms) used to control this type of networks. The superiority of the proposed measure over a known algebraic connectivity measure (proposed in [12]) in describing the connectivity of asymmetric networks is then illustrated by an example. An example is also given to demonstrate the counter-intuitive relationship between the generalized algebraic connectivity and the elements of the weight matrix of an asymmetric

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network. The generalized power iteration algorithm is then developed using the Krylov subspace approximation method, the Gram-Schmidt orthonormalization procedure, and a novel matrix transformation. Simulations confirm the efficacy of the results.

The remainder of the paper is organized as follows. Some background and definitions are given in Section II. The generalized algebraic connectivity of asymmetric networks is then introduced in Section III. Section IV presents the generalized power iteration algorithm to compute the proposed connectivity measure in a centralized fashion. The simulation results are subsequently presented in Section V, and conclusions are summarized in Section VI.

II. Preliminaries and Notations

Throughout this work, the set of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. Also $\mathbb{N}_n$ is the finite set of natural numbers $\{1, 2, \ldots, n\}$. The transpose and conjugate transpose of a vector $\mathrm{v} \in \mathbb{C}^n$ are represented by v^T and v^H, respectively. The inner product of two arbitrary vectors $\mathrm{v}, \mathrm{w} \in \mathbb{C}^n$ is denoted by $\langle \mathrm{v}, \mathrm{w} \rangle$. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$, and the all-one column vector of length n is represented by 1_n. Also, $\mathbb{1}_{\mathcal{A}} : \mathcal{X} \to \{0, 1\}$ is the characteristic function over a set $\mathcal{A} \subseteq \mathcal{X}$ defined by

$$\mathbb{1}_{\mathcal{A}}(a) = \begin{cases} 1, & \text{if } a \in \mathcal{A}, \\ 0, & \text{if } a \notin \mathcal{A}. \end{cases}$$

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathbf{P})$ represent a weighted directed graph (digraph) composed of n vertices with node set $\mathscr{V}$, edge set $\mathscr{E}$, and weight matrix $\mathbf{P} = [p_{ij}]$ such that

$$\mathscr{V} = \{1, 2, \ldots, n\},$$
$$\mathscr{E} = \{(i, j) \in \mathscr{V} \times \mathscr{V} \mid p_{ji} \neq 0\},$$

where p_{ji} is a finite positive real number for any ordered pair of distinct nodes $(i, j) \in \mathscr{E}$. The node j is said to belong to the neighbor set of node i, denoted by $\mathscr{N}_i$, if the directed edge pointing from j to i belongs to the edge set of $\mathscr{G}$, i.e., $(j, i) \in \mathscr{E}$. A digraph $\mathscr{G}$ is said to be *quasi-strongly connected* (QSC) if for every two distinct vertices i and j of $\mathscr{G}$, there exists a vertex k with a directed path from k to i and a directed path from k to j. The Laplacian of the weighted digraph $\mathscr{G}$ is an $n \times n$ real matrix $\mathbf{L} = [l_{ij}]$ whose elements are given by

$$l_{ij} = \begin{cases} -p_{ij}, & \text{if } (j, i) \in \mathscr{E}, \\ \sum_{k \neq i} p_{ik}, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\Psi(\mathbf{A})$ as a set of triplets composed of the eigenvalues and the right and left eigenvectors of an asymmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ given by $\Psi(\mathbf{A}) = \{(\lambda_i(\mathbf{A}), \mathrm{v}_i(\mathbf{A}), \mathrm{w}_i(\mathbf{A})) \mid i \in \mathbb{N}_n\}$, where $\lambda_i(\mathbf{A}) \in \mathbb{C}$, $\mathrm{v}_i(\mathbf{A}) \in \mathbb{C}^n$, and $\mathrm{w}_i(\mathbf{A}) \in \mathbb{C}^n$ denote the i^{th} eigenvalue of matrix $\mathbf{A}$, and the right and left eigenvectors associated with it, respectively. The triplets of $\Psi(\mathbf{A})$ are assumed to be indexed in increasing order in terms of the real parts of the eigenvalues, i.e., $\Re(\lambda_1(\mathbf{A})) \leq \Re(\lambda_2(\mathbf{A})) \leq \cdots \leq \Re(\lambda_n(\mathbf{A}))$. The spectrum of matrix $\mathbf{A}$ is defined as $\Lambda(\mathbf{A}) = \{\lambda_i(\mathbf{A}) \mid i \in \mathbb{N}_n\}$. Let also $\mathrm{Cond}(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$ denote the condition number of matrix $\mathbf{A}$ in the Euclidean norm.

III. Generalized Algebraic Connectivity

The notion of *generalized algebraic connectivity* is introduced in this section as a new measure of connectivity for asymmetric networks. Connectivity has a significant impact on the diffusion of information throughout a network where each node shares information only with its neighbors. Nodes communicate more effectively in a more connected network, in general, and thus the information propagates faster throughout the network. For instance, when the nodes are supposed to agree upon a value of interest as a cooperative goal, the agreement can be reached with a higher rate of convergence in a network with a higher degree of connectivity. Therefore, the rate of convergence to consensus in a networked control system is highly dependent on its degree of connectivity.

Given a weighted digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathbf{P})$ composed of n nodes, let $\mathrm{q}_i = [q_{i1} \ q_{i2} \ \cdots \ q_{id}]^\mathsf{T} \in \mathbb{R}^d$ denote the d-dimensional state vector of node $i \in \mathscr{V}$ with the following dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{q}_i(t) = \mathrm{u}_i(t), \tag{1}$$

where $\mathrm{u}_i(t) \in \mathbb{R}^d$ is the control vector of node i at time t. The following consensus control law can now be applied to the i^{th} node

$$\mathrm{u}_i(t) = -\sum_{j \in \mathscr{N}_i} p_{ij} (\mathrm{q}_i(t) - \mathrm{q}_j(t)). \tag{2}$$

Note that p_{ij} is a finite nonnegative real number, and if it is nonzero, it means that there is an edge from node j to node i in $\mathscr{G}$, as mentioned earlier. Now, let $\mathbf{q} = [\mathrm{q}_1^\mathsf{T} \ \mathrm{q}_2^\mathsf{T} \ \cdots \ \mathrm{q}_n^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nd}$ denote the augmented state vector of the entire network composed of n nodes, containing information of all nodes. The dynamics of the network driven by the consensus protocol (2) can then be represented as

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{q}(t) = -(\mathbf{L} \otimes 1_d) \, \mathbf{q}(t), \tag{3}$$

where $\mathbf{L} \in \mathbb{R}^{n \times n}$ denotes the Laplacian matrix of $\mathscr{G}$ and $\otimes$ denotes the Kronecker product. For simplicity of analysis, let the weighted digraph $\mathscr{G}$ be QSC, which means that the zero eigenvalue of $\mathbf{L}$ has multiplicity one [14]. By definition, $\mathrm{w}_1(\mathbf{L}) \in \mathbb{R}^n$ is the left eigenvector of the Laplacian matrix $\mathbf{L}$ corresponding to its zero eigenvalue such that $\langle \mathrm{w}_1(\mathbf{L}), 1_n \rangle = 1$. The final state vector that all nodes will converge to is denoted by $\tilde{\mathrm{q}} := [\tilde{q}_1 \ \tilde{q}_2 \ \cdots \ \tilde{q}_d]^\mathsf{T}$, where

$$\tilde{q}_j = \langle \mathrm{w}_1(\mathbf{L}), \mathrm{q}_{\bullet j}(0) \rangle, \tag{4}$$

and $\mathrm{q}_{\bullet j} := [q_{1j} \ q_{2j} \ \cdots \ q_{nj}]^\mathsf{T}$ for any $j \in \mathbb{N}_d$. To analyze the behavior of the network while converging to consensus, consider the following disagreement function

$$\mathfrak{d}_p(t) = \Gamma_p(\mathbf{q}(t)), \tag{5}$$

where $\Gamma_p : \mathbb{R}^{nd} \to \mathbb{R}$ is defined as $\Gamma_p(\mathbf{q}) = \|\mathbf{q} - (1_n \otimes \tilde{\mathrm{q}})\|_p$ for any $p \geq 1$. The above disagreement function represents the p-norm of the difference between the state vector of the network and the final agreement state at time t. Furthermore, if the nodes take random vectors as their initial states, one can investigate the expected behavior of the disagreement function in order to measure the connectivity of the network. This point will be clarified by two illustrative examples.

Example 1: Consider two asymmetric networks represented by weighted digraphs $\mathscr{G}_1$ and $\mathscr{G}_2$ in Fig. 1. Let the initial state of each network be a random vector with a uniform distribution over the unit sphere in $\mathbb{R}^3$. Assume that the nodes obey the dynamics (1), and apply the consensus control law (2). From the above discussion, one can investigate the connectivity of any digraph by inspecting the *expected disagreement function*, denoted by $\mathbb{E}\mathfrak{d}_2(t)$. Fig. 2 depicts the evolution of function $\mathbb{E}\mathfrak{d}_2(t)$ under the above consensus algorithm in networks 1 and 2. This figure indicates that the convergence to consensus in $\mathscr{G}_2$ is expected to be faster than $\mathscr{G}_1$, and thus, the weighted digraph $\mathscr{G}_2$ is more connected than $\mathscr{G}_1$.

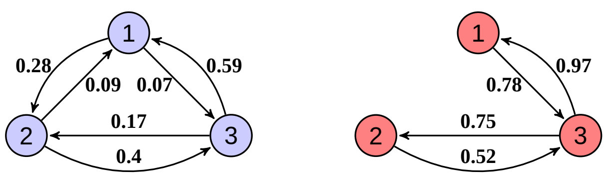

Fig. 1. The weighted digraphs $\mathscr{G}_1$ (left) and $\mathscr{G}_2$ (right) in Example 1.

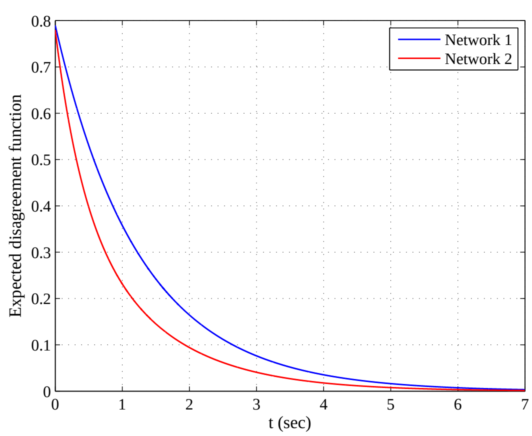

Fig. 2. Time evolution of the expected disagreement function in Example 1.

Remark 1: It is worth noting that despite the faster expected convergence of $\mathscr{G}_2$ to consensus in Example 1, the algebraic connectivity of $\mathscr{G}_1$ as defined in [12] is greater than that of $\mathscr{G}_2$. This implies that existing algebraic connectivity measure fails to accurately reflects the connectivity of asymmetric networks, signifying the need for developing a new algebraic connectivity measure for digraphs.

Example 2: Consider the weighted digraphs $\mathscr{G}_1$ and $\mathscr{G}_2$ shown in Fig. 3, representing two asymmetric networks. According to Fig. 3, $\mathscr{G}_1$ and $\mathscr{G}_2$ have the same node set but $\mathscr{G}_2$ has an additional edge pointing from node 2 to node 1 with weight 1. Intuitively, one would expect that $\mathscr{G}_2$ has a higher degree of connectivity compared to $\mathscr{G}_1$, and hence the expected disagreement function $\mathbb{E}\mathfrak{d}_2(t)$ should converge to zero faster over the network represented by $\mathscr{G}_2$. Let each node have a single-integrator dynamics given by (1), and apply the consensus control law (2) to every node. Fig. 4 depicts the evolution of $\mathbb{E}\mathfrak{d}_2(t)$ in both networks. Surprisingly, the expected convergence rate over the first network is faster than the second one. This counter-intuitive observation implies that adding a new link in an asymmetric network does not necessarily improve the connectivity because as this example shows, sometimes such additions might have a detrimental effect on the expected rate of convergence to consensus. It is important to note, however, that in symmetric networks, unlike the above example, adding a new (undirected) edge or increasing the weight of an existing edge will always increase the rate of convergence to consensus.

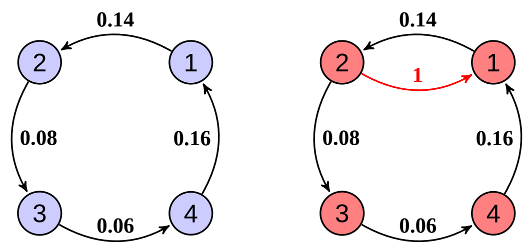

Fig. 3. The weighted digraphs $\mathscr{G}_1$ (left) and $\mathscr{G}_2$ (right) in Example 2.

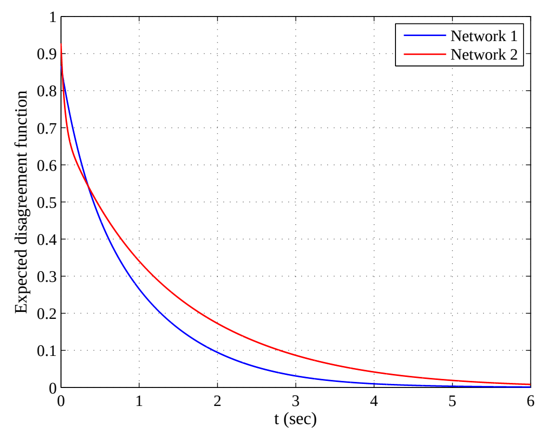

Fig. 4. Time evolution of the expected disagreement function in Example 2.

Since the weighted digraph $\mathscr{G}$ is QSC, it is guaranteed that $\mathbf{q}$ converges to $1_n \otimes \tilde{\mathrm{q}}$ as t goes to infinity, i.e., $\lim_{t\to\infty} \mathfrak{d}_p(t) = 0$. For a more detailed analysis of this convergence, one can check the *asymptotic rate* defined as $-\lim_{t\to\infty} t^{-1} \ln \mathfrak{d}_p(t)$, which represents the exponential speed of decreasing the disagreement function to zero in an asymptotic manner. To this end, assume for simplicity and without loss of generality that the eigenvalues of the Laplacian matrix $\mathbf{L}$ are distinct. Let the set of right eigenvectors and left eigenvectors of $\mathbf{L}$ be chosen such that $\langle \mathrm{w}_i(\mathbf{L}), \mathrm{v}_i(\mathbf{L})\rangle = 1$ for any $i \in \mathbb{N}_n$. Since $\mathbf{L}$ is a diagonally dominant matrix with nonnegative diagonal entries, there exists a strictly increasing sequence of m positive real scalars, denoted by $\{r_k\}_{k\in\mathbb{N}_m}$ for $m \in \mathbb{N}_{n-1}$, such that

$\{r_k \mid k \in \mathbb{N}_m\} = \{\Re(\lambda_i(\mathbf{L})) \mid \lambda_i(\mathbf{L}) \in \Lambda(\mathbf{L}), \lambda_i(\mathbf{L}) \neq 0\}$. Note that the equality $\mathrm{e}^{-\mathbf{L}t}\,\mathrm{v}_i(\mathbf{L}) = \mathrm{e}^{-\lambda_i(\mathbf{L})t}\,\mathrm{v}_i(\mathbf{L})$ holds at all times.

In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space, with Ω, $\mathcal{F}$ and $\mathbb{P}$ denoting the sampling space, σ-algebra and probability measure, respectively. Let the initial state $\mathbf{q}(0)$ be a random vector defined over $(\Omega, \mathcal{F})$, taking values in the unit sphere in $\mathbb{R}^{nd}$. This means that there exists a random variable $X_{ij} : (\Omega, \mathcal{F}) \to \mathbb{C}$ for any $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_d$, such that

$$\mathrm{q}_{\bullet j}(0) = \sum_{i \in \mathbb{N}_n} X_{ij} \mathrm{v}_i(\mathbf{L}). \tag{6}$$

It results from (3) that for any $j \in \mathbb{N}_d$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{q}_{\bullet j}(t) = -\mathbf{L}\, \mathrm{q}_{\bullet j}(t).$$

It then follows that $\mathrm{q}_{\bullet j}(t) = \mathrm{e}^{-\mathbf{L}t} \mathrm{q}_{\bullet j}(0)$ for any t. Also, one can easily verify that

$$\|\mathbf{q}(t) - (1_n \otimes \tilde{\mathrm{q}})\|_p^p = \sum_{j \in \mathbb{N}_d} \|\mathrm{q}_{\bullet j}(t) - \tilde{q}_j 1_n\|_p^p,$$

for any $p \geq 1$. Therefore,

$$\mathfrak{d}_p^p(t) = \sum_{j \in \mathbb{N}_d} \Big\| \sum_{i \in \mathbb{N}_n} X_{ij} \mathrm{e}^{-\mathbf{L}t} \mathrm{v}_i(\mathbf{L}) - \tilde{q}_j 1_n \Big\|_p^p. \tag{7}$$

From (4) and (6) and on noting that $\langle \mathrm{w}_1(\mathbf{L}), \mathrm{v}_i(\mathbf{L}) \rangle = 0$ for all $i \in \mathbb{N}_n \setminus \{1\}$ and $\langle \mathrm{w}_1(\mathbf{L}), \mathrm{v}_1(\mathbf{L}) \rangle = 1$, one arrives at

$$\tilde{q}_j = \sum_{i \in \mathbb{N}_n} X_{ij} \langle \mathrm{w}_1(\mathbf{L}), \mathrm{v}_i(\mathbf{L}) \rangle = X_{1j}. \tag{8}$$

Since $\lambda_1(\mathbf{L}) = 0$ and $\mathrm{v}_1(\mathbf{L}) = 1_n$, it follows from (7) and (8) that

$$\mathfrak{d}_p^p(t) = \sum_{j \in \mathbb{N}_d} \Big\| \sum_{i=2}^{n} X_{ij} \mathrm{e}^{-\lambda_i(\mathbf{L})t} \mathrm{v}_i(\mathbf{L}) \Big\|_p^p.$$

For any $k \in \mathbb{N}_m$, define the index set $\mathcal{I}_k$ as $\mathcal{I}_k = \{i \in \mathbb{N}_n \setminus \{1\} \mid \Re(\lambda_i(\mathbf{L})) = r_k\}$, which contains the index of any eigenvalue of $\mathbf{L}$ whose real part is equal to r_k. Let the event $\mathcal{A}_k \in \mathcal{F}$ be defined as $\mathcal{A}_k = \cup_{i \in \mathcal{I}_k, j \in \mathbb{N}_d} \{X_{ij} \neq 0\}$ for any $k \in \mathbb{N}_m$. An important property of $\mathcal{A}_k$ is that for at least one $j \in \mathbb{N}_d$, there exists an eigenvector $\mathrm{v}_i(\mathbf{L})$ such that $i \in \mathcal{I}_k$ and the inner product of $\mathrm{q}_{\bullet j}(0)$ and $\mathrm{v}_i(\mathbf{L})$ is nonzero. Now, define the events $\{\mathcal{B}_k\}_{k \in \mathbb{N}_m} \subset \mathcal{F}$ such that $\mathcal{B}_1 = \mathcal{A}_1$ and $\mathcal{B}_k = \mathcal{A}_k - \cup_{l=1}^{k-1} \mathcal{A}_l$ for any $k \in \mathbb{N}_m \setminus \{1\}$. Since the sets $\{\mathcal{B}_k\}_{k \in \mathbb{N}_m}$ are mutually disjoint and also $\Omega = \cup_{k=1}^m \mathcal{B}_k$, they partition the sample space Ω (which means that their intersection is empty and their union is the whole sample space). Let the random variable r, referred to as the *random rate*, be defined as $r := \sum_{k \in \mathbb{N}_m} r_k \, \mathbb{1}_{\mathcal{B}_k}$. Define also

$$S_{jk}(t) = \sum_{k' \geq k} \sum_{i \in \mathcal{I}_{k'}} X_{ij} \mathrm{e}^{-\lambda_i(\mathbf{L})t + r_k t}\, \mathrm{v}_i(\mathbf{L}), \tag{9}$$

for any $j \in \mathbb{N}_d$ and $k \in \mathbb{N}_m$. It can then be verified that

$$\begin{aligned}
(\mathrm{e}^{rt}\mathfrak{d}_p(t))^p &= \textstyle\sum_{j \in \mathbb{N}_d} \|\mathrm{e}^{rt} \sum_{i=2}^n X_{ij} \mathrm{e}^{-\lambda_i(\mathbf{L})t} \mathrm{v}_i(\mathbf{L})\|_p^p \\
&= \textstyle\sum_{j \in \mathbb{N}_d} \| \sum_{k \in \mathbb{N}_m} S_{jk}(t) \mathbb{1}_{\mathcal{B}_k} \|_p^p \qquad (10) \\
&= \textstyle\sum_{j \in \mathbb{N}_d} \| \sum_{k \in \mathbb{N}_m} S_{jk}(t) \|_p^p \, \mathbb{1}_{\mathcal{B}_k}.
\end{aligned}$$

Note that the real part of $-\lambda_i(\mathbf{L}) + r_k$ is nonpositive for any $k, k' \in \mathbb{N}_m$ where $k \leq k'$ and $i \in \mathcal{I}_{k'}$. It then follows from the triangle inequality that

$$\|S_{jk}(t)\|_p \leq \sum_{k' \geq k} \sum_{i \in \mathcal{I}_{k'}} |X_{ij}|\, \|\mathrm{v}_i(\mathbf{L})\|_p.$$

One can subsequently conclude that for any t

$$(\mathrm{e}^{rt}\mathfrak{d}_p(t))^p \leq \sum_{j \in \mathbb{N}_d} \Big(\sum_{k \in \mathbb{N}_m} \sum_{k' \geq k} \sum_{i \in \mathcal{I}_{k'}} |X_{ij}|\, \|\mathrm{v}_i(\mathbf{L})\|_p \Big)^p.$$

Lemma 1: Given a positive integer n, let $l \in \mathbb{N}_n$. Let also $\{\mathrm{v}_i\}_{i \in \mathbb{N}_l}$ denote a set of linearly independent unit vectors in $\mathbb{C}^n$, and $\{\gamma_i\}_{i \in \mathbb{N}_l}$ be a set of complex scalars with nonpositive real parts. Then, for any set of complex scalars $\{c_i\}_{i \in \mathbb{N}_l}$, $\liminf_{t \to \infty} \| \sum_{i \in \mathbb{N}_l} c_i \mathrm{e}^{\gamma_i t} \mathrm{v}_i \|_p = 0$ if and only if $c_i = 0$ for any $i \in \mathbb{N}_l$ such that $\Re(\gamma_i) = 0$.

Proof: The proof is omitted due to space limitations and can be found in [15]. ■

Since $\|\mathbf{q}(0)\|_p \neq 0$ for any initial state, there exists $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_d$ for which $X_{ij} \neq 0$. Let $k \in \mathbb{N}_m$ be such that $i \in \mathcal{I}_k$. It then follows from (9) and Lemma 1 that $\liminf_{t \to \infty} \|S_{jk}(t)\|_p > 0$. It can be concluded from (10) that $\liminf_{t \to \infty} \mathrm{e}^{rt}\mathfrak{d}_p(t) > 0$. It then results from the boundedness of $\mathrm{e}^{rt}\mathfrak{d}_p(t)$ that $\lim_{t \to \infty} t^{-1} \ln(\mathrm{e}^{rt}\mathfrak{d}_p(t)) = 0$ or $-\lim_{t \to \infty} t^{-1} \ln \mathfrak{d}_p(t) = r$. If the initial state $\mathbf{q}(0)$ has a continuous probability distribution, one can show that $\mathbb{P}[\mathcal{B}_1] = 1$, which yields

$$\mathbb{P}[-\lim_{t \to \infty} \frac{\ln \mathfrak{d}_p(t)}{t} = r_1] = 1.$$

The next theorem follows from the previous discussion.

Theorem 1: Let $\mathscr{G}$ be a QSC weighted digraph with Laplacian matrix $\mathbf{L}$ whose eigenvalues are distinct, and $p \geq 1$ denote a real scalar. Let also $\mathbf{q}$ denote the state vector of a dynamical system described by (3), whose initial state $\mathbf{q}(0)$ is a random vector with a continuous probability distribution over the unit sphere in $\mathbb{R}^{nd}$. Consider the disagreement function $\mathfrak{d}_p$ given in (5), and define $\tilde{\lambda}(\mathbf{L}) = \min_{\lambda_i(\mathbf{L}) \neq 0,\, \lambda_i(\mathbf{L}) \in \Lambda(\mathbf{L})} \Re(\lambda_i(\mathbf{L}))$. Then, the asymptotic rate of convergence to consensus is almost surely equal to $\tilde{\lambda}(\mathbf{L})$, i.e., $\mathbb{P}[-\lim_{t \to \infty} t^{-1} \ln \mathfrak{d}_p(t) = \tilde{\lambda}(\mathbf{L})] = 1$, and the expectation of asymptotic rate of convergence to consensus equals $\tilde{\lambda}(\mathbf{L})$, i.e., $\mathbb{E}[-\lim_{t \to \infty} t^{-1} \ln \mathfrak{d}_p(t)] = \tilde{\lambda}(\mathbf{L})$.

Proof: The proof of this theorem is a direct consequence of the above discussion. ■

Theorem 1 shows that the expected rate of convergence to consensus is well described by $\tilde{\lambda}(\mathbf{L})$, as defined in the statement of the theorem, when the initial state vector of network is chosen randomly. This motivates the introduction of a new connectivity measure for weighted digraphs.

Definition 1: Given a weighted digraph $\mathscr{G}$ with Laplacian matrix $\mathbf{L}$, the *generalized algebraic connectivity* of $\mathscr{G}$, denoted by $\tilde{\lambda}(\mathbf{L})$, is defined as

$$\tilde{\lambda}(\mathbf{L}) = \min_{\lambda_i(\mathbf{L}) \neq 0,\, \lambda_i(\mathbf{L}) \in \Lambda(\mathbf{L})} \Re(\lambda_i(\mathbf{L})).$$

The generalized algebraic connectivity (GAC) reflects the expected asymptotic convergence rate of cooperative consensus-based algorithms in an asymmetric network represented by a weighted digraph. It can be verified that in Example 1, $\tilde{\lambda}(\mathbf{L}_1) < \tilde{\lambda}(\mathbf{L}_2)$, and in Example 2, $\tilde{\lambda}(\mathbf{L}_1) > \tilde{\lambda}(\mathbf{L}_2)$, which are in accordance with the corresponding convergence results. In the next section, an algorithm is presented to compute the generalized algebraic connectivity.

IV. Computation of the Generalized Algebraic Connectivity

An algorithm is presented in this section to compute the GAC of a weighted digraph. The algorithm is an extension of the well-known power iteration method which has been used extensively in the literature to compute the algebraic connectivity of undirected graphs in both centralized and distributed fashions [10], [12]. The power iteration algorithm computes the dominant eigenvalue of a matrix (i.e., an eigenvalue with maximum magnitude) as well as the eigenvector associated with it, under certain conditions [16]. Some of the challenges in using the power iteration method to compute the GAC are as follows:

(C.1) the power iteration method computes the eigenvector corresponding to the eigenvalue with maximum (not minimum) magnitude (not real part), and

(C.2) the convergence of the procedure is not guaranteed when there are two eigenvalues, as a complex conjugate pair, with largest magnitude.

In order to address the above challenges, some useful results are developed in the sequel.

Theorem 2: Let $\mathbf{L}$ be the Laplacian matrix of a QSC weighted digraph $\mathscr{G}$ composed of n nodes, and define the modified Laplacian matrix of the digraph as

$$\tilde{\mathbf{L}} = \mathrm{e}^{\mathbf{I}_n - \epsilon \mathbf{L}} - \exp(1)\, \mathrm{w}_1(\mathbf{L})\, \mathrm{w}_1^{\mathsf{T}}(\mathbf{L}), \tag{11}$$

where $\epsilon < \Delta^{-1}$ and $\Delta = 2\max\limits_{i\in\mathscr{V}} \sum_{j\in\mathscr{N}_i} p_{ij}$. It then follows that

$$\tilde{\lambda}(\mathbf{L}) = \frac{1}{\epsilon}\big(1 - \ln(\max_{\lambda_i(\tilde{\mathbf{L}})\in\Lambda(\tilde{\mathbf{L}})} |\lambda_i(\tilde{\mathbf{L}})|)\big). \tag{12}$$

Proof: The proof is omitted due to space limitations and can be found in [15]. ■

The generalized power iteration (GPI) algorithm is developed next as an extension of the power iteration method to compute the GAC of asymmetric networks. To address the challenge (C.1), the Laplacian matrix $\mathbf{L}$ is transformed into the modified Laplacian matrix $\tilde{\mathbf{L}}$ defined in (11). It then follows from Theorem 2 that the problem of finding $\tilde{\lambda}(\mathbf{L})$ is reduced to the problem of finding the dominant eigenvalue of $\tilde{\mathbf{L}}$ with maximum magnitude. To address the challenge (C.2), a variant of the *Krylov subspace method* is used to develop the GPI algorithm in order to compute a relatively good approximation of the GAC, no matter if the corresponding eigenvalue is real or complex. The Krylov subspace procedure is briefly introduced next, followed by the proposed generalized power iteration algorithm.

A. Krylov Subspace Method [16]

The m-dimensional Krylov subspace of $\mathbb{C}^n$ w.r.t. the vector $\mathrm{x}_0 \in \mathbb{C}^n$ and matrix $\tilde{\mathbf{L}} \in \mathbb{R}^{n\times n}$, denoted by $\mathbb{K}_m(\mathrm{x}_0, \tilde{\mathbf{L}})$, is defined as

$$\mathbb{K}_m(\mathrm{x}_0, \tilde{\mathbf{L}}) = \mathrm{span}\{\mathrm{x}_0, \tilde{\mathbf{L}}\mathrm{x}_0, \ldots, \tilde{\mathbf{L}}^{m-1}\mathrm{x}_0\}.$$

Let $\mathbf{Q}_m = [\hat{\mathrm{x}}_1 \ \cdots \ \hat{\mathrm{x}}_m]$ denote a $n \times m$ matrix whose m columns constitute an orthonormal basis of $\mathbb{K}_m(\mathrm{x}_0, \tilde{\mathbf{L}})$ such that $\mathbf{Q}_m^{\mathsf{H}}\, \mathbf{Q}_m = \mathbf{I}_m$. Let also $\mathbf{R}_m$ represent the orthogonal projection of $\tilde{\mathbf{L}}$ onto the m-dimensional Krylov subspace such that

$$\mathbf{R}_m = \mathbf{Q}_m^{\mathsf{H}}\, \tilde{\mathbf{L}}\, \mathbf{Q}_m.$$

The Krylov subspace method approximates s desired eigenpairs of $\tilde{\mathbf{L}}$ by those of the reduced matrix $\mathbf{R}_m$ of order m, where $s \le m < n$. Then, the eigenpair $(\hat{\lambda}, \mathbf{Q}_m\hat{\mathrm{v}})$ provides an approximation of the desired eigenpair (λ, v) of $\tilde{\mathbf{L}}$, where

$$\mathbf{R}_m\, \hat{\mathrm{v}} = \hat{\lambda}\, \hat{\mathrm{v}}.$$

To handle the case where the GAC of a network is associated with a pair of complex conjugate dominant eigenvalues of $\tilde{\mathbf{L}}$, it is required to compute the eigenpairs of the first two dominant eigenvalues of $\tilde{\mathbf{L}}$. To this end, one can consider $s = 2$ and $m \ge 2$ in the GPI algorithm. To generate the orthonormal columns of $\mathbf{Q}_m$ using the Gram-Schmidt process, define $\hat{\mathrm{x}}_1 = \frac{\mathrm{x}_0}{\|\mathrm{x}_0\|}$, where the initial vector x_0 is chosen randomly. Then, $\hat{\mathrm{x}}_{j+1}$ is obtained recursively by

$$\hat{\mathrm{x}}_{j+1} = \frac{\tilde{\mathbf{L}}\hat{\mathrm{x}}_j - \sum_{i=1}^{j}\langle \tilde{\mathbf{L}}\hat{\mathrm{x}}_j, \hat{\mathrm{x}}_i\rangle \hat{\mathrm{x}}_i}{\|\tilde{\mathbf{L}}\hat{\mathrm{x}}_j - \sum_{i=1}^{j}\langle \tilde{\mathbf{L}}\hat{\mathrm{x}}_j, \hat{\mathrm{x}}_i\rangle \hat{\mathrm{x}}_i\|}, \tag{13}$$

for any $j \in \mathbb{N}_{m-1}$. The accuracy of the approximate eigenpair $(\hat{\lambda}, \mathbf{Q}_m\hat{\mathrm{v}})$ for the original matrix $\tilde{\mathbf{L}}$ is assessed by the residual norm ρ, which is defined as

$$\rho = \|(\tilde{\mathbf{L}} - \hat{\lambda}\mathbf{I}_n)\mathbf{Q}_m\hat{\mathrm{v}}\|.$$

The Krylov subspace method is used in the GPI algorithm to iteratively solve the maximization problem described in (12). A procedure for centralized implementation of the GPI algorithm is elaborated in Algorithm 1.

Theorem 3: Let $\tilde{\lambda}(k)$ denote the output of the GPI algorithm after k iterations over a QSC weighted digraph $\mathscr{G}$ with Laplacian matrix $\mathbf{L}$. Assume that the modified Laplacian matrix $\tilde{\mathbf{L}}$ is transformed to diagonal form using the nonsingular matrix $\mathbf{X} \in \mathbb{C}^{n\times n}$ such that $\mathbf{X}^{-1}\tilde{\mathbf{L}}\mathbf{X}$ is diagonal. Let also ρ_0 be a threshold on the residual norm which verifies if the desired accuracy is achieved in order to stop the iterations. It then follows that Algorithm 1 is convergent and the approximation error of the GAC is upper-bounded as

$$|\tilde{\lambda}(\mathbf{L}) - \tilde{\lambda}(k)| \le \frac{1}{\epsilon}\rho_0 \mathrm{Cond}(\mathbf{X}).$$

Proof: The proof is omitted due to space limitations and can be found in [15]. ■

V. Simulation Results

Example 3: The efficacy of Algorithm 1 in approximating the GAC of the network is demonstrated in this section.

Algorithm 1 Centralized Generalized Power Iteration

1: Choose $s=2$, $m \in \mathbb{N}_n \backslash \mathbb{N}_{s-1}$, and let $\mathrm{x}_0 = \mathrm{rand}(n,1)$ be the initial right eigenvector associated with the eigenvalue representing the GAC of $\mathscr{G}$.
2: Set $k=1$ and let ρ_0 be a prescribed small value used in the termination criterion.
3: **while** $\rho(k) > \rho_0$ **do**
4: Construct the matrix $\mathbf{Q}_m(k)$ based on the Gram-Schmidt procedure (13) to represent the Krylov subspace w.r.t. vector x_0 and matrix $\tilde{\mathbf{L}}$.
5: Construct the matrix $\mathbf{R}_m(k) = \mathbf{Q}_m^{\mathsf{H}}(k)\,\tilde{\mathbf{L}}\,\mathbf{Q}_m(k)$, which is the projection of $\tilde{\mathbf{L}}$ onto the subspace spanned by the columns of $\mathbf{Q}_m(k)$.
6: Compute $\hat{\Lambda}(k) = \mathrm{Diag}(\hat{\lambda}_1(k),\ldots,\hat{\lambda}_m(k))$ and $\hat{\mathbf{V}}(k) = [\hat{\mathrm{v}}_1(k) \cdots \hat{\mathrm{v}}_m(k)]$ such that $\mathbf{R}_m(k)\,\hat{\mathbf{V}}(k) = \hat{\mathbf{V}}(k)\,\hat{\Lambda}(k)$.
7: $\tilde{\lambda}(k) = \frac{1}{\epsilon}(1-\ln(|\hat{\lambda}_{i^*}(k)|))$ and $\tilde{\mathrm{v}}(k) = \mathbf{Q}_m(k)\,\hat{\mathrm{v}}_{i^*}(k)$ where $i^* = \mathrm{argmax}_{i\in\mathbb{N}_m} |\hat{\lambda}_i(k)|$.
8: $\rho(k) = \|\tilde{\mathbf{L}}\,\tilde{\mathrm{v}}(k) - \hat{\lambda}_{i^*}(k)\,\tilde{\mathrm{v}}(k)\|$.
9: $\mathrm{x}_0 = \tilde{\mathrm{v}}(k)$; $k = k+1$.
10: **end while**
11: **return** $\tilde{\lambda}(k)$.

Consider a network composed of six nodes with a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathbf{P})$ whose weight matrix $\mathbf{P} = [p_{ij}]$ is given by

$$\mathbf{P} = \begin{bmatrix} 0 & 0.1 & 0.3 & 0.8 & 0.9 & 0.8 \\ 0.2 & 0 & 1 & 0.1 & 0.5 & 0.1 \\ 0.1 & 0.9 & 0 & 0.7 & 0.9 & 0.1 \\ 1 & 0.5 & 0.6 & 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.3 & 0.1 & 0 & 1 \\ 0.5 & 0.4 & 0.6 & 0.1 & 0.1 & 0 \end{bmatrix}.$$

Moreover, $0 < p_{ij} \le 1$ for any $(j,i) \in \mathscr{E}$. For this network, $\tilde{\lambda}(\mathbf{L}) = 2.1916$ represents the smallest nonzero real part of the eigenvalues of $\mathbf{L}$ which corresponds to a pair of complex conjugate eigenvalues $\lambda_{2,3}(\mathbf{L}) = 2.1916 \pm \mathrm{j}\,0.4313$. Consider $m=3$ and $\rho_0 = 10^{-3}$ as the parameters used in the proposed centralized GPI algorithm. Fig. 5 depicts the approximate GAC measure versus the iteration number for this example.

VI. Conclusions

The notion of generalized algebraic connectivity is introduced in this paper as a novel connectivity measure for weighted asymmetric networks. This measure represents the expected asymptotic convergence rate of cooperative algorithms used to control the network. The proposed measure is, in fact, an extension of the existing notion of algebraic connectivity for symmetric networks based on the Laplacian of the network graph. An analytical formulation is provided for the proposed measure using an appropriate matrix transformation. The generalized power iteration algorithm is then introduced to compute this measure in a centralized fashion based on the Krylov subspace approximation method and Gram-Schmidt orthonormalization procedure. The efficacy of

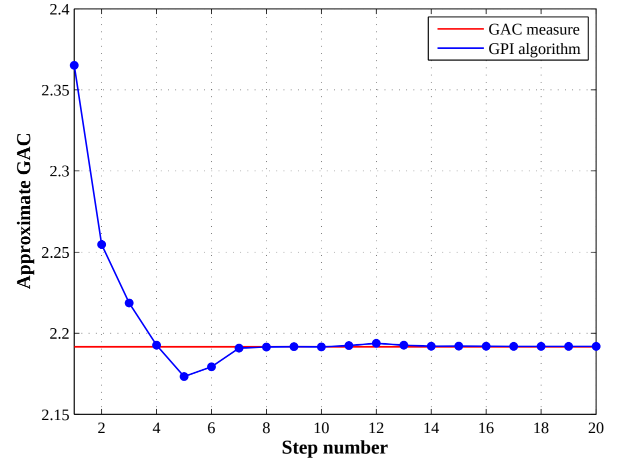

Fig. 5. The approximate GAC measure versus step number in Example 3.

the proposed algorithm is subsequently verified by simulations.

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