Variations in Graph Energy: A Measure for Network Resilience

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***Abstract*—There are many models and metrics developed to study the resilience of networks. Eigenvalues are the roots of the characteristic polynomial for a given graph and are mathematically rigorous compared to a statistical measure such as degree distribution. The graph energy is the sum of absolute values of eigenvalues; there is a subtle difference between the adjacency, Laplacian, and normalized Laplacian graph energy calculations. Our primary objective in this paper is to understand what different graph energy mean from a network resilience point of view. We calculate the adjacency, Laplacian, and normalized Laplacian graph energies on four backbone networks under targeted node and link attack scenarios. While adjacency and Laplacian graph energy decrease with node and link attacks, the normalized Laplacian energy increases with link attacks converging to a maximum value equal to the network order. The structural similarities of physical-level topologies is revealed by the close values of adjacency and Laplacian energies.**

***Index Terms*—Graph energy, adjacency energy, Laplacian energy, normalized Laplacian energy, eigenvalue, betweenness, closeness, degree, attack, backbone network**

I. Introduction and Motivation

Communication networks have become a critical infrastructure that applications depend upon. The utility of networks, on the other hand, has become an attractive target for malicious users to exploit [1]. There have been many papers exploring multiple facets of communication networks using traditional graph metrics [2]–[4]. More recently, *spectral graph properties* of communication networks have been studied [5]–[11], rather than relying on a single metric value to evaluate a network's performance. The network science community has been studying different metrics and developing models on how to improve the resilience of networks.

Eigenvalues, along with their multiplicities, specify spectra of a graph. There are several data structures to represent connectivity in a given graph, such as adjacency matrix, incidence matrix, Laplacian matrix, and normalized Laplacian matrix. There are several crucial eigenvalues that represent how well a graph is connected, such as the second largest eigenvalue of the Laplacian [12] and number of multiplicities along eigenvalue of 1 and 0 for the normalized Laplacian [9], [13], [14]. Another feature of the eigenvalues is the sum of absolute values of eigenvalues, that is, graph energy [15], [16].

In this paper, we seek to answer the following question: *What does graph energy mean for network resilience?* We analyze the adjacency, Laplacian, and normalized Laplacian energy of US backbone networks. We study the energy of backbone networks as nodes, and links are intelligently removed based on centrality measures – betweenness, closeness, and degree – simulating targeted attack scenarios. Our results indicate that while adjacency and Laplacian energy of graphs decrease with attacks, the normalized Laplacian energy increases for link-based attacks. We show that energy can be a useful metric to analyze, and it is relatively easy to calculate due to the availability of functions in existing open-source packages such as Python NumPy [17].

The paper is organized as follows: The background and past work about the energy of graphs is presented in Section II. The description of our methodology, including the dataset used, is presented in Section III. In Section IV, we analyze energy depletions under intelligent attacks on backbone networks, then we conclude and showcase future work in Section V.

II. Background and Related Work

Let $G = (V, E)$ be an unweighted, undirected graph with n vertices and l edges. The vertex set is denoted by $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and the edge set is denoted by $E = \{e_0, e_1, \ldots, e_{l-1}\}$ for G. The graph connectivity can be represented by several structures such as an adjacency matrix, Laplacian matrix, and normalized Laplacian matrix. $A(G)$ is the symmetric adjacency matrix with no self-loops, where $a_{ii} = 0$, $a_{ij} = a_{ji} = 1$ if there is a link between $\{v_i,v_j\}$, and $a_{ij} = a_{ji} = 0$ if there is no link between $\{v_i,v_j\}$. The Laplacian matrix of G is: $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix of node degrees, $d_{ii} = \deg(v_i)$. Given the degree of a node is $d_i = d(v_i)$, the normalized Laplacian matrix $\mathcal{L}(G)$ is denoted:

$$\mathcal{L}(G)(i,j) = \begin{cases} 1, & \text{if } i = j \text{ and } d_i \neq 0 \\ -\dfrac{1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Let M be a symmetric matrix of order n and I be the identity matrix of order n. Then, *eigenvalues* (λ) and the *eigenvector* ($\mathbf{x}$) of M satisfy $M\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq 0$, viz., eigenvalues are the roots of the characteristic polynomial, $\det(M - \lambda I) = 0$. The set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ together with their multiplicities (number of occurrences of an eigenvalue λ_i) define the *spectrum* of M.

Energy of a graph, $\mathcal{E}$, is captured as the sum of absolute values of its eigenvalues [15], [16]. Given a graph, its adjacency energy, $\mathcal{E}_A$, Laplacian energy, $\mathcal{E}_L$, and normalized Laplacian energy, $\mathcal{E}_{\mathcal{L}}$, differ slightly for each type of graph representation's respective eigenvalues ($\lambda(A), \lambda(L), \lambda(\mathcal{L})$). Given an adjacency matrix of a graph, $A(G)$, the graph energy is $\mathcal{E}_A(G)$ [15]:

$$\mathcal{E}_A(G) = \sum_{i=1}^{n} |\lambda_i(A)| \tag{1}$$

The graph energy of the Laplacian matrix, $L(G)$, is $\mathcal{E}_L(G)$ [18]:

$$\mathcal{E}_L(G) = \sum_{i=1}^{n} |\lambda_i(L) - 2l/n| \tag{2}$$

in which l is the number of links and n is the number of nodes. Lastly, given the normalized Laplacian graph, $\mathcal{L}(G)$, its energy $\mathcal{E}_{\mathcal{L}}(G)$ [19] is:

$$\mathcal{E}_{\mathcal{L}}(G) = \sum_{i=1}^{n} |\lambda_i(\mathcal{L}) - 1| \tag{3}$$

Regarding edge removals, it was found that as the edges are removed, the energy of an adjacency graph reduced, increased, or remained unchanged for different cases [20]. Our objective in this paper is to evaluate the all three types of energies of US backbone networks against targeted attacks. We believe that this is the first study that analyses graph energy of all three types (i.e. adjacency, Laplacian, normalized Laplacian) under node and link removal scenarios.

III. Methodology

We use the following packages: the Python NumPy package for numerical calculations [17], Python NetworkX library [21] for simulating network attacks, and Python Matplotlib 2D graphics package for plotting figures [22].

A. Dataset

We use backbone networks that are geographically located within the contiguous US to experiment with our graph energy analysis methodology. We use three commercial service provider networks: AT&T, Level 3, and Sprint, as well as the Internet2 research network. We analyze both the physical-level topologies (e.g. fiber) and logical-level (e.g. PoP-level) topologies of these networks. We note that our objective is not to compare the performance of different service provider networks, but rather investigate the adjacency, Laplacian, and normalized Laplacian energy of these backbone networks under challenged conditions. The properties – number of nodes, number of links, average node degree, closeness, maximum node betweenness, and maximum link betweenness – of these four backbone networks are shown in Table I. Note that L1 represents physical-level and L3 represents logical-level topologies in Table I.

TABLE I
TOPOLOGICAL PROPERTIES OF BACKBONE NETWORKS

Network	Node	Link	Avg. Deg.	Clo.	Max. Node Between.	Max. Link Between.
AT&T L1	383	488	2.6	0.1	17011	14466
AT&T L3	107	140	2.6	0.3	2168	661
Level 3 L1	99	130	2.6	0.1	1628	1046
Level 3 L3	38	376	19.8	0.7	59	37
Sprint L1	264	312	2.4	0.1	11275	9570
Sprint L3	28	76	5.4	0.5	100	27
Internet2 L1	57	65	2.3	0.2	630	521
Internet2 L3	9	13	2.9	0.5	9	11

We obtain the realistic topological data from the KU-Topview Network Topology Tool [23], of which an in-depth discussion of structural and visual properties of these graphs are explained in our earlier work [8], [9], [24]. We believe that the dataset we analyze sufficiently presents the methodology and conclusions in this paper.

B. Metrics for removals

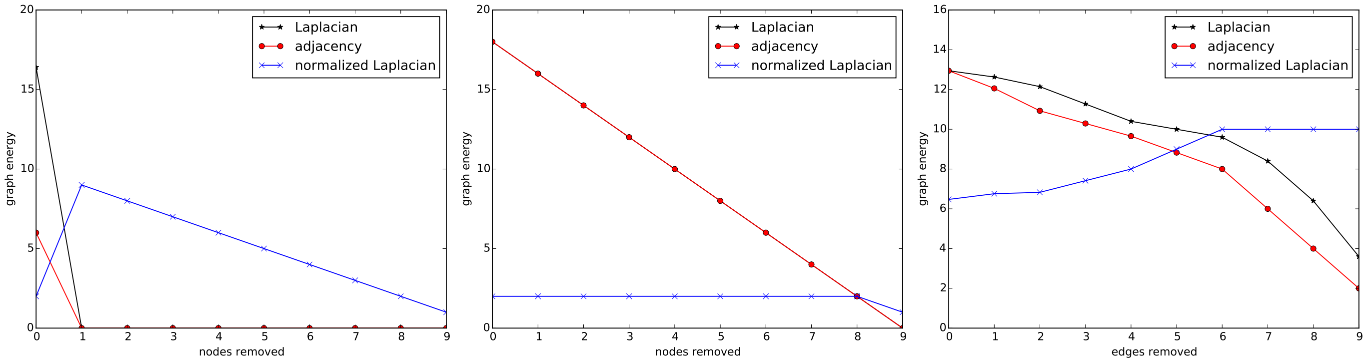
We use three node centrality measures for removal of nodes – betweenness, closeness, and degree. For link removals, we use the edge-betweenness measure [25]. The betweenness metric captures the number of shortest paths that flow through a node or a link. Closeness is the reciprocal of farness and is computed as the reciprocal of the sum of the distances from the given node to all the other nodes. Degree is calculated as the sum of the number of incident links to a node. While many other measures exist [4], we want to observe how the backbone networks behave under attack scenarios using the well-known graph metrics. Investigating graph energy under different graph measures will be part of our future work.

C. Removal sequence

The sequence of steps involved in the connectivity analysis is as follows. First we calculate the given graph metric for all nodes (i.e. node betweenness, closeness, or degree) or links (link betweenness). We sort the nodes or links in descending order and remove the highest node or link. We iterate this until we reach to a set number of nodes or links. For this study, we set the number of components to be removed at: 10, 20, 30, 50, 100, 200, and 300. The removal scenarios are representative of intelligent targeted attacks with topological information.

D. Energy depletion calculation

We calculate the graph energy by finding the sum of the absolute values of the eigenvalues in the network (i.e. Equations 1, 2, 3). For our energy analysis, we plot the energy of the graph across the number of removals we perform using metric-based removal algorithms. Each point on the plot marks the energy of a graph after reaching the respective number of removals for each step in a removal sequence.



(a) Star with degree-based node removals (b) Mesh with betweenness-based node removals (c) Ring with betweenness-based link removals

Fig. 1. Energy characteristics of toy networks (star, full-mesh, and ring) with 10 nodes

IV. Results

We analyze star, full mesh, and ring graphs to understand the behavior of energy variations when removing the nodes and links. Next, we analyze the energy of backbone networks in targeted attack scenarios.

A. Energy analysis of toy networks

In an effort to understand the energy characteristics, we study several toy networks including star, full-mesh, bus, ring, and tree with node numbers 10 and 100. Among these toy network structures being analyzed, we only present (due to space) three cases – star, mesh, and ring – with 10 nodes and links removal scenarios as shown in Figure 1. Before explaining the energies of toy networks, we note that the eigenvalue multiplicities for adjacency, Laplacian, and normalized Laplacian matrices of star and full-mesh type graphs are shown in Table II. The adjacency and Laplacian eigenvalue multiplicities [14], as well as the normalized Laplacian eigenvalue multiplicities [26] vary across different graph structures.

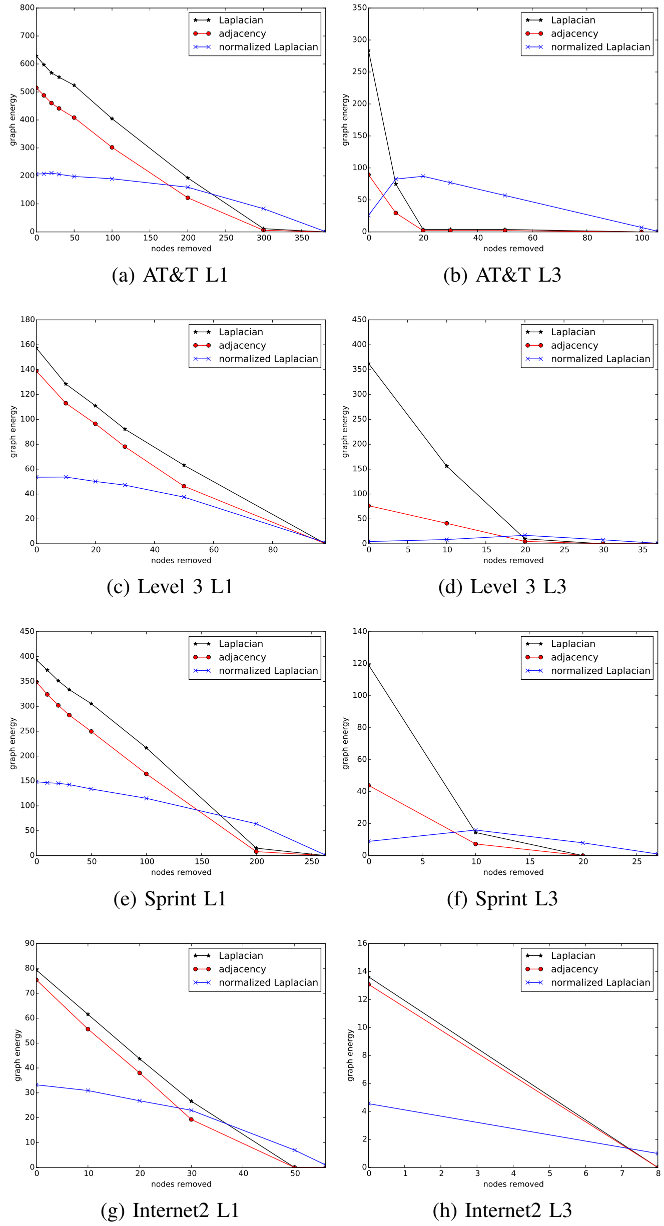
TABLE II
Eigenvalue multiplicities of star and full-mesh graph types

Matrix types		**Multiplicities**
$A(G)$	star	$\pm\sqrt{n-1}, 0^{n-2}$
	mesh	$(-1)^{n-1}, (n-1)^1$
$L(G)$	star	$0^1, 1^{n-2}, n^1$
	mesh	$0^1, n^{n-1}$
$\mathcal{L}(G)$	star	$0^1, 1^{n-2}, 2^1$
	mesh	$0^1, (\frac{n}{(n-1)})^{n-1}$

Figure 1a shows the three types of energies for a 10-node star graph (1 root node and 9 leaf nodes) as the highest degree nodes are removed iteratively. From the Table II, a star graph has its eigenvalues at $\{-\sqrt{10-1}, 0, 0, 0, 0, 0, 0, 0, 0, \sqrt{10-1}\}$. When these eigenvalues are inserted to Equation 1, we obtain an initial adjacency energy of 6. Once the highest degree node is removed, the empty graph with isolated edges have eigenvalues of 0, thus the adjacency energy becomes 0. While a similar decreasing pattern can be observed for the Laplacian energy, the normalized Laplacian energy starts from 2, peaks to 9, and then starts a decreasing energy pattern as shown in Figure 1a. The eigenvalues of a 10-node star graph are: $\{0, 1.11, 1.11, 1.11, 1.11, 1.11, 1.11, 1.11, 1.11, 1.11\}$, and when we plug these eigenvalues to Equation 3, we end up with a starting energy of 2. Once the highest degree node is removed, the eigenvalues become 0, but due to -1 term in Equation 3, the empty graph's energy converges to $n-1$. Adjacency and Laplacian energies behave the same in a full-mesh graph and drop at an equal rate as nodes are removed (we note that $\mathcal{E}_A(G) = \mathcal{E}_L(G)$ for regular graphs [16]), whereas the normalized Laplacian energy remains the same at 2, as shown in Figure 1b. Finally, we look into the removal of links for a ring graph (intuitively, the connectivity of a ring lies between a star and a full-mesh) as shown in Figure 1c. In the case of the ring toy network, the starting adjacency energy (i.e. 13 from Figure 1c) lies between a star (i.e. 6 from Figure 1a) and full-mesh (i.e. 18 from Figure 1b). The normalized Laplacian energy starts around 6 and ends with an energy of 10. As the links are removed, eventually the graph becomes an empty graph with isolated nodes, each node having an eigenvalue of 0. Due to the -1 term in Equation 3, the normalized Laplacian energy converges to n for an empty graph with all isolated nodes.

B. Energy analysis of backbone networks

To help alleviate possible targeted attacks on backbone communication networks, we can use the energy of the networks as a metric to analyze the breakdown of network structures. The energy of a network can be calculated in several different ways, each yielding unique results as discussed in Section II. In order to provide multiple perspectives of this metric, we will look at three different energy calculations: the adjacency energy, Laplacian energy, and normalized Laplacian energy. To better understand how a network breaks down under an attack scenario, we can simulate an attack by removing nodes or links using various metric-based attack algorithms and calculate the three energy types across each of these simulated attacks: node-betweenness-based, node-closeness-based, node-degree-based, and edge-betweenness-based (Figures 2, 3, 4, and 5, respectively).



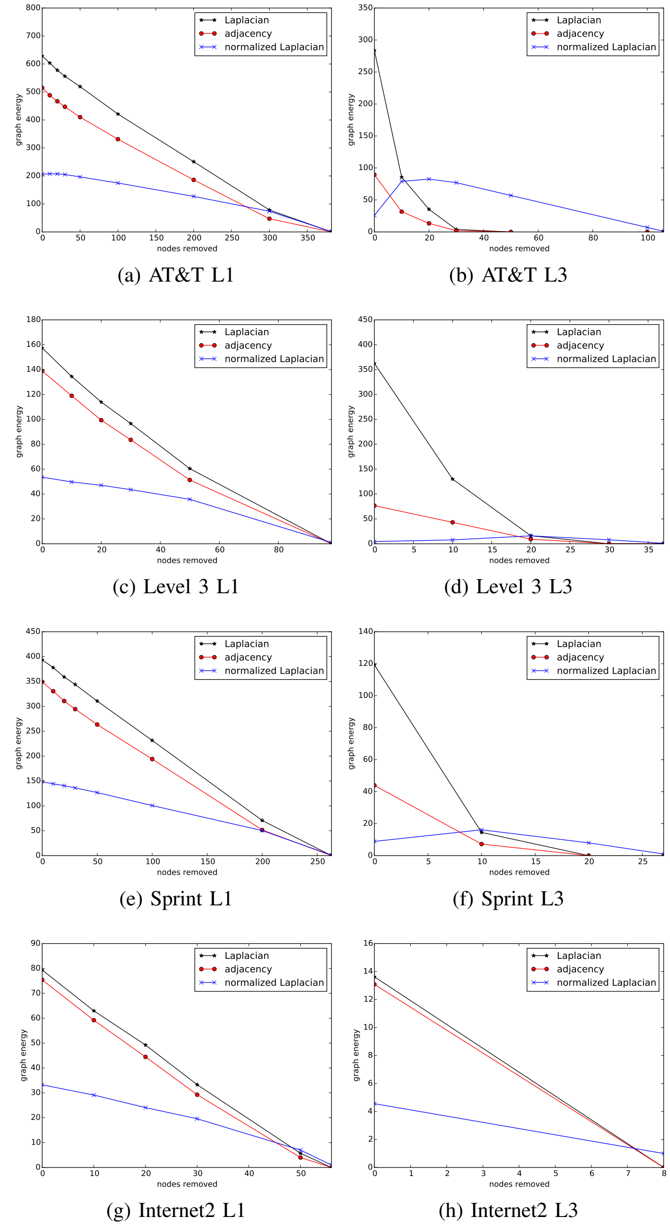
(a) AT&T L1 (b) AT&T L3

(c) Level 3 L1 (d) Level 3 L3

(e) Sprint L1 (f) Sprint L3

(g) Internet2 L1 (h) Internet2 L3

Fig. 2. Energy of US backbone networks based on node betweenness removal



(a) AT&T L1 (b) AT&T L3

(c) Level 3 L1 (d) Level 3 L3

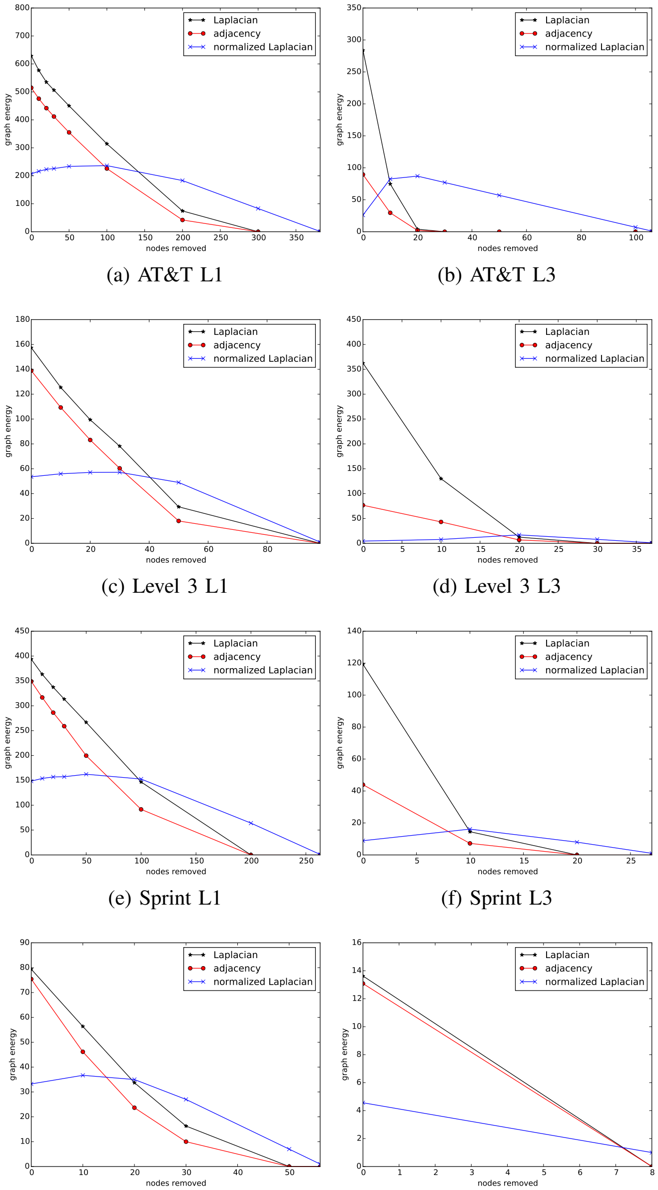
(e) Sprint L1 (f) Sprint L3

(g) Internet2 L1 (h) Internet2 L3

Fig. 3. Energy of US backbone networks based on node closeness removal

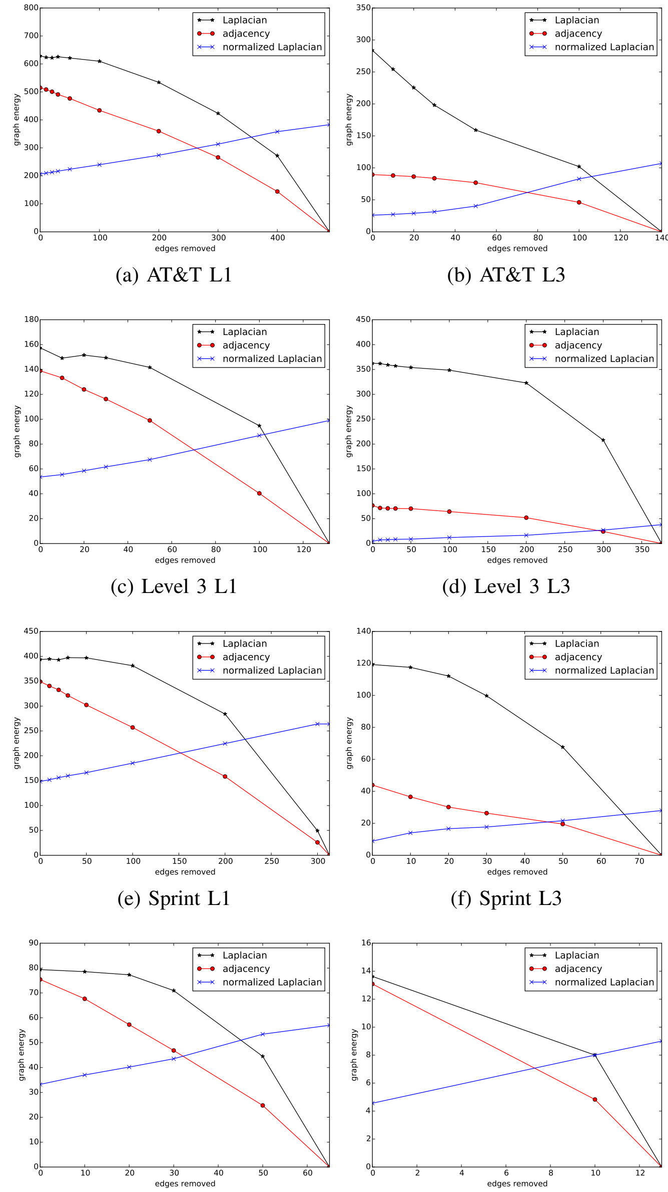
When comparing the energy of the graphs, we can use the three different energy plots ($\mathcal{E}_A$, $\mathcal{E}_L$, and $\mathcal{E}_{\mathcal{L}}$) to draw conclusions as to how the network structures react to having components removed, each of the different energy calculations providing us with different notable features. For instance, we can note that the adjacency energy only decreases as we remove edges or nodes, due to the removal of edge connections, by edge or node removal, directly decreasing the eigenvalues. Moreover, sharp drops in the adjacency energy for node removals (Seen in Figure 4b) represent the termination of high-degree nodes, while sharp drops in the adjacency energy for edge removals (Seen in Figure 5h) represent node disconnections.

The Laplacian energy appears to take on a depletion rate (i.e., the rate at which the energy decreases) proportional to that of the adjacency energy for node removals (as seen in Figure 4c), but not for edge removals. The edge removals for the Laplacian energy appear to show an upward bulge at certain points (Seen in Figure 5c), which may represent a point at which components are becoming disconnected and clustering occurs, but the overall average degree is still relatively high or close to its original value. This observed bulge appears to occur prominently in networks that have a large link betweenness, in which the removal of these high betweenness links would cause an increase in clustering, without causing a significant change in the average node degree.



(a) AT&T L1 (b) AT&T L3 (c) Level 3 L1 (d) Level 3 L3 (e) Sprint L1 (f) Sprint L3 (g) Internet2 L1 (h) Internet2 L3

Fig. 4. Energy of US backbone networks based on node degree removal



(a) AT&T L1 (b) AT&T L3 (c) Level 3 L1 (d) Level 3 L3 (e) Sprint L1 (f) Sprint L3 (g) Internet2 L1 (h) Internet2 L3

Fig. 5. Energy of US backbone networks based on link betweenness removal

The normalized Laplacian energy plot appears to increase and decrease, with peak energy occurring when the majority of eigenvalues are valued near zero or two (Seen in Figure 4f). Due to this feature, edge removals only result in an increase in normalized Laplacian energy due to the eigenvalues only decreasing towards zero as edges are removed, whereas node removals cause the energy to eventually decrease to zero as the contributing nodes are removed entirely. A peak in the normalized Laplacian energy appears to represent a point where a large amount of clustering occurs, along with an increased presence of near-zero-valued eigenvalues.

If we observe the energy plots, we see that the AT&T L1 network demonstrates a relatively linear energy depletion for all three energy types across node-betweenness-based and node-closeness-based removals (Figures 2a and 3a, respectively), but appears to show an increase in clustering for the normalized Laplacian energy plot when node-degree-based removals are used (shown in Figure 4a). Edge-betweenness-based removals for the AT&T L1 (shown in Figure 5a) show that the network maintains a stable energy depletion for the majority of its edge removals, only exhibiting a sharp drop in energy across the last third of removals. We can also observe that the AT&T L3 network exhibits a sharp depletion of Laplacian and adjacency energy to near-zero, along with a normalized Laplacian energy peak, after the first 20 node removals for node-betweenness-based, node-closeness-based, and node-degree-based removals (Figures 2b, 3b, and 4b, respectively). The AT&T L3 network also shows a gradually increasing normalized Laplacian energy that coincides with a slight bulge in the Laplacian energy for edge-betweenness-based removals (Figure 5b), indicating that clustering and the removal of high betweenness links occurs after the removal of nearly 60 edges based on link-betweenness.

The Level 3 L1 network shows a relatively linear energy depletion for all three energy types across node-betweenness-based and node-closeness-based removals (Figures 2c and 3c, respectively), besides a small amount of plateauing over the first couple node-betweenness-based removal steps. For node-degree-based removals (Figure 4c), the Laplacian and adjacency energy plots have a major decrease in depletion rate at 50 nodes removed, coinciding with the falling edge of the normalized Laplacian energy peak, meaning that clustering has occurred and most of the high-degree nodes have already been removed. The edge-betweenness-based removal plot (Figure 5c) shows that the Level 3 L1 network's Laplacian energy exhibits a bulge starting around ten edge removals, telling us that clustering is starting to occur, but the average node degree is largely unaffected. Moreover, the Level 3 L3 network's Laplacian and adjacency energy sharply deplete to near-zero after 20 node removals, while the normalized Laplacian energy reaches its peak, for node-betweenness-based, node-closeness-based, and node-degree-based removals (Figures 2d, 3d, and 4d, respectively). This tells us that at 20 nodes removed, the majority of edge connections have been terminated and there is a large amount of zero-valued or near-zero-valued eigenvalues present. The edge-betweenness-based removal plot (Figure 5d) prominently features a sharp decline in Laplacian energy around 200 edge removals, telling us that there is an increase in the amount of eigenvalues valued near the average degree and that the average degree and average eigenvalue both start to decrease rapidly after 200 edge removals.

The Sprint L1 network shows a linear depletion rate for both Laplacian and adjacency energy for node-betweenness-based removals (Figure 2e), with a major reduction in depletion rate at 200 nodes removed, signifying that the majority of high-degree nodes have been removed and clustering is occurring significantly. Node-closeness-based removals (Figure 3e) exhibit much less prominent evidence of clustering and disconnection, having a mainly linear depletion rate for Laplacian and adjacency energy, with a slight reduction in depletion rate at 200 nodes removed. Node-degree-based removals (Figure 4e) show signs of clustering occurring at 100 nodes removed; the Laplacian and adjacency energy show a reduction in their depletion rate at 100 nodes removed, coinciding with the falling edge of the normalized Laplacian energy peak, telling us that clustering is occurring and most of the high-degree nodes have been removed. The edge-betweenness-based removal plot (Figure 5e) shows a bulge in the Laplacian energy starting to occur near 30 nodes removed, telling us that clustering is occurring while the average degree remains relatively constant, a sign that large sections of the network are becoming isolated clusters as high-betweenness connections are removed. Furthermore, the Sprint L3 network's Laplacian and adjacency energy sharply deplete after 10 node removals, while the normalized Laplacian energy reaches its peak, for node-betweenness-based, node-closeness-based, and node-degree-based removals (Figures 2f, 3f, and 4f, respectively). This tells us that at 10 nodes removed, the majority of edge connections have been terminated and there are a large amount of zero-valued or near-zero-valued eigenvalues present. The edge-betweenness-based removal plot (Figure 5f) prominently features a sharp decline in Laplacian energy around 30 edge removals, telling us that there is an increase in the amount of eigenvalues valued near the average degree, and that the average degree and average eigenvalue both start to quickly decrease after 30 edge removals.

The Internet2 L1 network's node-betweenness-based removal plot (Figure 2g) shows that the Laplacian and adjacency energy deplete to zero at 50 node removals, before the normalized Laplacian energy, meaning that all links are removed by 50 node-removals. The node-closeness-based removal plot (Figure 3g) exhibits a relatively linear depletion rate for all three energy types, with clustering evident by the change in the depletion rate if the Laplacian and adjacency energy at 50 removals. Also, the node-degree-based removal plot's (Figure 4g) Laplacian and adjacency energy show a major decrease in depletion rate at 30 nodes removed before finally depleting to zero at 50 removals. This change in depletion rate appears to coincide with the falling edge of the normalized Laplacian energy peak occurring near 10 nodes removed, meaning that a noticeable amount of clustering has occurred and most of the high-degree nodes have been removed; all edges are removed by 50 node removals. The edge-betweenness-based removal plot (Figure 5g) shows that the Internet2 L1 network's Laplacian energy exhibits a bulge starting around 20 edge removals, indicating that clustering is starting to occur, but the average node degree is largely unaffected until major node disconnection takes place near 50 edge removals. The Internet2 L3 network features a linear depletion rate for all three energy types for node-based-removals (Figures 2h, 3h, and 4h, respectively), but prominently shows the occurrence of node disconnection where the Laplacian and adjacency energy sharply decrease near 10 edges removed on the edge-betweenness-based removal plot (Figure 5h).

The energy plots can provide a different perspective for visualizing the eigenvalue spectrum of networks and how they react to the removal of core components. Each of the three energy calculation methods give rise to different prominent plot features we can use when making observations about network resilience. The analysis of the adjacency, Laplacian, and normalized Laplacian energy, along with the trends that appear in the respective energy plots could provide insight into how node clusters maintain connections with one another despite component removals during a targeted attack and how they may be optimized against such an attack.

V. Conclusions and Future Work

Eigenvalues of a graph could provide insight into how well a given graph is connected. Graph energy – the sum of absolute values – is used to measure the connectivity of graphs. We analyze the adjacency, Laplacian, and normalized Laplacian graph energy of four backbone networks against centrality-based attacks. Notably, each removal algorithm tends to produce energy plots with certain characteristics. Among the three graph energy types, adjacency and Laplacian energies always decrease as the network connectivity become weaker while nodes or links are removed. For the normalized Laplacian, the energy behavior presents an increasing pattern converging to a maximum value equal to the number of nodes when links are removed. The normalized Laplacian energy peaks as nodes are removed – the peak point represents an empty graph with isolated nodes.

We also conclude that the adjacency and Laplacian energy of physical level (i.e. L1 backbone networks) graphs appear to be close under the node-based attacks. The physical-level networks have been shown to have similar grid-like pseudo-regular structures [9], [24]. It was also shown that the adjacency and Laplacian energy values of regular graphs are the same [16]. We believe that the close values of adjacency and Laplacian energies reaffirm that physical-level networks across different backbone providers have similar grid-like network structures. On the other hand, the logical-level networks (i.e. L3 backbone networks) represent star-like structures [9], [24]. The peak-and-drop phenomenon we observe for the normalized Laplacian energy is particularly for the logical-level (i.e. L3) networks, which we believe represents the removal of the centralized nodes in these star-like L3 network structures.

For our future work, we will investigate different attack strategies using different graph metrics. We will also compare the adjacency, Laplacian, and normalized Laplacian graph energies of different type of networks such as wireless and social networks to determine which type of graph energy provides more accurate connectivity information. Finally, we will utilize the methodology presented in this paper to harden networks against attacks.

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