

Reconstruction of Directed Networks from Consensus Dynamics

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Abstract—This paper addresses the problem of identifying the topology of an unknown, weighted, directed network running a consensus dynamics. We propose a methodology to reconstruct the network topology from the dynamic response when the system is stimulated by a wide-sense stationary noise of unknown power spectral density. The method is based on a node-knockout, or grounding, procedure wherein the grounded node broadcasts zero without being eliminated from the network. In this direction, we measure the empirical cross-power spectral densities of the outputs between every pair of nodes for both grounded and ungrounded consensus to reconstruct the unknown topology of the network. We also establish that in the special cases of undirected and purely unidirectional networks, the reconstruction does not need grounding. Finally, we extend our results to the case of a directed network assuming a general dynamics, and prove that the developed method can detect edges and their direction.

I. INTRODUCTION

In recent years, complex dynamical networks have attracted considerable attention [1]. The power grid, the Internet, the World Wide Web, as well as many other biological, social and economic networks [2], are examples of networked dynamic systems that motivate this interest. The availability of datasets describing the structure of many real-world networks has allowed to detect the presence of common patterns in a large variety of networks [3], [4]. In this paper, we address the problem of reconstructing the structure of an unknown network of dynamical nodes from observations of its input-output behavior.

The problem of network reconstruction is crucial in a wide variety of disciplines such as biology [5]–[8], physics [9]–[11] and finance [12]. A wide collection of approaches have been proposed to solve the network reconstruction problem. For example, we find in the literature several papers that approach this problem using an optimization framework, such as [8], [11], [13]. In these papers, the reconstruction problem is stated as the optimization problem of finding the network that maximizes a function that measures the sparsity of the network (e.g. ℓ_1 -norm) while conforming to known a priori structural information. Although the assumption of sparsity is well justified in some applications (e.g. biological networks), this assumptions might lead to unsuccessful topology inference in other cases, as illustrated in [14] and [15]. When the unknown network is known to be a tree, several techniques for network reconstruction were proposed in [12], [16] and [17]. More recently, Materassi and Salapaka proposed in [18] a methodology for reconstruction

of directed networks using Wiener filters. Although effective for many networks, this methodology is not exact when two nonadjacent nodes are connected to a common node with directed edges pointing towards the common node. In [19], Nabi-Abdolyousefi and Mesbahi proposed a technique to extract structural information, such as node degrees, of an undirected network running a consensus dynamics. Subsequently, they find a collection of undirected graphs that are consistent with this structural information. Furthermore, the work in [20] provides a method that performs a complete identification of undirected networks via a procedure called node knock-out.

In this paper, we propose an approach to reconstruct the structure and weights of a directed network from the output of an agreement dynamics run on the network. We also revisit the problem of network reconstruction for more general dynamical systems. We develop methodologies to unveil the network structure from the dynamic response of the network when the system is driven by stochastic inputs. More specifically, we assume that the system is stimulated by a collection of wide-sense stationary noises with unknown power spectral densities. Considering several cases, we propose methodologies to recover the network topology from the empirical cross-power spectral densities of the outputs between every pair of nodes. We first consider the case of undirected networks running a consensus dynamics, and propose an algorithm to reconstruct its unknown topology. Using the node knockout procedure proposed in [20], we extend our results to the directed networks. In this scenario, the node knockout is equivalent to state grounding, where the node broadcasts a zero state to its neighbors without being eliminated from the network. We also prove that for purely unidirectional networks (networks with no reciprocity [21]), there is no need to run the grounded consensus to perform the network identification. Finally, we consider the reconstruction of directed networks assuming a general dynamics. We establish that, without the knowledge of the power spectral densities of the input vector, a Boolean reconstruction is still possible, i.e., we can detect edges and their direction, but not their weights.

The rest of the paper is organized as follows. In section II, we provide some nomenclature needed in our exposition and describe the network reconstruction problem under consideration. Assuming the system is driven by wide-sense stationary noise, in section III.A, we provide a technique to recover the structure of an undirected network running consensus dynamics. Section III.B covers the extension of the problem to a directed network. Section III.C addresses the reconstruction of directed networks following general

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dynamics. Section IV concludes.

II. PRELIMINARIES & PROBLEM DESCRIPTION

In this section, we introduce a series of definitions used throughout the paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an unweighted, undirected graph, where $\mathcal{V} = \{v_1, \dots, v_n\}$ denotes a set of n nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes a set of m undirected edges. If $\{v_i, v_j\} \in \mathcal{E}$, we call nodes v_i and v_j *adjacent* (or first-neighbors), which we denote by $v_i \sim v_j$. A *weighted*, undirected graph is defined as the triad $\mathcal{W} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$, where \mathcal{V} and \mathcal{E} are the sets of nodes and edges in \mathcal{W} , and the function $\mathcal{F} : \mathcal{E} \rightarrow \mathbb{R}$ associates real weights to the edges. Similarly, a weighted, *directed* graph is defined as the triad $\mathcal{D} = (\mathcal{V}, \mathcal{E}_d, \mathcal{F}_d)$, where \mathcal{V} is the set of nodes and \mathcal{E}_d is the set of directed edges in \mathcal{D} , where a directed edge from node v_i to node v_j is defined as an ordered pair (v_i, v_j) . Furthermore, \mathcal{F}_d is a weight function $\mathcal{F}_d : \mathcal{E}_d \rightarrow \mathbb{R}$.

In an unweighted, undirected graph \mathcal{G} , the *degree* of a vertex v_i , denoted by $\deg(v_i)$, is the number of nodes adjacent to it, i.e., $\deg(v_i) = |\{v_j \in \mathcal{V} : \{v_i, v_j\} \in \mathcal{E}\}|$. This definition can be generalized to both weighted and directed graphs. For weighted graphs, the weighted degree of node v_i is equal to $\deg(v_i) = \sum_{j: \{v_i, v_j\} \in \mathcal{E}} \mathcal{F}(\{v_i, v_j\})$, i.e., the sum of the weights associated to edges connected to v_i . For weighted, directed networks, we define the weighted *in-degree* of node v_i as $\deg_{in}(v_i) = \sum_{j: (v_j, v_i) \in \mathcal{E}_d} \mathcal{F}_d((v_j, v_i))$.

The *adjacency matrix* of an unweighted, undirected graph \mathcal{G} , denoted by $A_{\mathcal{G}} = [a_{ij}]$, is a $n \times n$ Boolean symmetric matrix defined entry-wise as $a_{ij} = 1$ if nodes v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. We define the *Laplacian matrix* $L_{\mathcal{G}}$ of a graph \mathcal{G} as $L_{\mathcal{G}} = D_{\mathcal{G}} - A_{\mathcal{G}}$ where $D_{\mathcal{G}}$ is the diagonal matrix of degrees, $D_{\mathcal{G}} = \text{diag}((\deg(v_i))_{i=1}^n)$. For simple graphs, $L_{\mathcal{G}}$ is a symmetric positive semidefinite matrix, which we denote by $L_{\mathcal{G}} \succeq 0$ [22]. Thus, $L_{\mathcal{G}}$ has a full set of n real and orthogonal eigenvectors with real nonnegative eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$.

Similarly, the weighted adjacency of a weighted graph \mathcal{W} is defined as $A_{\mathcal{W}} = [w_{ij}]$, where $w_{ij} = \mathcal{F}(\{v_i, v_j\})$ for $\{v_i, v_j\} \in \mathcal{E}$, and $w_{ij} = 0$ if $\{v_i, v_j\} \notin \mathcal{E}$. We define the *degree matrix* of a weighted graph \mathcal{W} as the diagonal matrix $D_{\mathcal{W}} = \text{diag}((\deg(v_i))_{i=1}^n)$. The Laplacian matrix of a weighted, undirected graph \mathcal{W} , is defined as $L_{\mathcal{W}} = D_{\mathcal{W}} - A_{\mathcal{W}}$. Furthermore, the adjacency matrix of a weighted, directed graph \mathcal{D} is defined as $A_{\mathcal{D}} = [d_{ij}]$, where $d_{ij} = \mathcal{F}_d((v_j, v_i))$ for $(v_j, v_i) \in \mathcal{E}_d$, and $d_{ij} = 0$ if $(v_j, v_i) \notin \mathcal{E}_d$. We define the *in-degree matrix* of a directed graph \mathcal{D} as the diagonal matrix $D_{\mathcal{D}} = \text{diag}((\deg_{in}(v_i))_{i=1}^n)$. The Laplacian matrix of \mathcal{D} is then defined as $L_{\mathcal{D}} = D_{\mathcal{D}} - A_{\mathcal{D}}$. Note that the Laplacian matrix satisfies $L_{\mathcal{G}}\mathbf{1} = L_{\mathcal{W}}\mathbf{1} = L_{\mathcal{D}}\mathbf{1} = \mathbf{0}$, i.e., the vector of all ones is an eigenvector of the Laplacian matrix with corresponding eigenvalue 0.

In this paper, we focus on the continuous-time non-autonomous model of consensus dynamics in directed networks, described as

$$\dot{x}(t) = -L_{\mathcal{D}}x(t) + \mathbf{w}(t), \quad y(t) = x(t), \quad (1)$$

where $L_{\mathcal{D}}$ is the Laplacian matrix of a weighted, directed network \mathcal{D} ; and the vectors $x(t), \mathbf{w}(t), y(t) \in \mathbb{R}^n$ are the state, input, and output vectors, respectively. We assume that we do not have access to the network structure, i.e., $L_{\mathcal{D}}$ is unknown. In this context, we consider the problem of identifying the topology of the directed network \mathcal{D} when the (cross-)power spectral densities of the output vector $y(t)$ are measured empirically while the stochastic input vector $\mathbf{w}(t)$, injected at the nodes of the network, has unknown power spectral characteristics. We also investigate the reconstruction problem in the case that (1) follows a general dynamics $G_{\mathcal{D}}$ in lieu of $-L_{\mathcal{D}}$. In any case, we assume the input $\mathbf{w}(t) = [w_i(t)]$ is a vector of uncorrelated wide-sense stationary processes.

Definition 1 (Wide-Sense Stationary): A continuous-time scalar random process $w(t)$ is wide-sense stationary (WSS), if it satisfies the following properties:

$$\text{P1.} \quad \mu_w(t) \triangleq \mathbb{E}(w(t)) = \mu_w(t + \tau) \text{ for any } \tau \in \mathbb{R}.$$

$$\text{P2.} \quad R_w(t_1, t_2) \triangleq \mathbb{E}(w(t_1)w(t_2)) = R_w(t_1 + \tau, t_2 + \tau) = R_w(t_1 - t_2, 0) \text{ for any } \tau \in \mathbb{R}.$$

The reconstruction methods proposed in this paper take the output vector $y(t) = [y_i(t)]$ and deliver the network structure as a function of the (cross-)power spectral densities.

Definition 2 ((Cross-)Power Spectral Density): The cross-power spectral density of two WSS signals, $y_i(t)$ and $y_j(t)$, is the Fourier transform of their cross-correlation function, i.e.,

$$S_{y_i y_j}(\omega) \triangleq \mathcal{F}\{R_{y_i y_j}(\tau) \triangleq \mathbb{E}(y_i(t)y_j(t - \tau))\},$$

where $\mathcal{F}\{\cdot\}$ is the Fourier transform operator. The power spectral density of $y_i(t)$ is defined as

$$S_{y_i}(\omega) \triangleq \mathcal{F}\{R_{y_i}(\tau) \triangleq \mathbb{E}(y_i(t)y_i(t - \tau))\}.$$

In order to detect the links through observation of outputs, we need to ascertain that system (1) is driven by a collection of noises with nonzero power spectral densities. To capture this idea, we commence with the following definition:

Definition 3 (Excitation Frequency Interval): The excitation frequency interval of a vector $\mathbf{w}(t)$ of wide-sense stationary processes is defined as an interval $(-\Omega, \Omega)$, with $\Omega > 0$, such that the spectral densities of the input components $w_i(t)$ satisfy $S_{w_i}(\omega) > 0$ for all $\omega \in (-\Omega, \Omega)$, and all $i \in \{1, 2, \dots, n\}$.

We end this section by stating our assumptions. Throughout the paper we impose the following conditions on the input vector:

$$\text{A1.} \quad \text{The collection of signals } \{w_i(t), i = 1, \dots, n\} \text{ are uncorrelated zero-mean WSS processes such that, for any } t, \tau \in \mathbb{R},$$

$$\mathbb{E}(w_i(t)w_j(\tau)) = 0, \text{ for } i \neq j,$$

and

$$R_{w_i}(\tau) = \mathbb{E}(w_i(t)w_i(t + \tau)) = R_w(\tau).$$

- A2. There exists a nonempty excitation frequency interval $(-\Omega, \Omega)$.

III. MAIN RESULTS

We now consider several cases and present methodologies to reconstruct the structure of an unknown network from observations of its temporal response (1). We provide reconstruction techniques for weighted, undirected networks (Sect. III.A), and weighted, directed networks (Sect. III.B) following consensus dynamics. We briefly, revisit the reconstruction of directed networks following general dynamics (Sect. III.C).

For the consensus dynamics (as well as general dynamics) of weighted, directed networks, the following lemma provides an explicit relationship between the cross-power spectral densities of two outputs, $y_i(t)$ and $y_j(t)$, and the power spectral density of the input $w_k(t)$ (which we assume to be identical, i.e., $S_{w_k}(\omega) = S_w(\omega)$ for all k):

Lemma 4: Given assumptions (A1)-(A2), the following identity holds

$$\frac{S_{y_i y_j}(\omega)}{S_w(\omega)} = \mathbf{e}_i^T (\omega^2 I - \mathbf{j}\omega(L_{\mathcal{D}} - L_{\mathcal{D}}^T) + L_{\mathcal{D}}^T L_{\mathcal{D}})^{-1} \mathbf{e}_j, \quad (2)$$

for any $\omega \in (-\Omega, \Omega)$, where

$$S_{y_i y_j}(\omega) \triangleq \mathcal{F}\{R_{y_i y_j}(\tau)\} \quad \text{and} \quad S_w(\omega) \triangleq \mathcal{F}\{R_w(\tau)\},$$

for any $1 \leq i, j \leq n$.

Proof: The transfer function corresponding to the state-space equations (1) is $H(\omega) = (\mathbf{j}\omega I + L_{\mathcal{D}})^{-1}$. The transfer function from the k -th input $w_k(t)$ to the i -th output $y_i(t)$ is defined as

$$H_{ki}(\omega) \triangleq \mathbf{e}_i^T (\mathbf{j}\omega I + L_{\mathcal{D}})^{-1} \mathbf{e}_k.$$

Therefore, the power spectral density of the i -th output $y_i(t)$ when the input is $\mathbf{w}(t) = w_k(t) \mathbf{e}_k$ (i.e., a WSS noise on the k -th node) is equal to

$$S_{y_i}(\omega) = H_{ki}(\omega) H_{ki}^*(\omega) S_{w_k}(\omega). \quad (3)$$

On the other hand, the transfer functions from input $w_k(t)$ to the outputs $y_i(t)$ and $y_j(t)$ are, respectively, $Y_i(\omega)/W_k(\omega) = H_{ki}(\omega)$ and $Y_j(\omega)/W_k(\omega) = H_{kj}(\omega)$. Hence, $Y_j(\omega)/Y_i(\omega) = H_{ki}^{-1}(\omega) H_{kj}(\omega)$ which implies

$$S_{y_i y_j}(\omega) = \left(H_{kj}(\omega) H_{ki}^{-1}(\omega) \right)^* S_{y_i}(\omega). \quad (4)$$

Since $S_{w_k}(\omega) = S_w(\omega)$ for all k , we can combine (3) and (4) to obtain

$$S_{y_i y_j}(\omega) = H_{ki}(\omega) H_{kj}^*(\omega) S_w(\omega). \quad (5)$$

When the input is $\mathbf{w}(t) = \sum_{k=1}^n w_k(t) \mathbf{e}_k$, with $\mathbb{E}(w_i(t) w_j(\tau)) = 0$ for $i \neq j$, we can superpose (5) over $1 \leq k \leq n$, to obtain the following for any $\omega \in (-\Omega, \Omega)$:

$$\begin{aligned} \frac{S_{y_i y_j}(\omega)}{S_w(\omega)} &= \sum_{k=1}^n H_{kj}^*(\omega) H_{ki}(\omega) \\ &= \sum_{k=1}^n \mathbf{e}_j^T (-\mathbf{j}\omega I + L_{\mathcal{D}})^{-1} \mathbf{e}_k \mathbf{e}_k^T (\mathbf{j}\omega I + L_{\mathcal{D}}^T)^{-1} \mathbf{e}_i \\ &= \sum_{k=1}^n \mathbf{e}_j^T (-\mathbf{j}\omega I + L_{\mathcal{D}})^{-1} \mathbf{e}_k \mathbf{e}_k^T (\mathbf{j}\omega I + L_{\mathcal{D}}^T)^{-1} \mathbf{e}_i \\ &= \mathbf{e}_j^T (-\mathbf{j}\omega I + L_{\mathcal{D}})^{-1} \left(\sum_{k=1}^n \mathbf{e}_k \mathbf{e}_k^T \right) (\mathbf{j}\omega I + L_{\mathcal{D}}^T)^{-1} \mathbf{e}_i. \end{aligned}$$

As $\sum_{k=1}^n \mathbf{e}_k \mathbf{e}_k^T = I$, we can simplify the last equation to

$$\begin{aligned} \frac{S_{y_i y_j}(\omega)}{S_w(\omega)} &= \mathbf{e}_j^T (-\mathbf{j}\omega I + L_{\mathcal{D}})^{-1} (\mathbf{j}\omega I + L_{\mathcal{D}}^T)^{-1} \mathbf{e}_i \\ &= \mathbf{e}_i^T (\omega^2 I - \mathbf{j}\omega L_{\mathcal{D}} + \mathbf{j}\omega L_{\mathcal{D}}^T + L_{\mathcal{D}}^T L_{\mathcal{D}})^{-1} \mathbf{e}_j, \end{aligned}$$

which is the desired relation. \blacksquare

Corollary 5: Given assumptions (A1)-(A2), and substituting $-L_{\mathcal{D}}$ in (1) with any negative semi-definite $G_{\mathcal{D}}$, the input-output power spectra relationship is as following

$$\frac{S_{y_i y_j}(\omega)}{S_w(\omega)} = \mathbf{e}_i^T (\omega^2 I - \mathbf{j}\omega(G_{\mathcal{D}}^T - G_{\mathcal{D}}) + G_{\mathcal{D}}^T G_{\mathcal{D}})^{-1} \mathbf{e}_j,$$

for any $\omega \in (-\Omega, \Omega)$.

Proof: In the proof of Lemma 4, we did not use any properties of Laplacian, so we only need to replace $-L_{\mathcal{D}}$ by any negative semi-definite $G_{\mathcal{D}}$ in (2) to derive the result. \blacksquare

In our analysis in the next subsections, we frequently invoke the result of Lemma 4 and Corollary 5.

A. Undirected Laplacian Identification from Stochastic Inputs

In this subsection, we propose an approach to reconstruct the topology of an unknown weighted, undirected network \mathcal{W} from the output of (1). The formal statement of the reconstruction problem for weighted, undirected network can be stated as follows:

Problem 6: Consider the dynamical network in (1), with $L_{\mathcal{D}} \equiv L_{\mathcal{W}}$ where \mathcal{W} is an unknown weighted, undirected graph. Find the structure of \mathcal{W} , from the empirical (cross-)power spectral densities¹ of the outputs, i.e., $S_{y_i}(\omega)$ and $S_{y_i y_j}(\omega)$ for all $1 \leq i, j \leq n$.

In the case of weighted, undirected networks, we can use the result of Lemma 4 in the form of the following corollary:

Corollary 7: Consider the network dynamics (1), when \mathcal{D} is a weighted, undirected network \mathcal{W} . Given assumptions (A1)-(A2), the (cross-)power spectral densities satisfy

$$\frac{S_{y_i y_j}(\omega)}{S_w(\omega)} = \mathbf{e}_i^T (\omega^2 I + L_{\mathcal{W}}^2)^{-1} \mathbf{e}_j, \quad (6)$$

for any $1 \leq i, j \leq n$.

Proof: Since $L_{\mathcal{W}}$ is symmetric, replacing $L_{\mathcal{D}}$ in (2) by $L_{\mathcal{W}}$, the proof follows immediately. \blacksquare

¹There are several methods to empirically measure the (cross-)power spectral densities (see e.g., Welch's method [23]).

We now proceed with a theorem that can be used to reconstruct a weighted, undirected Laplacian $L_{\mathcal{W}}$, from the empirical (cross-)power spectral densities $S_{y_i}(\omega)$ and $S_{y_i y_j}(\omega)$ for all $1 \leq i, j \leq n$.

Theorem 8: Consider the network dynamics (1), when \mathcal{D} is a weighted, undirected network \mathcal{W} . Let us define the matrix of cross-correlations as $\mathbf{S}(\omega) = [S_{y_i y_j}(\omega)]$. Then, given assumptions (A1)-(A2), we can recover the weighted, undirected Laplacian $L_{\mathcal{W}}$ as

$$L_{\mathcal{W}} = \omega \left(\frac{\mathbf{S}^{-1}(\omega)}{[\mathbf{S}^{-1}(\omega)\mathbf{1}]_i} - I \right)^{1/2}, \quad (7)$$

for any i , and for any ω in the excitation frequency interval $(-\Omega, \Omega)$.

Proof: According to Corollary 7, we have

$$\mathbf{S}(\omega) = S_w(\omega)(\omega^2 I + L_{\mathcal{W}}^2)^{-1},$$

which yields

$$\mathbf{S}^{-1}(\omega) = \frac{1}{S_w(\omega)}(\omega^2 I + L_{\mathcal{W}}^2). \quad (8)$$

Hence,

$$L_{\mathcal{W}}^2 = S_w(\omega)\mathbf{S}^{-1}(\omega) - \omega^2 I. \quad (9)$$

We can derive an expression for $S_w(\omega)$ in terms of $\mathbf{S}^{-1}(\omega)$ and ω , as follows. Post-multiplying (8) by the vector $\mathbf{1}$, we get

$$\mathbf{S}^{-1}(\omega)\mathbf{1} = \frac{1}{S_w(\omega)}(\omega^2 I + L_{\mathcal{W}}^2)\mathbf{1} = \frac{\omega^2}{S_w(\omega)}\mathbf{1}, \quad (10)$$

since $L_{\mathcal{W}}^2 \mathbf{1} = 0$. Therefore,

$$S_w(\omega) = \frac{\omega^2}{[\mathbf{S}^{-1}(\omega)\mathbf{1}]_i}, \quad (11)$$

for any i . Substituting (11) in (9), we obtain the statement of our Theorem. (Notice that $L_{\mathcal{W}}$ is the Laplacian of an undirected network, hence, it is positive semidefinite). ■

In practice, since $[\mathbf{S}^{-1}(\omega)\mathbf{1}]_i$ is a noisy measurement, according to (10) one can average out noise by replacing $[\mathbf{S}^{-1}(\omega)\mathbf{1}]_i$ with $\mathbf{1}^T \mathbf{S}^{-1}(\omega) \mathbf{1} / n$, to achieve improved empirical results. Hence, (7) can be rewritten as

$$L_{\mathcal{W}} = \omega \left(\frac{\mathbf{S}^{-1}(\omega)}{\mathbf{1}^T \mathbf{S}^{-1}(\omega) \mathbf{1} / n} - I \right)^{1/2}.$$

Remark 9: To derive node degrees computationally efficient, one can circumvent the implicit eigenvalue decomposition involved in the result of Theorem 8 for computing $L_{\mathcal{W}}$ from $L_{\mathcal{W}}^2$, since

$$\begin{aligned} [L_{\mathcal{W}}^2]_{ii} &= [(D_{\mathcal{W}} - A_{\mathcal{W}})^2]_{ii} \\ &= [D_{\mathcal{W}}^2 - D_{\mathcal{W}} A_{\mathcal{W}} - A_{\mathcal{W}} D_{\mathcal{W}} + A_{\mathcal{W}}^2]_{ii} \\ &= \deg^2(v_i) + \deg(v_i). \end{aligned}$$

Therefore, computing $L_{\mathcal{W}}^2$, we obtain the degrees by solving the quadratic equation above for any i .

B. Directed Laplacian Identification from Stochastic Inputs

In this section we address the problem of reconstructing a weighted, directed Laplacian $L_{\mathcal{D}}$ when the input is a vector of uncorrelated wide-sense stationary processes $\mathbf{w}(t)$.

Problem 10: Consider the dynamical network in (1), where \mathcal{D} is an unknown weighted, directed graph. Reconstruct \mathcal{D} , from the empirical (cross-)power spectral densities of the outputs, i.e., $S_{y_i}(\omega)$ and $S_{y_i y_j}(\omega)$ for all $1 \leq i, j \leq n$.

In what follows, we propose a reconstruction approach based on a grounded consensus dynamics, similar to the one proposed in [20] for the reconstruction of undirected graphs. The grounded consensus is defined as follows:

Definition 11 (Grounded Consensus): The consensus dynamics with node v_j grounded takes the form

$$\dot{x}(t) = -\tilde{L}_{\mathcal{D}_j} x(t) + \mathbf{w}(t), \quad y(t) = x(t), \quad (12)$$

where $\tilde{L}_{\mathcal{D}_j} \in \mathbb{R}^{(n-1) \times (n-1)}$ is obtained by eliminating the j -th row and the j -th column from $L_{\mathcal{D}}$.

The consensus dynamics with node v_j grounded describes the evolution of the network when we force the state of node v_j to be $x_j(t) \equiv 0$. We now state a lemma and a corollary to extract $L_{\mathcal{D}}^T L_{\mathcal{D}}$ from the ungrounded consensus (1), and $\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j}$ from the grounded consensus (12), respectively.

Lemma 12: Consider the network dynamics (1), when \mathcal{D} is a weighted, directed network. Let us define the matrix of cross-correlations as $\mathbf{S}(\omega) = [S_{y_i y_j}(\omega)]$. Then, given assumptions (A1)-(A2), we can compute $L_{\mathcal{D}}^T L_{\mathcal{D}}$ and $L_{\mathcal{D}} - L_{\mathcal{D}}^T$ as

$$\begin{aligned} L_{\mathcal{D}}^T L_{\mathcal{D}} &= \omega^2 \left(\frac{\text{Re}\{\mathbf{S}^{-1}(\omega)\}}{[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_i} - I \right) \\ L_{\mathcal{D}} - L_{\mathcal{D}}^T &= -\omega \left(\frac{\text{Im}\{\mathbf{S}^{-1}(\omega)\}}{[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_i} \right), \end{aligned}$$

for any i , and for any ω in the excitation frequency interval $(-\Omega, \Omega)$.

Proof: Let us consider the matrix $\mathbf{S}(\omega) = [S_{y_i y_j}(\omega)]$. According to (2), we can separate the real and imaginary parts of its inverse as

$$\text{Re}\{\mathbf{S}^{-1}(\omega)\} = \frac{1}{S_w(\omega)}(\omega^2 I + L_{\mathcal{D}}^T L_{\mathcal{D}}) \quad (13)$$

$$\text{Im}\{\mathbf{S}^{-1}(\omega)\} = -\frac{1}{S_w(\omega)}(\omega L_{\mathcal{D}} - \omega L_{\mathcal{D}}^T). \quad (14)$$

Post-multiplying (13) by $\mathbf{1}$, and taking into consideration that $L_{\mathcal{D}}^T L_{\mathcal{D}} \mathbf{1} = \mathbf{0}$, we obtain

$$S_w(\omega) = \frac{\omega^2}{[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_i}, \quad (15)$$

for any i . Plugging (15) into (13) and (14), the proof follows immediately. ■

Corollary 13: Consider the network dynamics (12), where $\tilde{L}_{\mathcal{D}_j}$ is the grounded Laplacian matrix of the weighted, directed network \mathcal{D} grounded at v_j . Let us define the matrix of cross-correlations as $\tilde{\mathbf{S}}(\omega) = [\tilde{S}_{y_i y_k}(\omega)]$ for all $i, k \neq j$

j . Then, given assumptions (A1)-(A2), we can compute $\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j}$ as

$$\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j} = \omega^2 \left(\frac{\text{Re}\{\tilde{\mathbf{S}}^{-1}(\omega)\}}{[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_i} - I \right),$$

for any $i \neq j$, and for any ω in the excitation frequency interval $(-\Omega, \Omega)$.

Proof: By Corollary 5, it holds that

$$\frac{\tilde{S}_{y_i y_k}(\omega)}{S_w(\omega)} = \mathbf{e}_i^T (\omega^2 I - \mathbf{j}\omega(\tilde{L}_{\mathcal{D}_j} - \tilde{L}_{\mathcal{D}_j}^T) + \tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j})^{-1} \mathbf{e}_k.$$

Since $\tilde{\mathbf{S}}(\omega) = [\tilde{S}_{y_i y_k}(\omega)]$, we have that

$$\text{Re}\{\tilde{\mathbf{S}}^{-1}(\omega)\} = \frac{1}{S_w(\omega)} (\omega^2 I + \tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j}). \quad (16)$$

Substituting the expression for $S_w(\omega)$ from Lemma 12, we have

$$\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j} = \omega^2 \left(\frac{\text{Re}\{\tilde{\mathbf{S}}^{-1}(\omega)\}}{[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_i} - I \right).$$

In general, the results of Lemma 12 might not be informative enough to extract the underlying structure of the network without running the grounded consensus. We will see at the end of this section how running both ungrounded and grounded consensus will lead to exact reconstruction of the network. However, imposing certain conditions on the network graph, allows us to perform the reconstruction without recourse to grounding. For instance, for a network that does not contain any bidirectional edge, i.e., if $(v_j, v_i) \in \mathcal{E}_d$, then $(v_i, v_j) \notin \mathcal{E}_d$, we can employ Lemma 12 to identify the network. The adjacency matrix of a purely unidirectional network satisfies

$$\text{tr}(A_{\mathcal{D}}^2) = 0. \quad (17)$$

We now proceed to the identification technique of a purely unidirectional network.

Corollary 14: Suppose the conditions in the Lemma 12 hold. Also, suppose the weighted, directed network satisfies (17). Then, the entries of adjacency are recovered as

$$[A_{\mathcal{D}}]_{ij} = \max \left\{ \omega n \left(\frac{[\text{Im}\{\mathbf{S}^{-1}(\omega)\}]_{ij}}{\mathbf{1}^T \text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}} \right), 0 \right\}.$$

Proof: For any k , by (15) we have

$$[\text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}]_k = \frac{1}{n} (\mathbf{1}^T \text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}).$$

Therefore, under purview of Lemma 12 we obtain

$$L_{\mathcal{D}} - L_{\mathcal{D}}^T = -\omega n \left(\frac{\text{Im}\{\mathbf{S}^{-1}(\omega)\}}{\mathbf{1}^T \text{Re}\{\mathbf{S}^{-1}(\omega)\}\mathbf{1}} \right).$$

If the ij -th entry of the right hand side in the above equation is negative, since there is no bidirectional edge, $[A_{\mathcal{D}}]_{ij} \neq 0$ and $[A_{\mathcal{D}}]_{ji} = 0$. If the ij -th entry of the right hand side in the above equation is positive, the ji -th entry would be negative which implies $[A_{\mathcal{D}}]_{ji} \neq 0$ and $[A_{\mathcal{D}}]_{ij} = 0$. In case

the ij -th entry is zero, no directed edge between v_i and v_j exists, and the proof is complete. ■

In the next theorem, we show using grounding, there is no need for structural conditions to solve Problem 10.

Theorem 15: Consider the network dynamics (1), where \mathcal{D} is a weighted, directed network. Let us also consider the grounded consensus dynamics in (12), when node v_j is grounded. Then, given assumptions (A1)-(A2), we can recover the entries of the weighted, directed adjacency matrix $A_{\mathcal{D}}$ as

$$[A_{\mathcal{D}}]_{ji} = \begin{cases} \sqrt{[L_{\mathcal{D}}^T L_{\mathcal{D}}]_{ii} - [\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j}]_{ii}} & \text{for } i < j, \\ \sqrt{[L_{\mathcal{D}}^T L_{\mathcal{D}}]_{ii} - [\tilde{L}_{\mathcal{D}_j}^T \tilde{L}_{\mathcal{D}_j}]_{i-1, i-1}} & \text{for } i > j, \end{cases}$$

for any $i \neq j$.

Proof: For simplicity, we consider the case $j = n$ (for any $j \neq n$, we can transform the problem to the case $j = n$ via a simple reordering of rows and columns). Running the ungrounded consensus (1) and applying Lemma 12, we recover $L_{\mathcal{D}}^T L_{\mathcal{D}}$. Also, running the grounded consensus (12) and applying Corollary 13, we can recover $\tilde{L}_{\mathcal{D}_n}^T \tilde{L}_{\mathcal{D}_n}$. Hence, for any $i < n$

$$[L_{\mathcal{D}}^T L_{\mathcal{D}}]_{ii} - [\tilde{L}_{\mathcal{D}_n}^T \tilde{L}_{\mathcal{D}_n}]_{ii} = \sum_k [L_{\mathcal{D}}]_{ki}^2 - \sum_{k \neq n} [L_{\mathcal{D}}]_{ki}^2 = [L_{\mathcal{D}}]_{ni}^2.$$

Hence, we can recover the adjacency as

$$[A_{\mathcal{D}}]_{ni} = \sqrt{[L_{\mathcal{D}}^T L_{\mathcal{D}}]_{ii} - [\tilde{L}_{\mathcal{D}_n}^T \tilde{L}_{\mathcal{D}_n}]_{ii}},$$

for any $i < n$. The same analysis holds for any other $j \neq n$. ■

C. Boolean Reconstruction of General Dynamics

In this section, we study the reconstruction of a general dynamics $G_{\mathcal{D}}$ in weighted, directed networks, characterized as

$$\dot{x}(t) = G_{\mathcal{D}} x(t) + \mathbf{w}(t), \quad y(t) = x(t), \quad (18)$$

where $G_{\mathcal{D}}$ is a negative semi-definite matrix.

It turns out that for the general dynamics (18), complete identification including weights is not possible. However, we can propose an algorithm for a Boolean reconstruction of the system which detects the existence and direction of edges in the directed network (18).

Problem 16: Consider the dynamical network in (18), where \mathcal{D} is an unknown weighted, directed graph. Reconstruct an unweighted version of \mathcal{D} , from the empirical (cross-)power spectral densities of the outputs, i.e., $S_{y_i}(\omega)$ and $S_{y_i y_j}(\omega)$ for all $1 \leq i, j \leq n$.

To solve Problem 16, similar to Definition 11, we define the *Grounded Dynamics* at node v_j as

$$\dot{x}(t) = \tilde{G}_{\mathcal{D}_j} x(t) + \mathbf{w}(t), \quad y(t) = x(t), \quad (19)$$

where $\tilde{G}_{\mathcal{D}_j} \in \mathbb{R}^{(n-1) \times (n-1)}$ is obtained by eliminating the j -th row and the j -th column from $G_{\mathcal{D}}$. We now state a theorem to reconstruct an unweighted version of \mathcal{D} .

Theorem 17: Consider the network dynamics (18), where \mathcal{D} is a weighted, directed network. Let us also consider the grounded consensus dynamics in (19), when node v_j is grounded. Then, given assumptions (A1)-(A2), we can recover a scaled version of off diagonal entires for matrix $G_{\mathcal{D}}$ as

$$[G_{\mathcal{D}}]_{ji} = \begin{cases} \sqrt{S_w[\text{Re}\{\mathbf{S}^{-1}_{ii} - [\tilde{\mathbf{S}}^{-1}]_{ii}\}]} & \text{for } i < j, \\ \sqrt{S_w[\text{Re}\{\mathbf{S}^{-1}_{ii} - [\tilde{\mathbf{S}}^{-1}]_{i-1,i-1}\}]} & \text{for } i > j, \end{cases}$$

for any $i \neq j$.

Proof: Without loss of generality, let $j = n$ as in the proof of Theorem 15. By Corollary 5, it holds that

$$\begin{aligned} S_w(\omega)\text{Re}\{\mathbf{S}^{-1}(\omega)\} &= \omega^2 I + G_{\mathcal{D}}^T G_{\mathcal{D}} \\ S_w(\omega)\text{Re}\{\tilde{\mathbf{S}}^{-1}(\omega)\} &= \omega^2 I + \tilde{G}_{\mathcal{D}_n}^T \tilde{G}_{\mathcal{D}_n}, \end{aligned}$$

which entails, for any $i < n$

$$\begin{aligned} [G_{\mathcal{D}}]_{ni}^2 &= [G_{\mathcal{D}}^T G_{\mathcal{D}}]_{ii} - [\tilde{G}_{\mathcal{D}_n}^T \tilde{G}_{\mathcal{D}_n}]_{ii} \\ &= S_w(\omega) \left([\text{Re}\{\mathbf{S}^{-1}(\omega)\}]_{ii} - [\text{Re}\{\tilde{\mathbf{S}}^{-1}(\omega)\}]_{ii} \right), \end{aligned}$$

which implies the result of the theorem. The proof for $j \neq n$ follows with similar arguments. ■

Remark 18: Results of previous sections regarding Laplacian (Sect. III.A-B), could be extended to any negative semi-definite $G_{\mathcal{D}}$ with a nonzero nullity, as long as a nonzero vector in the kernel of $G_{\mathcal{D}}$ is known.

Theorems 8, 15 and 17 are functional for any frequency $\omega \neq 0$, and we do not require the knowledge of entire power spectral densities. In fact, we only need the power spectral densities evaluated at one frequency, which dramatically reduces the computational complexity. One can also evaluate the spectra at several frequencies to average out the measurement noise and make a robust reconstruction.

IV. CONCLUSIONS

In this paper, we addressed the problem of identifying the topology of an unknown, weighted, directed network running a consensus dynamics when the system is stimulated by a wide-sense stationary noise of unknown power spectral density. We proposed a methodology for network reconstruction based on sequentially grounding nodes in the system. We showed how to reconstruct the topology of the network from the empirical cross-power spectral densities of the outputs between every pair of nodes. We also established that, in the special cases of undirected (i.e., perfect edge reciprocity) or purely unidirectional networks (i.e., no edge reciprocity), our reconstruction procedure does not require any grounding. Finally, we extended our results to the case of a directed network with a general dynamics, and proved that the developed method can detect the presence of edges and their direction.

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