Natural Connectivity of Complex Networks

Article *in* Chinese Physics Letters · July 2010

DOI: 10.1088/0256-307X/27/7/078902

Jun Wu
National University of Defense Technology

Mauricio Barahona
Imperial College London

Hong-Zhong Deng
National University of Defense Technology

Natural Connectivity of Complex Networks *

WU Jun(吴俊)[1,2**], Mauricio Barahona[2,3], TAN Yue-Jin(谭跃进)[1], DENG Hong-Zhong(邓宏钟)[1**]
[1]*College of Information Systems and Management, National University of Defense Technology, Changsha 410073*
[2]*Institute for Mathematical Sciences, Imperial College London, London SW7 2PG, United Kingdom*
[3]*Department of Bioengineering, Imperial College London, London SW7 2AZ, United Kingdom*

(Received 22 April 2010)

The concept of natural connectivity is reported as a robustness measure of complex networks. The natural connectivity has a clear physical meaning and a simple mathematical formulation. It is shown that the natural connectivity can be derived mathematically from the graph spectrum as an average eigenvalue and that it changes strictly monotonically with the addition or deletion of edges. By comparing the natural connectivity with other typical robustness measures within a scenario of edge elimination, it is demonstrated that the natural connectivity has an acute discrimination which agrees with our intuition.

PACS: 89.75.Hc, 89.75.Fb DOI: 10.1088/0256-307X/27/7/078902

Complex networks rely for their function and performance on their robustness, i.e., the ability of a network to maintain its connectivity when a fraction of its vertices is damaged. Because of its broad application, the network robustness has received growing attention[1−8] and has been one of the most central topics in the complex network research.[9−11]

Simple and effective measures of robustness are essential for improved design in different areas of systems science. A variety of measures, based on different heuristics, have been proposed to quantify the robustness of networks. For instance, the vertex (edge) connectivity of a graph is an important, and probably the earliest, measure of robustness of a network.[12] However, the connectivity only partly reflects the ability of graphs to retain connectedness after vertex (or edge) deletion. Other improved measures were introduced and studied, such as toughness,[13] scattering number,[14] integrity,[15] tenacity,[16] etc. In contrast to vertex (edge) connectivity, these measures consider both the cost to damage a network and how badly the network is damaged. From an algorithmic point of view, it is unfortunate that the problem of recognizing these measures of general graphs is NP-complete.[17] This implies that these measures are of no great practical use within the context of complex networks. Another remarkable measure to unfold the robustness of a network is the second smallest (first non-zero) eigenvalue of the Laplacian matrix, also known as the algebraic connectivity.[18] Fiedler showed that the magnitude of the algebraic connectivity reflects how well the overall graph is connected; the larger the algebraic connectivity is, the more difficult it is to cut a graph into independent components. However, the algebraic connectivity is equal to zero for all disconnected networks. Therefore, it is too coarse a measure for complex networks.

An alternative formulation of robustness within the context of complex networks emerged from random graph theory[19] and was stimulated by the work of Albert *et al.*[1] Instead of a strict mathematical extreme, the critical removal fraction of vertices (edges) for the disintegration of networks is generally used to measure the robustness of complex networks. As the fraction of removed vertices (or edges) increases, the performance of the network will eventually collapse at a critical fraction f_c. Although we can obtain the analytical critical removal fraction for some special networks analytically,[20−25] the critical removal fraction of vertices (edges) can only be calculated by simulations in general.

In this Letter, we propose a new robustness measure of complex networks based on graph spectra, which has a clear physical meaning and a simple mathematical formulation.

A complex network can be viewed as a simple undirected graph $G(V,E)$, where V is the set of vertices, and $E \subseteq V \times V$is the set of edges. Let $N = |V|$ and $M = |E|$ be the number of vertices and the number of edges, respectively. Let d_i be the degree of vertex v_i, $d_{\min}$ be the minimum degree and $d_{\max}$ be the maximum degree of G. Let $A(G) = (a_{ij})_{N\times N}$ be the adjacency matrix of G, where $a_{ij} = a_{ji} = 1$ if vertex v_i and v_j are adjacent, and $A_{ij} = A_{ji} = 0$ otherwise. It follows immediately that $A(G)$ is a real symmetric matrix with real eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$, which are usually also called the eigenvalues of the graph G itself. The set $\{\lambda_1, \lambda_2, \cdots \lambda_N\}$ is called the spectrum of G. A walk of length k in a graph G is an alternating sequence of vertices and edges $v_0e_1v_1e_2\cdots e_kv_k$, where $v_i \in V$ and $e_i = (v_{i-1}, v_i) \in E$. A walk is closed if $v_0 = v_k$. The number of closed walks is an important index for complex networks.

An intuitive notion of graph robustness can be expressed in terms of the redundancy of routes between vertices. If we consider a source vertex and a termi-

*Supported by the National Natural Science Foundation of China under Grant Nos 60904065 and 70771111, and the Specialized Research Fund for the Doctoral Program of Higher Education under Grant No 20094307120001.
**To whom correspondence should be addressed. Email: wujunpla@hotmail.com

nation vertex, there may be several routes between them. When one route fails, the two vertices can still communicate through other alternative routes. It is intuitive that the more the alternative routes present, the more robust the connectedness between the two vertices. This observation leads us to consider the redundancy of alternative paths as a measure of the robustness of networks, since this ensures that the connection between vertices remains possible in spite of damage to the network. Although it would be ideal to define this redundancy as the number of alternative routes of different lengths for all pairs of vertices, this measure is very difficult to calculate. Here we propose to measure the redundancy of alternative routes as the scaled number of closed walks of all lengths.

Closed walks are directly related to the subgraphs of the graph. For instance, a closed walk of length $k = 2$ corresponds to an edge and a closed walk of length $k = 3$ represents a triangle. Note that a closed walk can be trivial, i.e., containing repeated vertices and leading to the length of a closed walk being infinite. Considering that shorter closed walks have more influence on the redundancy of alternative routes than longer closed walks and to avoid the divergence of the number of closed walks of all lengths,[27] we scale the contribution of closed walks by dividing them by the factorial of the length k. That is, we define a weighted sum of numbers of closed walks $S = \sum_{k=0}^{\infty} n_k/k!$, where n_k is the number of closed walks of length k. This scaling ensures that the weighted sum does not diverge and it also means that S can be obtained from the powers of the adjacency matrix,

$$n_k = \sum_{i=1}^{N} \lambda_i^k = \mathrm{trace}(A^k) = \sum \lambda_i^k. \qquad (1)$$

Using Eq. (1), we can obtain

$$S = \sum_{k=0}^{\infty} \frac{n_k}{k!} = \sum_{k=0}^{\infty} \sum_{i=1}^{N} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{N} \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{N} e^{\lambda_i}. \qquad (2)$$

Hence the proposed weighted sum of closed walks of all lengths can be derived from the graph spectrum. We remark that Eq. (2) corresponds to the Estrada index of the graph,[26] which has been used in several contexts in graph theory, including subgraph centrality[27] and bipartivity[28] Equation (2) suggests that the weighted sum of closed walks of different lengths can be derived from the graph spectra. Note that S will be a large number for large N, we consider to scale S and denote it by $\bar{\lambda}$,

$$\bar{\lambda} = \ln\left(\frac{1}{N} \sum_{i=1}^{N} e^{\lambda_i}\right), \qquad (3)$$

which corresponds to an "average eigenvalue" of the graph adjacency matrix. We propose to call it the natural connectivity or natural eigenvalue.

The proposed natural connectivity has some desirable features. In particular, $\bar{\lambda}$ changes monotonically when edges are added or deleted. To prove this, considering a graph G, let $G + e$ be the graph obtained by adding the edge e to G and $\hat{n}_k = \hat{n}'_k + \hat{n}''_k$ be the number of closed walks of length k in $G+e$, where $\hat{n}'_k$ is the number of closed walks of length k containing e and $\hat{n}''_k$ is the number of closed walks of length k without containing e. Note that $\hat{n}''_k = n_k$ and $\hat{n}'_k \geq 0$, thus we can obtain that $\bar{\lambda}(G+e) \geq \bar{\lambda}(G)$. It is easy to show that $\hat{n}_k > \hat{n}''_k = n_k$ for some k, e.g., $\hat{n}_2 = n_2 + 2$. Consequently, $\bar{\lambda}(G + e) > \bar{\lambda}(G)$, indicating that the natural connectivity increases strictly monotonically as edges are added. It then follows that, given the number of vertices N, the empty graph has the minimum natural connectivity and the complete graph has the maximum natural connectivity. It is known that.[29] $\lambda_1 = \lambda_2 = \cdots = \lambda_N = 0$ for an empty graph, and $\lambda_1 = N - 1$, $\lambda_2 = \cdots = \lambda_N = -1$ for a complete graph. Hence we obtain the bound of natural connectivity,

$$0 \leq \bar{\lambda} \leq \ln((N-1)e^{-1} + e^{N-1}) - \ln N, \qquad (4)$$

with asymptotic behavior as $N \to \infty$, given by

$$0 \leq \bar{\lambda} \leq N - \ln N. \qquad (5)$$

The natural connectivity agrees with our intuitive expectation of robustness and provides a sensitive means to detect its changes. For instance, consider the two simple graphs with six vertices in Fig. 1, where Fig. 1(b) is obtained by adding an edge to Fig. 1(a). Our intuition suggests that Fig. 1(b) should be more robust than Fig. 1(a). This agrees with our measure: the natural connectivity of Figs. 1(a) and 1(b) are 1.0878 and 1.3508, respectively. However, some of the traditional robustness measures mentioned in the introduction can not distinguish the two graphs. For example, both the graphs have identical edge connectivity 2 and identical algebraic connectivity 0.7369.

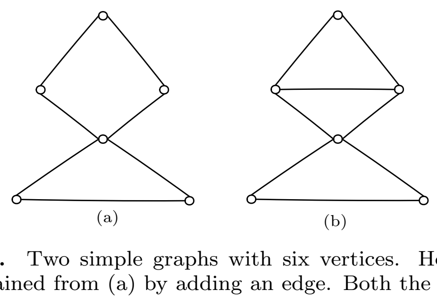

Fig. 1. Two simple graphs with six vertices. Here (b) is obtained from (a) by adding an edge. Both the graphs have the identical edge connectivity and identical algebraic connectivity, but can be distinguished by our proposed natural connectivity.

To explore in depth the natural connectivity measure and compare it with other robustness measures, we consider a scenario of edge elimination. As edges are deleted, we expect the decrease of the robustness measure, and we also expect different behaviors for different edge elimination strategies. We generate an initial network with a power-law degree distributions using the BA model,[30] where $N = 1000$ and avrage degree $\langle k \rangle = 6$. We remark that the type of network has no effect on the analysis and conclusions.

We consider four edge elimination strategies: (i) deleting the edges randomly (random strategy); (ii) deleting the edges connecting high-degree to high-degree vertices in descending order of $d_i \cdot d_j$, where d_i and d_j are the degrees of the end vertices of an edge (rich-rich strategy); (iii) deleting the edges connecting low-degree to low-degree vertices in ascending order of $d_i \cdot d_j$ (poor-poor strategy); (iv) deleting the edges connecting high-degree to low-degree vertices in descending order of $|d_i - d_j|$ (rich-poor strategy). Along with the natural connectivity, we investigate other three robustness measures: edge connectivity $\kappa_E(G)$, algebraic connectivity $a(G)$, critical removal fraction of vertices under random failure f_c^R. To find the critical removal fraction of vertices, we choose $\kappa \equiv \langle k^2 \rangle / \langle k \rangle < 2$ as the criterion for the disintegration of networks.[21] The results shown in Fig. 2 correspond to the averages over 100 realizations of a BA network.

Our numerical resusults in Figs. 2(a) and 2(b) show the similar behaviour for $\kappa_E(G)$ and $a(G)$. The first observation is that deleting a small quantity of rich-to-rich edges has no obvious effect on the robustness measured by the edge or algebraic connectivity. On the other hand, the robustness drops rapidly under the poor-poor strategy. It is generally believed that the edges between high-degree vertices are important, and the edges between low-degree vertices are inessential for the global network robustness. For example, in the Internet, the failure of the links between core routers will bring a disaster, but there is no effect on the network robustness if we disconnect two terminal computers. Clearly, robustness measures based on edge or algebraic connectivity do not agree with our intuition. These unexpected features can be explained by the bound $a(G) \leq \kappa_E(G) \leq d_{\min}(G)$, also known as Fiedler's inequality.[18] In fact, we find that the probability of $\kappa_E(G) = d_{\min}(G)$ almost approaches 1. The edge connectivity drops quickly in the poor-poor strategy since $d_{\min}(G)$ decreases rapidly after a few poor-poor edges are deleted. On the other hand, $d_{\min}(G)$ is preserved under the rich-rich strategy. Moreover, we find that, for all the four strategies, the edge or algebraic connectivity is equal to zero after particular edges are deleted, even in the case where only a few vertices are separated from the largest cluster. This means that both the edge connectivity and the algebraic connectivity lose discrimination when the network is disconnected.

Figure 2(c) shows the critical removal fraction of vertices f_c^R as a function of the number of deleted edges. Contrary to the result of edge or algebraic connectivity and in agreement with our intuition, we observe that the rich-rich strategy is the most effective edge elimination strategy and the poor-poor strategy is the least effective to induce the collapse of the network. However, our numerical results highlight the irregular behavior of the curves as edges are deleted even after averaging over many realizations. This indicates that the critical removal fraction is not a sensitive measure of robustness, especially for small sized networks.

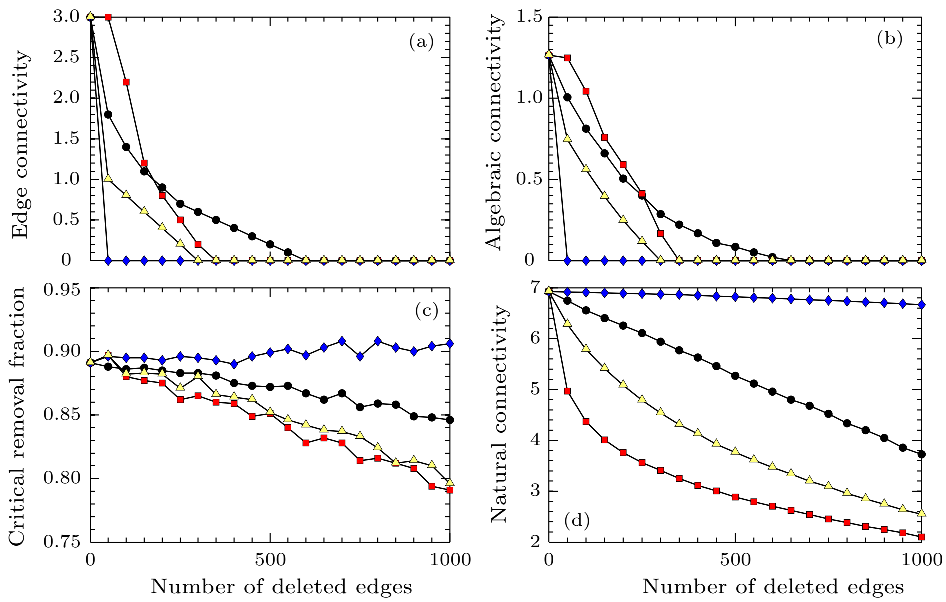

Fig. 2. The robustness measured by edge connectivity (a), algebraic connectivity (b), critical removal fraction of vertices (c) and natural connectivity (d), vesus the number of deleted edges for four edge elimination strategies: random strategy (circus), rich-rich strategy (squares), poor-poor strategy (diamonds) and rich-poor strategy (triangles). The initial network is generated using the BA model, with $N = 1000$ and $\langle k \rangle \approx 6$. Each quantity is an average over 100 realizations. Lines are guide to the eyes.

In Fig. 2(d), we display the results of the natural connectivity. We find a clear variation of the measure with distinct differences between the four edge elimination strategies, showing a clear ranking for the four edge elimination strategies: rich-rich strategy $\succ$ rich-poor strategy $\succ$ random strategy $\succ$ poor-poor strategy, which agrees with our intuition. For the random strategy, we observe a linear decrease of the natural connectivity. For the rich-rich strategy or rich-poor strategy, the natural connectivity decreases rapidly with the edge elimination. For poor-poor strategy, deleting a small quantity of poor-poor edges has a weak effect on the robustness. Moreover, the curves of the natural connectivity are smooth, a consequence of the strict monotonicity of the measure. This indicates that the natural connectivity can measure the robustness of complex networks stably even for very small sized networks. In fact, we find that the curves for natural connectivity are also smooth even without averaging over 100 realizations, viz. for each individual network. In contrast, in the case of individual networks, we find stepped curves for the edge or algebraic connectivity and large fluctuations for the critical removal fraction.

In summary, we have proposed the concept of natural connectivity as a novel spectral measure of robustness in complex networks. Other than the external measures, the natural connectivity roots in the inherent structure property of a network. The theoretical motivation of our measure arises from the fact the robustness of a network comes from the redundancy of alternate paths. It is a holistic robustness measure, which considers random failures and intentional attacks synthetically. The natural connectivity is expressed in mathematical form as a special case of average eigenvalue and changes strictly monotonically with the addition or deletion of edges. We have tested our natural connectivity measure and compared it to other measures within a scenario of edge elimination. We have demonstrated that the natural connectivity has the strong discrimination in measuring the robustness of complex networks and can exhibit the variation of robustness sensitively, even for disconnected networks. The rich information about the topological structure and diffusion processes can be extracted from the spectral analysis of the networks. The natural connectivity sets up a bridge between the graph spectra and the robustness of complex networks and then associates the robustness with other network structure and dynamical property. It is of great theoretical and practical significance to the network design and optimization.

References

[1] Albert R, Jeong H and Barabási A-L 2000 *Nature* **406** 378
[2] Holme P, Kim B J, Yoon C N and Han S K 2002 *Phys. Rev.* E **65** 056109
[3] Bollobás B and Riordan O 2003 *Internet Math.* **1** 1
[4] Chi L P, Yang C B and Cai X 2006 *Chin. Phys. Lett.* **23** 263
[5] Sun K and Ouyang Q 2001 *Chin. Phys. Lett.* **18** 452
[6] Wang J W and Rong L L 2008 *Chin. Phys. Lett.* **25** 3826
[7] Hu B, Li F and Zhou H S 2009 *Chin. Phys. Lett.* **26** 128901
[8] Liu J G, Wang Z T and Dang Y Z 2006 *Mod. Phys. Lett.* B **20** 815
[9] Albert R and Barabási A-L 2002 *Rev. Mod. Phys.* **74** 47
[10] Newman M E J 2003 *SIAM Rev.* **45** 167
[11] Wang X F 2002 *Int. J. Bifurcation &Chaos* **12** 885
[12] Frank H and Frisch I T 1970 *IEEE Trans. Commun. Tech.* **18** 567
[13] Chvatal V 1973 *Discr. Math.* **5** 215
[14] Jung H A 1978 *J. Combin. Theor.* B **24** 125
[15] Barefoot C A, Entringer R and Swart H 1987 *J. Combin. Math. Combin. Comput.* **1** 13
[16] Cozzens M, Moazzami D and Stueckle S 1995 *Seventh International Conference on the Theory and Applications of Graphs* (New York: Wiley) p 1111
[17] Kratsch D 1997 *Discr. App. Math.* **77** 259
[18] Fiedler M 1973 *Czech. Math. J.* **23** 298
[19] Bollobás B 1985 *Random Graphs* ???? ???? (New York: Academic)
[20] Cohen R, Erez K, ben-Avraham D and Havlin S 2001 *Phys. Rev. Lett.* **86** 3682
[21] Cohen R, Erez K, ben-Avraham D and Havlin S 2000 *Phys. Rev. Lett.* **85** 4626
[22] Callaway D S, Newman M E J, Strogatz S H and Watts D J 2000 *Phys. Rev. Lett.* **85** 5468
[23] Wu J, Deng H Z, Tan Y J, Li Y and Zhu D Z 2007 *Mod. Phys. Lett.* B **21** 1007
[24] Wu J, Deng H Z, Tan Y J and Zhu D Z 2007 *J. Phys.* A **40** 2665
[25] Wu J, Deng H Z, Tan Y J and Li Y 2007 *Chin. Phys. Lett.* **24** 2138
[26] Estrada E 2000 *Chem. Phys. Lett.* **319** 713
[27] Estrada E and Rodríguez-Velázquez J A 2005 *Phys. Rev.* E **71** 056103
[28] Estrada E and Rodríguez-Velázquez J A 2005 *Phys. Rev.* E **72** 046105
[29] Cvetković D M, Doob M and Sachs H 1979 *Spectra of Graphs* (New York: Academic)
[30] Barabási A L and Albert R 1999 *Science* **286** 509