Consensus, Flocking and Opinion Dynamics

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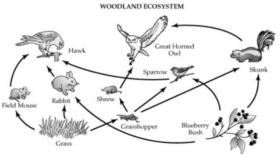
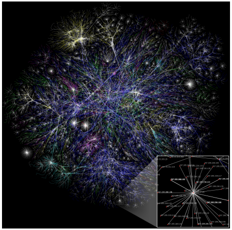
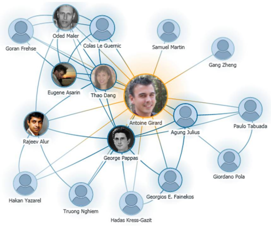
Scientific context

Networks everywhere:

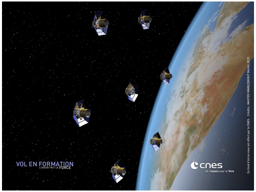
- Biological networks (genetic regulation, ecosystems...)
- Technological networks (internet, sensor networks...)
- Economical networks (production and distribution networks, financial networks...)
- Social networks (scientific collaboration networks, Facebook...)





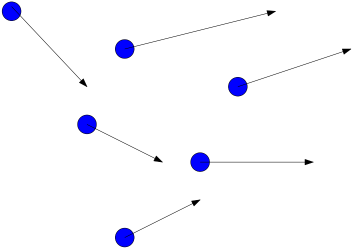


- Distributed decision making in a network.
- Each agent collaborates/negotiates locally with its neighbors in a network.
- The process succeeds if all agents eventually agree globally on some quantity of interest.
- Examples: bird flocks, fish schools, market prices...



Example: flocking in mobile networks

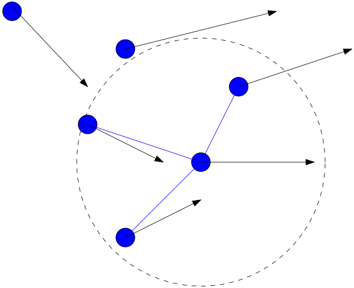
Consider a set of agents willing to move in a common direction:



Agent i is characterized by its position x_i and velocity v_i.

Example: flocking in mobile networks

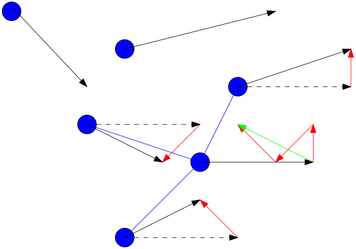
Consider a set of agents willing to move in a common direction:



Agent i has limited communication or sensing capabilities.

Example: flocking in mobile networks

Consider a set of agents willing to move in a common direction:



Agent i tries to align its velocity on its neighbors: $\dot{v}_i = \sum_{j \in N_i} (v_j - v_i)$.

Example: flocking in mobile networks

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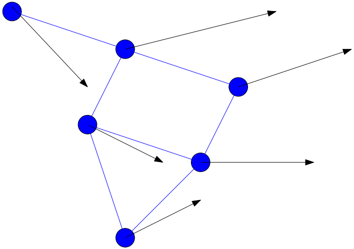

The communication network is described by a (dynamic) graph.

Example: flocking in mobile networks

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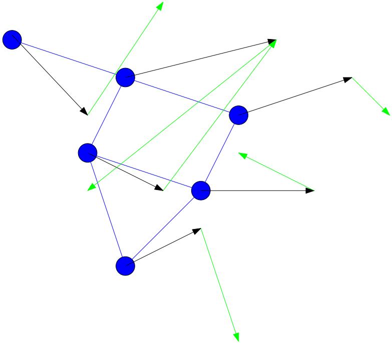
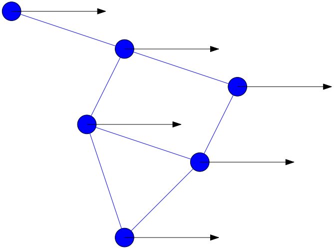


Global linear dynamics with structure given by the graph: $\dot{v} = -Lv$.

Example: flocking in mobile networks

Consider a set of agents willing to move in a common direction:



Do the agents eventually agree on a common velocity?

What we will see in this lecture

This lecture is not meant to give an exhaustive description of the area...

Instead, we will provide a deeper insight on a small number of representative results.

Main references used while preparing the lecture:

- C. Godsil & G. Royle, *Algebraic Graph Theory*, Springer 2001.
- R. Olfati-Saber, J.A. Fax & R.M. Murray, *Consensus and cooperation in networked multi-agent systems*, Proc. IEEE, 2007.
- V.D. Blondel, J.M. Hendrickx, A. Olshevsky & J.N. Tsitsiklis, *Convergence in multiagent coordination, consensus, and flocking*, Proc. CDC, 2005.
- L. Moreau, *Stability of continuous-time distributed consensus algorithms*, Proc. CDC, 2004.
- S. Martin & A. Girard, *Sufficient conditions for flocking via graph robustness analysis*, Proc. CDC, 2010.
- C. Morarescu & A. Girard, *Opinion dynamics with decaying confidence: application to community detection in graphs*, ArXiv, 2009.

Lecture outline

1. Algebraic graph theory
    - Basic graph notions
    - Laplacian matrix
    - Normalized Laplacian matrix
2. Consensus Algorithms:
    - Discrete time and continuous time
    - Agreement in networks with fixed topology
    - Agreement in networks with dynamic topology
3. Applications:
    - Flocking in mobile networks
    - Opinion dynamics and community detection in social networks

Graphs

Definition

A graph is couple $G = (V, E)$ consisting of:

- A finite set of vertices $V = \{1, \dots, n\}$;
- A set of edges, $E \subseteq V \times V$.

We assume G has no self-loops $(\forall i \in V, (i, i) \notin E)$ and is undirected $(\forall i, j \in V,\ (i, j) \in E \iff (j, i) \in E)$.

Definition

In an undirected graph $G = (V, E)$:

- The neighborhood of a vertex $i \in V$ is the set

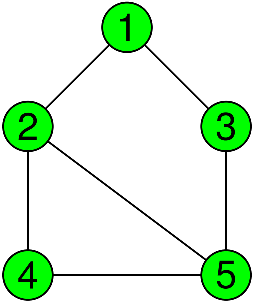
$$N_i = \{j \in V |\ (i, j) \in E\}.$$

- The degree of a vertex $i \in V$ is $d_i = |N_i|$.

Example

A simple graph:

$$
\begin{aligned}
V &= \{1, 2, 3, 4, 5\} \\
E &= \{(1,2), (1,3), (2,4), (2,5), (3,5), (4,5) \ldots \\
&\quad (2,1), (3,1), (4,2), (5,2), (5,3), (5,4)\}
\end{aligned}
$$



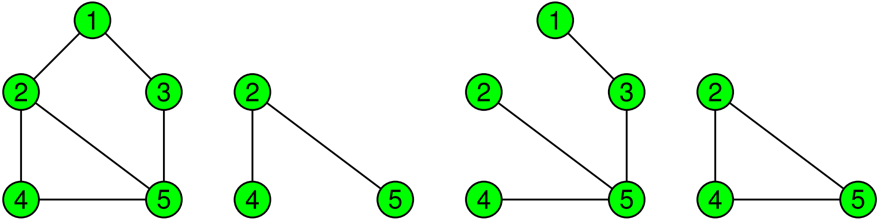
$$
\begin{aligned}
N_1 &= \{2, 3\}, d_1 = 2 \\
N_2 &= \{1, 4, 5\}, d_2 = 3 \\
N_3 &= \{1, 5\}, d_3 = 2 \\
N_4 &= \{2, 5\}, d_4 = 2 \\
N_5 &= \{2, 3, 4\}, d_5 = 3
\end{aligned}
$$

Subgraphs

Definition

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. In addition, if $V' = V$ then G' is a spanning subgraph of G.

The subgraph of G induced by a set of vertices $V' \subseteq V$ is the graph $G' = (V', E')$ where $E' = E \cap V' \times V'$.



initial graph subgraph spanning subgraph induced subgraph

Connectivity notions

Definition

A path in a graph $G = (V, E)$ is a finite sequence of edges $(i_1, i_2), (i_2, i_3), \dots, (i_p, i_{p+1})$ such that $(i_k, i_{k+1}) \in E$ for all $k \in \{1, \dots, p\}$.

Definition

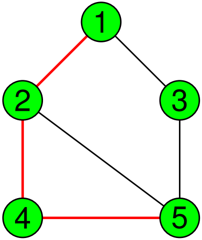
In a graph $G = (V, E)$, two vertices $i, j \in V$ are connected if there exists a path joining i and j (i.e. $i_1 = i$, $i_{p+1} = j$).

G is connected if for all $i, j \in V$, i and j are connected.

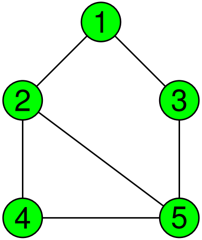
A subset of vertices $V' \subseteq V$ is a connected component of G if:

1. For all $i, j \in V'$, i and j are connected;
2. For all $i \in V'$, for all $j \in V \setminus V'$, i and j are not connected.

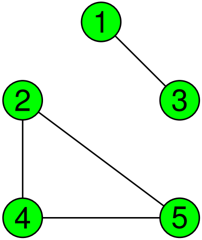
Example



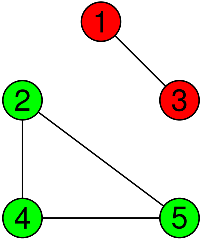
Path joining 1 and 5



Graph is connected



Graph is not connected



Connected components

Adjacency and degree matrices

Definition

The adjacency matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $A = (a_{ij})$ given for all $i, j \in V$ by:

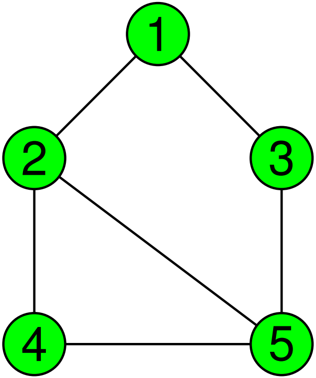
$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition

The degree matrix of G is the $n \times n$ diagonal matrix $D = (d_{ij})$ given for all $i, j \in V$ by:

$$d_{ij} = \begin{cases} d_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Example



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

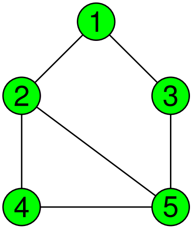
Laplacian matrix

Definition

The Laplacian matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $L = (l_{ij})$ given for all $i, j \in V$ by:

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

We have $L = D - A$.



$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

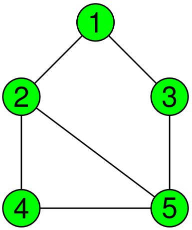
Normalized Laplacian matrix

Definition

The normalized Laplacian matrix of a graph $G = (V, E)$ is the $n \times n$ symmetric matrix $\mathcal{L} = (\ell_{ij})$ given for all $i, j \in V$ by:

$$\ell_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } d_i \neq 0, \\ -1/\sqrt{d_i d_j} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

If $d_i > 0$ for all $i \in V$, then $\mathcal{L} = I - D^{-1/2} A D^{-1/2} = D^{-1/2} L D^{-1/2}$.



$$\mathcal{L} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{6}} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & 1 & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 1 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{3} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 1 \end{pmatrix}$$

Fundamental property of the Laplacian matrix

Theorem (Sum of squares property)

Let L be the Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top L x = \frac{1}{2} \sum_{(i,j) \in E} (x_i - x_j)^2.$$

Proof: For all $x \in \mathbb{R}^n$,

$$\begin{aligned} x^\top L x &= \sum_{i \in V} x_i \sum_{j \in V} l_{ij} x_j = \sum_{i \in V} x_i \Big(d_i x_i - \sum_{(i,j) \in E} x_j\Big) \\ &= \sum_{i \in V} x_i \sum_{(i,j) \in E} (x_i - x_j) = \sum_{i \in V} \sum_{(i,j) \in E} (x_i^2 - x_i x_j) \\ &= \sum_{(i,j) \in E} (x_i^2 - x_i x_j) \end{aligned}$$

Fundamental property of the Laplacian matrix

Theorem (Sum of squares property)

Let L be the Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top L x = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Proof: Since whenever $(i, j) \in E$, $(j, i) \in E$ we have

$$\sum_{(i,j)\in E} (x_i^2 - x_i x_j) = \sum_{(i,j)\in E} (x_j^2 - x_i x_j).$$

It follows that

$$x^\top L x = \frac{1}{2} \sum_{(i,j)\in E} (x_i^2 - 2x_i x_j + x_j^2) = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Laplacian matrix of a subgraph

Corollary

Let $G = (V, E)$ be a graph and L its Laplacian matrix, let $G' = (V, E')$ be a spanning subgraph of G and L' its Laplacian matrix. Then, for all $x \in \mathbb{R}^n$,

$$x^\top Lx \geq x^\top L'x$$

Proof: By the SOS property,

$$\begin{aligned} x^\top Lx &= \frac{1}{2}\sum_{(i,j)\in E}(x_i - x_j)^2 \\ &= \frac{1}{2}\sum_{(i,j)\in E'}(x_i - x_j)^2 + \frac{1}{2}\sum_{(i,j)\in E\setminus E'}(x_i - x_j)^2 \\ &\geq \frac{1}{2}\sum_{(i,j)\in E'}(x_i - x_j)^2 = x^\top L'x. \end{aligned}$$

Eigenvalues of the Laplacian matrix

Proposition

The Laplacian matrix L of an undirected graph $G = (V, E)$ is symmetric positive. 0 is an eigenvalue of L with associated eigenvector $\mathbf{1}_n$.

Proof: Symmetry is consequence of G being undirected. By the SOS property,

$$\forall x \in \mathbb{R}^n,\ x^T L x = \frac{1}{2} \sum_{(i,j)\in E} (x_i - x_j)^2 \geq 0$$

which gives L positive.
Let $y = L\mathbf{1}_n$, then

$$y_i = \sum_{j\in V} l_{ij} = d_i + \sum_{(i,j)\in E} -1 = d_i - d_i = 0.$$

Eigenvalues of the Laplacian matrix

Proposition

0 is a simple eigenvalue of L if and only if the graph G is connected.

Proof: ($\Longrightarrow$) Assume the graph is not connected, then there exists a connected component of G, $V' \subsetneq V$.
Let $x \in \mathbb{R}^n$, such that $x_i = 1$ if $i \in V'$ and $x_i = 0$ otherwise, let $y = Lx$.
Then,

$$\forall i \in V', \qquad y_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V'} l_{ij} = \sum_{j \in V} l_{ij} - \sum_{j \in V \setminus V'} l_{ij} = 0$$

$$\forall i \in V \setminus V', \qquad y_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V'} l_{ij} = 0.$$

Then $x \neq \alpha \mathbf{1}_n$ is an eigenvector of L for eigenvalue 0. 0 is not simple.

Eigenvalues of the Laplacian matrix

Proposition

0 is a simple eigenvalue of L if and only if the graph G is connected.

Proof: ($\Longleftarrow$) Assume the graph is connected. Let x such that $Lx = 0$. Then the SOS property gives:

$$\sum_{(i,j)\in E} (x_i - x_j)^2 = x^\top L x = 0.$$

Thus, for all $(i,j) \in E$, $x_i = x_j$. Let $i \neq 1$, since G is connected there exists a path $(i_1, i_2), \dots, (i_p, i_{p+1})$ joining 1 and i. It follows that

$$x_1 = x_{i_1} = x_{i_2} = \dots = x_{i_{p+1}} = x_i.$$

The eigenvector x is of the form $\alpha \mathbf{1}_n$. 0 is simple.

Eigenvalues of the Laplacian matrix

Proposition

We denote the eigenvalues of L by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.
Let $\Delta = \max_{i \in V} d_i$, then, for all k, $\lambda_k \leq 2\Delta$.

Proof: Let λ be an eigenvalue of L and x an associated eigenvector. Let $i \in V$, such that for all $j \in V$, $|x_i| \geq |x_j|$. Then,

$$\lambda x_i = \sum_{j \in V} l_{ij} x_j = \sum_{j \in V \setminus \{i\}} l_{ij} x_j + d_i x_i.$$

It follows that

$$|\lambda - d_i| \leq \sum_{j \in V \setminus \{i\}} |l_{ij}| \frac{|x_j|}{|x_i|} \leq \sum_{j \in V \setminus \{i\}} |l_{ij}| = d_i.$$

Then $\lambda \leq 2d_i \leq 2\Delta$.

Algebraic connectivity

Definition

The second smallest eigenvalue λ_2 of L is referred to as the algebraic connectivity of the graph G.

Theorem

$\lambda_2 > 0$ if and only if G is connected. It satisfies:

$$\lambda_2 = \min_{x \perp \mathbf{1}_n} \frac{x^\top L x}{x^\top x}.$$

Proof: Let $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors. Then

$$\forall x \perp \mathbf{1}_n,\ x^\top L x = \sum_{k=2}^{n} \lambda_k (v_k^\top x)^2 \geq \lambda_2 \sum_{k=2}^{n} (v_k^\top x)^2 = \lambda_2 x^\top x.$$

Moreover, for $x = v_2$, $v_2^\top L v_2 = \lambda_2$.

Algebraic connectivity of a subgraph

Theorem

Let $G = (V, E)$ be a graph and λ_2 its algebraic connectivity, let $G' = (V, E')$ be a spanning subgraph of G and λ_2' its algebraic connectivity. Then, $\lambda_2 \geq \lambda_2'$.

Proof: Since for all $x \in \mathbb{R}^n$, $x^\top L x \geq x^\top L' x$, we have

$$\min_{x \perp \mathbf{1}_n} \frac{x^\top L x}{x^\top x} \geq \min_{x \perp \mathbf{1}_n} \frac{x^\top L' x}{x^\top x}$$

from which follows $\lambda_2 \geq \lambda_2'$.

Remark

Removing (adding) edges to graph can only make the algebraic connectivity decrease (increase).

Normalized Laplacian matrix

Using the fact the $\mathcal{L} = D^{-1/2} L D^{-1/2}$ we can obtain similar results.

Theorem (Sum of squares property)

Let $\mathcal{L}$ be the normalized Laplacian matrix of a graph $G = (V, E)$ then, for all $x \in \mathbb{R}^n$:

$$x^\top \mathcal{L} x = \frac{1}{2} \sum_{(i,j) \in E} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2 .$$

Proposition

The normalized Laplacian matrix L of an undirected graph $G = (V, E)$ is symmetric positive. 0 is an eigenvalue of L with associated eigenvector $D^{1/2}\mathbf{1}_n$.

Normalized Laplacian matrix

Proposition

0 is a simple eigenvalue of $\mathcal{L}$ if and only if the graph G is connected.

Proposition

We denote the eigenvalues of $\mathcal{L}$ by $0 = \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_n$. Then, for all k, $\tilde{\lambda}_k \leq 2$.

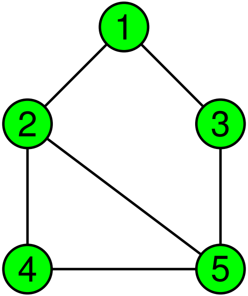
Theorem

$\tilde{\lambda}_2 > 0$ if and only if G is connected. It satisfies:

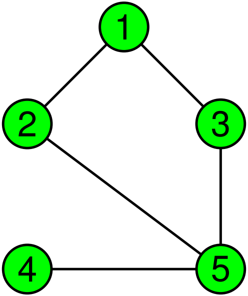
$$\tilde{\lambda}_2 = \min_{x \perp D^{1/2}\mathbf{1}_n} \frac{x^\top \mathcal{L} x}{x^\top x} = \min_{x \perp D\mathbf{1}_n} \frac{x^\top L x}{x^\top D x}.$$

Example

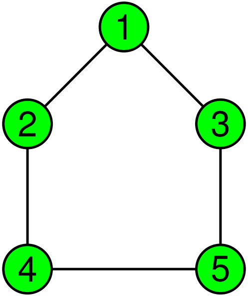
Second smallest eigenvalues of L and $\mathcal{L}$ for a graph and 2 subgraphs:



$\lambda_2 = 1.38$
$\tilde{\lambda}_2 = 0.67$



$\lambda_2 = 0.83$
$\tilde{\lambda}_2 = 0.59$



$\lambda_2 = 1.38$
$\tilde{\lambda}_2 = 0.69$

Summary

- We have defined fundamental notions such as graphs, subgraphs, pathes, connectivity...
- We have introduced Laplacian and normalized Laplacian matrices:
  - We have shown the SOS property for both types of matrices.
  - We have shown that the connectivity of a graph can be determined by second smallest eigenvalue of these matrices (λ_2 and $\tilde{\lambda}_2$).
- Some results that are valid for the Laplacian matrix cannot be extended to the normalized Laplacian matrix e.g.:

$$\textit{If } G' \textit{ is a subgraph of } G \textit{ then } \lambda_2' \leq \lambda_2.$$

- Though, in practice, the eigenvalue $\tilde{\lambda}_2$ is often a good measure of the connectivity of the graph (less dependent to the size of the graph).

Lecture outline

1. Algebraic graph theory
   - Basic graph notions
   - Laplacian matrix
   - Normalized Laplacian matrix
2. Consensus Algorithms:
   - Discrete time and continuous time
   - Agreement in networks with fixed topology
   - Agreement in networks with dynamic topology
3. Applications:
   - Flocking in mobile networks
   - Opinion dynamics and community detection in social networks

Consensus algorithms

Consider a set of agents V organized in a network with (possibly dynamic) topology described by an undirected graph $G(t) = (V, E(t))$.

Each agent $i \in V$ has a state $x_i(t) \in \mathbb{R}$ which is updated according to a simple local rule, e.g.

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t)),$$

or in matrix form

$$\dot{x}(t) = -L(t)x(t).$$

We say that the agents achieve a consensus if

$$\lim_{t \to +\infty} x_1(t) = \cdots = \lim_{t \to +\infty} x_n(t).$$

Various types of consensus algorithms

We will consider the following consensus algorithms:

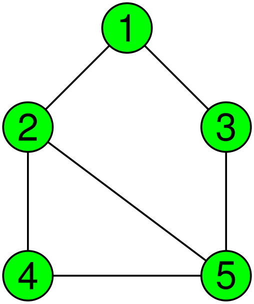
continuous time	discrete time
$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$	$x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$
$\dot{x}_i(t) = \frac{1}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$	$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$

Equivalent algorithms in matrix form:

continuous time	discrete time
$\dot{x}(t) = -L(t)x(t)$	$x(t+1) = (I - \varepsilon L(t))x(t)$
$\dot{x}(t) = -D^{-1}(t)L(t)x(t)$	$x(t+1) = (I - \varepsilon D^{-1}(t)L(t))x(t)$

Example

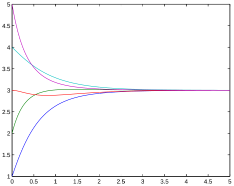
Consider the following network with fixed topology:



We run continuous time consensus algorithms over this network.

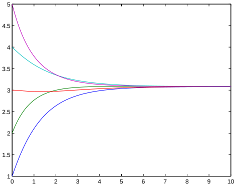
Standard consensus algorithm:

$$\dot{x}(t) = -Lx(t)$$



Normalized consensus algorithm:

$$\dot{x}(t) = -D^{-1}Lx(t)$$



Consensus in networks with fixed topology

We assume that for all $t \in \mathbb{R}$, $G(t) = (V, E)$ and consider the following consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -Lx(t)$.

Lemma

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ is invariant.

Proof: Let us compute the derivative

$$\begin{aligned} \frac{d}{dt}\left(\mathbf{1}_n^\top x(t)\right) &= \mathbf{1}_n^\top \dot{x}(t) \\ &= -\mathbf{1}_n^\top (Lx(t)) = -(\mathbf{1}_n^\top L)x(t) = 0. \end{aligned}$$

Consensus value

Proposition

Let $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, *if a consensus is achieved, then*

$$\lim_{t\to+\infty} x(t) = x^*\mathbf{1}_n.$$

Proof: If a consensus is achieved, then for all $i \in V$,

$$\lim_{t\to+\infty} x_i(t) = \lim_{t\to+\infty} \frac{1}{n}\mathbf{1}_n^\top x(t) = \frac{1}{n}\mathbf{1}_n^\top x(0) = x^*.$$

Remark

The consensus value is the average of the initial values. It is independent of the topology of the graph $G = (V, E)$.

Convergence

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq e^{-\lambda_2 t} \|x(0) - x^*\mathbf{1}_n\| .$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$\begin{aligned} \frac{d}{dt}V(t) &= 2\left(x(t) - x^*\mathbf{1}_n\right)^\top \dot{x}(t) = -2\left(x(t) - x^*\mathbf{1}_n\right)^\top Lx(t) \\ &= -2\left(x(t) - x^*\mathbf{1}_n\right)^\top L\left(x(t) - x^*\mathbf{1}_n\right). \end{aligned}$$

Let us remark that

$$\mathbf{1}_n^\top\left(x(t) - x^*\mathbf{1}_n\right) = \mathbf{1}_n^\top x(t) - x^*\mathbf{1}_n^\top\mathbf{1}_n = \mathbf{1}_n^\top x(0) - x^* n = 0$$

which gives $\left(x(t) - x^*\mathbf{1}_n\right) \perp \mathbf{1}_n$.

Convergence

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0, \; \|x(t) - x^* \mathbf{1}_n\| \leq e^{-\lambda_2 t} \|x(0) - x^* \mathbf{1}_n\| .$$

Proof: Therefore,

$$(x(t) - x^* \mathbf{1}_n)^\top L (x(t) - x^* \mathbf{1}_n) \geq \lambda_2 \|x(t) - x^* \mathbf{1}_n\|^2$$

which gives

$$\frac{d}{dt} V(t) \leq -2\lambda_2 \|x(t) - x^* \mathbf{1}_n\|^2 = -2\lambda_2 V(t).$$

Thus, $V(t) \leq V(0) e^{-2\lambda_2 t}$. If the graph G is connected then $\lambda_2 > 0$ and the consensus is achieved.

Consensus in discrete time

We assume that for all $t \in \mathbb{N}$, $G(t) = (V, E)$, let $\varepsilon \in (0, \frac{1}{2\Delta})$, we consider the consensus algorithm:

$$\forall i \in V,\ x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = Px(t)$ where $P = I - \varepsilon L$.

Proposition

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ is invariant. Let $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, if a consensus is achieved, then

$$\lim_{t \to +\infty} x(t) = x^* \mathbf{1}_n.$$

Proof. Let us remark that

$$\mathbf{1}_n^\top x(t+1) = \mathbf{1}_n^\top (I - \varepsilon L) x(t) = \mathbf{1}_n^\top x(t) - \varepsilon \mathbf{1}_n^\top L x(t) = \mathbf{1}_n^\top x(t).$$

Convergence

Lemma

We have $P\mathbf{1}_n = \mathbf{1}_n$ and for all $x \perp \mathbf{1}_n$, $\|Px\| \leq (1 - \varepsilon\lambda_2)\|x\|$.

Proof: First, $P\mathbf{1}_n = (I - \varepsilon L)\mathbf{1}_n = \mathbf{1}_n - \varepsilon L\mathbf{1}_n = \mathbf{1}_n$.
Let $x \perp \mathbf{1}_n$ and $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors of L.
Then,

$$x = \sum_{k=2}^{n} (v_k^\top x) v_k \text{ and } Px = \sum_{k=2}^{n} (v_k^\top x)(1 - \varepsilon\lambda_k) v_k.$$

Thus

$$\|Px\|^2 = \sum_{k=2}^{n} (v_k^\top x)^2 (1 - \varepsilon\lambda_k)^2 \leq \max_{k=2}^{n} (1 - \varepsilon\lambda_k)^2 \|x\|^2.$$

Moreover, from $0 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2\Delta$ and $\varepsilon \in (0, \frac{1}{2\Delta})$, we obtain

$$\max_{k=2}^{n} (1 - \varepsilon\lambda_k)^2 = (1 - \varepsilon\lambda_2)^2 < 1.$$

Theorem

If the graph G is connected, then the consensus is achieved. Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq (1-\varepsilon\lambda_2)^t \|x(0) - x^*\mathbf{1}_n\|.$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$V(t+1) = \|Px(t) - x^*\mathbf{1}_n\|^2 = \|P(x(t) - x^*\mathbf{1}_n)\|^2$$

Let us remark that

$$\mathbf{1}_n^\top (x(t) - x^*\mathbf{1}_n) = \mathbf{1}_n^\top x(t) - x^*\mathbf{1}_n^\top \mathbf{1}_n = \mathbf{1}_n^\top x(0) - x^* n = 0$$

which gives $(x(t) - x^*\mathbf{1}_n) \perp \mathbf{1}_n$. Then,

$$V(t+1) \leq (1-\varepsilon\lambda_2)^2 \|x(t) - x^*\mathbf{1}_n\|^2 = (1-\varepsilon\lambda_2)^2 V(t).$$

Other consensus algorithms

Using the fact the $D^{-1}L = D^{-1/2}\mathcal{L}D^{1/2}$ we obtain similar results for the consensus algorithm:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j\in N_i}(x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -D^{-1}Lx(t)$.

Theorem

If the graph G is connected, then the consensus is achieved. The consensus value is

$$x^* = \frac{\sum_{i\in V} d_i x_i(0)}{\sum_{i\in V} d_i}.$$

Moreover,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\|_D \leq e^{-\tilde{\lambda}_2 t}\|x(0) - x^*\mathbf{1}_n\|_D.$$

Other consensus algorithms

For the discrete time case, let $\varepsilon \in (0, \frac{1}{2})$ and consider the consensus algorithm:

$$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i} \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = (I - \varepsilon D^{-1} L) x(t)$.

Theorem

If the graph G is connected, then the consensus is achieved. The consensus value is

$$x^* = \frac{\sum_{i \in V} d_i x_i(0)}{\sum_{i \in V} d_i}.$$

Moreover,

$$\forall t \geq 0,\ \|x(t) - x^* \mathbf{1}_n\|_D \leq (1 - \varepsilon \tilde{\lambda}_2)^t \, \|x(0) - x^* \mathbf{1}_n\|_D \, .$$

Summary

- We have considered several consensus algorithms in continuous or discrete time for networks with fixed topology.
- For the standard consensus algorithm, the consensus value is independent of the network topology (average of initial values) whereas for the normalized consensus algorithm, it depends on the network topology (weighted average where the weights are the degrees of the vertices of the network).
- Consensus is achieved if the network is connected. The consensus is approached at exponential speed. Moreover, the convergence speed is determined by the second smallest eigenvalue of the Laplacian or normalized Laplacian matrix: the more connected the network the faster the consensus.

Consensus in networks with dynamic topology

We now assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$$

or in matrix form: $\dot{x}(t) = -L(t)x(t)$.
A lot of results are actually similar to the fixed topology case:

Proposition

The quantity $\mathbf{1}_n^\top x(t) = \sum_{i \in V} x_i(t)$ is invariant. Let $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$, if a consensus is achieved, then

$$\lim_{t \to +\infty} x(t) = x^* \mathbf{1}_n.$$

Convergence

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. Moreover, for $\underline{\lambda}_2 \leq \min_{t\in\mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq e^{-\underline{\lambda}_2 t} \|x(0) - x^*\mathbf{1}_n\| .$$

Proof: Let $V(t) = \|x(t) - x^*\mathbf{1}_n\|^2$. Then,

$$\begin{aligned} \frac{d}{dt}V(t) &= 2\left(x(t) - x^*\mathbf{1}_n\right)^\top \dot{x}(t) = -2\left(x(t) - x^*\mathbf{1}_n\right)^\top Lx(t) \\ &= -2\left(x(t) - x^*\mathbf{1}_n\right)^\top L(t)\left(x(t) - x^*\mathbf{1}_n\right) \\ &\leq -2\lambda_2(t)\|x(t) - x^*\mathbf{1}_n\|^2 \leq -2\underline{\lambda}_2 V(t) \end{aligned}$$

Thus, $V(t) \leq V(0)e^{-2\underline{\lambda}_2 t}$. If the graph G is connected, we can choose $\underline{\lambda}_2 > 0$ and the consensus is achieved.

Discrete time algorithm

For the discrete time case, let $\varepsilon \in (0, \frac{1}{2(n-1)})$ and consider the consensus algorithm:

$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j \in N_i} (x_j(t) - x_i(t))$$

or in matrix form: $x(t+1) = (I - \varepsilon L)x(t)$.

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. The consensus value is $x^* = \frac{1}{n}\mathbf{1}_n^\top x(0)$.
Moreover, for $\underline{\lambda}_2 \leq \min_{t \in \mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0,\ \|x(t) - x^*\mathbf{1}_n\| \leq (1 - \varepsilon\underline{\lambda}_2)^t \|x(0) - x^*\mathbf{1}_n\|.$$

Some remarks

- For the standard Laplacian consensus algorithm, it is straightforward to extend the results from fixed topology to dynamic topology under the assumption that the graph $G(t)$ remains connected for all t.
- For the normalized Laplacian consensus algorithm, the results cannot be extended in a straightforward manner. Indeed, even the consensus value is dependent on the graph sequence... Another approach is needed!
- The assumption that the graph remains connected for all time is actually quite strong. Is it possible to relax this assumption ?

A general consensus algorithm

Let us assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following discrete time consensus algorithm:

$$x_i(t+1) = \sum_{j \in V} p_{ij}(t) x_j(t)$$

under the following assumptions (for some $\alpha > 0$):

Assumption

- $p_{ii}(t) \geq \alpha$, $\forall i \in V$, $t \in \mathbb{N}$.
- $p_{ij}(t) \neq 0$ *if and only if* $(i,j) \in E(t)$, $\forall i,j \in V$, $i \neq j$, $t \in \mathbb{N}$.
- $p_{ij}(t) \in \{0\} \cup [\alpha, 1]$, $\forall i,j \in V$, $i \neq j$, $t \in \mathbb{N}$.
- $\sum_{j \in V} p_{ij}(t) = 1$, $\forall i \in V$, $t \in \mathbb{N}$.

Let us remark that the previous discrete time consensus algorithms satisfy these assumptions.

Convergence analysis

For a subset of agents $V' \subseteq V$, let $m_{V'}(t) = \min_{i \in V'} x_i(t)$ and $M_{V'}(t) = \max_{i \in V'} x_i(t)$.

Proposition

Let $V' \subseteq V$ such that for all $i \in V$, for all $j \in V \setminus V'$, $(i,j) \notin E(t)$. Then,

$$m_{V'}(t+1) \geq m_{V'}(t) \text{ and } M_{V'}(t+1) \leq M_{V'}(t).$$

Proof. Let $i \in V'$, then

$$\begin{aligned} x_i(t+1) &= \sum_{j \in V} p_{ij}(t) x_j(t) = \sum_{j \in V'} p_{ij}(t) x_j(t) \\ &\geq m'_V(t) \sum_{j \in V'} p_{ij}(t) = m_{V'}(t). \end{aligned}$$

Then, $m_{V'}(t+1) \geq m_{V'}(t)$. Similarly, $M_{V'}(t+1) \leq M_{V'}(t)$.

Remark

The sequences $M_V(t)$ and $m_V(t)$ are monotonic and bounded, therefore convergent.

The consensus is achieved if and only if

$$\lim_{t\to+\infty} M_V(t) = \lim_{t\to+\infty} m_V(t)$$

or equivalently

$$\lim_{t\to+\infty} M_V(t) - m_V(t) = 0.$$

We will prove this under an assumption of asymptotic connectivity:

Assumption

For all $t \in \mathbb{N}$, the graph $(V, \cup_{s\geq t} E(s))$ is connected.

Lemma

For all $t \in \mathbb{N}$ there exists $T \geq t$ such that

$$M_V(T) - m_V(T) \leq (1-\alpha^n)(M_V(t) - m_V(t)).$$

Proof: Let us remark that for all $s \geq t$, $i \in V$, $m_V(t) \leq x_i(s) \leq M_V(t)$.
Consider the following property for $k \in \{1, \dots, n\}$:

$$P_k : \left\{ \begin{array}{l} \exists t_k \geq t,\ V_k \subseteq V, \ \text{ such that } |V_k| = k \text{ and} \\ m_{V_k}(t) \geq m_V(t) + \alpha^k (M_V(t) - m_V(t)). \end{array} \right.$$

If P_n is true then necessarily $V_n = V$ and since $M_V(t_n) \leq M_V(t)$:

$$\begin{array}{rcl} M_V(t_n) - m_V(t_n) & \leq & M_V(t) - m_V(t) - \alpha^n (M_V(t) - m_V(t)) \\ & \leq & (1-\alpha^n)(M_V(t) - m_V(t)). \end{array}$$

Convergence analysis

Proof: Let $i_1 \in V$ such that $x_{i_1}(t) = M_V(t)$. Let $t_1 = t$ and $V_1 = \{i_1\}$, then

$$m_{V_1}(t_1) = x_{i_1}(t) = M_V(t) \geq m_V(t) + \alpha(M_V(t) - m_V(t)).$$

Thus, P_1 is true. Assume P_k is true for some $k \in \{1, \dots, n-1\}$.

Let $T_k \geq t_k$ be the first time such that

$$\exists (i_{k+1}, j_{k+1}) \in E(T_k), \quad \text{for some } i_{k+1} \in V \setminus V_k,\ j_{k+1} \in V_k.$$

For all $t_k \leq s \leq T_k - 1$, $i \in V_k$, $j \in V \setminus V_k$, $(i,j) \notin E(s)$. Then,

$$\forall i \in V_k,\ x_i(T_k) \geq m_{V_k}(T_k) \geq m_{V_k}(t_k) \geq m_V(t) + \alpha^k(M_V(t) - m_V(t)).$$

Convergence analysis

Proof: Then, for all $i \in V_k$,

$$\begin{aligned}
x_i(T_k+1) - m_V(t) &= \left(\sum_{j \in V} p_{ij}(T_k) x_j(T_k)\right) - m_V(t) \\
&= \sum_{j \in V} p_{ij}(T_k)\left(x_j(T_k) - m_V(t)\right) \\
&\geq p_{ii}(T_k)\left(x_i(T_k) - m_V(t)\right) \\
&\geq \alpha\left(x_i(T_k) - m_V(t)\right) \\
&\geq \alpha\left(m_V(t) + \alpha^k(M_V(t) - m_V(t)) - m_V(t)\right) \\
&\geq \alpha^{k+1}(M_V(t) - m_V(t)).
\end{aligned}$$

Proof: Moreover,

$$
\begin{aligned}
x_{i_{k+1}}(T_k+1) - m_V(t) &= \left(\sum_{j\in V} p_{i_{k+1}j}(T_k)x_j(T_k)\right) - m_V(t) \\
&= \sum_{j\in V} p_{i_{k+1}j}(T_k)\left(x_j(T_k) - m_V(t)\right) \\
&\geq p_{i_{k+1}j_{k+1}}(T_k)\left(x_{j_{k+1}}(T_k) - m_V(t)\right) \\
&\geq \alpha\left(x_{j_{k+1}}(T_k) - m_V(t)\right) \\
&\geq \alpha\left(m_V(t) + \alpha^k(M_V(t) - m_V(t)) - m_V(t)\right) \\
&\geq \alpha^{k+1}(M_V(t) - m_V(t)).
\end{aligned}
$$

Hence, P_{k+1} holds for $t_{k+1} = T_k + 1$ and $V_{k+1} = V_k \cup \{i_{k+1}\}$.
Then, P_n holds.

Convergence analysis

Theorem

If, for all $t \in \mathbb{N}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected, then the consensus is achieved.

Proof: From the previous lemma, there exists an increasing sequence $T_k \in \mathbb{N}$ such that $T_0 = 0$ and

$$\forall k \in \mathbb{N},\ 0 \leq M_V(T_k) - m_V(T_k) \leq (1-\alpha^n)^k (M_V(0) - m_V(0)).$$

Therefore,

$$\lim_{k \to +\infty} M_V(T_k) - m_V(T_k) = 0.$$

Since in addition $M_V(t)$ and $m_V(t)$ are convergent it necessarily follows that

$$\lim_{t \to +\infty} M_V(t) - m_V(t) = 0.$$

Continuous time consensus

The previous result is not valid for continuous time algorithms.

Consider e.g.:

$$\dot{x}_i(t) = \sum_{j \in N_i(t)} (x_j(t) - x_i(t)),$$

in a network of two agents with dynamic topology $G(t) = (V, E(t))$ where

$$V = \{1,2\},\ E(t) = \begin{cases} \{(1,2),(2,1)\}, & \text{if } t \in [k, k+1/2^k) \\ \emptyset, & \text{if } t \in [k+1/2^k, k+1) \end{cases},\ k \in \mathbb{N}.$$

Let us remark that for all $t \in \mathbb{R}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected.

Continuous time consensus

It follows that,

$$\dot{x}_2(t) - \dot{x}_1(t) = \begin{cases} -2(x_2(t) - x_1(t)), & \text{if } t \in [k, k + 1/2^k) \\ 0, & \text{if } t \in [k + 1/2^k, k + 1) \end{cases}$$

Then, for all $k \in \mathbb{N}$:

$$\begin{aligned} x_2(k+1) - x_1(k+1) &= x_2(k + 1/2^k) - x_1(k + 1/2^k) \\ &= (x_2(k) - x_1(k))e^{-2/2^k} \\ &= (x_2(0) - x_1(0))e^{-2}e^{-2/2}e^{-2/2^2} \dots e^{-2/2^k} \\ &= (x_2(0) - x_1(0))e^{-2(1+1/2+1/2^2+\dots+1/2^k)} \\ &= (x_2(0) - x_1(0))e^{-4(1-1/2^{k+1})}. \end{aligned}$$

It follows that

$$\lim_{k\to+\infty} x_2(k+1) - x_1(k+1) = (x_2(0) - x_1(0))e^{-4} \neq 0.$$

- The consensus is not achieved despite the fact that for all $t \in \mathbb{R}$, the graph $(V, \cup_{s\geq t} E(s))$ is connected.
- The key observation is that even though the edge $(1,2)$ appears infinitely often, it is present only for a finite time...
- If one impose a dwell time (when an edge appears it remains present for a duration at least $\tau > 0$), one can avoid this kind of phenomenom.

A general consensus algorithm

Let us assume that the graph is time-varying $G(t) = (V, E(t))$ and consider the following continuous time consensus algorithm:

$$\dot{x}_i(t) = \sum_{j \in V \setminus \{i\}} p_{ij}(t)(x_j(t) - x_i(t))$$

under the following assumptions (for some $\alpha > 0$, $\beta > 0$):

Assumption

- $p_{ij}(t) \neq 0$ *if and only if* $(i, j) \in E(t)$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{R}$.
- $p_{ij}(t) \in \{0\} \cup [\alpha, \beta]$, $\forall i, j \in V$, $i \neq j$, $t \in \mathbb{R}$.
- $\sum_{j \in V \setminus \{i\}} p_{ij}(t) \leq \beta$, $\forall i \in V$, $t \in \mathbb{R}$.

Let us remark that the previous continuous time consensus algorithms satisfy these assumptions.

A general consensus algorithm

We add the following dwell time assumption:

Assumption

For all $i, j \in V$, if the edge (i, j) appears at time t, i.e.:

$$(i,j) \in E(t) \text{ and } \exists \varepsilon > 0,\ \forall s \in [t-\varepsilon, t),\ (i,j) \notin E(s)$$

then (i, j) remains at for a duration τ, i.e.:

$$\forall s \in [t, t+\tau],\ (i,j) \in E(s).$$

Then, we have the convergence result:

Theorem

If, for all $t \in \mathbb{R}$, the graph $(V, \cup_{s \geq t} E(s))$ is connected, then the consensus is achieved.

Summary

Discrete time consensus algorithms:

Algorithm	Topology	Consensus if	Convergence rate
$x(t+1) = (I - \varepsilon L)x(t)$	fixed	G is connected	$(1-\varepsilon\lambda_2)^t$
$x(t+1) = (I - \varepsilon D^{-1} L)x(t)$	fixed	G is connected	$(1-\varepsilon\tilde{\lambda}_2)^t$
$x(t+1) = (I - \varepsilon L)x(t)$	dynamic	$G(t)$ is connected for all $t \in \mathbb{R}$	$(1-\varepsilon\underline{\lambda}_2)^t$ with $\underline{\lambda}_2 \leq \min_{t\in\mathbb{N}} \lambda_2(t)$
$x(t+1) = (I - \varepsilon L)x(t)$	dynamic	$(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-
$x(t+1) = (I - \varepsilon D^{-1} L)x(t)$	dynamic	$(V, \cup_{s\geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-

- Connectivity properties are crucial for convergence of consensus algorithms.
- Some convergence rates are determined by the second smallest eigenvalue of the Laplacian or normalized Laplacian matrix: the more connected the network the faster the consensus.

Summary

Continuous time consensus algorithms:

Algorithm	Topology	Consensus if	Convergence rate
$\dot{x}(t) = -Lx(t)$	fixed	G is connected	$e^{-\lambda_2 t}$
$\dot{x}(t) = -D^{-1}Lx(t)$	fixed	G is connected	$e^{-\tilde{\lambda}_2 t}$
$\dot{x}(t) = -Lx(t)$	dynamic	$G(t)$ is connected for all $t \in \mathbb{R}$	$e^{-\underline{\lambda}_2 t}$ with $\underline{\lambda}_2 \leq \min_{t \in \mathbb{R}^+} \lambda_2(t)$
$\dot{x}(t) = -Lx(t)$	dynamic	Dwell time and $(V, \cup_{s \geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-
$\dot{x}(t) = -D^{-1}Lx(t)$	dynamic	Dwell time and $(V, \cup_{s \geq t} E(s))$ is connected for all $t \in \mathbb{R}$	-

- Results are very similar to those for discrete time algorithms.
- For dynamic topologies, a supplementary dwell time assumption is needed in order to prove that the consensus is achieved.

Lecture outline

1. Algebraic graph theory
   - Basic graph notions
   - Laplacian matrix
   - Normalized Laplacian matrix

2. Consensus Algorithms:
   - Discrete time and continuous time
   - Agreement in networks with fixed topology
   - Agreement in networks with dynamic topology

3. Applications:
   - Flocking in mobile networks
   - Opinion dynamics and community detection in social networks

Flocking in mobile agents network

Flocking is the behavior exhibited when a group of birds are in flight:



More precisely, a flocking behavior is characterized by three properties:

1. Alignement: the birds have the same velocity.
2. Cohesion: the birds remain together.
3. Separation: there is a minimum distance between birds.

In the following, we focus on the first and second property.

A flocking model

- We consider a set of mobile agents $V = \{1, \dots, n\}$: for each $i \in V$, $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ denote its position and its velocity.
- An agent can communicate only with agents that are sufficiently close: the interaction graph $G(t) = (V, E(t))$ is given by

$$E(t) = \{(i,j) \in V \times V |\ i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq R\}.$$

- The agents try to agree on their velocity using the continuous time consensus algorithm:

$$\dot{v}_i(t) = \sum_{j \in N_i(t)} (v_j(t) - v_i(t)).$$

A flocking model

The considered model is then:

$$\begin{cases} \dot{x}_i(t) & = & v_i(t) \\ \dot{v}_i(t) & = & \sum_{j \in N_i(t)} (v_j(t) - v_i(t)) \end{cases}$$

We want to determine a set of initial conditions ensuring that a flocking behavior is achieved.

We have already shown the following result:

Theorem

If the graph $G(t)$ is connected for all $t \in \mathbb{R}$, then the consensus is achieved. Moreover, for $\underline{\lambda}_2 \leq \min_{t \in \mathbb{R}^+} \lambda_2(t)$,

$$\forall t \geq 0,\ \|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\underline{\lambda}_2 t} \|v(0) - \mathbf{1}_n \otimes v^*\|$$

with $v^* = \sum_{i \in V} v_i(0)$.

- We define a notion of robustness for interaction graphs.
- For $i \in V$, let $x_i \in \mathbb{R}^d$ be the position of agent i, let $x = (x_1, \dots, x_n)$, we define the associated graph $G_x = (V, E_x)$ with

$$E_x = \{(i,j) \in V \times V |\ i \neq j \text{ and } \|x_i - x_j\| \leq R\}.$$

- Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration. Assuming that G_{x^0} is a connected graph, we are interested in characterizing a neighborhood of x^0 such that for any perturbed configuration y in this neighborhood the graph G_y is connected.
- We introduce a measure of robustness for G_{x^0} which allows us to identify such a neighborhood.

Graph robustness analysis

- Let $(i_1, i_2), \dots, (i_p, i_{p+1})$ be a path of G_{x^0}, we define the slackening of the path:

$$s((i_1, i_2), \dots, (i_p, i_{p+1})) = \min_{k=1}^{p} (R - \|x^0_{i_k} - x^0_{i_{k+1}}\|).$$

  If the distances between agents do not change more than $s((i_1, i_2), \dots, (i_p, i_{p+1}))$ then the path is preserved.

- The path-robustness ρ_{ij} between two agents $i, j \in V$ with $i \neq j$, is the maximal slackening of all paths between i and j:

$$\rho_{ij} = \max_{(i_1,i_2),\dots,(i_p,i_{p+1}) \in \mathrm{Paths}(i,j)} s((i_1, i_2), \dots, (i_p, i_{p+1})).$$

  If the distances between agents do not change more than ρ_{ij} there remains a path between i and j. We set $\rho_{ii} = R$.

Graph robustness analysis

- The robustness of the graph $\rho_{G_{x^0}}$ is the minimal path-robustness between all pair of nodes:

$$\rho_{G_{x^0}} = \min_{(i,j)\in V^2} \rho_{ij}$$

If the distances between agents do not change more than $\rho_{G_{x^0}}$ then the graph remains connected.

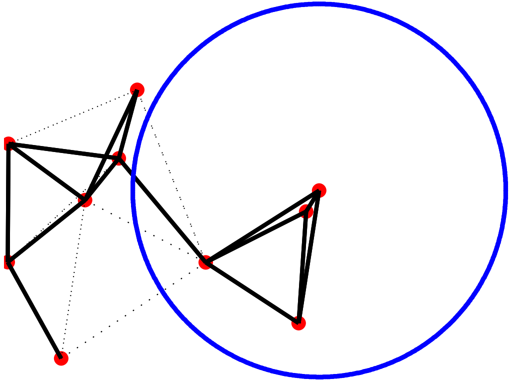
- The core robust subgraph of G_{x^0} is the graph $\mathcal{M}(G_{x^0}) = (V, \mathcal{M}(E_{x^0}))$ where

$$\mathcal{M}(E_{x^0}) = \left\{(i,j) \in V \times V |\ i \neq j \text{ and } \|x_i - x_j\| \leq R - \rho_{G_{x^0}}\right\}.$$

Since $\rho_{G_{x^0}} \geq 0$, $\mathcal{M}(G_{x^0})$ is clearly a subgraph of G_{x^0}.

Graph robustness analysis

Example of a core robust subgraph:



Lemma

Let $x^0 \in \mathbb{R}^{nd}$ be a reference configuration such that the graph G_{x^0} is connected. Then, the core robust subgraph $\mathcal{M}(G_{x^0})$ is connected.

Proof: Let $i, j \in V$, then $\rho_{ij} \geq \rho_{G_{x^0}}$. Let $(i_1, i_2), \dots, (i_p, i_{p+1})$ be a path of G_{x^0} between i and j with maximal slackening, then

$$s((i_1, i_2), \dots, (i_p, i_{p+1})) = \rho_{ij}.$$

Then, for all $k \in \{1, \dots, p\}$,

$$R - \|x^0_{i_k} - x^0_{i_{k+1}}\| \geq s((i_1, i_2), \dots, (i_p, i_{p+1})) \geq \rho_{G_{x^0}}.$$

Therefore, for all $k \in \{1, \dots, p\}$, $(i_k, i_{k+1}) \in \mathcal{M}(E_{x^0})$.
Then, $(i_1, i_2), \dots, (i_p, i_{p+1})$ is a path of $\mathcal{M}(G_{x^0})$ between i and j.
Thus, $\mathcal{M}(G_{x^0})$ is connected.

Graph robustness analysis

Proposition

Let $x^0 \in \mathbb{R}^{n\times d}$ be a reference configuration such that the associated graph G_{x^0} is connected. Let $y \in \mathbb{R}^{n\times d}$ be a perturbed configuration such that

$$\|y - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, $\mathcal{M}(G_{x^0})$ is a subgraph of G_y and G_y is connected.

Proof: Let $z = (z_1, \ldots, z_n)$ such that $z = y - x^0$.
For all $i, j \in V$, we have $-2z_i^\top z_j \leq \|z_i\|^2 + \|z_j\|^2$. Then,

$$\begin{aligned} \|z_i - z_j\|^2 &= \|z_i\|^2 + \|z_j\|^2 - 2z_i^\top z_j \leq 2(\|z_i\|^2 + \|z_j\|^2) \\ &\leq 2\|z\|^2 = 2\|y - x^0\|^2 \leq \rho^2_{G_{x^0}}. \end{aligned}$$

Then, $\|z_i - z_j\| \leq \rho_{G_{x^0}}$.

Graph robustness analysis

Proposition

Let $x^0 \in \mathbb{R}^{n\times d}$ be a reference configuration such that the associated graph G_{x^0} is connected. Let $y \in \mathbb{R}^{n\times d}$ be a perturbed configuration such that

$$\|y - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, $\mathcal{M}(G_{x^0})$ is a subgraph of G_y and G_y is connected.

Proof: Let $(i,j) \in \mathcal{M}(\mathcal{E}_{x^0})$, then $\|x_i^0 - x_j^0\| \leq R - \rho_{G_{x^0}}$. Thus,

$$\begin{aligned} \|y_i - y_j\| &= \|x_i^0 - x_j^0 + z_i - z_j\| \leq \|x_i^0 - x_j^0\| + \|z_i - z_j\| \\ &\leq R - \rho_{G_{x^0}} + \rho_{G_{x^0}} = R. \end{aligned}$$

Then, $(i,j) \in E_y$. Therefore $\mathcal{M}(G_{x^0})$ is a subgraph of G_y.
Since $\mathcal{M}(G_{x^0})$ is connected, so is G_y.

Graph robustness computation

Algorithm (Computation of the robustness $\rho_{G_{x^0}}$)

// Initialization:

$\forall (i,j) \in V^2,\ \rho_{ij}^0 \leftarrow R - \|x_i^0 - x_j^0\|;$

// Main loop:

for $k \in V$ **do**

  for $i \in V$ **do**

    for $j \in V$ **do**

      $\rho_{ij}^k \leftarrow \max\left(\rho_{ij}^{k-1}, \min\left(\rho_{ik}^{k-1}, \rho_{kj}^{k-1}\right)\right);$

    end for

  end for

end for

// Computation of robustness:

$\rho_{G_{x^0}} = \min_{(i,j)\in V^2} \rho_{ij}^n;$

Sufficient conditions for flocking

Theorem

Let $x(0) \in \mathbb{R}^{nd}$ *be a vector of initial positions of the agents such that the associated graph* $G_{x(0)}$ *is connected and its robustness* $\rho_{G_{x(0)}} > 0$.

Let $v(0) \in \mathbb{R}^{nd}$ *be a vector of initial velocities such that*

$$\|v(0) - \mathbf{1}_n \otimes v^*\| \leq \frac{\lambda_2^* \rho_{G_{x(0)}}}{\sqrt{2}}$$

where λ_2^* *is the algebraic connectivity of* $\mathcal{M}(G_{x(0)})$.

Then, for all $t \in \mathbb{R}^+$, $\mathcal{M}(G_{x(0)})$ *is a subgraph of* $G(t)$. *Moreover,*

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\| .$$

Proof: Let Π be the set of graphs with n nodes which have $\mathcal{M}(G_{x(0)})$ as a subgraph. Since $G_{x(0)}$ is connected, we have that $\mathcal{M}(G_{x(0)})$ is connected. Therefore, all graphs in Π are connected and since $\mathcal{M}(G_{x(0)}) \in \Pi$,

$$\min_{G \in \Pi} \lambda_2(G) = \lambda_2(\mathcal{M}(G_{x(0)})) = \lambda_2^* > 0.$$

Let us assume that there exists $t > 0$ such that $\mathcal{M}(G_{x(0)})$ is not a subgraph of $G(t)$ (i.e. $G(t) \notin \Pi$). Let

$$t^* = \inf\{t \in \mathbb{R}^+;\ G(t) \notin \Pi\}.$$

For $i \in V$, let $y_i(t) = x_i(t) - v^* t$. Let us remark that for all $i, j \in V$, $y_i(t) - y_j(t) = x_i(t) - x_j(t)$. Therefore, for all $t \in \mathbb{R}^+$, $G(t) = G_{y(t)}$.

Sufficient conditions for flocking

Proof: If $t^* > 0$, it follows that for all $t \in [0, t^*)$

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \frac{\lambda_2^* \rho_{G_{x(0)}}}{\sqrt{2}}.$$

By remarking that

$$y(t) = x^0 + \int_0^t (v(t) - \mathbf{1}_n \otimes v^*) \, ds$$

we have for all $t \in [0, t^*)$

$$\|y(t) - x^0\| \leq \frac{\lambda_2^* \rho_{G_{x^0}}}{\sqrt{2}} \int_0^t e^{-\lambda_2^* s} ds < \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Then, by continuity of y, there exists $\varepsilon > 0$ such that for all $t \in [0, t^* + \varepsilon]$,

$$\|y(t) - x^0\| \leq \frac{\rho_{G_{x^0}}}{\sqrt{2}}.$$

Sufficient conditions for flocking

Proof: If $t^* = 0$, since $y(0) = x^0$ and by continuity of y, the same kind of property holds.

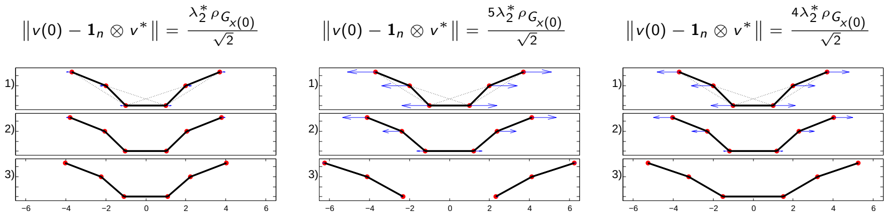
Then, we have for all $t \in [0, t^* + \varepsilon]$, $G(t) = G_{y(t)} \in \Pi$. This contradicts the definition of t^*. Therefore, for all $t \in \mathbb{R}^+$, $G(t) \in \Pi$. Thus for all $t \in \mathbb{R}^+$, $\mathcal{M}(G_{x(0)})$ as a subgraph of $G(t)$. This proves the first part of the theorem.

Moreover, it follows that for all $t \in \mathbb{R}^+$, $\lambda_2(t) \geq \lambda_2^*$, and then

$$\|v(t) - \mathbf{1}_n \otimes v^*\| \leq e^{-\lambda_2^* t} \|v(0) - \mathbf{1}_n \otimes v^*\|.$$

Example

Example in a network of 6 agents with communication radius $R = 3.2$:



- The bound that we compute is often conservative as shown on the example above.
- However, we can find examples where the bound is tight.

Summary

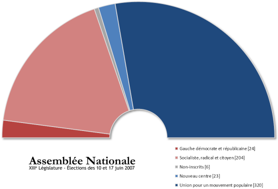
- We have considered a model of flocking behavior based on communication graphs given by a proximity rule.
- We have established a set of initial conditions for which the flocking behavior is achieved.
- The conditions are only sufficient but can be checked algorithmically.
- The main concepts are those of graph robustness and the associated core robust subgraph that remains for all time ensuring that the communication graph remains connected for all time.

Lecture outline

1. Algebraic graph theory
    - Basic graph notions
    - Laplacian matrix
    - Normalized Laplacian matrix
2. Consensus Algorithms:
    - Discrete time and continuous time
    - Agreement in networks with fixed topology
    - Agreement in networks with dynamic topology
3. Applications:
    - Flocking in mobile networks
    - Opinion dynamics and community detection in social networks

Opinion dynamics in social networks

Opinion dynamics studies the emergence of consensus in social networks:



- In real social networks, it is often the case that a global consensus cannot be reached. Instead, several consensus are reached locally by subsets of agents forming communities.
- Can we propose a model of opinion dynamics reproducing this phenomenom ? Can we learn something on real social networks using this model ?

A model of opinion dynamics

We consider a set of agents $i \in V$ in a network $G = (V, E)$.
Each agent $i \in V$ has an opinion $x_i(t) \in \mathbb{R}$.
We propose a discrete time model of opinion dynamics as follows:

- At each time step, agent i receives the opinion of its neighbors in G.
- Agent i gives confidence only to his neighbor that have an opinion close from its own. Agent i updates its opinion accordingly to its confidence neighborhood.
- Due to loss of patience, the confidence of each agent decreases at each time step: agent i requires that, at each negotiation round, the opinion of agent j moves significantly towards its opinion in order to keep negotiating with j.

A model of opinion dynamics

Formally, the opinion dynamics model is described as follows:

$$x_i(t+1) = x_i(t) + \frac{\varepsilon}{d_i(t)} \sum_{j \in N_i(t)} (x_j(t) - x_i(t))$$

where the interaction graph at time t is $G(t) = (V, E(t))$ with

$$E(t) = \left\{(i,j) \in E | \ \left(|x_i(t) - x_j(t)| \leq R\rho^t\right)\right\}$$

where $\varepsilon \in (0, 1/2)$, $R \geq 0$ and $\rho \in (0, 1)$ are model parameters.

The parameter ρ characterizes the confidence decay of the agents.

Convergence analysis

Proposition

For all $i \in V$, the sequence $(x_i(t))_{t\in\mathbb{N}}$ is convergent. We denote x_i^* its limit. Furthermore, we have for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq \frac{\varepsilon R}{1-\rho}\rho^t.$$

Proof: Let $i \in V$, $t \in \mathbb{N}$, we have

$$\begin{aligned}
|x_i(t+1) - x_i(t)| &= \left| \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} (x_j(t) - x_i(t)) \right| \\
&\leq \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} |x_j(t) - x_i(t)| \\
&\leq \frac{\varepsilon}{d_i(t)} \sum_{j\in N_i(t)} R\rho^t = \varepsilon R \rho^t.
\end{aligned}$$

Convergence analysis

Proposition

For all $i \in V$, the sequence $(x_i(t))_{t\in\mathbb{N}}$ is convergent. We denote x_i^* its limit. Furthermore, we have for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq \frac{\varepsilon R}{1-\rho}\rho^t.$$

Proof: Let $t \in \mathbb{N}$, $\tau \in \mathbb{N}$, then

$$\begin{aligned} |x_i(t+\tau) - x_i(t)| &\leq \sum_{k=0}^{\tau-1} |x_i(t+k+1) - x_i(t+k)| \leq \sum_{k=0}^{\tau-1} \varepsilon R \rho^{t+k} \\ &\leq \frac{\varepsilon R}{1-\rho}\rho^t \end{aligned}$$

which shows, since $\rho \in (0,1)$, that the sequence $(x_i(t))_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, it is convergent. The inequality of the proposition is obtained by letting τ go to $+\infty$.

Communities

Definition

A community is subset of agent $C \subseteq V$ such that for all $i, j \in C$, $x_i^* = x_j^*$. The graph of a community C is $G_C = (C, E_C)$ where

$$E_C = \{(i,j) \in E |\ i \in C,\ j \in C\}.$$

The set of communities is $\mathscr{C}$, it is a partition of V. The graph of communities is $G_{\mathscr{C}} = (V, E_{\mathscr{C}})$ where

$$E_{\mathscr{C}} = \{(i,j) \in E |\ x_i^* = x_j^*\}.$$

Let us remark that

$$E_{\mathscr{C}} = \bigcup_{C \in \mathscr{C}} E_C.$$

Characterization of communities

Before giving an algebraic characterization of communities, we need to make a supplementary assumption.

- We have seen that the opinions converge at a speed $O(\rho^t)$.
- In practice, we observe that the convergence is often slightly faster and this motivates the following assumption:

Assumption (Fast convergence)

There exists $\underline{\rho} < \rho$ and $M \geq 0$ such that for all $i \in V$, for all $t \in \mathbb{N}$,

$$|x_i(t) - x_i^*| \leq M\underline{\rho}^t.$$

Characterization of communities

Proposition

There exists $T \in \mathbb{N}$, such that for all $t \geq T$, $E(t) = E_{\mathscr{C}}$.

Proof: $(E(t) \subseteq E_{\mathscr{C}})$

E can be splitted into two subets: E^f consists of edges that disappear in finite time, E^∞ consists of edges that appear infinitely often.
Then, there exists T_1 such that for all $t \geq T_1$, $E(t) \subseteq E^\infty$.

Let $(i,j) \in E^\infty$ then there exists an unbounded sequence of times τ_k such that $(i,j) \in E(\tau_k)$. This gives

$$|x_i(\tau_k) - x_j(\tau_k)| \leq R\rho^t$$

and $x_i^* = x_j^*$. Therefore $(i,j) \in E_{\mathscr{C}}$.
Hence, for all $t \geq T_1$, $E(t) \subseteq E^\infty \subseteq E_{\mathscr{C}}$.

Characterization of communities

Proposition

There exists $T \in \mathbb{N}$, such that for all $t \geq T$, $E(t) = E_{\mathscr{C}}$.

Proof: $(E_{\mathscr{C}} \subseteq E(t))$

Let $(i,j) \in E_{\mathscr{C}}$, then for all $t \in \mathbb{N}$

$$\begin{array}{rcl} |x_i(t) - x_j(t)| & \leq & |x_i(t) - x_i^*| + |x_i^* - x_j^*| + |x_j^* - x_j(t)| \\ & \leq & |x_i(t) - x_i^*| + |x_j^* - x_j(t)| \leq 2M\underline{\rho}^t. \end{array}$$

Since $\underline{\rho} < \rho$, there exists T_2 such that for all $t \geq T_2$, $2M\underline{\rho}^t \leq R\rho^t$. This implies that for all $t \geq T_2$, $(i,j) \in E(t)$.
Hence, for all $t \geq T_2$, $E_{\mathscr{C}} \subseteq E(t)$.

Remark

We have shown that after a finite time, the graph $G(t)$ is fixed.

Characterization of communities

Theorem

For almost all vectors of initial opinions, for all communities $C \in \mathscr{C}$, such that $|C| \geq 2$,

$$\tilde{\lambda}_2(G_C) > (1-\rho)/\varepsilon.$$

Main ideas of the proof:
Let $C \in \mathscr{C}$ and let us assume that $\tilde{\lambda}_2(G_C) \leq (1-\rho)/\varepsilon$.

Let $x_C(t)$ denote the vector of opinions of agents in C. For all $t \geq T$, since $G(t) = G_{\mathscr{C}}$, we have

$$x_C(t+1) = (I - \varepsilon D(G_C)^{-1} L(G_C)) x_C(t).$$

We have seen that the rate of convergence of $x_C(t)$ is $1 - \varepsilon \tilde{\lambda}_2(G_C) \geq \rho$ except if $x_C(T)$ and thus $x(T)$ belongs to a specific subspace of zero measure H_C.

Characterization of communities

Theorem

For almost all vectors of initial opinions, for all communities $C \in \mathscr{C}$, such that $|C| \geq 2$,

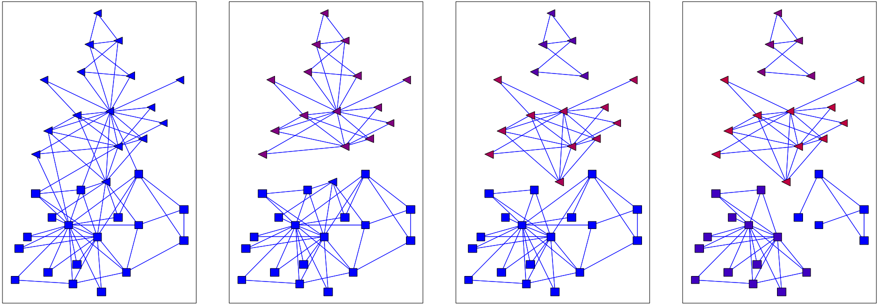
$$\tilde{\lambda}_2(G_C) > (1-\rho)/\varepsilon.$$

Main ideas of the proof:
By assumption the rate of convergence of $x_C(t)$ is smaller than $\underline{\rho} < \rho$. Thus $x(T)$ necessarily belongs to H_C.

Then, by remarking that all matrices $(I - \varepsilon D(t)^{-1} L(t))$ are invertible, we can move backward in time and show that the initial conditions leading to H_C at time T are included in a set of zero measure (consisting of a countable union of subspaces) that is independent of C and T.

Example: karate club network

A social network of 34 agents:



$(1-\rho)/\varepsilon = 0.1$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.12$

$(1-\rho)/\varepsilon = 0.2$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.25$

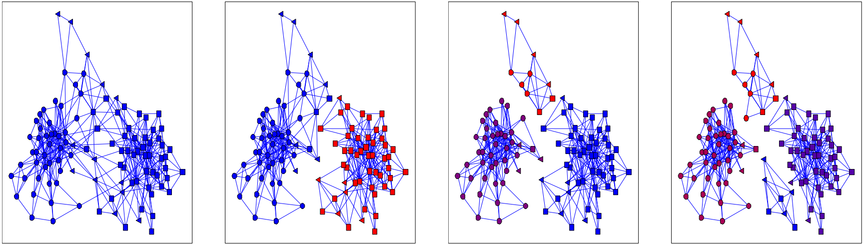
$(1-\rho)/\varepsilon = 0.3$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.33$

$(1-\rho)/\varepsilon = 0.4$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.57$

The second partition is almost that observed by Zachary in its original study (1973).

Example: book network

A network of 105 books on American politics:



Original network | $(1-\rho)/\varepsilon = 0.1$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.13$ | $(1-\rho)/\varepsilon = 0.15$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.18$ | $(1-\rho)/\varepsilon = 0.2$, $\min_{C \in \mathscr{C}} \tilde{\lambda}_2(G_C) = 0.27$

One recovers the information on the political alignment (democrat, republican, centrist) of the books.

Summary

- We have introduced a model of opinion dynamics with decaying confidence.
- In this model, a global consensus may not be achieved and only local agreements can be reached. The agents organize themselves in communities.
- The analysis of the model provided an algebraic characterization of the communities under a fast convergence assumption.
- The experimental results tend to confirm the validity of the algebraic characterization. Moreover, the communities that are obtained on real examples are meaningful and allows to uncover information hidden in the network structure.

Conclusions

- We have presented a collection of representative results on consensus algorithms in continuous and discrete time, and we have shown the relation to graph theory.
- We have shown two different applications of consensus algorithms:
  - Flocking in mobile networks
  - Opinion dynamics in social networks
- Consensus algorithms are currently pretty well understood...
  but there are still a lot of things to do on consensus applications.

Thank you for your attention!