The projective to Einstein correspondence and a kink on a wormhole

Figs/University_Crest.pdf

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Abstract

The first part of this thesis is concerned with recent work by Dunajski and Mettler. They show that a class of neutral signature Einstein manifolds M can be canonically constructed as rank n affine bundles over projective structures in dimension n. These have the same symmetry group as the underlying projective manifold, and are also endowed with a natural symplectic form, which is related to the metric by an endomorphism of the tangent bundle that squares to the identity. Consequently, the manifolds M carry an almost para–Kähler structure.

We show that every metric within the class is a Kaluza–Klein reduction of an Einstein metric on an $\mathbb{R}^*$ bundle over M. We also show that the structures are para–c–projectively compact in the sense of Čap–Gover, and intepret the compactification in terms of the tractor bundle of the projective structure.

In dimension four, the manifolds M have anti–self–dual conformal curvature, and are thus associated with a twistor space. In the presence of a symmetry, they can be reduced to Einstein–Weyl structures in dimension three via the Jones–Tod correspondence. Because M is also Einstein with non–zero scalar curvature, these Einstein–Weyl structures are determined by solutions of the $SU(\infty)$–Toda equation.

We classify the Einstein–Weyl structures which can be obtained in this way in terms of the symmetry group of the underlying projective surface. Several examples are considered in detail, resulting in new, explicit solutions of the $SU(\infty)$–Toda equation. We focus in particular on the case where the projective structure is $\mathbb{RP}^n$, additionally describing the Jones–Tod reduction from the twistor perspective.

Finally, we study ϕ^4 field theory on a wormhole spacetime in $3+1$ dimensions. This spacetime has two asymptotically flat ends connected by a spherical throat of radius a. We show that the theory possesses a kink solution which is linearly stable, and compare its discrete spectrum to that of the ϕ^4 kink on $\mathbb{R}^{1,1}$. We present some preliminary results on the non–linear resonant coupling between the discrete and continuous spectra in the range of a where there is exactly one discrete mode.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

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Acknowledgements

And I would like to acknowledge ...

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Chapter 1

Introduction

This thesis splits naturally into two parts. The first 4 chapters are concerned with a class of Einstein manifolds which can be canonically constructed from projective structures as shown in recent work by Dunajski and Mettler [22]. Chapter 5 discusses a solitonic solution of the equations of motion associated with ϕ^4–field theory on a wormhole spacetime. Chapter 5 is self–contained, and possesses its own introduction and notation. The remainder of this chapter will introduce some preliminaries for the first part of the thesis, culminating in the results of [22] on the almost para–Kähler Einstein manifolds which will form the basis of chapters 2–4.

aw: Projective geometry, based on geodesics as unparametrised curves, initiated in ... by ...

aw: General point - should be more than in abstract but not repeating the actual intro below.

Given a projective structure on a manifold N of dimension n, Dunajski and Mettler [22] construct a neutral signature Einstein metric g with non-zero scalar curvature on a certain rank n affine bundle $M \to N$. The $2n$–dimensional space M also carries a natural symplectic form Ω, and an endomorphism $J : TM \to TM$ which is such that J^2 is the identity and $g(\cdot\,,\cdot) = \Omega(\cdot\,, J\cdot)$. This makes (M, g, Ω) a so–called *almost para–Kähler* structure. It is interesting for a number of different reasons.

aw: It expresses the second order object which is the projective structure as a first order object on a larger space. See if you can find the slides of the talk Thomas gave in damtp where he talked about this? If I put in Thomas' explicit construction of g (i.e. define the second order coframe bundle) this point would be more relevant.

Firstly, (M, g) is interesting by virtue of being an Einstein space. In fact, it turns out that g arises as the Kaluza–Klein reduction of an Einstein metric $\mathcal{G}$ on an $\mathbb{R}^*$ bundle $\sigma : \mathcal{Q} \to M$ which has curvature form $\sigma^*(\Omega)$. In chapter 2 we will construct $\mathcal{G}$

explicitly, and give an interpretation of the manifold $(\mathcal{Q}, \mathcal{G})$ in terms of the projective geometry on N.

It also turns out that (M, g, Ω) can be thought of as compactifiable in a certain sense. Recall that a (psuedo–)Riemannian manifold (M, g) is said to be *conformally compact* if there is a smooth positive function T such that $T^2 g$ smoothly extends to a manifold with boundary $\overline{M} = M \cup \partial M$, and the set $\{p \in \overline{M} : T(p) = 0\}$ is a hypersurface which coincides with the boundary ∂M. This is a useful concept because $(M, T^2 g)$ has the same conformal structure, and hence the same *causal* structure, as (M, g). It has been used to study said causal structure in both general relativity [76] and quantum field theory [78]. It is also useful for formulating the boundary conditions of conformally invariant field equations such as those arising in Yang–Mills theory [77].

Recent work by Čap and Gover [65, 66] has generalised this idea to other geometrical structures which admit some weakening which extends to a manifold with boundary. In particular, on an almost complex manifold (M, J) with complex connection ∇, one can define the c–projective equivalence class $[\nabla]$ to which ∇ belongs, and show that the c–projective structure $(M, J, [\nabla])$ extends to a manifold with boundary $\overline{M}$ [66]. The main goal of chapter 3 is to adapt the work of [66] to the para–c–projective case, and to show that the almost complex structure J on M has a complex connection which admits a so–called para–c–projective compactification. The result of this is that the manifolds (M, g, Ω) can be thought of as para–c–projectively compact.

Another reason this construction is interesting is that for $n = 2$ (so that M has dimension 4), the conformal curvature of g is anti-self-dual. Recall that the Hodge operator $\star$ defined by a Euclidean or neutral signature metric in four dimensions is an involution on two–forms (i.e. squares to the identity). It thus has eigenvalues ± 1, and the space of two-forms splits into the corresponding eigenspaces, which are referred to as self–dual (SD) or anti–self–dual (ASD) respectively. Due to its index symmetries, the Weyl tensor can be thought of as a map from two-forms to two-forms, and therefore has a corresponding decomposition. Since the Weyl tensor encodes the conformal curvature, we say that a conformal or (psuedo–)Riemannian manifold whose Weyl tensor is ASD is equipped with an *ASD conformal structure*.

The field equations corresponding to anti-self-duality of the Weyl tensor in four dimensions can be solved by a twistor construction, and are thus *integrable* [58]. This means that any systems of differential equations which can be obtained from them by symmetry reduction should also be integrable (see [43] for a review). In particular, the class of dispersionless integrable systems in 2+1 and 3 dimensions arise in this way. The construction [22] provides some examples of ASD conformal structures in neutral signature which, in the presence of a (non–null) symmetry, give rise to solutions of an integrable system called the $SU(\infty)$–Toda field equation via $2+1$–dimensional Einstein-Weyl structures. In chapter 4 we discuss the extraction of all possible Toda solutions obtainable in this way.

aw: some notation and conventions e.g. $i, j, k = 1, ..., n$ are indices for coordinates on n? and Γ are connection components which are defined by... Curvature is defined by

- We denote the bundle of anti–symmetrised covariant tensors of degree m as Λ^m, and call sections of this bundle m–forms.
- Note that our conventions are $(d\omega)_{ab\ldots c} = \partial_{[a}\omega_{b\ldots c]}$, $(\eta \wedge \omega)_{a\ldots d} = \eta_{[a\ldots b}\omega_{c\ldots d]}$, $\omega = \omega_{a\ldots b} dx^a \wedge \cdots \wedge dx^b$, and $F_{ab} dx^a \wedge dx^b = F_{[ab]} dx^a \otimes dx^b$ implying $dx^a \wedge dx^b = \frac{1}{2}(dx^a \otimes dx^b - dx^b \otimes dx^a)$.
- N is oriented.

  aw: Although $\mathbb{RP}^n$ is not oriented right?? also something about which part of $SL(n+1)$ we mean?
- Probably a convention for the Hodge operator
- Probably a convention for ε_{AB} and matching indices descending to the right

1.1 Projective Geometry

Our discussion follows Eastwood [1].

Definition 1.1.1. *A projective structure $(N, [\nabla])$ on a manifold N is an equivalence class $[\nabla]$ of torsion–free affine connections on N which have the same unparametrised geodesics.*

The following proposition converts definition 1.1.1 to a to a more operational form.

Proposition 1.1.2. *Two torsion–free connections ∇ and $\overline{\nabla}$ belong to the same projective class if and only if their components Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ are related by*

$$\overline{\Gamma}^i_{jk} - \Gamma^i_{jk} = \delta^i_j \Upsilon_k + \delta^i_k \Upsilon_j \tag{1.1}$$

for some one–form Υ.

Proof. We denote by V the vertical sub–bundle of $T(TN)$, where $\nu : TN \to N$ is the tangent bundle to N. A connection defines a splitting of the exact sequence

$$0 \longrightarrow V \longrightarrow T(TN) \longrightarrow \nu^* TN \longrightarrow 0 \tag{1.2}$$

so that each $X^i \in T_pM$ has a unique pull–back in the horizontal sub–bundle complementary to V. The integral curves of these pull–backs, when projected down to N, then define the geodesics of the connection.

Any two connections are related by some $\delta\Gamma^k_{ij}$, which satisfies $\delta\Gamma^k_{ij} = \delta\Gamma^k_{(ij)}$ as long as both connections are torsion–free. A change of connection is equivalent to a change in the splitting of (1.2). At $X^i \in T_pN$, the change is given by the homomorphism from T_pN to $T_pN = V_p$ defined by the contraction $X^i\Gamma^k_{ij}$. Thus the two connections define the same geodesics if and only if $X^iX^j\Gamma^k_{ij}$ is a multiple of X^k for all X^i. This is true if and only if there is a one–form Υ_i such that (1.1) is satisfied.[1]

$\square$

One can show that the curvature of a connection ∇ in the projective class can be uniquely decomposed as

$$R_{ijk}{}^l = W_{ijk}{}^l + 2\delta^l_{[i}\mathrm{P}_{j]k} - 2\mathrm{P}_{[ij]}\delta^l_k, \tag{1.3}$$

where the Weyl projective curvature tensor, $W_{ijk}{}^l$, is trace free, and the Schouten tensor, P_{ij}, is given in terms of the Ricci tensor by

$$\mathrm{P}_{ij} = \frac{1}{n-1}R_{(ij)} + \frac{1}{n+1}R_{[ij]}.$$

The objects $W_{ijk}{}^l$ and P_{ij} transform as

$$\overline{W}_{ijk}{}^l = W_{ijk}{}^l, \qquad \overline{\mathrm{P}}_{ij} = \mathrm{P}_{ij} - \nabla_i\Upsilon_j + \Upsilon_i\Upsilon_j \tag{1.4}$$

under a change of representative connection (1.1). Note that for $n = 2$ the Weyl tensor is always vanishing.

A projective structure in dimension n is said to be flat if it is diffeomorphic to the real projective space $\mathbb{RP}^n$ with its standard flat projective structure.

[1] To see this, take some one–form ω_i and note that $2X^iX^j\delta^k_{(i}\Upsilon_{j)}\omega_k$ vanishes if and only if $X^k\omega_k$ does.

Definition 1.1.3. *The real projective space* $\mathbb{RP}^n$ *of dimension* n *is the space of un-oriented lines through the origin in* $\mathbb{R}^{n+1}$*, thought of as* $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^*$*, with geodesics given by straight lines in* $\mathbb{R}^{n+1}$ *under the projection* $\kappa : \mathbb{R}^{n+1} \to \mathbb{RP}^n$*.*

Let X denote a non–zero point in $\mathbb{R}^{n+1}$ with coordinates $(X^0, X^1, \ldots, X^n)^T$, and let $[X]$ denote the corresponding point in $\mathbb{RP}^n$, labelled by homogeneous coordinates. In a patch where $X^0 \neq 0$, we can write $[X] = [1, X^1/X^0, \ldots, X^n/X^0]$ and define inhomogeneous coordinates on $\mathbb{RP}^n$ by

$$(x^1, \ldots, x^n) = (X^1/X^0, \ldots, X^n/X^0).$$

If we combine this with coordinate patches where $X^i \neq 0,\ i = 1, \ldots, n$, we can build an atlas for $\mathbb{RP}^n$.

Real projective space can be viewed as homogeneous space as follows. The group $SL(n+1, \mathbb{R})$ acts from the left via the fundamental representation on coordinates $(X^0, \ldots, X^n)^T$ in $\mathbb{R}^{n+1}$, and this descends to a transitive action on $\mathbb{RP}^n$. By the orbit stabiliser theorem, $\mathbb{RP}^n = SL(n+1, \mathbb{R})/H$, where H is a subgroup stabilising a point. If we choose the point $[1, 0, \ldots, 0]$, the elements of H are matrices of the general form

$$\begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}$$

for some $a \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}_n$.

It can be shown that a projective structure in dimension $n > 2$ is flat if and only if its Weyl projective curvature tensor vanishes. The necessary and sufficient condition in dimension $n = 2$ is the vanishing of the Cotton tensor $\nabla_{[i}\mathrm{P}_{j]k}$ for any choice of representative connection.

1.1.1 The Cartan bundle

One way of understanding the construction of (M, g) in [22] is via the Cartan bundle [2] of the projective structure $(N, [\nabla])$ (see also [4]). Cartan geometries generalise Klein's Erlangen programme [3], a study of homogeneous spaces G/H, to the curved case, in which the total space G is replaced by a principal right H-bundle over a manifold N such that the tangent space to N at every point is isomorphic to the Lie algebra quotient $\mathfrak{g}/\mathfrak{h}$. Since projective structures are modelled on $\mathbb{RP}^n$, which can be viewed as a homogeneous space, they constitute a type of Cartan geometry.

In the Riemannian case, the model space is $\mathbb{R}^n \cong \mathrm{Euc}(n)/SO(n)$. The corresponding Cartan geometry is a general, curved Riemannian manifold. One has an obvious subclass of frames which are "adapted" to the metric, i.e. those which are orthonormal. We can thus think of a curved Riemannian manifold as a principal $SO(n)$ bundle whose tangent spaces are modelled on $\mathbb{R}^n \cong \mathfrak{Euc}(n)/\mathfrak{so}(n)$. We say that Riemannian manifolds are Cartan geometries of type $(\mathrm{Euc}(n), SO(n))$.

The theory of Cartan geometries was developed as part of Cartan's *method of moving frames*. The idea is to pick out some adapted frames for manifolds equipped with some non-metric structure. The bundle of such frames over a manifold is then a principal bundle $\pi : P \to N$ with structure group H. In the projective case, the notion of a *second order frame* must be introduced to obtain an object which is correctly adapted to the projective structure.

The bundle P is equipped with a $\mathfrak{g}$-valued one-form θ called the Cartan connection. It defines an isomorphism $\theta : T_pP \to \mathfrak{g}$ at every point $p \in P$ such that the vertical subspace $V_pP \subset T_pP$ is mapped to $\mathfrak{h}$ and the horizontal subpace $H_pP \subset T_pP$ is defined as the inverse image of $\mathfrak{g}/\mathfrak{h}$. Note that it is not a connection in the usual sense of a principal bundle connection, since it takes value in a Lie algebra larger than that of the structure group. Further details can be found in [5].

In the projective case, if we choose the point which is stabilised by H to be $[1, 0, \ldots, 0]$, the Cartan connection can be written as a matrix

$$\theta = \begin{pmatrix} -\mathrm{tr}\phi & \eta \\ \omega & \phi \end{pmatrix}, \tag{1.5}$$

where ω, η and ϕ are one-forms valued in $\mathbb{R}^n$, $\mathbb{R}_n$ and $\mathfrak{gl}(n, \mathbb{R})$ respectively. We will refer to the components of ω and η with respect to the natural basis of $\mathfrak{sl}(n+1, \mathbb{R})$ as $\{\omega^{(i)}\}$ and $\{\eta_{(i)}\}$, so that $\omega^{(i)}$ and $\eta_{(i)}$ are both one-forms.

Definition 1.1.4. *The Cartan geometry of a projective structure $(N, [\nabla])$ consists of a principal right H–bundle $\pi_P : P \to N$, where the right–action of some $h \in H$ on P is denoted by R_h, carrying a one–form θ called the Cartan connection, which takes values in $\mathfrak{sl}(n+1, \mathbb{R})$, can be written in the form (1.5) and has the following properties:*

1. *$\theta_u : T_uP \to \mathfrak{sl}(n+1, \mathbb{R})$ is an isomorphism for all $u \in P$;*

2. *$\theta(X_v) = v$ for all fundamental vector fields X_v on P;*

3. *$R_h^*\theta = \mathrm{Ad}(h^{-1})\theta = h^{-1}\theta h$ for all $h \in H$.*

4. *If X is a vector field on P with the property that $\eta(X) = \phi(X) = 0$ and $\omega(X) \in \mathbb{R}^n \backslash \{0\}$, then the integral curve of X projects down to a geodesic on N and conversely every geodesic of $[\nabla]$ arises in this way.*

5. *The $\mathfrak{sl}(n+1, \mathbb{R})$-valued curvature two-form Θ satisfies*

$$\Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & L(\omega \wedge \omega) \\ 0 & W(\omega \wedge \omega) \end{pmatrix}, \tag{1.6}$$

where L and W are smooth curvature functions valued in $\mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, R_n)$ *and* $\mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, R_n \otimes \mathbb{R}^n)$ *respectively. The function W represents the Weyl projective curvature tensor appearing in (1.3).*

Remark 1.1.5. The Cartan geometry of a projective structure is unique in the sense that for any two Cartan geometries $(\widehat{\pi}_P : \widehat{P} \to N, \widehat{\theta})$ and $(\pi_P : P \to N, \theta)$ of type $(SL(n+1, \mathbb{R}), H)$ satisfying the above properties there is a H–bundle isomorphism $\psi : P \to \widehat{P}$ such that $\psi^* \widehat{\theta} = \theta$.

Remark 1.1.6. For every open set $\mathcal{U} \subset N$, projective vector fields on $\mathcal{U}$ are in one-to-one correspondence with vector fields on $\pi^{-1}(\mathcal{U})$ which preserve θ and are equivariant

aw: what do I mean by this

under the principal H-action.

Remark 1.1.7. The Cartan connection also gives us a unique connection on any bundle associated to P via some H–module. In particular, let E be a vector space and $\rho : H \to GL(E)$ a representation of H acting on E. We can construct an *associated bundle*

$$\pi_E : P \times_\rho E \to N$$

where points in $P \times_\rho E$ are equivalence classes of pairs $[u, v]$, where $u \in P$ and $v \in E$, up to the equivalence relation

$$(u_1, v_1) \sim (u_2, v_2) \quad \Leftrightarrow \quad \exists\, h \text{ such that } u_2 = u_1 h,\ v_2 = \rho(h^{-1}) v_1.$$

We thus obtain a vector bundle over N whose fibers are diffeomorphic to E. A section $s : N \to P \times_\rho E$ is represented by a map $\sigma_s : P \to E$ which is *equivariant* in the sense that $\sigma_s(uh) = \rho(h^{-1}) \sigma_s(u)$ for all $h \in H$. Importantly, any such bundle inherits a connection from the Cartan connection θ on P.

1.1.2 The cotractor bundle

A particularly important example of a vector bundle associated to P is the *cotractor bundle*, which defined by the canonical right action of H on $\mathbb{R}_{n+1}$ given by $(h, v) \mapsto vh^{-1}$. We call this bundle $\pi : \mathcal{T}^* \to N$. In order to describe its connection, we consider a section represented by $\sigma : P \to \mathbb{R}_{n+1}$ and define the one–form

$$d\sigma - \sigma\theta. \tag{1.7}$$

This turns out to be a *semi–basic*[2] one–form satisfying

$$R_h^*(d\sigma - \sigma\theta) = (d\sigma - \sigma\theta)\rho(h^{-1}),$$

making $\sigma \mapsto d\sigma - \sigma\theta$ an equivariant connection on $\mathcal{T}^*$.

Although this construction of $\mathcal{T}^*$ relies on the Cartan bundle, it is possible to construct it independently. In order to do so we need the notion of a *projective density*.

Projective densities

From the projective change of connection (1.1) we can derive the the corresponding change in $\nabla\chi$ for some m–form χ on N:

$$\overline{\nabla}_i \chi_{jk\ldots l} = \nabla_i \chi_{jk\ldots l} - (m+1)\Upsilon_i \chi_{jk\ldots l} - (m+1)\Upsilon_{[i}\chi_{jk\ldots l]}. \tag{1.8}$$

In particular, for a volume form $(m = n)$ we find

$$\overline{\nabla}_i \chi_{jk\ldots l} = \nabla_i \chi_{jk\ldots l} - (n+1)\Upsilon_i \chi_{jk\ldots l},$$

where the final term in (1.8) has vanished because it contains a symmetrisation over $n+1$ indices. We can write this in a more compact way as

$$\overline{\nabla}_i \chi = \nabla_i \chi - (n+1)\Upsilon_i \chi.$$

Note that for sections τ of the bundle $\mathcal{E}(w) := (\Lambda^n)^{-w/(n+1)}$ we have

$$\overline{\nabla}_i \tau = \nabla_i \tau + w\Upsilon_i \tau. \tag{1.9}$$

[2]Recall that a semi–basic form on a fiber bundle $P \to N$ is a form which is a linear combination, with coefficients parametrised by the fibers, of basic forms on P (i.e. forms which are the pull-backs of forms on N).

We called such sections *projective densities of weight* w, and for any vector bundle $\mathcal{B} \to N$ we write $\mathcal{B}(w)$ for the tensor product of $\mathcal{B}$ with $\mathcal{E}(w)$. For example, $T^*N(w)$ is the bundle of one–forms with projective weight w, and for sections μ_i of $T^*N(w)$ we have

$$\overline{\nabla}_i \mu_j = \nabla_i \mu_j + (w-1)\Upsilon_i \mu_j - \Upsilon_j \mu_i. \tag{1.10}$$

We can now define the cotractor bundle $\pi : \mathcal{T}^* \to N$. For a choice of connection in the projective class we identify

$$\mathcal{T}^* = \mathcal{E}(1) \oplus T^*N(1), \tag{1.11}$$

so that a section can be represented by a pair

$$\begin{pmatrix} \tau \\ \mu_i \end{pmatrix}. \tag{1.12}$$

Under a change of projective connection (1.1), this splitting changes according to

$$\overline{\begin{pmatrix} \tau \\ \mu_i \end{pmatrix}} = \begin{pmatrix} \tau \\ \mu_i + \Upsilon_i \tau \end{pmatrix}. \tag{1.13}$$

Note the exact sequence

$$0 \longrightarrow T^*N(1) \longrightarrow \mathcal{T}^* \xrightarrow{X} \mathcal{E}(1) \longrightarrow 0, \tag{1.14}$$

where we call the map X the projective *canonical tractor*. A choice of connection in the projective class defines a splitting (1.11) of (1.14).[3]

The bundle $\mathcal{T}^*$ admits a projectively invariant *tractor connection* given by

$$\nabla_i^{\mathcal{T}} \begin{pmatrix} \tau \\ \mu_j \end{pmatrix} = \begin{pmatrix} \nabla_i \tau - \mu_i \\ \nabla_i \mu_j + \mathrm{P}_{ij} \tau \end{pmatrix}, \tag{1.15}$$

[3] In fact X is a section of a bundle $\mathcal{T}(1)$, where $\mathcal{T}$ can be identified with a direct sum $\mathcal{E}(1) \oplus TN(1)$ given a choice of connection in the projective class. The natural pairing between $\mathcal{T}$ and $\mathcal{T}^*$ defines the map $X : \mathcal{T}^* \to \mathcal{E}(1)$.

which turns out to agree with (1.7). Under a change of projective connection (1.1), we find

$$
\begin{aligned}
\overline{\nabla}_i^{\mathcal{T}} \overline{\begin{pmatrix} \tau \\ \mu_j \end{pmatrix}} &= \overline{\nabla}_i^{\mathcal{T}} \begin{pmatrix} \tau \\ \mu_j + \Upsilon_j \tau \end{pmatrix} \\
&= \begin{pmatrix} \overline{\nabla}_i \tau - (\mu_i + \Upsilon_i \tau) \\ \overline{\nabla}_i (\mu_j + \Upsilon_j \tau) + \overline{\mathrm{P}}_{ij} \tau \end{pmatrix} \\
&= \begin{pmatrix} \nabla_i \tau + \Upsilon_i \tau - (\mu_i + \Upsilon_i \tau) \\ \nabla_i (\mu_j + \Upsilon_j \tau) - \Upsilon_j (\mu_i + \Upsilon_i \tau) + (\mathrm{P}_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j) \tau \end{pmatrix},
\end{aligned}
$$

where we have used (1.13) in the first line, (1.15) in the second and (1.9), (1.10) and (1.4) in the third. After some cancellation, we identify

$$
\begin{aligned}
\overline{\nabla}_i^{\mathcal{T}} \overline{\begin{pmatrix} \tau \\ \mu_j \end{pmatrix}} &= \begin{pmatrix} \nabla_i \tau - \mu_i \\ \nabla_i \mu_j + \Upsilon_j \nabla_i \tau - \Upsilon_j \mu_i + \mathrm{P}_{ij} \tau \end{pmatrix} \\
&= \overline{\begin{pmatrix} \nabla_i \tau - \mu_i \\ \nabla_i \mu_j + \mathrm{P}_{ij} \tau \end{pmatrix}} = \overline{\nabla_i^{\mathcal{T}} \begin{pmatrix} \tau \\ \mu_j \end{pmatrix}}
\end{aligned}
$$

using (1.15) and the change of splitting (1.13) adapted to the tensor product of T^*N and $\mathcal{T}^*$ of which the derivative is a section.

aw: With a bit more work Eastwood calculates the curvature of the tractor connection and proves the statements about when projective structures are flat which I made above. Could add this.

Any bundle associated to P via some H–module is equipped with a tractor connection which is inherited from the connection on the standard cotractor bundle, or equivalently from the Cartan connection. It is this connection, with its special equivariance property, that allows us to construct an Einstein metric as in invariant of the projective structure. This construction is the subject of the following section.

1.2 The projective to Einstein correspondence

The work of [22] can be understood from two perspectives, one of which is based on the Cartan bundle of the projective structure and one of which is based on the cotractor bundle. We will make use of both, since the Cartan and tractor approaches are most natural in chapters 2 and 3 respectively. We begin by describing the viewpoint based on the Cartan geometry.

1.2.1 Cartan perspective

Consider a quotient of the total space P of the Cartan bundle by $GL(n,\mathbb{R})$, which is embedded in H in the obvious way:

$$GL(n,\mathbb{R}) \ni a \longmapsto \begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix} \in H. \tag{1.16}$$

It is easily verified that

$$\begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & \eta \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \eta a^{-1} \det a^{-1} \\ (\det a) a\omega & 0 \end{pmatrix},$$

for any $a \in GL(n,\mathbb{R})$, meaning that due to the equivarience property of the Cartan connection, the natural contraction $\eta\omega := \sum_i \eta_{(i)} \otimes \omega^{(i)}$ defined by θ is presevered by the adjoint action of this $GL(n,\mathbb{R})$ subgroup. It thus descends to a naturally defined object on the quotient $M = P/GL(n,\mathbb{R})$.

Theorem 1.2.1. *[22] There exist a metric and two-form (g,Ω) on $M = P/GL(n,\mathbb{R})$ such that the quotient map $q : P \to M$ gives*

$$q^* g = \mathrm{Sym}(\eta\omega) \tag{1.17}$$
$$q^* \Omega = \mathrm{Ant}(\eta\omega), \tag{1.18}$$

where Sym *and* Ant *denote the symmetric and anti-symmetric parts of the $(0,2)$ tensor $\eta\omega$. Moreover, Ω is closed as a consequence of the Bianchi identity, g is Einstein with non-zero scalar curvature, and the two are related by an endomorphism J satisying $J^2 = \mathrm{id}$. Hence (g,Ω) is an almost para-Kahler structure on M.*

Note that the full proof of theorem 1.2.1 only appears explicitly in [22] in the case $n = 2$, although it can be generalised to $n > 2$. This generalisation is discussed in their appendix. They show that the Ricci scalar of g is 24 in the case $n = 2$. In chapter 2 we will need the Ricci scalar for general n. We will calculate this under the assumption (stated without proof in [22]) that g is Einstein.

aw: Motivate chapter 2 here by mentioned the alternative quotient?

A coordinate expression for the metric g and two–form Ω can be obtained by writing out the Cartan connection θ explicitly for some choice of connection in the projective class. Alternatively, one can take a tractor approach using the tractor connection (1.15). This approach is a product of [24].

aw: Might be good to add Thomas' construction explicitly. This would be better for justifying the comments made in chapter 2 about the lifted metric being a different quotient of the Cartan bundle.

1.2.2 Tractor perspective

From the tractor perspective, the space M will turn out to be the projectivised cotractor bundle of N with an $\mathbb{RP}^{n-1}$ sub–bundle removed from each fiber. We can see this as follows.

On the total space of $\mathcal{T}^*$ we pullback $\pi : \mathcal{T}^* \to N$ along π to get $\pi^*(\mathcal{T}^*) \to \mathcal{T}^*$ as a vector bundle over the total space $\mathcal{T}^*$. By construction this bundle has a tautological section $U \in \Gamma(\pi^*(\mathcal{T}^*))$. We also have $\pi^*(\mathcal{T}(w))$ for any weight w, and we shall write simply $X \in \Gamma(\pi^*(\mathcal{T}(1)))$ for the pullback to $\mathcal{T}^*$ of the canonical tractor X on N.

Now define

$$\kappa : \mathcal{T}^* \longrightarrow \mathcal{M} := \mathbb{P}(\mathcal{T}^*) \tag{1.19}$$

by the fibrewise projectivisation, and use $\pi_{\mathcal{M}}$ for the map

$$\pi_{\mathcal{M}} : \mathcal{M} \to N.$$

We denote by $\mathcal{E}_{\mathcal{T}^*}(w')$, for $w' \in \mathbb{R}$, the line bundle on $\mathbb{P}(\mathcal{T}^*)$ whose sections correspond to functions $f : \pi^*\mathcal{T}^* \to \mathbb{R}$ that are homogeneous of degree w' in the fibres of $\pi^*\mathcal{T}^* \to \mathbb{P}(\mathcal{T}^*)$. For any weight w we also have $\mathcal{E}(w)$ on N and its pull back to the bundle $\pi^*_{\mathcal{M}}\mathcal{E}(w) \to \mathbb{P}(\mathcal{T}^*)$. We define the product of these two density bundles on $\mathcal{M}$ as

$$\mathcal{E}(w, w') := \pi^*\mathcal{E}(w) \otimes \mathcal{E}_{\mathcal{T}^*}(w').$$

On $\mathcal{T}^*$ there is a canonical density $\tau \in \Gamma(\pi^*\mathcal{E}(1))$ given by

$$\tau := X \lrcorner U.$$

Note that τ is homogeneous of degree 1 up the fibres of the map $\mathcal{T}^* \to \mathcal{M}$. Thus τ determines, and is equivalent to, a section (that we also denote) τ of the density bundle $\mathcal{E}(1, 1)$. So $\mathcal{M}$ is stratified according to whether or not τ is vanishing, and we write $\mathcal{Z}(\tau)$ to denote, in particular, the zero locus of τ. With these tools we can recover the construction in [22].

Theorem 1.2.2. *[24] There is a metric g and two–form Ω on $\mathcal{M} \setminus \mathcal{Z}(\tau)$ determined by the canonical pairing of the horizontal and vertical subspaces of $T(\mathcal{T}^*)$. The pair (g, Ω) agrees with (1.17, 1.18).*

Proof. Considering first the total space $\mathcal{T}^*$ and then its tangent bundle, note that there is an exact sequence

$$0 \to \pi^*\mathcal{T}^* \to T(\mathcal{T}^*) \to \pi^*TN \to 0, \tag{1.20}$$

where we have identified $\pi^*\mathcal{T}^*$ as the vertical sub-bundle of $T(\mathcal{T}^*)$. The tractor connection on the vector bundle $\mathcal{T}^* \to N$ is equivalent to a splitting of this sequence, identifying π^*TN with a distinguished sub-bundle of horizontal subspaces in $T(\mathcal{T}^*)$ so that we have

$$T(\mathcal{T}^*) = \pi^*TN \oplus \pi^*\mathcal{T}^*. \tag{1.21}$$

We move now to the total space of $\mathbb{P}\mathcal{T}^*$, and we note that again the tractor (equivalently, Cartan) connection determines a splitting of the tangent bundle $T(\mathbb{P}\mathcal{T}^*)$ in which the second term of the display (1.21) is replaced by a quotient of $\pi^*\mathcal{T}^*(0,1)$ [67]. Indeed, if we work at a point $p \in \mathbb{P}(\mathcal{T}^*)$, observe that $\pi^*\mathcal{T}^*(0,1)$ has a filtration

$$0 \to \mathcal{E}(0,0)_p \xrightarrow{U_p} \pi^*_{\mathcal{M}}\mathcal{T}^*(0,1)|_p \to \pi^*_{\mathcal{M}}\mathcal{T}^*(0,1)|_p/\langle U_p\rangle \to 0 \tag{1.22}$$

where, as usual, U is the canonical section. But away from $\mathcal{Z}(\tau)$, we have that U canonically splits the appropriately re-weighted pull back of the sequence (1.14)

$$0 \to \pi^*_{\mathcal{M}}T^*N(1,1) \to \pi^*_{\mathcal{M}}\mathcal{T}^*(0,1) \xrightarrow{X/\tau} \mathcal{E}(0,0) \to 0.$$

This identifies the quotient in (1.22), and thus we have canonically

$$T(\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)) = \pi^*_{\mathcal{M}}TN \oplus \pi^*_{\mathcal{M}}T^*N(1,1).$$

It follows that on $\mathcal{M}$ there is canonically a metric $\boldsymbol{g}$ and symplectic form $\boldsymbol{\Omega}$ taking values in $\mathcal{E}(1,1)$, given by

$$\begin{aligned} \boldsymbol{g}(w_1, w_2) &= \frac{1}{2}\Big(\Pi_H(w_1) \lrcorner \Pi_V(w_2) + \Pi_H(w_2) \lrcorner \Pi_V(w_1)\Big) \quad \text{and} \\ \boldsymbol{\Omega}(w_1, w_2) &= \frac{1}{2}\Big(\Pi_H(w_1) \lrcorner \Pi_V(w_2) - \Pi_H(w_2) \lrcorner \Pi_V(w_1)\Big) \end{aligned}$$

where

$$\Pi_H : T(\mathcal{M} \setminus \mathcal{Z}(\tau)) \to \pi^*_{\mathcal{M}} TN \quad \text{and} \quad \Pi_V : T(\mathcal{M} \setminus \mathcal{Z}(\tau)) \to \pi^*_{\mathcal{M}} T^*N(1,1)$$

are the projections. Then we obtain the metric and symplectic form by

$$g := \frac{1}{\tau}\boldsymbol{g} \qquad \text{and} \qquad \Omega := \frac{1}{\tau}\boldsymbol{\Omega}. \tag{1.23}$$

What remains to be done, is to show that (1.23) agrees with the form obtained in [22] once a trivialisation of $\mathcal{T}^* \to N$ has been chosen.

Let $p \in N$ and let $\mathcal{W} \subset N$ be an open neighbourhood of p with local coordinates $(x^1, \ldots, x^n)$ such that $T_pN = \mathrm{span}(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. The connection (**??**) gives a splitting of $T(\mathcal{T}^*)$ into the horizontal and vertical sub-bundles

$$T(\mathcal{T}^*) = H(\mathcal{T}^*) \oplus V(\mathcal{T}^*),$$

as in (1.21). To obtain the explicit form of this splitting, let $V_\alpha,\ \alpha = 0, 1, \ldots, n$ be components of a local section of $\mathcal{T}^*$ in the trivialisation over $\mathcal{W}$. Then

$$\nabla^{\mathcal{T}^*} V_\beta = dV_\beta - \gamma^\alpha_\beta V_\alpha,$$

where $\gamma^\beta_\alpha = \gamma^\beta_{i\alpha} dx^i$, and the components of the co-tractor connection $\gamma^\beta_{i\alpha}$ are given in terms of the connection ∇ on N, and its Schouten tensor, and can be read–off from (1.15):

$$\gamma^0_{i0} = 0, \quad \gamma^j_{i0} = \delta^j_i, \quad \gamma^k_{ij} = \Gamma^k_{ij}, \quad \gamma^0_{ij} = -\mathrm{P}_{ij}.$$

In terms of these components we can write

$$H(\mathcal{T}^*) = \mathrm{span}\Big(\frac{\partial}{\partial x^i} + \gamma^\beta_{i\alpha} V_\beta \frac{\partial}{\partial V_\alpha}, i = 1, \ldots, n\Big),$$
$$V(\mathcal{T}^*) = \mathrm{span}\Big(\frac{\partial}{\partial V_\alpha}, \alpha = 0, 1, \ldots, n\Big).$$

Setting $\zeta_i = V_i/V_0$, where $\tau = V_0 \neq 0$ on the complement of $\mathcal{Z}(\tau)$, we can compute the push forwards of these subspaces to $\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)$:

$$\kappa_* H(\mathcal{T}^*) = \mathrm{span}\Big(h_i \equiv \frac{\partial}{\partial x^i} - (\mathrm{P}_{ij} + \zeta_i\zeta_j - \Gamma^k_{ij}\zeta_k)\frac{\partial}{\partial \zeta_j}\Big), \quad \kappa_* V(\mathcal{T}^*) = \mathrm{span}\Big(v^i \equiv \frac{\partial}{\partial \zeta_i}\Big).$$

The non–zero components of the metric (1.23) are given by

$$g(v^i, h_j) = \delta^i{}_j,$$

or in local coordinates $(x^1, \dots, x^n, \zeta_1, \dots, \zeta_n)$ on $\mathcal{M} \setminus \mathcal{Z}(\tau)$,

$$\begin{aligned} g &= d\zeta_i \odot dx^i - (\Gamma^k_{ij}\zeta_k - \zeta_i\zeta_j - \mathrm{P}_{ij})dx^i \odot dx^j, \qquad (1.24) \\ \Omega &= d\zeta_i \wedge dx^i + \mathrm{P}_{ij} dx^i \wedge dx^j, \qquad i, j = 1, \dots, n. \end{aligned}$$

This is identical to the form appearing in [22].

□

Remark 1.2.3. The expressions (1.24) are projectively invariant in the sense that a different choice of $\nabla \in [\nabla]$ corresponds to shifting the fiber coordinates ζ_i, i.e. metrics corresponding to different representative connections are isometric. Explicitly, a projective transformation (1.1) corresponds to a change

$$\zeta_i \longrightarrow \zeta_i + \Upsilon_i, \qquad (1.25)$$

as can be seen from (1.13) and the definitions of V_i and ζ_i.

Remark 1.2.4. aw: If I add an explicit construction from Thomas, this might be easier to motivate from his perspective.

Next we observe that $\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)$ is an affine bundle modelled on T^*N. Given a connection in the projective class and hence a decomposition (1.11), there is a smooth fibre bundle isomorphism

$$\iota : T^*N \to \mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau). \qquad (1.26)$$

given by

$$T^*_pN \ni \zeta_i \mapsto [(1, \zeta_i)] = [(\tau, \tau\zeta_i)] \in \mathbb{P}(\mathcal{T}^*_p) \setminus \mathcal{Z}(\tau). \qquad (1.27)$$

aw: Make a comment about the model case? Motivated by chapter 3 and maybe 4?

Remark 1.2.5. In fact, the metric and symplectic form (1.24) turn out to belong to a one-parameter family $\{g_\Lambda\}$, which can be written in local coordinates as

$$\begin{aligned} g_\Lambda &= d\zeta_i \odot dx^i - (\Gamma^k_{ij}\zeta_k - \Lambda\zeta_i\zeta_j - \Lambda^{-1}\mathrm{P}_{ij})dx^i \odot dx^j \qquad (1.28) \\ \Omega_\Lambda &= d\zeta_i \wedge dx^i + \frac{1}{\Lambda}\mathrm{P}_{ij} dx^i \wedge dx^j, \qquad i, j = 1, \dots, n. \qquad (1.29) \end{aligned}$$

They are all Einstein with non–zero scalar curvature 24Λ, but for $\Lambda \neq 1$ the relation to projective geometry is lost. For the remainder of the thesis we will write g for $g_{\Lambda=1}$ unless stated otherwise. Note that $\{g_\Lambda\}$ will be the subject of chapter 2, whilst in chapters 3 and 4 we will restrict our attention to g because the projective geometry is a key aspect of the content of these chapters.

1.2.3 Symmetries of $(M, g_\Lambda, \Omega_\Lambda)$

Recall that a projective vector field on any manifold with a connection generates a 1-parameter family of transformations which preserve the geodesics of that connection up to parametrisation. Projective vectors fields thus naturally arise as the symmetries of a projective structure. Explicitly, a vector field $\tilde{K}$ is projective if it satisfies

$$\mathcal{L}_{\tilde{K}}\Gamma^k_{ij} = \delta^k_i \Upsilon_j + \delta^k_j \Upsilon_i \tag{1.30}$$

for some 1-form Υ, where Γ^k_{ij} are the connection components, and their Lie derivative is defined (see [61]) by

$$\mathcal{L}_{\tilde{K}}\Gamma^k_{ij} \equiv \frac{\partial^2 \tilde{K}^k}{\partial x^i \partial x^j} + \tilde{K}^m \frac{\partial \Gamma^k_{ij}}{\partial x^m} - \Gamma^m_{ij} \frac{\partial \tilde{K}^k}{\partial x^m} + \Gamma^k_{im} \frac{\partial \tilde{K}^m}{\partial x^j} + \Gamma^k_{mj} \frac{\partial \tilde{K}^m}{\partial x^i}. \tag{1.31}$$

One consequence of the symmetry property of the Cartan connection discussed in remark 1.1.6 is that for every open set $\mathcal{U} \subset N$ we have an isomorphism between the Lie algebra of projective vector fields on $\mathcal{U}$ and the Lie algebra of vector fields on $\pi_P^{-1}(\mathcal{U})$ preserving the natural contraction $\eta\omega$. Such vector fields must descend to vector fields on $\pi_{\mathcal{M}}^{-1}(\mathcal{U})$ preserving (g, Ω). In fact, it can be shown that every Killing vector field of (M, g_Λ) is also symplectic with respect to Ω_Λ and is therefore the lift of a projective vector field on $(N, [\nabla])$.

Explicitly, for every projective vector field $\tilde{K}$ of $(N, [\nabla])$ there is a corresponding Killing vector K of (M, g_Λ) given in local coordinates by

$$K = \tilde{K} - \zeta_i \frac{\partial \tilde{K}^j}{\partial x^i} \frac{\partial}{\partial \zeta_j} + \frac{1}{\Lambda} \Upsilon_i \frac{\partial}{\partial \zeta_i}, \tag{1.32}$$

where Υ_i is defined by (1.30).

1.2.4 Anti–Self–Duality for $n = 2$

aw: Probs be good to expand this section. Not sure what the right balance is between putting stuff here and putting it in chapter 4. Probs want to motivate that chapter but not say things that the reader will have forgotten by the time they get there.

For $n = 2$, a local of characterisation of the sapces M is provided in [22]: they show that any 4-dimensional anti-self-dual Einstein space with scalar curvature -24Λ and a parallel anti-self-dual totally null distribution can be considered as the total space of a rank 2 affine bundle T^*N over a projective surface N of the form (1.24). The anti–self-duality property, in combination with the correspondence of symmetries of (M, g) with symmetries of $(N, [\nabla])$, is important in the context of the applications of the work [22] to integrability. It means that if we start with a projective surface with at least one projective vector field, we will find an ASD Einstein space with at least one Killing vector field, and thus will be able to perform a symmetry reduction to obtain an Einstein-Weyl structure in $2 + 1$ dimensions and a corresponding solution to the $SU(\infty)$-Toda field equation. This is the subject of chapter 4.

Chapter 2

An Einstein metric on an S^1–bundle over $(M, g_\Lambda, \Omega_\Lambda)$

In this chapter, we show that there is a canonical Einstein of metric on an $\mathbb{R}^*$–bundle over M, with a connection whose curvature is the pull–back of the symplectic structure from M. This metric is interesting in the context of Kaluza-Klein theory. The material covered here is based on material appearing in [23].

aw: I would like to understand this section from the tractor perspective. Try comparing my lifted metric to what can be obtained by slightly modifying the tractor construction of g.

2.1 Flat case

We first note that taking the flat projective structure on $\mathbb{RP}^n$ results in a one–parameter family of $2n$-dimensional Einstein spaces M which are a Kaluza-Klein reductions of quadrics in $\mathbb{R}^{n+1,n+1}$. For $N = \mathbb{RP}^n$ the metric and symplectic form on M reduce to

$$\begin{aligned} g_\Lambda &= dx^i \odot d\zeta_i + \Lambda(\zeta_i dx^i)^2 \\ \Omega_\Lambda &= d\zeta_i \wedge dx^i. \end{aligned}$$

Proposition 2.1.1. *The Einstein spaces M corresponding to $\mathbb{RP}^n$ are projections from the $2n+1$-dimensional quadrics $Q \subset \mathbb{R}^{n+1,n+1}$ given by $X^\alpha Y_\alpha = \frac{1}{\Lambda}$, where $X, Y \in \mathbb{R}^{n+1}$ are coordinates on $\mathbb{R}^{n+1,n+1}$ such that the metric is given by*

$$G = dX^\alpha dY_\alpha,$$

under the embedding

$$\begin{aligned} X^\alpha &= \begin{cases} x^i \mathrm{e}^t, & \alpha = i = 1, \dots, n \\ \mathrm{e}^t, & \alpha = n+1 \end{cases} \\ Y_\alpha &= \begin{cases} \zeta_i \mathrm{e}^{-t}, & \alpha = i = 1, \dots, n \\ \mathrm{e}^{-t}\Big(\frac{1}{\Lambda} - x^k \zeta_k\Big), & \alpha = n+1 \end{cases} \end{aligned} \tag{2.1}$$

following Kaluza-Klein reduction by the vector $\frac{\partial}{\partial t}$.

Proof. We find the basis of coordinate 1-forms $\{dX^\alpha, dY_\alpha\}$ to be

$$\begin{aligned} dX^\alpha &= \begin{cases} \mathrm{e}^t(dx^i + x^i dt), & \alpha = i = 1, \dots, n \\ \mathrm{e}^t dt, & \alpha = n+1 \end{cases} \\ dY_\alpha &= \begin{cases} \mathrm{e}^{-t}(d\zeta_i - \zeta_i dt), & \alpha = i = 1, \dots, n \\ -\mathrm{e}^{-t}\Big[\Big(\frac{1}{\Lambda} - x^k \zeta_k\Big)dt + x^k d\zeta_k + \zeta_k dx^k\Big], & \alpha = n+1. \end{cases} \end{aligned}$$

The metric is then given by

$$\begin{aligned} G &= \mathrm{e}^t(dx^i + x^i dt)\mathrm{e}^{-t}(d\zeta_i - \zeta_i dt) \;-\; \mathrm{e}^t dt \mathrm{e}^{-t}\Big[\Big(\frac{1}{\Lambda} - x^k\zeta_k\Big)dt + x^k d\zeta_k + \zeta_k dx^k\Big] \\ &= dx^i d\zeta_i \;+\; (x^i d\zeta_i - \zeta_i dx^i)dt \;-\; (x^i\zeta_i)dt^2 \;-\; dt\Big[\Big(\frac{1}{\Lambda} - x^k\zeta_k\Big)dt + x^k d\zeta_k + \zeta_k dx^k\Big] \\ &= dx^i d\zeta_i \;-\; \frac{1}{\Lambda}dt^2 \;-\; 2\zeta_i dx^i dt \\ &= dx^i d\zeta_i \;+\; \Lambda(\zeta_i dx^i)^2 \;-\; \Lambda\Big(\frac{dt}{\Lambda} + \zeta_i dx^i\Big)^2, \end{aligned}$$

which is clearly going to give g_Λ under Kaluza-Klein reduction by $\frac{\partial}{\partial t}$.

$\square$

Note that the symplectic form Ω is the exterior derivative of the potential term $\zeta_i dx^i$, implying a possible generalisation to the curved case.

2.2 Curved case

We now return to a general projective structure $(N, [\nabla])$. Since symplectic form picks out the antisymmetric part of the Schouten tensor, it has the fairly simple form

$$\Omega_\Lambda = d\zeta_i \wedge dx^i - \frac{\partial_{[i}\Gamma^k_{j]k}}{\Lambda(n+1)} dx^i \wedge dx^j.$$

By inspection, this can be written $\Omega_\Lambda = d\mathcal{A}$, where

$$\mathcal{A} = \zeta_i dx^i - \frac{\Gamma^k_{ik}}{\Lambda(n+1)} dx^i.$$

This is a trivialisation of the Kaluza-Klein bundle which we are about to construct. Note that for $\Lambda = 1$, under a change of projective connection (1.1) the corresponding change in the fiber coordinates (1.25) ensures that Ω and $\mathcal{A}$ are unchanged.

Motivated by the Kaluza-Klein reduction in the flat case, we consider the following metric.

Theorem 2.2.1. *The metric*

$$\mathcal{G}_\Lambda = g_\Lambda - \Lambda\left(\frac{dt}{\Lambda} + \mathcal{A}\right)^2 \tag{2.2}$$

on an $\mathbb{R}^*$*–bundle* $\kappa_{\mathcal{Q}} : \mathcal{Q} \to M$ *is Einstein, with Ricci scalar* $2n(2n+1)\Lambda$.

Proof. We prove this using the Cartan formalism. Our treatment parallels a calculation by Kobayashi [37], who considered principal circle bundles over Kähler manifolds in order to study the topology of the base. For the remainder of this chapter we will suppress the constant Λ, writing $\mathcal{G} \equiv \mathcal{G}_\Lambda$, $g \equiv g_\Lambda$ and $\Omega \equiv \Omega_\Lambda$ since the proof applies to any choice $\Lambda \neq 0$ within this family.

Consider a frame

$$e^a = \begin{cases} dx^i, & a = i = 1, \dots, n \\ d\zeta_i - (\Gamma^k_{ij}\zeta_k - \Lambda\zeta_i\zeta_j - \Lambda^{-1}\mathrm{P}_{ij})dx^j, & a = i + n = n+1, \dots 2n. \end{cases} \tag{2.3}$$

In this basis the metric takes the form

$$g = e^1 \odot e^{n+1} + \cdots + e^n \odot e^{2n}. \tag{2.4}$$

We are interested in the metric

$$\mathcal{G} = -e^0 \odot e^0 + g,$$

where

$$e^0 = \sqrt{\Lambda}\left(\frac{d\lambda}{\Lambda} + \mathcal{A}\right).$$

aw: Can we have $\Lambda < 0$? I think so. Then what do we mean by $\sqrt{\Lambda}$?

We reserve Roman indices $a, b, \ldots$ for the $2n$-metric components $1, \ldots, 2n$ and allow greek indices $\mu, \nu, \ldots$ to take values $0, 1, \ldots, 2n$. The dual basis to $\{e^\mu\}$ will be denoted $\{E_\mu\}$ and will act on functions as vector fields in the usual way. We wish to find the new connection 1-forms $\hat{\psi}^\mu{}_\nu$ (defined by $de^\mu = -\hat{\psi}^\mu{}_\nu \wedge e^\nu$) in terms of the old ones $\psi^a{}_b$ (defined by $de^a = -\psi^a{}_b \wedge e^b$). Hence we examine de^0 to find $\hat{\psi}^0{}_a$.

$$de^0 = \sqrt{\Lambda} d\mathcal{A} = \sqrt{\Lambda}\Omega_{ab} e^a \wedge e^b = -\hat{\psi}^0{}_a \wedge e^a \qquad \Longrightarrow$$
$$\hat{\psi}^0{}_a = \sqrt{\Lambda}\Omega_{[ab]} e^b = \sqrt{\Lambda}\Omega_{ab} e^b, \qquad \hat{\psi}^a{}_0 = \sqrt{\Lambda}\Omega^a{}_b e^b.$$

Since de^a is unchanged, we have that

$$\hat{\psi}^a{}_0 \wedge e^0 + \hat{\psi}^a{}_b \wedge e^b = \psi^a{}_b \wedge e^b,$$

thus

$$\hat{\psi}^a{}_b \wedge e^b = \psi^a{}_b \wedge e^b - \sqrt{\Lambda}\Omega^a{}_b e^b \wedge e^0 \qquad \Longrightarrow \qquad \hat{\psi}^a{}_b = \psi^a{}_b + \sqrt{\Lambda}\Omega^a{}_b e^0.$$

We now calculate the curvature 2-forms $\hat{\Psi}^\mu{}_\nu = d\hat{\psi}^\mu{}_\nu + \hat{\psi}^\mu{}_\rho \wedge \hat{\psi}^\rho{}_\nu = \frac{1}{2}\mathcal{R}_{\rho\sigma\nu}{}^\mu e^\rho \wedge e^\sigma$ in terms of $\Psi^a{}_b = d\psi^a{}_b + \psi^a{}_c \wedge \psi^c{}_b$, where $\mathcal{R}_{\rho\sigma\nu}{}^\mu$ is the Riemann tensor of $\mathcal{Q}$. Note that we use the notation $\psi^a{}_b = \psi^a{}_{bc} e^c$.

$$\begin{aligned}
\hat{\Psi}^a{}_b &= d\hat{\psi}^a{}_b + \hat{\psi}^a{}_c \wedge \hat{\psi}^c{}_b + \hat{\psi}^a{}_0 \wedge \hat{\psi}^0{}_b \\
&= d\psi^a{}_b + \sqrt{\Lambda} d(\Omega^a{}_b e^0) + \psi^a{}_c \wedge \psi^c{}_b + \sqrt{\Lambda}\Omega^a{}_c e^0 \wedge \psi^c{}_b + \sqrt{\Lambda}\Omega^c{}_b \psi^a{}_c \wedge e^0 + \Lambda \Omega^a{}_{[c}\Omega_{|b|d]} e^c \wedge e^d \\
&= \Psi^a{}_b + \sqrt{\Lambda} E_c(\Omega^a{}_b) e^c \wedge e^0 + \Lambda(\Omega^a{}_b \Omega_{cd} + \Omega^a{}_{[c}\Omega_{|b|d]}) e^c \wedge e^d + \sqrt{\Lambda}(\Omega^c{}_b \psi^a{}_{cd} - \Omega^a{}_c \psi^c{}_{bd}) e^d \wedge e^0 \\
&= \Psi^a{}_b + \sqrt{\Lambda}\nabla_c \Omega^a{}_b e^c \wedge e^0 + \Lambda(\Omega^a{}_b \Omega_{cd} + \Omega^a{}_{[c}\Omega_{|b|d]}) e^c \wedge e^d. \\
\hat{\Psi}^0{}_a &= d\hat{\psi}^0{}_a + \hat{\psi}^0{}_b \wedge \hat{\psi}^b{}_a \\
&= \sqrt{\Lambda} E_{[c}(\Omega_{|a|b]}) \theta^c \wedge \theta^b - \sqrt{\Lambda}\Omega_{ab}\psi^b{}_c \wedge e^c + \sqrt{\Lambda}\Omega_{bc} e^c \wedge (\psi^b{}_a + \sqrt{\Lambda}\Omega^b{}_a e^0) \\
&= \sqrt{\Lambda}(E_{[d}(\Omega_{|a|b]}) - \Omega_{ac}\psi^c{}_{[bd]} + \Omega_{c[d}\psi^c{}_{|a|b]}) e^d \wedge e^b + \Lambda \Omega_{bc}\Omega^b{}_a e^c \wedge e^0 \\
&= \sqrt{\Lambda}\nabla_{[c}\Omega_{|a|d]} e^c \wedge e^d + \Lambda \Omega_{bc}\Omega^b{}_a e^c \wedge e^0.
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\mathcal{R}_{cdb}{}^a &= R_{cdb}{}^a + 2\Lambda(\Omega^a{}_b \Omega_{cd} + \Omega^a{}_{[c}\Omega_{|b|d]}) \\
\mathcal{R}_{c0b}{}^a &= \sqrt{\Lambda}\nabla_c \Omega^a{}_b \\
\mathcal{R}_{cda}{}^0 &= 2\sqrt{\Lambda}\nabla_{[c}\Omega_{|a|d]} \\
\mathcal{R}_{c0a}{}^0 &= \Lambda \Omega_{bc}\Omega^b{}_a,
\end{aligned}$$

and thus, using $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu\nu}{}^\rho$,

$$\begin{aligned}
\mathcal{R}_{00} &= \Lambda \Omega_{bc}\Omega^{bc} = -2n\Lambda = 2n\Lambda \mathcal{G}_{00} \\
\mathcal{R}_{b0} &= \sqrt{\Lambda}\nabla_c \Omega^c{}_b = 0 \\
\mathcal{R}_{db} &= R_{db} + 2\Lambda(\Omega^c{}_b \Omega_{cd} + \frac{1}{2}\Omega^c{}_c \Omega_{bd} - \frac{1}{2}\Omega^c{}_d \Omega_{bc}) - \Lambda \Omega_{cd}\Omega^c{}_b \\
&= R_{db} + 2\Lambda \Omega_b{}^c \Omega_{dc} \\
&= 2(n+1)\Lambda g_{db} - 2\Lambda g_{db} = 2n\Lambda g_{db} = 2n\Lambda \mathcal{G}_{db}.
\end{aligned}$$

Note that we have used the facts that g is Einstein with Ricci scalar $4n(n+1)\Lambda$ and the symplectic form Ω is divergence–free; these are justified in lemmas 2.2.3 and 2.2.2 below. Since $\mathcal{G}_{a0} = 0$, we conclude that

$$\mathcal{R}_{\mu\nu} = 2n\Lambda \mathcal{G}_{\mu\nu} = \frac{\mathcal{R}}{2n+1}\mathcal{G}_{\mu\nu},$$

i.e. $\mathcal{G}$ is Einstein with Ricci scalar $2n(2n+1)\Lambda$.

□

Physically, this is a Kaluza-Klein reduction with constant dilation field and where the Maxwell two-form is related to the reduced metric by $\Omega_a{}^c\Omega_{cb} = g_{ab}$. This is what allows both the reduced and lifted metric to be Einstein. A more general discussion can be found in [48].

From the Cartan perspective, $\mathcal{G}_{\Lambda=1}$ can be thought of as a metric on the $2n+1$-dimensional space obtained by taking a quotient $\tilde{q} : P \mapsto P/SL(n,\mathbb{R}) = \mathcal{Q}$ of the Cartan bundle, where we embed $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ in H as in (1.16) but with a now denoting an element of $SL(n,\mathbb{R})$ (so that $\det a^{-1} = 1$). This new subgroup acts adjointly on θ as

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} -\mathrm{tr}\phi & \eta \\ \omega & \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} -\mathrm{tr}\phi & \eta a^{-1} \\ a\omega & \phi \end{pmatrix},$$

so not only is the inner product $\eta\omega$ invariant but also the $(0,0)$-component $\theta^0{}_0 = -\mathrm{tr}\phi$, which is a scalar one-form whose exterior derivative is constrained by (1.6) to be $d\theta^0{}_0 = -\theta^0{}_i \wedge \theta^i{}_0 = -\mathrm{Ant}(\eta\wedge\omega)$. Thus, denoting by A the object on $\mathcal{Q} = P/SL(n,\mathbb{R})$ which is such that $\tilde{q}^*A = \mathrm{tr}\phi$, we have that $dA = \Omega$ (where we are now taking Ω and g to be defined on $\mathcal{Q}$ by $\tilde{q}^*\Omega = \mathrm{Ant}(\eta\wedge\omega)$ and $\tilde{q}^*g = \mathrm{Sym}(\eta\wedge\omega)$ respectively, or equivalently redefining $\tilde{\Omega} = \kappa_{\mathcal{Q}}^*\Omega$ and $\tilde{g} = \kappa_{\mathcal{Q}}^* g$).

We then have a natural way of constructing a metric $\mathcal{G}$ on $\mathcal{Q}$ as a linear combination of g and $e^0 \odot e^0$, where e^0 is A up to addition of some exact one-form. It turns out that there is choice of linear combination such that $\mathcal{G}$ is Einstein:

$$\mathcal{G} = -e^0 \odot e^0 + g.$$

The fact that this metric is exactly (2.2) can be verified by constructing the Cartan connection of $(N, [\nabla])$ explicitly in terms of a representative connection $\nabla \in [\nabla]$.

aw: Would be nice to verify this explicitly, if I had written out Thomas' explicit construction of g in the intro.

2.2.1 Ricci scalar of g_Λ and divergence of Ω_Λ

Lemma 2.2.2. *The symplectic form $\Omega \equiv \Omega_\Lambda$ on M is divergence–free. In index notation,*

$$\nabla^c\Omega_{cb} = 0.$$

Proof. In the basis (2.3) we have g as above (2.4) and

$$\Omega = \sum_{i=1}^{n} e^i \wedge e^{i+n} \qquad \Longrightarrow \qquad \Omega_{ab} = \sum_{i=1}^{n} \delta^i_{[a}\delta^{i+n}_{b]}.$$

Note that from now on we will omit the summation sign and use the summation convention regardless of whether i, j-indices are up or down. As in section 2.1, we look for ${\psi^a}_b$ by considering de^a (recall that $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, 2n$):

$$\begin{aligned}
de^i &= 0 \\
de^{i+n} &= -(E_l(\Gamma^k_{ij})\zeta_k - \Lambda^{-1}E_l(\mathrm{P}_{ij}))dx^l \wedge dx^j - (\Gamma^k_{ij} - 2\Lambda\zeta_{(i}\delta^k_{j)})d\zeta_k \wedge dx^j \\
&= -(E_l(\Gamma^k_{ij})\zeta_k - \Lambda^{-1}E_l(\mathrm{P}_{ij}))e^l \wedge e^j \\
&\quad -(\Gamma^k_{ij} - 2\Lambda\zeta_{(i}\delta^k_{j)})(e^{k+n} + (\Gamma^l_{km}\zeta_l - \Lambda\zeta_k\zeta_m - \Lambda^{-1}\mathrm{P}_{km})e^m) \wedge e^j \\
&= \Big[\Lambda^{-1}E_m(\mathrm{P}_{ij}) - E_m(\Gamma^k_{ij})\zeta_k + \Lambda^{-1}\Gamma^k_{ij}\mathrm{P}_{km} - \Gamma^k_{ij}\Gamma^l_{km}\zeta_l + \Lambda\Gamma^k_{ij}\zeta_m\zeta_k \\
&\quad +2\Lambda\zeta_{(i}(\Gamma^l_{j)m}\zeta_l - \Lambda\zeta_{j)}\zeta_m - \Lambda^{-1}\mathrm{P}_{j)m})\Big]e^m \wedge e^j + (2\Lambda\zeta_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^{k+n} \wedge e^j \\
&= \Big[\Lambda^{-1}D_m\mathrm{P}_{ij} - (D_m\Gamma^k_{ij})\zeta_k - 2\zeta_{(i}\mathrm{P}_{j)m}\Big]e^m \wedge e^j + (2\Lambda\zeta_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^{k+n} \wedge e^j
\end{aligned}$$

Note that we have used D to denote the chosen connection on N with components Γ^i_{jk}. Next we wish to read off the spin connection ${\psi^a}_b$ such that $de^a = -{\psi^a}_b \wedge e^b$ and the following index symmetries are satisfied:

$$\begin{aligned}
{\psi^i}_j &= \frac{1}{2}\psi_{i+n\,j} = -\frac{1}{2}\psi_{j\,i+n} = -\psi^{j+n}_{\ \ i+n} \\
{\psi^i}_{j+n} &= \frac{1}{2}\psi_{i+n\,j+n} = -\frac{1}{2}\psi_{j+n\,i+n} = -{\psi^j}_{i+n} \\
\psi^{i+n}_{\ \ j} &= \frac{1}{2}\psi_{i\,j} = -\frac{1}{2}\psi_{j\,i} = -\psi^{j+n}_{\ \ i}
\end{aligned}$$

We find that

$$\begin{aligned}
\psi^{i+n}_{\ \ k+n} &= (2\Lambda\zeta_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^j = -{\psi^k}_i \\
\psi^{i+n}_{\ \ j} &= \Big[2(D_{[i}\Gamma^l_{j]k})\zeta_l - 2\Lambda^{-1}D_{[i}\mathrm{P}^{\mathrm{S}}_{j]k} - \Lambda^{-1}D_k\mathrm{P}^{\mathrm{A}}_{ij} + 2\zeta_{(j}\mathrm{P}_{k)i} - 2\zeta_{(i}\mathrm{P}_{k)j}\Big]e^k =: A_{ijk}e^k \\
{\psi^i}_{j+n} &= 0.
\end{aligned}$$

One can check that these satisfy both the index symmetries above and are such that $de^a = -{\psi^a}_b \wedge e^b$, and we know from theory that there is a unique set of ${\psi^a}_b$ that have both of these properties. Note that we have used P^{S} and P^{A} to denote the symmetric and antisymmetric parts of P in order to avoid too much confusion from having multiple symmetrisation brackets in the indices.

We are now ready to calculate the divergence of Ω. Since it is covariantly constant in this basis, we obtain

$$\nabla_c \Omega_{ab} = -\psi^d{}_{ac}\Omega_{db} - \psi^d{}_{bc}\Omega_{ad} = -\psi^d{}_{ac}\Omega_{db} + \psi^d{}_{bc}\Omega_{da} = 2\Omega_{d[a}\psi^d{}_{b]c}.$$

We can split the right hand side into

$$\begin{aligned}
\Omega_{da}\psi^d{}_{bc} &= \Omega_{ka}\psi^k{}_{bc} + \Omega_{k+n\,a}\psi^{k+n}_{bc} \\
&= \delta^i_{[k}\delta^{i+n}_{a]}\psi^k{}_{bc} + \delta^i_{[k+n}\delta^{i+n}_{a]}\psi^{k+n}_{bc} \\
&= \frac{1}{2}\Big(-\delta^{k+n}_a\delta^i_b\delta^j_c(2\Lambda\zeta_{(i}\delta^k_{j)} - \Gamma^k_{ij}) - \delta^k_a\delta^{l+n}_b\delta^j_c(2\Lambda\zeta_{(k}\delta^l_{j)} - \Gamma^l_{kj}) - \delta^k_a\delta^l_b\delta^m_c A_{klm}\Big).
\end{aligned}$$

The first two terms are the same but with $a \leftrightarrow b$, so are lost in the antisymmetrisation. Thus

$$\nabla_c \Omega_{ab} = -\delta^k_{[a}\delta^l_{b]}\delta^m_c A_{klm}.$$

Tracing amounts to contracting this with g^{ac}:

$$\nabla^c \Omega_{cb} = -\delta^k_{[a}\delta^l_{b]}g^{ac}\delta^m_c A_{klm} = -\delta^k_{[a}\delta^l_{b]}g^{am}A_{klm},$$

but g^{am} is non-zero only when $a = m + n > n$ and $\delta^k_{[a}\delta^l_{b]}$ is non-zero only when $a = k \leq n$ or $a = l \leq n$. We can therefore conclude that the right hand side is zero and Ω is divergence-free.

□

Lemma 2.2.3. *The metric $g \equiv g_\Lambda$ corresponding to a projective structure $(N, [\nabla])$ in dimension n has Ricci scalar*

$$R = 4n(n + 1)\Lambda.$$

Proof. We calculate the Ricci scalar of g (given that it's Einstein, as stated in the appendix of [22]) via the curvature two-forms $\Psi^a{}_b = d\psi^a{}_b + \psi^a{}_c \wedge \psi^c{}_b = \frac{1}{2}R_{cdb}{}^a e^c \wedge e^d$. We are only interested in non-zero components of the Ricci tensor such as $R_{i\,j+n} = R_{ci\,j+n}{}^c$. In fact, we will calculate only $R_{m+n\,j}$, for which we need to consider $R_{l\,m+n\,j}{}^i$ and $R_{k+n\,m+n\,j}{}^{l+n}$, i.e. we need only calculate $\Psi^i{}_j$ and $\Psi^{l+n}{}_j$.

$$\Psi^i{}_j = d\Big((\Gamma^i_{jk} - 2\Lambda\zeta_{(j}\delta^i_{k)})e^k\Big) + \psi^i{}_k \wedge \psi^k{}_j + \psi^i{}_{k+n} \wedge \psi^{k+n}_j.$$

The last term vanishes since $\psi^{k+n}_{\ \ j} = 0$, and the middle term only has components that look like $\frac{1}{2}R_{lmj}{}^{i}e^l \wedge e^m$, so the only term we are interested in is

$$-2\Lambda d\zeta_{(j}\delta^i_{k)}e^k = -2\Lambda\delta^i_{(k}(e^{j)+n} + (\Gamma^l_{j)m}\zeta_l - \Lambda\zeta_{j)}\zeta_m - \Lambda^{-1}\mathrm{P}_{j)m})e^m) \wedge e^k.$$

Again, discarding the $e^m \wedge e^k$ term gives

$$-\Lambda(e^{j+n} \wedge e^k + \delta^i_j e^{k+n} \wedge e^k) = \frac{1}{2}R_{l\,m+n\,j}{}^{i}e^l \wedge e^{m+n} + \frac{1}{2}R_{m+n\,l\,j}{}^{i}e^{m+n} \wedge e^l,$$

so we conclude

$$R_{l\,m+n\,j}{}^{i} = \Lambda(\delta^i_j\delta^m_l + \delta^i_l\delta^m_j).$$

The other Riemann tensor component we need to know to calculate $R_{m+n\,j} = R_{c\,m+n\,j}{}^{c}$ is $R_{l+n\,m+n\,j}{}^{i+n}$, so we examine

$$\Psi^{i+n}_{\ \ \ j} = d\psi^{i+n}_{\ \ \ j} + \psi^{i+n}_{\ \ \ k} \wedge \psi^{k}_{\ j} + \psi^{i+n}_{\ \ \ k+n} \wedge \psi^{k+n}_{\ \ \ j},$$

but none of these terms have $e^{l+n} \wedge e^{m+n}$ components, so $R_{l+n\,m+n\,j}{}^{i+n} = 0$. Hence

$$R_{m+n\,j} = \delta^l_i R_{l\,m+n\,j}{}^{i} = \Lambda(\delta^m_j + n\delta^m_j) = \Lambda(n+1)\delta^m_j.$$

Setting this equal to $\frac{R}{2n}g_{m+n\,j} = \frac{R}{4n}\delta^m_j$ we find

$$R = 4n(n+1)\Lambda,$$

as required.

$\square$

Chapter 3

Para–c–projective compactification of (M, g, Ω)

aw: Remaining non–trivial notes:

- How to justify the "well–known fact" that there is a para–CR structure on the projectivised cotangent bundle.
- Relating to the note about proving the main theorem using tractor methods: what does the Cartan/tractor bundle of a para–c–projective structure look like? What is it in this case? And put in some background about orbit decompositions from [67] and how they relate to compactifications.
- On the last section about the model - I think maybe some of it needs to go in the introduction, and maybe some of it chucked depending on the outcome of the item above.
- There is a 2 paragraph proof in [66] that the Nijenhuis condition is equivalent to Hermiticity of $d\theta$ on ∂M which I could adapt to the para case and put in section 3.1.4.
- It would be nice also to check that my explicit calculations on J agree with the convention for how g and Ω are related and there is a similar uncertainty about h_H and $d\theta_0$.

aw: In fact in the theorem we are not talking about h and $d\theta$ restricted to H, we are talking about h and $d\theta$ on ∂M, so maybe what I showed below is not the full condition?

In [66] the concept of c–projective compactification was defined. It is based on almost c–projective geometry [69], an analogue of projective geometry defined for almost complex manifolds, i.e., even–dimensional manifolds M carrying a smooth endomorphism J of TM which satisfies $J^2 = -Id$. In c–projective geometry, the equivalence class of torsion–free connections is replaced by an equivalence class of connections which are adapted to the almost complex structure J in a natural way. In this chapter we discuss a notion of compactification which is modified to the *"para"* case, i.e. where the endomorphism J squares to Id rather than $-Id$. We show that the natural almost para–complex structure J on any manifold M arising in the projective to Einstein correspondence admits a type of compactification which we call *para–c–projective*. The content of this chapter is based on material appearing in [24].

3.1 Background and definitions

3.1.1 Almost (para–)complex geometry

The purpose of this section is to introduce the definitions which are required to state the main results of [66].

Definition 3.1.1. *The Nijenhuis tensor of an endomorphism J of TM is defined by*

$$\mathcal{N}(X, Y) := [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]), \tag{3.1}$$

where X, Y are vector fields on M and $[\cdot\,,\cdot]$ denotes the Lie bracket of vector fields. This is equivalent to

$$\mathcal{N}^a_{bc} = J^d{}_{[b}\partial_{|d|}J^a{}_{c]} - J^d{}_{[b}\partial_{c]}J^a{}_d. \tag{3.2}$$

Let M be a complex manifold of (complex) dimension n, in the sense of having complex coordinates and complex transition functions. Then multiplication of the coordinates by i defines an endomorphism J of TM which squares to $-Id$, so complex manifolds are a subset of almost complex manifolds. In this case, J has eigenvalues $\pm i$, and the corresponding splitting of TM into eigen–bundles is Frobenius integrable, meaning that the Lie bracket of any two sections of either eigen–bundle is also a section of that eigen–bundle, or equivalently that each of the two sub–bundles is tangent to a foliation of sub–manifolds of real dimension n at every point. The Newlander–Nirenberg theorem describes complex manifolds in terms of the Nijenhuis tensor (3.1).

Theorem 3.1.2 ([66])**.** *An almost complex manifold (M, J) is a complex manifold if and only if the Nijenhuis tensor of J vanishes. In this case, we call the almost complex structure J integrable.*

An endomorphism J which squares to Id defines an analogous splitting of the tangent bundle into sub–bundles with eigenvalues ± 1, and this splitting is also Frobenius integrable if and only if the Nijenhuis tensor of J vanishes. We thus call an almost para–complex structure J with vanishing Nijenhuis tensor a para–complex structure, and say that in this case J is integrable. In all the definitions below, the word *almost* can be removed if the (para–)complex structure J is integrable.

Definition 3.1.3. *A (para–)Hermitian metric on an almost (para–)complex manifold (M, J) is a metric g satisfying*

$$g(J\cdot\,, J\cdot) = \pm g(\cdot\,, \cdot),$$

where the minus sign corresponds to the "para" case. The triple (M, J, g) then defines an almost (para–)Hermitian manifold.

Note that every (para–)Hermitian manifold has a naturally defined two–form $\Omega(\cdot\,, \cdot) = g(\cdot\,, J\cdot)$ which is (para–)Hermitian in the sense that

$$\Omega(J\cdot\,, J\cdot) = \pm\Omega(\cdot\,, \cdot),$$

and can alternatively be specified as (M, J, Ω) or (M, g, Ω). An almost (para–)Kähler manifold (M, J, g) is a (para–)Hermitian manifold whose associated two–form is closed, meaning M carries compatible complex, pseudo–Riemannian and symplectic structures. The manifolds M arising in the projective to Einstein correspondence are almost para–Kähler, and para–Kähler when the underlying projective structure is flat [22].

3.1.2 Almost (para–)CR structures and contact distributions

aw: I think I should probably specialise to the para case at this point.

Definition 3.1.4. *An almost (para–)CR structure $(\mathcal{Z}, H, J)$ on a manifold $\mathcal{Z}$ is a sub–bundle $H \subset T\mathcal{Z}$ of the tangent bundle together with a fiber–preserving endomorphism $J : H \to H$ which satisfies $J^2 = Id$ or $J^2 = -Id$ depending on whether or not we are talking about the "para" case.*

We will be interested in the case where H is a hyperplane distribution on $\mathcal{Z}$; then $(\mathcal{Z}, H, J)$ is called an almost (para–)CR structure of *hypersurface type*. An almost (para–)complex structure (M, J) of dimension $2n$ defines an almost (para–)CR structure of hypersurface type on any hypersurface $\mathcal{Z} \subset M$ given by the restriction of J to the hyperplane distribution $H := T\mathcal{Z} \cap J(T\mathcal{Z})$ on $\mathcal{Z}$. Note that this distribution must have dimension $2n-2$. An almost (para–)CR structure is a (para–)CR structure if and only if the splitting of H into eigen–bundles induced by J is Frobenius integrable.

We can define the notion of non–degeneracy for an almost (para–)CR structure as follows. The Lie bracket of vector fields induces an antisymmetric $\mathbb{R}$–bilinear operator $\Gamma(H) \times \Gamma(H) \to \Gamma(T\mathcal{Z}/H)$ which in fact is also bilinear over smooth functions on $\mathcal{Z}$. This means it is induced by a bundle map $\mathcal{L} : H \times H \to T\mathcal{Z}/H$ which is called the *Levi bracket*. Since it takes values in a line bundle it can be thought of as an antisymmetric bilinear form called the *Levi form*. Degeneracy (or not) of the almost (para–)CR structure is defined as degeneracy (or not) of the Levi form. Note that the Levi form also defines a *symmetric* bilinear form $h(\cdot\,,\cdot) = \mathcal{L}(\cdot\,, J\cdot)$ as long as $\mathcal{L}$ is (para–)Hermitian with respect to J, and that this symmetric bilinear form is non–degenerate if and only if $\mathcal{L}$ is.

Definition 3.1.5. *A contact structure on a manifold $\mathcal{Z}$ of dimension $2n-1$ is a hyperplane distribution $H \subset T\mathcal{Z}$ specified as the kernel of a one–form θ on $\mathcal{Z}$ which satisfies the complete non–integrability condition*

$$\theta \wedge \underbrace{(d\theta \wedge \cdots \wedge d\theta)}_{n-1 \text{ times}} \neq 0. \tag{3.3}$$

aw: I also use θ for the Cartan connection.

The complete non–integrability condition can be thought of as the opposite of Frobenius integrability of the hyperplane distribution, see for example [6].

3.1.3 Connections and (para–)c–projective equivalence

Definition 3.1.6. *A connection on an almost (para–)complex manifold (M, J) is called complex if it preserves J.*

Note that, in contrast to a metric connection, it is not always possible to define a complex connection which is torsion–free. In fact, this is possible if and only if the Nijenhuis tensor (3.1) of J vanishes. However, one can always define a complex connection whose torsion is equal to the Nijenhuis tensor of J up to a constant multiplicative factor [69]. Such connections are called *minimal*.

Definition 3.1.7. *Two affine connections ∇ and $\overline{\nabla}$ on an almost (para–)complex manifold (M, J) are called (para–)c–projectively equivalent if there is a one–form Υ_a on M such that their components Γ^a_{bc} and $\overline{\Gamma}^a_{bc}$ are related by*

$$\overline{\Gamma}^a_{bc} - \Gamma^a_{bc} = \delta^a_b \Upsilon_c + \delta^a_c \Upsilon_b \pm (\Upsilon_d J^d{}_b J^a{}_c + \Upsilon_d J^d{}_c J^a{}_b), \tag{3.4}$$

where the $+$ corresponds to the case $J^2 = Id$ and the $-$ corresponds to the case $J^2 = -Id$.

Note that the para–c–projective change of connection differs from the c–projective case in the signs of some of the terms, to account for the fact that J squares to the Id rather than $-Id$. It is easy to show that if ∇ is complex then so is $\overline{\nabla}$, and the index symmetry of the right hand side of (3.4) means that if ∇ is minimal then so is $\overline{\nabla}$. An almost (para–)c–projective structure on a manifold M comprises an almost (para–)complex structure J and a (para–)c–projective equivalence class $[\nabla]$ of complex minimal connections.

3.1.4 Para–c–projective compactification

We now specialise to the "para" case, where $J^2 = Id$. Note that all the corresponding results for $J^2 = -Id$ can be found in the original paper [66].

Definition 3.1.8. *Let (M, J) be an almost para–complex manifold, and let ∇ be a complex minimal connection. The structure (M, J) admits a para–c–projective compactification to a manifold with boundary $\overline{M} = M \cup \partial M$ if there exists a function $T : \overline{M} \to \mathbb{R}$ such that $\mathcal{Z}(T)$ is the boundary $\partial M \subset \overline{M}$, the differential dT does not vanish on ∂M, and the connection $\overline{\nabla}$ related to ∇ by (3.4) with $\Upsilon = dT/(2T)$ extends to $\overline{M}$.*

It follows easily from this definition that the endomorphism J on M naturally extends to all of $\overline{M}$ by parallel transport with respect to $\overline{\nabla}$. It thus defines an almost para–CR structure on the hyperplane distribution H defined by $H_x := T_x\partial M \cap J(T_x \partial M)$ for all $x \in \partial M$. It can be shown (see lemma 5 of [66] and modify to the case $J^2 = Id$) that this almost para–CR structure is non–degenerate if and only if for any local defining function T the one–form $\theta = dT \circ J$, whose restriction to ∂M has kernel H, satisfies the complete non–integrability condition (3.3) making H a contact distribution on ∂M.

To see this, first note that $\theta(X) = 0 \;\forall\; X \in \Gamma(H)$ implies $d\theta(\cdot\,,\cdot) = -\theta([\cdot\,,\cdot])$, so the restriction of $d\theta$ to $H \times H$ represents the Levi form $\mathcal{L}$. This means that the almost para–CR structure on ∂M is non–degenerate if and only if the restriction of $d\theta(X, \cdot)$ to H is non–zero for all non–zero $X \in \Gamma(H)$. But this is equivalent to the non–integrability condition (3.3).

Another result of lemma 5 of [66] is that $d\theta$ is Hermitian on ∂M if and only if the Nijenhuis tensor (3.1) of J takes so–called *asymptotically tangential values*. This is equivalent to the following statement in index notation:

$$\left(\mathcal{N}^a{}_{bc}\nabla_a T\right)\Big|_{T=0} = 0. \tag{3.5}$$

Note in particular that Hermiticity of $d\theta$ on ∂M implies Hermiticity of $d\theta$ on H, and hence the existence of a non–degenerate metric $h(\cdot\,,\cdot) = d\theta(\cdot\,, J\cdot)|_H$ on H. Both of these facts also apply in the "para" case.

aw: Could check this and maybe put the proof in. It is only a couple of paragraphs.

Although c–projective compactification is defined for any almost complex manifold, the definition can be applied to pseudo–Riemannian metrics g which are Hermitian with respect to the almost complex structure so long as there exists a connection which preserves both g and J and has minimal torsion. Such Hermitian metrics are said to be *admissible*. Note that such a connection, if it exists, is uniquely defined, since the conditions that it be complex and minimal determine its torsion. It is thus given by the Levi–Civita connection of g plus a constant multiple of the Nijenhuis tensor (3.1) of J.

The first main result of [66] is Theorem 8 in this reference, which gives a local form for an admissible Hermitian metric which is sufficient for the corresponding c–projective structure to be c–projectively compact. The theorem is stated below, adapted to the para–c–projective case. The proof can be obtained by a trivial adaptation of the arguments in [66], and so further details may be obtained from that source.

Theorem 3.1.9 ([66])**.** *Let $\overline{M}$ be a smooth manifold with boundary ∂M and interior M. Let J be an almost para–complex structure on $\overline{M}$, such that ∂M is non–degenerate and the Nijenhuis tensor $\mathcal{N}$ of J has asymptotically tangential values. Let g be an admissible pseudo–Riemannian Hermitian metric on M. For a local defining function T for the boundary defined on an open subset $\mathcal{U} \subset \overline{M}$, put $\theta = dT \circ J$ and, given a non–zero real constant C, define a Hermitian tensor field $h_{T,C}$ on $\mathcal{U} \cap M$ by*

$$h_{T,C} := Tg + \frac{C}{T}(dT^2 - \theta^2).$$

Suppose that for each $x \in \partial M$ there is an open neighbourhood $\mathcal{U}$ of x in $\overline{M}$, a local defining function T defined on $\mathcal{U}$, and a non–zero constant C such that

- *$h_{T,C}$ admits a smooth extension to all of $\mathcal{U}$*
- *for all vector fields X, Y on U with $dT(Y) = \theta(Y) = 0$, the function $h_{T,C}(X, JY)$ approaches $Cd\theta(X,Y)$ at the boundary.*

Then g is c–projectively compact.

Note that the statement in Theorem 3.1.9 does not depend on the choice of T. Different choices of T result in rescalings of the contact form θ on the boundary by a nowhere vanishing function.

3.2 Compactifying the Dunajski–Mettler Class

In section 1.2.2 it was shown that the manifolds M arising in the projective to Einstein correspondence can be identified with the projectivised cotractor bundle of N where the zero locus of the canonical density τ has been removed. Note that by construction in section 1.2.2 it is easily verified that the zero locus of τ is a smoothly embedded hypersurface in $\mathcal{M}$, and from (1.27) it follows at once that this may be identified with the total space of the fibrewise projectivisation $\mathbb{P}(T^*N)$ (which is well known to have an almost para–CR structure).

aw: Is there a way to justify this "well known" fact?

In the special case where $N = \mathbb{RP}^n$, and $[\nabla]$ is projectively flat the manifold $M = SL(n+1,\mathbb{R})/GL(n,\mathbb{R})$ can be identified with the projection of $\mathbb{R}^{n+1} \times \mathbb{R}_{n+1} \setminus \mathcal{Z}$, where $\mathcal{Z}$ denotes the set of incident pairs (point, hyperplane).

aw: Do I need to explain why this is? possibly in the intro?

The compactification procedure described in the theorem 3.2.1 below will, for the model, attach these incident pairs back to M, and more generally (in case of a curved projective structure $(N, [\nabla])$) will attach the zero locus of τ back into $\mathbb{P}(\mathcal{T}^*)$. The boundary $\partial M \cong \mathcal{Z}(\tau)$ from definition 3.1.8 will play a role of a submanifold separating two open sets in $\mathbb{P}(\mathcal{T}^*)$ which have $\tau > 0$ and $\tau < 0$ respectively. The method of the proof will be to show that near the boundary $\mathcal{Z}(\tau) = 0$ of $\overline{M}$ the metric (1.24) can be put in the local normal form of theorem 3.1.9.

Theorem 3.2.1. *The Einstein almost para–Kähler metric (M, g, Ω) given by (1.24) admits a para–c–projective compactification $\overline{M}$. The structure on the $(2n-1)$–dimensional* *boundary $\partial M \cong \mathbb{P}(T^*N)$ of $\overline{M}$ includes a contact structure together with a conformal structure and a para–CR structure defined on the contact distribution.*

aw: Here we stated that $\partial M \cong \mathbb{P}(T^*N)$ but unless I can justify why this carries a para–CR structure it might be better to leave this out or put $\partial M \cong \mathcal{Z}(\tau)$.

Proof. In the proof below we shall explicitly construct the boundary ∂M together with the contact structure and the associated conformal structure on the contact distribution. We shall first deal with the model $M = SL(n+1)/GL(n)$, and then explain how the curvature of $(N, [\nabla])$ modifies the compactification.

In the model case we can define coordinates x^i on $N = \mathbb{RP}^n$ by taking $X = (1, x^1, \dots, x^n)$, where $(X^0, \dots, X^n)$ are homogeneous coordinates and we are working in an open set where $X^0 \neq 0$. The x^i are flat coordinates, so the connection components (and hence the Schouten tensor) vanish and (1.24) reduces to

$$g = d\zeta_i \odot dx^i + \zeta_i \zeta_j dx^i \odot dx^j, \qquad \Omega = d\zeta_i \wedge dx^i \quad \text{where} \quad i, j = 1, \dots, n. \tag{3.6}$$

As noted in section 1.2.2, we can use (1.27) relate the affine coordinates ζ_i on the fibres of T^*N to the tractor coordinates (1.12) by setting $\zeta_i = \mu_i/\tau$ on the complement of the zero locus $\mathcal{Z}(\tau)$ of τ.

Now consider an open set $\mathcal{U} \subset M$ given by $\zeta_i x^i > 0$, and define the function T on $\mathcal{U}$ by

$$T = \frac{1}{\zeta_i x^i}. \tag{3.7}$$

We shall attach a boundary $\partial\mathcal{U}$ to the open set $\mathcal{U}$ such that T extends to a function $\overline{T}$ on $\mathcal{U} \cup \partial\mathcal{U}$, and $\overline{T}$ is the defining function for this boundary. We then investigate the geometry on M in the limit $T \to 0$. It is clear from above that the zero locus of $\overline{T}$ will be contained in the zero locus $\mathcal{Z}(\tau)$ of τ, and therefore belongs to the boundary of $\overline{M}$. We will use $\overline{T}$ as a defining function for $\overline{M}$ in an open set $\overline{\mathcal{U}} \subset \overline{M}$. The strategy of the proof is to extend T to a coordinate system on $\mathcal{U}$, such that near the boundary the metric g takes a form as in Theorem 3.1.9.

First define $\theta \in \Lambda^1(\overline{M})$ $\overline{M}$ by

$$V \lrcorner\, \theta = J(V) \lrcorner\, dT, \quad \text{or equivalently} \quad \theta_a = \Omega_{ac} g^{bc} \nabla^{\mathbf{g}}{}_b T, \quad a, b, c = 1, \dots, 2n \tag{3.8}$$

where J is the para–complex structure of (g, Ω). Using (3.6) this gives

$$\theta = 2T(1-T)\zeta_i dx^i - dT.$$

We need n open sets $U_1, \dots, U_n$ such that $\zeta_k \neq 0$ on U_k to cover the zero locus of T. Here we chose $k = n$, and use a coordinate system given by

$$(T, Z_1, \dots, Z_{n-1}, X^1, \dots, X^{n-1}, Y),$$

where T is given by (3.7) and

$$Z_A = \frac{\zeta_A}{\zeta_n}, \quad X^A = x^A, \quad Y = x^n, \quad \text{where} \quad A = 1, \dots, n-1.$$

We compute

$$\theta = 2(1-T)\frac{dY + Z_A dX^A}{K} - dT, \quad \zeta_n = \frac{1}{KT}, \quad \text{where} \quad K \equiv Y + Z_A X^A,$$

and substitute

$$\zeta_i dx^i = \frac{1}{KT}(dY + Z_A dX^A)$$

into (3.6). This gives

$$g = \frac{\theta^2 - dT^2}{4T^2} + \frac{1}{T}h, \tag{3.9}$$

where

$$h = \frac{1}{4(1-T)}(\theta^2 - dT^2) + \frac{1}{K}\Big(dZ_A \odot dX^A - \frac{1}{2(1-T)}X^A dZ_A \odot (\theta + dT)\Big)$$

is regular at the boundary $T = 0$. This is in agreement with the asymptotic form in Theorem 3.1.9 (see [66] for further details).

The restriction h to ∂M gives a metric on a distribution $H = \text{Ker}(\theta|_{T=0})$

$$\theta|_{T=0} = 2\frac{dY + Z_A dX^A}{Y + Z_A X^A},$$

$$h_0 = \frac{1}{4}(\theta|_{T=0})^2 + \frac{1}{2(Y + Z_A X^A)}(2dZ_A \odot dX^A - X^A dZ_A \odot (\theta|_{T=0})). \tag{3.10}$$

Note that T is only defined up to multiplication by a positive function. Changing the defining function in this way results in a conformal rescalling of $\theta|_{T=0}$, thus the metric on the contact distribution is also defined up to an overall conformal scale. We shall choose the scale so that the contact form is given by $\theta_0 \equiv K\theta|_{T=0}$ on $T(\partial M)$, with the metric on H given by

$$h_H = dZ_A \odot dX^A. \tag{3.11}$$

We now move on to deal with the curved case where the metric on M is given by (1.24). The coordinate system (T, Z_A, X^A, Y) is as above, and the one–form θ in (3.8) is given by

$$\theta = 2T(1-T)\zeta_i dx^i - dT + 2T^2(\mathrm{P}_{ij} - \Gamma^k_{ij}\zeta_k)x^i dx^j,$$

or in the (T, Z_A, X^A, Y) coordinates,

$$\begin{aligned}\theta = \ & 2(1-T)\frac{Z_A dX^A + dY}{K} - dT \\ & + 2T^2\Bigg[\left(\mathrm{P}_{AB} - \frac{\Gamma^C_{AB}Z_C + \Gamma^n_{AB}}{TK}\right)X^A dX^B + \left(\mathrm{P}_{nB} - \frac{\Gamma^C_{nB}Z_C + \Gamma^n_{nB}}{TK}\right)Y dX^B \\ & + \left(\mathrm{P}_{An} - \frac{\Gamma^C_{An}Z_C + \Gamma^n_{An}}{TK}\right)X^A dY + \left(\mathrm{P}_{nn} - \frac{\Gamma^C_{nn}Z_C + \Gamma^n_{nn}}{TK}\right)Y dY\Bigg].\end{aligned}$$

Guided by the formula (3.9) we define

$$h = Tg - \frac{1}{4T}(\theta^2 - dT^2),$$

which we find to be

$$\begin{aligned} h = & \frac{1}{4(1-T)}(\theta^2 - dT^2) + \frac{1}{K}\left(dZ_A \odot dX^A - \frac{1}{2(1-T)}X^A dZ_A \odot (\theta + dT)\right) \\ & - \frac{1}{K}\Big((\Gamma^C_{AB}Z_C + \Gamma^n_{AB})dX^A \odot dX^B + (\Gamma^C_{nn}Z_C + \Gamma^n_{nn})dY \odot dY \\ & \qquad + 2(\Gamma^C_{An}Z_C + \Gamma^n_{An})dX^A \odot dY\Big) \\ & + T(\mathrm{P}_{AB}dX^A \odot dX^B + 2\mathrm{P}_{An}dX^A \odot dY + \mathrm{P}_{nn}dY \odot dY).\end{aligned}$$

This is smooth as $T \to 0$.

Restricting h to $T = 0$ yields a metric which differs from (3.10) by the curved contribution given by the components of the connection, but not the Schouten tensor. Substituting $dY = K\theta|_{T=0}/2 - Z_A dX^A$, disregarding the terms involving $\theta|_{T=0}$ in h, and conformally rescalling by K yields the metric

$$\begin{aligned} h_H &= (dZ_A - \Theta_{AB}dX^B) \odot dX^A, \quad \text{where} \qquad (3.12) \\ \Theta_{AB} &= \Gamma^C_{AB}Z_C + \Gamma^n_{AB} + (\Gamma^C_{nn}Z_C + \Gamma^n_{nn})Z_A Z_B - 2(\Gamma^C_{An}Z_C + \Gamma^n_{An})Z_B \end{aligned}$$

defined on the contact distribution $H = \mathrm{Ker}(\theta_0)$, where $\theta_0 = 2(dY + Z_A dX^A)$.

We now invoke theorem 3.1.9, verifying by explicit computation that the remaining two conditions are satisfied. The first of these conditions is that the metric h_H is compatible with the Levi–form of the almost para–CR structure induced on ∂M by J, i.e.

$$h_H(X,Y) = C d\theta_0(JX,Y), \quad \text{for} \quad X \in H. \tag{3.13}$$

aw: I think this is slightly different but I guess equivalent to the form given in the theorem. Might need a comment?

aw: Okay I'm not sure if C should be negative, i.e. if the statement in the theorem is the better one because $d\theta$ is like $-\mathcal{L}$ but the statement in the theorem is the same as in [66] i.e. in the case $J^2 = -Id$. Would be good to check this.

aw: In fact in the theorem we are not talking about h and $d\theta$ restricted to H, we are talking about h and $d\theta$ on ∂M, so maybe what I showed here is not the full condition?

The second is that the Nijenhuis tensor takes asymptotically tangential values, i.e. that (3.5) is satisfied.

Both of these can be checked by computing the almost para–complex structure J in the (T, Z_A, X^A, Y) coordinates. We find

$$\begin{aligned} J|_{T=0} = & -\frac{\partial}{\partial X^A} \otimes dX^A + \frac{\partial}{\partial Y} \otimes dY + \frac{\partial}{\partial Z_A} \otimes dZ_A + \frac{\partial}{\partial T} \otimes dT \\ & - \frac{Z_B}{K} \frac{\partial}{\partial T} \otimes dX^B - \frac{1}{K} \frac{\partial}{\partial T} \otimes dY \\ & - \left(\Gamma^D_{AB} Z_D + \Gamma^n_{AB}\right) \frac{\partial}{\partial Z_A} \otimes dX^B + \left(\Gamma^D_{nB} Z_D + \Gamma^n_{nB}\right) Z_C \frac{\partial}{\partial Z_C} \otimes dX^B \\ & - \left(\Gamma^D_{An} Z_D + \Gamma^n_{An}\right) \frac{\partial}{\partial Z_A} \otimes dY + \left(\Gamma^D_{nn} Z_D + \Gamma^n_{nn}\right) Z_C \frac{\partial}{\partial Z_C} \otimes dY. \end{aligned} \tag{3.14}$$

Restricting to vectors in H amounts to substituting $dY = \theta_0/2 - Z_A dX^A$ and disregarding the terms involving θ_0 as above, so that

$$\begin{aligned} J|_H = & -\frac{\partial}{\partial X^A} \otimes dX^A + Z_A \frac{\partial}{\partial Y} \otimes dX^A + \frac{\partial}{\partial Z_A} \otimes dZ_A + \frac{\partial}{\partial T} \otimes dT \\ & - \frac{2Z_B}{K} \frac{\partial}{\partial T} \otimes dX^B - \Theta_{AB} \frac{\partial}{\partial Z_A} \otimes dX^B \end{aligned}$$

and (3.13) is satisfied.

For the Nijenhuis condition, we use the formula (3.2). Note that we need only consider components of this with $a = T$, and thus only need to work with the $\partial/\partial T$ components of J to find the terms which look like ∂J. This is a one–form which we shall call $J^{(T)}$ and find to be

$$\begin{aligned} J^{(T)} =& \Bigg(- \frac{Z_B}{K} + \frac{T[2Z_B + (\Gamma^D_{AB} Z_D + \Gamma^n_{AB}) X^A + (\Gamma^D_{nB} Z_D + \Gamma^n_{nB}) Y]}{K} \\ & - T^2 [\mathrm{P}_{AB} X^A + \mathrm{P}_{nB} Y] \Bigg) dX^B \\ & + \Bigg(- \frac{1}{K} + \frac{T[2 + (\Gamma^D_{An} Z_D + \Gamma^n_{An}) X^A + (\Gamma^D_{nn} Z_D + \Gamma^n_{nn}) Y]}{K} \\ & - T^2 [\mathrm{P}_{An} X^A + \mathrm{P}_{nn} Y] \Bigg) dY. \end{aligned}$$

Note that this agrees with (3.14) when $T = 0$. We use it to calculate $\mathcal{N}^a{}_{bc} \nabla_a T$, dropping terms which vanish when $T = 0$ to verify (3.5).

□

3.2.1 Two–dimensional projective structures

In the case if $n = 2$ the coordinates on ∂M are (X, Y, Z), and (3.12) yields

$$h_H = dZ \odot dX - [\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12}) Z + (\Gamma^2_{22} - 2\Gamma^2_{12}) Z^2 + \Gamma^1_{22} Z^3] dX \odot dX,$$

which is transparently invariant under the projective changes

$$\Gamma^k_{ij} \longrightarrow \Gamma^k_{ij} + \delta^k_i \Upsilon_j + \delta^k_j \Upsilon_i$$

of ∇. In the two-dimensional case the projective structures $(N, [\nabla])$ are equivalent to second order ODEs which are cubic in the first derivatives (see, e.g. [12])

$$\frac{d^2 Y}{dX^2} = \Gamma^1_{22} \Big(\frac{dY}{dX} \Big)^3 + (2\Gamma^1_{12} - \Gamma^2_{22}) \Big(\frac{dY}{dX} \Big)^2 + (\Gamma^1_{11} - 2\Gamma^2_{12}) \Big(\frac{dY}{dX} \Big) - \Gamma^2_{11}, \qquad (3.15)$$

where the integral curves of (3.15) are the unparametrised geodesics of ∇. The integral curves C of (3.15) are integral submanifolds of a differential ideal $\mathcal{I} =< \theta_0, \theta_1 >$, where

$$\theta_0 = dY + Z dX, \quad \theta_1 = dZ - \Big(\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12}) Z + (\Gamma^2_{22} - 2\Gamma^1_{12}) Z^2 + \Gamma^1_{22} Z^3 \Big) dX$$

are one–forms on a three–dimensional manifold $B = \mathbb{P}(T^* N)$ with local coordinates (X, Y, Z). If $f : C \to B$ is an immersion, then $f^*(\theta_0) = 0, f^*(\theta_1) = 0$ is equivalent to

(3.15) as long as $\theta_2 \equiv dX$ does not vanish. In terms of these three one–forms the contact structure, and the metric on the contact distribution are given by $\theta_0, h_H = \theta_1 \odot \theta_2$.

3.3 An alternative approach to theorem 3.2.1

aw: If I am going to keep this section, I probably need to put in some background about the orbit decompositions in [67] and how these different compactifications arise from them/are related to them. I would also want to understand enough about the Cartan/tractor bundle of a para–c–projective geometry to justify the statements about it made below.

It would be possible to show that the structures (M, g, Ω) arising in the projective to Einstein correspondence are para–c–projectively compact using a tractor approach. By our construction above it follows that $\mathcal{M}$ has a canonical para–c–projective geometry. In the notation of section 1.2.2, $\pi^*\mathcal{T} \oplus \pi^*\mathcal{T}^*$ is the corresponding para–c–projective tractor bundle and this has a canonical tractor connection that trivially extends (in fibre directions) the pull back of the projective connection (that is available in horizontal directions). The dual pairing between $\pi^*\mathcal{T}$ and $\pi^*\mathcal{T}^*$ determines a fibre metric and compatible symplectic form on the bundle $\pi^*\mathcal{T} \oplus \pi^*\mathcal{T}^*$ and this is obviously preserved by the connection. What remains is to show that the tractor connection so constructed satisfies properties that mean that it is *normal* in the sense defined in e.g. [68]. With this established then the main results then follow from the general holonomy theory in [67].

3.3.1 The alternative approach applied in the model case

aw: This section is nice in that it applies orbit decomposition [67] to the model. But only to show g is Einstein, not that it's para–Kahler, and it doesn't mention the boundary geometry. So I'm not really sure if it's worth keeping. Also some part of it might be better placed in the intro.

In this section we describe here the flat (in the sense of parabolic geometries) model [19, 22] of our construction in tractor terms.

The flat projective structure on $N = \mathbb{RP}^n$ gives rise to the neutral signature para–Kähler Einstein metric on $M = SL(n+1)/GL(n)$

$$g = d\zeta_i \odot dx^i + (\zeta_i dx^i)^2, \quad \Omega = d\zeta_i \wedge dx^i, \quad \text{where} \quad i, j, \dots = 1, \dots, n. \tag{3.16}$$

In [22], §7.1 it was explained how this homogeneous model corresponds to the projectivised co-tractor bundle of $\mathbb{RP}^n$, with an $\mathbb{RP}_{n-1}$ removed from each $\mathbb{RP}_n$ fiber. This $\mathbb{RP}_{n-1}$ corresponds to incident pairs of points and hyperplanes in $\mathbb{R}^{n+1} \times \mathbb{R}_{n+1}$.

Here we shall instead take N to be the sphere S^n with its standard projective structure as this is orientable in all dimensions and, more importantly, on this (double cover of $\mathbb{RP}^n$) the tractor bundle is trivial, and this simplifies the discussion. The underlying space of the (compactified) model of dimension $2n$ is $S^n \times S_n$ where both S_n and S^n denote spheres that are dual as we shall explain.

Consider first two vector spaces each isomorphic to $\mathbb{R}^{n+1}$:

$$V \cong \mathbb{R}^{n+1} \qquad \mathrm{W} \cong \mathbb{R}^{n+1}$$

and view each as a representation space for an $SL(n+1, \mathbb{R})$ action. So $G := SL(V) \times SL(W)$ acts on $V \times W$. (Note that we may wlog consider V and W as respectively the ± 1 eigenspaces of the single vector space $\mathbb{V} := V \oplus W$ equipped with a $\mathbb{J}$ s.t. $\mathbb{J}^2 = 1$.)

Now the action of $SL(V)$ descends to a transitive action on the ray projectivisation $\mathbb{P}_+(V)$ and similarly $SL(W)$ acts transitively on $\mathbb{P}_+(W)$. Thus $G := SL(V) \times SL(W)$ acts transitively on the manifold

$$\mathcal{M} := \mathbb{P}_+(V) \times \mathbb{P}_+(W).$$

We can represent an element of $\mathcal{M}$ in terms of pairs of homogeneous coordinates $([Y], [Z])$ where $0 \neq Y \in V$ and $0 \neq Z \in W$.

Note that as a smooth manifold $\mathcal{M} = S^n \times S^n$, but as a homogeneous manifold it is

$$G/P = \Big(SL(V)/P_X\Big) \times \Big(SL(W)/P_U\Big)$$

where P_X (resp. P_U) is the parabolic subgroup in $SL(V)$ that stabilises a point $[X]$ in $\mathbb{P}_+(V)$ (resp. $[U] \in \mathbb{P}_+(W)$), and P is the group product $P_X \times P_W$ which itself is a parabolic subgroup of the semisimple group G.

Now introduce an additional structure which breaks the G symmetry. Namely we fix an isomorphism

$$I : W \to V^*$$

where V^* denotes the dual space to V. The subgroup $H \cong SL(n+1, \mathbb{R})$ of G that fixes this may be identified with $SL(V)$ which acts on a pair $(Y, Z) \in V \times V^*$ by the defining representation and on the first factor and by the dual representation on the second factor.

Given this structure we may now (suppress I and) view $\mathcal{M}$ as consisting of pairs $([X],[U])$ where $0 \neq X \in V$ and $0 \neq U \in V^*$. That is

$$\mathcal{M} = \mathbb{P}_+(V) \times \mathbb{P}_+(V^*).$$

This is useful as follows: Each element $[U]$ in $\mathbb{P}_+(V^*)$ determines an oriented hyperplane in V and each $[X] \in \mathbb{P}_+(V)$ an oriented line in V. So now we consider the H action on M. This has two open orbits and a closed orbit. The last is the incidence space

$$\mathcal{Z} = \{([X],[U]) \in \mathcal{M} \mid U(X) = 0\}$$

which sits as smooth orientable separating hypersurface in $\mathcal{M}$. Then there are the open orbits

$$M_+ = \{([X],[U]) \in \mathcal{M} \mid U(X) > 0\} \qquad \text{and} \qquad M_- = \{([X],[U]) \in \mathcal{M} \mid U(X) < 0\}.$$

We may think of $\mathcal{Z}$ as the 'boundary' (at infinity) for the open orbits $M_\pm$.

We now describe the geometries on the orbits. The claim is that there are Einstein metrics in $M_\pm$, while $\mathcal{Z}$ is well known as the model for so-called contact Langrangian (or sometimes called para–CR) geometry, this is a real analogue of hypersurface type CR geometry.

First observe that $N_V := \mathbb{P}_+(V)$ is the flat model of projective geometry. So in particular we have

$$0 \to \mathcal{E}_V(-1) \xrightarrow{X} \mathcal{T}_V \to TN_V(-1) \to 0$$

where $\mathcal{T}_V$ is the projective tractor bundle on N_V and X is the tautological section of $\mathcal{T}(1)$, which coincides with the canonical tractor. Similarly there a sequence on $N^W := \mathbb{P}_+(V^*)$

$$0 \to \mathcal{E}^W(-1) \xrightarrow{U} \mathcal{T}^W \to TN^W(-1) \to 0. \tag{3.17}$$

There is a natural tractor bundle $\mathcal{T} := \mathcal{T}_V \oplus \mathcal{T}^W$ on M. Where X and U are not incident this induces a metric on M as follows. Observe that, at a point $([X],[U])$ where $X \lrcorner U \neq 0$, the tractor field U splits the first sequence by $\nu \in \Gamma(\mathcal{E}(-1,0))$ defined by

$$\nu := U/\tau$$

with $\tau := X \lrcorner U$ (and where we have used an obvious weight notation). This follows as $X \lrcorner \nu = 1$. Similarly

$$x := X/\tau \in \Gamma(\mathcal{E}(0,-1))$$

splits the second short exact sequence because $x \lrcorner U = 1$. Thus we obtain a neutral signature metric on $TN_V \oplus TN^W$ by these two steps: First, using these splittings yields a bundle monomorphism

$$TN_V(-1,0) \oplus TN^W(0,-1) \to \mathcal{T}_V \oplus \mathcal{T}^W.$$

Second, this gives a symmetric form $\boldsymbol{g}$ and symplectic form $\boldsymbol{\Omega}$ on $TN_V(-1,0) \oplus TN^W(0,-1)$ by then using the canonical metric and symplectic form on $\mathcal{T}_V \oplus \mathcal{T}^W$ given by the duality of $\mathcal{T}_V$ and $\mathcal{T}^W$. Thus $\boldsymbol{g} \in \Gamma(S^2T^*M(1,1))$ and $\boldsymbol{\Omega} \in \Gamma(\Lambda^2T^*M(1,1))$. Then set

$$g := \frac{1}{\tau}\boldsymbol{g} \qquad \text{and} \qquad \Omega := \frac{1}{\tau}\boldsymbol{\Omega}.$$

The metric g is easily seen to have neutral signature. It is Einstein because the tractor metric on $\mathcal{T}$ is parallel for the tractor connection (see [67] for the analogous c-projective case). The tractor connection arises from the usual parallel transport on the vector space $V \oplus V^*$ viewed as an affine manifold.

Chapter 4

Einstein–Weyl structures and $SU(\infty)$–Toda fields

In this chapter, we focus on the four–dimensional Einstein manifolds that arise from the projective to Einstein correspondence in the case $n = 2$. As discussed in chapter 1, it is shown in [22] that the Einstein manifolds in this subclass have anti–self–dual Weyl tensor, and are therefore associated with a twistor space [46]. If they also carry a Killing vector field arising from a symmetry of the underlying projective surface, one can extract solutions of the $SU(\infty)$–Toda equation via symmetry reduction to Lorentzian Einstein–Weyl structures in $2 + 1$ dimensions [36, 54].

The aim of this chapter is to exhibit the Einstein–Weyl structures obtainable in this way, resulting in several examples of new, explicit solutions of the Toda equation. We also give an explicit criterion for a vector field that generates a symmetry of a Weyl structure, and prove some results about the Einstein manifold and corresponding twistor space arising from the flat projective surface $\mathbb{RP}^2$. The content of this chapter is based on some of the work in [23].

In the case $n = 2$ we will write the metric and symplectic form as

$$g = \quad dz_{A'} \odot dx^{A'} - (\Gamma^{C'}_{A'B'} z_{C'} - z_{A'} z_{B'} - \mathrm{P}_{A'B'}) dx^{A'} \odot dx^{B'}, \tag{4.1}$$

$$\Omega = \quad dz_{A'} \wedge dx^{A'} + \mathrm{P}_{A'B'} dx^{A'} \wedge dx^{B'}, \qquad A', B', C' = 0, 1. \tag{4.2}$$

where we have replaced $\{\zeta_i\}$ with $\{z_{A'}\}$ and shifted the indices from $i, j = 1, 2$ to $A', B' = 0, 1$. This is helpful for the twistorial calculations because it agrees with the usual notation for two–component spinor indices. Note that a change of projective connection is now given by

$$\Gamma^{C'}_{A'B'} \to \Gamma^{C'}_{A'B'} + \delta^{C'}_{A'}\Upsilon_{B'} + \delta^{C'}_{B'}\Upsilon_{A'}, \qquad z_{A'} \to z_{A'} + \Upsilon_{A'}, \qquad A', B', C' = 0, 1. \quad (4.3)$$

4.1 Background

4.1.1 A classification of projective surfaces

Recall (see, for example, [12]) that a projective structure on a surface can be locally specified by a single second order ODE: taking coordinates (x, y) on the surface we find that geodesics on which $\dot{x} \neq 0$ can be written as unparametrised curves $y(x)$ such that

$$y'' + a_0(x, y) + 3a_1(x, y)y' + 3a_2(x, y)(y')^2 + a_3(x, y)(y')^3 = 0, \quad (4.4)$$

where the coefficients $\{a_i\}$ are given by the projectively invariant formulae

$$a_0 = \Gamma^1_{00}, \quad 3a_1 = -\Gamma^0_{00} + 2\Gamma^1_{01}, \quad 3a_2 = -2\Gamma^0_{01} + \Gamma^1_{11}, \quad a_3 = -\Gamma^0_{11}.$$

As we saw in chapter 1, the maximally symmetric projective surface $\mathbb{RP}^2$ has symmetry group $SL(3, \mathbb{R})$. In fact, the possible symmetry groups of projective surfaces are $SL(3, \mathbb{R})$, $SL(2, \mathbb{R})$, the two–dimensional affine group, and $\mathbb{R}$. A partial classification is given in [11].

1. On **the flat projective surface** $\mathbb{RP}^2$ described in section 1.1.3, geodesics $y(x)$ are described in inhomogeneous coordinates $(x, y) = (X^0/X^2, X^1/X^2)$ by the ODE
$$y'' = 0.$$

2. **The punctured plane** $\mathbb{R}^2 \backslash \{0\}$ has symmetry group $SL(2, \mathbb{R})$ acting via its fundamental representation. In this case there is a one–parameter family of projective structures falling into three distinct equivalence classes, with geodesics $y(x; \mu)$ described by the differential equation
$$y'' = -\mu(y - xy')^3, \quad (4.5)$$
where (x, y) are standard Euclidean coordinates on $\mathbb{R}^2$ and μ is a constant parameter. The two non–flat equivalence classes are given by $\mu > 0$ and $\mu < 0$ respectively. For simplicity we will consider only the first class, taking the representative with $\mu = 1$.

3. **The two–dimensional Lie group of affine transformations on** $\mathbb{R}$, which we denote Aff(1), is generated by the unique non–abelian two–dimensional Lie algebra $\{v_1, v_2\}$, where we choose a basis such that $[v_1, v_2] = v_1$. We can choose coordinates on Aff(1) such that these correspond to vector fields

$$\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

and using invariance under these vector fields, the geodesic equation can be cast in the form [28]

$$y'' = e^{-2x}(y')^3 + Ay' + Be^x,$$

where A and B are constants.

4. **The general projective surface with a symmetry**, after a choice of coordinates such that the symmetry is $\frac{\partial}{\partial x}$, corresponds to a set of geodesics $y(x)$ which satisfy an ODE that can be written uniquely in the form [28]

$$y'' = A(y)(y')^3 + B(y)(y')^2 + 1. \tag{4.6}$$

Note that each of these classes of projective structures forms a subset of the next, and this can be seen explicitly by some changes of coordinates. For example, the general projective surface with a symmetry is flat when $A(y) = B(y) = 0$.

4.1.2 Anti–self–duality, spinors and totally null distributions

Let M be an oriented four–dimensional manifold with a metric g of signature $(2, 2)$. The Hodge operator $*$ on the space of two forms is an involution, and induces a decomposition [7]

$$\Lambda^2(T^*M) = \Lambda^2_-(T^*M) \oplus \Lambda^2_+(T^*M) \tag{4.7}$$

of two-forms into anti-self-dual (ASD) and self-dual (SD) components, which only depends on the conformal class of g. The Riemann tensor of g can be thought of as a map $\mathcal{R} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$

aw: Be more explicit about the index symmetries once you have put the convention in the intro.

which admits a decomposition under (4.7):

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ - 2\Lambda & \phi \\ \hline \phi & C_- - 2\Lambda \end{array}\right), \tag{4.8}$$

where $C_\pm$ are the SD and ASD parts of the conformal Weyl tensor[1], ϕ is the trace-free Ricci curvature, and -24Λ is the scalar curvature which acts by scalar multiplication. The metric g is Einstein if $\phi = 0$, and the corresponding conformal structure $[g]$ is ASD if $C_+ = 0$. We will call g *conformally ASD* if it has ASD Weyl tensor. If both of these conditions are satisfied then the Riemann tensor is also anti-self-dual.

Two–component spinors

The symmetry group of a metric in signature $(+,+,-,-)$ decomposes under the Lie group isomorphism

$$SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2.$$

Locally[2] there exist real rank two vector bundles $\mathbb{S}, \mathbb{S}'$ (spin bundles) over M equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that [47]

$$TM \cong \mathbb{S} \otimes \mathbb{S}' \tag{4.9}$$

is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2) \tag{4.10}$$

for $v_1, v_2 \in \Gamma(\mathbb{S})$ and $w_1, w_2 \in \Gamma(\mathbb{S}')$.

aw: Huggett and Tod introduce ε (after taking tensor products of $\mathbb{S}, \mathbb{S}'$) as the unique non–zero two index spinor up to scale, then use symmetry arguments to show $g = \varepsilon \otimes \varepsilon'$...

We will use the usual notation $v^A \in \Gamma(\mathbb{S}), w^{A'} \in \Gamma(\mathbb{S}')$, then in index notation we can write (4.10) as

$$g_{ab}U^aV^b = \varepsilon_{AB}\varepsilon_{A'B'}U^{AA'}V^{BB'}. \tag{4.11}$$

[1]Note this is a different object to the projective Weyl tensor discussed in chapter 1.
[2]$\mathbb{S}, \mathbb{S}'$ are defined globally if M satisfies some topological restrictions; see [47].

We identify $\mathbb{S}$ with its dual according to

$$v_A = v^B \varepsilon_{BA}, \qquad v^A = \varepsilon^{AB} v_B,$$

and similarly for $\mathbb{S}'$. Note that the contraction is always over adjacent indices descending to the right, and $\varepsilon^{AB}\varepsilon_{CB} = \delta_C^A$. In higher valence spinors, the relative order of primed and unprimed indices is unimportant.

A vector $V \in \Gamma(TM)$ is called null if $g(V, V) = 0$. For $U = V$ the right hand side of (4.11) is just the determinant of $V^{AA'}$ viewed as a matrix, so any null vector is of the form $V = v \otimes w$ where v, and w are sections of $\mathbb{S}$ and $\mathbb{S}'$ respectively. The antisymmetry of ε means that $\varepsilon(v, v) = 0$ for any section $v \in \Gamma(\mathbb{S})$ (and similarly for any $w \in \mathbb{S}'$), so the converse is also true (i.e. any vector that can be written $V = v \otimes w$ is null).

Since $\mathbb{S}$ and $\mathbb{S}'$ have dimension two, the symplectic structures $\varepsilon_{AB}, \varepsilon_{A'B'}$ are the unique skew–symmetric two–index spinors up to scale. Any spinor of valence n which is skew on a pair of indices can thus be factorised as the tensor product of a spinor of valence $n - 2$ and either ε or ε'. This leads to the decomposition of a two–form $F_{ab} = F_{AA'BB'} = F_{ABA'B'}$ as

$$F_{ABA'B'} = \varepsilon_{AB}\Phi_{A'B'} + \varepsilon_{A'B'}\Psi_{AB},$$

where $\Phi_{A'B'} = \Phi_{(A'B')}$ and $\Psi_{AB} = \Psi_{(AB)}$. One can show using an analogous decomposition of the volume form that $\Phi_{A'B'}$ and Ψ_{AB} are the SD and ASD parts of F_{ab} respectively.

The nonlinear graviton

Any two–dimensional distribution on a four–manifold M can be expressed as the kernel of a two–form, and we define a distribution to be (A)SD if the corresponding two–form is (A)SD. Taking any $\iota^A \in \Gamma(\mathbb{S})$, the two–form $\iota_A\iota_B\varepsilon_{A'B'}$ defines an ASD distribution $D_\beta = \{\iota^A w^{A'}, w^{A'} \in \Gamma(\mathbb{S}')\}$ which is totally null in the sense that $g(U, V) = 0$ for all $U, V \in \Gamma(D)$. We call this a β–distribution. Note that it is only defined up to scale, which means that there is a $\mathbb{CP}^1$ worth of β–planes at every point in M. Given any $\pi^{A'} \in \Gamma(\mathbb{S}')$, one can similarly define a totally null SD distribution called an α–distribution by

$$D_\alpha = \{\iota^A \pi^{A'}, \iota^A \in \Gamma(\mathbb{S})\} = \text{span}\{\pi^{A'} \mathsf{e}_{AA'}\}, \tag{4.12}$$

where $\{\mathsf{e}_{AA'}\}$ is a null tetrad of vector fields.

aw: Do I need to write this in terms of an orthonormal basis for $\mathbb{S}, \mathbb{S}'$?

An α–surface (respectively β–surface) is a two–dimensional surface in M which is tangent to an α–(β–)distribution at every point. Penrose's Nonlinear Graviton Theorem [46] states that a maximal, three dimensional family of α–surfaces exists in M if and only if its conformal curvature is ASD, i.e. $C_+ = 0$.

In fact, Penrose considers four dimensional *complex* manifolds[3] M carrying a metric which is *holomorphic* in the sense that the metric components depend on the coordinates on M and not on their complex conjugates. Then $\mathbb{S}, \mathbb{S}'$ are complex vector bundles over M and $\varepsilon, \varepsilon'$ are holomorphic symplectic forms. A real conformally ASD metric in a given signature can then be obtained by choosing the correct reality conditions. In neutral signature this amounts to identifying spinors with their complex conjugates.

aw: ??? What is the correct choice in neutral signature? I'm not sure I understand what reality conditions are when we are not in Minkowski spacetime.

The *twistor space* $\mathscr{T}$ of M is defined as the three–dimensional complex manifold comprising the set of all α–surfaces in M. Each point $p \in M$ corresponds to a subset $\mathscr{L}_p \subset \mathscr{T}$ of α–surfaces which pass through p. Since an α–surface at p is defined by a $\pi^{A'} \in \mathbb{S}'|_p$ up to scale, $\mathscr{L}_p$ is an embedding $\mathbb{CP}^1 \subset \mathscr{T}$. We denote the lift of the distribution (4.12) to $\mathbb{S}'$ by

$$\mathcal{D} = \operatorname{span}\{L_A := \pi^{A'} \tilde{\mathsf{e}}_{AA'}\}, \tag{4.13}$$

where $\{\tilde{\mathsf{e}}_{AA'}\}$ are the lifts of the null tetrad $\{\mathsf{e}_{AA'}\}$ to $\mathbb{S}'$ (which are uniquely defined from the connection on $\mathbb{S}'$ inherited from the Levi–Civita connection of g). We call $\mathcal{D}$ the *twistor distribution*.

The Nonlinear Graviton allows us to express an ASD conformal structure in terms of the algebraic geometry of $\mathscr{T}$. First note that if two points $p_1, p_2 \in M$ are null–separated, then the corresponding curves $\mathscr{L}_{p_1}, \mathscr{L}_{p_2}$ intersect at a single point. This is because any null geodesic must have a tangent vector field of the form $\iota^A \pi^{A'}$ for some sections $\iota^A \in \Gamma(\mathbb{S})$ and $\pi^{A'} \in \Gamma(\mathbb{S}')$, and thus the geodesic is contained within the unique α–surface spanned by $\pi^{A'} \mathsf{e}_{AA'}$. This unique α–surface corresponds to the point in $\mathscr{T}$ where the curves $\mathscr{L}_{p_1}, \mathscr{L}_{p_2}$ meet.

In order to understand this correspondence at an infinitesimal level and thereby recover an ASD conformal structure from $\mathscr{T}$, we need to understand the *normal bundle*

[3] Note that familiar facts from real geometry such as a unique Levi–Civita connection and the Frobenius theorem carry over to holomorphic geometry. See [40] for details.

$N(\mathscr{L}_p) := \cup_{Z\in\mathscr{L}_p}\{T_Z\mathscr{T}/T_Z\mathscr{L}_p\}$ over a $\mathbb{CP}^1$ embedding $\mathscr{L}_p$. This is evidently a complex vector bundle, and in fact it is a *holomorphic* vector bundle, meaning that the total space is a complex manifold and the projection $N(\mathscr{L}_p) \to \mathscr{L}_p$ is holomorphic. It is thus subject to the following theorem due to Birkhoff and Grothendieck (see for example [44] for a proof).

Theorem 4.1.1 (Birkhoff–Grothendieck)**.** *Any rank k holomorphic vector bundle is isomorphic to a direct sum of k complex line bundles $\mathcal{O}(m_i), 1 \leq i \leq k$, each with first Chern class m_i.*

The first Chern class completely classifies complex line bundles topologically. For us, $\mathcal{O}(n)$ will mean a line bundle over $\mathbb{CP}^1$ with transition functions λ^{-n}, where λ is the inhomogeneous coordinate on $\mathbb{CP}^1$. A section of $\mathcal{O}(n)$ is given by functions $s(\lambda), \tilde{s}(\tilde{\lambda})$ on patches $\mathcal{U}, \tilde{\mathcal{U}}$ which are related by

$$s(\lambda) = \lambda^n \tilde{s}(\tilde{\lambda}).$$

If we expand these as power series in the local coordinates and use the fact that $\tilde{\lambda} = \lambda^{-1}$, we find by equating coefficients that they are polynomials of degree at most n, making the space of holomorphic sections $n+1$–dimensional.

We now define the *correspondence space* $\mathcal{F} = M \times \mathbb{CP}^1$ with local coordinates $(x^a, \lambda) := (x^a, \pi_{0'}/\pi_{1'})$, where $\pi^{A'}$ parametrises the set of α–surfaces through the point in M with coordinates x^a. Note that $\mathcal{F}$ can be obtained from the primed spin bundle $\mathbb{S}' \to M$ by projectivising each fibre. Now consider a holomorphic function on $\mathbb{S}'$ which is homogeneous of degree n in the fibres. This corresponds to a section of $\mathcal{O}(n)$ over the $\mathbb{CP}^1$ factor of $\mathcal{F}$.

The correspondence space has the alternative definition

$$\mathcal{F} = \{(Z, p) \in \mathscr{T} \times M \ : \ Z \in L_p\},$$

leading to the double fibration

$$M \xleftarrow{r} \mathcal{F} \xrightarrow{q} \mathscr{T}.$$

We are now ready to state the following lemma.

Lemma 4.1.2 ([46])**.** *The normal bundle of the holomorphic curves $L_p = q(r^{-1}(p))$ corresponding to points $p \in M$ can be identified with $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. The double fibration picture allows us to identify the normal bundle with the quotient $r^*(T_pM)/\mathrm{span}\{L_A\}$. In their homogeneous form the operators L_A have weight one, and the distribution spanned by them is isomorphic to the bundle $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. The definition of the normal bundle as a quotient gives the exact sequence

$$0 \to \mathbb{C}^2 \otimes \mathcal{O}(-1) \to \mathbb{C}^4 \to N \to 0$$

and thus $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$, since the last map is given explicitly by $V^{AA'} \mapsto V^{AA'}\pi_{A'}$ in spinor notation.

aw: I don't understand this proof. I basically copied it out of Maciej's book. It might be better to just explain it rather than stating it formally as a lemma.

□

The Nonlinear Graviton Theorem can now be stated as follows.

Theorem 4.1.3 ([46])**.** *There is a one–to–one correspondence between complex ASD conformal structures and three–dimensional complex manifolds containing a four–parameter family of* $\mathbb{CP}^1$ *embeddings with normal bundle* $\mathcal{O}(1) \oplus \mathcal{O}(1)$*.*

From the results of Kodaira [38] we have that a vector at a point $p \in M$ corresponds to a holomorphic section of the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of the curve $\mathscr{L}_p$ in $\mathscr{T}$. Penrose shows that we obtain an ASD conformal structure from $\mathscr{T}$ by defining a vector to be null if the corresponding holomorphic section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ has a zero. Note that the vanishing of such a section is a quadratic condition, since $V^{AA'}\pi_{A'}$ can be solved for $\pi_{A'}$ if $\det(V^{AA'}) = 0$.

(Anti–)self–duality in the sense of Calderbank

Although a β–distribution is intrinsically ASD, there are two subclasses of β–distribution which we shall call SD or ASD *in the sense of Calderbank* [13] (see also [60]). Let D_β be a β–distribution defined by an ASD two–form $\Sigma_{ab} = \iota_A \iota_B \epsilon_{A'B'}$, and such that the spinor ι_A satisfies

$$\nabla_{A'(A}\iota_{B)} = \mathcal{A}_{A'(A}\iota_{B)} \tag{4.14}$$

where $d\mathcal{A}$ is an (A)SD Maxwell field. Then D_β is (A)SD in the sense of Calderbank.

aw: Add the justification from West's thesis that integrability implies (4.14), if you can understand why his condition about $\iota\nabla\iota$ is equivalent to integrability.

Local characterisation of the Einstein manifolds (M, g)

A general ASD metric depends, in the real–analytic category, on six arbitrary functions of the variables. Theorem 1.2.1 gives an explicit subclass of such metrics which are additionally constrained by the following local characterisation.

Theorem 4.1.4. *[22] Let (M, g) be an real ASD Einstein manifold with scalar curvature 24 admitting a totally null distribution D which is ASD in the sense of Calderbank and parallel in the sense that ${}^g\nabla_X Y \in \Gamma(D)$ for all $X \in \Gamma(TM)$, $Y \in \Gamma(D)$. Then (M, g) is conformally flat, or it is locally isometric to (4.1).*

Note that a parallel distribution is necessarily Frobenius integrable, since

$$[X, Y] = {}^g\nabla_X Y - {}^g\nabla_Y X \in \Gamma(D) \quad \text{for all} \quad X, Y \in \Gamma(D).$$

In our coordinates D is the kernel of the two–form $\Sigma = dx^{0'} \wedge dx^{1'}$, and can be written as $D = \text{span}\{\partial/\partial z_{0'}, \partial/\partial z_{1'}\}$. We find that

$$^g\nabla\Sigma = 6\mathcal{A} \otimes \Sigma, \tag{4.15}$$

where $d\mathcal{A} = \Omega$, and Ω is the symplectic form on M. Writing $\Sigma_{ab} = \iota_A \iota_B \epsilon_{A'B'}$, (4.15) implies (4.14) for a rescaling of $\mathcal{A}$, so it is the anti–self–duality of Ω which makes D ASD in the sense of Calderbank.

aw: I believe this is the first time we have mentioned the ASD'ty of Ω. Can we also justify this?

In section 4.4 we will consider the model case where M is constructed from the flat projective structure on $\mathbb{RP}^2$. In this case, we can explicitly describe the ASD Maxwell two–form Ω in terms of the twistor space of M, and we will find that M carries a so–called *pseudo–hyper–Hermitian* structure, in which $\mathcal{A}$ plays an important role.

4.1.3 Einstein–Weyl structures

Definition 4.1.5. *A Weyl Structure $(W, \mathscr{D}, [h])$ is a conformal equivalence class of metrics $[h]$ on a manifold W along with a fixed torsion–free affine connection $\mathscr{D}$ which preserves any representative $h \in [h]$ up to conformal class. That is, for some one-form ω,*

$$\mathscr{D}h = \omega \otimes h.$$

aw: h is also the name of a metric in c-proj...

A pair (h, ω) uniquely defines the connection and hence the Weyl structure, so we can alternatively specify a Weyl structure as a triple (W, h, ω). However, there is an equivalence class of such pairs which define the same Weyl structure. These are related by transformations

$$h \to \rho^2 h, \quad \omega \to \omega + 2d\ln(\rho), \tag{4.16}$$

where ρ is a smooth, non-zero function on W. Physically, the Weyl condition in Lorentzian signature corresponds to the statement that null geodesics of the conformal structure $[h]$ are also geodesics of the connection $\mathscr{D}$.

If additionally the symmetric part of the Ricci tensor of $\mathscr{D}$ is a scalar multiple of h, then W is said to carry an Einstein-Weyl structure. This condition is invariant under (4.16). A trivial Einstein–Weyl structure is one whose one–form ω is closed, so that it is locally exact and thus may be set to zero by a change of scale (4.16). Then $\mathscr{D}$ is the Levi–Civita connection of some representative $h \in [h]$, and this representative is Einstein.

The Einstein–Weyl equations give a set of five non–linear PDEs on the pair (h, ω). These equations are integrable by the twistor transform of Hitchin [35], which can be regarded as a reduction of Penrose's twistor transform [46] for ASD conformal structures by the following theorem of Tod.

Theorem 4.1.6. *[36]*

1. *Let (M, g) be a neutral signature, conformally ASD four–manifold with a conformal Killing vector K. Let*

$$h = |K|^{-2} g - |K|^{-4} \mathbf{K} \odot \mathbf{K}, \qquad \omega = \frac{2}{|K|^2} \star (\mathbf{K} \wedge d\mathbf{K}), \tag{4.17}$$

where $|K|^2 = g(K, K)$, $\mathbf{K} = g(K, \cdot)$ and $\star$ is the Hodge operator defined by g. Then (h, ω) is a solution of the Einstein–Weyl equations on the space of orbits W of K in M.

2. *Given an Einstein–Weyl structure (W, h, ω) there is a one–to–one correspondence between solutions (V, α) to the Abelian monopole equation*

$$dV + \frac{1}{2}\omega V = \star_h d\alpha. \tag{4.18}$$

on W, where V is a function and α is a one–form, and conformally ASD four–metrics

$$g = Vh - V^{-1}(dx + \alpha)^2 \tag{4.19}$$

over W with an isometry $K = \partial/\partial x$.

4.1.4 The $SU(\infty)$–Toda equation

The $SU(\infty)$–Toda equation is given by

$$U_{XX} + U_{YY} = \epsilon(e^U)_{ZZ}, \quad \text{where} \quad U = U(X, Y, Z), \quad \text{and} \;\; \epsilon = \pm 1 \tag{4.20}$$

Equation (4.20) has originally arisen in the context of complex general relativity [29, 10, 50], and then in Einstein–Weyl [59] and (in Riemannian context, with $\epsilon = -1$) scalar–flat Kähler geometry [39]. It belongs to a class of dispersionless systems integrable by the twistor transform [43, 27, 8], the method of hydrodynamic reduction [30], and the Manakov–Santini approach [41]. The equation is nevertheless not linearisable and most known explicit solutions admit Lie point or other symmetries (there are exceptions - see [15, 16, 42, 51]).

The $SU(\infty)$–Toda equation is related to a subclass of Einstein–Weyl structures by the following result of Tod which improved the earlier result of Przanowski [50].

Theorem 4.1.7. *[54] Let (W, h, ω) be an Einstein–Weyl structure arising from the first part of Theorem 4.1.6, under the additional assumption that the ASD conformal structure $(M, [g])$ has a representative $g \in [g]$ which is Einstein with non–zero Ricci scalar.*

1. *The Einstein–Weyl structure admits a shear–free, twist–free geodesic congruence.*
2. *There exists $h \in [h]$, and coordinates (X, Y, Z) on an open set in W such that (assuming the signature of h is $(2, 1)$ and the congruence is time–like)*

$$h = e^U(dX^2 + dY^2) - dZ^2, \quad \omega = 2U_Z dZ \tag{4.21}$$

and the function $U = U(X, Y, Z)$ satisfies the $SU(\infty)$–Toda equation (4.20) with $\epsilon = 1$.

aw: Need to define a congruence and its shear and twist, or modify the statement of Tod's theorem to be independent of the congruence.

4.1.5 From projective surfaces to $SU(\infty)$–Toda fields

The whole construction can now be summarised in the following diagram

$$\begin{array}{rcl} \text{Projective structure with symmetry} & \overset{\text{thm 1.2.1}}{\longrightarrow} & \text{ASD Einstein with symmetry} \\ \downarrow & & \downarrow\text{thm 4.1.6} \\ \text{Solution to } SU(\infty) \text{ Toda} & \overset{\text{thm 4.1.7}}{\longleftarrow} & \text{Einstein–Weyl.} \end{array} \tag{4.22}$$

4.2 The general projective surface with symmetry

Consider the most general Einstein–Weyl structure arising from the combination of Theorem 1.2.1 and Theorem 4.1.6. Because of the correspondence (1.32) between symmetries of (M, g) and symmetries of the projective surface $(N, [\nabla])$, the construction must begin with the general projective surface with at least one symmetry.

By trial and error, we chose a representative connection for (4.6) such that the metric (4.1) had the simplest possible form. The choice of connection we took was

$$\Gamma^0_{11} = A(y), \quad \Gamma^1_{00} = -1, \quad \Gamma^1_{11} = -B(y)$$

with all other components vanishing. Note that this choice of connection has a symmetric Ricci tensor, so the Schouten tensor is also symmetric and the symplectic form (4.2) pulls back to just $dz_{A'} \wedge dx^{A'}$. Thus we can write the Maxwell potential $\mathcal{A}$ which is such that $d\mathcal{A} = \Omega$ as $\mathcal{A} = z_{A'}dx^{A'}$. Writing $x^{A'} = (x, y)$, $z_{A'} = (p, q)$, the resulting metric (4.1) is

$$g = (B(y)+p^2+q)dx^2+2(pq+A(y))dxdy+(-A(y)p+B(y)q+q^2)dy^2+dxdp+dydq. \tag{4.23}$$

Factoring by $K = \frac{\partial}{\partial x}$ following the algorithm of Theorem 4.1.6, equation (4.17) gives the following form for the Einstein–Weyl structure.

Proposition 4.2.1. *The most general Einstein–Weyl structure arising from the procedure (4.22) is locally equivalent to*

$$\begin{aligned} h &= \frac{1}{V}\Big((Bq - Ap + q^2)dy + dq\Big)dy - \Big(pq + Ady + \frac{1}{2}dp\Big)^2, \qquad (4.24) \\ \omega &= V(4dq + 2pdp), \quad \textit{where} \quad V = (B + p^2 + q)^{-1}. \end{aligned}$$

Here (p, q, y) are local coordinates on W, $A(y), B(y)$ are arbitrary functions of y, and the corresponding solution to the monopole equation (4.18) is the pair (V, α), where

$$\alpha = V(pq + A)dy + \frac{V}{2}dp.$$

4.2.1 Solution to the $SU(\infty)$–Toda equation

The procedure for extracting the corresponding solution to the $SU(\infty)$–Toda equation is given in [54] (see also [39] and [25]). It involves finding the coordinates (X, Y, Z) that put the metric (4.24) in the form (4.21). Given an ASD Einstein metric (M, g) with a Killing vector K

1. The conformal factor $c : M \to \mathbb{R}^+$ given by

   $$c = |d\mathbf{K} + *_g d\mathbf{K}|_g{}^{-1/2}$$

   has a property that the rescaled self–dual derivative of K

   $$\Theta \equiv c^3 \Big(\frac{1}{2}(d\mathbf{K} + *_g d\mathbf{K})\Big)$$

   is parallel with respect to $\hat{g} = c^2 g$. The metric $\hat{g}$ is Kähler with self–dual Kähler form Θ, and admits a Killing vector K, as $\mathcal{L}_K(c) = 0$.

2. Define a function $Z : M \to \mathbb{R}$ to be the moment map:

   $$dZ = K \lrcorner \Theta. \tag{4.25}$$

   It is well defined, as the Kähler form is Lie–derived along K.

3. Construct the Einstein–Weyl structure of Theorem 4.1.6 by factoring $(M, \hat{g})$ by K. Restrict the metric h to a surface $Z = Z_0 = \text{const}$, and construct isothermal coordinates (X, Y) on this surface:

   $$\gamma \equiv h|_{Z=Z_0} = e^U(dX^2 + dY^2), \quad U = U(X, Y, Z_0).$$

   To implement this step chose an orthonormal basis of one–forms such that $\gamma = {e_1}^2 + {e_2}^2$. Now (X, Y) are solutions to the linear system of 1st order PDEs

   $$(e_1 + ie_2) \wedge (dX + idY) = 0.$$

4. Extend the coordinates (X, Y) from the surface $Z = Z_0$ to W. This may involve a Z–dependent affine transformation of (X, Y).

Implementing the steps 1–4 on MAPLE we find that if $A = 0$, and $B = B(y)$ is arbitrary, then the $SU(\infty)$–Toda solution is given implicitly by

$$\begin{aligned} X &= -\frac{8\mathrm{e}^{-2\int B(y)dy}Z^3 p}{(Z^2p^2+4)^2}, \quad Y = \int \mathrm{e}^{-2\int B(y)dy}dy + \frac{\mathrm{e}^{-2\int B(y)dy}(-2Z^4p^2+8Z^2)}{(Z^2p^2+4)^2}. \\ U &= \ln\Big(\frac{(Z^2p^2+4)^3}{64Z^2}\Big) + 4\int B(y)dy. \end{aligned} \tag{4.26}$$

We can check that this is indeed a solution using the fact that the $SU(\infty)$–Toda equation is equivalent to $d \star_h dU = 0$. We have also checked by performing a coordinate transformation of (4.20) to the coordinates (y, p, Z).

To simplify the form of (4.26) set

$$G = \int \exp\Big(-2\int B(y)dy\Big), \quad T = \frac{2Z^2}{Z^2p^2+4}.$$

Then (4.26) becomes

$$e^U = \frac{Z^4}{8T^3(G')^2}, \quad Y = G + G'T\Big(\frac{4T}{Z^2} - 1\Big), \quad X^2 = \frac{4T^4(G')^2}{Z^2}\Big(\frac{2}{T} - \frac{4}{Z^2}\Big).$$

Eliminating (T, y) between these three equations gives one relation between (X, Y, Z) and U which is our implicit solution. The elimination can be carried over explicitly if $G = y^k$ for any integer k, or if $G = \exp y$. In the later case the solution is given by

$$4Y^2e^U(e^UX^2 - Z^2)^3 + (2e^{2U}X^4 - 3e^UX^2Z^2 + Z^4 + 2Z^2)^2 = 0.$$

We can also consider the flat projective structure with $A = B = 0$, in which case the coordinate p can be eliminated between

$$e^U = \Big(\frac{(Z^2p^2+4)^3}{64Z^2}\Big), \quad X = -\frac{8Z^3p}{(Z^2p^2+4)^2}$$

by taking a resultant. This yields

$$e^U(e^UX^2 - Z^2)^3 + Z^4 = 0.$$

Note that even the flat projective surface can yield a non–trivial solution to the Toda equation; further discussion can be found in section 4.4.5.

4.2.2 Two monopoles

The Einstein–Weyl structures (4.24) we have constructed in Proposition 4.2.1 are special, as they belong to the $SU(\infty)$–Toda class, and so (as shown by Tod [52]) admit a non–null geodesic congruence which has vanishing shear and twist. The general solution to the $SU(\infty)$–Toda equation depends (in the real analytic category) on two arbitrary functions of two variables, but the solutions of the form (4.24) depend on two functions of one variable. The additional constraints on the solutions can be traced back to the four dimensional ASD conformal structures which give rise (by the Jones–Tod construction) to (4.24).

As discussed above, in addition to their being ASD and Einstein they are characterised [22] by a β–distribution which is parallel with respect to the Levi–Civita connection and ASD in the sense of Calderbank [13]. The corresponding β–surfaces do not generically intersect with a given α–surface, however if they do intersect then they will intersect in curves (null geodesics) which descend to the Einstein–Weyl structures, and give rise to another (in addition to the Tod shear–free, twist–free) geodesic congruence. In what follows we shall point out how some of this structure arises from a couple of solutions to the Abelian monopole equation on EW backgrounds.

Let us call the solution (V, α) arising in Proposition 4.2.1 the Einstein monopole, as the resulting conformal class contains an Einstein metric (4.23). The second solution (V_M, α_M) (which we shall call the Maxwell monopole) arises as a symmetry reduction of the ASD Maxwell potential

$$\mathcal{A} = pdx + qdy = -V_M K + \alpha_M,$$

where $K = K_\mu dx^\mu$ is the Killing one–form, and we find

$$V_M = -pV, \quad \alpha_M = qdy - p\alpha.$$

aw: This is not related back to the congruence, so I need to either delete the congruence chat above or related this monopole solution to the congruence.

aw: Also I don't quite understand how this second monopole arises from $\mathcal{A}$.

4.3 The submaximally symmetric projective surface

aw: Say something about Aff(1)?

Choosing a representative connection from the projective class defined by (4.5), we obtain from (4.1) an Einstein metric

$$\begin{aligned} g = (p^2 - xy^2p - y^3q + 4y^2)dx^2 + 2(pq + x^2yp + xy^2q - 4xy)dxdy \\ +(q^2 - x^3p - x^2yq + 4x^2)dy^2 + dxdp + dydq \end{aligned} \tag{4.27}$$

on M, again with $z_0 =: p$, $z_1 =: q$, having Killing vectors

$$K_1 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p} - y\frac{\partial}{\partial y} + q\frac{\partial}{\partial q}, \quad K_2 = x\frac{\partial}{\partial y} - q\frac{\partial}{\partial p}, \quad K_3 = y\frac{\partial}{\partial x} - p\frac{\partial}{\partial q}.$$

These are lifts of the projective vector fields corresponding to the $\mathfrak{sl}(2)$ elements

$$T_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}.$$

aw: There is a comment about conjugacy classes to be made here.

Factoring by K_3 and choosing coordinates

$$u = \frac{p^2}{y^2}, \quad v = 2\ln(y^2), \quad w = xp + yq,$$

we obtain an Einstein-Weyl structure

$$\begin{aligned} h &= -du^2 - 2dudw - w(w^2 + u - 5w + 4)dv^2 + 2(u - w + 4)dvdw, \\ \omega &= \frac{1}{u - w + 4}du - \frac{3w}{u - w + 4}dv - \frac{4}{u - w + 4}dw. \end{aligned} \tag{4.28}$$

The solution to the $SU(\infty)$–Toda equation (4.20) which determines the Einstein-Weyl structure (4.28) is described by an algebraic curve $f(\mathrm{e}^U, X, Y, Z) = 0$ of degree six in e^U and degree twelve in the other coordinates. This solution has been found following the Steps 1-4 in §4.2.1, and is given by

$$\begin{aligned}
64\mathrm{e}^{6U}X^6(X+Y)^3(X-Y)^3 - 92\mathrm{e}^{5U}X^4Z^2(X+Y)^3(X-Y)^3 \\
+48\mathrm{e}^{4U}X^2Z^2(5X^6Z^2 - 14X^4Y^2Z^2 + 13X^2Y^2Z^2 - 4Y^4Z^2 + 9X^4 + 27X^2) \\
+8\mathrm{e}^{3U}Z^4(-20X^6Z^2 + 48X^4Y^2Z^2 - 36X^2Y^4Z^2 + 8Y^6Z^2 - 81X^4 - 243X^2Y^2) \\
+3\mathrm{e}^{2U}Z^4(20X^4Z^4 - 36X^2Y^2Z^4 + 16Y^4Z^4 + 108X^2Z^2 + 216Y^2Z^2 + 243) \\
+6\mathrm{e}^{U}Z^8(-2X^2Z^2 + 2Y^2Z^2 - 9) + Z^{12} \\
= 0.
\end{aligned}$$

Note that the formulae (4.28) are independent of the coordinate v, and therefore have a symmetry. This was unexpected because there is no other symmetry of (M, g) that commutes with K_3. However, it is possible for symmetries to appear in the Einstein-Weyl structure without a corresponding symmetry of the ASD conformal structure. This can be seen from the general formula (4.17); the function V may depend on the coordinate v so that g depends on v even though h does not. For example, the Gibbons-Hawking metrics [31] give a trivial Einstein–Weyl structure with the maximal symmetry group, but the four-metric is in general not so symmetric. Our discovery of this unexpected symmetry motivated a more concrete description of a symmetry of a Weyl structure.

Definition 4.3.1. *An infinitesimal symmetry of a Weyl structure* $(W, \mathscr{D}, [h])$ *is a vector field* $\mathcal{K}$ *which is both an affine vector field with respect to the connection*[4] $\mathscr{D}$ *and a conformal Killing vector with respect to the conformal structure* $[h]$.

Proposition 4.3.2. *Given an infinitesimal symmetry* $\mathcal{K}$ *of a Weyl structure* $(W, \mathscr{D}, [h])$ *in dimension* N*, and a representative* $h \in [h]$ *such that* $\mathscr{D}h = \omega \otimes h$*, there exists a smooth function* $f : W \to \mathbb{R}$ *such that*

$$\mathcal{L}_{\mathcal{K}}h = fh, \qquad \mathcal{L}_{\mathcal{K}}\omega = \frac{1}{N}d[\mathcal{K} \lrcorner\, d(\ln(\det(h)))]. \tag{4.29}$$

Proof. The first equation follows immediately from the fact that $\mathcal{K}$ is a conformal Killing vector of h. It remains to evaluate the Lie derivative of the one–form ω along the flow of $\mathcal{K}$ given that $\mathcal{L}_{\mathcal{K}}h = fh$ and $\mathcal{L}_{\mathcal{K}}\Gamma^i_{jk} = 0$, where Γ^i_{jk} are the components of

[4] Recall that an affine vector field of a connection $\mathscr{D}$ is one which preserves its components, i.e. $\mathcal{L}_{\mathcal{K}}\Gamma^i_{jk} = 0$.

the connection $\mathscr{D}$. We do this by considering the Lie derivative of $\mathscr{D}h$:

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}(\mathscr{D}_i h_{jk}) &= \mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) - \mathcal{L}_{\mathcal{K}}(\Gamma^l_{ji} h_{lk} + \Gamma^l_{ki} h_{jl}) \\ &= \mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) - f(\Gamma^l_{ji} h_{lk} + \Gamma^l_{ki} h_{jl}).\end{aligned}$$

Now

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) &= \mathcal{K}^l \partial_l \partial_i h_{jk} + (\partial_i \mathcal{K}^l)\partial_l h_{jk} + (\partial_j \mathcal{K}^l)\partial_i h_{lk} + (\partial_k \mathcal{K}^l)\partial_i h_{jl} \\ &= \partial_i[\mathcal{K}^l \partial_l h_{jk} + (\partial_j \mathcal{K}^l) h_{lk} + (\partial_k \mathcal{K}^l) h_{jl}] - (\partial_i \partial_j \mathcal{K}^l) h_{lk} - (\partial_i \partial_k \mathcal{K}^l) h_{jl}.\end{aligned}$$

The term with square brackets is just

$$\partial_i(\mathcal{L}_{\mathcal{K}} h_{jk}) = \partial_i(f h_{jk}) = f\partial_i h_{jk} + \partial_i f h_{jk},$$

so we have

$$\mathcal{L}_{\mathcal{K}}(\mathscr{D}_i h_{jk}) = f\mathscr{D}_i h_{jk} + \partial_i f h_{jk} - (\partial_i \partial_j \mathcal{K}^l) h_{lk} - (\partial_i \partial_k \mathcal{K}^l) h_{jl}.$$

Setting this equal to $\mathcal{L}_{\mathcal{K}}(\omega_i h_{jk}) = (\mathcal{L}_{\mathcal{K}}\omega_i) h_{jk} + f\omega_i h_{jk}$ and cancelling $f\omega_i h_{jk}$ with $f\mathscr{D}_i h_{jk}$, we find

$$\begin{aligned}(\mathcal{L}_{\mathcal{K}}\omega_i) g_{jk} &= \partial_i f h_{jk} - (\partial_i \partial_j \mathcal{K}^l) h_{lk} - (\partial_i \partial_k \mathcal{K}^l) h_{jl} \\ \implies \mathcal{L}_{\mathcal{K}}\omega_i &= \partial_i f - \frac{2}{N}\partial_i \partial_j \mathcal{K}^j.\end{aligned} \tag{4.30}$$

Finally, we note that

$$\partial_i \partial_j \mathcal{K}^j = \frac{N}{2}\partial_i f - \frac{1}{2}\partial_i[\mathcal{K} \lrcorner\, d(\ln(\det(h))].$$

This follows from tracing the expression $\mathcal{L}_{\mathcal{K}} h_{ij} = f h_{ij}$:

$$\begin{aligned}\mathcal{L}_{\mathcal{K}} h_{ij} = \mathcal{K}^k \partial_k h_{ij} + (\partial_i \mathcal{K}^k) h_{kj} + (\partial_j \mathcal{K}^k) h_{ik} &= f h_{ij} \\ \implies \quad \mathcal{K}^k h^{ij} \partial_k h_{ij} + 2\partial_k \mathcal{K}^k &= Nf \\ \implies \quad 2\partial_i \partial_k \mathcal{K}^k &= N\partial_i f - \partial_i(\mathcal{K}^k h^{jl} \partial_k h_{jl})\end{aligned}$$

and recalling that $h^{jl}\partial_k h_{jl} = \partial_k \ln(\det(h))$. Substituting into (4.30) then yields the result.

□

We can easily verify the invariance of (4.29) under Weyl transformations. Let $(\hat{h}, \hat{\omega})$ be a new metric and one–form related to the old ones by (4.16). Then

$$\mathcal{L}_{\mathcal{K}}\hat{\omega} = \mathcal{L}_{\mathcal{K}}\omega + 2\mathcal{K} \lrcorner\, d\ln(\rho)$$

from (4.16), and from (4.29) we have

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}\hat{\omega} &= \frac{1}{N} d[\mathcal{K} \lrcorner\, d(\ln(\rho^{2N}\det(h)))] \\ &= \frac{1}{N} d[\mathcal{K} \lrcorner\, d(\ln(\det(h)))] + \frac{2N}{N}\mathcal{K} \lrcorner\, d\ln(\rho) \\ &= \mathcal{L}_{\mathcal{K}}\omega + 2\mathcal{K} \lrcorner\, d\ln(\rho),\end{aligned}$$

as above. Note that the function f in (4.29) will change according to

$$\hat{f} = f + 2\mathcal{K} \lrcorner\, d\ln\rho.$$

In the case of the Weyl structure (4.28), the infinitesimal symmetry is

$$\mathcal{K} = \frac{\partial}{\partial v}.$$

Since we have chosen a scale such that $\mathcal{K}$ is in fact a Killing vector of h, we have that $\mathcal{K} \lrcorner\, d(\ln(\det(h)) = 0$, so the one–form ω is also preserved by $\mathcal{K}$. This is consistent with the fact that it has no explicit v–dependence.

4.4 The flat projective surface

In the following section we discuss the four–manifold (M, g) obtained from the maximally symmetric flat projective surface $N = \mathbb{RP}^2$. In this case, g is the indefinite analogue of the Fubini–Study metric given by the quotient $SL(3)/GL(2)$, and is not only almost para–Kähler but in fact para–Kähler, since the symplectic form Ω is parallel with respect to the Levi–Civita connection of g. Chosing a representative connection with $\Gamma^C_{AB} = 0$ gives g as

$$g = dz_{A'} \odot dx^{A'} + z_{A'}z_{B'}dx^{A'} \odot dx^{B'}. \tag{4.31}$$

We begin by discussing the conformal structure of (4.31), both explicitly and in terms of its twistor space. We then note a pseudo–hyper–Hermiticity property which is unique to the model case, and find that the twistor space fibers holomorphicaly over $\mathbb{CP}^1$. Finally, we present a classification of the Einstein–Weyl structures which can be obtained by Jones–Tod factorisation of $SL(3)/GL(2)$ and exhibit an explicit example of such a factorisation from the twistor perspective, reconstructing the conformal structure on W from minitwistor curves.

4.4.1 Conformal Structure

aw: Should I put this in the intro?

We first discuss the notion of duality in projective geometry. Consider the set of planes through the origin in $\mathbb{R}^3$. These can be specified by their normal vector, which is defined only up to multiplication by $\mathbb{R}^*$. Let us denote such a plane by a non–zero row vector $L \in \mathbb{R}_3$. A point $P \in \mathbb{R}^3$ lies in the plane defined by L if and only if $L \cdot P = 0$.

When we projectivise the $\mathbb{R}^3$, any $P \neq 0$ descends to a point $[P] \in \mathbb{RP}^2$, and any plane descends to a line $[L] \subset \mathbb{RP}^2$. The incidence relation $L \cdot P = 0$ is now equivalent to the point $[P]$ lying in the line $[L]$. The homogeneous coordinates $[L]$ parametrise a second projective surface which we think of as the dual to the $\mathbb{RP}^2$ parametrised by $[P]$, and denote $\mathbb{RP}^{2*}$.

In what follows we will drop the square brackets and understand points $[P] \in \mathbb{RP}^2$ and lines $[L] \in \mathbb{RP}^{2*}$ to be represented by vectors $P \in \mathbb{R}^3$ and $L \in \mathbb{R}_3$. Let $M \subset \mathbb{RP}^2 \times \mathbb{RP}^{2*}$ be the set of non–incident pairs (P, L).

Proposition 4.4.1. *Two pairs (P, L) and $(\widetilde{P}, \widetilde{L})$ are null–separated with respect to the conformal structure (4.31) if there exists a line which contains three points $(P, \widetilde{P}, L \cap \widetilde{L})$.*

Proof. First not that the null condition of Proposition 4.4.1 defines a co–dimension one cone in TN: generically there is no line through three given points. To make explicit the condition for such a line to exist, consider two pairs (P, L) and $(\widetilde{P}, \widetilde{L})$ of non–incident points and lines. By thinking of $L, \widetilde{L}$ as normal vectors to planes in $\mathbb{R}^3$, we see that $L + t\widetilde{L}$ is a plane which intersects L and tL at their intersection, thus defining a line in $\mathbb{RP}^2$ which intersects the lines $L, \widetilde{L} \subset \mathbb{RP}^2$ at their intersection.

If $P, \widetilde{P}, L \cap \widetilde{L}$ are co–linear then there exists t such that both P and $\widetilde{P}$ lie on $L + t\widetilde{L}$, i.e.

$$P \cdot (L + t\widetilde{L}) = 0, \quad \widetilde{P} \cdot (L + t\widetilde{L}) = 0. \tag{4.32}$$

Eliminating t from (4.32) gives

$$(P \cdot L)(\widetilde{P} \cdot \widetilde{L}) - (\widetilde{P} \cdot L)(P \cdot \widetilde{L}) = 0.$$

Setting $\widetilde{P} = P + dP, \widetilde{L} = L + dL$ yields a metric g representing the conformal structure

$$g = \frac{dP \cdot dL}{P \cdot L} - \frac{1}{(P \cdot L)^2}(L \cdot dP)(P \cdot dL).$$

We can use the normalisation $P \cdot L = 1$, so that $P \cdot dL = -L \cdot dP$, and

$$g = dP \cdot dL + (L \cdot dP)^2. \tag{4.33}$$

We take affine coordinates

$$P = [x^{A'}, 1], \quad L = [z_{A'}, 1 - x^{A'} z_{A'}] \tag{4.34}$$

with a normalisation $P \cdot L = 1$ to recover the metric (4.31).

□

4.4.2 Twistor space

aw: As I understand it, this section could be rephrased as a proposition that the conformal structure described above is reproduced by the twistor space F_{12}.

To understand $(M, [g])$ from the twistor perspective, we need to move to the complex picture. In what follows, we will view M as the set of non–incident pairs in $\mathbb{CP}^2 \times \mathbb{CP}^{2*}$.

aw: Is it obvious that this is the right thing to do?

Let $F_{12}(\mathbb{C}^3) \subset \mathbb{CP}^2 \times \mathbb{CP}^{2*}$ be set of incident pairs (p, l), so that $p \cdot l = 0$. Note that, since l and p correspond to planes and lines in $\mathbb{C}^3$ respectively, and since $p \cdot l = 0$ is the condition for the line p lying in the plane l, $F_{12}(\mathbb{C}^3)$ coincides with the flag manifold of type $(1, 2)$ in $\mathbb{C}^3$, i.e. the collection of one– and two–dimensional vector subspaces (p, l) in $\mathbb{C}^3$ such that $p \subset l$. This is the twistor space of (M, g). A $\mathbb{CP}^1$ embedding corresponding to a point $(P, L) \in M$ consists of all lines l thorough P, and all points $p = l \cap L$:

$$P \cdot l = 0, \quad p \cdot L = 0, \quad p \cdot l = 0. \tag{4.35}$$

Let (P, L) and $(\widetilde{P}, \widetilde{L})$ be points in M. These uniquely define a point $p = L \cap \widetilde{L} \in \mathbb{CP}^2$ and line $l \subset \mathbb{CP}^2$ such that $P, \widetilde{P} \in l$ given by

$$p = L \wedge \widetilde{L}, \quad l = P \wedge \widetilde{P},$$

where $[L \wedge \widetilde{L}]^\alpha = \epsilon^{\alpha\beta\gamma} L_\alpha \widetilde{L}_\beta$ etc. The pair (p, l) lies in F_{12} if p lies on l, i.e. if $p \cdot l = 0$, so that (P, L) and $(\widetilde{P}, \widetilde{L})$ are null–separated with respect to the conformal structure (4.32). Then (p, l) is the intersection of the $\mathbb{CP}^1$ embeddings corresponding to (P, L) and $(\widetilde{P}, \widetilde{L})$. The contact structure on F_{12} is $(l \cdot dp - p \cdot dl)/2 = p \cdot dl$.

aw: Contact structure on F_{12}?

We shall now give an explicit parametrisation of twistor lines, and show how the metric (4.33) arises from the Penrose condition [46, 58]. Let $P \in \mathbb{CP}^2$. The corresponding $l \in \mathbb{CP}^{2*}$ is represented by some normal vector which is perpendicular to P in $\mathbb{C}^3$, i.e.

$$l = P \wedge \pi, \quad \text{where} \quad \pi \sim a\pi + bP, \tag{4.36}$$

where $a \in \mathbb{R}^*, b \in \mathbb{R}$.

aw: Should these $\mathbb{R}$s be $\mathbb{C}$s?

Thus π parametrises a projective line $\mathbb{CP}^1$, and by making a choice of b we can take $\pi = [\pi^0, \pi^1, 0]$, where $\pi^A = [\pi^0, \pi^1] \in \mathbb{CP}^1$. The constraint $P \cdot l = 0$ now holds. To satisfy the remaining constraints in (4.35) we take

$$p = L \wedge l = (L \cdot \pi)P - (L \cdot P)\pi. \tag{4.37}$$

Substituting (4.34) gives the corresponding twistor line parametrised by $[\pi] \in \mathbb{CP}^1$

$$p^\alpha = [(z \cdot \pi)x^{A'} - \pi^A, z \cdot \pi], \quad l_\alpha = [\pi_A, -\pi \cdot x], \tag{4.38}$$

where the spinor indices are raised and lowered with ϵ^{AB} and its inverse, and $z \cdot x \equiv z_{A'} x^{A'}$.

We shall now derive the expression for the conformal structure. According to the Nonlinear Graviton prescription of Penrose [46] a vector $V \in \Gamma(T_m M)$ is null if the corresponding section of the normal bundle $N(L_m) = \mathcal{O}(1) \oplus \mathcal{O}(1)$ has a single zero. To compute the normal bundle, let $([l(\pi, P, L)], [p(\pi, P, L)])$ be the twistor line corresponding to a point $m = (P, L)$ in M. The neighbouring line is $([l + \delta l], [p + \delta p])$, where from (4.36) and (4.37) we have

$$\delta l = \delta P \wedge \pi, \quad \delta p = (\delta L \cdot \pi)P + (L \cdot \pi)\delta P - \delta(L \cdot P)\pi.$$

The lines $(l+\delta l, p+\delta p)$ and (l, p) intersect if there exists some $[\pi]$ such that $l+\delta l \sim l$ and $p + \delta p \sim p$ (note that this point, if it exists, is unique, since projective lines in

$\mathbb{RP}^2$ cannot meet more than once). We thus find

$$l + \delta l \sim l \qquad \Longleftrightarrow \qquad \pi \sim \delta P = [\delta x^1, \delta x^2, 0].$$

And $p + \delta p \sim p \qquad \Longleftrightarrow$

$$0 = p \wedge \delta p = (L\cdot\pi)^2 P\wedge\delta P - (L\cdot P)(\delta L\cdot\pi)\pi\wedge P - (L\cdot\pi)\delta(L\cdot P)P\wedge\pi - (L\cdot P)(L\cdot\pi)\pi\wedge\delta P.$$

Substituting $\pi \sim \delta P$, we find that all terms on the RHS are proportional to $P \wedge \delta P = [0, 0, x \cdot dx]$, with

$$(L\cdot\delta P)^2 - (L\cdot\delta P)\delta(L\cdot P) + (L\cdot P)(\delta L\cdot\delta P) = 0.$$

Setting $L \cdot P = 1$ this gives the conformal structure (4.33).

4.4.3 Pseudo–hyper–Hermitian structure

A pseudo–hyper–complex structure on a four manifold M is a triple of endomorphisms I, S, T of TM which satisfy

$$I^2 = -Id, \quad S^2 = T^2 = Id, \quad IST = Id,$$

and such that $aI + bS + cT$ is an integrable complex structure for any point on the hyperboloid $a^2 - b^2 - c^2 = 1$. A neutral signature metric g on a pseudo–hyper–complex four–manifold is pseudo–hyper–Hermitian if

$$g(V, V) = g(IV, IV) = -g(SV, SV) = -g(TV, TV)$$

for any vector field V on M.

Given a pseudo–hyper–complex structure $(M, \{I, S, T\})$ and any vector field V on M, the frame (V, IV, SV, TV) defines a conformal structure on M. With a natural choice of orientation which makes the fundamental two–forms of I, S, T self–dual, this conformal structure is anti–self–dual.

aw: Can we justify this?

Let $\Sigma^{A'B'}$ be a basis of SD two–forms on M. The following result is proved in [26] (see also [9]) in the Riemannian (i.e. hyper–complex) case.

Theorem 4.4.2 ([26])**.** *A four–manifold M equipped with a neutral signature metric g is pseudo–hyper–Hermitian if there exists a one–form A depending only on g such that*

$$d\Sigma^{A'B} + A \wedge \Sigma^{A'B'} = 0.$$

In fact, this condition is necessary and sufficient for hyper–Hermiticity [26, 9]. Given some (M, g) which is conformally ASD, it can also be shown (see Lemma 2 in [26] and Theorem 7.1 in [14]) that a lack of vertical $\partial/\partial\pi$ terms in the twistor distribution (4.13) implies hyper–Hermiticity of (M, g).

aw: This is not clear to me from a brief look at [26, 14]...

Proposition 4.4.3. *The Einstein metric (4.31) on $SL(3)/GL(2)$ is pseudo–hyper–Hermitian.*

Proof. The null frame for the 4-metric is

$$e^{0A'} = dx^{A'}, \quad e^{1A'} = dz^{A'} + z^{A'}(z \cdot dx), \quad \text{so that} \quad g = \epsilon_{A'B'}\epsilon_{AB}e^{AA'}e^{BB'}. \tag{4.39}$$

Thus the forms $\Sigma = dx^{0'} \wedge dx^{1'}$ and $\Omega = dz_{A'} \wedge dx^{A'}$ are ASD. The basis of SD two forms is spanned by

$$dx \wedge dq + q^2 dx \wedge dy, \quad dx \wedge dp - dy \wedge dq + 2pq dx \wedge dy, \quad -dy \wedge dp + p^2 dx \wedge dy$$

or, in a more compact notation, by $\Sigma^{A'B'} = dx^{(A'} \wedge dz^{B')} + z^{A'}z^{B'}\Sigma$. We can verify that

$$d\Sigma^{A'B'} + 2\mathcal{A} \wedge \Sigma^{A'B'} = 0, \tag{4.40}$$

where $\mathcal{A} = z_{A'}dx^{A'}$ is such that $d\mathcal{A} = \Omega$, so from Theorem 4.4.2 we have that M carries a hyper–Hermitian structure, and in fact the corresponding ASD Maxwell field $d\mathcal{A} = \Omega$ coincides with the one arising from the para–Kähler structure on M via (4.15).

Alternatively, note that the twistor distribution (4.13), having chosen the basis (4.39), is given by

$$L_0 = \pi \cdot \frac{\partial}{\partial x} + (z \cdot \pi) z \cdot \frac{\partial}{\partial z}, \quad L_1 = \pi \cdot \frac{\partial}{\partial z}, \tag{4.41}$$

which have no vertical terms. We can easily verify that it is Frobenius integrable, as $[L_0, L_1] = -(\pi \cdot z)L_1$. The SD part of the spin connection is given in terms of $\mathcal{A}$ as $\Gamma_{AA'B'C'} = -2\mathcal{A}_{A(B'}\epsilon_{C')A'}$.

□

In the next section we shall show how to encode $\mathcal{A}$ in the twisted–photon Ward bundle over the twistor space of (M, g).

4.4.4 The twisted photon

The twistor space F_{12} described in §4.4.2 is the projectivised tangent bundle $T(\mathbb{CP}^2)^*$ of the minitwistor space of the flat projective structure: a point in F_{12} consists of $l \in \mathbb{CP}^2$, and a direction through l. Thus the twistor space of M is the correspondence space of $\mathbb{CP}^2$ and $\mathbb{CP}^{2*}$. There are many open sets needed to cover $\mathbb{CP}(T\mathbb{CP}^2)$, but it is sufficient to consider two: U, where $(l_1, \neq 0, p^2 \neq 0)$, and $(l_2/l_1, l_3/l_1, p^3/p^2)$ are coordinates, and $\widetilde{U}$ where $(l_1 \neq 0, p^3 \neq 0)$, and $(l_2/l_1, l_3/l_1, p^2/p^3)$ are coordinates. Now consider the total space of $T\mathbb{CP}^2$ (or perhaps it is $T\mathbb{CP}^2$ tensored with some power of the canonical bundle to make it trivial on twistor lines), and restrict it to the intersection of (pre–images in $T\mathbb{CP}^2$ of) U and $\widetilde{U}$. The coordinates on $T\mathbb{CP}^2$ in these region are $(l_2/l_1, l_3/l_1, p^2/p^1, p^3/p^1)$, and the fiber coordinates over τ over U and $\tilde{\tau}$ over $\widetilde{U}$ are related by[5]

$$\tilde{\tau} = \exp(F)\tau, \quad \text{where} \quad F = \ln(p_2/p_3).$$

Now we follow the procedure of [56]: restrict F to a twistor line, and split it. The holomorphic splitting is $F = H - \widetilde{H}$, where $H = \ln(p_2)$ is holomorphic in the pre–image of U in the correspondence space, and $\widetilde{H} = \ln(p_3)$ is holomorphic in the pre–image of $\widetilde{U}$. Note that F is a twistor function, but $H, \widetilde{H}$ are not. Therefore $L_A F = 0$, where the twistor distribution L_A is given by (4.41). This, together with the Liouville theorem implies that

$$L_A H = L_A \widetilde{H} = \pi^{A'} \mathcal{A}_{AA'}$$

for some one–form $\mathcal{A}$ on M, as the LHS is holomorphic on $\mathbb{CP}^1$ and homogeneous of degree one. To construct this one–form recall the parametrisation of twistor curves (4.38). This gives

$$H = \ln(z \cdot \pi), \quad \widetilde{H} = \ln((z \cdot \pi)x^{1'} - \pi^{1'})$$

and

$$L_1(H) = L_1(\widetilde{H}) = 0, \quad L_0(H) = L_0(\widetilde{H}) = \pi \cdot z.$$

Therefore $\mathcal{A}_{1A'} = 0, \mathcal{A}_{0A'} = z_{A'}$ which gives $\mathcal{A} = z_{A'}dx^{A'}$, and $d\mathcal{A}$ is indeed the ASD para–Kähler structure.

[5] Here we are following Ward [56], and thinking of a $\mathbb{C}^*$ bundle.

4.4.5 Factoring $SL(3)/GL(2)$ to Einstein-Weyl

If a metric with ASD Weyl tensor has more than one conformal symmetry, then distinct Einstein–Weyl structures are obtained on the space of orbits of conformal Killing vectors which are not conjugate with respect to an isometry [45]. We can thus classify the Einstein–Weyl structures obtainable from $SL(3)/GL(2)$ by first classifying its symmetries up to conjugation.

Proposition 4.4.4. *The non–trivial Einstein–Weyl structures obtainable from the ASD Einstein metric (4.31) on $SL(3)/GL(2)$ by the Jones–Tod correspondence consist of a two–parameter family, and two additional cases which do not belong to this family.*

Proof. Since we have an isomorphism between the Lie algebra of projective vector fields on $(N, [\nabla])$ and the Lie algebra of Killing vectors on (M, g), the problem of classifying the symmetries of (4.31) on $M = SL(3)/GL(2)$ is reduced to a classification of the infinitesimal projective symmetries of $\mathbb{RP}^2$, i.e. the near–identity elements of $SL(3)$, up to conjugation.

Non–singular complex matrices are determined up to similarity by their Jordan normal form (JNF). While real matrices do not have such a canonical form, all of the information they contain is determined (up to similarity) by the JNF that they would have if they were considered as complex matrices. Thus we can still discuss the JNF of a real matrix, even if it cannot always be obtained from the real matrix by a real similarity transformation. The possible non–trivial Jordan normal forms of matrices in $SL(3)$ are shown below.

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1/\lambda\mu \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

It is possible that two matrices in $SL(3)$ with the same JNF may be related by a complex similarity transformation, and thus not conjugate in $SL(3)$. However, if the JNF is a real matrix, then the required similarity transformation just consists of the eigenvectors and generalised eigenvectors of the matrix, which must also be real since they are defined by real linear simultaneous equations. This means we only have to worry about matrices with complex eigenvalues, and since these occur in complex conjugate pairs, they will only be a problem when we have three distinct eigenvalues.

In this case, we can always make a real similarity transformation such that the matrix is block diagonal, with the real eigenvalue in the bottom right. Then we have limited choice from the 2×2 matrix in the top left. Let us parametrise such a 2×2 matrix by a, b, c, $d \in \mathbb{R}$ as follows:

$$\begin{pmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix}.$$

This has characteristic polynomial

$$\chi(\lambda) = \lambda^2 - (2 + \epsilon(a+d))\lambda + 1 + (a+d)\epsilon + (ad - bc)\epsilon^2.$$

Evidently the important degrees of freedom are $a + d$ and $ad - bc$, so we can use these to encode every near–identity element of the class with three distinct eigenvalues. The bottom–right entry will be determined by our choice of $a + d$ and $ad - bc$.

Taking a projective vector field on $\mathbb{RP}^2$, we can find the corresponding Killing vector on $SL(3)/GL(2)$ using (1.32), and factor to Einstein–Weyl using (4.17). We find by explicit calculation that vector fields arising from the second and fourth JNFs above give trivial Einstein–Weyl structures, so restricting to the non–trivial cases we have a two–parameter family of Einstein–Weyl structures coming from the first class, and two additional Einstein–Weyl structures coming from the third and fifth, as claimed.

□

aw: Why does the third class not give a one–parameter family?

aw: Give the example corresponding to the mini–twistor factorisation below.

4.4.6 Mini–twistor correspondence

Below we investigate a one–parameter subfamily of the two–parameter family. We use the holomorphic vector field on the twistor space F_{12} (see §4.4.2) corresponding to the chosen symmetry, and reconstruct the conformal structure $[h]$ on N using minitwistor curves (in the sense of [35]) on the space of orbits. Take $a \in \mathbb{R}$ and

$$K = P^1 \frac{\partial}{\partial P^1} - L_1 \frac{\partial}{\partial L_1} + aP^2 \frac{\partial}{\partial P^2} - aL_2 \frac{\partial}{\partial L_2}, \tag{4.42}$$

In order to preserve the relations

$$p \cdot L = 0, \quad P \cdot l = 0, \quad p \cdot l = 0,$$

the corresponding holomorphic action on (p,l) must be $p \mapsto Mp$, $l \mapsto M^{-1}l$, thus the holomorphic vector field $\mathcal{K}$ on F_{12} is

$$\mathcal{K} = p^1\frac{\partial}{\partial p^1} - l_1\frac{\partial}{\partial l_1} + ap^2\frac{\partial}{\partial p^2} - al_2\frac{\partial}{\partial l_2}.$$

In order to factor F_{12} by this vector field, we must find invariant minitwistor coordinates (Q,R). In addition to satisfying $\mathcal{K}(Q) = \mathcal{K}(R) = 0$, they must be homogeneous of degree zero in (P,L). We choose

$$Q = \frac{p^1 l_1}{p^2 l_2}, \quad R = \frac{(l_1)^a}{l_2(l_3)^{a-1}}.$$

Substituting in our parametrisation (4.38) and using the freedom to perform a Mobius transformation on π, we obtain

$$\begin{aligned} Q &= \frac{(\lambda t - u - 1)\lambda}{v\lambda + \lambda - \frac{uv}{t}} \\ R &= \lambda^a\Big(-\lambda - \frac{v}{t}\Big)^{1-a}, \end{aligned} \tag{4.43}$$

where we have defined $\lambda = \pi_{0'}/\pi_{1'}$, and the Einstein-Weyl coordinates

$$u = xp, \quad v = yq, \quad t = x^a q.$$

Note these are invariants of the Killing vector (4.42).

Next we wish to use these minitwistor curves to reconstruct the conformal structure of the Einstein-Weyl space. In doing so we follow [45]. The tangent vector field to a fixed curve is given by

$$T = \frac{\partial Q}{\partial\lambda}\frac{\partial}{\partial Q} + \frac{\partial R}{\partial\lambda}\frac{\partial}{\partial R},$$

Hence we can write the normal vector field as

$$\begin{aligned} N &= dQ\frac{\partial}{\partial Q} + dR\frac{\partial}{\partial R} \mod T \\ &= \left(\frac{\partial R}{\partial\lambda}\right)^{-1}\left(dQ\frac{\partial R}{\partial\lambda} - dR\frac{\partial Q}{\partial\lambda}\right)\frac{\partial}{\partial Q}, \end{aligned}$$

where

$$dQ = \frac{\partial Q}{\partial u}du + \frac{\partial Q}{\partial v}dv + \frac{\partial Q}{\partial t}dt$$

and similarly for dR. Calculating N using (4.43), we find

$$N \propto (A\lambda^2 + B\lambda + C)\frac{\partial}{\partial Q},$$

where

$$\begin{aligned} A &= t^2(v+1)dt - t^3 dv, \\ B &= -2tuvdt + t^2(a+2u)dv - t^2 du, \\ C &= uv(1+u)dt - tu(1+u)dv - atvdu. \end{aligned}$$

The discriminant of this quadratic in λ then gives a representative $h \in [h]$ of our conformal structure:

$$\begin{aligned} h = 4(u^2v + uv^2 + uv)dt^2 - 4tv(a(v+1)+u)dtdu + 4tu(u - av + 2v + 1)dtdv \\ -t^2du^2 + 2t^2(2av + a + 2u)dvdu - t^2(a^2 + 4u(a-1))dv^2. \end{aligned} \tag{4.44}$$

This is the same conformal structure that we obtain by Jones-Tod factorisation of $SL(3)/GL(2)$ by (4.42) using the formula (4.17).

Chapter 5

The ϕ^4 kink on a wormhole spacetime

The soliton resolution conjecture [92] states that solutions to solitonic equations with generic initial data should, after some non–linear behaviour, eventually resolve into a finite number of solitons plus a radiative term. This conjecture is intimately tied to soliton stability, which has been investigated for a number of solitonic equations, including that of ϕ^4 theory on $\mathbb{R}^{1,1}$. In this chapter, we study a modification of this theory on a $3+1$ dimensional wormhole spacetime which has a spherical throat of radius a, with a focus on the stability properties of the modified kink. In particular, we prove that the modified kink is linearly stable, and compare its discrete spectrum to that of the ϕ^4 kink on $\mathbb{R}^{1,1}$. We also study the resonant coupling between the discrete modes and the continuous spectrum for small but non–linear perturbations. Some numerical and analytical evidence for asymptotic stability is presented for the range of a where the kink has exactly one discrete mode. This chapter is almost identical to the preprint [79].

5.1 Introduction: the ϕ^4 kink on $\mathbb{R}^{1,1}$

One dimensional ϕ^4 theory is well–documented in the literature (see for example [87]). The aim of this section is to introduce some notation and some ideas about stability which will be useful when we come to consider the modified theory.

5.1.1 Topological Stability and the Kink Solution

The action takes the form

$$S = \int_{\mathbb{R}} \left(\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}(1-\phi^2)^2 \right) dx,$$

where $x^a = (t, x)$ are coordinates on $\mathbb{R}^{1,1}$ and η^{ab} is the Minkowski metric with signature $(-,+)$. The potential $U(\phi) = (1-\phi^2)^2/2$ is shown in figure 5.1. Note that it has two vacua, given by $\phi = \pm 1$. Finiteness of the associated conserved energy

$$E = \int_{\mathbb{R}} \left(\frac{1}{2}(\phi_t)^2 + \frac{1}{2}(\phi_x)^2 + \frac{1}{2}(1-\phi^2)^2 \right) dx,$$

requires that the field lies in one of these two vacua in the limits $\phi_\pm = \lim_{x\to\pm\infty}[\phi(x)]$. We can thus classify finite energy solutions in terms of their topological charge $N = (\phi_+ - \phi_-)/2$, which takes values in $\{-1, 0, 1\}$.

The equations of motion are

$$\phi_{tt} = \phi_{xx} + 2\phi(1-\phi^2) \tag{5.1}$$

and we find a static solution $\phi = \tanh(x-c)$ which we call the flat kink. It interpolates between the two vacua and thus has topological charge $N = 1$. The constant of integration c can be thought of as the position of the kink. We will henceforth use Φ_0 to denote the static kink at the origin, that is, $\Phi_0(x) = \tanh(x)$. It is evident that no finite energy deformation can affect N. For this reason, we say that the kink is *topologically stable*.

5.1.2 Linear Stability

A second notion of stability which will be important to our discussion is linear stability. On discarding non–linear terms, we find that small pertubations $\phi(t,x) = \Phi_0(x) + \mathrm{e}^{i\omega t}v_0(x)$ satisfy the Schrödinger equation

$$L_0 v_0 := -v_0'' - 2(1-3\Phi_0^2)v_0 = \omega_0^2 v_0. \tag{5.2}$$

The potential $V_0(x) = -2[1-3\Phi_0(x)^2]$ exhibits a so–called "mass gap", meaning that it takes a finite positive value in the limits $x \to \pm\infty$. In this case, $V_0(\pm\infty) = 4$. For $\omega^2 > 4$, (5.2) admits a continuous spectrum of wave–like solutions.

potential.pdf

Fig. 5.1 A plot of the potential $U(\phi)$.

In addition to its continuum states, the Schrödinger operator in (5.2) has two discrete eigenvalues with normalisable solutions given by

$$\Big(v_0(x), \omega_0\Big) = \left(\frac{\sqrt{3}}{2}\mathrm{sech}^2(x), 0\right) \quad \text{and} \quad \Big(v_0(x), \omega_0\Big) = \left(\frac{\sqrt{3}}{\sqrt{2}}\mathrm{sech}(x)\mathrm{tanh}(x), \sqrt{3}\right), \tag{5.3}$$

where we have chosen the normalisation constant such that $\int_{-\infty}^{\infty} v_0^2(x)dx = 1$.

The first of these is the zero mode of the kink. Its existence is guaranteed by the translation invariance of (5.1), and up to a multiplicative constant it is equal to $\Phi_0'(x)$. Excitation of this state corresponds to performing a Lorentz boost. In the non–relativistic limit, this amounts to replacing $\Phi_0(x)$ with $\Phi_0(x - vt)$ for some $v \ll 1$ [87].

The second normalisable solution, called an *internal mode*, has non–zero frequency ω, and is thus time periodic. In the full non–linear theory, it decays through resonant coupling to the continuous spectrum [86]. This phenomenon is of considerable interest in non–linear PDEs, and was studied in a more general setting in [90]. The corresponding process in the modified theory will be discussed in section 5.4.

Linear stability of the kink is equivalent to the Schrödinger operator L_0 in (5.2) having no negative eigenvalues, so that linearised perturbations cannot grow exponentially with time. One way to see that the kink is linearly stable is via the Sturm oscillation theorem:

Theorem 5.1.1 (Sturm)**.** *Let L be a differential operator of the form*

$$L = -\frac{d^2}{dx^2} + V(x)$$

on the smooth square integrable functions u on the interval $[0, \infty)$, with the boundary condition $u(0) = 0$ (corresponding to even parity) or $u'(0) = 0$ (corresponding to odd parity). Let ω^2 be an eigenvalue of L with associated eigenfunction $u(x;\omega)$. Then the number of eigenvalues of L (subject to the appropriate boundary conditions) which are strictly below ω^2 is the number of zeros of $u(x;\omega)$ in $(0, \infty)$.

Note that the symmetry of (5.2) under $x \mapsto -x$ means that any solution on the interval $[0, \infty)$ has a corresponding solution on the interval $(-\infty, 0]$, and these solutions can be pieced together to make a smooth solution on $(-\infty, \infty)$ as long as the boundary conditions at $x = 0$ are chosen to ensure parity ± 1. Thus there is a one–to–one correspondence between solutions on $[0, \infty)$ and solutions on $(-\infty, \infty)$ which are smooth at $x = 0$. Since the eigenfunctions (5.3) have no zeros on the interval $[0, \infty)$, it follows that there can be no eigenfunctions with $\omega^2 < \omega_0^2 = 0$, and thus the kink is linearly stable.

5.1.3 Asymptotic stability

The final notion of stability that we will consider is that of asymptotic stability. Stated simply, asymptotic stability of the kink means that for sufficiently small initial pertubations, solutions of (5.1) will converge locally to $\Phi_0(r)$ or its Lorentz boosted counterpart. This was proved in [85] for odd perturbations, but has not been proved in the general case.

5.1.4 Derrick's Scaling Argument

Generalisation of the finite energy ϕ^4 kink to higher dimensional Minkowski spacetimes is prohibited by a scaling argument due to Derrick. Suppose $\Phi_d(\mathbf{x})$ is a static, finite energy solution to the equation of motion of the ϕ^4 theory on $\mathbb{R}^{1,n}$. Then it is a minimiser of the (static) energy

$$E(\Phi_n) = \int \Big(\nabla\Phi_n(\mathbf{x}) \cdot \nabla\Phi_n(\mathbf{x}) + U(\Phi_n)\Big) d^n x =: E_1 + E_2,$$

where we have split E into the two components coming from the two different terms in the integrand. Now consider a spatial rescaling $\mathbf{x} \to \mu\mathbf{x}$, $\mu > 0$ and define

$$\begin{aligned} e(\mu) = E(\Phi_n(\mu\mathbf{x})) &= \int \Big(\nabla(\Phi_n(\mu\mathbf{x})) \cdot \nabla(\Phi_n(\mu\mathbf{x})) + U(\Phi_n(\mu\mathbf{x}))\Big) d^d x \\ &= \int \Big(\mu^2\nabla\Phi_n(\mu\mathbf{x}) \cdot \nabla\Phi_n(\mu\mathbf{x}) + U(\Phi_n(\mu\mathbf{x}))\Big) d^d x \\ &= \mu^{2-n}E_1 + \mu^{-n}E_2, \end{aligned}$$

where we have obtained the last line by a change of variables from $\mathbf{x}$ to $\mu\mathbf{x}$.

If $\Phi_n(\mathbf{x})$ is a minimiser of E then it $\mu = 1$ must also be a stationary point of $e(\mu)$. Evaluating the derivative yields

$$e'(\mu) = \begin{cases} -n\mu^{-n-1}E_2, & \text{if } n = 2 \\ (2-n)\mu^{1-n}E_1 - n\mu^{-n-1}E_2, & \text{otherwise.} \end{cases}$$

Since E_1, E_2 and μ are all positive, the derivative can only have a zero only when n and $2 - n$ have the same sign, which only happens when $n = 1$. We thus conclude that no static, finite energy solutions to the equations of motion exist for $n > 1$.

In order to construct a higher dimensional ϕ^4 kink, we must add curvature. In the next section we introduce a curved background, and show that a modified ϕ^4 kink exists on this background. We will also examine a limit in which the modified kink reduces to the flat kink. In section 5.3 we consider linearised perturbations around the modified kink, proving that it is linearly stable and comparing its discrete spectrum to that of the flat kink. In section 5.4 we examine the mode of decay to the modified kink in the full non–linear theory, in particular the resonant coupling of its internal modes to the continuous spectrum.

5.2 The static kink on a wormhole

We now replace the flat $\mathbb{R}^{1,1}$ background with a wormhole spacetime (M, g), where

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

for some constant $a > 0$, and $-\infty < r < \infty$. This spacetime was first studied by Ellis [83] and Bronnikov [82], and has featured in a number of recent studies about kinks and their stability [80, 81]. Note the presence of asymptotically flat ends as $r \to \pm\infty$, connected by a spherical throat of radius a at $r = 0$.

Our action is the modified by the presence of a non-flat metric:

$$S = \int \left(\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}(1-\phi^2)^2\right)\sqrt{-g}dx,$$

where x^a are now local coordinates on M. Variation with respect to ϕ gives

$$\Box_g\phi + 2\phi(1-\phi^2) = 0 \tag{5.4}$$

where $\Box_g\phi = \frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\phi)$. We assume ϕ is independent of the angular coordinates (ϑ, φ), so (5.4) can be written explicitly as

$$\phi_{tt} = \phi_{rr} + \frac{2r}{r^2+a^2}\phi_r + 2\phi(1-\phi^2). \tag{5.5}$$

The conserved energy in the theory is given by

$$E = \int_{-\infty}^{+\infty}\left(\frac{1}{2}(\phi_t)^2 + \frac{1}{2}(\phi_r)^2 + \frac{1}{2}(1-\phi^2)^2\right)(r^2+a^2)dr,$$

which we require to be finite. This imposes the condition $\phi^2 \to 1$ as $r \to \pm\infty$, so that the field lies at one of the two vacua at both asymptotically flat ends.

Static solutions $\phi(r)$ satisfy

$$\phi'' + \frac{2r}{r^2+a^2}\phi' = -\frac{d}{d\phi}\left(-\frac{1}{2}(1-\phi^2)^2 \right), \tag{5.6}$$

which, imagining r as a time coordinate, can be thought of as a Newtonian equation of motion for a particle at position ϕ moving in a potential $\mathcal{U}(\phi) = -(1-\phi^2)/2$, with a time dependent friction term.

In addition to the two vacuum solutions, we have a single soliton solution which interpolates between the saddle points at $(-1,0)$ and $(1,0)$ in the in the (ϕ,ϕ') plane. Its existence and uniqueness among odd parity solutions follow from a shooting argument: suppose the particle lies at $\phi = 0$ when $r = 0$. If its velocity $\phi'(0)$ is too small, it will never reach the local maximum of the potential at $\phi = 1$, but if $\phi'(0)$ is too large it will overshoot the maximum so that $\mathcal{U}(\phi) \to -\infty$ as $r \to \infty$, thus having infinite energy. Continuity ensures that there is some critical velocity $\phi'(0)$ such that the particle reaches $\phi = 1$ in infinite time and has zero velocity upon arrival. This corresponds to the non–trivial kink solution, which we call $\Phi(r)$. Time reversal implies that $\phi \to -1$ as $r \to -\infty$, and that the anti–kink $\phi(r) = -\Phi(r)$ is also a solution.

We can find $\Phi(r)$ numerically using a shooting method for the gradient at $r = 0$. method for the gradient at $r = 0$. Figure 5.2 shows such numerically generated kinks for several values of a. Note that the absolute value of $\Phi(r)$ is always greater than or equal to that of the flat kink $\Phi_0(r)$, and that at fixed non–zero r, the absolute value of Φ decreases as a increases. The reason for this will become clear in section 5.3. In section 5.2.1 we examine $\Phi(r)$ in the limit where a is large, finding that it reduces to the flat kink $\Phi_0(r)$, and examining its departure from the flat kink at first order in $1/a^2$.

We again label the values at the boundary as

$$\phi_\pm := \lim_{r\to\pm\infty} \Phi(r) \in \{\pm 1\}.$$

Since no finite energy deformation can change the value of the topological charge $N = (\phi_+ - \phi_-)/2 \in \{-1,0,1\}$, we again conclude that $\Phi(r)$ is topologically stable.

several_kinks.pdf

Fig. 5.2 The kink solution for several values of a, along with the flat kink $\Phi_0(r)$.

5.2.1 Large a limit

As $a \to \infty$, equation (5.5) becomes the standard equation (5.1) for the flat kink. It is thus helpful to expand the modified kink in $\epsilon^2 := 1/a^2$ for small ϵ^2, since we can then solve both (5.5) and (5.9) analytically up to $\mathcal{O}(\epsilon^4)$. We shall denote the static kink by $\Phi_\epsilon(r)$ in this limit. It satisfies

$$\Phi_\epsilon'' + \frac{2r\epsilon^2}{\epsilon^2 r^2 + 1}\Phi_\epsilon' = -2\Phi_\epsilon(1 - \Phi_\epsilon^2). \tag{5.7}$$

Setting $\Phi_\epsilon(r) = \Phi_0(r) + \epsilon^2\Phi_1(r) + \mathcal{O}(\epsilon^4)$ we obtain at order zero the equation (5.1) of a static kink on $\mathbb{R}^{1,1}$. This has solution $\Phi_0(r)$, where we have chosen the kink at the origin to restrict to solutions with odd parity.

At order ϵ^2 we find that $\Phi_1(r)$ must satisfy

$$\Phi_1'' + 2r\mathrm{sech}^2 r = 2\Phi_1(2 - 3\mathrm{sech}^2 r).$$

The unique solution which is odd and decays as $r \to \pm\infty$ is given by

$$\Phi_1(r) = \frac{1}{24}\mathrm{sech}^2 r(f_1(r) + f_2(r) + f_3(r)),$$

where

$$\begin{aligned} f_1(r) &= r\Big[3 - 8\cosh(2r) - \cosh(4r)\Big], \\ f_2(r) &= \sinh(2r)\Big[8\log(2\cosh(r)) - 1\Big] + \sinh(4r)\log(2\cosh(r)), \\ f_3(r) &= \frac{\pi^2}{2} + 6r^2 + 6\mathrm{Li}_2(-\mathrm{e}^{-2r}), \end{aligned}$$

and $\mathrm{Li}_2(z)$ is the dilogarithm function.

To show that $\Phi_1(r)$ is odd, note that $\mathrm{sech}^2 r$ is an even function, and that f_1 and f_2 are constructed from products of even and odd functions, and hence are odd. To see that f_3 is also odd, we use Landen identity for the dilogarithm:

$$\begin{aligned} \mathrm{Li}_2(-\mathrm{e}^{-2r}) + \mathrm{Li}_2(-\mathrm{e}^{2r}) &= -\frac{\pi^2}{6} - \frac{1}{2}\Big[\log(\mathrm{e}^{-2r})\Big]^2 \\ &= -\frac{\pi^2}{6} - 2r^2, \end{aligned}$$

thus verifying $f_3(r) + f_3(-r) = 0$.

We now turn to the behaviour of $\Phi_1(r)$ as $r \to \infty$. Since $\mathrm{sech}^2 r \sim 4\mathrm{e}^{-2r}$ for large r, we need only consider terms in the $\{f_i\}$ of order e^{2r} or higher. We first note that

$$\begin{aligned}\log(2\cosh r) &= \log(\mathrm{e}^r(1+\mathrm{e}^{-2r})) = r + \log(1+\mathrm{e}^{-2r}) \\ &= r + \mathrm{e}^{-2r} + \mathcal{O}(\mathrm{e}^{-4r}).\end{aligned}$$

Then

$$\begin{aligned}f_1(r) &= -4r\mathrm{e}^{2r} - \frac{r}{2}\mathrm{e}^{4r} + \mathcal{O}(\mathrm{e}^r) \\ f_2(r) &= \frac{1}{2}\mathrm{e}^{2r}(8r + 8\mathrm{e}^{-2r} - 1) + \frac{1}{2}\mathrm{e}^{4r}(r + \mathrm{e}^{-2r}) + \mathcal{O}(\mathrm{e}^r) \\ &= 4r\mathrm{e}^{2r} + \frac{r}{2}\mathrm{e}^{4r} + \mathcal{O}(\mathrm{e}^r),\end{aligned}$$

so $f_1(r) + f_2(r) = \mathcal{O}(\mathrm{e}^r)$. Since $f_3(r) = \mathcal{O}(r^2)$ for large r, we see that $\Phi_1(r)$ vanishes as $r \to \infty$, as we expect. Note that its vanishing as $r \to -\infty$ then follows using parity. A plot of $\Phi_1(r)$ is shown in figure 5.3.

5.3 Linearised perturbations around the kink

To study the linear stability of the kink, we first plug

$$\phi(t,r) = \Phi(r) + w(t,r) \tag{5.8}$$

into equation (5.5), discarding terms non–linear in w. Imposing the fact that $\Phi(r)$ satifies (5.6), we find

$$w_{tt} = w_{rr} + \frac{2r}{r^2+a^2}w_r + 2w(1-3\Phi^2).$$

For $w(t,r) = \mathrm{e}^{i\omega t}(r^2+a^2)^{-1/2}v(r)$, this becomes a one–dimensional Schrödinger equation

$$Lv := (-\partial_r\partial_r + V(r))v = \omega^2 v, \tag{5.9}$$

where the potential is given by

$$V(r) = \frac{a^2}{(r^2+a^2)^2} - 2(1-3\Phi^2). \tag{5.10}$$

Phi_1.pdf

Fig. 5.3 The order ϵ^2 perturbation to the static kink on $\mathbb{R}^{1,1}$.

Figure 5.4 shows the potential $V(r)$ for several values of a. Note that for large a it has a single well with a minimum at $r = 0$, close to the potential V_0 corresponding to the flat kink. As a decreases, the critical point at $r = 0$ becomes a maximum with minima on either side, creating a double well. We find numerically that this happens at about $a = 0.55$.

Proposition 5.3.1. *The kink solution $\Phi(r)$ is linearly stable.*

Proof. We first decompose the potential $V(r)$ in (5.9) as $V = V_0 + V_1 + V_a$, where

$$V_0 = -2[1 - 3\Phi_0(r)^2], \qquad V_1 = 6[\Phi(r)^2 - \Phi_0(r)^2], \qquad V_a = \frac{a^2}{(r^2 + a^2)^2}.$$

As discussed above, we know that the operator $L_0 = -\partial_r\partial_r + V_0$ has no negative eigenvalues. It then follows that L itself has no negative eigenvalues as long as the functions $V_1(r)$ and $V_a(r)$ are everywhere non–negative.

The latter is obvious; to prove the former we recall that we can think of $\Phi(r)$ and $\Phi_0(r)$ as the trajectories of particles moving in a potential $\mathcal{U}(\phi)$, where r is imagined as the time coordinate. The particle corresponding to $\Phi(r)$ suffers an increased frictional force compared to $\Phi_0(r)$, i.e.

$$\Phi_0'' = -\left.\frac{\partial \mathcal{U}}{\partial \phi}\right|_{\phi=\Phi_0}, \qquad \Phi'' + \frac{2r}{r^2 + a^2}\Phi' = -\left.\frac{\partial \mathcal{U}}{\partial \phi}\right|_{\phi=\Phi}. \tag{5.11}$$

Both Φ and Φ_0 interpolate between the maxima of $\mathcal{U}$ at $\phi = \pm 1$; reaching the minimum ($\phi = 0$) when $r = 0$.

Multiplying the equations (5.11) by Φ_0' and Φ' respectively, then integrating from r to ∞, we have that at every instant of time

$$\frac{1}{2}(\Phi_0')^2 + \mathcal{U}(\Phi_0) = 0, \qquad \frac{1}{2}(\Phi')^2 + \mathcal{U}(\Phi) = \int_r^\infty \frac{2r}{r^2 + a^2}(\Phi')^2 dr. \tag{5.12}$$

These equations are equivalent to conservation of energy for each of the particles. Note that the integral on the RHS is non–negative for $r \geq 0$, and vanishes only at $r = \infty$. In particular, when $r = 0$ we have $\mathcal{U}(\Phi) = \mathcal{U}(\Phi_0) = -1/2$, so $\Phi'(0) > \Phi_0'(0)$. This means $V_1(r)$ is initially increasing from zero.

For $V_1(r)$ to return to zero at some finite $r = r_0$, we would need that $\Phi(r_0) = \Phi_0(r_0)$ at a point where $\Phi'(r_0) \leq \Phi_0'(r_0)$. However, this is made impossible by equations (5.12), since at such a point $\mathcal{U}(\Phi) = \mathcal{U}(\Phi_0)$ and the integral on the RHS is positive. Hence $V_1(r)$ remains non–negative for all $r > 0$, and thus for all r since it is even in r.

several_potentials.pdf

Fig. 5.4 The potential of the 1–dimensional quantum mechanics problem arising from the study of stability of the soliton for values of a between $a = 10$ and $a = 0.3$. In particular, note that those with $a < 1/\sqrt{2}$ are everywhere positive.

□

5.3.1 Finding internal modes numerically

Bound states of the potential (5.10) correspond to internal modes of the kink like the odd solution of (5.2) in (5.3). In contrast, for frequencies greater than $\omega = 2$, solutions to (5.9) are interpreted as radiation. It is possible to search for bound states of (5.10) numerically. The method for this is as follows:

1. For the chosen value of a, generate the soliton $\Phi(r)$ as described above.
2. Calculate the potential $V(r)$.
3. For some initial guess of the eigenvalue ω^2, integrate equation (5.9) numerically, setting $v(0) = 1$ and $v'(0) = 0$ to obtain even bound states and $v(0) = 0, v'(0) = 1$ to obtain odd bound states.
4. Use a bisection method to find the value of ω^2 for which a bound state exists.

This procedure will only be effective within the range of r for which $\Phi(r)$ is calculated.

For large a, the potential has both an even and an odd bound state which look qualitatively similar to the internal modes (5.3) of the ϕ^4 kink on $\mathbb{R}^{1,1}$. The bound states for several values of a can be found in figures 5.5 and 5.6. As a decreases, the eigenvalues ω^2 of the bound states increase, until they disappear into the continuous spectrum ($\omega^2 > 4$). This disappearance will be further discussed in section 5.3.3. Their frequencies are plotted against a in figure 5.7.

aw: At the moment I have not motivated the choice of axis ticks in section 5.4, as I state in the caption of figure 5.7.

5.3.2 Large a limit

We can also perturbatively expand the eigenvalues of the eigenvalue problem (5.9). Consider solutions to (5.5) of the form[1] $\phi_\epsilon(r) = \Phi_\epsilon(r) + e^{i\omega t} v_\epsilon(r)$, where v_ϵ is small. These satisfy

$$v_\epsilon'' + \frac{2r\epsilon^2}{\epsilon^2 r^2 + 1} v_\epsilon' + 2(1 - 3\Phi_\epsilon^2) v_\epsilon = -\omega_\epsilon^2 v_\epsilon. \tag{5.13}$$

[1] Note that in section 5.3 we considered perturbations $v(r)$ which differ from $v_\epsilon(r)$ by a factor of $(r^2 + a^2)^{-1/2}$, since such perturbations are described by a Schrödinger problem. Here it will be simpler to remove this factor; however there is a one–to–one correspondence between $v(r)$ and $v_\epsilon(r)$.

several_even_modes.pdf

Fig. 5.5 Even bound states of the potential $V_0(r)$ and of the potential $V(r)$ for two different values of a.

several_odd_modes.pdf

Fig. 5.6 Odd bound states of the potential $V_0(r)$ and of the potential $V(r)$ for two different values of a.

omega_vs_a.pdf

Fig. 5.7 The frequencies of the internal modes of the kink plotted against the wormhole radius a. The choice of axis ticks will be motivated in section 5.4.

Let $(v_\epsilon, \omega_\epsilon^2)$ be a solution to (5.13) with

$$\omega_\epsilon^2 = \omega_0^2 + \epsilon^2\xi + \mathcal{O}(\epsilon^4) \quad \text{and} \quad v_\epsilon(r) = v_0(r) + \epsilon^2 v_1(r) + \mathcal{O}(\epsilon^4).$$

Our aim will be to find ξ. Substituting into (5.13), at zero order we obtain the equation (5.2) which controls the linear stability analysis of the ϕ^4 kink on $\mathbb{R}^{1,1}$.

The terms of order ϵ^2 in (5.13) give us

$$v_1'' + 2rv_0' + 2(1 - 3\Phi_0^2)v_1 - 12\Phi_0\Phi_1 v_0 = -\omega_0^2 v_1 - \xi v_0. \tag{5.14}$$

We multiply equation (5.14) by v_0, and subtract from this v_1 multiplied by equation (5.2). Integrating the result from $r = -\infty$ to $r = \infty$, we find

$$\int_{-\infty}^{\infty} \Big(v_1'' v_0 - v_0'' v_1\Big) dr + \int_{-\infty}^{\infty} 2rv_0' v_0 dr - 12 \int_{-\infty}^{\infty} \Phi_0\Phi_1 v_0^2 dr = -\xi.$$

In the first term the integrand is a total derivative, and the second term is easily found to be -1 using integration by parts. We thus obtain

$$\xi = 1 + 12 \int_{-\infty}^{\infty} \Phi_0\Phi_1 v_0^2 dr, \tag{5.15}$$

which we can evaluate for each of the solutions (5.3) using the symbolic computation facility in Mathematica. We find $\xi = 2$ in the case of the zero mode and $\xi = \pi^2 - 7$ in the case of the first non–trivial vibrational mode. We can check these values by finding (v, ω) numerically for a range of small values of ϵ and comparing ω^2 to the $\omega_0^2 + \xi\epsilon^2$ predicted here. The corresponding plots are shown in figures 5.8 and 5.9.

5.3.3 Critical values of a

It is interesting to investigate the values of a at which the internal modes disappear into the continuous spectrum. The larger of these, at which the odd internal mode disappears, we shall call a_1. The smaller one, at which the even internal mode disappears, we shall call a_0.

The most convenient method of estimating a_0 and a_1 is based on the Sturm Oscillation Theorem 5.1.1. The points at which the even and odd internal modes disappear into the continuous spectrum are the points at which the zeros of the even and odd eigenfunctions of L with $\omega^2 = 4$ disappear. We can thus examine the number of zeros of the odd eigenfunction with $\omega^2 = 4$ to determine the number of odd bound states with $\omega^2 < 4$. The critical value a_1 which we are searching for can then be found xi_zero_mode.pdf

Fig. 5.8 A comparison of the predicted and numerical calculations for the energy of the zero mode as a function of ϵ^2 for small ϵ. The numerical calculations were executed by finding the even bound states and their energies as described in section 5.3.1.

xi_other_mode.pdf

Fig. 5.9 A comparison of the predicted and numerical calculations for the energy of the odd vibrational mode as a function of ϵ^2 for small ϵ. The numerical calculations were executed by finding the odd bound states and their energies as described in section 5.3.1.

using a bisection method. An equivalent method using even bound states will yield an estimate of a_0.

One problem with this method is that we need the number of zeros in the interval $(0, \infty)$, and the shooting method we use to generate $\Phi(r)$ and $V(r)$ is only accurate up to a finite value of r. Since zeros of the eigenfunction with $\omega^2 = 4$ disappear at $r = \infty$, this limits the accuracy with which we can determine a_0 and a_1.

For the finite integration range which is accessible based on the shooting method, the odd state disappears at $a_1 \approx 0.8$ and the even state disappears at $a_0 \approx 0.3$.

It is well known that the condition

$$I := \int_{-\infty}^{\infty} \mathcal{V}(r) dr < 0 \tag{5.16}$$

is sufficient to ensure that the potential $\mathcal{V}(r)$ has at least one bound state. In fact, the condition $I \leq 0$ is sufficient [88]. However, (5.16) is not a necessary condition: there are potentials which have at least one bound state where (5.16) is not satisfied. It is interesting to investigate the disappearance of our ground state in this context.

Note that $\mathcal{V}(r)$ must go to zero as $r \to \pm\infty$ to ensure that the integral converges, meaning that the relevant choice for us is $\mathcal{V}(r) = V(r) - 4$. We then examine the value of this integral for the critical value $a = a_0$ when the ground state disappears. We find that $I \approx 0$ at the critical value of $a_0 \approx 0.3$ given above. We can also search numerically for the value of a at which $I = 0$; this also occurs at around $a_0 \approx 0.3$. Thus our results would be consistent with the conjecture that (5.9) has no bound states for $I > 0$.

5.4 Resonant Coupling of the Internal Modes to the Continuous Spectrum

We now move on to consider time dependent perturbations of the form

$$\phi(t, r) = \Phi(r) + (r^2 + a^2)^{-1/2} w(t, r),$$

where we consider non–linear terms in $w(t, r)$. Substituting into (5.5) we find

$$w_{tt} = -Lw + f(w), \tag{5.17}$$

where we have defined

$$f(w) = -\frac{6w^2\Phi}{\sqrt{a^2 + r^2}} - \frac{2w^3}{a^2 + r^2}, \tag{5.18}$$

suppressing the dependence of f on r to simplify the notation. We will not have much need for the expression for f other than to note that it contains terms which are quadratic and cubic in w.

If a is large enough to allow internal modes, then these can only decay through resonant coupling to the continuous spectrum of L. The analogous process of decay to the ϕ^4 kink on $\mathbb{R}^{1,1}$ was discussed in [86], and the general theory was developed in [90]. In the following sections we investigate this decay in the case of a single internal mode, before comparing our result with numerical data.

5.4.1 Conjectured decay rate in the presence of a single internal mode

In this section we follow the analysis in [81]. Looking at figure 5.7, we note that for $a \in (0.3, 0.8)$ we have

$$\operatorname{spec} L = \{\omega^2\} \cap [m^2, \infty), \qquad \omega^2 < m^2 < 4\omega^2 \tag{5.19}$$

where $m^2 = 4$. As above, we denote the unique normalised eigenfunction of L by v, so that $Lv = \omega^2 v$. We will use $\langle \cdot, \cdot \rangle$ to denote the usual inner product on $\mathbb{R}$.

We decompose the perturbation as

$$w(t, r) = \alpha(t)v(r) + \eta(t, r), \tag{5.20}$$

where $v(r)$ refers to the single even internal mode of the kink and η is a superposition of states from the continuous spectrum of L. Where there is only one internal mode present, its frequency ω always lies in the upper half of the mass gap: $1 < \omega < 2$. This is important because it means that 2ω lies within the continuous spectrum.

We substitute this into (5.17) and project onto and away from the internal mode direction, obtaining the following equations for α and η:

$$\ddot{\alpha} + \omega^2 \alpha = \langle v, f(\alpha v + \eta) \rangle \tag{5.21}$$

$$\ddot{\eta} + L\eta = P^{\perp} f(\alpha v + \eta), \tag{5.22}$$

where $P^{\perp}$ is the projection onto the space of eigenstates of L which are orthogonal to v, given by

$$P^{\perp}\psi = \psi - \langle v, \psi \rangle v. \tag{5.23}$$

These equations have initial conditions $\alpha(0)$ and $\eta(0,r)$ such that

$$\phi(0,r) = \Phi(r) + (r^2+a^2)^{-1/2}(\alpha(0)v(r) + \eta(0,r)), \quad \text{and}$$
$$\dot\phi(0,r) = (r^2+a^2)^{-1/2}(\dot\alpha(0)v(r) + \dot\eta(0,r)).$$

In the following analysis we investigate decay of $\alpha(t)$. Equation (5.21) has a homogeneous solution consisting of oscillations with frequency ω. Since 2ω lies within the continuous spectrum of L, there will be a resonant interaction between the these oscillations and the radiation modes in η with frequencies $\pm 2\omega$, arising from the term of order α^2 in the RHS of (5.22). Thus, to leading order, (5.22) is a driven wave equation with driving frequency 2ω. This resonant part of η will have a back–reaction on α through (5.21), which will result in decay of the internal mode oscillations.

We now define $\alpha_1 = \alpha,\ \omega\alpha_2 = \dot\alpha_1$ so that (5.21) becomes

$$\begin{cases} \dot\alpha_1 = \omega\alpha_2, \\ \dot\alpha_2 = -\omega\alpha_1 + \frac{1}{\omega}\langle v, f(\alpha_1 v + \eta)\rangle, \end{cases}$$

or equivalently

$$\dot A = -i\omega A + \frac{i}{\omega}\left\langle v, f\left(\frac{1}{2}(A+\bar A)v + \eta\right)\right\rangle, \tag{5.24}$$

where $A = \alpha_1 + i\alpha_2$. Next we write $\eta_1 = \eta,\ \eta_2 = \dot\eta_1$, converting (5.22) to

$$\begin{cases} \dot\eta_1 = \eta_2, \\ \dot\eta_2 = -L\eta_1 + P^\perp f\left(\frac{1}{2}(A+\bar A)v + \eta_1\right). \end{cases}$$

We will regard the right hand sides of (5.24) and (5.4.1) as power series in A and η. Terms which we expect to be higher order will not be treated rigorously; for this reason, our analysis will produce only a conjecture about the decay rate. Numerical evidence concerning the conjecture will be discussed in section 5.4.2.

It will be helpful to introduce the notation $\mathcal{O}_p(A,\eta)$ to mean terms of at least order p in $A, \bar A, \eta_1, \eta_2$, so that A^2, η_1^2 and $\bar A\eta_1$ are all examples of terms which are $\mathcal{O}_2(A,\eta)$. Currently, the coupling between (5.24) and (5.4.1) is $\mathcal{O}_2(A,\eta)$. We will write

$$f\left(\frac{1}{2}(A+\bar A)v + \eta_1\right) = \sum_{k+l\geq 2} f_{kl}A^k\bar A^l + \sum_{\substack{k+l\geq 1\\ n\geq 1}} f_{kln}\eta_1 A^k \bar A^l$$

where k, l, n are non–negative, to elucidate the lowest order terms in (5.4.1). Note that f_{kl} and f_{kln} are decaying functions of r defined by (5.18). We can then write

$$P^\perp\left[f\left(\frac{1}{2}(A+\bar{A})v+\eta_1\right)\right] = \sum_{k+l=2} P^\perp[f_{kl}]A^k\bar{A}^l + \sum_{k+l=1} P^\perp[f_{kl1}\eta_1]A^k\bar{A}^l + \mathcal{O}_3(A,\eta).$$

Terms in (5.24) with imaginary coefficients correspond to rotation in the complex plane, and thus to oscillatory behaviour in α. At first order, A oscillates with frequency ω. This is exactly the behaviour expected in the linearised theory discussed in section 5.3. In fact, a priori, all the terms in the power series for $\dot{A}$ have coefficients which are purely imaginary.

The next step in our analysis will be to attempt a change of variable $\eta_i \mapsto \tilde{\eta}_i$ in (5.4.1) so that its right hand side is $\mathcal{O}_3(A,\tilde{\eta})$, meaning $\tilde{\eta}$ is $\mathcal{O}(A^3)$. It will turn out that the required change of variables is complex. The result will be a term in (5.24) which is $\mathcal{O}(A^3)$ and has a real coefficient. This will be the lowest order term with a real coefficient, and thus the key resonant damping term.

We write the change of variables as

$$\eta_1 = \tilde{\eta}_1 + \sum_{k+l=2} b_{kl}A^k\bar{A}^l, \qquad \eta_2 = \tilde{\eta}_2 + \sum_{k+l=2} c_{kl}A^k\bar{A}^l, \tag{5.25}$$

where b_{kl} and c_{kl} are functions of r which are so far undetermined. Differentiating with respect to time and using (5.24), we find

$$\dot{\eta}_1 = \dot{\tilde{\eta}}_1 - i\omega \sum_{k+l=2} b_{kl}(k-l)A^k\bar{A}^l + \mathcal{O}_3(A,\tilde{\eta}),$$

$$\dot{\eta}_2 = \dot{\tilde{\eta}}_2 - i\omega \sum_{k+l=2} c_{kl}(k-l)A^k\bar{A}^l + \mathcal{O}_3(A,\tilde{\eta}).$$

We equate these to the right hand sides of (5.4.1), substituting from (5.25) and requiring that

$$\dot{\tilde{\eta}}_1 = \tilde{\eta}_2 + \mathcal{O}_3(A,\tilde{\eta}), \qquad \dot{\tilde{\eta}}_2 = -L\tilde{\eta}_1 + \mathcal{O}_3(A,\tilde{\eta}). \tag{5.26}$$

This yields

$$-i\omega b_{kl}(k-l) = c_{kl} \qquad \text{and} \qquad -i\omega c_{kl}(k-l) = -Lb_{kl} + P^\perp[f_{kl}]$$

for $k+l=2$, where we have discarded

$$\begin{aligned}\sum_{k+l=1} P^{\perp}[f_{kl1}\eta_1]A^k\bar{A}^l &= \sum_{k+l=1} P^{\perp}[f_{kl1}\tilde{\eta}_1]A^k\bar{A}^l + \sum_{\substack{k+l=1\\p+q=2}} P^{\perp}[f_{kl1}b_{pq}]A^{k+p}\bar{A}^{l+q}\\ &= \mathcal{O}_3(A)\end{aligned}$$

because $\tilde{\eta}$ is at least third order in A.

The change of variables (5.25) is now given by the solution to

$$\Big(L-\omega^2(k-l)^2\Big)b_{kl} = P^{\perp}[f_{kl}]. \tag{5.27}$$

Because of the spectrum of L given in (5.19), for $(k,l)\in(2,0)\cup(0,2)$ the solution b_{kl} is in general a complex function of r, whilst for $k=l=1$ the solution is real and decaying. The reason for this can be understood using the variation of parameters method for inhomogeneous ODEs.

Let $g(r)$ be such that $\langle g,g\rangle$ is finite, and $\lambda\geq 0$ a constant. The general solution of

$$(L-\lambda^2)b(r) = g(r)$$

is given by

$$b(r) = Z_2(r)\int_{-\infty}^{r}\frac{1}{W(r')}Z_1(r')g(r')dr' + Z_1(r)\int_{r}^{\infty}\frac{1}{W(r')}Z_2(r')g(r')dr',$$

where $\{Z_1,Z_2\}$ is a basis for solutions to the homogeneous equation with Wronskian $W(r)=Z_1Z_2'-Z_2Z_1'$. The basis must be chosen so that the above integrals converge.

For $k=l=1$, so that $\lambda^2=0$ and hence $\lambda^2<m^2$, we can choose a basis such that $W=1$ and Z_1,Z_2 are both real, and they decay to zero in the limits $r\to-\infty$ and $r\to\infty$ respectively. Then

$$b_{11}(r) = Z_2(r)\int_{-\infty}^{r} Z_1(r')P^{\perp}[f_{11}](r')dr' + Z_1(r)\int_{r}^{\infty} Z_2(r')P^{\perp}[f_{11}](r')dr'. \tag{5.28}$$

For $\lambda^2\geq m^2$, we cannot choose a real solution in general. In the case $(k,l)\in(2,0)\cup(0,2)$, we take as a basis the Jost functions $\{j_\pm\}$, defined by

$$j_\pm(r)\sim e^{\pm i\xi r} \text{ as } r\to\infty,$$

where $\xi = \sqrt{4\omega^2 - m^2}$. Their Wronskian is then $W(j_+, j_-) = -2i\xi$, and we write the solution as

$$b_{02}(r) = b_{20}(r) = \frac{ij_-(r)}{2\xi}\int_{-\infty}^{r} j_+(r')P^\perp[f_{20}](r')dr' + \frac{ij_+(r)}{2\xi}\int_r^\infty j_-(r')P^\perp[f_{20}](r')dr'. \tag{5.29}$$

Finally, we use (5.28) and (5.29) to change variable $\eta_i \mapsto \tilde{\eta}_i$ in (5.24), obtaining

$$\dot{A} = -i\omega A + \frac{i}{\omega}\left(\sum_{2\le k+l\le 3}\langle v, f_{kl}\rangle A^k\bar{A}^l + \sum_{\substack{k+l=1\\ p+q=2}}\langle v, f_{kl1}b_{pq}\rangle A^{k+p}\bar{A}^{l+q} + \mathcal{O}(A^4)\right), \tag{5.30}$$

where we have ignored terms containing $\tilde{\eta}_1$ since these are at least fourth order in A. We can now see that, of the terms which we have written explicitly, the only ones that can give a real contribution to $\dot{A}$ are those containing b_{02} and b_{20}. We thus find

$$\begin{aligned}
\frac{d}{dt}|A|^2 &= \dot{A}\bar{A} + A\dot{\bar{A}} = 2\mathbb{R}\mathrm{e}[\dot{A}\bar{A}] \\
&= \frac{2}{\omega}\mathbb{R}\mathrm{e}\left[i\sum_{k+l=1}\left(\langle v, f_{kl1}b_{20}\rangle A^{k+2}\bar{A}^{l+1} + \langle v, f_{kl1}b_{02}\rangle A^k\bar{A}^{l+3}\right)\right] + \mathcal{O}(A^5) \\
&= -\frac{2}{\omega}\mathbb{I}\mathrm{m}\left[\langle v, f_{101}b_{20}\rangle(A^3\bar{A} + A^2\bar{A}^2 + A\bar{A}^3 + \bar{A}^4)\right] + \mathcal{O}(A^5).
\end{aligned}$$

In particular, the term $A^2\bar{A}^2 = |A|^4$ is real and non–oscillating, giving a contribution

$$\frac{d}{dt}|A|^2 \sim -\frac{2}{\omega}\mathbb{I}\mathrm{m}\Big[\langle v, f_{101}b_{20}\rangle\Big]|A|^4.$$

The terms $A^3\bar{A}$, $A\bar{A}^3$ and $\bar{A}^4$, on the other hand, would be expected to oscillate at frequencies 2ω and 4ω at first order, and thus time average to zero.

Hence we conclude

$$|A| \sim \left(\Gamma t + \frac{1}{|A(0)|^2}\right)^{-1/2}, \qquad \Gamma := \frac{2}{\omega}\mathbb{I}\mathrm{m}\Big[\langle v, f_{101}b_{20}\rangle\Big]. \tag{5.31}$$

The constant Γ is a function of a which can be calculated explicitly. Using (5.18) and (5.29) gives

$$\langle v, f_{101}b_{20}\rangle = \int_{-\infty}^{\infty} dr\ v(r)f_{101}(r)\Bigg(\frac{ij_-(r)}{2\xi}\int_{-\infty}^{r} j_+(r')P^{\perp}[f_{20}](r')dr' + \frac{ij_+(r)}{2\xi}\int_{r}^{\infty} j_-(r')P^{\perp}[f_{20}](r')dr'\Bigg).$$

We now use the facts that $f_{20} = vf_{101}/4$, and $P^{\perp}[f_{20}] = f_{20} - \langle v, f_{20}\rangle v$. Note that f_{20} is an odd function of r, so in fact $\langle v, f_{20}\rangle = 0$ and so $P^{\perp}[f_{20}] = f_{20}$. We thus obtain

$$\langle v, f_{101}b_{20}\rangle = \frac{2i}{\xi}\Bigg(\int_{-\infty}^{\infty} dr\int_{-\infty}^{r} dr' f_{20}(r)j_-(r)f_{20}(r')j_+(r') + \int_{-\infty}^{\infty} dr\int_{r}^{\infty} dr' f_{20}(r)j_+(r)f_{20}(r')j_-(r')\Bigg).$$

The two double integrals are integrals over complementary halves of the (r, r') plane, and thus sum to a single integral over the full plane. Hence

$$\langle v, f_{101}b_{20}\rangle = \frac{2i}{\xi}\int_{-\infty}^{\infty} f_{20}(r)j_+(r)dr\int_{-\infty}^{\infty} f_{20}(r')j_-(r')dr' = \frac{2i}{\xi}|\langle f_{20}, j_+\rangle|^2,$$

since $j_\pm$ are complex conjugates.

Combining this with (5.31) gives

$$\Gamma = \frac{4}{\omega\xi}|\langle f_{20}, j_+\rangle|^2.$$

The so–called Fermi Golden Rule then reads

$$|\langle f_{20}, j_+\rangle| \neq 0.$$

5.4.2 Numerical investigation of the conjectured decay rate

In order to integrate the PDE (5.5) to large times t, we employ the method of hyperboloidal foliations and scri–fixing [95]. Following [80, 81], we define

$$s = \frac{t}{a} - \sqrt{\frac{r^2}{a^2} + 1}, \qquad y = \arctan\Big(\frac{r}{a}\Big),$$

ic_comparison_0002.pdf

Fig. 5.10 The decay of internal mode oscillations for various initial conditions when $a = 0.5$. Note that $\phi(0, s)$ is used as a proxy for the internal mode amplitude, and we use a log–log scale to elucidate the dependence on $s^{-1/2}$ in the large s limit. The lines are labelled in the legend by the initial conditions which produced them, with the exception of the gradient line $4.2s^{-1/2}$.

resulting in a hyperbolic equation

$$\partial_s\partial_s F + 2\sin(y)\partial_y\partial_s F + \frac{1+\sin^2(y)}{\cos(y)}\partial_s F = \cos^2(y)\partial_y\partial_y F + 2a^2\frac{F(1-F^2)}{\cos^2(y)}. \qquad (5.32)$$

for $F(s,y) = \phi(t,r)$.

We solve the corresponding initial value problem at space–like hypersurfaces of constant s, specifying $\phi(s=0,y)$ and $\partial_s\phi(s=0,y)$. No boundary conditions are required, since the principal symbol of (5.32) degenerates to $\partial_s(\partial_s \pm 2\partial_y)$ as $y \to \pm\pi/2$, so there are no ingoing characteristics. This reflects the fact that no information comes in from future null infinity.

Following [80, 96] we define the auxiliary variables

$$\Psi = \partial_y F, \qquad \Pi = \partial_s F + \sin y \partial_y F$$

to obtain the first order symmetric hyperbolic system

$$\partial_s F = \Pi - \Psi \sin y \qquad (5.33)$$

$$\partial_s \Psi = \partial_y(\Pi - \Psi \sin y) \qquad (5.34)$$

$$\partial_s \Pi = \partial_y(\Psi - \Pi \sin y) + 2\tan y(\Psi - \Pi\sin y) + 2a^2\frac{F(1-F^2)}{\cos^2 y}, \qquad (5.35)$$

which we solve numerically using the method of lines. Kreiss–Oliger dissipation is required to reduce unphysical high–frequency noise. We also add the term $-0.1(\Psi - \partial_y F)$ to the right hand side of equation (5.34) to suppress violation of the constraint $\Psi = \partial_y F$.

We are interested in the range of values $a_0 < a < a_1$ for which the kink has exactly one internal mode. We find that, for fixed but arbitrary y, $F(s,y)$ oscillates in s with a frequency close to the internal mode frequency, and that these oscillations tend towards a decay rate of $s^{-1/2}$, as we expect from section 5.4.1. Plots demonstrating this decay at $y = 0$ for $a = 0.5$ are shown in figure 5.10. Note that the constant 4.2 is related to Γ as defined in (5.31).

5.5 Summary and Discussion

We have found that the modified kink is topologically and linearly stable, and investigated its asymptotic stability for the range of a where exactly one discrete mode is present. It would be interesting to expand the investigation in section 5.4 to the case when both discrete modes are present. This problem is much more complicated because of the extra terms in (5.22) and (5.21) coming from the amplitude of the second internal mode. Similar problems have been discussed in [93], although no such analysis has been done for non–linear Klein–Gordon equation of this type with two discrete modes. The ϕ^4 theory on the wormhole presents a useful setting to undertake such analysis because the kink has exactly two discrete modes for any $a > a_1$, and because their frequencies can be controlled by the parameter a.

This model shares an interesting property with its sine–Gordon counterpart in that we expect a discontinuous change in decay behaviour when a moves out of the range $a_0 < a < a_1$. Insight from the ϕ^4 case may help to elucidate the character of such discontinuous changes.

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