A class of almost para–Kähler Einstein manifolds and a kink on a wormhole spacetime

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Abstract

The first part of this thesis is concerned with recent work by Dunajski and Mettler. They show that a class of almost para–Kähler manifolds M can be canonically constructed as rank n affine bundles over projective structures in dimension n. These have the same symmetry group as the underlying projective manifold, and the associated metrics have neutral signature and satisfy the vacuum Einstein equations with non–zero scalar curvature.

We show that every metric within the class is a Kaluza–Klein reduction of an Einstein metric on an $\mathbb{R}^*$ bundle over M. We also show that the structures are para–c–projectively compact in the sense of Čap–Gover, and intepret the compactification in terms of the tractor bundle of the projective structure.

In dimension four, the manifolds M have anti–self–dual conformal curvature, and are thus associated with a twistor space. In the presence of a symmetry, they can be reduced to Einstein–Weyl structures in dimension three via the Jones–Tod correspondence. Because M is also Einstein with non–zero scalar curvature, these Einstein–Weyl structures are determined by solutions of the $SU(\infty)$–Toda equation.

We classify the Einstein–Weyl structures which can be obtained in this way in terms of the symmetry group of the underlying projective structure. Several examples are considered in detail, resulting in new, explicit solutions of the $SU(\infty)$–Toda equation. We focus in particular on the case where the projective structure is $\mathbb{RP}^n$, additionally describing the Jones–Tod reduction from the twistor perspective.

Finally, we study ϕ^4 field theory on a wormhole spacetime in $3+1$ dimensions. This spacetime has two asymptotically flat ends connected by a spherical throat of radius a. We show that the theory possesses a kink solution which is linearly stable, and compare its discrete spectrum to that of the ϕ^4 kink on $\mathbb{R}^{1,1}$. We present some preliminary results on the non–linear resonant coupling between the discrete and continuous spectra in the range of a where there is exactly one discrete mode.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

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And I would like to acknowledge ...

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Chapter 1

Introduction

This thesis splits naturally into two parts. The first 4 chapters are concerned with a class of Einstein manifolds which can be canonically constructed from projective structures as shown in recent work by Dunajski and Mettler [16]. Chapter 5 discusses a solitonic solution of the equations of motion associated with ϕ^4–field theory on a wormhole spacetime. Chapter 5 is self–contained, and possesses its own introduction. The remainder of this chapter will introduce some preliminaries for the first part of the thesis, culminating in the results of [16] on the almost para–Kähler Einstein manifolds which will form the basis of chapters 2–4.

Definition 1.0.1 *A projective structure* $(N, [\nabla])$ *on a manifold* N *is an equivalence class* $[\nabla]$ *of torsion–free affine connections on* N *which have the same unparametrised geodesics. For two sets of connection components* Γ^i_{jk} *to represent connections in the same projective class, they must be related by*

$$\delta\Gamma^i_{jk} = \delta^i_j \Upsilon_k + \delta^i_k \Upsilon_j \tag{1.1}$$

for some 1-form Υ.

Given a projective structure on a manifold N of dimension n, Dunajski and Mettler [16] construct a neutral signature Einstein metric g with non-zero scalar curvature on a certain rank n affine bundle $M \to N$. The $2n$–dimensional space M also carries a natural symplectic form Ω, and an endomorphism $J : TM \to TM$ which is such that J^2 is the identity and $g(\cdot\ ,\cdot) = \Omega(\cdot\ , J\cdot)$. This makes (M, g, Ω) a so–called *almost para–Kähler* structure. It is interesting for a number of different reasons.

Firstly, (M, g) is interesting by virtue of being an Einstein space. In fact, it turns out that g arises as the Kaluza–Klein reduction of an Einstein metric $\mathcal{G}$ on an $\mathbb{R}^*$

bundle $\sigma : \mathcal{Q} \to M$ which has curvature form $\sigma^*(\Omega)$. In chapter 2 we will construct $\mathcal{G}$ explicitly, and give an interpretation of the manifold $(\mathcal{Q}, \mathcal{G})$ in terms of the projective geometry on N.

It also turns out that (M, g, Ω) can be thought of as compactifiable in a certain sense. Recall that a (psuedo–)Riemannian manifold (M, g) is said to be *conformally compact* if there is a smooth positive function T such that $T^2 g$ smoothly extends to a manifold with boundary $\overline{M} = M \cup \partial M$, and the set $\{p \in \overline{M} : T(p) = 0\}$ is a hypersurface which coincides with the boundary ∂M. This is a useful concept because $(M, T^2 g)$ has the same conformal structure, and hence the same *causual* structure, as (M, g). It has been used to study said causual structure in both general relativity [64] and quantum field theory [66]. It is also useful for formulating the boundary conditions of conformally invariant field equations such as those arising in Yang–Mills theory [65].

A mathematical summary of conformal compactification is that some weakening of the (psuedo–)Riemannian geometry on M extends to a manifold with boundary $\overline{M}$. Recent work by Čap and Gover [53, 54] has generalised this idea to other geometrical structures which admit some weakening which extends to a manifold with boundary. In particular, on an almost complex manifold (M, J) with complex connection ∇, one can define the c–projective equivalence class $[\nabla]$ to which ∇ belongs, and show that the c–projective structure $(M, J, [\nabla])$ extends to a manifold with boundary $\overline{M}$ [54]. The main goal of chapter 3 is to adapt the work of [54] to the para–c–projective case, and to show that the almost complex structure J on M has a complex connection which admits a so–called para–c–projective compactification. The result of this is that the manifolds (M, g, Ω) can be thought of as para–c–projectively compact.

Another reason this construction is interesting is that for $n = 2$ (so that M has dimension 4), the conformal curvature of g is anti-self-dual. Recall that the Hodge operator $\star$ defined by a Euclidean or neutral signature metric in four dimensions is an involution on two–forms (i.e. squares to the identity). It thus has eigenvalues ± 1, and the space of two-forms splits into the corresponding eigenspaces, which are referred to as self–dual (SD) or anti–self–dual (ASD) respectively. Due to its index symmetries, the Weyl tensor can be thought of as a map from two-forms to two-forms, and therefore has a corresponding decomposition. Since the Weyl tensor encodes the conformal curvature, we say that a conformal or (psuedo–)Riemannian manifold whose Weyl tensor is ASD is equipped with an *ASD conformal structure*.

The field equations corresponding to anti-self-duality of the Weyl tensor in four dimensions can be solved by a twistor construction, and are thus *integrable* [46]. This means that any systems of differential equations which can be obtained from them by symmetry reduction should also be integrable (see [33] for a review). In particular, the class of dispersionless integrable systems in 2+1 and 3 dimensions arise in this way. The construction [16] provides some examples of ASD conformal structures in neutral signature which, in the presence of a (non–null) symmetry, give rise to solutions of an integrable system called the $SU(\infty)$–Toda field equation via $2+1$–dimensional Einstein-Weyl structures. In chapter 4 we discuss the extraction of all possible Toda solutions obtainable in this way.

1.1 The Cartan Geometry of a Projective Structure

One way of understanding the construction of (M, g) in [16] is via the Cartan bundle of the projective structure $(N, [\nabla])$. By virtue of being modelled on $\mathbb{RP}^n$, which can be viewed as a homogeneous space, projective structures admit a description as Cartan geometries. These generalise Klein's Erlangen programme, a study of homogeneous spaces G/H, to the curved case, in which the total space G is replaced by a principal right H-bundle over a manifold N such that the tangent space to N is isomorphic to the Lie algebra quotient $\mathfrak{g}/\mathfrak{h}$. In the Riemannian case, this corresponds to generalising $\mathbb{R}^n \cong \mathrm{Euc}(n)/SO(n)$ to a general, curved Riemannian manifold, whose orthonormal frame bundle is a principal $SO(n)$ bundle and whose tangent spaces are modelled on $\mathbb{R}^n \cong \mathfrak{Euc}(n)/\mathfrak{so}(n)$.

The theory of Cartan geometries was developed as part of Cartan's *method of moving frames.* In Riemannian geometry, one has an obvious subclass of frames which are "adapted" to the metric, i.e. those which are orthonormal. The idea of Cartan's method is to pick out some adapted frames for manifolds equipped with some non-metric structure. The bundle of such frames over a manifold is then a principal bundle $\pi : P \to S$ with structure group H.

In addition, P is equipped with a $\mathfrak{g}$-valued one-form θ called the Cartan connection. It satisfies a number of properties, in particular equivariance, i.e. $R_h^*\theta = \mathrm{Ad}(h^{-1})\theta$ for all $h \in H$. It also defines an isomorphism $\theta : T_pP \to \mathfrak{g}$ every point $p \in P$ such that the vertical subspace $V_pP \subset T_pP$ is mapped to $\mathfrak{h}$ and the horizontal subpace $H_pP \subset T_pP$ is defined as the inverse image of $\mathfrak{g}/\mathfrak{h}$. Note that it is not a connection in the usual sense of a principal bundle connection, since it takes value in a Lie algebra larger than that of the structure group. Further details can be found in [**?**].

In the case of a projective surface, the model is $\mathbb{RP}^n \cong SL(n+1, \mathbb{R})/H$, where $SL(n+1, \mathbb{R})$ acts via the fundamental representation on homogenous coordinates

$[X_1, X_2, \ldots, X_n]$ in $\mathbb{RP}^n$, and H is the stabiliser subgroup of the point $[1, 0, \ldots, 0]$, whose elements are matrices of the general form

$$\begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}$$

for some $a \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}_n$. We say that the Cartan geometry of a projective surface is of type $(SL(n+1, \mathbb{R}), H)$. We call it $(\pi : P \to N, \theta)$, where θ is the Cartan connection and takes values in $\mathfrak{sl}(n+1, \mathbb{R})$. It can be written as a matrix

$$\theta = \begin{pmatrix} -\mathrm{tr}\phi & \eta \\ \omega & \phi \end{pmatrix},$$

where ω, η and ϕ are one-forms valued in $\mathbb{R}^n$, $\mathbb{R}_n$ and $\mathfrak{gl}(n, \mathbb{R})$ respectively. We will refer to the components of ω and η with respect to the natural basis of $\mathfrak{sl}(n+1, \mathbb{R})$ as $\{\omega^{(i)}\}$ and $\{\eta_{(i)}\}$, so that $\omega^{(i)}$ and $\eta_{(i)}$ are both one-forms. We also note for later use that the $\mathfrak{sl}(n+1, \mathbb{R})$-valued curvature two-form Θ satisfies

$$\Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & L(\omega \wedge \omega) \\ 0 & W(\omega \wedge \omega) \end{pmatrix},$$

where L and W are smooth curvature functions.

1.2 The Dunajski–Mettler Construction $(M, g_\Lambda, \Omega_\Lambda)$

We can think of M as a quotient of the total space P of the Cartan geometry by $GL(n, \mathbb{R})$, which is embedded in H in the obvious way:

$$GL(n, \mathbb{R}) \ni a \longmapsto \begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix} \in H.$$

It is easily verified that

$$\begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & \eta \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \eta a^{-1} \det a^{-1} \\ (\det a) a\omega & 0 \end{pmatrix},$$

for any $a \in GL(n, \mathbb{R})$, meaning that due to the equivarience property of the Cartan connection, the natural contraction $\eta\omega := \sum_i \eta_{(i)} \otimes \omega^{(i)}$ defined by θ is presevered by the adjoint action of this $GL(n,\mathbb{R})$ subgroup. It thus descends to a naturally defined object on the quotient $M = P/GL(n,\mathbb{R})$.

Theorem 1.2.1 *[16] There exist a metric and two-form (g,Ω) on $M = P/GL(n,\mathbb{R})$ such that the quotient map $q : P \to M$ gives*

$$\begin{aligned} q^* g &= \mathrm{Sym}(\eta\omega) \\ q^*\Omega &= \mathrm{Ant}(\eta\omega), \end{aligned}$$

where Sym *and* Ant *denote the symmetric and anti-symmetric parts of the* $(0,2)$ *tensor* $\eta\omega$*. Moreover,* Ω *is closed as a consequence of the Bianchi identity,* g *is Einstein with non-zero scalar curvature, and the two are related by an endomorphism* J *satisying* $J^2 = \mathrm{id}$*. Hence* (g,Ω) *is an almost para-Kahler structure on* M*.*

Note that the full proof of theorem 1.2.1 only appears explicitly in [16] in the case $n = 2$, although it can be generalised to $n > 2$. This generalisation is discussed in their appendix.

The quotient M turns out to be an affine bundle over N with structure group H, i.e. H acts affinely on the fibers of $\rho : M \to N$, and sections of this bundle are in one-to-one correspondence with representative connections $\nabla \in [\nabla]$. This means that given some choice of connection $\nabla \in [\nabla]$ we have a diffeomorphism $\varphi : T^*N \to M$ with which we can pull back the pair (g,Ω). In canonical local coordinates (x^i, p_i) on the cotangent bundle, we find

$$\begin{aligned} \varphi^* g &= dp_i \odot dx^i - (\Gamma^k_{ij} p_k - p_i p_j - \mathrm{P}_{ij}) dx^i \odot dx^j, \\ \varphi^*\Omega &= dp_i \wedge dx^i + \mathrm{P}_{ij} dx^i \wedge dx^j, \qquad i,j = 1,\dots,n. \end{aligned} \tag{1.2}$$

Here Γ^i_{jk} are the connection components of the representative connection ∇ that we chose, and its Schouten tensor[1] is denoted P_{ij}. This can be shown to be projectively invariant in the sense that a different choice of $\nabla \in [\nabla]$ corresponds to shifting the fiber coordinates p_i, i.e. metrics on T^*N resulting from pulling back g using different representative connections are isometric. Explicitly, a projective transformation (1.1) corresponds to a change

$$\mathrm{P}_{ij} \longrightarrow \mathrm{P}_{ij} + \Upsilon_i \Upsilon_j - \nabla_i \Upsilon_j, \qquad p_i \longrightarrow p_i + \Upsilon_i. \tag{1.3}$$

[1] Recall that the Schouten tensor is given in terms of the Ricci tensor R_{ij} by $P_{ij} = \frac{1}{n-1} R_{(ij)} + \frac{1}{n+1} R_{[ij]}$ where n is the dimension of the manifold.

In fact, the metric and symplectic form (1.2) turn out to belong to a one-parameter family $\{g_\Lambda\}$, which can be written in local coordinates as

$$g_\Lambda = dp_i \odot dx^i - (\Gamma^k_{ij}p_k - \Lambda p_i p_j - \Lambda^{-1}\mathrm{P}_{ij})dx^i \odot dx^j \tag{1.4}$$

$$\Omega_\Lambda = dp_i \wedge dx^i + \frac{1}{\Lambda}\mathrm{P}_{ij}dx^i \wedge dx^j, \qquad i,j = 1,\ldots,n. \tag{1.5}$$

They are all Einstein and, for $n = 2$, all ASD, but for $\Lambda \neq 1$ the relation to projective geometry is lost. For the remainder of the thesis we will write g for $g_{\Lambda=1}$ unless stated otherwise. Note that $\{g_\Lambda\}$ will be the subject of chapter 2, whilst in chapters 3 and 4 we will restrict our attention to g because the projective geometry is a key aspect of the content of these chapters.

1.2.1 Symmetries of $(M, g_\Lambda, \Omega_\Lambda)$

Recall that a projective vector field on any manifold with a connection generates a 1-parameter family of transformations which preserve the geodesics of that connection up to parametrisation. Projective vectors fields thus naturally arise as the symmetries of a projective structure. Explicitly, a vector field K is projective if it satisfies

$$\mathcal{L}_K\Gamma^k_{ij} = \delta^k_i\Upsilon_j + \delta^k_j\Upsilon_i \tag{1.6}$$

for some 1-form Υ, where Γ^k_{ij} are the connection components, and their Lie derivative is defined (see [?]) by

$$\mathcal{L}_K\Gamma^k_{ij} \equiv \frac{\partial^2 K^k}{\partial x^i \partial x^j} + K^m\frac{\partial\Gamma^k_{ij}}{\partial x^m} - \Gamma^m_{ij}\frac{\partial K^k}{\partial x^m} + \Gamma^k_{im}\frac{\partial K^m}{\partial x^j} + \Gamma^k_{mj}\frac{\partial K^m}{\partial x^i}. \tag{1.7}$$

One consequence of the above theorem is that for every open set $\mathcal{U} \subset N$ we have an isomorphism between the Lie algebra of projective vector fields on $\mathcal{U}$ and the Lie algebra of vector fields on $\rho^{-1}(\mathcal{U})$ which preserve both g and Ω. This follows from a standard result about Cartan geometries: projective vector fields on $\mathcal{U}$ are in one-to-one correspondence with vector fields on $\pi^{-1}(\mathcal{U})$ which preserve θ and are equivariant under the principal H-action. Such vector fields thus preserve the natural contraction $\eta\omega$ and must descend to vector fields on M preserving (g, Ω). In fact, it can be shown that every Killing vector field of (M, g) is also symplectic with respect to Ω and is therefore the lift of a projective vector field on $(N, [\nabla])$.

Explicitly, for every projective vector field K of $(N, [\nabla])$ there is a corresponding Killing vector $\mathcal{K}$ of (M, g_Λ) given in local coordinates by

$$\mathcal{K} = K - p_i \frac{\partial K^j}{\partial x^i} \frac{\partial}{\partial p_j} + \frac{1}{\Lambda} \Upsilon_i \frac{\partial}{\partial p_i}, \tag{1.8}$$

where Υ_i is defined by (1.6).

1.2.2 Anti–Self–Duality for $n = 2$

For $n = 2$, a local of characterisation of the sapces M is provided in [16]: they show that any 4-dimensional anti-self-dual Einstein space with scalar curvature -24Λ and a parallel anti-self-dual totally null distribution can be considered as the total space of a rank 2 affine bundle T^*S over a projective surface S of the form (1.2). The anti–self-duality property, in combination with the correspondence of symmetries of (M, g) with symmetries of $(N, [\nabla])$, is important in the context of the applications of the work [16] to integrability. It means that if we start with a projective surface with at least one projective vector field, we will find an ASD Einstein space with at least one Killing vector field, and thus will be able to perform a symmetry reduction to obtain an Einstein-Weyl structure in $2 + 1$ dimensions and a corresponding solution to the $SU(\infty)$-Toda field equation. This is the subject of chapter 4.

Chapter 2

An Einstein metric on an S^1–bundle over $(M, g_\Lambda, \Omega_\Lambda)$

In this chapter, we show that there is a canonical Einstein of metric on an $\mathbb{R}^*$–bundle over M, with a connection whose curvature is the pull–back of the symplectic structure from M. This metric is interesting in the context of Kaluza-Klein theory.

2.1 Flat case

We first note that taking the flat projective structure on $\mathbb{RP}^n$ results in a one–parameter family of $2n$-dimensional Einstein spaces M which are a Kaluza-Klein reductions of quadrics in $\mathbb{R}^{n+1,n+1}$. For $N = \mathbb{RP}^n$ the metric and symplectic form on M reduce to

$$\begin{aligned} g_\Lambda &= dx^i dp_i + \Lambda(p_i dx^i)^2 \\ \Omega_\Lambda &= dp_i \wedge dx^i. \end{aligned}$$

Proposition 2.1.1 *The Einstein spaces M corresponding to $\mathbb{RP}^n$ are projections from the $2n+1$-dimensional quadrics $Q \subset \mathbb{R}^{n+1,n+1}$ given by $X^\alpha Y_\alpha = \frac{1}{\Lambda}$, where $X, Y \in \mathbb{R}^{n+1}$ are coordinates on $\mathbb{R}^{n+1,n+1}$ such that the metric is given by*

$$G = dX^\alpha dY_\alpha,$$

under the embedding

$$
\begin{aligned}
X^\alpha &= \begin{cases} x^i \mathrm{e}^\tau, & \alpha = i = 1, \dots, n \\ \mathrm{e}^\tau, & \alpha = n+1 \end{cases} \\
Y_\alpha &= \begin{cases} p_i \mathrm{e}^{-\tau}, & \alpha = i = 1, \dots, n \\ \mathrm{e}^{-\tau}\Big(\frac{1}{\Lambda} - x^k p_k\Big), & \alpha = n+1 \end{cases}
\end{aligned} \tag{2.1}
$$

following Kaluza-Klein reduction by the vector $\frac{\partial}{\partial\tau}$.

Proof. We find the basis of coordinate 1-forms $\{dX^\alpha, dY_\alpha\}$ to be

$$
\begin{aligned}
dX^\alpha &= \begin{cases} \mathrm{e}^\tau(dx^i + x^i d\tau), & \alpha = i = 1, \dots, n \\ \mathrm{e}^\tau d\tau, & \alpha = n+1 \end{cases} \\
dY_\alpha &= \begin{cases} \mathrm{e}^{-\tau}(dp_i - p_i d\tau), & \alpha = i = 1, \dots, n \\ -\mathrm{e}^{-\tau}\Big[\Big(\frac{1}{\Lambda} - x^k p_k\Big)d\tau + x^k dp_k + p_k dx^k\Big], & \alpha = n+1. \end{cases}
\end{aligned}
$$

The metric is then given by

$$
\begin{aligned}
G &= \mathrm{e}^\tau(dx^i + x^i d\tau)\mathrm{e}^{-\tau}(dp_i - p_i d\tau) \; - \; \mathrm{e}^\tau d\tau \mathrm{e}^{-\tau}\Big[\Big(\frac{1}{\Lambda} - x^k p_k\Big)d\tau + x^k dp_k + p_k dx^k\Big] \\
&= dx^i dp_i \; + \; (x^i dp_i - p_i dx^i)d\tau \; - \; (x^i p_i)d\tau^2 \; - \; d\tau\Big[\Big(\frac{1}{\Lambda} - x^k p_k\Big)d\tau + x^k dp_k + p_k dx^k\Big] \\
&= dx^i dp_i \; - \; \frac{1}{\Lambda}d\tau^2 \; - \; 2p_i dx^i d\tau \\
&= dx^i dp_i \; + \; \Lambda(p_i dx^i)^2 \; - \; \Lambda\Big(\frac{d\tau}{\Lambda} + p_i dx^i\Big)^2,
\end{aligned}
$$

which is clearly going to give g_Λ under Kaluza-Klein reduction by $\frac{\partial}{\partial\tau}$.

□

Note that the symplectic form Ω is the exterior derivative of the potential term $p_i dx^i$, implying a possible generalisation to the curved case.

2.2 Curved case

We now return to a general projective structure $(N, [\nabla])$. Since symplectic form picks out the antisymmetric part of the Schouten tensor, it has the fairly simple form

$$\Omega_\Lambda = dp_i \wedge dx^i - \frac{\partial_{[i}\Gamma^k_{j]k}}{\Lambda(n+1)} dx^i \wedge dx^j.$$

By inspection, this can be written $\Omega_\Lambda = d\mathcal{A}$, where

$$\mathcal{A} = p_i dx^i - \frac{\Gamma^k_{ik}}{\Lambda(n+1)} dx^i.$$

This is a trivialisation of the Kaluza-Klein bundle which we are about to construct. Note that for $\Lambda = 1$, under a change of projective connection (1.1) the corresponding change in the fiber coordinates (1.3) ensures that Ω and $\mathcal{A}$ are unchanged.

Motivated by the Kaluza-Klein reduction in the flat case, we consider the following metric.

Theorem 2.2.1 *The metric*

$$\mathcal{G}_\Lambda = g_\Lambda - \Lambda\left(\frac{d\tau}{\Lambda} + \mathcal{A}\right)^2 \tag{2.2}$$

on a principal circle bundle $\sigma : \mathcal{Q} \to M$ is Einstein, with Ricci scalar $2n(2n+1)\Lambda$.

Proof. We prove this using the Cartan formalism. Our treatment parallels a calculation by Kobayashi [**?**], who considered principal circle bundles over Kahler manifolds in order to study the topology of the base. Note that we temporarily surpress the constant Λ, writing $\mathcal{G} \equiv \mathcal{G}_\Lambda$ and $g \equiv g_\Lambda$, since the proof applies to any choice $\Lambda \neq 0$ within this family. consider a frame

$$e^a = \begin{cases} dx^i, & a = i = 1, \ldots, n \\ dp_i - (\Gamma^k_{ij} p_k - \Lambda p_i p_j - \Lambda^{-1} P_{ij}) dx^j, & a = i + n = n+1, \ldots 2n. \end{cases} \tag{2.3}$$

In this basis the metric takes the form

$$g = e^1 \odot e^{n+1} + \cdots + e^n \odot e^{2n}. \tag{2.4}$$

We are interested in the metric

$$\mathcal{G} = -e^0 \odot e^0 + g,$$

where

$$e^0 = \sqrt{\Lambda}\left(\frac{d\lambda}{\Lambda} + \mathcal{A}A\right).$$

We reserve Roman indices $a, b, \ldots$ for the $2n$-metric components $1, \ldots, 2n$ and allow greek indices $\mu, \nu, \ldots$ to take values $0, 1, \ldots, 2n$. The dual basis to $\{e^\mu\}$ will be denoted $\{E_\mu\}$ and will act on functions as vector fields in the usual way. We wish to find the new connection 1-forms $\hat{\psi}^\mu{}_\nu$ (defined by $de^\mu = -\hat{\psi}^\mu{}_\nu \wedge e^\nu$) in terms of the old ones $\psi^a{}_b$ (defined by $de^a = -\psi^a{}_b \wedge e^b$). Hence we examine [1] de^0 to find $\hat{\psi}^0{}_a$.

$$de^0 = \sqrt{\Lambda} d\mathcal{A} = \sqrt{\Lambda}\Omega_{ab} e^a \wedge e^b = -\hat{\psi}^0{}_a \wedge e^a \qquad \Longrightarrow \qquad \hat{\psi}^0{}_a = \sqrt{\Lambda}\Omega_{[ab]} e^b = \sqrt{\Lambda}\Omega_{ab} e^b, \qquad \hat{\psi}^a{}_0 = \sqrt{\Lambda}\Omega^a{}_b e^b.$$

Since de^a is unchanged, we have that

$$\hat{\psi}^a{}_0 \wedge e^0 + \hat{\psi}^a{}_b \wedge e^b = \psi^a{}_b \wedge e^b,$$

thus

$$\hat{\psi}^a{}_b \wedge e^b = \psi^a{}_b \wedge e^b - \sqrt{\Lambda}\Omega^a{}_b e^b \wedge e^0 \qquad \Longrightarrow \qquad \hat{\psi}^a{}_b = \psi^a{}_b + \sqrt{\Lambda}\Omega^a{}_b e^0.$$

We now calculate the curvature 2-forms $\hat{\Psi}^\mu{}_\nu = d\hat{\psi}^\mu{}_\nu + \hat{\psi}^\mu{}_\rho \wedge \hat{\psi}^\rho{}_\nu = \frac{1}{2}\mathcal{R}_{\rho\sigma\nu}{}^\mu e^\rho \wedge e^\sigma$ in terms of $\Psi^a{}_b = d\psi^a{}_b + \psi^a{}_c \wedge \psi^c{}_b$, where $\mathcal{R}_{\rho\sigma\nu}{}^\mu$ is the Riemann tensor of $\mathcal{Q}$. Note that we use the notation $\psi^a{}_b = \psi^a{}_{bc} e^c$

$$\begin{aligned}
\hat{\Psi}^a{}_b &= d\hat{\psi}^a{}_b + \hat{\psi}^a{}_c \wedge \hat{\psi}^c{}_b + \hat{\psi}^a{}_0 \wedge \hat{\psi}^0{}_b \\
&= d\psi^a{}_b + \sqrt{\Lambda} d(\Omega^a{}_b e^0) + \psi^a{}_c \wedge \psi^c{}_b + \sqrt{\Lambda}\Omega^a{}_c e^0 \wedge \psi^c{}_b + \sqrt{\Lambda}\Omega^c{}_b \psi^a{}_c \wedge e^0 + \Lambda \Omega^a{}_{[c}\Omega_{|b|d]} e^c \wedge e^d \\
&= \Psi^a{}_b + \sqrt{\Lambda} E_c(\Omega^a{}_b) e^c \wedge e^0 + \Lambda(\Omega^a{}_b \Omega_{cd} + \Omega^a{}_{[c}\Omega_{|b|d]}) e^c \wedge e^d + \sqrt{\Lambda}(\Omega^c{}_b \psi^a{}_{cd} - \Omega^a{}_c \psi^c{}_{bd}) e^d \wedge e^0 \\
&= \Psi^a{}_b + \sqrt{\Lambda}\nabla_c \Omega^a{}_b e^c \wedge e^0 + \Lambda(\Omega^a{}_b \Omega_{cd} + \Omega^a{}_{[c}\Omega_{|b|d]}) e^c \wedge e^d. \\
\hat{\Psi}^0{}_a &= d\hat{\psi}^0{}_a + \hat{\psi}^0{}_b \wedge \hat{\psi}^b{}_a \\
&= \sqrt{\Lambda} E_{[c}(\Omega_{|a|b]})\theta^c \wedge \theta^b - \sqrt{\Lambda}\Omega_{ab}\psi^b{}_c \wedge e^c + \sqrt{\Lambda}\Omega_{bc} e^c \wedge (\psi^b{}_a + \sqrt{\Lambda}\Omega^b{}_a e^0) \\
&= \sqrt{\Lambda}(E_{[d}(\Omega_{|a|b]}) - \Omega_{ac}\psi^c{}_{[bd]} + \Omega_{c[d}\psi^c{}_{|a|b]}) e^d \wedge e^b + \Lambda \Omega_{bc}\Omega^b{}_a e^c \wedge e^0 \\
&= \sqrt{\Lambda}\nabla_{[c}\Omega_{|a|d]} e^c \wedge e^d + \Lambda\Omega_{bc}\Omega^b{}_a e^c \wedge e^0.
\end{aligned}$$

[1]Note that our conventions are $(d\omega)_{ab\ldots c} = \partial_{[a}\omega_{b\ldots c]}$, $(\eta \wedge \omega)_{a\ldots d} = \eta_{[a\ldots b}\omega_{c\ldots d]}$, $\omega = \omega_{a\ldots b} dx^a \wedge \cdots \wedge dx^b$, and $F_{ab} dx^a \wedge dx^b = F_{[ab]} dx^a \otimes dx^b$ implying $dx^a \wedge dx^b = \frac{1}{2}(dx^a \otimes dx^b - dx^b \otimes dx^a)$.

Hence we have that

$$\begin{aligned}
\mathcal{R}_{cdb}{}^{a} &= R_{cdb}{}^{a} + 2\Lambda(\Omega^{a}{}_{b}\Omega_{cd} + \Omega^{a}{}_{[c}\Omega_{|b|d]}) \\
\mathcal{R}_{c0b}{}^{a} &= \sqrt{\Lambda}\nabla_{c}\Omega^{a}{}_{b} \\
\mathcal{R}_{cda}{}^{0} &= 2\sqrt{\Lambda}\nabla_{[c}\Omega_{|a|d]} \\
\mathcal{R}_{c0a}{}^{0} &= \Lambda\Omega_{bc}\Omega^{b}{}_{a},
\end{aligned}$$

and thus, using $\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu\nu}{}^{\rho}$,

$$\begin{aligned}
\mathcal{R}_{00} &= \Lambda\Omega_{bc}\Omega^{bc} = -2n\Lambda = 2n\Lambda\mathcal{G}_{00} \\
\mathcal{R}_{b0} &= \sqrt{\Lambda}\nabla_{c}\Omega^{c}{}_{b} = 0 \\
\mathcal{R}_{db} &= R_{db} + 2\Lambda(\Omega^{c}{}_{b}\Omega_{cd} + \frac{1}{2}\Omega^{c}{}_{c}\Omega_{bd} - \frac{1}{2}\Omega^{c}{}_{d}\Omega_{bc}) - \Lambda\Omega_{cd}\Omega^{c}{}_{b} \\
&= R_{db} + 2\Lambda\Omega_{b}{}^{c}\Omega_{dc} \\
&= 2(n+1)\Lambda g_{db} - 2\Lambda g_{db} = 2n\Lambda g_{db} = 2n\Lambda\mathcal{G}_{db}.
\end{aligned}$$

Note that we have used the facts that g is Einstein with Ricci scalar $4n(n+1)\Lambda$ and the symplectic form Ω is divergence-free; these are justified in the appendix. Since $\mathcal{G}_{a0} = 0$, we conclude that

$$\mathcal{R}_{\mu\nu} = 2n\Lambda\mathcal{G}_{\mu\nu} = \frac{\mathcal{R}}{2n+1}\mathcal{G}_{\mu\nu},$$

i.e. $\mathcal{G}$ is Einstein with Ricci scalar $2n(2n+1)\Lambda$.

□

Physically, this is a Kaluza-Klein reduction with constant dilation field and where the Maxwell two-form is related to the reduced metric by $\Omega_{a}{}^{c}\Omega_{cb} = g_{ab}$. This is what allows both the reduced and lifted metric to be Einstein. A more general discussion can be found in [36].

From the Cartan perspective, $\mathcal{G}_{\Lambda=1}$ can be thought of as a metric on the $2n+1$-dimensional space obtained by taking a quotient $\tilde{q} : P \mapsto P/SL(n,\mathbb{R})$ of the Cartan bundle, where we embed $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$ in H as in (**??**) but with a now denoting an element of $SL(n,\mathbb{R})$ (so that $\det a^{-1} = 1$). This new subgroup acts adjointly on θ as

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} -\mathrm{tr}\phi & \eta \\ \omega & \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} -\mathrm{tr}\phi & \eta a^{-1} \\ a\omega & \phi \end{pmatrix},$$

so not only is the inner product $\eta\omega$ invariant but also the $(0,0)$-component $\theta^0{}_0 = -\mathrm{tr}\phi$, which is a scalar one-form whose exterior derivative is constrained by (**??**) to be $d\theta^0{}_0 = -\theta^0{}_i \wedge \theta^i{}_0 = -\mathrm{Ant}(\eta \wedge \omega)$. Thus, denoting by A the object on $\mathcal{Q} = P/SL(n,\mathbb{R})$ which is such that $\tilde{q}^* A = \mathrm{tr}\phi$, we have that $dA = \Omega$ (where we are now taking Ω and g to be defined on $\mathcal{Q}$ by $\tilde{q}^*\Omega = \mathrm{Ant}(\eta \wedge \omega)$ and $\tilde{q}^* g = \mathrm{Sym}(\eta \wedge \omega)$ respectively, or equivalently redefining $\tilde{\Omega} = \sigma^*\Omega$ and $\tilde{g} = \sigma^* g$).

We then have a natural way of constructing a metric $\mathcal{G}$ on $\mathcal{Q}$ as a linear combination of g and $e^0 \odot e^0$, where e^0 is A up to addition of some exact one-form. It turns out that the choice of linear combination such that $\mathcal{G}$ is Einstein is

$$\mathcal{G} = -e^0 \odot e^0 + g.$$

The fact that this metric is exactly (2.2) can be verified by constructing the Cartan connection of $(N, [\nabla])$ explicitly in terms of a representative connection $\nabla \in [\nabla]$.

2.2.1 Ricci scalar of g_Λ and divergence of Ω_Λ

We calculate these using the Cartan formalism, again using the basis (2.3). In this basis we have g as above (2.4) and

$$\Omega = \sum_{i=1}^{n} e^i \wedge e^{i+n} \qquad \Longrightarrow \qquad \Omega_{ab} = \sum_{i=1}^{n} \delta^i_{[a}\delta^{i+n}_{b]}.$$

Note that from now on we will omit the summation sign and use the summation convention regardless of whether i,j-indices are up or down. As in section **??**, we look for $\psi^a{}_b$ by considering de^a (recall that $i,j = 1,\ldots,n$ and $a,b = 1,\ldots,2n$):

$$\begin{aligned}
de^i &= 0 \\
de^{i+n} &= -(E_l(\Gamma^k_{ij})p_k - \Lambda^{-1}E_l(P_{ij}))dx^l \wedge dx^j - (\Gamma^k_{ij} - 2\Lambda p_{(i}\delta^k_{j)})dp_k \wedge dx^j \\
&= -(E_l(\Gamma^k_{ij})p_k - \Lambda^{-1}E_l(P_{ij}))e^l \wedge e^j \\
&\quad -(\Gamma^k_{ij} - 2\Lambda p_{(i}\delta^k_{j)})(e^{k+n} + (\Gamma^l_{km}p_l - \Lambda p_k p_m - \Lambda^{-1}P_{km})e^m) \wedge e^j \\
&= \Big[\Lambda^{-1}E_m(P_{ij}) - E_m(\Gamma^k_{ij})p_k + \Lambda^{-1}\Gamma^k_{ij}P_{km} - \Gamma^k_{ij}\Gamma^l_{km}p_l + \Lambda\Gamma^k_{ij}p_m p_k \\
&\quad +2\Lambda p_{(i}(\Gamma^l_{j)m}p_l - \Lambda p_{j)}p_m - \Lambda^{-1}P_{j)m})\Big]e^m \wedge e^j + (2\Lambda p_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^{k+n} \wedge e^j \\
&= \Big[\Lambda^{-1}D_m P_{ij} - (D_m\Gamma^k_{ij})p_k - 2p_{(i}P_{j)m}\Big]e^m \wedge e^j + (2\Lambda p_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^{k+n} \wedge e^j
\end{aligned}$$

Note that we have used D to denote the chosen connection on N with components Γ^i_{jk}. Next we wish to read off the spin connection $\psi^a{}_b$ such that $de^a = -\psi^a{}_b \wedge e^b$ and the following index symmetries are satisfied:

$$\begin{aligned}
\psi^i{}_j &= \frac{1}{2}\psi_{i+n\,j} = -\frac{1}{2}\psi_{j\,i+n} = -\psi^{j+n}_{i+n} \\
\psi^i{}_{j+n} &= \frac{1}{2}\psi_{i+n\,j+n} = -\frac{1}{2}\psi_{j+n\,i+n} = -\psi^j{}_{i+n} \\
\psi^{i+n}_{j} &= \frac{1}{2}\psi_{i\,j} = -\frac{1}{2}\psi_{j\,i} = -\psi^{j+n}_{i}
\end{aligned}$$

We find that

$$\begin{aligned}
\psi^{i+n}_{k+n} &= (2\Lambda p_{(i}\delta^k_{j)} - \Gamma^k_{ij})e^j = -\psi^k{}_i \\
\psi^{i+n}_{j} &= \Big[2(D_{[i}\Gamma^l_{j]k})p_l - 2\Lambda^{-1}D_{[i}P^{\mathrm{S}}_{j]k} - \Lambda^{-1}D_kP^{\mathrm{A}}_{ij} + 2p_{(j}P_{k)i} - 2p_{(i}P_{k)j}\Big]e^k =: A_{ijk}e^k \\
\psi^i{}_{j+n} &= 0.
\end{aligned}$$

One can check that these satisfy both the index symmetries above and are such that $de^a = -\psi^a{}_b \wedge e^b$, and we know from theory that there is a unique set of $\psi^a{}_b$ that have both of these properties. Note that we have used P^{S} and P^{A} to denote the symmetric and antisymmetric parts of P in order to avoid too much confusion from having multiple symmetrisation brackets in the indices.

We are now ready to calculate the divergence of Ω. Since it is covariantly constant in this basis, we obtain

$$\nabla_c\Omega_{ab} = -\psi^d{}_{ac}\Omega_{db} - \psi^d{}_{bc}\Omega_{ad} = -\psi^d{}_{ac}\Omega_{db} + \psi^d{}_{bc}\Omega_{da} = 2\Omega_{d[a}\psi^d{}_{b]c}.$$

We can split the right hand side into

$$\begin{aligned}
\Omega_{da}\psi^d{}_{bc} &= \Omega_{ka}\psi^k{}_{bc} + \Omega_{k+n\,a}\psi^{k+n}_{bc} \\
&= \delta^i_{[k}\delta^{i+n}_{a]}\psi^k{}_{bc} + \delta^i_{[k+n}\delta^{i+n}_{a]}\psi^{k+n}_{bc} \\
&= \frac{1}{2}\Big(-\delta^{k+n}_a\delta^i_b\delta^j_c(2\Lambda p_{(i}\delta^k_{j)} - \Gamma^k_{ij}) - \delta^k_a\delta^{l+n}_b\delta^j_c(2\Lambda p_{(k}\delta^l_{j)} - \Gamma^l_{kj}) - \delta^k_a\delta^l_b\delta^m_c A_{klm}\Big).
\end{aligned}$$

The first two terms are the same but with $a \leftrightarrow b$, so are lost in the antisymmetrisation. Thus

$$\nabla_c\Omega_{ab} = -\delta^k_{[a}\delta^l_{b]}\delta^m_c A_{klm}.$$

Tracing amounts to contracting this with g^{ac}:

$$\nabla^c \Omega_{cb} = -\delta^k_{[a}\delta^l_{b]} g^{ac} \delta^m_c A_{klm} = -\delta^k_{[a}\delta^l_{b]} g^{am} A_{klm},$$

but g^{am} is non-zero only when $a = m + n > n$ and $\delta^k_{[a}\delta^l_{b]}$ is non-zero only when $a = k \leq n$ or $a = l \leq n$. We can therefore conclude that the right hand side is zero and Ω is divergence-free.

Finally, we calculate the Ricci scalar of g (given that it's Einstein) via the curvature two-forms $\Psi^a{}_b = d\psi^a{}_b + \psi^a{}_c \wedge \psi^c{}_b = \frac{1}{2} R_{cdb}{}^a e^c \wedge e^d$. We are only interested in non-zero components of the Ricci tensor such as $R_{i\,j+n} = R_{ci\,j+n}{}^c$. In fact, we will calculate only $R_{m+n\,j}$, for which we need to consider $R_{l\,m+n\,j}{}^i$ and $R_{k+n\,m+n\,j}{}^{l+n}$, i.e. we need only calculate $\Psi^i{}_j$ and $\Psi^{l+n}{}_j$.

$$\Psi^i{}_j = d\Big((\Gamma^i_{jk} - 2\Lambda p_{(j}\delta^i_{k)})e^k\Big) + \psi^i{}_k \wedge \psi^k{}_j + \psi^i{}_{k+n} \wedge \psi^{k+n}_j.$$

The last term vanishes since $\psi^{k+n}_j = 0$, and the middle term only has components that look like $\frac{1}{2} R_{lmj}{}^i e^l \wedge e^m$, so the only term we are interested in is

$$-2\Lambda dp_{(j}\delta^i_{k)} e^k = -2\Lambda \delta^i_{(k}(e^{j)+n} + (\Gamma^l_{j)m} p_l - \Lambda p_{j)} p_m - \Lambda^{-1} P_{j)m})e^m) \wedge e^k.$$

Again, discarding the $e^m \wedge e^k$ term gives

$$-\Lambda(e^{j+n} \wedge e^k + \delta^i_j e^{k+n} \wedge e^k) = \frac{1}{2} R_{l\,m+n\,j}{}^i e^l \wedge e^{m+n} + \frac{1}{2} R_{m+n\,l\,j}{}^i e^{m+n} \wedge e^l,$$

so we conclude

$$R_{l\,m+n\,j}{}^i = \Lambda(\delta^i_j \delta^m_l + \delta^i_l \delta^m_j).$$

The other Riemann tensor component we need to know to calculate $R_{m+n\,j} = R_{c\,m+n\,j}{}^c$ is $R_{l+n\,m+n\,j}{}^{i+n}$, so we examine

$$\Psi^{i+n}_j = d\psi^{i+n}_j + \psi^{i+n}_k \wedge \psi^k{}_j + \psi^{i+n}{}_{k+n} \wedge \psi^{k+n}_j,$$

but none of these terms have $e^{l+n} \wedge e^{m+n}$ components, so $R_{l+n\,m+n\,j}{}^{i+n} = 0$. Hence

$$R_{m+n\,j} = \delta^l_i R_{l\,m+n\,j}{}^i = \Lambda(\delta^m_j + n\delta^m_j) = \Lambda(n+1)\delta^m_j.$$

Setting this equal to $\frac{R}{2n} g_{m+n\,j} = \frac{R}{4n}\delta^m_j$ we find

$$R = 4n(n+1)\Lambda,$$

as required.

Chapter 3

Para–c–projective compactification of (M, g, Ω)

In [54] the concept of c–projective compactification was defined. It is based on almost c–projective geometry [57], an analogue of projective geometry defined for almost complex manifolds (M, J) in which the equivalence class of connections defining the c–projective structure must be *complex* and *minimal*.[1] Here we review the definition of c–projective compactification, modifying to the "para" case where the para–almost–complex structure squares to Id rather than $-Id$. We then show that the $2n$–dimensional, neutral signature Einstein metrics (1.2) admit a compactification which we call *para–c–projective*.

Although c–projective compactification is defined for any almost complex manifold, the definition can be applied to pseudo–Riemannian metrics g which are Hermitian with respect to the almost complex structure so long as there exists a connection which preserves both g and J and has minimal torsion. Such Hermitian metrics are said to be *admissible*. Note that such a connection, if it exists, is uniquely defined, since the conditions that it be complex and minimal determine its torsion. It is thus given by the Levi–Civita connection of g plus a constant multiple of the Nijenhuis tensor of J.

We make the following definition in the para–c–projective case.

Definition 3.0.1 *Let (M, g, J) be a para–Hermitian manifold, and let ∇^L be a connection which preserves both g and J and has minimal torsion. The structure (M, g, J) admits a para–c–projective compactification to a manifold with boundary $\overline{M} = M \cup \partial M$, if there exists a function $T : \overline{M} \to \mathbb{R}$ such that $\mathcal{Z}(T)$ is the boundary $\partial M \subset \overline{M}$, the*

[1] Recall that a connection on an almost complex manifold (M, J) is called *complex* if it preserves J and *minimal* if the torsion is just the Nijenhuis tensor of J up to a constant factor.

differential dT does not vanish on ∂M, and the connection

$$\overline{\nabla^L}_X Y = \nabla^L{}_X Y + \Upsilon(X)Y + \Upsilon(JX)JX + \Upsilon(Y)X + \Upsilon(JY)JX, \quad \textit{where} \quad \Upsilon = \frac{dT}{2T},$$

extends to $\overline{M}$.

Note that the para–c–projective change of connection $\nabla^L \to \overline{\nabla^L}$ differs from the c–projective case in the signs of some of the terms, to account for the fact that J squares to the Id rather than $-Id$.

It follows easily from this definition that the endomorphism J on M naturally extends to all of $\overline{M}$ by parallel transport with respect to $\overline{\nabla^L}$. It thus defines an almost para–CR structure on the hypersurface distribution $\mathcal{D}$ defined by $\mathcal{D}_x := T_x\partial M \cap J(T_x\partial M)$. It can be shown (see Lemma 5 of [54] and modify to the case $J^2 = Id$) that this almost para–CR structure is non–degenerate if and only if for any local defining function T the one–form $\theta = dT \circ J$ restricts to a contact form on ∂M.

The first main result of [54] is Theorem 8 in this reference, which gives a local form for an admissible Hermitian metric which is sufficient for the corresponding c–projective structure to be c–projectively compact. The theorem is stated below, adapted to the para–c–projective case. Note that this includes an assumption that the Nijenhuis tensor $\mathcal{N}$ of J takes so–called *asymptotically tangential values*. This is equivalent to the following statement in index notation:

$$\Big(\mathcal{N}^a{}_{bc}\nabla_a T\Big)|_{T=0} = 0. \tag{3.1}$$

The following result arises by a trivial adaption of the arguments in [54] for the almost complex case, and so further details may be obtained from that source.

Theorem 3.0.2 ([54]) *Let $\overline{M}$ be a smooth manifold with boundary ∂M and interior M. Let J be an almost para–complex structure on $\overline{M}$, such that ∂M is non–degenerate and the Nijenhuis tensor $\mathcal{N}$ of J has asymptotically tangential values. Let g be an admissible pseudo–Riemannian Hermitian metric on M. For a local defining function T for the boundary defined on an open subset $\mathcal{U} \subset \overline{M}$, put $\theta = dT \circ J$ and, given a non–zero real constant C, define a Hermitian $\binom{0}{2}$–tensor field $h_{T,C}$ on $\mathcal{U} \cap M$ by*

$$h_{T,C} := Tg + \frac{C}{T}(dT^2 - \theta^2).$$

Suppose that for each $x \in \partial M$ there is an open neighbourhood $\mathcal{U}$ of x in $\overline{M}$, a local defining function T defined on $\mathcal{U}$, and a non–zero constant C such that

- *$h_{T,C}$ admits a smooth extension to all of $\mathcal{U}$*
- *for all vector fields X, Y on U with $dT(Y) = \theta(Y) = 0$, the function $h_{T,C}(X, JY)$ approaches $Cd\theta(X, Y)$ at the boundary.*

Then g is c–projectively compact.

The statement in Theorem 3.0.2 does not depend on the choice of T. Different choices of T result in rescalings of the contact form θ on the boundary by a nowhere vanishing function.

3.1 Compactifying the Dunajski–Mettler Class

In [16] it was shown that the manifold M can be identified with the complement of an $\mathbb{RP}^{n-1}$ sub-bundle in the projectivisation $\mathbb{P}(\mathcal{T}^*)$ of a certain rank $(n+1)$ vector bundle (the so called co-tractor bundle) $\mathcal{T}^*$ over N. In the special case where $N = \mathbb{RP}^n$, and $[\nabla]$ is projectively flat the manifold $M = SL(n+1, \mathbb{R})/GL(n, \mathbb{R})$ can be identified with the projection of $\mathbb{R}^{n+1} \times \mathbb{R}_{n+1} \setminus \mathcal{Z}$, where $\mathcal{Z}$ denotes the set of incident pairs (point, hyperplane).

The compactification procedure described in the Theorem 3.3.1 will, for the model, attach these incident pairs back to M, and more generally (in case of a curved projective structure on N) will attach the $\mathbb{RP}^{n-1}$ sub-bundle of $\mathbb{P}(\mathcal{T}^*)$. The boundary ∂M from Definition 3.0.1 will play a role of a submanifold manifold separating two open sets in $\mathbb{P}(\mathcal{T}^*)$ in the sense described in the next sub-Section.

3.2 The Dunajski–Mettler construction in tractor terms

Let the projective structure $(N, [\nabla])$, be represented by some torsion free affine connection ∇ on N, where the latter has dimension at least 2. Let $\mathcal{E}(1) \to N$ be the line bundle which is the standard $-2(n+1)$th root of the square of the canonical bundle of N (which, note, is canonically oriented). For any vector bundle $\mathcal{B}$ and line bundle $\mathcal{E}(w)$ we write $\mathcal{B}(w)$ as a shorthand for $\mathcal{B} \otimes \mathcal{E}(w)$.

Canonically on the projective manifold $(N, [\nabla])$ there is the rank $(n+1)$ cotractor bundle [50]

$$\mathcal{T}^* \to N.$$

This has a composition sequence

$$0 \to T^*N(1) \to \mathcal{T}^* \xrightarrow{X} \mathcal{E}(1) \to 0, \tag{3.2}$$

where the map $X \in \Gamma(\mathcal{T}(1))$ is called the (projective) *canonical tractor*. Choosing a connection in $[\nabla]$ determines a splitting of this sequence and so then have $\mathcal{T}^* = \mathcal{E}(1) \oplus T^*N(1)$, and we can represent an element V of $\mathcal{T}^*$ as a pair $(\sigma, \mu) = [V]_\nabla$ (see e.g. [53]). Any other connection $\overline{\nabla}$ in $[\nabla]$ is related to ∇ by (**??**), for some 1-form field Υ on N, and the corresponding transformation

$$[V]_\nabla = (\sigma, \mu) \mapsto (\sigma, \mu + \sigma\Upsilon) = [V]_{\overline{\nabla}}. \tag{3.3}$$

The main importance of $\mathcal{T}^*$ is that it admits a canonical projectively invariant *tractor connection* $\nabla^{\mathcal{T}^*}$ given by

$$\nabla^{\mathcal{T}}{}_i \begin{pmatrix} \sigma \\ \mu_j \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + \mathrm{P}_{ij}\sigma \end{pmatrix}. \tag{3.4}$$

We shall now present two variants of the construction of [16], and then its compactification (to be made precise in Theorem 3.3.1). We begin with compactification of the construction in [16]. For simplicity let us assume that N is orientable.

3.2.1 Compactification by line projectiviation:

On the total space of $\mathcal{T}^*$ we pullback $\pi : \mathcal{T}^* \to N$ along π to get $\pi^*(\mathcal{T}^*) \to \mathcal{T}^*$ as a vector bundle over the total space $\mathcal{T}^*$. By construction this bundle has a tautological section $U \in \Gamma(\pi^*(\mathcal{T}^*))$. We also have $\pi^*(\mathcal{T}(w))$ for any weight w, and we shall write simply $X \in \Gamma(\pi^*(\mathcal{T}(1)))$ for the pullback to $\mathcal{T}^*$ of the canonical tractor X on N.

There is a canonical density $\tau \in \Gamma(\pi^*\mathcal{E}(1))$ given by

$$\tau := X \lrcorner\, U.$$

Now define

$$\kappa : \mathcal{T}^* \longrightarrow \mathcal{M} := \mathbb{P}(\mathcal{T}^*) \tag{3.5}$$

by the fibrewise projectivisation, and use also π for the map to N:

$$\pi : \mathcal{M} \to N.$$

Note that τ is homogeneous of degree 1 up the fibres of the map $\mathcal{T}^* \to \mathcal{M}$. Thus τ determines, and is equivalent to, a section (that we also denote) τ of a certain density bundle $\pi^*(\mathcal{E}(1)) \otimes \mathcal{E}_{\mathcal{T}^*}(1)$, on $\mathcal{M}$ that for simplicity we shall denote $\mathcal{E}(1,1)$. So $\mathcal{M}$ is stratified according to whether or not τ is vanishing, and we write $\mathcal{Z}(\tau)$ to denote, in particular, the zero locus of τ.

To elaborate on the densities used here, and their generalisation to arbitrary weights: By $\mathcal{E}_{\mathcal{T}^*}(w')$, for $w' \in \mathbb{R}$, we mean the line bundle on $\mathbb{P}(\mathcal{T}^*)$ whose sections correspond to functions $f : \pi^*\mathcal{T}^* \to \mathbb{R}$ that are homogeneous of degree w in the fibres of $\pi^*\mathcal{T}^* \to \mathbb{P}(\mathcal{T}^*)$. Then for any weight w we also have $\mathcal{E}(w)$ on N and its pull back to the bundle $\pi^*\mathcal{E}(w) \to \mathbb{P}(\mathcal{T}^*)$. Then

$$\mathcal{E}(w, w') := \pi^*\mathcal{E}(w) \otimes \mathcal{E}_{\mathcal{T}^*}(w').$$

Using these tools we can recover the metric of [16]:

Theorem 3.2.1 *There is a neutral signature metric on $\mathcal{M} \setminus \mathcal{Z}(\tau)$ determined by the canonical pairing of the horizontal and vertical subspaces of $T(\mathcal{T}^*)$. This metric is Einstein, with non–zero Ricci scalar, and agrees with (**??**).*

Proof. Considering first the total space $\mathcal{T}^*$ and then its tangent bundle, note that there is an exact sequence

$$0 \to \pi^*\mathcal{T}^* \to T(\mathcal{T}^*) \to \pi^*TN \to 0, \tag{3.6}$$

where we have identified $\pi^*\mathcal{T}^*$ as the vertical sub-bundle of $T(\mathcal{T}^*)$. A connection on the vector bundle $\mathcal{T}^* \to N$ is equivalent to a splitting of this sequence; a connection identifies π^*TN with a distinguished sub-bundle of horizontal subspaces in $T(\mathcal{T}^*)$. Thus, in particular, the projective tractor connection on $\mathcal{T}^* \to N$ gives a canonical splitting of the sequence (3.6). So we have

$$T(\mathcal{T}^*) = \pi^*TN \oplus \pi^*\mathcal{T}^*. \tag{3.7}$$

We move now to the total space of $\mathbb{P}\mathcal{T}^*$, and we note that again the tractor (equivalently, Cartan) connection determines a splitting of the tangent bundle $T(\mathbb{P}\mathcal{T}^*)$, see [55]. From the usual Euler sequence of projective space (or see (3.26) in the last Section) it follows that for $T(\mathbb{P}\mathcal{T}^*)$ the second term of the display (3.7) is replaced by a quotient of $\pi^*\mathcal{T}^*(0,1)$. Indeed, if we work at a point $p \in \mathbb{P}(\mathcal{T}^*)$, observe that

$\pi^*\mathcal{T}^*(0,1)$ has a filtration

$$0 \to \mathcal{E}(0,0)_p \xrightarrow{U_p} \pi^*\mathcal{T}^*(0,1)|_p \to \pi^*\mathcal{T}^*(0,1)|_p/\langle U_p\rangle \to 0 \tag{3.8}$$

where, as usual, U is the canonical section. But away from $\mathcal{Z}(\tau)$, we have that U canonically splits the appropriately re-weighted pull back of the sequence (3.2)

$$0 \to \pi^*T^*N(1,1) \to \pi^*\mathcal{T}^*(0,1) \xrightarrow{X/\tau} \mathcal{E}(0,0) \to 0.$$

This identifies the quotient in (3.8), and thus we have canonically

$$T(\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)) = \pi^*TN \oplus \pi^*T^*N(1,1).$$

It follows that on $\mathcal{M}$ there is canonically a metric $\boldsymbol{g}$ and symplectic form $\boldsymbol{\Omega}$ taking values in $\mathcal{E}(1,1)$, given by

$$\begin{aligned} \boldsymbol{g}(w_1,w_2) &= \frac{1}{2}\Big(\Pi_H(w_1) \lrcorner \Pi_V(w_2) + \Pi_H(w_2) \lrcorner \Pi_V(w_1)\Big) \quad \text{and} \\ \boldsymbol{\Omega}(w_1,w_2) &= \frac{1}{2}\Big(\Pi_H(w_1) \lrcorner \Pi_V(w_2) - \Pi_H(w_2) \lrcorner \Pi_V(w_1)\Big) \end{aligned}$$

where

$$\Pi_H : T(\mathcal{M} \setminus \mathcal{Z}(\tau)) \to \pi^*TN \quad \text{and} \quad \Pi_V : T(\mathcal{M} \setminus \mathcal{Z}(\tau)) \to \pi^*T^*N(1,1)$$

are the projections. Then we obtain the metric and symplectic form by

$$g := \frac{1}{\tau}\boldsymbol{g} \qquad \text{and} \qquad \Omega := \frac{1}{\tau}\boldsymbol{\Omega}. \tag{3.9}$$

What remains to be done, is to show that (3.9) agrees with the normal form (**??**) once a trivialisation of $\mathcal{T}^* \to N$ has been chosen.

Let $p \in N$ and let $\mathcal{W} \subset N$ be an open neighbourhood of p with local coordinates $(x^1, \dots, x^n)$ such that $T_pN = \mathrm{span}(\partial/\partial x^1, \dots, \partial/\partial x^n)$. The connection (3.4) gives a splitting of $T(\mathcal{T}^*)$ into the horizontal and vertical sub-bundles

$$T(\mathcal{T}^*) = H(\mathcal{T}^*) \oplus V(\mathcal{T}^*),$$

as in (3.7). To obtain the explicit form of this splitting, let $V_\alpha, \alpha = 0, 1, \ldots, n$ be components of a local section of $\mathcal{T}^*$ in the trivialisation over $\mathcal{W}$. Then

$$\nabla^{\mathcal{T}^*} V_\beta = dV_\beta - \gamma^\alpha_\beta V_\alpha,$$

where $\gamma^\beta_\alpha = \gamma^\beta_{i\alpha} dx^i$, and the components of the co-tractor connection $\gamma^\beta_{i\alpha}$ are given in terms of the connection ∇ on N, and its Schouten tensor, and can be read–off from (3.4):

$$\gamma^0_{i0} = 0, \quad \gamma^j_{i0} = \delta^j_i, \quad \gamma^k_{ij} = \Gamma^k_{ij}, \quad \gamma^0_{ij} = -\mathrm{P}_{ij}.$$

In terms of these components we can write

$$H(\mathcal{T}^*) = \mathrm{span}\Big(\frac{\partial}{\partial x^i} + \gamma^\beta_{i\alpha} V_\beta \frac{\partial}{\partial V_\alpha}, i = 1, \ldots, n\Big), \quad V(\mathcal{T}^*) = \mathrm{span}\Big(\frac{\partial}{\partial V_\alpha}, \alpha = 0, 1, \ldots, n\Big).$$

Setting $\xi_i = V_i/V_0$, where $\tau = V_0 \neq 0$ [2] on the complement of $\mathcal{Z}(\tau)$, we can compute the push forwards of these spaces to $\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)$:

$$\kappa_* H(\mathcal{T}^*) = \mathrm{span}\Big(h_i \equiv \frac{\partial}{\partial x^i} - (\mathrm{P}_{ij} + \xi_i\xi_j - \Gamma^k_{ij}\ \xi_k)\frac{\partial}{\partial \xi_j}\Big), \quad \kappa_* V(\mathcal{T}^*) = \mathrm{span}\Big(v^i \equiv \frac{\partial}{\partial \xi_i}\Big).$$

The non–zero components of the metric (3.9) are given by

$$g(v^i, h_j) = \delta^i{}_j$$

which indeed agrees with (**??**) which is known to be Einstein [16].

□

Next we observe that $\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)$ is an affine bundle modelled on T^*N. The point is that given ∇ in the projective class there is a there is a smooth fibre bundle isomorphism

$$\iota : T^*N \to \mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau). \tag{3.10}$$

First, given ∇, we can represent an element $U \in \mathcal{T}^*_p$ $(p \in N)$ by the pair $(\tau, \mu) \in \mathcal{E}(1)_p \oplus T^*_p N(1)$, or, if we choose coordinates on N, by collection

$$U = (\tau, \mu_i), \quad i = 1, \ldots, n. \tag{3.11}$$

Then, dropping the choice $\nabla \in [\nabla]$, $U \in \mathcal{T}^*_p$ is an equivalence class of such pairs by the equivalence relation (3.3) that covers the equivalence relation between elements of $[\nabla]$.

[2] Rod @ Maciej: I have added $V_0 = \tau$. You agree, right!?

Thus, given ∇, and from the naturality of all maps, it follows that the total space of T^*N can be identified with $\mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau)$ by (for each $p \in N$)

$$T_p^*N \ni \xi_i \mapsto [(1, \xi_i)] = [(\tau, \tau\xi_i)] \in \mathbb{P}(\mathcal{T}^*) \setminus \mathcal{Z}(\tau). \tag{3.12}$$

Thus we may view $\mathcal{M}$ as a compactification of T^*N and, by construction, this is a closed manifold iff N is closed.

Note that by this construction it is easily verified that the zero locus of τ is a smoothly embedded hypersurface in $\mathcal{M}$, and from (3.2) it follows at once that this may be identified with the total space of the fibrewise projectivisation $\mathbb{P}(T^*N)$ (which is well known to have a para-CR structure).

A feature of this construction is that in each dimension n (of N) either the hypersurface $\mathcal{Z}(\tau)$ (if n odd) is not orientable, or $\mathcal{M}$ (if n even) is not orientable.

3.2.2 Compactification by ray-projectiviation:

Instead we may follow the construction above but instead define $\mathcal{M} := \pi^*(\mathbb{P}_+(\mathcal{T}^*))$, where $\mathbb{P}_+(\mathcal{T}^*)$ is the ray-projectivisation of $\mathcal{T}^*$ (i.e. the fibres of $\mathcal{T}^* \to \mathbb{P}_+(\mathcal{T}^*)$ are isomorphic to $\mathbb{R}_+$). Then the bundles $\mathcal{E}(w, w')$ should also be defined via ray homogeneity. In this case, for N orientable, both $\mathcal{Z}(\tau)$ and $\mathcal{M}$ are orientable, and again $\mathcal{M}$ is closed iff N is. Now from (3.2) we have that $\mathcal{Z}(\tau)$ may be identified with the fibrewise ray-projectivisation $\mathbb{P}_+(T^*N)$. In this variant of the construction there are two copies $M_\pm$ of T^*N in $\mathcal{M}$ according to the sign of τ. Moreover each of $\mathcal{M} \setminus M_\mp$ is a manifold that is globally a para-c-projective compactification of $M_\pm$ in sense of Theorem 3.0.2.

3.2.3 Remark on continuing the tractor approach

It would be possible to achieve our main aims by continuing the tractor approach. We will not pursue this here as we want to emphasise that with little effort the main result now follows directly form the properties of the metric. However we sketch just the basic idea: By our construction above it follows that $\mathcal{M}$ has a canonical para-c-projective geometry. In the notation as above, $\pi^*\mathcal{T} \oplus \pi^*\mathcal{T}^*$ is the corresponding para-c-projective tractor bundle and this has a canonical tractor connection that trivially extends (in fibre directions) the pull back of the projective connection (that is available in horizontal directions). The dual pairing between $\pi^*\mathcal{T}$ and $\pi^*\mathcal{T}^*$ determines a fibre metric and compatible symplectic form on the bundle $\pi^*\mathcal{T} \oplus \pi^*\mathcal{T}^*$ and this is obviously preserved by the connection. What remains is to show that the tractor connection so constructed satisfies properties that mean that it is *normal* in the sense defined in e.g. [56]. With this established then the main results then follow from the general holonomy theory in [55].

3.3 The main theorem

In the previous subsection 3.2.2 we have presented a candidate $\overline{M} \equiv \mathcal{M} \setminus M_{\mp}$ for a para-c-projective compactification of the Einstein para-Kahler manifold (M, g, Ω) given by (**??**). What remains to be done is to show that near the boundary $\mathcal{Z}(\tau) = 0$ of $\overline{M}$ the metric (**??**) can be put in the local normal form of Theorem 3.0.2.

The endomorphism $J : TM \to TM$ defined by $\Omega(X, Y) = g(JX, Y)$ satisfies $J^2 = Id$, and the associated Libermann connection ∇^L [60] is given by

$$\nabla^L{}_a X_b = \nabla^{\mathbf{g}}{}_a X_b - G^c{}_{ab} X_c, \quad \text{where} \quad G^c{}_{ab} = -\Omega^{cd} \nabla^{\mathbf{g}}{}_d \Omega_{ab} \tag{3.13}$$

and $\nabla^{\mathbf{g}}$ is the Levi–Civita connection of g. This connection is metric, has minimal torsion, and preserves the almost para–complex structure J. It thus belongs to a para–c–projective equivalence class which we will show to be compactifiable in the sense of Theorem 3.0.2.

Theorem 3.3.1 *The Einstein almost para–Kähler metric (M, g, Ω) given by (**??**) admits a para–c–projective compactification $\overline{M}$. The structure on the $(2n-1)$–dimensional boundary $\partial M \cong \mathbb{P}(T^*N)$ of $\overline{M}$ includes a contact structure together with a conformal structure and a para–CR structure defined on the contact distribution.*

Proof. In the proof below we shall explicitly construct the boundary ∂M together with the contact structure and the associated conformal structure on the contact distribution. We shall first deal with the model $M = SL(n+1)/GL(n)$, and then explain how the curvature of $(N, [\nabla])$ modifies the compactification.

In the model case we can define coordinates x^i on $N = \mathbb{RP}^n$ by taking $X = (1, x^1, \dots, x^n)$, where $(X^0, \dots, X^n)$ are homogeneous coordinates and we are working in an open set where $X^0 \neq 0$. The x^i are flat coordinates, so the connection components (and hence the Schouten tensor) vanish and (**??**) reduces to

$$g = d\xi_i \odot dx^i + \xi_i \xi_j dx^i \odot dx^j, \qquad \Omega = d\xi_i \wedge dx^i \quad \text{where} \quad i, j = 1, \dots, n. \tag{3.14}$$

We can relate [16] the affine coordinates ξ_i on the fibres of T^*N to the tractor coordinates (3.11) by setting $\xi_i = \mu_i/\tau$ on the complement of the zero locus $\mathcal{Z}(\tau)$ of τ.

Now consider an open set $\mathcal{U} \subset M$ given by $\xi_i x^i > 0$, and define the function T on $\mathcal{U}$ by

$$T = \frac{1}{\xi_i x^i}. \tag{3.15}$$

We shall attach a boundary $\partial\mathcal{U}$ to the open set $\mathcal{U}$ such that T extends to a function $\overline{T}$ on $\mathcal{U} \cup \partial\mathcal{U}$, and $\overline{T}$ is the defining function for this boundary. We then investigate the geometry on M in the limit $T \to 0$. It is clear from above that the zero locus of $\overline{T}$ will be contained in the zero locus $\mathcal{Z}(\tau)$ of τ, and therefore belongs to the boundary of $\overline{M}$. We will use $\overline{T}$ as a defining function for $\overline{M}$ in an open set $\overline{\mathcal{U}} \subset \overline{M}$. The strategy of the proof is to extend T to a coordinate system on $\mathcal{U}$, such that near the boundary the metric g takes a form as in Theorem 3.0.2.

First define $\theta \in \Lambda^1(\overline{M})$ by

$$V \lrcorner\, \theta = J(V) \lrcorner\, dT, \quad \text{or equivalently} \quad \theta_a = \Omega_{ac} g^{bc} \nabla^{\mathbf{g}}{}_b T, \quad a, b, c = 1, \ldots, 2n \tag{3.16}$$

where J is the para–complex structure of (g, Ω). Using (3.14) this gives

$$\theta = 2T(1-T)\xi_i dx^i - dT.$$

We need n open sets $U_1, \ldots, U_n$ such that $\xi_k \neq 0$ on U_k to cover the zero locus of T. Here we chose $k = n$, and adapt a coordinate system (which we will prove to be Pfaff) given by

$$(T, Z_1, \ldots, Z_{n-1}, X^1, \ldots, X^{n-1}, Y),$$

where T is given by (3.15) and

$$Z_A = \frac{\xi_A}{\xi_n}, \quad X^A = x^A, \quad Y = x^n, \quad \text{where} \quad A = 1, \ldots, n-1.$$

We compute

$$\theta = 2(1-T)\frac{dY + Z_A dX^A}{K} - dT, \quad \xi_n = \frac{1}{KT}, \quad \text{where} \quad K \equiv Y + Z_A X^A,$$

and substitute

$$\xi_i dx^i = \frac{1}{KT}(dY + Z_A dX^A)$$

into (3.14). This gives

$$g = \frac{\theta^2 - dT^2}{4T^2} + \frac{1}{T}h, \tag{3.17}$$

where

$$h = \frac{1}{4(1-T)}(\theta^2 - dT^2) + \frac{1}{K}\Big(dZ_A \odot dX^A - \frac{1}{2(1-T)}X^A dZ_A \odot (\theta + dT)\Big)$$

is regular at the boundary $T = 0$. This is in agreement with the asymptotic form in Theorem 3.0.2 (see [54] for further details).

The restriction h to ∂M gives a metric on a distribution $\mathcal{D} = \mathrm{Ker}(\theta|_{T=0})$

$$\theta|_{T=0} = 2\frac{dY + Z_A dX^A}{Y + Z_A X^A}, \quad h_0 = \frac{1}{4}(\theta|_{T=0})^2 + \frac{1}{2(Y + Z_A X^A)}(2dZ_A \odot dX^A - X^A dZ_A \odot (\theta|_{T=0})). \tag{3.18}$$

Note that T is only defined up to multiplication by a positive function. Changing the defining function in this way results in a conformal rescalling of $\theta|_{T=0}$, thus the metric on the contact distribution is also defined up to an overall conformal scale. We shall choose the scale so that the contact form is given by $\theta_0 \equiv K\theta|_{T=0}$ on $T(\partial M)$, with the metric on $\mathcal{D}$ given by

$$h_{\mathcal{D}} = dZ_A \odot dX^A. \tag{3.19}$$

We now move on to deal with the curved case where the metric on M is given by (**??**). The coordinate system (T, Z_A, X^A, Y) is as above, and the one–form θ in (3.16) is given by

$$\theta = 2T(1-T)\xi_i dx^i - dT + 2T^2(\mathrm{P}_{ij} - \Gamma^k_{ij}\xi_k)x^i dx^j,$$

or in the (T, Z_A, X^A, Y) coordinates,

$$\begin{aligned}\theta = \;& 2(1-T)\frac{Z_A dX^A + dY}{K} - dT \\ & + 2T^2\Bigg[\Bigg(\mathrm{P}_{AB} - \frac{\Gamma^C_{AB}Z_C + \Gamma^n_{AB}}{TK}\Bigg)X^A dX^B + \Bigg(\mathrm{P}_{nB} - \frac{\Gamma^C_{nB}Z_C + \Gamma^n_{nB}}{TK}\Bigg)Y dX^B \\ & + \Bigg(\mathrm{P}_{An} - \frac{\Gamma^C_{An}Z_C + \Gamma^n_{An}}{TK}\Bigg)X^A dY + \Bigg(\mathrm{P}_{nn} - \frac{\Gamma^C_{nn}Z_C + \Gamma^n_{nn}}{TK}\Bigg)Y dY\Bigg].\end{aligned}$$

Guided by the formula (3.17) we define

$$h = Tg - \frac{1}{4T}(\theta^2 - dT^2),$$

which we find to be

$$
\begin{aligned}
h =& \frac{1}{4(1-T)}(\theta^2 - dT^2) + \frac{1}{K}\Big(dZ_A \odot dX^A - \frac{1}{2(1-T)} X^A dZ_A \odot (\theta + dT)\Big) \\
& - \frac{1}{K}\Big((\Gamma^C_{AB} Z_C + \Gamma^n_{AB}) dX^A \odot dX^B + (\Gamma^C_{nn} Z_C + \Gamma^n_{nn}) dY \odot dY + 2(\Gamma^C_{An} Z_C + \Gamma^n_{An}) dX^A \odot dY\Big) \\
& + T(\mathrm{P}_{AB} dX^A \odot dX^B + 2\mathrm{P}_{An} dX^A \odot dY + \mathrm{P}_{nn} dY \odot dY).
\end{aligned} \tag{3.20}
$$

This is smooth as $T \to 0$.

Restricting h to $T = 0$ yields a metric which differs from (3.18) by the curved contribution given by the components of the connection, but not the Schouten tensor. Substituting $dY = K\theta|_{T=0}/2 - Z_A dX^A$, disregarding the terms involving $\theta|_{T=0}$ in h, and conformally rescalling by K yields the metric

$$
\begin{aligned}
h_{\mathcal{D}} &= (dZ_A - \Theta_{AB} dX^B) \odot dX^A, \quad \text{where} \\
\Theta_{AB} &= \Gamma^C_{AB} Z_C + \Gamma^n_{AB} + (\Gamma^C_{nn} Z_C + \Gamma^n_{nn}) Z_A Z_B - 2(\Gamma^C_{An} Z_C + \Gamma^n_{An}) Z_B
\end{aligned} \tag{3.21}
$$

defined on the contact distribution $\mathcal{D} = \mathrm{Ker}(\theta_0)$, where $\theta_0 = 2(dY + Z_A dX^A)$.

We now invoke Theorem 3.0.2, verifying by explicit computation that the remaining two conditions are satisfied. The first of these conditions is that the metric $h_{\mathcal{D}}$ is compatible with the Levi–form of the almost para–CR structure induced on ∂M by J, i.e.

$$
h_{\mathcal{D}}(X, Y) = d\theta_0(JX, Y), \quad \text{for} \quad X \in \mathcal{D}. \tag{3.22}
$$

The second is that the Nijenhuis tensor takes asymptotically tangential values, i.e. that (3.1) is satisfied.

Both of these follow from computing the para–complex structure J in the (T, Z_A, X^A, Y) coordinates. We find

$$
\begin{aligned}
J|_{T=0} = & -\frac{\partial}{\partial X^A} \otimes dX^A + \frac{\partial}{\partial Y} \otimes dY + \frac{\partial}{\partial Z_A} \otimes dZ_A + \frac{\partial}{\partial T} \otimes dT \\
& - \frac{Z_B}{K} \frac{\partial}{\partial T} \otimes dX^B - \frac{1}{K} \frac{\partial}{\partial T} \otimes dY \\
& - \Big(\Gamma^D_{AB} Z_D + \Gamma^n_{AB}\Big) \frac{\partial}{\partial Z_A} \otimes dX^B + \Big(\Gamma^D_{nB} Z_D + \Gamma^n_{nB}\Big) Z_C \frac{\partial}{\partial Z_C} \otimes dX^B \\
& - \Big(\Gamma^D_{An} Z_D + \Gamma^n_{An}\Big) \frac{\partial}{\partial Z_A} \otimes dY + \Big(\Gamma^D_{nn} Z_D + \Gamma^n_{nn}\Big) Z_C \frac{\partial}{\partial Z_C} \otimes dY.
\end{aligned} \tag{3.23}
$$

Restricting to vectors in $\mathcal{D}$ amounts to substituting $dY = \theta_0/2 - Z_A dX^A$ and disregarding the terms involving θ_0 as above, so that

$$J|_{\mathcal{D}} = -\frac{\partial}{\partial X^A} \otimes dX^A + Z_A \frac{\partial}{\partial Y} \otimes dX^A + \frac{\partial}{\partial Z_A} \otimes dZ_A + \frac{\partial}{\partial T} \otimes dT - \frac{2Z_B}{K}\frac{\partial}{\partial T} \otimes dX^B - \Theta_{AB}\frac{\partial}{\partial Z_A} \otimes dX^B$$

and (3.22) is satisfied.

For the Nijenhuis condition, we use the formula

$$\mathcal{N}^a_{bc} = J^d{}_{[b}\partial_{|d|}J^a{}_{c]} - J^d{}_{[b}\partial_{c]}J^a{}_d.$$

Note that we need only consider components of this with $a = T$, and thus only need to work with the $\partial/\partial T$ components of J to find the terms which look like ∂J. This is a one–form which we shall call $J^{(T)}$ and find to be

$$J^{(T)} = \left(-\frac{Z_B}{K} + \frac{T[2Z_B + (\Gamma^D_{AB}Z_D + \Gamma^n_{AB})X^A + (\Gamma^D_{nB}Z_D + \Gamma^n_{nB})Y]}{K} - T^2[\mathrm{P}_{AB}X^A + \mathrm{P}_{nB}Y] \right) dX^B \\ \left(-\frac{1}{K} + \frac{T[2 + (\Gamma^D_{An}Z_D + \Gamma^n_{An})X^A + (\Gamma^D_{nn}Z_D + \Gamma^n_{nn})Y]}{K} - T^2[\mathrm{P}_{An}X^A + \mathrm{P}_{nn}Y] \right) dY.$$

Note that this agrees with (3.23) when $T = 0$. We use it to calculate $\mathcal{N}^a{}_{bc}\nabla_a T$, dropping terms which vanish when $T = 0$ to verify (3.1).

□

3.3.1 Two–dimensional projective structures

In the case if $n = 2$ the coordinates on ∂M are (X, Y, Z), and (3.21) yields

$$h_{\mathcal{D}} = dZ \odot dX - [\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12})Z + (\Gamma^2_{22} - 2\Gamma^2_{12})Z^2 + \Gamma^1_{22}Z^3]dX \odot dX,$$

which is transparently invariant under the projective changes

$$\Gamma^k_{ij} \longrightarrow \Gamma^k_{ij} + \delta^k_i \Upsilon_j + \delta^k_j \Upsilon_i$$

of ∇. In the two-dimensional case the projective structures $(N, [\nabla])$ are equivalent to second order ODEs which are cubic in the first derivatives (see, e.g. [6])

$$\frac{d^2Y}{dX^2} = \Gamma^1_{22}\Big(\frac{dY}{dX}\Big)^3 + (2\Gamma^1_{12} - \Gamma^2_{22})\Big(\frac{dY}{dX}\Big)^2 + (\Gamma^1_{11} - 2\Gamma^2_{12})\Big(\frac{dY}{dX}\Big) - \Gamma^2_{11}, \qquad (3.24)$$

where the integral curves of (3.24) are the unparametrised geodesics of ∇. The integral curves C of (3.24) are integral submanifolds of a differential ideal $\mathcal{I} =< \theta_0, \theta_1 >$, where

$$\theta_0 = dY + ZdX, \quad \theta_1 = dZ - \Big(\Gamma^2_{11} + (\Gamma^1_{11} - 2\Gamma^2_{12})Z + (\Gamma^2_{22} - 2\Gamma^1_{12})Z^2 + \Gamma^1_{22}Z^3\Big)dX$$

are one–forms on a three–dimensional manifold $B = \mathbb{P}(T^*N)$ with local coordinates (X, Y, Z). If $f : C \to B$ is an immersion, then $f^*(\theta_0) = 0, f^*(\theta_1) = 0$ is equivalent to (3.24) as long as $\theta_2 \equiv dX$ does not vanish. In terms of these three one–forms the contact structure, and the metric on the contact distribution are given by $\theta_0, h_{\mathcal{D}} = \theta_1 \odot \theta_2$.

3.4 The model via an orbit decomposition

In this section we describe here the flat (in the sense of parabolic geometries) model [13, 16] of our construction in tractor terms.

The flat projective structure on $N = \mathbb{RP}^n$ gives rise to the neutral signature para–Kähler Einstein metric on $M = SL(n+1)/GL(n)$

$$g = d\xi_i \odot dx^i + (\xi_i dx^i)^2, \quad \Omega = d\xi_i \wedge dx^i, \quad \text{where} \quad i, j, \dots = 1, \dots, n. \qquad (3.25)$$

In [16], §7.1 it was explained how this homogeneous model corresponds to the projectivised co-tractor bundle of $\mathbb{RP}^n$, with an $\mathbb{RP}_{n-1}$ removed from each $\mathbb{RP}_n$ fiber. This $\mathbb{RP}_{n-1}$ corresponds to incident pairs of points and hyperplanes in $\mathbb{R}^{n+1} \times \mathbb{R}_{n+1}$.

Here we shall instead take N to be the sphere S^n with its standard projective structure as this is orientable in all dimensions and, more importantly, on this (double cover of $\mathbb{RP}^n$) the tractor bundle is trivial, and this simplifies the discussion. The underlying space of the (compactified) model of dimension $2n$ is $S^n \times S_n$ where both S_n and S^n denote spheres that are dual as we shall explain.

Consider first two vector spaces each isomorphic to $\mathbb{R}^{n+1}$:

$$V \cong \mathbb{R}^{n+1} \qquad \mathrm{W} \cong \mathbb{R}^{n+1}$$

and view each as a representation space for an $SL(n+1,\mathbb{R})$ action. So $G := SL(V) \times SL(W)$ acts on $V \times W$. (Note that we may wlog consider V and W as respectively the ± 1 eigenspaces of the single vector space $\mathbb{V} := V \oplus W$ equipped with a $\mathbb{J}$ s.t. $\mathbb{J}^2 = 1$.)

Now the action of $SL(V)$ descends to a transitive action on the ray projectivisation $\mathbb{P}_+(V)$ and similarly $SL(W)$ acts transitively on $\mathbb{P}_+(W)$. Thus $G := SL(V) \times SL(W)$ acts transitively on the manifold

$$\mathcal{M} := \mathbb{P}_+(V) \times \mathbb{P}_+(W).$$

We can represent an element of $\mathcal{M}$ in terms of pairs of homogeneous coordinates $([Y],[Z])$ where $0 \neq Y \in V$ and $0 \neq Z \in W$.

Note that as a smooth manifold $\mathcal{M} = S^n \times S^n$, but as a homogeneous manifold it is

$$G/P = \Big(SL(V)/P_X\Big) \times \Big(SL(W)/P_U\Big)$$

where P_X (resp. P_U) is the parabolic subgroup in $SL(V)$ that stabilises a point $[X]$ in $\mathbb{P}_+(V)$ (resp. $[U] \in \mathbb{P}_+(W)$), and P is the group product $P_X \times P_W$ which itself is a parabolic subgroup of the semisimple group G.

Now introduce an additional structure which breaks the G symmetry. Namely we fix an isomorphism

$$I : W \to V^*$$

where V^* denotes the dual space to V. The subgroup $H \cong SL(n+1,\mathbb{R})$ of G that fixes this may be identified with $SL(V)$ which acts on a pair $(Y,Z) \in V \times V^*$ by the defining representation and on the first factor and by the dual representation on the second factor.

Given this structure we may now (suppress I and) view $\mathcal{M}$ as consisting of pairs $([X],[U])$ where $0 \neq X \in V$ and $0 \neq U \in V^*$. That is

$$\mathcal{M} = \mathbb{P}_+(V) \times \mathbb{P}_+(V^*).$$

This is useful as follows: Each element $[U]$ in $\mathbb{P}_+(V^*)$ determines an oriented hyperplane in V and each $[X] \in \mathbb{P}_+(V)$ an oriented line in V. So now we consider the H action on M. This has two open orbits and a closed orbit. The last is the incidence space

$$\mathcal{Z} = \{([X],[U]) \in \mathcal{M} \mid U(X) = 0\}$$

which sits as smooth orientable separating hypersurface in $\mathcal{M}$. Then there are the open orbits

$$M_+ = \{([X],[U]) \in \mathcal{M} \mid U(X) > 0\} \qquad \text{and} \qquad M_- = \{([X],[U]) \in \mathcal{M} \mid U(X) < 0\}.$$

We may think of $\mathcal{Z}$ as the 'boundary' (at infinity) for the open orbits $M_\pm$.

We now describe the geometries on the orbits. The claim is that there are Einstein metrics in $M_\pm$, while $\mathcal{Z}$ is well known as the model for so-called contact Langrangian (or sometimes called para–CR) geometry, this is a real analogue of hypersurface type CR geometry.

First observe that $N_V := \mathbb{P}_+(V)$ is the flat model of projective geometry. So in particular we have

$$0 \to \mathcal{E}_V(-1) \xrightarrow{X} \mathcal{T}_V \to TN_V(-1) \to 0$$

where $\mathcal{T}_V$ is the projective tractor bundle on N_V and X is the tautological section of $\mathcal{T}(1)$, which coincides with the canonical tractor. Similarly there a sequence on $N^W := \mathbb{P}_+(V^*)$

$$0 \to \mathcal{E}^W(-1) \xrightarrow{U} \mathcal{T}^W \to TN^W(-1) \to 0. \tag{3.26}$$

There is a natural tractor bundle $\mathcal{T} := \mathcal{T}_V \oplus \mathcal{T}^W$ on M. Where X and U are not incident this induces a metric on M as follows. Observe that, at a point $([X],[U])$ where $X \lrcorner U \neq 0$, the tractor field U splits the first sequence by $\nu \in \Gamma(\mathcal{E}(-1,0))$ defined by

$$\nu := U/\tau$$

with $\tau := X \lrcorner U$ (and where we have used an obvious weight notation). This follows as $X \lrcorner \nu = 1$. Similarly

$$x := X/\tau \in \Gamma(\mathcal{E}(0,-1))$$

splits the second short exact sequence because $x \lrcorner U = 1$. Thus we obtain a neutral signature metric on $TN_V \oplus TN^W$ by these two steps: First, using these splittings yields a bundle monomorphism

$$TN_V(-1,0) \oplus TN^W(0,-1) \to \mathcal{T}_V \oplus \mathcal{T}^W.$$

Second, this gives a symmetric form $\boldsymbol{g}$ and symplectic form $\boldsymbol{\Omega}$ on $TN_V(-1,0) \oplus TN^W(0,-1)$ by then using the canonical metric and symplectic form on $\mathcal{T}_V \oplus \mathcal{T}^W$ given by the duality of $\mathcal{T}_V$ and $\mathcal{T}^W$. Thus $\boldsymbol{g} \in \Gamma(S^2T^*M(1,1))$ and $\boldsymbol{\Omega} \in \Gamma(\Lambda^2T^*M(1,1))$.

Then set

$$g := \frac{1}{\tau}\boldsymbol{g} \qquad \text{and} \qquad \Omega := \frac{1}{\tau}\boldsymbol{\Omega}.$$

The metric g is easily seen to have neutral signature. It is Einstein because the tractor metric on $\mathcal{T}$ is parallel for the tractor connection (see [55] for the analogous c-projective case). The tractor connecction arises from the usual parallel transport on the vector space $V \oplus V^*$ viewed as an affine manifold.

Chapter 4

Einstein–Weyl structures from (M, g, Ω)

In this chapter, we focus on Einstein manifolds in the Dunajski–Mettler class which arise from projective surfaces with at least one symmetry. As discussed in chapter 1, it is shown in [16] that metrics in this subclass have anti–self–dual Weyl tensor and at least one Killing vector field. From the results of [28] and [42], this means that they are associated to solutions of the $SU(\infty)$–Toda equation via Lorentzian Einstein–Weyl structures in 2+1 dimensions. The aim of this chapter is to exhibit every Einstein–Weyl structure obtainable from the Dunajski–Mettler class, resulting in several examples of new, explicit solutions of the Toda equation, and their associated mini–twistor spaces. Note that in the case $n = 2$ we will write the metric and symplectic form as

$$g = \quad dz_A \odot dx^A - (\Gamma^C_{AB} z_C - z_A z_B - \mathrm{P}_{AB}) dx^A \odot dx^B, \tag{4.1}$$

$$\Omega = \quad dz_A \wedge dx^A + \mathrm{P}_{AB} dx^A \wedge dx^B, \qquad A, B, C = 0, 1. \tag{4.2}$$

where we have replaced $\{p_i\}$ with $\{z_A\}$ and shifted the indices from $i, j = 1, 2$ to $A, B = 0, 1$. This is helpful for the twistorial calculations because it agree with the usual notation for two–component spinor indices. Note that a change of projective connection is now given by

$$\Gamma^C_{AB} \to \Gamma^C_{AB} + \delta^C_A \Upsilon_B + \delta^C_B \Upsilon_A, \quad z_A \to z_A + \Upsilon_A, \quad A, B, C = 0, 1. \tag{4.3}$$

4.0.1 The $SU(\infty)$–Toda equation

The $SU(\infty)$–Toda equation is given by

$$U_{XX} + U_{YY} = \epsilon(e^U)_{ZZ}, \quad \text{where} \quad U = U(X, Y, Z), \quad \text{and } \epsilon = \pm 1 \tag{4.4}$$

Equation (4.4) has originally arisen in the context of complex general relativity [21, 4, 38], and then in Einstein–Weyl [47] and (in Riemannian context, with $\epsilon = -1$) scalar–flat Kähler geometry [30]. It belongs to a class of dispersionless systems integrable by the twistor transform [33, 19, 2], the method of hydrodynamic reduction [22], and the Manakov–Santini approach [31]. The equation is nevertheless not linearisable and most known explicit solutions admit Lie point or other symmetries (there are exceptions - see [9, 10, 32, 39]). The solutions we find depend on two arbitrary functions of one variable, and arise from an essentially linear procedure, where no non–linear PDEs/ODEs have to be solved. An example of a solution in our class is given by an implicit relation[1]

$$4Y^2 e^U (e^U X^2 - Z^2)^3 + (2e^{2U} X^4 - 3e^U X^2 Z^2 + Z^4 + 2Z^2)^2 = 0, \tag{4.5}$$

where the level sets of U in $\mathbb{R}^3$ are real algebraic surfaces.

4.0.2 The neutral signature version of the Jones–Tod correspondence

Theorem 4.0.1 *[28] Let (M, g) be a four–manifold with a neutral signature metric with ASD Weyl tensor, and a conformal Killing vector K. Let*

$$h = |K|^{-2} g - |K|^{-4} \mathbf{K} \odot \mathbf{K}, \qquad \omega = \frac{2}{|K|^2} \star (\mathbf{K} \wedge d\mathbf{K}), \tag{4.6}$$

where $|K|^2 = g(K, K)$, $\mathbf{K} = g(K, \cdot)$ and $\star$ is the Hodge operator defined by g. Then (h, ω) is a solution of the Einstein–Weyl equations on the space of orbits W of K in M. All Lorentzian Einstein–Weyl structures arise from some anti-self-dual (M, g, K).

The final step is the occurrence of the $SU(\infty)$–Toda equation (4.4). This is a consequence of the following result of Tod

Theorem 4.0.2 *[42] Let (h, ω) be the Einstein–Weyl structure arising from Theorem 4.0.1, under the additional assumption that the ASD conformal structure (M, g) is Einstein, and with non–zero Ricci scalar.*

[1] MD: this one has a symmetry

1. *The Einsten–Weyl structure admits a shear–free, twist–free geodesic congruence.*
2. *There exists $h \in [h]$, and coordinates (X, Y, Z) on an open set in W such that (assuming the signature of h is $(2, 1)$ and the congruence is time–like)*

$$h = e^U(dX^2 + dY^2) - dZ^2, \quad \omega = 2U_Z dZ \tag{4.7}$$

and the function $U = U(X, Y, Z)$ satisfies the $SU(\infty)$–Toda equation (4.4) with $\epsilon = 1$.

The whole construction can now be summarised in the following diagram

$$\begin{array}{rcl} \text{Projective structure with symmetry} & \stackrel{\text{Thm ??}}{\longrightarrow} & \text{ASD Einstein with symmetry} \\ \downarrow & & \downarrow \text{Thm 4.0.1} \\ \text{Solution to } SU(\infty) \text{ Toda} & \stackrel{\text{Thm 4.0.2}}{\longleftarrow} & \text{Einstein–Weyl} \end{array} \tag{4.8}$$

The paper is organised as follows. In the next section we summarise the basic facts and relevant formulae underlying Theorems **??** and 4.0.1. In §4.2, Proposition 4.2.1 we present the most general class of EW spaces arising from our construction, and show how to associate solutions of the $SU(\infty)$–Toda equation with this class. In §4.3 and §4.4.5 we give several examples corresponding to $SL(2, \mathbb{R})$, and $SL(3, \mathbb{R})$ invariant projective structures. In the latter case the four–manifold (M, g) of Theorem **??** is $SL(3)/GL(2)$, and the mini–twistor space of the $SU(\infty)$–Toda equation can be constructed explicitly by quotiening the flag manifold $F_{12}(\mathbb{C}^3)$ by a $\mathbb{C}^*$ action. In Poposition 4.3.2 we give an explicit criterion, in terms of the representative metric $h \in [h]$ and the one form ω for a vector field the generate a symmetry of the Weyl structure. In Proposition 4.4.2 we show that the Einstein metric on $SL(3)/GL(2)$ is also pseudo–hyper–Hermitian, and its twistor space fibers holomorphicaly over $\mathbb{CP}^1$. In §**??** we make contact with the Cartan approach to Einstein–Weyl geometry via special 3rd order ODEs. In §2 we shall prove (Theorem **??**) that the $2n$–dimensional analogue of the Einstein metric (4.1) canonically lifts to an Einstein metric of signature $(n, n+1)$ on the $\mathbb{R}^*$ bundle $\mathcal{Q}$ over M with a connection whose curvature is the pullback of the symplectic form from M to $\mathcal{Q}$. Some calculations underlying the proof of Theorem **??** are relegated to Appendix A. In the Appendix B we shall present a solution to the elliptic $SU(\infty)$–Toda equation corresponding to an ALH gravitational instanton.

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4.1 Projective, Einstein, and Weyl geometries

Here we summarise basic facts about projective, Einstein–Weyl, and anti–self–dual geometries.

4.1.1 Projective structures

A two–dimensional projective structure is called flat if the Cotton tensor $\nabla_{[A}P_{B]C}$ vanishes for any choice of the representative connection.

Let (N, ∇) be a manifold with an affine connection. A projective vector field k is a generator of a one–parameter group of transformations mapping unparametrised geodesics of ∇ to unparametrised geodesics. At the infinitesimal level the projective condition is

$$\mathcal{L}_k \Gamma_{AB}^C = \delta_A^C \Upsilon_B + \delta_B^C \Upsilon_A, \tag{4.9}$$

where the Lie derivate of the connection components is defined as in [49]. In general no projective vector field exist on (N, ∇). The possible Lie algebras of projective vector fields on a surface are $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{a}_2$ (the two–dimensional affine Lie algebra) or $\mathbb{R}$. See [5].

4.1.2 Einstein–Weyl structures

Definition 4.1.1 *A Weyl Structure $(W, \mathcal{D}, [h])$ is a conformal equivalence class of metrics $[h]$ on a manifold W along with a fixed torsion–free affine connection $\mathcal{D}$ which preserves any representative $h \in [h]$ up to conformal class. That is, for some one-form ω,*

$$\mathcal{D}h = \omega \otimes h.$$

A pair (h, ω) uniquely defines the connection and hence the Weyl structure, but there is an equivalence class of such pairs which define the same Weyl structure. These are related by transformations

$$h \to \rho^2 h, \quad \omega \to \omega + 2d\ln(\rho), \tag{4.10}$$

where ρ is a smooth, non-zero function on W. Physically, the Weyl condition in Lorentzian signature corresponds to the statement that null geodesics of the conformal structure $[h]$ are also geodesics of the connection $\mathcal{D}$.

If additionally the symmetric part of the Ricci tensor of $\mathcal{D}$ is a scalar multiple of h, then W is said to carry an Einstein-Weyl structure. This condition is invariant under (4.10). The Einstein–Weyl equations give a set of five non–linear PDEs on the pair (h, ω). These equations are integrable by the twistor transform of Hitchin [27], which (by Theorem 4.0.1) can be regarded as a reduction of Penrose's twistor transform [35] for ASD conformal structures. A trivial Einstein–Weyl structure is one whose one–form ω is closed, so that it is locally exact and thus may be set to zero by a change of scale (4.10). Then $\mathcal{D}$ is the Levi–Civita connection of some representative $h \in [h]$, and this representative is Einstein.

4.1.3 Anti–self–dual Einstein metrics

Let M be an oriented four–dimensional manifold with a metric g of signature $(2, 2)$. The Hodge $*$ operator on the space of two forms is an involution, and induces a decomposition [1]

$$\Lambda^2(T^*M) = \Lambda^2_-(T^*M) \oplus \Lambda^2_+(T^*M) \tag{4.11}$$

of two-forms into anti-self-dual (ASD) and self-dual (SD) components, which only depends on the conformal class of g. The Riemann tensor of g can be thought of as a map $\mathcal{R} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ which admits a decomposition under (4.11):

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ - 2\Lambda & \phi \\ \hline \phi & C_- - 2\Lambda \end{array} \right), \tag{4.12}$$

where $C_\pm$ are the SD and ASD parts of the Weyl tensor, ϕ is the trace-free Ricci curvature, and -24Λ is the scalar curvature which acts by scalar multiplication. The metric g is ASD and Einstein if $C_+ = 0$ and $\phi = 0$. In this case the Riemann tensor is also anti-self-dual.

Locally there exist real rank-two vector bundles $\mathbb{S}, \mathbb{S}'$ (spin-bundles) over M equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that

$$TM \cong \mathbb{S} \otimes \mathbb{S}' \tag{4.13}$$

is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2)$$

for $v_1, v_2 \in \Gamma(\mathbb{S})$ and $w_1, w_2 \in \Gamma(\mathbb{S}')$. A vector $V \in \Gamma(TM)$ is called null if $g(V, V) = 0$. Any null vector is of the form $V = \lambda \otimes \pi$ where λ, and π are sections of $\mathbb{S}$ and $\mathbb{S}'$ respectively. An α–plane (respectively a β–plane) is a two–dimensional plane in T_pM spanned by null vectors of the above form with π (respectively λ) fixed, and an α–surface (β–surface) is a two–dimensional surface in M such that its tangent plane at every point is an α–plane (β–plane). Penrose's Nonlinear Graviton Theorem [35] states that a maximal, three dimensional, family of α–surfaces exists in M iff $C_+ = 0$.

ASD Einstein metrics from projective structures

A general ASD metric depends, in the real–analytic category, on six arbitrary functions of the variables. Theorem **??** gives an explicit subclass of such metrics which are additionaly Einstein, and depend on two arbitrary functions of two variables. Any projective structure $(N, [\nabla])$ gives rise to an ASD Einstein metric.

There is an additional structure on four–manifolds arising from Theorem **??** given by a para–Hermitian structure. For any choice of a connection in a projective class $[\nabla]$ there exists diffeomorphism[2] $\varphi : T^*N \to M$ via which we can pull back the metric g and symplectic form Ω to obtain, in canonical local coordinates (x^A, z_A) on the cotangent bundle, the metric g given by (4.1) and the two–form Ω

$$\varphi^*\Omega = dz_A \wedge dx^A + P_{AB} dx^A \wedge dx^B. \tag{4.14}$$

The pair (g, Ω) is projectively invariant under the changes (4.9) if $z_A \to z_A + \Upsilon_A$.

If k is a projective vector field on (N, ∇), then the corresponding Killing vector field on (M, g) is symplectic, and is given in local coordinates by

$$K = k - z_A \frac{\partial k^B}{\partial x^A} \frac{\partial}{\partial z_B} + \Upsilon_A \frac{\partial}{\partial z_A}. \tag{4.15}$$

[2] In [17] it is shown how to extend this metric to a c–projective compactification.

ASD β–foliation

It follows from the general construction of Calderbank [7] and West [48] that any ASD conformal structure arising from Theorem **??** carries a foliation by β–surfaces defined by an ASD two–form $\Sigma_{ab} = \iota_A \iota_B \epsilon_{A'B'}$, and such that the spinor ι_A satisfies

$$\nabla_{A'(A}\iota_{B)} = \mathcal{A}_{A'(A}\iota_{B)} \tag{4.16}$$

where $d\mathcal{A}$ is an ASD Maxwell field.

We shall call such foliations ASD β–surface foliations. In our coordinates $\Sigma = dx^0 \wedge dx^1$ and $D = \text{span}\{\partial/\partial z_0, \partial/\partial z_1\}$. We find that

$$\nabla\Sigma = 6\mathcal{A} \otimes \Sigma, \tag{4.17}$$

where $d\mathcal{A} = \Omega$, and Ω is the symplectic form on M, whose anti-self-duality implies (4.16) for a rescaling of $\mathcal{A}$. In §4.4 we will consider the model case where $M = SL(3)/GL(2)$, for which we can find the Ward transform of this ASD Maxwell two-form to the twistor space $F_{12}(\mathbb{C}^3)$ explicitly, and we will find that there is a second ASD β–surface foliation with a different two–form.

4.2 From projective structures to $SU(\infty)$ Toda fields

Recall (see, e.g. [6]) that a projective structure on a surface can be locally specified by a single 2nd order ODE: taking coordinates (x, y) on the surface we find that geodesics on which $\dot{x} \neq 0$ can be written as unparametrised curves $y(x)$ such that

$$y'' + a_0(x, y) + 3a_1(x, y)y' + 3a_2(x, y)(y')^2 + a_3(x, y)(y')^3 = 0, \tag{4.18}$$

where the coefficients $\{a_i\}$ are given by the projectively invariant formulae

$$a_0 = \Gamma^1_{00}, \quad 3a_1 = -\Gamma^0_{00} + 2\Gamma^1_{01}, \quad 3a_2 = -2\Gamma^0_{01} + \Gamma^1_{11}, \quad a_3 = -\Gamma^0_{11}.$$

Consider the most general Einstein–Weyl structure arising from the combination of Theorem **??** and Theorem 4.0.1. Because of the correspondence (Theorem **??**, part 2.) between symmetries of (M, g) and symmetries of the projective surface $(N, [\nabla])$, the construction must begin with the general projective surface with at least one symmetry. In this case, the unparametrised geodesics can be written uniquely as integral curves of the ODE

$$y'' = A(y)(y')^3 + B(y)(y')^2 + 1, \tag{4.19}$$

(there are other, equivalent, choices. The one we made has been explored in [20]) where we have made a choice of coordinates such that the symmetry is $k = \frac{\partial}{\partial x}$. The resulting projective structure is flat iff both A and B are constant. By trial and error, we chose a representative connection for (4.19) such that (4.1) had the simplest possible form. The choice of connection we took was

$$\Gamma^0_{11} = A(y), \quad \Gamma^1_{00} = -1, \quad \Gamma^1_{11} = -B(y)$$

with all other components vanishing. Note that this choice of connection has a symmetric Ricci tensor, so the Schouten tensor is also symmetric and the symplectic form (4.14) pulls back to just $dz_A \wedge dx^A$. Thus we can write the Maxwell potential $\mathcal{A}$ which is such that $d\mathcal{A} = \Omega$ as $\mathcal{A} = z_A dx^A$. Writing $x^A = (x, y)$, $z_A = (p, q)$, the resulting metric (4.1) is

$$g = (B(y)+p^2+q)dx^2+2(pq+A(y))dxdy+(-A(y)p+B(y)q+q^2)dy^2+dxdp+dydq. \tag{4.20}$$

Factoring by $K = \frac{\partial}{\partial x}$, and following the algorithm of Theorem 4.0.1 gives the following

Proposition 4.2.1 *The most general Einstein–Weyl structure arising from the procedure (4.8) is locally equivalent to*[3]

$$\begin{aligned} h &= \frac{1}{V}\Big((Bq - Ap + q^2)dy + dq\Big)dy - \Big(pq + Ady + \frac{1}{2}dp\Big)^2, \\ \omega &= V(4dq + 2pdp), \quad \textit{where} \quad V = (B + p^2 + q)^{-1} \end{aligned} \tag{4.21}$$

where (p, q, y) are local coordinates on W, and A, B are arbitrary functions of y.

4.2.1 Solution to the $SU(\infty)$–Toda equation

The procedure for extracting the corresponding solution to the $SU(\infty)$–Toda equation is given in [42] (see also [30] and[4] []). It involves finding the coordinates (X, Y, Z) that put the metric (4.21) in the form (4.7). Given an ASD Einstein metric (M, g) with a Killing vector K

[3]MD: something wrong

[4]MD: me and paul

1. The conformal factor $c : M \to \mathbb{R}^+$ given by

$$c = |d\mathbf{K} + *_g d\mathbf{K}|_g{}^{-1/2}$$

has a property that the rescaled self–dual derivative of K

$$\Theta \equiv c^3\Big(\frac{1}{2}(d\mathbf{K} + *_g d\mathbf{K})\Big)$$

is parallel with respect to $\hat{g} = c^2 g$. The metric $\hat{g}$ is Kähler with self–dual Kähler form Θ, and admits a Killing vector K, as $\mathcal{L}_K(c) = 0$.

2. Define a function $Z : M \to \mathbb{R}$ to be the moment map:

$$dZ = K \lrcorner \Theta. \tag{4.22}$$

It is well defined, as the Kähler form is Lie–derived along K.

3. Construct the Einstein–Weyl structure of Theorem 4.0.1 by factoring $(M, \hat{g})$ by K. Restrict the metric h to a surface $Z = Z_0 = \text{const}$, and construct isothermal coordinates (X, Y) on this surface:

$$\gamma \equiv h|_{Z=Z_0} = e^U(dX^2 + dY^2), \quad U = U(X, Y, Z_0).$$

To implement this step chose an orthonormal basis of one–forms such that $\gamma = {e_1}^2 + {e_2}^2$. Now (X, Y) are solutions to the linear system of 1st order PDEs

$$(e_1 + ie_2) \wedge (dX + idY) = 0.$$

4. Extend the coordinates (X, Y) from the surface $Z = Z_0$ to W. This may involve a Z–dependent affine transformation of (X, Y).

Implementing the steps 1–4 on MAPLE we find that if $A = 0$, and $B = B(y)$ is arbitrary, then the $SU(\infty)$–Toda solution is given implicitly by

$$\begin{aligned} X &= -\frac{8\mathrm{e}^{-2\int B(y)dy}Z^3 p}{(Z^2p^2+4)^2}, \quad Y = \int \mathrm{e}^{-2\int B(y)dy}dy + \frac{\mathrm{e}^{-2\int B(y)dy}(-2Z^4p^2+8Z^2)}{(Z^2p^2+4)^2}. \\ U &= \ln\Big(\frac{(Z^2p^2+4)^3}{64Z^2}\Big) + 4\int B(y)dy. \end{aligned} \tag{4.23}$$

We can check that this is indeed a solution using the fact that the $SU(\infty)$–Toda equation is equivalent to $d \star_h dU = 0$. We have also checked by performing a coordinate transformation of (4.4) to the coordinates (y, p, Z).

Example 1.

Consider the flat projective structure with $A = B = 0$, in which case the coordinate p can be eliminated between

$$e^U = \left(\frac{(Z^2p^2+4)^3}{64Z^2} \right), \quad X = -\frac{8Z^3p}{(Z^2p^2+4)^2}$$

by taking a resultant. This yields

$$e^U(e^U X^2 - Z^2)^3 + Z^4 = 0.$$

Example 2.

To simplify the form of (4.23) set

$$G = \int \exp\Big(-2 \int B(y)dy \Big), \quad T = \frac{2Z^2}{Z^2p^2+4}$$

Then (4.23) becomes

$$e^U = \frac{Z^4}{8T^3(G')^2}, \quad Y = G + G'T\Big(\frac{4T}{Z^2} - 1\Big), \quad X^2 = \frac{4T^4(G')^2}{Z^2}\Big(\frac{2}{T} - \frac{4}{Z^2}\Big). \tag{4.24}$$

Eliminating (T, y) between these three equations gives one relation between (X, Y, Z) and U - this is our implicit solution. The elimination can be carried over explicitly if $G = y^k$ for any integer k, or if $G = \exp y$. In the later case the solution is given by (4.5).

4.2.2 Two monopoles

The Einstein–Weyl structures (4.21) we have constructed in Proposition 4.2.1 are special, as they belong to the $SU(\infty)$–Toda class, and so (as shown by Tod [40]) admit a non–null geodesic congruence which has vanishing shear and twist. The general solution to the $SU(\infty)$–Toda equation depends (in the real analytic category) on two arbitrary functions of two variables, but the solutions of the form (4.21) depend on two functions of one variable. The additional constraints on the solutions can be traced back to the four dimensional ASD conformal structures which give rise (by the Jones–Tod construction) to (4.21). In addition to their being ASD and Einstein they are characterised [16] by a β–distribution which is parallel with respect to the Levi–Civita connection and ASD in the sense of Calderbank [7]. The corresponding β–surfaces do not generically intersect with a given α–surface, however if they do intersect then they will intersect in curves (null geodesics) which descend to the Einstein–Weyl structures, and give rise to another (in addition to the Tod shear–free, twist–free) geodesic congruence. In what follows we shall point out how some of this structure arises from a couple of solutions to the Abelian monopole equation on EW backgrounds.

Jones and Tod [28] show that there is a correspondence between conformally ASD four-metrics over an Einstein-Weyl structure $(W, \mathcal{D}, [h])$, and solutions to the monopole equation on W. Hence the symmetry reduction of Theorem 4.0.1 gives us a solution to the monopole equation. In fact, since we have an ASD Maxwell field on M, the reduction gives us a second monopole. In this subsection we compute these explicitly. Given an EW structure (h, ω) in 2+1 dimension, the Abelian monopole consits of a pair (V, α), where V is a function, and α is a one–form subject to an equation

$$dV + \frac{1}{2}\omega V = \star_h d\alpha.$$

The inverse Jones–Tod correspondence [28] associates a neutral signature ASD conformal structure

$$g = Vh - V^{-1}(dx + \alpha)^2$$

with an isometry $K = \partial/\partial x$ with any solution of the monopole equation.

The conformal gauge in the EW geometry of Proposition 4.2.1 is chosen so that

$$\alpha = V(pq + A)dy + \frac{V}{2}dp.$$

Let us call this solution the Einstein monopole, as the resulting conformal class contains an Einstein metric (4.20). The second solution (V_M, α_M) (which we shall call the Maxwell monopole) arises as a symmetry reduction of the ASD Maxwell potential

$$\mathcal{A} = pdx + qdy = -V_M K + \alpha_V,$$

where $K = K_\mu dx^\mu$ is the Killing one–form, and we find

$$V_M = -pV, \quad \alpha_V = qdy - p\alpha.$$

4.3 An example from the submaximally symmetric projective surface

The submaximally symmetric projective surface is the punctured plane $N = \mathbb{R}^2 \backslash 0$, with the symmetry group $SL(2)$ acting via its fundamental representation. Here we have a one parameter family of projective structures falling into three distinct equivalence classes, with geodesics described by the differential equation

$$y'' = -\mu(y - xy')^3, \tag{4.25}$$

where (x, y) are coordinates on $\mathbb{R}^2$ and μ is a constant parameter. The equivalence class that a given projective structure falls into depends on the value of μ: those with $\mu > 0$ form one of the classes, those with $\mu < 0$ form another, and those with $\mu = 0$ form the third. Further details can be found in [5]. For simplicity, we choose $\mu = 1$.

Choosing a representative connection from the projective class defined by (4.25), we obtain from (4.1) an Einstein metric

$$\begin{aligned} g = (p^2 - xy^2p - y^3q + 4y^2)dx^2 + 2(pq + x^2yp + xy^2q - 4xy)dxdy \\ +(q^2 - x^3p - x^2yq + 4x^2)dy^2 + dxdp + dydq \end{aligned} \tag{4.26}$$

on M, again with $z_0 =: p$, $z_1 =: q$, having Killing vectors

$$K_1 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p} - y\frac{\partial}{\partial y} + q\frac{\partial}{\partial q}, \quad K_2 = x\frac{\partial}{\partial y} - q\frac{\partial}{\partial p}, \quad K_3 = y\frac{\partial}{\partial x} - p\frac{\partial}{\partial q}.$$

These are lifts of the projective vector fields corresponding to the $\mathfrak{sl}(2)$ elements

$$T_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}.$$

Factoring by K_3 and choosing coordinates

$$u = \frac{p^2}{y^2}, \quad v = 2\ln(y^2), \quad w = xp + yq,$$

we obtain an Einstein-Weyl structure

$$\begin{aligned} h &= -du^2 - 2dudw - w(w^2 + u - 5w + 4)dv^2 + 2(u - w + 4)dvdw, \\ \omega &= \frac{1}{u - w + 4}du - \frac{3w}{u - w + 4}dv - \frac{4}{u - w + 4}dw. \end{aligned} \tag{4.27}$$

The solution to the $SU(\infty)$–Toda equation (4.4) which determines the Einstein-Weyl structure (4.27) is described by an algebraic curve $f(\mathrm{e}^U, X, Y, Z) = 0$ of degree six in e^U and degree twelve in the other coordinates. This solution has been found following th Steps 1-4 in §4.2.1, and is given by

$$\begin{aligned}
64\mathrm{e}^{6U}X^6(X+Y)^3(X-Y)^3 - 92\mathrm{e}^{5U}X^4Z^2(X+Y)^3(X-Y)^3 \\
+48\mathrm{e}^{4U}X^2Z^2(5X^6Z^2 - 14X^4Y^2Z^2 + 13X^2Y^2Z^2 - 4Y^4Z^2 + 9X^4 + 27X^2) \\
+8\mathrm{e}^{3U}Z^4(-20X^6Z^2 + 48X^4Y^2Z^2 - 36X^2Y^4Z^2 + 8Y^6Z^2 - 81X^4 - 243X^2Y^2) \\
+3\mathrm{e}^{2U}Z^4(20X^4Z^4 - 36X^2Y^2Z^4 + 16Y^4Z^4 + 108X^2Z^2 + 216Y^2Z^2 + 243) \\
+6\mathrm{e}^{U}Z^8(-2X^2Z^2 + 2Y^2Z^2 - 9) + Z^{12} \\
= 0.
\end{aligned}$$

Note that the formulae (4.27) are independent of the coordinate v, and therefore have a symmetry. This was unexpected because there is no other symmetry of (M, g) that commutes with K_3. However, it is possible for symmetries to appear in the Einstein-Weyl structure without a corresponding symmetry of the ASD conformal structure. This can be seen from the general formula (4.6); the function V may depend on the coordinate v so that g depends on v even though h does not. For example, the Gibbons-Hawking metrics [23] give a trivial Einstein–Weyl structure with the maximal symmetry group, but the four-metric is in general not so symmetric. Our discovery of this unexpected symmetry motivated a more concrete description of a symmetry of a Weyl structure.

Definition 4.3.1 *An infinitesimal symmetry of a Weyl structure* $(W, \mathcal{D}, [h])$ *is a vector field* $\mathcal{K}$ *which is both an affine vector field with respect to the connection*[5] $\mathcal{D}$ *and a conformal Killing vector with respect to the conformal structure* $[h]$*.*

Proposition 4.3.2 *Given an infinitesimal symmetry* $\mathcal{K}$ *of a Weyl structure* $(W, \mathcal{D}, [h])$ *in dimension* N*, and a representative* $h \in [h]$ *such that* $\mathcal{D}h = \omega \otimes h$*, there exists a smooth function* $f : W \to \mathbb{R}$ *such that*

$$\mathcal{L}_{\mathcal{K}}h = fh, \qquad \mathcal{L}_{\mathcal{K}}\omega = \frac{1}{N}d[\mathcal{K} \lrcorner\, d(\ln(\det(h)))]. \tag{4.28}$$

Proof. The first equation follows immediately from the fact that $\mathcal{K}$ is a conformal Killing vector of h. It remains to evaluate the Lie derivative of the one–form ω along

[5]Recall that an affine vector field of a connection $\mathcal{D}$ is one which preserves its components, i.e. $\mathcal{L}_{\mathcal{K}}\Gamma^i_{jk} = 0$.

the flow of $\mathcal{K}$ given that $\mathcal{L}_{\mathcal{K}} h = fh$ and $\mathcal{L}_{\mathcal{K}}\Gamma^i_{jk} = 0$, where Γ^i_{jk} are the components of the connection $\mathcal{D}$. We do this by considering the Lie derivative of $\mathcal{D}h$:

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}(\mathcal{D}_i h_{jk}) &= \mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) - \mathcal{L}_{\mathcal{K}}(\Gamma^l_{ji} h_{lk} + \Gamma^l_{ki} h_{jl})\\ &= \mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) - f(\Gamma^l_{ji} h_{lk} + \Gamma^l_{ki} h_{jl}).\end{aligned}$$

Now

$$\begin{aligned}\mathcal{L}_{\mathcal{K}}(\partial_i h_{jk}) &= \mathcal{K}^l\partial_l\partial_i h_{jk} + (\partial_i\mathcal{K}^l)\partial_l h_{jk} + (\partial_j\mathcal{K}^l)\partial_i h_{lk} + (\partial_k\mathcal{K}^l)\partial_i h_{jl}\\ &= \partial_i[\mathcal{K}^l\partial_l h_{jk} + (\partial_j\mathcal{K}^l)h_{lk} + (\partial_k\mathcal{K}^l)h_{jl}] - (\partial_i\partial_j\mathcal{K}^l)h_{lk} - (\partial_i\partial_k\mathcal{K}^l)h_{jl}.\end{aligned}$$

The term with square brackets is just

$$\partial_i(\mathcal{L}_{\mathcal{K}} h_{jk}) = \partial_i(f h_{jk}) = f\partial_i h_{jk} + \partial_i f h_{jk},$$

so we have

$$\mathcal{L}_{\mathcal{K}}(\mathcal{D}_i h_{jk}) = f\mathcal{D}_i h_{jk} + \partial_i f h_{jk} - (\partial_i\partial_j\mathcal{K}^l)h_{lk} - (\partial_i\partial_k\mathcal{K}^l)h_{jl}.$$

Setting this equal to $\mathcal{L}_{\mathcal{K}}(\omega_i h_{jk}) = (\mathcal{L}_{\mathcal{K}}\omega_i)h_{jk} + f\omega_i h_{jk}$ and cancelling $f\omega_i h_{jk}$ with $f\mathcal{D}_i h_{jk}$, we find

$$\begin{aligned}(\mathcal{L}_{\mathcal{K}}\omega_i)g_{jk} &= \partial_i f h_{jk} - (\partial_i\partial_j\mathcal{K}^l)h_{lk} - (\partial_i\partial_k\mathcal{K}^l)h_{jl}\\ \implies \quad \mathcal{L}_{\mathcal{K}}\omega_i &= \partial_i f - \frac{2}{N}\partial_i\partial_j\mathcal{K}^j. \end{aligned} \tag{4.29}$$

Finally, we note that

$$\partial_i\partial_j\mathcal{K}^j = \frac{N}{2}\partial_i f - \frac{1}{2}\partial_i[\mathcal{K} \lrcorner\, d(\ln(\det(h))].$$

This follows from tracing the expression $\mathcal{L}_{\mathcal{K}} h_{ij} = f h_{ij}$:

$$\begin{aligned}\mathcal{L}_{\mathcal{K}} h_{ij} = \mathcal{K}^k\partial_k h_{ij} + (\partial_i\mathcal{K}^k)h_{kj} + (\partial_j\mathcal{K}^k)h_{ik} &= f h_{ij}\\ \implies \quad \mathcal{K}^k h^{ij}\partial_k h_{ij} + 2\partial_k\mathcal{K}^k &= Nf\\ \implies \quad 2\partial_i\partial_k\mathcal{K}^k &= N\partial_i f - \partial_i(\mathcal{K}^k h^{jl}\partial_k h_{jl})\end{aligned}$$

and recalling that $h^{jl}\partial_k h_{jl} = \partial_k \ln(\det(h))$. Substituting into (4.29) then yields the result.

$\square$

We can easily verify the invariance of (4.28) under Weyl transformations. Let $(\hat{h}, \hat{\omega})$ be a new metric and one–form related to the old ones by (4.10). Then

$$\mathcal{L}_{\mathcal{K}}\hat{\omega} = \mathcal{L}_{\mathcal{K}}\omega + 2\mathcal{K} \lrcorner\, d\ln(\rho)$$

from (4.10), and from (4.28) we have

$$\begin{aligned}
\mathcal{L}_{\mathcal{K}}\hat{\omega} &= \frac{1}{N} d[\mathcal{K} \lrcorner\, d(\ln(\rho^{2N}\det(h)))] \\
&= \frac{1}{N} d[\mathcal{K} \lrcorner\, d(\ln(\det(h)))] + \frac{2N}{N}\mathcal{K} \lrcorner\, d\ln(\rho) \\
&= \mathcal{L}_{\mathcal{K}}\omega + 2\mathcal{K} \lrcorner\, d\ln(\rho),
\end{aligned}$$

as above. Note that the function f in (4.28) will change according to

$$\hat{f} = f + 2\mathcal{K} \lrcorner\, d\ln\rho.$$

In the case of the Weyl structure (4.27), the infinitesimal symmetry is

$$\mathcal{K} = \frac{\partial}{\partial v}.$$

Since we have chosen a scale such that $\mathcal{K}$ is in fact a Killing vector of h, we have that $\mathcal{K} \lrcorner\, d(\ln(\det(h)) = 0$, so the one–form ω is also preserved by $\mathcal{K}$. This is consistent with the fact that it has no explicit v–dependence.

4.4 The model $SL(3)/GL(2)$ and its reductions

In the following section we discuss the four–manifold (M, g) obtained from the maximally symmetric flat projective surface $N = \mathbb{RP}^2$. In this case, g is the indefinite analogue of the Fubini–Study metric, and is not only bi–Lagrangian but also para–Kähler, since the symplectic form Ω is parallel with respect to the Levi–Civita connection of g. Chosing a representative connection with $\Gamma_{AB}^{C} = 0$ gives g as

$$g = dz_A \odot dx^A + z_A z_B dx^A \odot dx^B. \tag{4.30}$$

We begin by discussing the conformal structure of (4.30), both explicitly and in terms of twistor lines. We then note some structure which is unique to the model case:

hyper-hermiticity and a second foliation by β–surfaces which is ASD in the sense of Calderbank[6].

Finally, we present a classification of the Einstein–Weyl structures which can be obtained by Jones–Tod factorisation of $SL(3)/GL(2)$ and exhibit an explicit example of such a factorisation from the twistor perspective, reconstructing the conformal structure on W from minitwistor curves.

4.4.1 Conformal Structure

Let $M \subset \mathbb{P}^2 \times \mathbb{P}^{2^*}$ be set of non–incident pairs (P, L), where $P \in \mathbb{P}^2$, and $L \subset \mathbb{P}^2$ is a line.

Proposition 4.4.1 *Two pairs (P, L) and $(\widetilde{P}, \widetilde{L})$ are null–separated with respect to the conformal structure (4.30) if there exists a line which contains three points $(P, \widetilde{P}, L \cap \widetilde{L})$.*

Proof. The null condition of Proposition 4.4.1 defines a co–dimension one cone in TN: generically there is no line through three given points.

To find an analytic expression for the resulting conformal structure consider two pairs (P, L) and $(\widetilde{P}, \widetilde{L})$ of non–incident points and lines. Let $L + t\widetilde{L}$ be a pencil of lines. There exists t such that

$$P \cdot (L + t\widetilde{L}) = 0, \quad \widetilde{P} \cdot (L + t\widetilde{L}) = 0. \tag{4.31}$$

Eliminating t from (4.31) gives

$$(P \cdot L)(\widetilde{P} \cdot \widetilde{L}) - (\widetilde{P} \cdot L)(P \cdot \widetilde{L}) = 0.$$

Setting $\widetilde{P} = P + dP, \widetilde{L} = L + dL$ yields a metric g representing the conformal structure

$$g = \frac{dP \cdot dL}{P \cdot L} - \frac{1}{(P \cdot L)^2}(L \cdot dP)(P \cdot dL).$$

We can use the normalisation $P \cdot L = 1$, so that $P \cdot dL = -L \cdot dP$, and

$$g = dP \cdot dL + (L \cdot dP)^2. \tag{4.32}$$

We take affine coordinates

$$P = [x^A, 1], \quad L = [z_A, 1 - x^A z_A] \tag{4.33}$$

[6]MD: we do not actually do that

with a normalisation $P \cdot L = 1$ to recover the metric (4.1) which now takes the form (4.30).

□

4.4.2 Twistor space

Let $F_{12}(\mathbb{C}^3) \in \mathbb{P}^2 \times \mathbb{P}^{2*}$ be set of incident pairs (p, l), so that $p \cdot l = 0$. This is the twistor space of (M, g). A $\mathbb{P}^1$ embedding corresponding to a point (P, L) consists of all lines l thorough P, and all points $p = l \cap L$:

$$P \cdot l = 0, \quad p \cdot L = 0, \quad p \cdot l = 0. \tag{4.34}$$

Let (P, L) and $(\widetilde{P}, \widetilde{L})$ be null separated. The corresponding lines in F_{12} intersect at a point (p, l) given by

$$p = L \wedge \widetilde{L}, \quad l = P \wedge \widetilde{P},$$

where $[L \wedge \widetilde{L}]^\alpha = \epsilon^{\alpha\beta\gamma} L_\alpha \widetilde{L}_\beta$ etc. The incidence condition $p \cdot l = 0$ now gives the conformal structure (4.31). The contact structure on F_{12} is $1/2(l \cdot dp - p \cdot dl) = p \cdot dl$.

We shall now give an explicit parametrisation of twistor lines, and show how the metric (4.32) arises from the Penrose condition [35, 46]. Let $P \in \mathbb{P}^2$. The corresponding $l \in \mathbb{P}^{2*}$ is

$$l = P \wedge \pi, \quad \text{where} \quad \pi \sim a\pi + bP,$$

where $a \in \mathbb{R}^*, b \in \mathbb{R}$. Thus π parametrises a projective line $\mathbb{P}^1$, and by making a choice of b we can take $\pi = [\pi^0, \pi^1, 0]$, where $\pi^A = [\pi^0, \pi^1] \in \mathbb{P}^1$. The constraint $P \cdot l = 0$ now holds. To satisfy the remaining constraints in (4.34) we take

$$p = L \wedge l = (L \cdot \pi)P - (L \cdot P)\pi.$$

Substituting (4.33) gives the corresponding twistor line parametrised by $[\pi] \in \mathbb{P}^1$

$$p^\alpha = [(z \cdot \pi)x^A - \pi^A, z \cdot \pi], \quad l_\alpha = [\pi_A, -\pi \cdot x], \tag{4.35}$$

where the spinor indices are raised and lowered with ϵ^{AB} and its inverse, and $z \cdot x \equiv z_A x^A$.

We shall now derive the expression for the conformal structure. According to the Nonlinear Graviton prescription of Penrose [35] a vector $V \in \Gamma(T_m M)$ is null if the corresponding section of the normal bundle $N(L_m) = \mathcal{O}(1) \oplus \mathcal{O}(1)$ has a single zero. To compute the normal bundle, let $([l(\pi, P, L)], [p(\pi, P, L)])$ be the twistor line corresponding to a point $m = (P, L)$ in M. The neighbouring line is $([l + \delta l], [p + \delta p])$, where

$$\delta l = \delta P \wedge \pi, \quad \delta p = (\delta L \cdot \pi)P + (L \cdot \pi)\delta P - \delta(L \cdot P)\pi.$$

The lines $(l + \delta l, p + \delta p)$ and (l, p) intersect at one point which correspond to some particular value of π. Therefore

$$l + \delta l \sim l, \quad \text{so} \quad \pi \sim \delta P = [\delta x^1, \delta x^2, 0].$$

The other condition is $p + \delta p \sim p$ which holds iff

$$0 = p \wedge \delta p = (L \cdot \pi)^2 P \wedge \delta P - (L \cdot P)(\delta L \cdot \pi)\pi \wedge P - (L \cdot \pi)\delta(L \cdot P)P \wedge \pi - (L \cdot P)(L \cdot \pi)\pi \wedge \delta P.$$

Substituting $\pi \sim \delta P$, we find that all terms on the RHS are proportional to $P \wedge \delta P = [0, 0, x \cdot dx]$. Therefore

$$(L \cdot \pi)^2 - (L \cdot \pi)\delta(L \cdot P) + (L \cdot P)(\delta L \cdot \pi) = 0,$$

together with $L \cdot P = 1$. This gives the conformal structure (4.32).

4.4.3 Hyper–Hermitian structure

A pseudo–hyper–complex structure on a four manifold M is a triple of endomorphisms I, S, T of TM which satisfy

$$I^2 = -Id, \quad S^2 = T^2 = Id, \quad IST = Id,$$

and such that $aI + bS + cT$ is an integrable complex structure foo any point on the hyperboloid $a^2 - b^2 - c^2 = 1$. A neutral signature metric g on a pseudo–hyper–complex four–manifold is pseudo-hyper-Hermitian if

$$g(V, V) = g(IV, IV) = -g(SV, SV) = -g(TV, TV)$$

for any vector field V on M. There is a unique conformal structure compatible with each pseudo–hyper–complex structure. With a natural choice of orientation which makes the fundamental two–forms of I, S, T self–dual, this conformal structure is anti–self–dual.

Proposition 4.4.2 *The Einstein metric (4.30) on $SL(3)/GL(2)$ is pseudo–hyper–Hermitian.*

Proof. First a note about conventions. Our model metric is ASD, so it is the primed Weyl spinor which vanishes. This makes all indices on both the EW space and the projective surface primed. Lets therefore swap the role of primed and unprimed indices. The null frame for the 4-metric is

$$e^{0'A} = dx^A, \quad e^{1'A} = dz^A + z^A(z \cdot dx), \quad \text{so that} \quad g = \epsilon_{A'B'}\epsilon_{AB}e^{A'A}e^{B'B}.$$

Thus the forms $\Sigma = dx^0 \wedge dx^1$ and $\Omega = dz_A \wedge dx^A$ are ASD. The basis of SD two forms is spanned by

$$dx \wedge dq + q^2 dx \wedge dy, \quad dx \wedge dp - dy \wedge dq + 2pq dx \wedge dy, \quad -dy \wedge dp + p^2 dx \wedge dy$$

or, in a more compact notation, by $\Sigma^{AB} = dx^{(A} \wedge dz^{B)} + z^A z^B \Sigma$. We can verify that

$$d\Sigma^{AB} + 2\mathcal{A} \wedge \Sigma^{AB} = 0, \tag{4.36}$$

where $\mathcal{A} = z_A dx^A$ is such that $d\mathcal{A} = \Omega$. The condition (4.36) is neccesary and sufficient for (para) hyper–Hermiticity [3, 18]. Thus the ASD Maxwell fields arising from the para–Kähler structure on M, and the para hyper–Hermitian structure coincide. To this end note that the twistor distribution form (M, g) is

$$L_{0'} = \pi \cdot \frac{\partial}{\partial x} + (z \cdot \pi) z \cdot \frac{\partial}{\partial z}, \quad L_{1'} = \pi \cdot \frac{\partial}{\partial z}. \tag{4.37}$$

It is Frobenius integrable, as $[L_{0'}, L_{1'}] = -(\pi \cdot z) L_{1'}$. It also does not contain the vertical $\partial/\partial\pi$ terms which again confirms the hyper–Hermiticity of (M, g) (see Lemma 2 in [18] and Theorem 7.1 in [8]). The SD part of the spin connection is given in terms of $\mathcal{A}$ as $\Gamma_{A'ABC} = -2\mathcal{A}_{A'(B}\epsilon_{C)A}$.

□

In the next section we shall show how to encode $\mathcal{A}$ in the twisted–photon Ward bundle over the twistor space of (M, g).

4.4.4 The twisted photon

The twistor space F_{12} described in §4.4.2 is the projectivised tangent bundle $T(\mathbb{P}^2)^*$ of the minitwistor space of the flat projective structure: a point in F_{12} consists of $l \in \mathbb{P}^2$, and a direction through l. Thus the twistor space of M is the correspondence space of $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. There are many open sets needed to cover $\mathbb{P}(T\mathbb{P}^2)$, but it is sufficient to consider two: U, where $(l_1, \neq 0, p^2 \neq 0)$, and $(l_2/l_1, l_3/l_1, p^3/p^2)$ are coordinates, and $\widetilde{U}$ where $(l_1 \neq 0, p^3 \neq 0)$, and $(l_2/l_1, l_3/l_1, p^2/p^3)$ are coordinates. Now consider the total space of $T\mathbb{P}^2$ (or perhaps it is $T\mathbb{P}^2$ tensored with some power of the canonical bundle to make it trivial on twistor lines), and restrict it to the intersection of (pre–images in $T\mathbb{P}^2$ of) U and $\widetilde{U}$. The coordinates on $T\mathbb{P}^2$ in these region are $(l_2/l_1, l_3/l_1, p^2/p^1, p^3/p^1)$, and the fiber coordinates over τ over U and $\tilde{\tau}$ over $\widetilde{U}$ are related by[7]

$$\tilde{\tau} = \exp(F)\tau, \quad \text{where} \quad F = \ln(p_2/p_3).$$

Now we follow the procedure of [44]: restrict F to a twistor line, and split it. The holomorphic splitting is $F = H - \widetilde{H}$, where $H = \ln(p_2)$ is holomorphic in the pre–image of U in the correspondence space, and $\widetilde{H} = \ln(p_3)$ is holomorphic in the pre–image of $\widetilde{U}$. Note that F is a twistor function, but $H, \widetilde{H}$ are not. Therefore $L_{A'}F = 0$, where the twistor distribution $L_{A'}$ is given by (4.37). This, together with the Liouville theorem implies that

$$L_{A'}H = L_{A'}\widetilde{H} = \pi^A \mathcal{A}_{A'A}$$

for some one–form $\mathcal{A}$ on M, as the LHS is holomorphic on $\mathbb{CP}^1$ and homogeneous of degree one. To construct this one–form recall the parametrisation of twistor curves (4.35). This gives

$$H = \ln(z \cdot \pi), \quad \widetilde{H} = \ln((z \cdot \pi)x^1 - \pi^1)$$

and

$$L_{1'}(H) = L_{1'}(\widetilde{H}) = 0, \quad L_{0'}(H) = L_{0'}(\widetilde{H}) = \pi \cdot z.$$

Therefore $\mathcal{A}_{1'A} = 0, \mathcal{A}_{0'A} = z_A$ which gives $\mathcal{A} = z_A dx^A$, and $d\mathcal{A}$ is indeed the ASD para–Kähler structure.

4.4.5 Factoring $SL(3)/GL(2)$ to Einstein-Weyl

If a metric with ASD Weyl tensor has more than one conformal symmetry, then distinct Einstein–Weyl structures are obtained on the space of orbits of conformal Killing vectors which are not conjugate with respect to an isometry [34]. We can thus classify the Einstein–Weyl structures obtainable from $SL(3)/GL(2)$ by first classifying its symmetries up to conjugation.

[7] Here we are following Ward [44], and thinking of a $\mathbb{C}^*$ bundle.

Proposition 4.4.3 *The non–trivial Einstein–Weyl structures obtainable from $SL(3)/GL(2)$ by the Jones–Tod correspondence consist of a two–parameter family, and two additional cases which do not belong to this family.*

Proof. Since we have an isomorphism between the Lie algebra of projective vector fields on $(N, [\nabla])$ and the Lie algebra of Killing vectors on (M, g), the problem of classifying the symmetries of $M = SL(3)/GL(2)$ is reduced to a classification of the infinitesimal projective symmetries of $\mathbb{RP}^2$, i.e. the near–identity elements of $SL(3)$, up to conjugation. Non–singular complex matrices are determined up to similarity by their Jordan normal form (JNF). While real matrices do not have such a canonical form, all of the information they contain is determined (up to similarity) by the JNF that they would have if they were considered as complex matrices. Thus we can still discuss the JNF of a real matrix, even if it cannot always be obtained from the real matrix by a real similarity transformation. The possible non–trivial Jordan normal forms of matrices in $SL(3)$ are shown below.

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1/\lambda\mu \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

It is possible that two matrices in $SL(3)$ with the same JNF may be related by a complex similarity transformation, and thus not conjugate in $SL(3)$. However, if the JNF is a real matrix, then the required similarity transformation just consists of the eigenvectors and generalised eigenvectors of the matrix, which must also be real since they are defined by real linear simultaneous equations. This means we only have to worry about matrices with complex eigenvalues, and since these occur in complex conjugate pairs, they will only be a problem when we have three distinct eigenvalues.

In this case, we can always make a real similarity transformation such that the matrix is block diagonal, with the real eigenvalue in the bottom right. Then we have limited choice from the 2×2 matrix in the top left. Let us parametrise such a 2×2 matrix by a, b, c, $d \in \mathbb{R}$ as follows:

$$\begin{pmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix}.$$

This has characteristic polynomial

$$\chi(\lambda) = \lambda^2 - (2 + \epsilon(a + d))\lambda + 1 + (a + d)\epsilon + (ad - bc)\epsilon^2.$$

Evidently the important degrees of freedom are $a + d$ and $ad - bc$, so we can use these to encode every near–identity element of the class with three distinct eigenvalues. The bottom–right entry will be determined by our choice of $a + d$ and $ad - bc$.

Taking a projective vector field on $\mathbb{RP}^2$, we can find the corresponding Killing vector on $SL(3)/GL(2)$ using (4.15), and factor to Einstein–Weyl using (4.6). We find by explicit calculation that vector fields arising from the second and fourth JNFs above give trivial Einstein–Weyl structures, so restricting to the non–trivial cases we have a two–parameter family of Einstein–Weyl structures coming from the first class, and two additional Einstein–Weyl structures coming from the third and fifth, as claimed.

□

4.4.6 Mini–twistor correspondence

Below we investigate a one–parameter subfamily of the two–parameter family. We use the holomorphic vector field on the twistor space F_{12} (see §4.4.2) corresponding to the chosen symmetry, and reconstruct the conformal structure $[h]$ on N using minitwistor curves (in the sense of [27]) on the space of orbits. Take $a \in \mathbb{R}$ and

$$K = P^1 \frac{\partial}{\partial P^1} - L_1 \frac{\partial}{\partial L_1} + aP^2 \frac{\partial}{\partial P^2} - aL_2 \frac{\partial}{\partial L_2}, \tag{4.38}$$

In order to preserve the relations

$$p \cdot L = 0, \quad P \cdot l = 0, \quad p \cdot l = 0,$$

the corresponding holomorphic action on (p, l) must be $p \mapsto Mp$, $l \mapsto M^{-1}l$, thus the holomorphic vector field $\mathcal{K}$ on F_{12} is

$$\mathcal{K} = p^1 \frac{\partial}{\partial p^1} - l_1 \frac{\partial}{\partial l_1} + ap^2 \frac{\partial}{\partial p^2} - al_2 \frac{\partial}{\partial l_2}.$$

In order to factor F_{12} by this vector field, we must find invariant minitwistor coordinates (Q, R). In addition to satisfying $\mathcal{K}(Q) = \mathcal{K}(R) = 0$, they must be homogeneous of degree zero in (P, L). We choose

$$Q = \frac{p^1 l_1}{p^2 l_2}, \quad R = \frac{(l_1)^a}{l_2 (l_3)^{a-1}}.$$

Substituting in our parametrisation (4.35) and using the freedom to perform a Mobius transformation on π, we obtain

$$Q = \frac{(\lambda t - u - 1)\lambda}{v\lambda + \lambda - \frac{uv}{t}} \tag{4.39}$$
$$R = \lambda^a \Big(- \lambda - \frac{v}{t} \Big)^{1-a},$$

where we have defined $\lambda = \pi_0/\pi_1$, and the Einstein-Weyl coordinates

$$u = xp, \quad v = yq, \quad t = x^a q.$$

Note these are invariants of the Killing vector (4.38).

Next we wish to use these minitwistor curves to reconstruct the conformal structure of the Einstein-Weyl space. In doing so we follow [34]. The tangent vector field to a fixed curve is given by

$$T = \frac{\partial Q}{\partial \lambda}\frac{\partial}{\partial Q} + \frac{\partial R}{\partial \lambda}\frac{\partial}{\partial R},$$

Hence we can write the normal vector field as

$$\begin{aligned} N &= dQ\frac{\partial}{\partial Q} + dR\frac{\partial}{\partial R} \mod T \\ &= \left(\frac{\partial R}{\partial \lambda}\right)^{-1}\left(dQ\frac{\partial R}{\partial \lambda} - dR\frac{\partial Q}{\partial \lambda}\right)\frac{\partial}{\partial Q}, \end{aligned}$$

where

$$dQ = \frac{\partial Q}{\partial u}du + \frac{\partial Q}{\partial v}dv + \frac{\partial Q}{\partial t}dt$$

and similarly for dR. Calculating N using (4.39), we find

$$N \propto (A\lambda^2 + B\lambda + C)\frac{\partial}{\partial Q},$$

where

$$\begin{aligned} A &= t^2(v+1)dt - t^3 dv, \\ B &= -2tuvdt + t^2(a+2u)dv - t^2 du, \\ C &= uv(1+u)dt - tu(1+u)dv - atvdu. \end{aligned}$$

The discriminant of this quadratic in λ then gives a representative $h \in [h]$ of our conformal structure:

$$\begin{aligned} h = 4(u^2v + uv^2 + uv)dt^2 - 4tv(a(v+1) + u)dtdu + 4tu(u - av + 2v + 1)dtdv \\ -t^2du^2 + 2t^2(2av + a + 2u)dvdu - t^2(a^2 + 4u(a-1))dv^2. \end{aligned} \tag{4.40}$$

This is the same conformal structure that we obtain by Jones-Tod factorisation of $SL(3)/GL(2)$ by (4.38) using the formula (4.6).

Chapter 5

The ϕ^4 kink on a wormhole spacetime

The soliton resolution conjecture [78] states that solutions to solitonic equations with generic initial data should, after some non–linear behaviour, eventually resolve into a finite number of solitons plus a radiative term. This conjecture is intimately tied to soliton stability, which has been investigated for a number of solitonic equations, including that of ϕ^4 theory on $\mathbb{R}^{1,1}$. We study a modification of this theory on a $3+1$ dimensional wormhole with a spherical throat of radius a, with a focus on the stability properties of the modified kink. In particular, we prove that the modified kink is linearly stable, and compare its discrete spectrum to that of the ϕ^4 kink on $\mathbb{R}^{1,1}$. We also study the resonant coupling between the discrete modes and the continuous spectrum for small but non–linear perturbations. Numerical and analytical evidence for asymptotic stability is presented for the range of a where the kink has exactly one discrete mode.

5.1 Introduction: the ϕ^4 kink on $\mathbb{R}^{1,1}$

One dimensional ϕ^4 theory is well–documented in the literature (see for example [74]). The aim of this section is to introduce some notation and some ideas about stability which will be useful when we come to consider the modified theory.

5.1.1 Topological Stability and the Kink Solution

Our action takes the standard form

$$S = \int \left(\frac{1}{2} \eta^{ab} \partial_a \phi \partial_b \phi + U(\phi) \right) dx,$$

where η^{ab} is the Minkowski metric on $\mathbb{R}^{1,1}$, and the potential term is

$$U(\phi) = \frac{1}{2}(1 - \phi^2)^2,$$

which is plotted in figure 5.1. There are two vacua, given by $\phi = \pm 1$. Finiteness of the associated conserved energy

$$E = \int \left(\frac{1}{2}(\phi_t)^2 + \frac{1}{2}(\phi_x)^2 + U(\phi) \right) dx,$$

requires that the field lies in one of these two vacua in the limits $x \to \pm\infty$. We can thus classify finite energy solutions in terms of their topological charge

$$N = \frac{1}{2}\Big[\phi(\infty) - \phi(-\infty)\Big]. \tag{5.1}$$

The equations of motion are

$$\phi_{tt} = \phi_{xx} + 2\phi(1 - \phi^2) \tag{5.2}$$

and we find a static solution

$$\phi = \tanh(x - c) =: \Phi_0(x) \tag{5.3}$$

which we call the flat kink. It interpolates between the two vacua and thus has topological charge $N = 1$. The constant c is a constant of integration which can be thought of as the position of the kink. It is evident that no finite energy deformation can affect N. For this reason, we say that the kink is topologically stable.

5.1.2 Linear Stability

A second notion of stability which will be important to our discussion is linear stability. On discarding non–linear terms, we find that small pertubations $\phi(t, x) = \Phi_0(x) +$

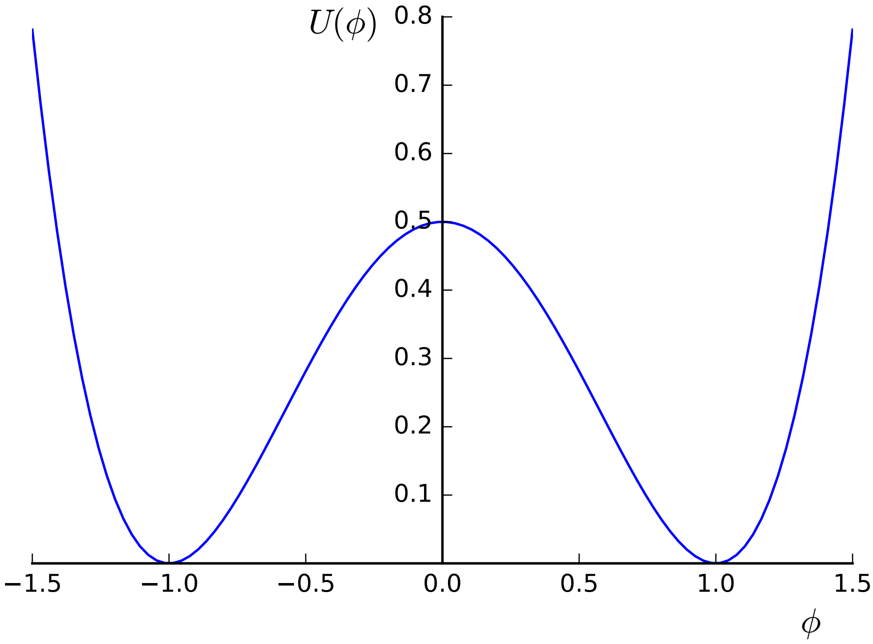

Fig. 5.1 A plot of the potential $U(\phi)$.

$e^{i\omega t}v_0(x)$ satisfy the Schrödinger equation

$$L_0 v_0 := -v_0'' - 2(1-3\Phi_0^2)v_0 = \omega_0^2 v_0. \tag{5.4}$$

The potential $V_0(x) = -2[1-3\Phi_0(x)^2]$ is shown in figure 5.2. It exhibits a so–called "mass gap", meaning that it takes a positive value in the limits $x \to \pm\infty$. In this case, $V_0(\pm\infty) = 4$. For $\omega^2 > 4$, (5.4) admits a continuous spectrum of wave–like solutions.

In addition to its continuum states, the Schrödinger operator in (5.4) has two discrete eigenvalues with normalisable solutions given by

$$\Big(v_0(x), \omega_0\Big) = \left(\frac{\sqrt{3}}{2}\mathrm{sech}^2(x), 0\right) \quad \text{and} \quad \Big(v_0(x), \omega_0\Big) = \left(\frac{\sqrt{3}}{\sqrt{2}}\mathrm{sech}(x)\tanh(x), \sqrt{3}\right), \tag{5.5}$$

where we have chosen the normalisation constant such that $\int_{-\infty}^{\infty} v_0^2(x)dx = 1$.

The first of these is the zero mode of the kink. Its existence is guaranteed by the translation invariance of (5.2), and up to a constant it is equal to $\Phi_0'(x)$. Excitation of this state corresponds to performing a Lorentz boost. In the non–relativistic limit, this amounts to replacing the constant c in (5.3) with a term vt for some $v \ll 1$.

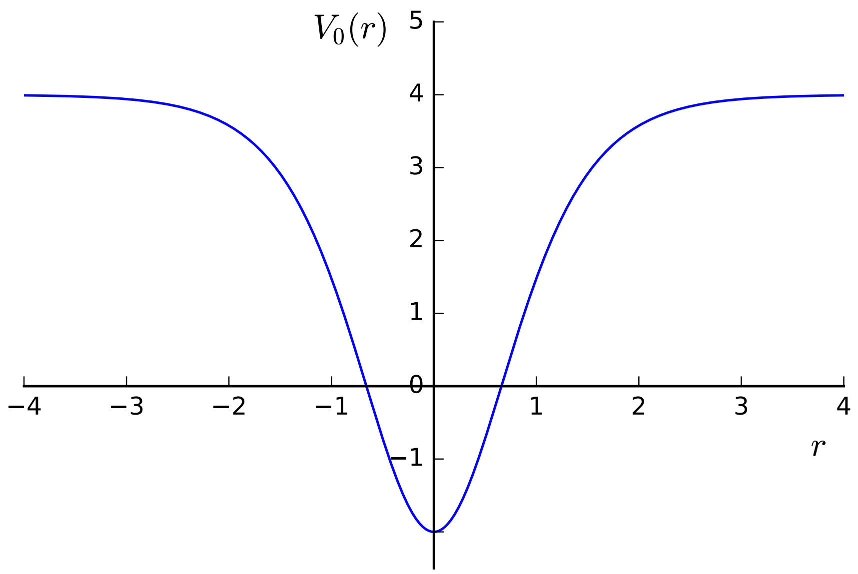

Fig. 5.2 The Schrödinger potential $V_0(r)$ appearing in the linear stability analysis of the ϕ^4 kink on $\mathbb{R}^{1,1}$
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The second normalisable solution, called an *internal mode*[1], has non–zero frequency ω, and is thus time periodic. In the full non–linear theory, it decays through resonant coupling to the continuous spectrum [73]. This phenomenon is of considerable interest in non–linear PDEs, and was studied in detail in [76]. The corresponding process in the modified theory will be discussed in section 5.4.

Linear stability of the kink is equivalent to this Schrödinger operator L_0 having no negative eigenvalues. This is desirable condition because if there were some solution to (5.11) with $\omega_0^2 < 0$, then the corresponding perturbation would grow exponentially with time. One way to see that the kink is linearly stable is via the Sturm oscillation theorem:

Theorem 5.1.1 (Sturm) *Let L be a differential operator of the form*

$$L = -\frac{d^2}{dx^2} + V(x)$$

[1]The presence of an internal mode sets this model apart from its friend the Sine–Gordon equation, whose associated discrete spectrum contains only the zero mode. However, the Sine–Gordon equation possesses a special solution called a wobbling kink, which can be thought of as a non–linear superposition of a kink and a breather. See [75] for more details.

on the square integrable functions u on the interval $(0, \infty)$, with the boundary condition $u(0) = 0$ (corresponding to even parity) or $u'(0) = 0$ (corresponding to odd parity). Let ω^2 be an eigenvalue of L with associated eigenfunction $u(x; \omega)$. Then the number of eigenvalues of L (subject to the appropriate boundary conditions) which are strictly below ω^2 is exactly the number of zeros of $u(x; \omega)$ in $(0, \infty)$.

Since the eigenfunctions (5.5) have no zeros on the interval $(0, \infty)$, it follows that there can be no eigenfunctions with $\omega^2 < \omega_0^2 = 0$, and thus the kink is linearly stable.

5.1.3 Asymptotic stability

The final notion of stability that we will consider is that of asymptotic stability. Stated simply, asymptotic stability of the kink means that for sufficiently small initial pertubations, solutions of (5.2) will converge locally to $\Phi_0(r)$ or its Lorentz boosted counterpart. This was proved in [72] for odd perturbations, but has not been proved in the general case.

5.1.4 Derrick's Scaling Argument

Generalisation of the finite energy ϕ^4 kink to higher dimensional Minkowski spacetimes is prohibited by a scaling argument due to Derrick. Suppose $\Phi_d(\mathbf{x})$ is a static, finite energy solution to the equation of motion of the ϕ^4 theory on $\mathbb{R}^{1,n}$. Then it is a minimiser of the (static) energy

$$E(\Phi_n) = \int \Big(\nabla\Phi_n(\mathbf{x}) \cdot \nabla\Phi_n(\mathbf{x}) + U(\Phi_n)\Big) d^n x =: E_1 + E_2,$$

where we have split E into the two components coming from the two different terms in the integrand. Now consider a spatial rescaling $\mathbf{x} \to \mu\mathbf{x}$, $\mu > 0$ and define

$$\begin{aligned} e(\mu) = E(\Phi_n(\mu\mathbf{x})) =& \int \Big(\nabla(\Phi_n(\mu\mathbf{x})) \cdot \nabla(\Phi_n(\mu\mathbf{x})) + U(\Phi_n(\mu\mathbf{x}))\Big) d^d x \\ =& \int \Big(\mu^2\nabla\Phi_n(\mu\mathbf{x}) \cdot \nabla\Phi_n(\mu\mathbf{x}) + U(\Phi_n(\mu\mathbf{x}))\Big) d^d x \\ =& \mu^{2-n} E_1 + \mu^{-n} E_2, \end{aligned}$$

where we have obtained the last line by a change of variables from $\mathbf{x}$ to $\mu\mathbf{x}$.

If $\Phi_n(\mathbf{x})$ is a minimiser of E then it $\mu = 1$ must also be a stationary point of $e(\mu)$. Evaluating the derivative yields

$$e'(\mu) = \begin{cases} -n\mu^{-n-1}E_2, & \text{if } n = 2 \\ (2-n)\mu^{1-n}E_1 - n\mu^{-n-1}E_2, & \text{otherwise.} \end{cases}$$

Since E_1, E_2 and μ are all positive, the derivative can only have a zero only when n and $2-n$ have the same sign, which only happens when $n = 1$. We thus conclude that no static, finite energy solutions to the equations of motion exist for $n > 1$.

In order to construct a higher dimensional ϕ^4 kink, we must add curvature. In the next section we introduce a curved background, and show that a modified ϕ^4 kink exists on this background. We will also examine a limit in which the modified kink reduces to the flat kink. In section 5.3 we consider linearised perturbations around the modified kink, proving that it is linearly stable and comparing its discrete spectrum to that of the flat kink. In section 5.4 we examine the mode of decay to the modified kink in the full non–linear theory, in particular the resonant coupling of its internal modes to the continuous spectrum.

5.2 The static kink on a wormhole

We now replace the flat $\mathbb{R}^{1,1}$ background with a wormhole spacetime (M, g), where

$$g = -dt^2 + dr^2 + (r^2 + a^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

for some constant $a > 0$, and $-\infty < r < \infty$. This spacetime was first studied by Ellis [70] and Bronnikov [69], and has featured in a number of recent studies about kinks and their stability [67, 68]. Note the presence of asymptotically flat ends as $r \to \pm\infty$, connected by a spherical throat of radius a at $r = 0$.

Our action is the modified by the presence of a non-flat metric:

$$S = \int \left(\frac{1}{2} g^{ab}\partial_a\phi\partial_b\phi + U(\phi) \right) \sqrt{-g} dx,$$

where x^a are now local coordinates on M. Variation with respect to ϕ gives

$$\Box_g \phi + 2\phi(1 - \phi^2) = 0 \tag{5.6}$$

where $\Box_g\phi = \frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\phi)$. We assume ϕ is independent of the angular coordinates (ϑ, φ), so (5.6) can be written explicitly as

$$\phi_{tt} = \phi_{rr} + \frac{2r}{r^2 + a^2}\phi_r + 2\phi(1 - \phi^2). \tag{5.7}$$

The conserved energy in the theory is given by

$$E = \int_{-\infty}^{+\infty} \left(\frac{1}{2}(\phi_t)^2 + \frac{1}{2}(\phi_r)^2 + \frac{1}{2}(1 - \phi^2)^2\right)(r^2 + a^2)dr,$$

which we require to be finite. This imposes the condition $\phi^2 \to 1$ as $r \to \pm\infty$, so that the field lies at one of the two vacua at both asymptotically flat ends.

Static solutions $\phi(r)$ satisfy

$$\phi'' + \frac{2r}{r^2 + a^2}\phi' = -\frac{d}{d\phi}\left(-\frac{1}{2}(1 - \phi^2)^2\right), \tag{5.8}$$

which, if we think of r as a time coordinate, can be thought of as a Newtonian equation of motion for a particle at position ϕ moving in a potential $\mathcal{U}(\phi) = -U(\phi)$, with a time dependent friction term.

In addition to the two vacuum solutions, we have a single soliton solution which interpolates between the saddle points at $(-1, 0)$ and $(1, 0)$ in the in the (ϕ, ϕ') plane. Its existence and uniqueness among odd parity solutions follow from a shooting argument: suppose the particle lies at $\phi = 0$ when $r = 0$. If its velocity $\phi'(0)$ is too small, it will never reach the local maximum of the potential at $\phi = 1$, but if $\phi'(0)$ is too large it will overshoot the maximum so that $\mathcal{U}(\phi) \to -\infty$ as $r \to \infty$, thus having infinite energy. Continuity ensures that there is some critical velocity $\phi'(0)$ such that the particle reaches $\phi = 1$ in infinite time and has zero velocity upon arrival. This corresponds to the non–trivial kink solution, which we call $\Phi(r)$. Time reversal implies that $\phi \to -1$ as $r \to -\infty$.

We can find $\phi'(0)$ numerically using a shooting method. For $a = 1$ we find $\phi'(0) = 1.379602$. A plot of the corresponding soliton is shown in figure 5.3. Since no finite energy deformation can change the value of the topological charge N in the curved version of (5.1), we again conclude that $\Phi(r)$ is topologically stable.

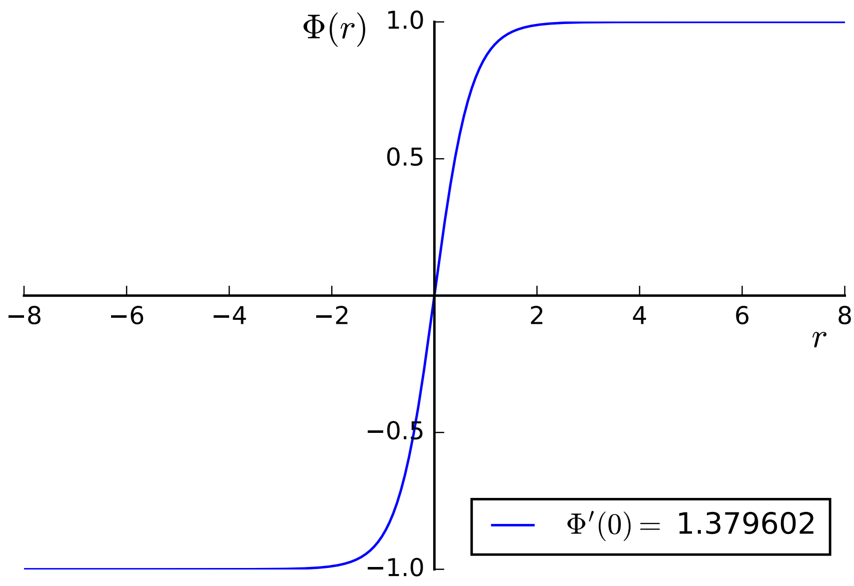

Fig. 5.3 The kink solution $\Phi(r)$ for $a = 1$.

5.2.1 Large a limit

As $a \to \infty$, equation (5.7) becomes the standard equation (5.2) for the flat kink. It is thus helpful to expand the modified kink in $\epsilon^2 := 1/a^2$ for small ϵ^2, since we can then solve both (5.7) and (5.11) analytically up to $\mathcal{O}(\epsilon^4)$. We shall denote the static kink by $\Phi_\epsilon(r)$ in this limit. It satisfies

$$\Phi_\epsilon'' + \frac{2r\epsilon^2}{\epsilon^2 r^2 + 1}\Phi_\epsilon' = -2\Phi_\epsilon(1 - \Phi_\epsilon^2). \tag{5.9}$$

Setting $\Phi_\epsilon(r) = \Phi_0(r) + \epsilon^2\Phi_1(r) + \mathcal{O}(\epsilon^4)$ we obtain at order zero the equation (5.2) of a static kink on $\mathbb{R}^{1,1}$. This has solution (5.3), where we set $c = 0$ to restrict to solutions with odd parity.

At order ϵ^2 we find that $\Phi_1(r)$ must satisfy

$$\Phi_1'' + 2r\text{sech}^2 r = 2\Phi_1(2 - 3\text{sech}^2 r).$$

The unique solution which is odd and decays as $r \to \pm\infty$ is given by

$$\Phi_1(r) = \frac{1}{24}\text{sech}^2 r(f_1(r) + f_2(r) + f_3(r)),$$

where

$$
\begin{aligned}
f_1(r) &= r\Big[3 - 8\cosh(2r) - \cosh(4r)\Big],\\
f_2(r) &= \sinh(2r)\Big[8\log(2\cosh(r)) - 1\Big] + \sinh(4r)\log(2\cosh(r)),\\
f_3(r) &= \frac{\pi^2}{2} + 6r^2 + 6\mathrm{Li}_2(-\mathrm{e}^{-2r}),
\end{aligned}
$$

and $\mathrm{Li}_2(z)$ is the dilogarithm function.

To show that $\Phi_1(r)$ is odd, note that $\mathrm{sech}^2 r$ is an even function, and that f_1 and f_2 are constructed from products of even and odd functions, and hence are odd. To see that f_3 is also odd, we use Landen identity for the dilogarithm:

$$
\begin{aligned}
\mathrm{Li}_2(-\mathrm{e}^{-2r}) + \mathrm{Li}_2(-\mathrm{e}^{2r}) &= -\frac{\pi^2}{6} - \frac{1}{2}\Big[\log(\mathrm{e}^{-2r})\Big]^2\\
&= -\frac{\pi^2}{6} - 2r^2,
\end{aligned}
$$

thus verifying $f_3(r) + f_3(-r) = 0$.

We now turn to the behaviour of $\Phi_1(r)$ as $r \to \infty$. Since $\mathrm{sech}^2 r \sim 4\mathrm{e}^{-2r}$ for large r, we need only consider terms in the $\{f_i\}$ of order e^{2r} or higher. We first note that

$$
\begin{aligned}
\log(2\cosh r) &= \log(\mathrm{e}^r(1 + \mathrm{e}^{-2r})) = r + \log(1 + \mathrm{e}^{-2r})\\
&= r + \mathrm{e}^{-2r} + \mathcal{O}(\mathrm{e}^{-4r}).
\end{aligned}
$$

Then

$$
\begin{aligned}
f_1(r) &= -4r\mathrm{e}^{2r} - \frac{r}{2}\mathrm{e}^{4r} + \mathcal{O}(\mathrm{e}^r)\\
f_2(r) &= \frac{1}{2}\mathrm{e}^{2r}(8r + 8\mathrm{e}^{-2r} - 1) + \frac{1}{2}\mathrm{e}^{4r}(r + \mathrm{e}^{-2r}) + \mathcal{O}(\mathrm{e}^r)\\
&= 4r\mathrm{e}^{2r} + \frac{r}{2}\mathrm{e}^{4r} + \mathcal{O}(\mathrm{e}^r),
\end{aligned}
$$

so $f_1(r) + f_2(r) = \mathcal{O}(\mathrm{e}^r)$. Since $f_3(r) = \mathcal{O}(r^2)$ for large r, we see that $\Phi_1(r)$ vanishes as $r \to \infty$, as we expect. Note that its vanishing as $r \to -\infty$ then follows using parity. A plot of $\Phi_1(r)$ is shown in figure 5.4.

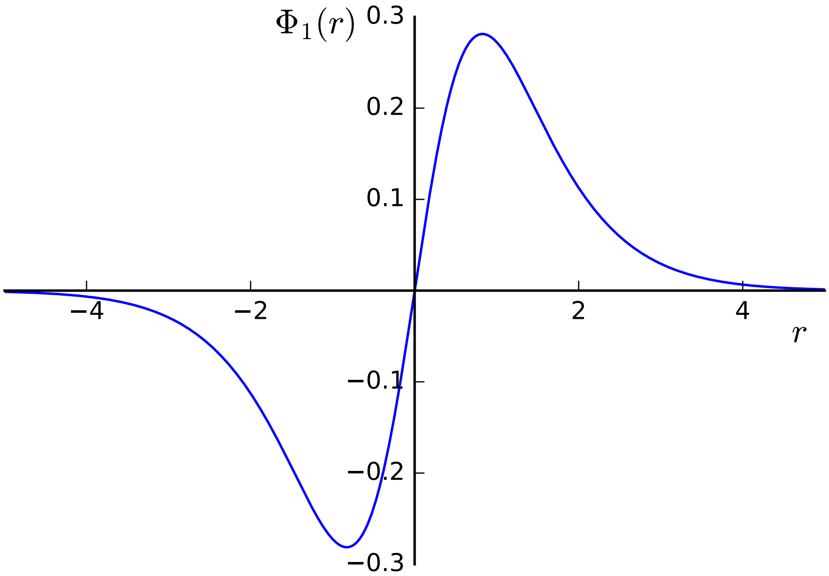

Fig. 5.4 The order ϵ^2 perturbation to the static kink on $\mathbb{R}^{1,1}$.

5.3 Linearised perturbations around the kink

To study the linear stability of the kink, we first plug

$$\phi(t,r) = \Phi(r) + w(t,r) \tag{5.10}$$

into equation (5.7), discarding terms non–linear in w. Imposing the fact that $\Phi(r)$ satifies (5.8), we find

$$w_{tt} = w_{rr} + \frac{2r}{r^2+a^2} w_r + 2w(1-3\Phi^2).$$

For $w(t,r) = \mathrm{e}^{i\omega t}(r^2+a^2)^{-1/2}v(r)$, this becomes a one–dimensional Schrödinger equation

$$Lv := (-\partial_r\partial_r + V(r))v = \omega^2 v, \tag{5.11}$$

where the potential is given by

$$V(r) = \frac{a^2}{(r^2+a^2)^2} - 2(1-3\Phi^2). \tag{5.12}$$

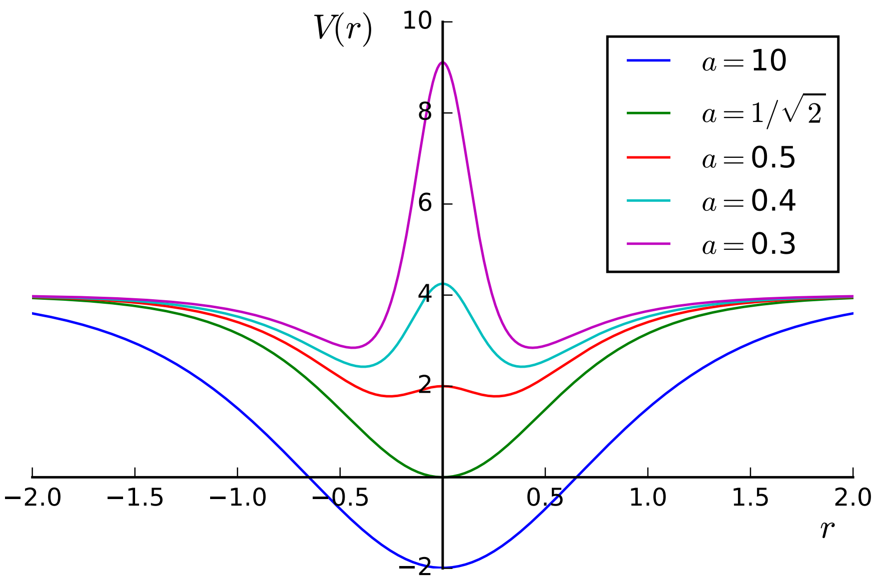

Fig. 5.5 The potential of the 1–dimensional quantum mechanics problem arising from the study of stability of the soliton for values of a between $a = 10$ and $a = 0.3$. In particular, note that those with $a < 1/\sqrt{2}$ are everywhere positive.

Figure 5.5 shows the potential $V(r)$ for several values of a. Note that for large a it has a single well with a minimum at $r = 0$, close to the potential in figure 5.2 corresponding to the flat kink. As a decreases, the critical point at $r = 0$ becomes a maximum with minima on either side, creating a double well. We find numerically that this happens at about $a = 0.55$.

Proposition 5.3.1 *The kink solution $\Phi(r)$ is linearly stable.*

Proof. We first decompose the potential $V(r)$ in (5.11) as $V = V_0 + V_1 + V_a$, where

$$V_0 = -2[1 - 3\Phi_0(r)^2], \qquad V_1 = 6[\Phi(r)^2 - \Phi_0(r)^2], \qquad V_a = \frac{a^2}{(r^2 + a^2)^2},$$

and $\Phi_0(r)$ is given by (5.3) with $c = 0$. As discussed above, we know that the operator $L_0 = -\partial_r\partial_r + V_0$ has no negative eigenvalues. It then follows that L itself has no negative eigenvalues as long as the functions $V_1(r)$ and $V_a(r)$ are everwhere non–negative.

The latter is obvious; to proove the former we recall that we can think of $\Phi(r)$ and $\Phi_0(r)$ as the trajectories of particles moving in the potential $\mathcal{U}(\phi)$, where r is imagined as the time coordinate. The particle corresponding to $\Phi(r)$ suffers an increased frictional force compared to $\Phi_0(r)$, i.e.

$$\Phi_0'' = -\frac{\partial \mathcal{U}}{\partial \phi}\bigg|_{\phi=\Phi_0}, \qquad \Phi'' + \frac{2r}{r^2+a^2}\Phi' = -\frac{\partial \mathcal{U}}{\partial \phi}\bigg|_{\phi=\Phi}. \tag{5.13}$$

Both Φ and Φ_0 interpolate between the maxima of $\mathcal{U}$ at $\phi = \pm 1$; reaching the minimum $(\phi = 0)$ when $r = 0$.

Multiplying the equations (5.13) by Φ_0' and Φ' respectively, then integrating from r to ∞, we have that at every instant of time

$$\frac{1}{2}(\Phi_0')^2 + \mathcal{U}(\Phi_0) = 0, \qquad \frac{1}{2}(\Phi')^2 + \mathcal{U}(\Phi) = \int_r^\infty \frac{2r}{r^2+a^2}(\Phi')^2 dr. \tag{5.14}$$

These equations are equivalent to conservation of energy for each of the particles. Note that the integral on the RHS is non–negative for $r \geq 0$, and vanishes only at $r = \infty$. In particular, when $r = 0$ we have $\mathcal{U}(\Phi) = \mathcal{U}(\Phi_0) = -1/2$, so $\Phi'(0) > \Phi_0'(0)$. This means $V_1(r)$ is initially increasing from zero.

For $V_1(r)$ to return to zero at some finite $r = r_0$, we would need that $\Phi(r_0) = \Phi_0(r_0)$ at a point where $\Phi'(r_0) \leq \Phi_0'(r_0)$. However, this is made impossible by equations (5.14), since at such a point $\mathcal{U}(\Phi) = \mathcal{U}(\Phi_0)$ and the integral on the RHS is positive. Hence $V_1(r)$ remains non–negative for all $r > 0$, and thus for all r since it is even in r.

□

5.3.1 Finding internal modes numerically

Bound states of the potential (5.12) correspond to internal modes of the kink like the odd solution of (5.4) in (5.5). In contrast, for frequencies greater than $\omega = 2$, solutions to (5.11) are interpreted as radiation. It is possible to search for bound states of (5.12) numerically. The method for this is as follows:

1. For the chosen value of a, find the required value of $\phi'(0)$ using the shooting method described above, generating the soliton $\Phi(r)$.

2. Calculate the potential $V(r)$.

3. For some initial guess of the eigenvalue ω^2, integrate equation (5.11) numerically, setting $v(0) = 1$ and $v'(0) = 0$ to obtain even bound states and $v(0) = 0, v'(0) = 1$ to obtain odd bound states.[2]

[2]Note that we can do this WLOG because the solution $v(r)$ is defined only up to overall scale.

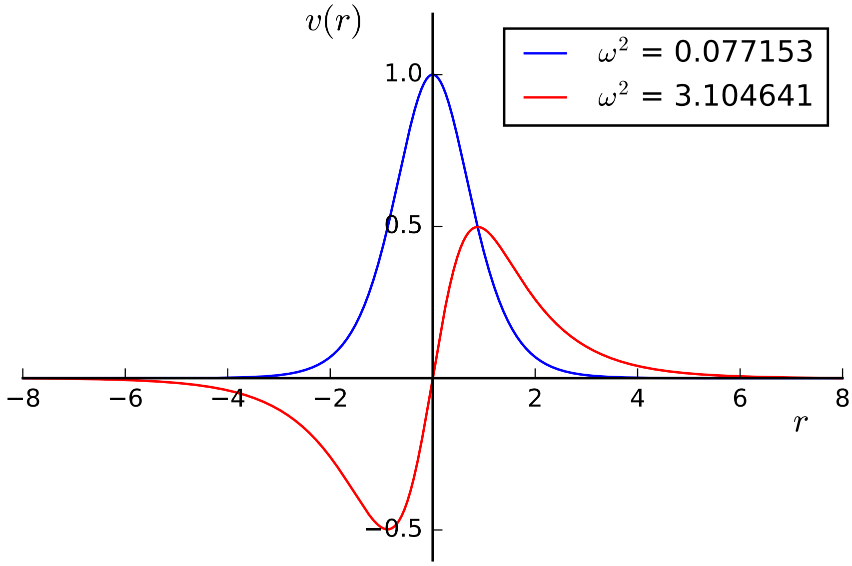

Fig. 5.6 The even and odd bound states of the potential for $a = 5$.

4. Tune the value of ω^2 until a bound state is obtained.

This procedure will only be effective within the range of r for which $\phi'(0)$ is calculated.

It seems that for large a, the potential has both an even and an odd bound state which look qualitatively similar to the internal modes (5.5) of the ϕ^4 kink on $\mathbb{R}^{1,1}$. As a decreases, the eigenvalues ω^2 of the bound states increase, until they disappear into the continuous spectrum ($\omega^2 > 4$). The odd state disappears around $a = 0.8$, and the even state disappears around $a = 0.3$. A plot of both the even and odd bound state for $a = 5$ shown in figure 5.6. For $a = 0.4$, a plot of the single even bound state is shown in figure 5.7. Note that for small values of a, there is a local maximum of the potential located at $r = 0$, with local minima on either side. For $a = 0.4$, this is reflected in the local maxima of the even bound state either side of $r = 0$.

5.3.2 Large a limit

We can also perturbatively expand the eigenvalues of the eigenvalue problem (5.11). Consider solutions to (5.7) of the form[3] $\phi_\epsilon(r) = \Phi_\epsilon(r) + e^{i\omega t} v_\epsilon(r)$, where v_ϵ is small.

[3]Note that in section 5.3 we considered perturbations $v(r)$ which differ from $v_\epsilon(r)$ by a factor of $(r^2 + a^2)^{-1/2}$, since such perturbations are described by a Schrödinger problem. Here it will be simpler to remove this factor; however there is a one–to–one correspondence between $v(r)$ and $v_\epsilon(r)$.

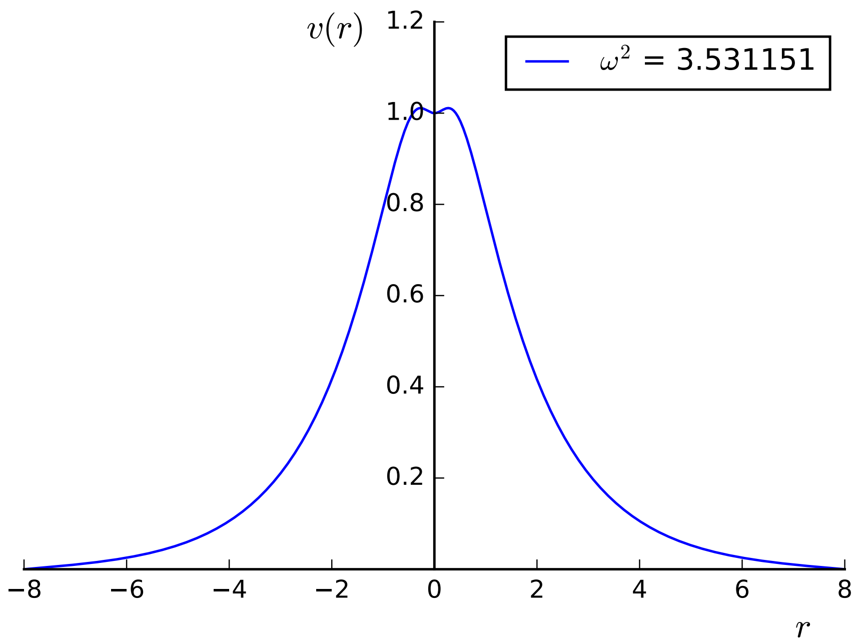

Fig. 5.7 The single even bound state of the potential $V(r)$ for $a = 0.4$.

These satisfy

$$v_\epsilon'' + \frac{2r\epsilon^2}{\epsilon^2 r^2 + 1} v_\epsilon' + 2(1 - 3\Phi_\epsilon^2) v_\epsilon = -\omega_\epsilon^2 v_\epsilon. \tag{5.15}$$

Let $(v_\epsilon, \omega_\epsilon^2)$ be a solution to (5.15) with

$$\omega_\epsilon^2 = \omega_0^2 + \epsilon^2 \xi + \mathcal{O}(\epsilon^4) \quad \text{and} \quad v_\epsilon(r) = v_0(r) + \epsilon^2 v_1(r) + \mathcal{O}(\epsilon^4).$$

Our aim will be to find ξ. Substituting into (5.15), at zero order we obtain the equation (5.4) which controls the linear stability analysis of the ϕ^4 kink on $\mathbb{R}^{1,1}$.

Note that the first solution in (5.5), the zero mode, is proportional to $\Phi_0'(r)$. Its existence follows from translation invariance of (5.4). It corresponds to an even bound state of the Schrödinger equation (5.11) in the limit $a \to \infty$. The second solution in (5.5) is the first non–trivial vibrational mode, and corresponds to an odd bound state of (5.11) in the limit $a \to \infty$.

The terms of order ϵ^2 in (5.15) give us

$$v_1'' + 2rv_0' + 2(1 - 3\Phi_0^2)v_1 - 12\Phi_0\Phi_1 v_0 = -\omega_0^2 v_1 - \xi v_0. \tag{5.16}$$

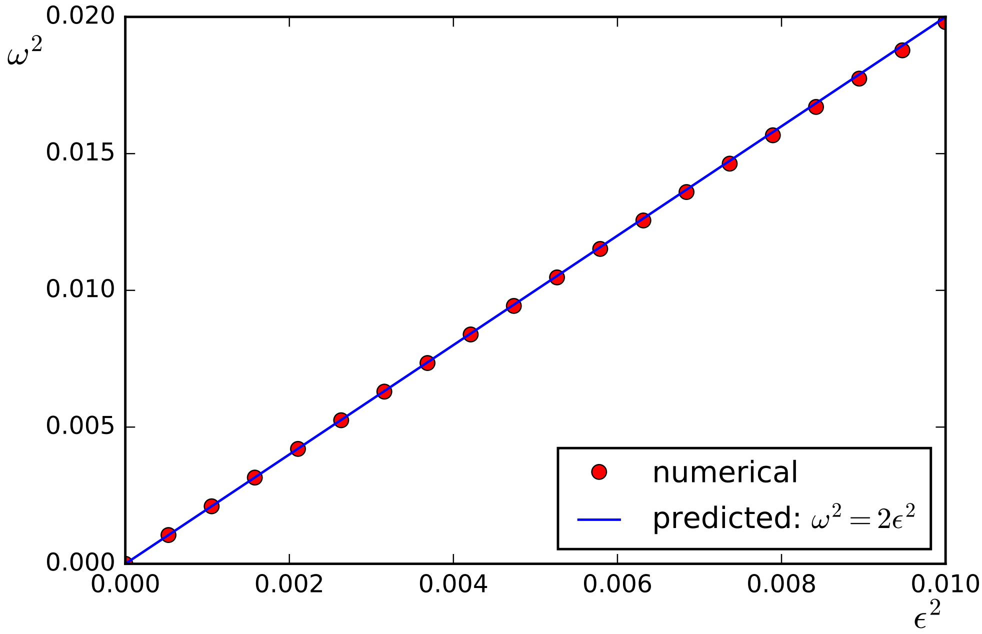

Fig. 5.8 A comparison of the predicted and numerical calculations for the energy of the zero mode as a function of ϵ^2 for small ϵ. The numerical calculations were executed by finding the even bound states and their energies as described in section 5.3.1.

We multiply equation (5.16) by v_0, and subtract from this v_1 multiplied by equation (5.4). Integrating the result from $r = -\infty$ to $r = \infty$, we find

$$\int_{-\infty}^{\infty} \left(v_1'' v_0 - v_0'' v_1 \right) dr + \int_{-\infty}^{\infty} 2 r v_0' v_0 dr - 12 \int_{-\infty}^{\infty} \Phi_0 \Phi_1 v_0^2 dr = -\xi.$$

In the first term the integrand is a total derivative, and the second term is easily found to be -1 using integration by parts. We thus obtain

$$\xi = 1 + 12 \int_{-\infty}^{\infty} \Phi_0 \Phi_1 v_0^2 dr, \tag{5.17}$$

which we can evaluate for each of the solutions (5.5) using symbolic computation in Mathematica. We find $\xi = 2$ in the case of the zero mode and $\xi = \pi^2 - 7$ in the case of the first non–trivial vibrational mode. We can check these values by finding (v, ω) numerically for a range of small values of ϵ and comparing ω^2 to the $\omega_0^2 + \xi\epsilon^2$ predicted here. The corresponding plots are shown in figures 5.8 and 5.9.

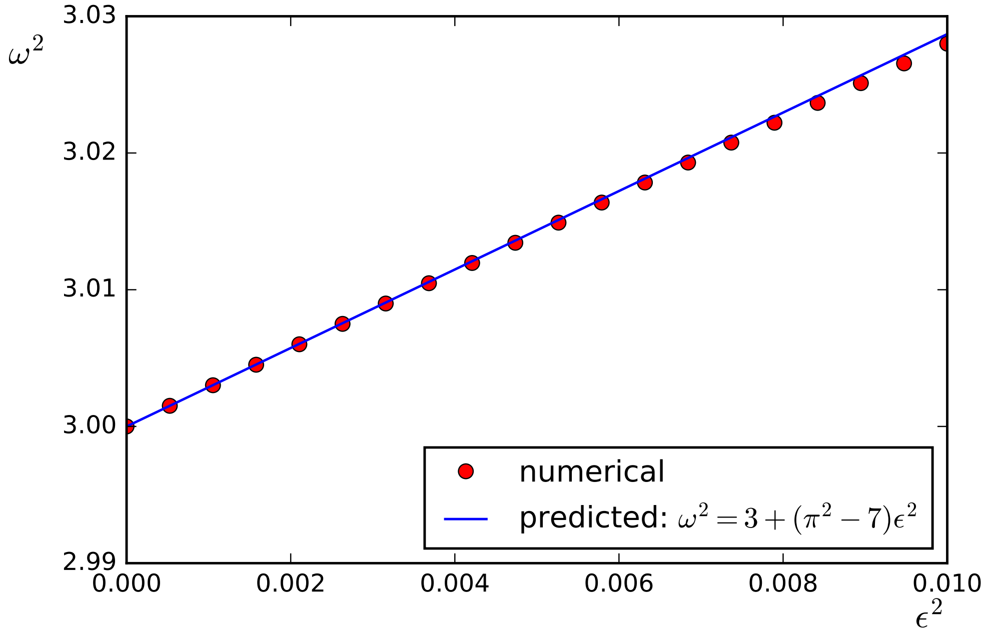

Fig. 5.9 A comparison of the predicted and numerical calculations for the energy of the odd vibrational mode as a function of ϵ^2 for small ϵ. The numerical calculations were executed by finding the odd bound states and their energies as described in section 5.3.1.

5.3.3 Critical values of a

It is interesting to investigate the values of a at which the internal modes disappear into the continuous spectrum. The larger of these, at which the odd internal mode disappears, we shall call a_1. The smaller one, at which the even internal mode disappears, we shall call a_0.

The most convenient method of estimating a_0 and a_1 is based on the Sturm Oscillation Theorem 5.1.1. The points at which the even and odd internal modes disappear into the continuous spectrum are the points at which the zeros of the even and odd eigenfunctions of L with $\omega^2 = 4$ disappear. We can thus examine the number of zeros of the odd eigenfunction with $\omega^2 = 4$ to determine the number of odd bound states with $\omega^2 < 4$. The critical value a_1 which we are searching for can then be found using a bisection method. An equivalent method using even bound states will yield an estimate of a_0.

One problem with this method is that we need the number of zeros in the interval $(0, \infty)$, and the shooting method we use to generate $\Phi(r)$ and $V(r)$ is only accurate up to a finite value of r. Since zeros of the eigenfunction with $\omega^2 = 4$ disappear at $r = \infty$, this limits the accuracy with which we can determine a_0 and a_1.

For the finite integration range which is accessible based on the shooting method, the odd state disappears at $a_1 \approx 0.8$ and the even state disappears at $a_0 \approx 0.3$.

It is well known that the condition

$$I := \int_{-\infty}^{\infty} \mathcal{V}(r) dr < 0 \tag{5.18}$$

is sufficient to ensure that the potential $\mathcal{V}(r)$ has at least one bound state. However, it is not known whether (5.18) is also a necessary condition: it is possible that a bound state could exist for a potential where (5.18) is not satisfied. We can use the disappearance of our ground state into the continuous spectrum to investigate this question.

Note that $\mathcal{V}(r)$ must go to zero as $r \to \pm\infty$ to ensure that the integral converges, meaning that the relevant choice for us is $\mathcal{V}(r) = V(r) - 4$. We then examine the value of this integral for the critical value $a = a_0$ when the ground state disappears. We find that $I \approx 0$ at the critical value of $a_0 \approx 0.3$ given above. We can also search numerically for the value of a at which $I = 0$; this also occurs at around $a_0 \approx 0.3$. Thus our results are consistent with the conjecture that no bound states can occur when (5.18) is not satisfied, i.e. that (5.18) is also a necessary condition for the potential $\mathcal{V}(r)$ to have a bound state.

5.4 Resonant Coupling of the Internal Modes to the Continuous Spectrum

We now move on to consider time dependent perturbations of the form

$$\phi(t,r) = \Phi(r) + (r^2+a^2)^{-1/2}w(t,r),$$

where we consider non–linear terms in $w(t,r)$. Substituting into (5.7) we find

$$w_{tt} + Lw + \frac{6w^2\Phi}{\sqrt{a^2+r^2}} + \frac{2w^3}{a^2+r^2} = 0. \tag{5.19}$$

If a is large enough to allow internal modes, then these can only decay through resonant coupling to the continuous spectrum of L. The analogous process of decay to the ϕ^4 kink on $\mathbb{R}^{1,1}$ was discussed in [73], and the general theory was developed in [76]. In the following sections we investigate this decay in the case of a single internal mode, before comparing our result with numerical data.

5.4.1 Conjectured decay rate in the presence of a single internal mode

We decompose the perturbation as

$$w(t,r) = \alpha(t)v(r) + \eta(t,r), \tag{5.20}$$

where $v(r)$ refers to the single even internal mode of the kink and η is a superposition of states from the continuous spectrum of L. Where there is only one internal mode present, its frequency ω always lies in the upper half of the mass gap: $1 < \omega < 2$. This is important because it means that 2ω lies within the continuous spectrum.

We substitute this into (5.19) and project onto and away from the internal mode direction, obtaining the following equations for α and η:

$$\ddot{\alpha} + \omega^2\alpha = -\left\langle v, \frac{6\Phi}{(r^2+a^2)^{1/2}}(\alpha v+\eta)^2 + \frac{2}{r^2+a^2}(\alpha v+\eta)^3 \right\rangle \tag{5.21}$$

$$\ddot{\eta} + L\eta = -P^{\perp}\left[\frac{6\Phi}{(r^2+a^2)^{1/2}}(\alpha v+\eta)^2 + \frac{2}{r^2+a^2}(\alpha v+\eta)^3\right], \tag{5.22}$$

where $P^{\perp}$ is the projection onto the space of eigenstates of L which are orthogonal to v, given by

$$P^{\perp}\psi = \psi - \langle v, \psi \rangle v,$$

and the inner product $\langle \cdot\,, \cdot \rangle$ is given by

$$\langle u, w \rangle = \int_{-\infty}^{\infty} u(r)w(r)(r^2 + a^2)dr.$$

These equations have initial conditions $\alpha(0)$ and $\eta(0, r)$ such that

$$\begin{aligned} \phi(0, r) &= \Phi(r) + (r^2 + a^2)^{-1/2}(\alpha(0)v(r) + \eta(0, r)), \quad \text{and} \\ \dot{\phi}(0, r) &= (r^2 + a^2)^{-1/2}(\dot{\alpha}(0)v(r) + \dot{\eta}(0, r)). \end{aligned}$$

In the following analysis we investigate decay of $\alpha(t)$. Equation (5.21) has a homogeneous solution consisting of oscillations with frequency ω. Since 2ω lies within the continuous spectrum of L, there will be a resonant interaction between the these oscillations and the radiation modes in η with frequencies $\pm 2\omega$, arising from the term of order α^2 in the RHS of (5.22). Thus, to leading order, (5.22) is a driven wave equation with driving frequency 2ω. This resonant part of η will have a back–reaction on α through (5.21), which will result in decay of the internal mode oscillations.

We proceed by solving (5.22) using a Green's function and plugging the result into (5.21). A similar calculation is performed by Soffer and Weinstein in [77]. They isolate the term corresponding to the key resonance, and show that the other terms decay more rapidly than the key resonant term. In contrast, here we will do only the former, isolating the term which we expect to contain the key resonance without analysing the others in detail. We will thus obtain only a conjecture for the decay rate of α. Numerical results which support the conjectured decay rate are given in section 5.4.2.

To isolate the resonant part of η, we write $\eta = \eta_0 + \eta_1 + \eta_2$, where

$$(\partial_{tt} + L)\eta_0 = 0, \qquad \eta_0(0, r) = \eta(0, r), \qquad \dot{\eta}_0(0, r) = \dot{\eta}(0, r); \tag{5.23}$$

$$(\partial_{tt} + L)\eta_1 = -6\alpha^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2 + a^2)^{1/2}}\right], \qquad \eta_1(0, r) = 0 = \dot{\eta}_1(0, r); \quad \text{and} \tag{5.24}$$

$$(\partial_{tt} + L)\eta_2 = -P^{\perp}\left[\frac{6\Phi}{(r^2 + a^2)^{1/2}}(2\alpha v\eta + \eta^2) + \frac{2}{r^2 + a^2}(\alpha v + \eta)^3\right], \tag{5.25}$$

with $\eta_2(0, r) = 0 = \dot{\eta}_2(0, r)$. The evolution of η_1 contains the leading order behaviour, since this is the part which is directly driven by the internal mode oscillations at order α^2. Since L is self–adjoint, by the spectral theorem we can think of it as a multiplication operator (see for example [71]). We can thus consider functions of L; in particular we can write $L = B^2$, since its eigenvalues are all positive for finite a. Then η_1 is given by

$$\eta_1 = -6\int_0^t \frac{\sin B(t-s)}{B}\alpha^2(s)P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds,$$

again using the fact that L has only positive eigenvalues.

We now turn to the effect of this resonant part of η on the oscillations in α. It is convenient to write

$$\alpha(t) = A(t)\mathrm{e}^{i\omega t} + \bar{A}(t)\mathrm{e}^{-i\omega t}$$

for complex valued function $A(t)$ with complex conjugate $\bar{A}(t)$, imposing the condition

$$\dot{A}\mathrm{e}^{i\omega t} + \dot{\bar{A}}\mathrm{e}^{-i\omega t} = 0$$

following [77]. This allows us to replace (5.21) with a first order ODE for $A(t)$:

$$\begin{aligned}
\ddot{\alpha} + \omega^2\alpha &= \partial_{tt}[A\mathrm{e}^{i\omega t} + \bar{A}\mathrm{e}^{-i\omega t}] + \omega^2[A\mathrm{e}^{i\omega t} + \bar{A}\mathrm{e}^{-i\omega t}] \\
&= \partial_t[\dot{A}\mathrm{e}^{i\omega t} + \dot{\bar{A}}\mathrm{e}^{-i\omega t}] + \partial_t[i\omega A\mathrm{e}^{i\omega t} - i\omega\bar{A}\mathrm{e}^{-i\omega t}] + \omega^2[A\mathrm{e}^{i\omega t} + \bar{A}\mathrm{e}^{-i\omega t}] \\
&= [i\omega\dot{A}\mathrm{e}^{i\omega t} - \omega^2 A\mathrm{e}^{i\omega t} - i\omega\dot{\bar{A}}\mathrm{e}^{-i\omega t} - \omega^2\bar{A}\mathrm{e}^{-i\omega t}] + \omega^2[A\mathrm{e}^{i\omega t} + \bar{A}\mathrm{e}^{-i\omega t}] \\
&= i\omega\dot{A}\mathrm{e}^{i\omega t} - i\omega\dot{\bar{A}}\mathrm{e}^{-i\omega t} \\
&= 2i\omega\dot{A}\mathrm{e}^{i\omega t}.
\end{aligned}$$

We can now write

$$\dot{A} = -\frac{\mathrm{e}^{-i\omega t}}{i\omega}\left[\alpha\left\langle v, \frac{6\Phi v\eta}{(r^2+a^2)^{1/2}}\right\rangle + \left\langle v, \frac{3\Phi}{(r^2+a^2)^{1/2}}(v^2\alpha^2+\eta^2) + \frac{(\alpha v+\eta)^3}{r^2+a^2}\right\rangle\right] \tag{5.26}$$

in place of (5.21), and

$$\eta_1 = -\frac{3}{iB}\int_0^t (\mathrm{e}^{iB(t-s)} - \mathrm{e}^{-iB(t-s)})(A\mathrm{e}^{i\omega s} + \bar{A}\mathrm{e}^{-i\omega s})^2 P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds.$$

The key resonance in (5.26) will come from the term which is linear in η_1:

$$\begin{aligned}
\dot{A_{res}} &= -\frac{1}{i\omega}(A + \bar{A}\mathrm{e}^{-2i\omega t})\left\langle v, \frac{6\Phi v\eta_1}{(r^2+a^2)^{1/2}}\right\rangle && (5.27)\\
&= -\frac{6}{i\omega}(A + \bar{A}\mathrm{e}^{-2i\omega t})\left\langle \frac{\Phi v^2}{(r^2+a^2)^{1/2}}\middle| \eta_1\right\rangle. && (5.28)
\end{aligned}$$

Expanding the brackets in the integrand of η_1 gives

$$
\begin{aligned}
\eta_1 =& -\frac{3}{iB}\int_0^t (\mathrm{e}^{iB(t-s)} - \mathrm{e}^{-iB(t-s)})(A\mathrm{e}^{i\omega s} + \bar{A}\mathrm{e}^{-i\omega s})^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds \\
=& -\frac{3}{iB}\int_0^t \mathrm{e}^{iBt}\mathrm{e}^{-is(B-2\omega)}A^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds \\
&+\frac{3}{iB}\int_0^t \mathrm{e}^{-iBt}\mathrm{e}^{is(B-2\omega)}\bar{A}^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds \\
&+\frac{3}{iB}\int_0^t \mathrm{e}^{iBt}(2\mathrm{e}^{-iBs}A\bar{A} + \mathrm{e}^{-is(B+2\omega)}\bar{A}^2) - \mathrm{e}^{-iBt}(2\mathrm{e}^{isB}A\bar{A} + \mathrm{e}^{is(B+2\omega)}A^2)P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds \\
=& \, I_1 + I_2 + I_3.
\end{aligned}
$$

We are looking for a part of I which will produce a real[4], non–oscillating coefficient in the power series for $\dot{A}$, so that it corresponds directly to decay rather than to oscillations. This can only happen when the exponent of e^{is} in the integral can vanish when it acts on the function

$$
P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right],
$$

which lies in the continuous spectrum of L. Since B has no negative eigenvalues, this cannot happen in I_3. However, because 2ω lies in the continuous spectrum, the exponent of e^{is} can vanish in I_1 and I_2.

We introduce the following regularisations:

$$
\begin{aligned}
I_1^{\epsilon} =& -\frac{3\mathrm{e}^{iBt}}{iB}\int_0^t \mathrm{e}^{-is(B-2\omega+i\epsilon)}A^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds, \\
I_2^{\epsilon} =& \frac{3\mathrm{e}^{-iBt}}{iB}\int_0^t \mathrm{e}^{is(B-2\omega+i\epsilon)}\bar{A}^2 P^{\perp}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds.
\end{aligned}
$$

[4]Note that any term in $\dot{A}$ whose coefficient is imaginary will have a corresponding term in $\dot{\bar{A}}$ which cancels its contribution to the time derivative of $|A|^2$.

Integrating by parts yields

$$\begin{aligned}
I_1^\epsilon =& -\frac{3e^{\epsilon t}}{B(B-2\omega+i\epsilon)}e^{2i\omega t}A^2P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right] \\
&+\frac{3e^{iBt}}{B(B-2\omega+i\epsilon)}(A(0))^2P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right] \\
&+\frac{6e^{iBt}}{B(B-2\omega+i\epsilon)}\int_0^t e^{-is(B-2\omega+i\epsilon)}A\dot{A}P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]ds \\
=& I_{11}^\epsilon+I_{12}^\epsilon+I_{13}^\epsilon
\end{aligned}$$

We now make use of the distributional identity

$$(x\pm i0)^{-1}:=\lim_{\epsilon\to 0}(x\pm i\epsilon)^{-1}=\mathrm{PV}x^{-1}\mp i\pi\delta(x)$$

to write

$$\lim_{\epsilon\to 0}(B-2\omega+i\epsilon)^{-1}=\mathrm{PV}(B-2\omega)^{-1}-i\pi\delta(B-2\omega).$$

We identify the key resonant term as the one which is local in t, i.e.

$$\lim_{\epsilon\to 0}I_{11}^\epsilon=-\frac{3A^2e^{2i\omega t}}{2\omega}\Big(\mathrm{PV}(B-2\omega)^{-1}-i\pi\delta(B-2\omega)\Big)P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right],$$

specifically the delta function part. This gives a contribution to $\dot{A}$ of

$$\begin{aligned}
&-\frac{9\pi A^2e^{2i\omega t}}{\omega^2}(A+\bar{A}e^{-2i\omega t})\left\langle\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\middle|\,\delta(B-2\omega)\,\middle|P^\perp\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right]\right\rangle \\
=&-\frac{9\pi A^2}{\omega^2}(Ae^{2i\omega t}+\bar{A})\left|\mathcal{F}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right](2\omega)\right|^2 \\
=&-\frac{9\pi A^2}{\omega^2}(Ae^{2i\omega t}+\bar{A})\Lambda^2,
\end{aligned}$$

where we have defined

$$\Lambda^2:=\left|\mathcal{F}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right](2\omega)\right|^2 \tag{5.29}$$

for some $\Lambda>0$. The so–called Fermi Golden Rule then reads

$$\mathcal{F}\left[\frac{\Phi v^2}{(r^2+a^2)^{1/2}}\right](2\omega)\neq 0.$$

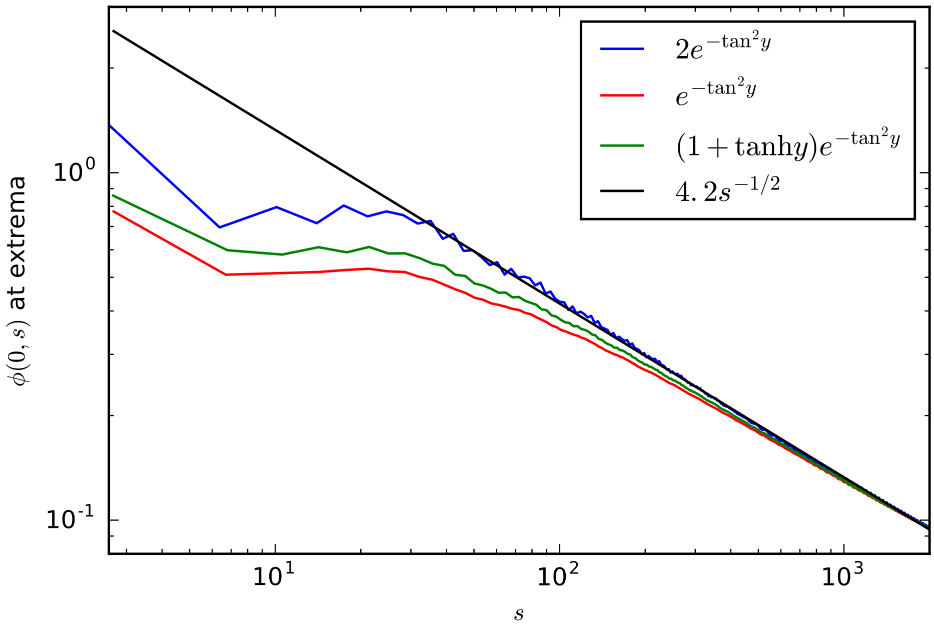

Fig. 5.10 The decay of internal mode oscillations for various initial conditions when $a = 0.5$. Note that $\phi(0, s)$ is used as a proxy for the internal mode amplitude, and we use a log–log scale to elucidate the dependence on $s^{-1/2}$ in the large s limit. The lines are labelled in the legend by the initial conditions which produced them, with the exception of the gradient line $4.2s^{-1/2}$.

The term $-9\pi\Lambda^2|A|^2A/\omega^2$ is the key resonant damping term. It yields

$$|\dot{A}| \approx -\Gamma|A|^3,$$

with $\Gamma > 0$, and hence $|A| \sim t^{-1/2}$ as $t \to \infty$. Note that a similar analysis of I_2 produces a term which oscillates as $\mathrm{e}^{-2i\omega t}$, which cannot give a contribution to $\dot{A}$ which is non–oscillating.

5.4.2 Numerical investigation of the conjectured decay rate

In order to integrate the PDE (5.7) to large times t, we employ the method of hyperboloidal foliations and scri–fixing [81]. Following [67, 68], we define

$$s = \frac{t}{a} - \sqrt{\frac{r^2}{a^2} + 1}, \qquad y = \arctan\left(\frac{r}{a}\right),$$

resulting in a hyperbolic equation

$$\partial_s\partial_s F + 2\sin(y)\partial_y\partial_s F + \frac{1+\sin^2(y)}{\cos(y)}\partial_s F = \cos^2(y)\partial_y\partial_y F + 2a^2\frac{F(1-F^2)}{\cos^2(y)}. \quad (5.30)$$

for $F(s,y) = \phi(t,r)$. The surfaces of constant s approach right future null infinity $\mathcal{J}_R^+$ along outgoing null cones of constant retarded time $t-r$, and left future null infinity $\mathcal{J}_L^+$ along outgoing null cones of constant advanced time $t+r$.

We solve the corresponding initial value problem at space–like hypersurfaces of constant s, specifying $\phi(s=0,y)$ and $\partial_s\phi(s=0,y)$. No boundary conditions are required, since the principal symbol of (5.30) degenerates to $\partial_s(\partial_s \pm 2\partial_y)$ as $y \to \pm\pi/2$, so there are no ingoing characteristics. This reflects the fact that no information comes in from future null infinity.

Following [67, 82] we define the auxiliary variables

$$\Psi = \partial_y F, \qquad \Pi = \partial_s F + \sin y\partial_y F$$

to obtain the first order symmetric hyperbolic system

$$\partial_s F = \Pi - \Psi\sin y \quad (5.31)$$

$$\partial_s\Psi = \partial_y(\Pi - \Psi\sin y) \quad (5.32)$$

$$\partial_s\Pi = \partial_y(\Psi - \Pi\sin y) + 2\tan y(\Psi - \Pi\sin y) + 2a^2\frac{F(1-F^2)}{\cos^2 y}, \quad (5.33)$$

which we solve numerically using the method of lines. Kreiss–Oliger dissipation is required to reduce unphysical high–frequency noise. We also add the term $-0.1(\Psi - \partial_y F)$ to the right hand side of equation (5.32) to surpress violation of the constraint $\Psi = \partial_y F$.

We are interested in the range of values $a_0 < a < a_1$ for which the kink has exactly one internal mode. We find that, for fixed but arbitrary y, $F(s,y)$ oscillates in s with a frequency close to the internal mode frequency, and that these oscillations tend towards a decay rate of $s^{-1/2}$, as we expect from section 5.4.1. Plots demonstrating this decay at $y=0$ for $a=0.5$ are shown in figure 5.10. Note that the constant 4.2 is related to Λ as defined in (5.29).

5.5 Summary and Discussion

We have found that the modified kink is topologically and linearly stable, and investigated its asymptotic stability for the range of a where exactly one discrete mode is present. It would be interesting to expand the investigation in section 5.4 to the case when both discrete modes are present. This problem is much more complicated because of the extra terms in (5.22) and (5.21) coming from the amplitude of the second internal mode. Similar problems have been discussed in [79], although no such analysis has been done for non–linear Klein–Gordon equation of this type with two discrete modes. The ϕ^4 theory on the wormhole presents a useful setting to undertake such analysis because the kink has exactly two discrete modes for any $a > a_1$, and because their frequencies can be controlled by the parameter a.

This model shares an interesting property with its sine–Gordon counterpart in that we expect a discontinuous change in decay behaviour when a moves out of the range $a_0 < a < a_1$. Insight from the ϕ^4 case may help to elucidate the character of such discontinuous changes.

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