Mathematics 220 Calculus II, Homework #1

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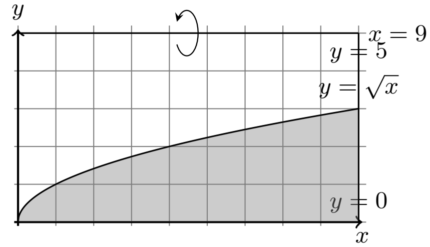
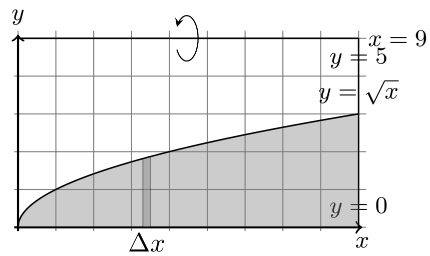
1

First, we should draw the functions $y = \sqrt{x}$, $y = 0$, and $x = 9$, so we can visualize the area we are working with, as well as the line $y = 5$ so we know what we are rotating about.



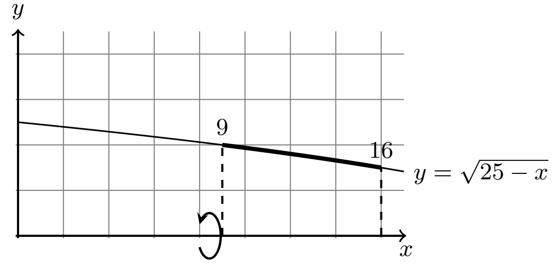
Here, we will use the washer method, with the outer radius being the distance of the X axis from $y = 5$, and the inner radius, the distance of $\sqrt{x}$ from $y = 5$. We can take a slice of the graph, at some x, with a thickness of Δx.



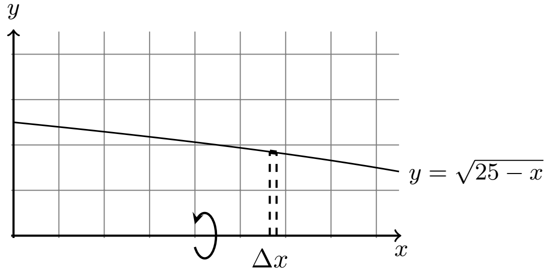
The area of this rectangle is clearly $\Delta x(9 - \sqrt{x})$. And the volume generated by rotating that rectangle would be $\pi \Delta x(9^2 - \sqrt{x}^2)$. And of course, the summation of those rectangles, as they become thinner and thinner and approxomate reality is the integral. So we get $\pi \int\limits_0^9 (81 - x)dx$. Which becomes $\pi(81x - \frac{x^2}{2})|_0^9$, and evaluates to $\pi(729 - \frac{81}{2})$.

2

Similarly to the first problem, we first draw the function, and note relevant information like the axis of rotation, and choose a slice at location x, with thickness ΔX.



Instead of approximating the area underneath with a rectangle, we're approximating the length with a straight line. We can pick an arbitrary point at x, with a length Δx. We can find that line easily once we notice that the top of that area forms a triangle.



And by the pythagorean theorem, we can easily see that the length of the long side is $\sqrt{\Delta x^2 + (f(x_1) - f(x_2))^2}$. By the mean value theorem, we know that there is a point such that $f'(x^*)\Delta x = (f(x_1) - f(x_2))$. We get, then $\sqrt{\Delta x^2 + [f'(x)]^2 \Delta x^2}$, which is the same as $\sqrt{(1 + [f'(x)]^2)}\Delta x$, which we can take that infinite sum of, resulting in the integral. $\int \sqrt{1 + [f'(x)]^2} dx$.

Let's quickly find $(y')^2$. Using the chain rule, y' is $-\frac{1}{2}\frac{1}{\sqrt{25-x}}$, and the square of that is $\frac{1}{4}\frac{1}{25-x}$.

Now, let's finally, using the length we've found, let's finally see what the area is. The formula for the surface area of a frustum is simple. $A = \pi(r_1 + r_2)l$, where r_1 and r_2 are the radii of the two ends, and l is the length of the frustum. Substitute in the correct values, add the infinite sum, and we get our integral, which we can now evaluate.

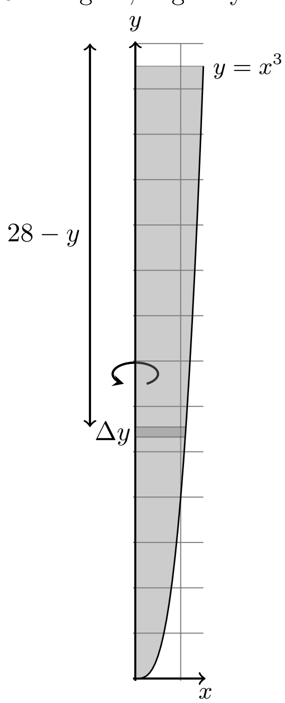
$2\pi \int_9^{16} \sqrt{25-x}\sqrt{1 + \frac{1}{100-4x}} dx$

With u substitution, $u = 25 - x, du = -dx$. In our integral, replace 9 with a 16, and 16 with a 9, and we get $-2\pi \int_{16}^{9} \sqrt{u}\sqrt{1 + \frac{1}{4u}} du$, which in turn can be factored into $-2\pi \int_{16}^{9} \frac{1}{2}\sqrt{4u+1} du$. Substitute again, $v = 4u + 1, dv = 4du, \frac{1}{4}dv = du$, correct the integral, 16 becomes 65, 9 becomes 37, for a final integral of $-2\pi \int_{65}^{37} \frac{1}{2}\sqrt{v}$.

$-\pi(\frac{2}{3}v^{\frac{3}{2}})|_{65}^{37} = -\pi(\frac{2}{3}37^{\frac{3}{2}} - \frac{2}{3}65^{\frac{3}{2}})$.

3

Once again, begin by drawing a graph.



Now, as we are rotating around the y axis, we want x in terms of y. This is simply $x = \sqrt[3]{y}$. Next, we want to find a formula for the volume of liquid at a height y. Since we can simply imagine this as a cylinder of height Δy, this is simply $\pi \sqrt[3]{y}^2 \Delta y$. Weight is simply volume∗density, and the work required to push 1 meter above the surface is the weight times the remaining distance. The total distance being $3^3 + 1$, so $(28 - y) * \rho * V$.

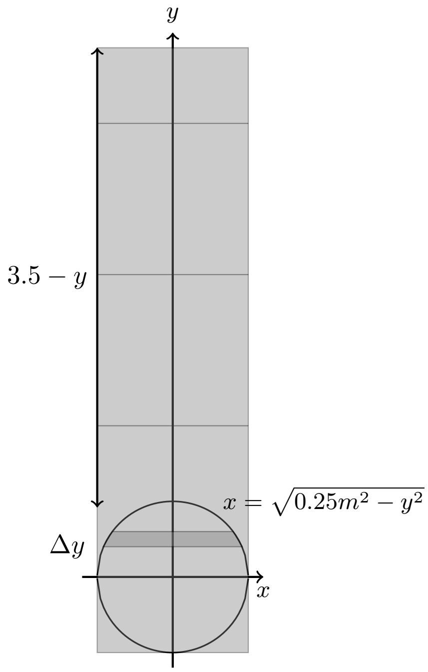
Take the infinite sum, and plug everything in, and we get our integral.

$30kg/m^3 \pi \int_{0m}^{27m} (28 - y)\sqrt[3]{y}^2 dy = 30kg/m^3 \pi \int_{0m}^{27m} (28y^{\frac{2}{3}} - y^{\frac{4}{3}}) dy$

Evaluate:

$30kg/m^3\pi(\frac{84}{5}y^{\frac{5}{3}} - \frac{3}{7}y^{\frac{7}{3}})|_{0m}^{27m} = 30kg/m^3\pi(\frac{20412}{5}m^{\frac{5}{3}} - \frac{6561}{7}m^{\frac{7}{3}}) = 30\pi(\frac{20412}{5} - \frac{6561}{7})J.$

4



The hydrostatic pressure at a given depth is simply $\rho g d$, where ρ is the density, g is the force of gravity, and d is the depth. The hydrostatic force on an area A is the pressure times A.
The equation for the circle at a given point x is simply $2\sqrt{r^2 - y^2}$. If we place the coordinate system at the center of the window, for simplycity's sake, then the top of the water is at $y = 3.5$. The pressure at any given depth y then is $(3.5 - y)\rho g$. The area of a strip, of height Δy and depth y is just $2\sqrt{0.25m^2 - ym^2}\Delta y$. The hydrostatic force is the limit of all these thin strips, so we get:
$2\rho g \int_{-0.5m}^{0.5m}(3.5 - y)\sqrt{0.25m^2 - y^2}dy$, which we split into two integrals:
$2\rho g(\int_{-0.5m}^{0.5m} 3.5\sqrt{0.25m^2 - y^2}dy - \int_{-0.5m}^{0.5m} y\sqrt{0.25m^2 - y^2}dy)$.
For the first integral, take out the $\frac{1}{4}$, so we get a $\int_{-0.5m}^{0.5m} 1.75\sqrt{1m^2 - 4y^2}dy$. Apply trig substitution, $y = \frac{1}{2}sin(u)$.
$1.75\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}cos^2(u)du$.
For the second integral, $u = y^2, du = 2y$, replace the -0.5m and 0.5m both with 0.25m, which evaluates to zero. So now all that's left is to evaluate the first integral:
$3.5\rho g\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}cos^2(u)du = 3.5\rho g(\frac{1}{4}(x + sin(x)cos(x)))|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} =$
$3.5\rho g(\frac{1}{4}(\frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4}))$.