Section 1.4
Systems of Difference Equations

Introduction

In general, one can model using changes to get a system of difference equation

$$x(n+1)-x(n)=changes$$

$$x(n+1)-y(n)=changes$$

So that we can get

$$x(n+1)=f(x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_0, y_0)$$

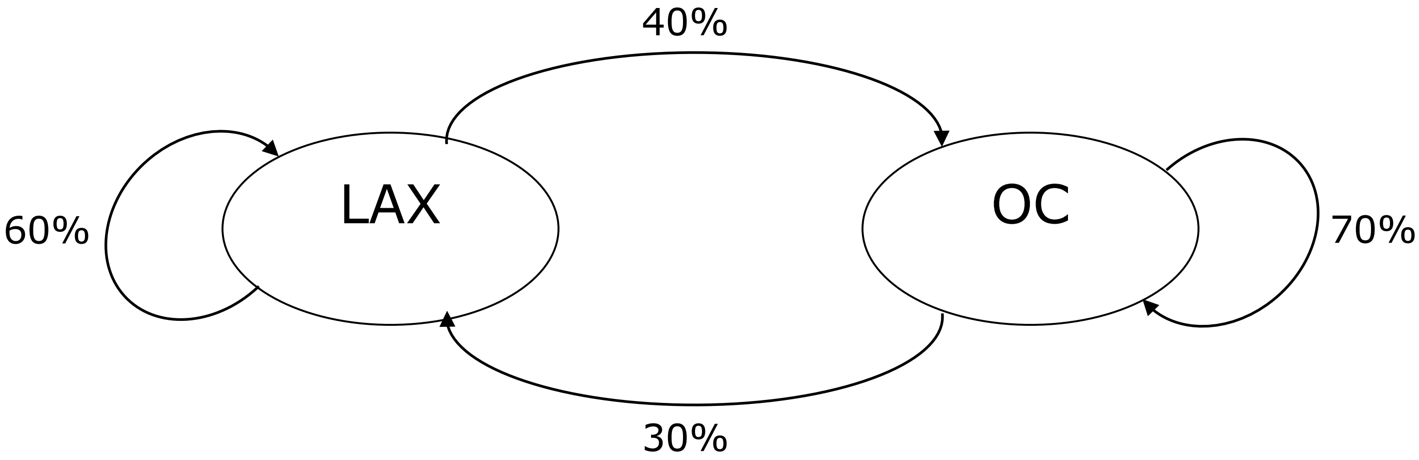
$$y(n+1)=g(x_n, y_n, x_{n-1}, y_{n-1}, \ldots, x_0, y_0)$$

$$with\, x_0 = x(0)$$

$$y_0 = y(0)$$

Example

Car rental company with two locations



Suppose 500 cars are at each location. How will the cars be distributed at the two locations? Will one site be attracting all the cars?

A model using changes

Let *O(n)* be the number of vehicles at the OC location at time n.
Let *L(n)* be the number of vehicles at LAX at time n.

$$O(n+1) - O(n) = Gains - Losses$$
$$= 0.40L(n) - 0.30O(n)$$
$$O(n+1) = 0.70O(n) + 0.40L(n)$$

Similarly,

$$L(n+1) - L(n) = Gains - Losses$$
$$= 0.30O(n) - 0.40L(n)$$
$$L(n+1) = 0.60L(n) + 0.30O(n)$$

A model using changes (cont.)

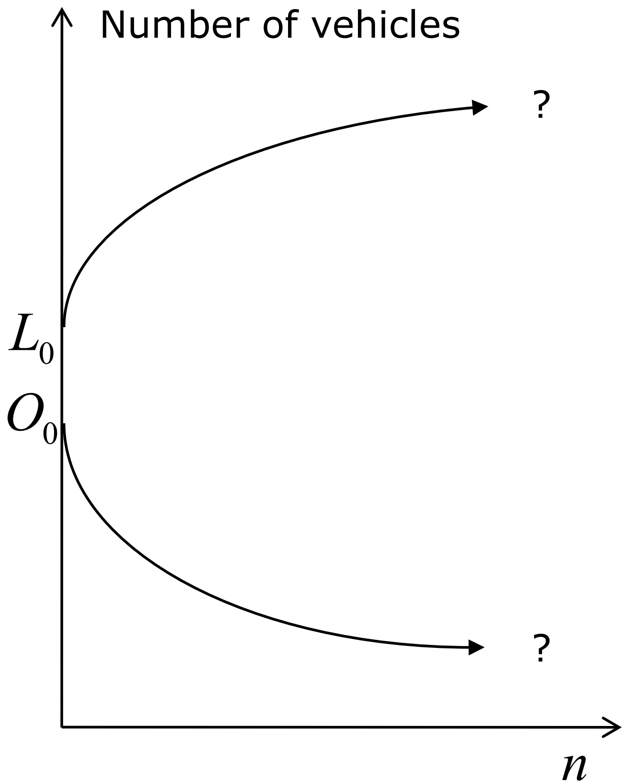
$$O(n+1) = 0.70O(n) + 0.40L(n)$$
$$L(n+1) = 0.30O(n) + 0.60L(n)$$

$$Let\, x(n) = \begin{bmatrix} O(n) \\ L(n) \end{bmatrix}$$

$$x(n+1) = M\,\vec{x}(n) = M(M\,\vec{x}(n-1)) \;\; where\, M = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

$$\vec{x}(n+1) = M^{n+1}\,\vec{x}(0) \quad where\,\vec{x}(0) = \begin{bmatrix} O_0 \\ L_0 \end{bmatrix}$$

A model using changes (cont.)



Finding Eigen Value and Vector

$$\det(M - \lambda I) = 0$$

$$\begin{bmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{bmatrix}$$

$$= (0.7-\lambda)(0.6-\lambda) - 0.12 = 0$$

Suppose λ_1, λ_2 are the Eigen values

and $\vec{\vee}_1, \vec{\vee}_2 \; R^2$ are the Eigen vectors

so $\vec{x}_0 = k_1 \vec{\vee}_1 + k_2 \vec{\vee}_2$

Eigen Value and Vector (cont.)

$$\vec{x}(n+1) = M^{n+1}\vec{x}_0$$

$$= M^{n+1}\left(k_1\vec{\vee}_1 + k_2\vec{\vee}\,2\right)$$

$$M\vec{\vee}_1 = \lambda_1\vec{\vee}_1$$

You can find k_1 and k_2 in a different way if $\vee_1$ and $\vee_2$ are orthogonal

$$k_1 M^{n+1}\vec{\vee}_1 + k_2 M^{n+1}\vec{\vee}_2$$

$$k_1\lambda^{n+1}\vec{\vee}_1 + k_2\lambda^{n+1}\vec{\vee}_2$$

Eigen Value and Vector (cont.)

Note: $k_2 \vec{\vee_2} = \vec{x}_0 - k_1 \vec{\vee_1}$

$$\vec{x}(n+1) = \lambda_1^{n+1} k_1 \vec{\vee_1} + \lambda_2^{n+1}\left(\vec{x}_0 - k_1 \vec{\vee_1}\right)$$

$$= k_1 \vec{\vee_1}\left(\lambda_1^{n+1} - \lambda_2^{n+1}\right) + \lambda_2^{n+1} \vec{x}_0$$

so that the long-term solution will depend on the values of

λ_1, λ_2, and $\vec{x}_0$

Steady-State Solution

$$\vec{x}* = M\,\vec{x}*$$

$$O * L* = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} O * \\ L * \end{bmatrix}$$

$$O* = 0.7O * + 0.4L *$$

$$L* = 0.3O * + 0.6L *$$

$$\begin{cases} 0.3O* = 0.4L * \\ 0.4L* = 0.3O * \end{cases}$$

$$0.3O* = 0.4L *$$

$$L* = \frac{3}{4}O *$$

Steady-State Solution (cont.)

So, if N = total of number of vehicles,

$$L^* + O^* = N$$

$$\frac{3}{4} O^* + O^* = N$$

$$O^* = \frac{4}{7} N$$

$$L^* = \frac{3}{7} N$$

That is, if there are 700 vehicles, the equilibrium state is

$O^* = 400$ and $L^* = 300$

Competitive Hunter Model

-Spotted Owls and Hawks

When in separated environments

$$\Delta O_n = k_1 O_n \quad \text{and} \quad \Delta H_n = k_2 H_n$$

When in the same environment

$$\Delta O_n = k_1 O_n - k_3 O_n H_n$$
$$\Delta H_n = k_2 H_n - k_4 O_n H_n$$

$$O_{n+1} = (1 + k_1) O_n - k_3 O_n H_n$$
$$H_{n+1} = (1 + k_2) H_n - k_4 O_n H_n$$

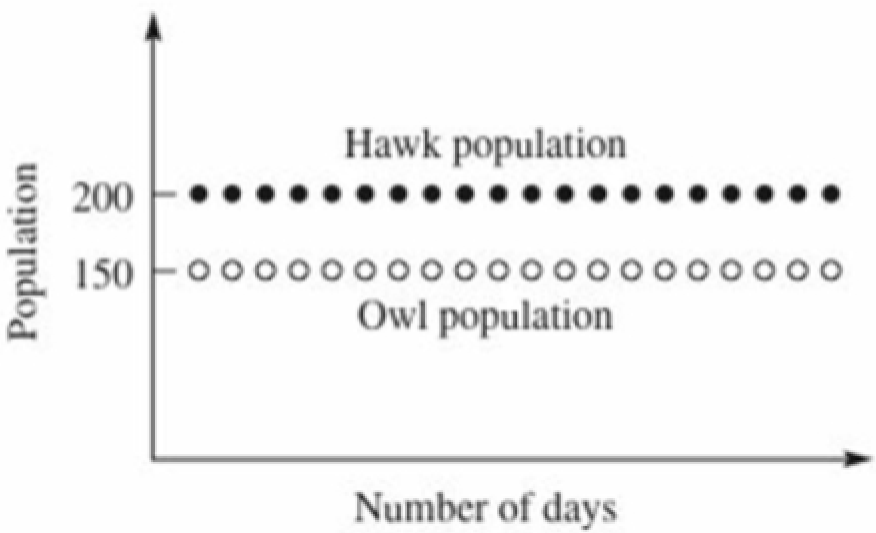
Competitive Hunter Model (Equilibrium point = Co-habitat)

$$O_{n+1} = (1 + k_1)O_n - k_3 O_n H_n$$
$$H_{n+1} = (1 + k_2)H_n - k_4 O_n H_n$$

Competitive Hunter Model (Equilibrium point = Co-habitat)

$$O_{n+1} = (1 + k_1)O_n - k_3 O_n H_n$$
$$H_{n+1} = (1 + k_2)H_n - k_4 O_n H_n$$

$$O = 1.2O - 0.001OH$$
$$H = 1.3H - 0.002OH$$



Sensitivity of Equilibrium Point

	Owls	Hawks
Case 1	151	199
Case 2	149	201
Case 3	10	10

Homework

Section 1.4 (4, 5, 6)