Dimension Analysis – Buckingham Theorem

Buckingham – Π Theorem:

For a physical system described by the variables

$$\{v_1, v_2, \ldots, v_n\}$$

generated by M *principal dimensions (e.g.* L, M, T, …*), the law governing the system can be described as the mathematical relation among at most (n-m) dimensionless π-groups,*

$$\left(\pi_1, \pi_2, \ldots, \pi_{(n-m)}\right).$$

Furthermore, there exist for each π-group, there is a function Φ_i *such that*

$$\pi_i = \Phi_i \cdot \left(\{\pi_1, \pi_2, \ldots, \pi_{(n-m)}\} \setminus \pi_i\right).$$

Example 2:

Suppose the dimensionless π-groups are $\{\pi_1, \pi_2, \pi_3\}$,

then there exist $\{\Phi_1, \Phi_2, \Phi_3\}$ such that

$$\pi_1 = \Phi_1(\pi_2, \pi_3),$$
$$\pi_2 = \Phi_2(\pi_1, \pi_3),$$
$$\pi_3 = \Phi_3(\pi_1, \pi_2).$$

Example 3:

From the pendulum submerged in a liquid problem earlier, we have the following 7 derived variables: {T, θ, r, m, g, μ, and ρ}, which, by examining their dimensions, are described by 3 principal dimensions: {M, L, and T}.

It is important that we pick 3 generators (derived variables) that are linearly independent

$\{m, r, \rho\}$ is a bad choice as it spans only the Mass and Length dimensions.

$\{m, r, \mu\}$ is one good choice, but not necessarily the only.

$\{m, r, T\}$ is also a bad choice because we cannot explicitly express T.

Example 3:

Using the dimensional analysis technique similarly to the Rayleigh method, we obtain the dimensionless pi-groups

$$\pi_1 = \theta$$

$$\pi_2 = \frac{m^2 \cdot g}{r^3 \cdot \mu^2}$$

$$\pi_3 = \frac{r^3 \cdot \rho}{m}$$

$$\pi_4 = \frac{\mathrm{T} \cdot r \cdot \mu}{m}$$

Example 3:

Using the dimensional analysis technique similarly to the Rayleigh method, we obtain the dimensionless pi-groups

So by the Buckingham – Π Theorem, we have

$$\pi_4 = \Phi_4(\pi_1, \pi_2, \pi_3)$$

$$\frac{\mathrm{T}}{\frac{m}{r \cdot \mu}} = \Phi_4(\pi_1, \pi_2, \pi_3)$$

$$\mathrm{T} = \frac{m}{r \cdot \mu} \cdot \Phi_4\left(\theta, \frac{m^2 \cdot g}{r^3 \cdot \mu^2}, \frac{r^3 \cdot \rho}{m}\right).$$

Example: Section 9.2 # 11

The lift force F on a missile depends on its length r, velocity v, diameter D, and initial angle with the horizon; it also depends on the density δ, viscosity μ , gravity g, and speed of sound s of the air. Show that,

$$F = \rho v^2 r \Psi\left(\frac{D}{r}, \theta, \frac{\mu}{\rho v r}, \frac{s}{v}, \frac{rg}{v^2}\right)$$

- F- force
- r- length
- v- velocity
- D- diameter
- θ - angle
- δ - density
- μ- viscosity
- g - gravity
- s- speed

Choose ρ, v, r as dimension basis elements

9 derived variables
3 principal variables
6 derived Π –group

$$F = \Pi_1 \rho^a v^b r^c$$

$$D = \Pi_2 \rho^a v^b r^c$$

$$\theta = \Pi_3 \rho^a v^b r^c$$

$$\mu = \Pi_4 \rho^a v^b r^c$$

$$g = \Pi_5 \rho^a v^b r^c$$

$$s = \Pi_6 \rho^a v^b r^c$$

Then we find the Pi-groups

For Diameter

$$\left[D\right] = L$$

$\Rightarrow$ -3a+b+c= 1 $\Rightarrow$ a= 0

a= 0 b= 0

b= 0 c= 1

$$\Pi_2 = \frac{D}{r} = \frac{D}{r}$$

For angle

$$\left[\theta\right] = 1$$

$\Rightarrow$ -3a+b+c= 0 $\Rightarrow$ a= 0

a= 0 b= 0

b= 0 c= 0

$$\Pi_3 = \theta$$

For viscosity

$$[\mu] = \frac{M}{LT}$$

$\Rightarrow$ -3a+b+c= 0 $\Rightarrow$ a= 1

a= 1 b= 1

b= 1 c= 1

$$\Pi_4 = \frac{\mu}{\rho v r}$$

For gravity

$$[g] = \frac{L}{T^2}$$

$\Rightarrow$ -3a+b+c= 1 $\Rightarrow$ a= 0

a= 0 b=2

b= 2 c= -1

$$\Pi_5 = \frac{gr}{v^2}$$

For speed

$$[s] = \frac{L}{T}$$

$\Rightarrow$ -3a+b+c= 1 $\Rightarrow$ b= 1

a= 0 b= 1

a= 0 c= 0

$$\Pi_6 = \frac{s}{v}$$

So,

$$F = \rho v^2 r \Psi\left(\frac{D}{r}, \theta, \frac{\mu}{\rho v r}, \frac{s}{v}, \frac{rg}{v^2}\right)$$

Remark:

(1) From dimensional analysis, it follows that a relation exists among dimensionless Π-groups. However, it does not give the exact form of this relationship.

i.e. $\Pi_i = \Phi_i(\{\Pi_1,\ldots,\Pi_{m-n}\} \setminus \Pi_i)$

Here, Φ can be found empirically.

(2) The advantage of writing the physical law in dimensionless form is the reduction of the number of independent variables, (n-m) instead of n.

(3) It is normal to manipulate the dimensionless Π-groups to obtain a new Π-group.

i.e. $\Pi_3 = {}^{\Pi_3}\!/_{\Pi_4}; \quad \Pi_5 = \sqrt{{}^{\Pi_5}\!/_{\Pi_7}}; \quad \Pi_1 = {}^{1}\!/_{\Pi_1}$

Algorithm for finding dimensionless dimensionless π - groups using the Buckingham π - Theorem

- Given V_1, V_2, … V_m, (m - derived variables)
- For each derived variable, we take its dimensions $[V_1]$, $[V_2]$, … $[V_m]$
- List the common principal dimensions, say n of them.
- Pick n derived variables out of the list of $\{V_1, V_2, \ldots V_m\}$ so that the n derived variables can generate any of the $\{V_1, V_2, \ldots V_m\}$ variables.
- Use the Rayleigh method to relate the remaining derived variables dimensionlessly proportional to the powers of the generators.

Homework –Section 14.2

- # 3-10