Section 1.1
Modeling Change with Difference Equations

Difference Equation

$$P(t+\Delta t) = P(t) + changes$$

Remarks:

1. *The model for P(t) can be found by solving the difference equation*
2. *Changes = Created – Destroyed*
3. *Above difference equation can also be understood as a dynamical system. (Discrete)*

Example: Mortgage Payment

$P(t)$ ~ Balance owed to the bank at time t

Δt ~ Time between payments (bi-weekly, monthly, etc.)

Model changes in this application:

Changes: Interest (Gain) = $I * P(t)$
Payments (Loss) = M

Assume $I=r/N$ *where* r ~ annual interest rate and N ~ number of times money is compounded per year (N = 12 for monthly program)

Mortgage Payment (Cont.)

$$
\begin{aligned}
P(t+\Delta t) &= P(t) + IP(t) - M \\
&= (1+I)P(t) - M
\end{aligned}
$$

Mortgage Payment (Cont.)

$$p(k+1) = (1+I)P(t) - M$$
$$= (1+I)[(1+I)P(t-1) - M] - M$$
$$= (1+I)^2 P(k-1) - M[1+(1+I)]$$
$$= (1+I)^3 P(k-2) - M[1+(1+I)+(1+I)^2]$$
$$= (1+I)^{k+1} P(0) - M \sum_{M=0}^{k} (1+I)^m$$

i. *Note: P(0) is how much we borrow initially*
ii. *Suppose term of the loan is n equal payments. Then P(n) = 0*

Mortgage Payment (Cont.)

$$0 = P(n) = (1+I)^n P(0) - M \sum_{M=0}^{n-1} (1+I)^m$$

So,

$$M = \frac{(1+I)^n P(0)}{\sum_{M=0}^{n-1} (1+I)^M}$$

Mortgage Payment (Cont.)

But recall that,

$$(1+x+x^2+\ldots+x^{n-1})(1-x)=1-x^n$$

So, $$(1+x+x^2+\ldots+x^{n-1})=\frac{1-x^n}{1-x}$$ where $x\neq 0$

So, $$\sum_{m=0}^{n-1}(1+I)^m=\frac{1-(1+I)^n}{1-(1+I)}$$ where $I\neq 0$

$$=\frac{(1+I)^n-1}{I}$$

Mortgage Payment (Cont.)

$$M = \frac{(1+I)^n P(0)}{(1+I)^n - 1} * I = \frac{IP(0)}{1-(1+I)^{-n}} \qquad \text{for} \quad I \neq 0$$

and $$M = \frac{P(0)}{n} \qquad \text{for} \quad I = 0$$

$$M = \begin{cases} \dfrac{IP(0)}{1-(1+I)^{-n}} & I \neq 0 \\[2ex] \dfrac{P(0)}{n} & I = 0 \end{cases}$$

Example:

Suppose a car dealer is selling you a vehicle with the following terms:
M =650
r = 3.99% annually for 5 years
How much was the vehicle sold for?

$$P(0)=\frac{M(1-(1+I)^{-n})}{I}$$

$$=650*\frac{\left[1-\left(1+\frac{0.0399}{12}\right)^{-60}\right]}{\left(\frac{0.0399}{12}\right)}$$

Retirement Account:

Suppose you are working and you plan to contribute monthly to your retirement account.
Changes: Interest (Gain) and Contribution (Gain)

Assume B(k) is the balance in the account at k-th month, with

$$B(0) = B_0$$
$$B(k+1) = B(k) + changes$$
$$B(k+1) = (1+I)B(k) + M$$
$$= (1+I)^{k+1}B(0) + M\sum_{M=0}^{k}(1+I)^M$$

If you have no initial deposit and want 1,000,000 by the time you retire, then

$$M = \frac{1000000}{\sum_{M=0}^{k}(1+I)^M}$$

Application:

Suppose a retirement account returns 6.25% annually.
Jane contributes $200 per month from age 25 to age 65.
Karen contributes $200 per month from age 45 to age 65.

1. How much will each have by age 65?
2. In order to catch up, how much more does Karen have to contribute?

$$B(n) = (1+I)^n B_0 + M\,\frac{(1+I)^n - 1}{I}$$

Note: $B_0 = 0$ in both cases

hence, we solve question 1 with the following equation:

$$B(n) = \frac{M}{I}\left[(1+I)^n - 1\right]$$

Application (Cont.)

Jane: $n_1 = 40 * 12 = 480$

Karen: $n_2 = 20 * 12 = 240$

$$I = \frac{0.0625}{12}, \quad M = 200$$

We solve question 2 with the following equation:

$$\frac{M}{I}\left[(1+I)^{n_1} - 1\right] = \frac{(M+C)}{I}\left[(1+I)^{n_2} - 1\right]$$

$Solve\ for\ C$

Homework –Section 1.1

- # 1-5c, 7, 8, 10