$$\boxed{Ax = b}$$

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Example

$$7x = 21$$

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Two ways:

$$x = \frac{21}{7} = 3$$

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or

$$x = 7^{-1} \times 21 = .142857 \times 21 = 2.99997$$

The MATLAB backslash operator:

$$Ax = b \qquad x = A\backslash b$$

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$$xA = b \qquad x = b/A$$

Example (A 3×3 example)

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{aligned} 10x_1 - 7x_2 &= 7 \\ -3x_1 + 2x_2 + 6x_3 &= 4 \\ 5x_1 - x_2 + 5x_3 &= 6 \end{aligned}$$

Example

Elimination:

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

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$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 2.5 \\ 6.2 \end{pmatrix}$$

Example

Back-substitution:

$$6.2x_3 = 6.2$$

$$2.5x_2 + (5)(1) = 2.5$$

$$10x_1 + (-7)(-1) = 7$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Example

LU factorization

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

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$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & -0.04 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$LU = PA$$

To solve

$$Ax = b$$

factor A as

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Solve

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in two steps:

$$\begin{aligned} Ly &= Pb && \text{by forward substitution} \\ Ux &= y && \text{by back substitution} \end{aligned}$$

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The heavy lifting is the factorization.

PERMUTATION AND TRIANGULAR MATRICES

Example

A permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Px = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \end{pmatrix}$$

PA moves row 4 to row 1, etc.

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PA moves row 4 to row 1, etc.

$$\boxed{P^{-1} = P^T}$$

Example

Upper triangular matrix:

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Unit lower triangular matrix:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

LU FACTORIZATION

$$U = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A$$

$$L_1 L_2 \cdots L_{n-1} U = P_{n-1} \cdots P_2 P_1 A$$

LU FACTORIZATION

$$U = M_{n-1}P_{n-1}\cdots M_2P_2M_1P_1A$$

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$$L = L_1L_2\cdots L_{n-1}$$
$$P = P_{n-1}\cdots P_2P_1$$

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$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.04 & 1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.3 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.04 & 1 \end{pmatrix}$$

WHY IS PIVOTING NECESSARY?

Change the last example slightly:

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3.901 \\ 6 \end{pmatrix}$$

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Solution is $x_1 = 0, x_2 = -1, x_3 = 1$, as before. If we proceed without pivoting using 5 significant digits, we have

$$\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.001 \\ 2.5 \end{pmatrix}$$

$$(5 + (2500)(6))x_3 = (2.5 + (2500)(6.001))$$

$$(5 + 15000)x_3 = (2.5 + 15002.5)$$

$$15005x_3 = 15004$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$-0.001x_2 + (6)(0.99993) = 6.001$$

$$x_2 = \frac{1500}{-1000} = -1.5$$

$$10x_1 + (-7)(-1.5) = 7$$

$$x_1 = -0.35$$

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$$x = \begin{pmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{pmatrix}, \quad \text{instead of} \quad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

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The problem is the large multiplier in the second step: 2500. This caused us to lose a significant digit in the next step and threw the whole computation off. Partial pivoting eliminates this problem.

PARTIAL PIVOTING

At each step switch rows to get the maximal element in the pivot position.

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In the previous example

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$$(6 + (.0004)(5))x_3 = (6.001 + (.0004)(2.5))$$

$$(6 + 0.002)x_3 = (6.001 + .001)$$

$$6.002x_3 = 6.002$$

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$$(6 + (.0004)(5))x_3 = (6.001 + (.0004)(2.5))$$

$$(6 + 0.002)x_3 = (6.001 + .001)$$

$$6.002x_3 = 6.002$$

So we get $x_3 = 1, x_2 = -1, x_1 = 0$, the exact solution.

Partial pivoting guarantees that the multipliers will all be less than or equal to 1. This guarantees a small residual, and a "satisfactory" solution in a sense that can be made rigorous.

EFFECT OF ROUNDOFF ERRORS

x_* – the computed solution.

Error:

$$e = x - x_*$$

Residual:

$$r = b - Ax_*$$

Consider the system:

$$\begin{aligned} .780x + .563y &= .217 \\ .913x + .659y &= .254 \end{aligned}$$

The exact solution is $(1, -1)$. If we compute the solution with partial pivoting using 3 significant digits, we find that the multiplier is

$$\frac{0.780}{0.913} = 0.854$$

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$$

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One step of elimination gives us

$$\begin{pmatrix} 0.913 & 0.659 \\ 0 & 0.001 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.254 \\ 0.001 \end{pmatrix}$$

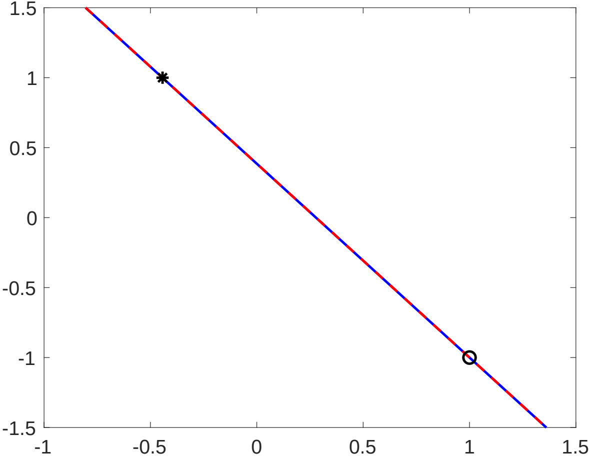
The second equation gives us $x_2 = 1$, and the first gives us

$$0.913x_1 + (0.659)(1) = 0.254 \quad x_1 = \frac{.254 - .659}{.913} = -.443$$

So our computed solution, error and residual are

$$x_* = \begin{pmatrix} -0.443 \\ 1 \end{pmatrix}, \qquad e = \begin{pmatrix} 1.443 \\ -2 \end{pmatrix}, \qquad r = \begin{pmatrix} -0.000460 \\ -0.000541 \end{pmatrix}$$

A small residual, but a huge error!


1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5

Notice that this matrix is nearly singular. Its determinant is

$$\det(A) = 10^{-6}$$

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However, it is not just a small determinant that makes a problem *ill-conditioned*.

Example

Consider

$$A = \begin{pmatrix} .01 & 0 \\ 0 & .01 \end{pmatrix}$$

Then

$$\det(A) = 10^{-4}$$

but A is perfectly well-conditioned. The computation of the solution of $Ax = b$ will be just fine even with a small precision.

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We need another measure of "ill-conditioned" than just smallness of the determinant.

NORMS AND CONDITION NUMBERS

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

$$\|x\|_\infty = \max_i |x_i|$$

These are three examples of *norms* that satisfy

$$\|x\| > 0 \quad \text{if } x \neq 0$$

$$\|0\| = 0$$

$$\|cx\| = |c|\|x\| \quad \text{for all scalars } c$$

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{(the \textit{triangle inequality})}$$

We define the norm of a matrix in the following way:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

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$\|A\|$ is the maximal "stretch" of a vector when A is applied. If A is invertible

$$\|A^{-1}\| = \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} \right)^{-1}$$

CONDITION NUMBER

We define the condition number κ of a matrix as

$$\kappa(A) = \|A\|\|A^{-1}\|$$

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$$\kappa(A) = \frac{\max\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}}{\min\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|}}$$

Call

$$M = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \qquad m = \min_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

So

$$\kappa(A) = \frac{M}{m}$$

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If $A\delta x = \delta b$

$$\|\delta b\| = \|A\delta x\| = \frac{\|A\delta x\|}{\|\delta x\|}\|\delta x\| \geq m\|\delta x\|$$

$$Ax = b, \qquad A(x + \delta x)$$

$$\|b\| \le M\|x\|, \qquad \|\delta b\| \ge m\|\delta x\|$$

$$\boxed{\frac{\|\delta x\|}{\|x\|} \le \kappa(A)\frac{\|\delta b\|}{\|b\|}}$$

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$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}}$$

Large condition number means ill-conditioned. A small change in b can lead to big changes in x. Bad news! For the A in the previous example,

$$\kappa(A) \approx 10^6$$

$$\kappa(A) \geq 1$$

$$\kappa(A) \geq 1$$

$$\kappa(P) = 1$$

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$$\kappa(P) = 1$$

$$\kappa(cA) = \kappa(A)$$

$$\kappa(A) \geq 1$$

$$\kappa(P) = 1$$

$$\kappa(cA) = \kappa(A)$$

If D is diagonal,

$$\kappa(D) = \frac{\max |d_{ii}|}{\min |d_{ii}|}$$

A small condition number means the matrix is well-conditioned (and hence, numerical results are reliable).

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Example

For

$$A = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$$

$\det(A) = 10^{-4}$, but $A = 0.01I$, so

$$\kappa(A) = 1$$

Thus, A is perfectly well-conditioned.

Example

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}, \quad b = \begin{pmatrix} 4.1 \\ 9.7 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 4.11 \\ 9.7 \end{pmatrix}$$

Example

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Then the solutions of $Ax = b$ and $A\tilde{x} = \tilde{b}$ are

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tilde{x} = \begin{pmatrix} 0.34 \\ 0.97 \end{pmatrix}$$

$$\|\delta b\| = 0.01, \qquad \|\delta x\| = 1.63$$

$$\frac{\|\delta b\|}{\|b\|} = 0.0007246, \qquad \frac{\|\delta x\|}{\|x\|} = 1.63$$

$$\kappa(A) \geq \frac{1.63}{0.0007246} = 2249.4$$

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Some condition numbers can be computed in terms of the entries of the matrix (and its inverse).

$$\|A\|_1 = \max_j \sum_i |a_{ij}|, \qquad \|A\|_\infty = \max_i \sum_j |a_{ij}|$$

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For others its not so easy. One important condition number is

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_n}$$

```
cond(A,p)

condest(A)
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