Section 11.1

Modeling with a Differential Equation

Population Growth

(I) Unlimited Growth/ Malthusian Models

ΔP = Changes = Gains– Losses

$$P(t+\Delta t)-P(t)=\beta\cdot P(t)\cdot\Delta t-\delta\cdot P(t)\cdot\Delta t$$

β = Birth rate
δ = Death rate

$$\frac{P(t+\Delta t)-P(t)}{\Delta t}=(\beta-\delta)P(t)$$

As $\Delta t \rightarrow 0$

$$\frac{dP}{dt}=rP(t) \quad ; \quad r=\beta-\delta$$

with an initial population $P(t_0)=P_0$

$$\frac{dP}{P}=r\cdot dT$$

Integrating, we get

$$\ln P = rt + C$$

Plug in

$$\ln P_0 = rt_0 + C$$

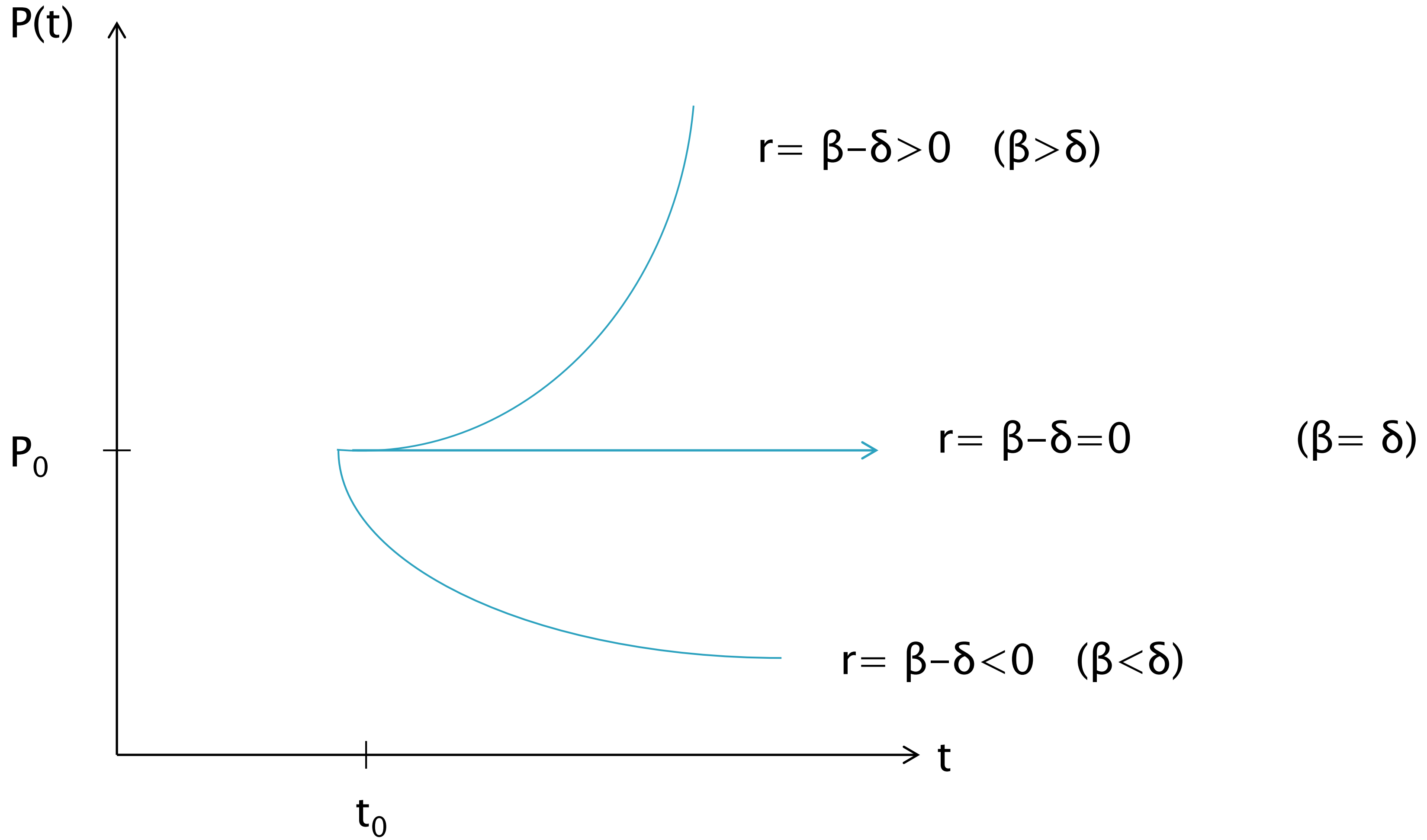
$P(t_0) = P_0$

$$\ln P - \ln P_0 = rt - rt_0$$

$$\ln\left(\frac{P}{P_0}\right) = r(t - t_0)$$

$$\frac{P}{P_0} = e^{r(t-t_0)}$$

$$\boxed{P(t) = P_0 e^{r(t-t_0)}}$$


P(t)
r= β−δ>0 (β>δ)
P0
r= β−δ=0 (β= δ)
r= β−δ<0 (β<δ)
t
t0

Remark: If a data set $\{t_i, P_i\}_{i=1}^{N}$ is given, then the growth/decay rate r can be approximated by $P(t) = P_0 e^{r(t-t_0)}$

One can choose a t_0 and P_0

$$P(t) = P_0 e^{r(t-t_0)}$$

$$\underbrace{\ln\left(\frac{P(t)}{P_0}\right)}_{\text{Y}} = \underbrace{r}_{\text{A}}\underbrace{(t - t_0)}_{\text{X}}$$

$$\boxed{A = \frac{\sum_{i=1}^{N} \mathrm{X}_i \mathrm{Y}_i}{\sum_{i=1}^{N} \mathrm{X}_i^{\,2}}}$$

In Matlab:

>> $t = [t_1 , \dots , t_N]$

>> $P = [P_1 , \dots , P_N]$

>> t0 = t_0 ; P0 = P_0

>> $X = t - t_0$

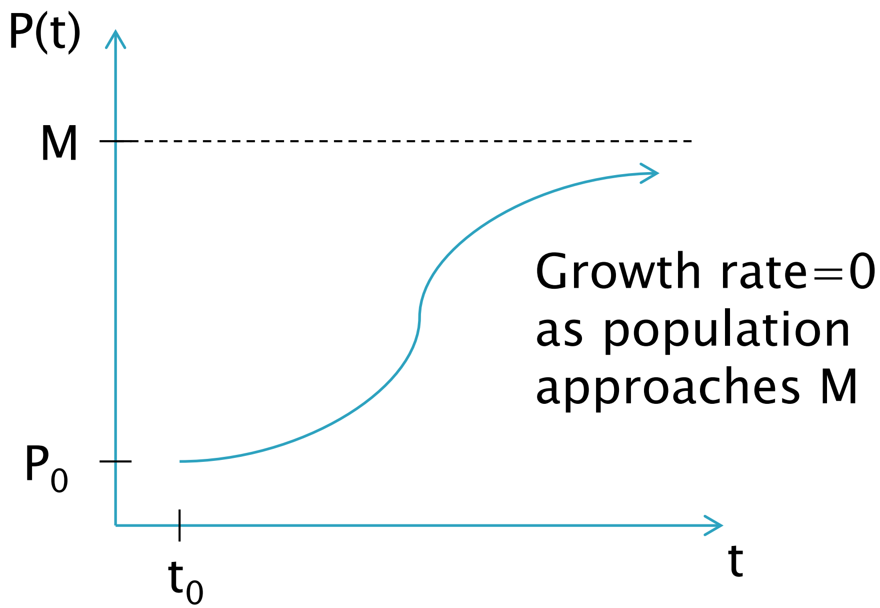
>> $Y = \ln(P/ P_0)$

>> r = sum (X.*Y) / sum (X.*X)

>> tt = linspace (mint(t) , max(t) , 50)

>> PP = P0 * exp (r * (tt – t0))

(II) Limited Resources Growth Model



$$P(t+\Delta t)-P(t)=r\cdot(M-P(t))\cdot P(t)\cdot\Delta t$$

Logistic Growth Model:

$$\frac{dP}{dt}=r(M-P)P$$

with $P(t_0) = P_0$ initial population

$$\frac{dP}{(M-P)\cdot P} = rdT$$

$$\left(\frac{A}{M-P} + \frac{B}{P}\right)dP = rdT$$

$$AP - BP + BM = 1$$

$$A = B; B = \frac{1}{M}$$

$$\left(\frac{1}{M-P} + \frac{1}{P}\right)dP = rMdT$$

Integrate $\quad \ln(P) - \ln|M-P| = rMt + C$

Plug in the initial condition.. $\quad \ln(P_0) - \ln|M-P_0| = rMt_0 + C$

Subtract the two equations $\quad \ln\left(\frac{P}{P_0}\right) - \ln\left(\frac{M-P}{M-P_0}\right) = rM(t-t_0)$

$$\ln\left(\frac{P}{M-P}\cdot\frac{M-P_0}{P_0}\right)=rM(t-t_0)$$

$$\frac{P}{M-P}\cdot\frac{M-P_0}{P_0}=e^{rM(t-t_0)}$$

$$P(M-P_0)=P_0(M-P)e^{rM(t-t_0)}$$

$$P(M-P_0)=P_0Me^{rM(t-t_0)}-PP_0e^{rM(t-t_0)}$$

$$P\left(1+\frac{P_0}{M-P_0}e^{rM(t-t_0)}\right)=\frac{P_0Me^{rM(t-t_0)}}{M-P_0}$$

$$P=\frac{P_0Me^{rM(t-t_0)}}{M-P_0}\cdot\frac{M-P_0}{(M-P_0)+P_0e^{rM(t-t_0)}}$$

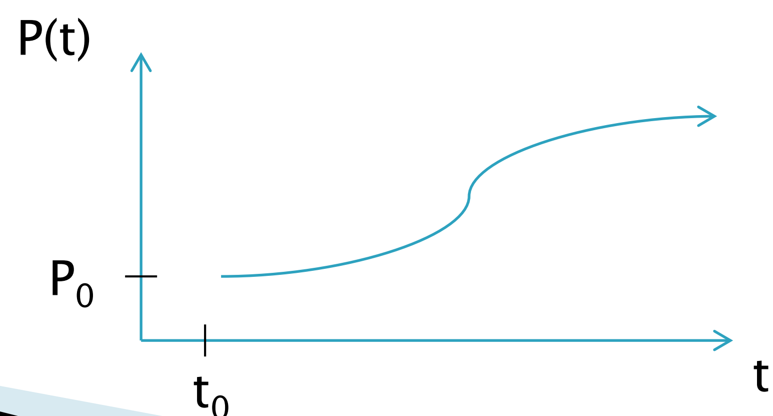
$$P=\frac{P_0M}{(M-P_0)e^{-rM(t-t_0)}+P_0}$$

$$P(t)=\frac{M\left(\frac{P_0}{M-P_0}e^{rM(t-t_0)}\right)}{\left(1+\frac{P_0}{M-P_0}e^{rM(t-t_0)}\right)}$$

$$=\frac{M}{\frac{M-P_0}{P_0}e^{-rM(t-t_0)}+1}$$

$$\boxed{P(t)=\frac{M}{1+\left(\frac{M-P_0}{P_0}\right)e^{-rM(t-t_0)}}} \quad ; \quad P(t_0) = P_0$$

Given (t_0 , P_0), r, and M, one can get the population profile

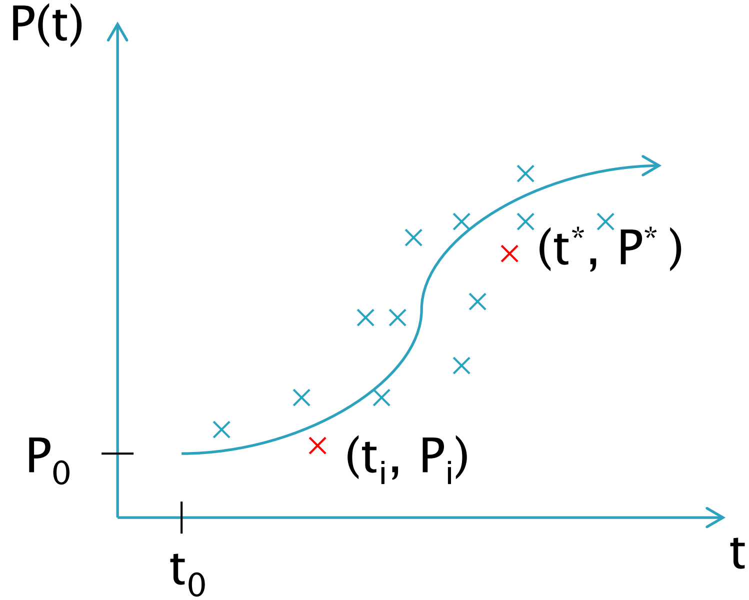


Modeling the Logistic Growth Model With Data

A) Suppose we can estimate M and select a (t^*, P^*) (estimate from data set)

(one parameter fit)

and we want to determine the value of r



(I) Directly from the model

$$P_i = P(t_i) = \frac{M}{1 + \frac{M - P^*}{P^*} e^{-rM(t - t^*)}}$$

$$P_i + P_i \left(\frac{M - P^*}{P^*} \right) e^{-rM(t - t^*)} = M$$

$$e^{-rM(t - t^*)} = \left(\frac{M - P_i}{P_i} \right) \frac{P^*}{M - P^*}$$

$$\underbrace{\ln \left(\frac{M - P_i}{P_i} \cdot \frac{P^*}{M - P^*} \right)}_{Y_i} = \underbrace{-rM}_{r} \cdot \underbrace{(t - t^*)}_{X_i}$$

$$Y_i \quad = \quad r \quad \cdot \quad X_i$$

$$\boxed{r = \frac{\sum X_i Y_i}{\sum X_i^{\ 2}}}$$

(II) Directly from the differential model

$$\frac{dP}{dt} = rP(M-P)$$

$$\frac{P(t_{i+1}) - P(t_i)}{t_{i+1} - t_i} = rP(t_i)(M - P(t_i))$$

$$\underbrace{\frac{P_{i+1} - P_i}{t_{i+1} - t_i}}_{Y_i} = \underbrace{r}_{r} \cdot \underbrace{P_i(M - P_i)}_{X_i}$$

$Y_i = r \cdot X_i$

t = 1, … , N-1

$$\boxed{r = \frac{\sum X_i Y_i}{\sum X_i^{\,2}}}$$

B. Suppose we can select (t^*, P^*) and we want to find the best possible values of r & M

Remark: Direct fit from the model is difficult due to the model being transcendental. Thus, only direct differential model will be implemented.

$$\frac{dP}{dt} = rP(M-P)$$

$$\frac{dP}{dt} = rMP - rP^2$$

$$\underbrace{\frac{1}{P}\frac{dP}{dt}}_{Y} = \underbrace{-r}_{A}\,\underbrace{P}_{X} + \underbrace{rM}_{B}$$

$$Y = AX + B$$

data points (t_i ,P_i)

no data points (M,r)

Matlab

$>> Y_i = \frac{1}{P_i} \frac{P_{i+1} - P_i}{t_{i+1} - t_i}$ i = 1, … , N−1

$>> X_i = P_i$

>> [A,B] = linefit (X,Y)

>> r = A

>> rM = B

=> M = B/r

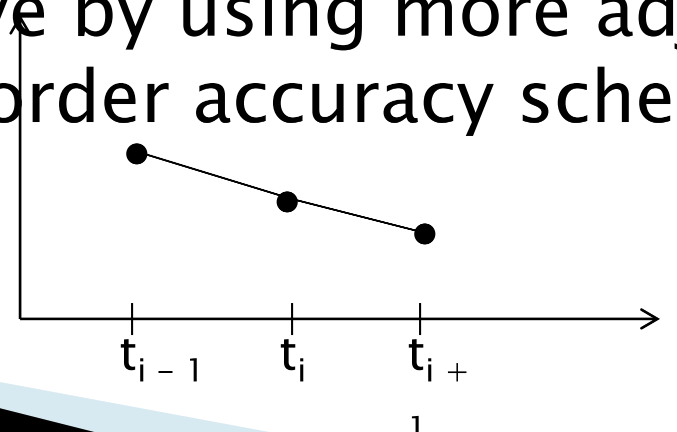
Remarks...

i. Since $\dfrac{dP_i}{dt} = \dfrac{P_{i+1} - P_i}{t_{i+1} - t_i}$, to avoid numerical errors, we need

to sort the data set in the monotone order in time.

ii. One can improve the accuracy of the derivative by using more adjacent points. (higher order accuracy schemes)

Ex:



$$\frac{dP_i}{dt} = \frac{P_{i+1} - P_{i-1}}{t_{i+1} - t_{i-1}}$$

Homework §11.1

1, 3, 4