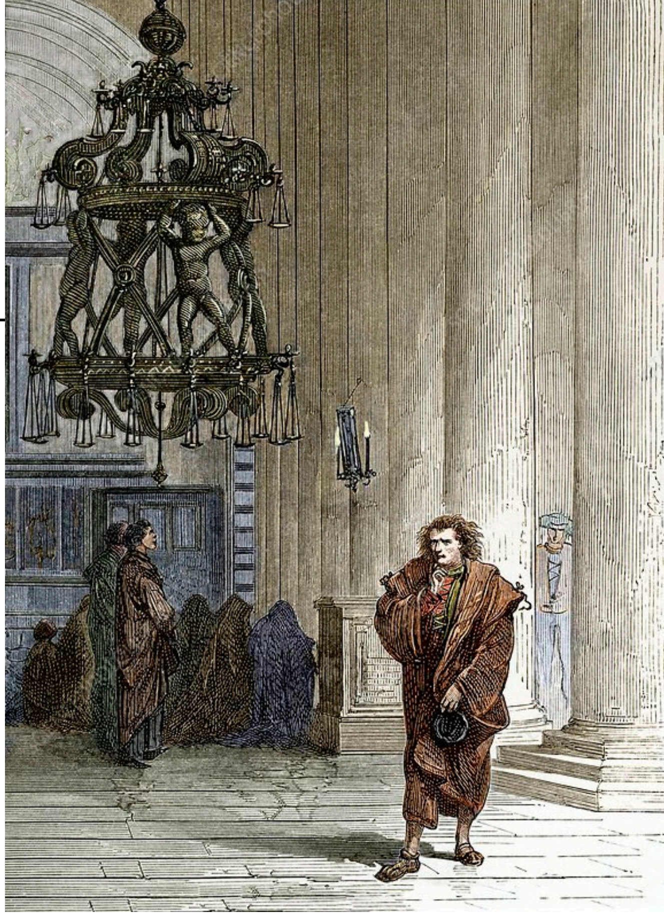
Section 14.1
Dimension Analysis

Galileo's Pendulum



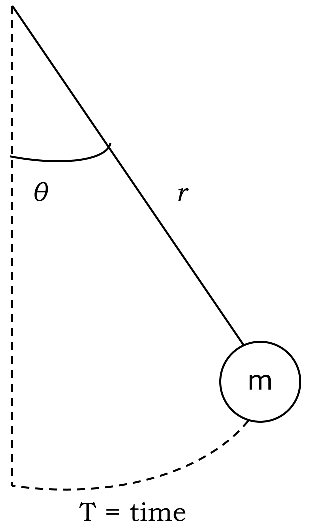
One day, in 1583, while in church at the Cathedral of Pisa, Galileo was watching a chandelier as it swung. Galileo found that:

- the time that it took for a pendulum to travel its arc length and back to its starting point was the same regardless of the arc length itself.
- If the pendulum was released from a higher point it would travel faster through the path and therefore it would take the same time as a pendulum released from a lower point that travels slower.
- the one variable that did effect the time, known as the period, was the length of the pendulum string.
- These experiments led to the first uses of pendulums for the powering of clocks.

Application of Dimensional Analysis

Swing of a Pendulem

T^2~Length (Galieo)
Model the time T
Other dependent variables:
r, m, g, θ



Using the Rayleigh Method, we get

$$[\mathrm{T}] = \left[k \cdot r^{\alpha} \cdot m^{\beta} \cdot g^{\gamma} \cdot \theta^{\delta}\right]$$

$$\mathrm{T} = \mathrm{L}^{\alpha} \cdot \mathrm{M}^{\beta} \cdot \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^{\gamma}$$

$$= \mathrm{L}^{\alpha+\gamma} \cdot \mathrm{M}^{\beta} \cdot \mathrm{T}^{-2\gamma}$$

Then we solve the following system of equations

Then our equation is:

$$\mathrm{T} = k \cdot \theta^{\delta} \cdot r^{\frac{1}{2}} \cdot m^{0} \cdot g^{-\frac{1}{2}}$$

$$= k \cdot \theta^{\delta} \sqrt{\frac{r}{g}}$$

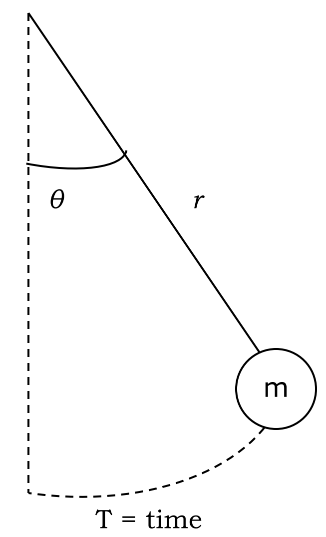
k and δ can be estimated using some experiments.

The Rayleigh Method can not give use these values.

Note

The true formulation for a period of a pendulum in a vacuum is:

$$T = \theta\sqrt{\frac{r}{g}}$$



And the formulation for a satellite orbits is:

$$T = 2\Pi\sqrt{\frac{r}{g}} = 2\Pi\sqrt{\frac{r^3}{GM}},$$

where G ~ Universal Gravitational Constant and
M ~ Mass of Earth..

Example 1:

Suppose the pendulum is submerged in a liquid with (bulk) viscosity μ at density ρ. Find the period of the Pendulum.

Solution:

$$[\mathrm{T}] = \left[k \cdot r^{\alpha} \cdot m^{\beta} \cdot g^{\gamma} \cdot \mu^{\varepsilon} \cdot \rho^{a} \cdot \theta^{\delta}\right]$$

$$\mathrm{T} = \mathrm{L}^{\alpha} \cdot \mathrm{M}^{\beta} \cdot \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^{\gamma} \cdot \left(\frac{\mathrm{M}}{\mathrm{LT}}\right)^{\varepsilon} \cdot \left(\frac{\mathrm{M}}{\mathrm{L}^3}\right)^{a}$$

$$= \mathrm{L}^{\alpha+\gamma-\varepsilon-3a} \cdot \mathrm{M}^{\beta+\varepsilon+a} \cdot \mathrm{T}^{-2\gamma-\varepsilon}$$

Then we solve the following system of equation:

Then substituting these into the original T equation gives us.

Sanity check:

$$[T] = T$$

$$\left[\frac{m}{r \cdot \mu}\right] = \frac{\mathrm{M}}{\mathrm{L} \cdot \left(\dfrac{\mathrm{M}}{\mathrm{L} \cdot \mathrm{T}}\right)} = \mathrm{T}$$

$$\left[\frac{m^2 \cdot g}{r^3 \cdot \mu}\right] = \frac{\mathrm{M}^2 \cdot \dfrac{\mathrm{L}}{\mathrm{T}^2}}{\mathrm{L}^3 \cdot \left(\dfrac{\mathrm{M}}{\mathrm{LT}}\right)^2} = 1$$

$$\left[\frac{r^3 \cdot \rho}{m}\right] = \frac{\mathrm{L}^3 \cdot \left(\dfrac{\mathrm{M}}{\mathrm{L}^3}\right)}{\mathrm{M}} = 1$$

Note:

$$T_{vac} = k \cdot \theta^{\delta} \cdot \sqrt{\frac{r}{g}}$$

$$T_{med} = k \cdot \theta^{\delta} \cdot \left[\frac{m}{r \cdot \mu}\right] \cdot \left(\frac{m^2 \cdot g}{r^3 \cdot \mu}\right)^{\gamma} \cdot \left(\frac{r^3 \cdot \rho}{m}\right)^{a}$$

Remarks:

1) When there are numerous variables, it is quite easy to make algebraic mistakes

2) The relationship derived from the Rayleigh Method is in terms of powers, which might not be necessary.

3) Regardless, the equation is dimensionally correct.

Example

- The height h that a fluid will rise in a capillary tube decreases as the diameter D of the tube increases. Use dimensional analysis to determine how h varies with D and the specific weight w and surface tension of the liquid.

h= f(D,W,σ)

$$h = k \cdot D^a \cdot w^b \cdot \sigma^c$$

$$[h] = [k \cdot D^a \cdot W^b \cdot \sigma^c]$$

$$L = L^a \left(\frac{M}{L^2 T^2} \right)^b \left(\frac{M}{T^2} \right)^c = \frac{L^{a-2b}}{T^{2b+2c}}$$

Solve for a, b, and c, we have

Homework –Section 14.1

- # 2, 4, 7, 8