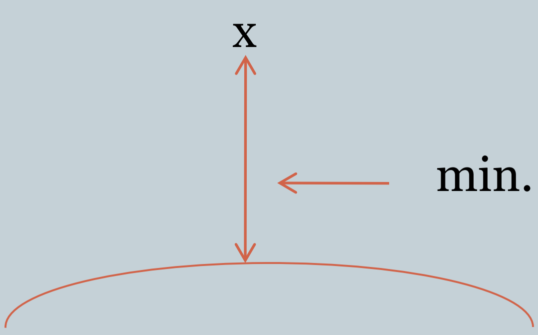
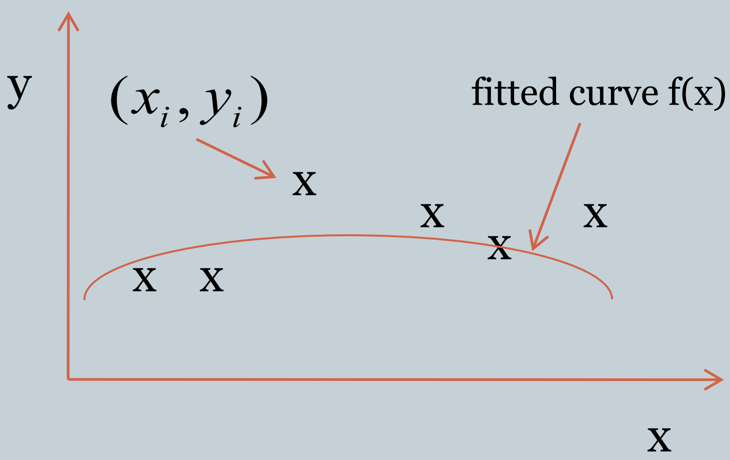
Section 3.3

LEAST SQUARE DATA FITTING

Least Square Fitting Model

Given a set of data points $(x_i, y_i)_{i=1}^{N}$, we need to find a model f(x) best fit the data points, and $f(x_i)$ is as close to y_i as possible.





Error of the Model

We measure the accuracy of the model by comparing $f(x_i)$ with y_i .

To find the error made at the points:

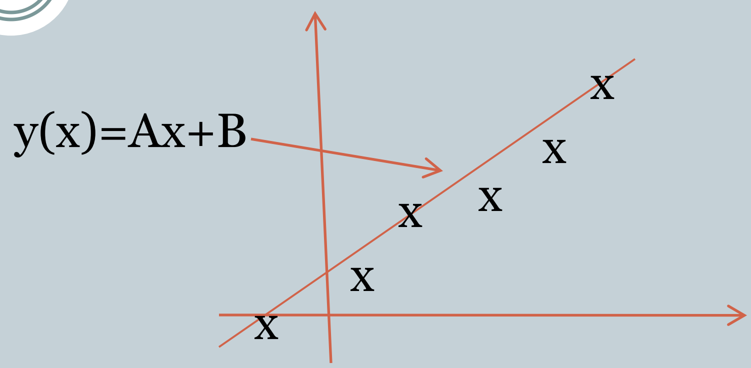
	ERROR =
1-norm	$\sum_{i=1}^{N} \lvert f(x_i) - y_i \rvert$
2-norm	$\sum_{i=1}^{N} (f(x_i) - y_i)^2$
∞-norm	$\max \lvert f(x_i) - y_i \rvert \, 1 \le i \le N$

Note: the 2-norm is more desirable because differentiability is smoother than the others.

Line fitting

The goal is to find:

1. *slope* ***A***
2. *y-intercept* ***B***



so the error is as small as possible.

Note that for each set of (A,B), we can measure its performance. Namely, the square error:

$$E(A,B)=\sum_{i=1}^{N}\left(f(x_i)-y_i\right)^2 \underset{\substack{linefit \\ f(x)=Ax+B}}{=} \sum_{i=1}^{N}(Ax_i+B-y_i)^2 \leftarrow \text{Square error}$$

Least Square Error

Remarks:

The optimal (A^*, B^*) will yield the Least Square Error(LSE)

E(A,B) = $\sum_{i=1}^{N}(Ax_i + B - y_i)^2$ is a 2-variables function

To find extreme of F(x,y)

1. Find the critical points, set $\frac{\partial F}{\partial x} = 0 \;\&\; \frac{\partial F}{\partial y} = 0 \Rightarrow (x^*, y^*)$
2. Form D(x,y) = $F_{xx}(x,y)F_{yy}(x,y) - F_{xx}^2\;(x,y)$

To find critical points:

$$\frac{\partial E}{\partial A} = \sum_{i=1}^{N} 2(Ax_i + B = y_i)x_i = 0 \Rightarrow A = \sum x_i^2 + B\sum x_i = \sum x_i y_i \text{....(1)}$$

$$\frac{\partial E}{\partial B} = 2\sum_{i=1}^{N} (Ax_i + B = y_i) \bullet 1 = 0 \Rightarrow B = \frac{1}{N}\sum y_i - \frac{1}{N} A\sum x_i \text{........(2)}$$

Substitute (2) into (1) :

To find critical points:

$$A^* = \frac{\sum x_i y_i - \frac{1}{N}\sum x_i y_i}{\sum x_i^2 - \frac{1}{N}(\sum x_i)^2} \text{............}(3)$$

$$B^* = \frac{1}{N}\frac{(\sum x_i)^2 \sum y_i - (\sum x_i y_i)\sum x_i}{\sum x_i^2 - \frac{1}{N}(\sum x_i)^2} \text{............}(4)$$

To minimal error, we show that

$$D(A^*,B^*) > 0 \; \& \; E_{AA}(A^*,B^*) > 0$$

$$E_{AA} = \frac{\partial}{\partial A}(2\sum Ax_i^2 + Bx_i - x_i y_i) = 2\sum x_i^2$$

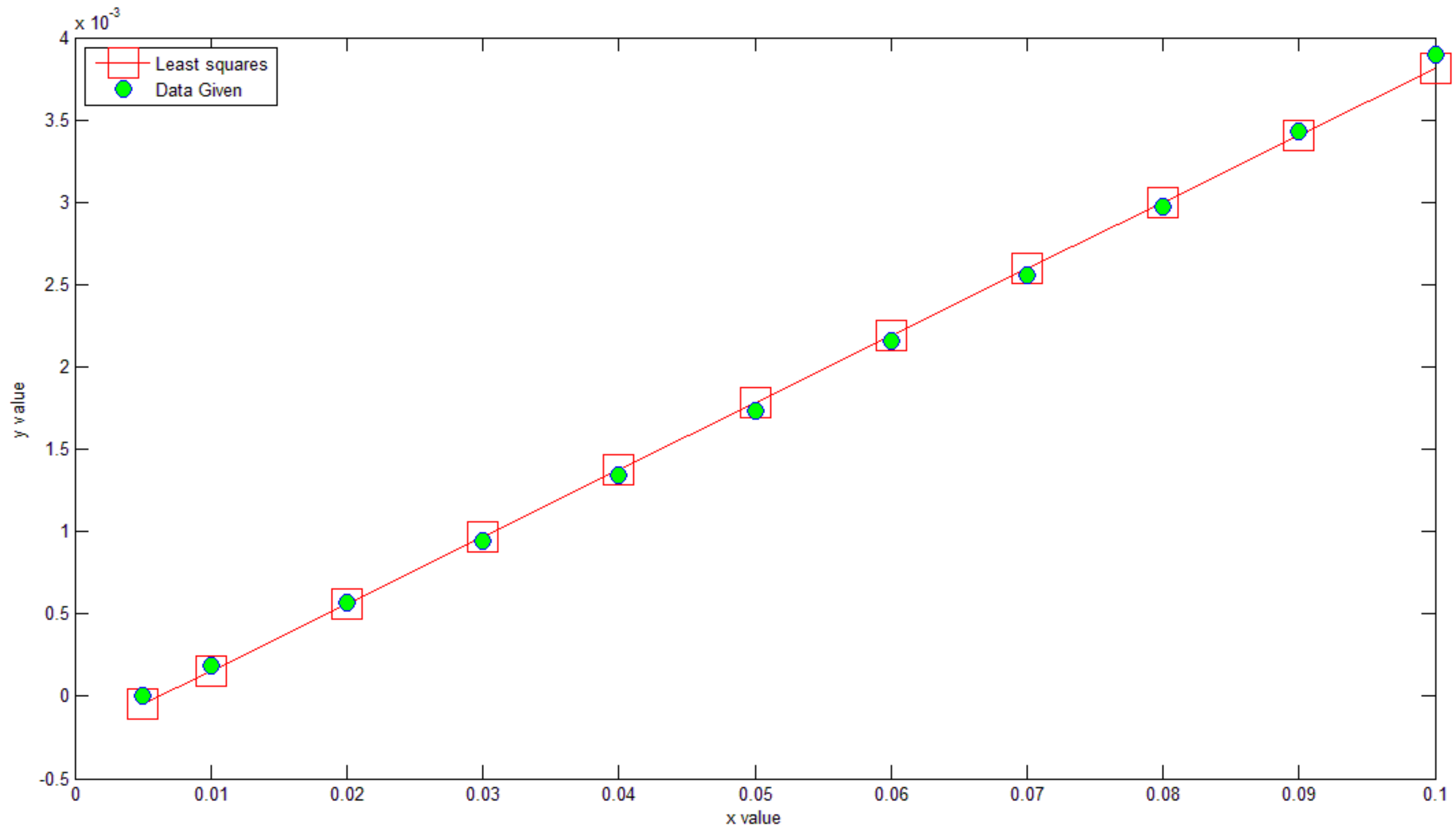
$$E_{BB} = \frac{\partial}{\partial B}(2\sum Ax_i + BN - \sum y_i) = 2N$$

$$E_{AB} = 2\sum x_i$$

Matlab –Line fit

- x = [.............];
- y = [.............];
- a = x;
- b = y;
- [A,B] = mylinefit(x,y)
- y=B + A*x;
- plot(x,y,'-rs‘); hold on
- plot(a,b,'bo');hold on
- title(.....)
- ylabel ('y value ')
- xlabel ('x value')
- legend ('Least squares','Data Given',2)

Linefit Graph



Homework:

1. Prove that $\mathrm{D}(\mathrm{A}^*, B^*) > 0$
2. For a one$-$parameter fit $f(x) = Ax$, derive the formulation for the optimal $A*$ such that $E(A) = \sum_{i=1}^{N} (Ax_i - y_i)^2$ yields the least mean square error
3. Write a MATLAB function that takes in $(x_i, y_i)_{i=1}^{N}$ and returns $A*$
4. Section 3.3 # 7a, 8b, 9a