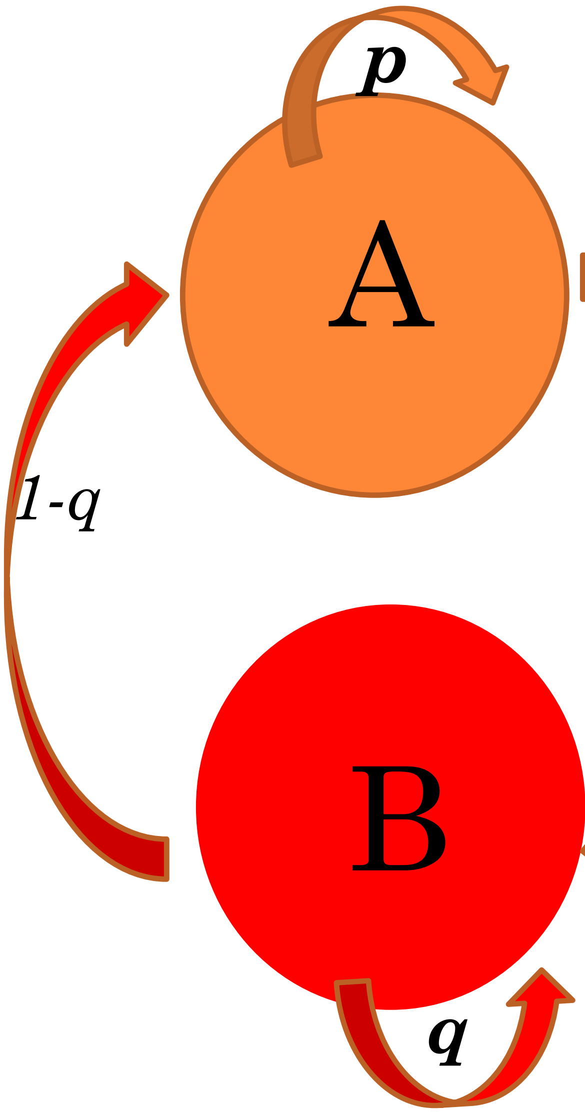
§ 6.1 PROBABILISTIC MODELING WITH DISCRETE SYSTEM

EXAMPLE

Two dining halls A and B

- Students will be dining in either A or B. At each time we count the number of students dining in the halls. *A(n)* or *B(n)* are the states.
- The students return to diner A with probability *p* and switch to diner with probability *1-p*. The students return to diner *B* with probability *q* and switch to diner with probability *1-q.*



REMARKS

1) States do not overlap. They are mutually exclusive (the student can only dine in A or B, but not in both at the same time).

2) The transitions between states are indicated by arrows with the transition probabilities.

3) The sum of the transition probabilities for each state must amount to one ($1+(1-p)=1$).

Probabilistic Model

$$A(n+1) - A(n) = changes$$
$$= gains - loses$$
$$= (1-Q)B(n) - (1-p)A(n)$$
$$= (1-Q)B(n) - A(n) + pA(n)$$

So,

$$A(n+1) = pA(n) + (1-Q)B(n)$$

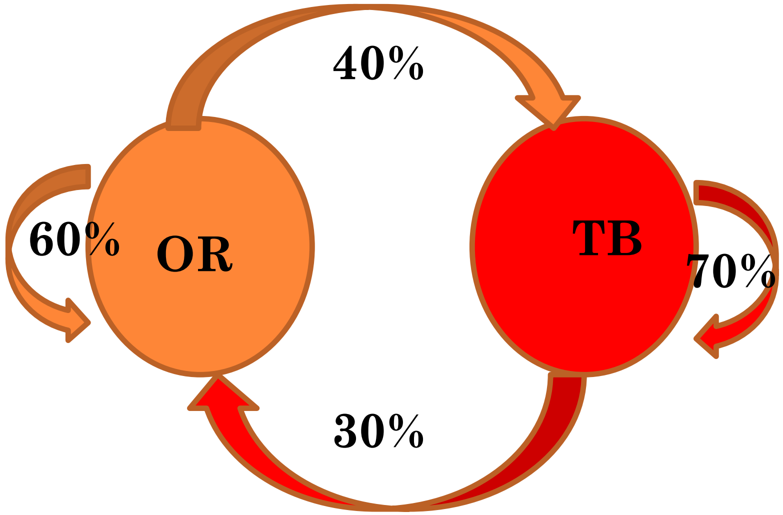
Similarly,

$$B(n+1) = (1-p)A(n) + QB(n)$$

Note: In other sections, the values for P and Q are given. In this section, if data is available $\{(A_n, B_n)\}_{n=1}^{N}$, one can perform a model fit to determine P and Q. This process is known as <u>parameter identification</u>

Ex: Car rental at Tampa Bay and Orlando

Let $\theta(n)$ be the number of rental vehicles at Orlando at time n. Similarly, let $\tau(n)$ be the number of vehicles at Tampa Bay at time n.



So,

$$\theta(n+1) = 0.6\theta(n) + 0.3\tau(n)$$

$$\tau(n+1) = 0.4\theta(n) + 0.7\tau(n)$$

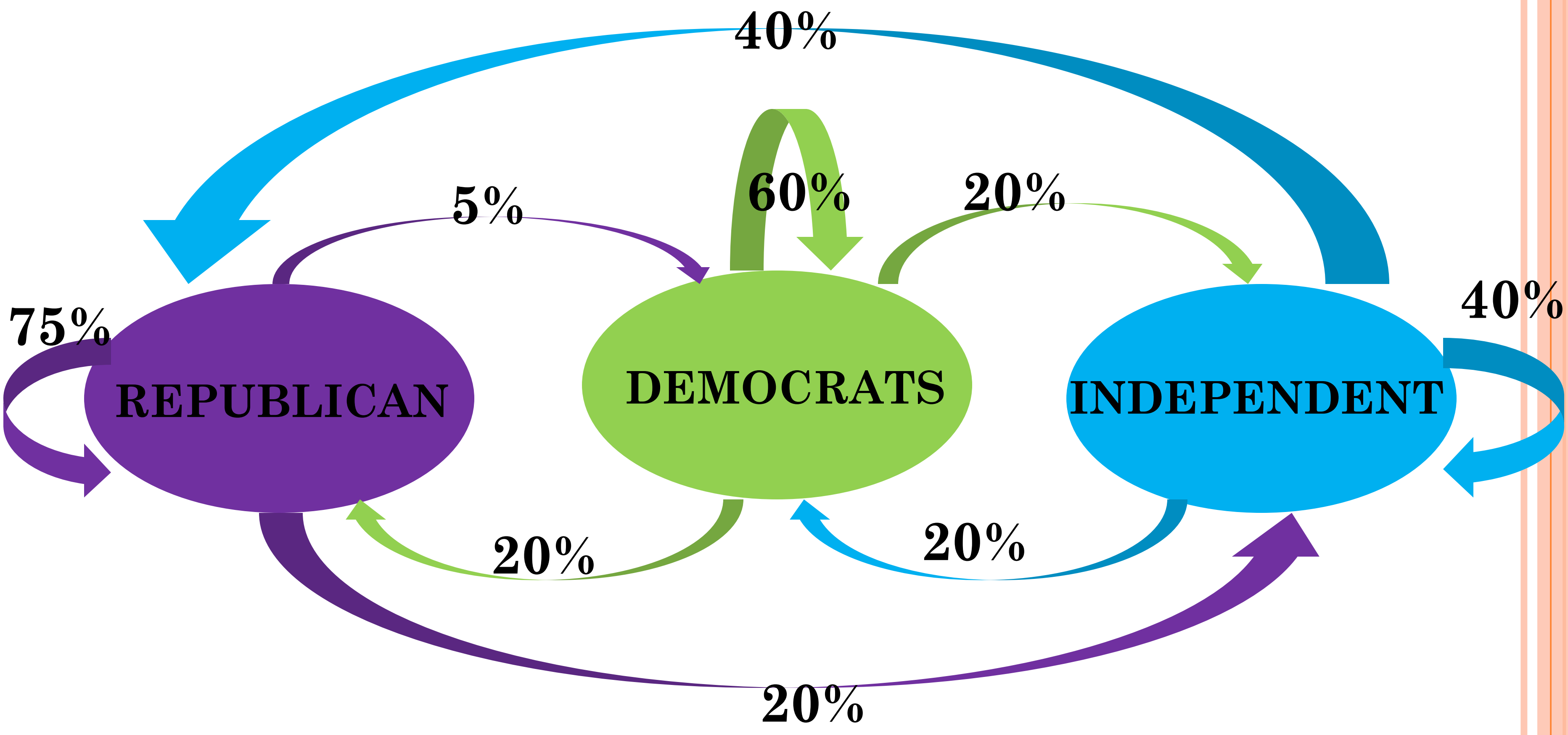
$$\begin{bmatrix}\theta \\ \tau\end{bmatrix}(n+1) = A\begin{bmatrix}\theta \\ \tau\end{bmatrix}(n)$$

Where,

$$A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$$

is the state transition matrix

EX: VOTER TENDENCIES



Let [R(n),D(n),I(n)] be the state of the voter tendencies

R(n+1) - R(n)= GAINS – LOSSES

= 0.2 D(n) + 0.4 I(n) – 0.25 R(n)

-----------------------------------o-----------------------------------

R(n+1) = 0.75 R(n) + 0.2 D(n) + 0.4 I(n)

Similarly,

D(n+1)= 0.05 R(n) + 0.6 D(n) + 0.2 I(n)

I (n+1)= 0.2 R(n) + 0.2 D(n) + 0.4 I(n)

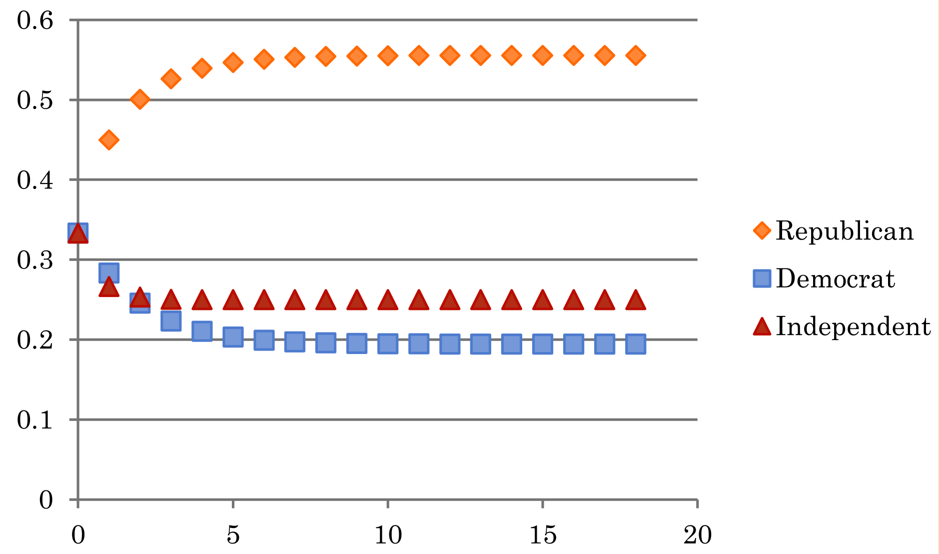
$$Let\ \vec{x}(n) = \begin{bmatrix} R_n \\ D_n \\ I_n \end{bmatrix} \quad A = \begin{bmatrix} 0.75 & 0.20 & 0.40 \\ 0.05 & 0.60 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{bmatrix}$$

$$\bar{x}(n+1) = A\bar{x}(n) \quad with\ \bar{x}(0) = [R_0\ D_0\ \ I_0]$$

Model solution

Assume that initially 1/3 of the voters are Republicans, 1/3 of the voters are Democrats, 1/3 of the voters are Independents

n	Republican	Democrat	Independent	SUM
0	0.333333333	0.333333333	0.333333333	1
1	0.45	0.283333333	0.266666667	1
2	0.500833333	0.245833333	0.253333333	1
3	0.526125	0.223208333	0.250666667	1
4	0.539502083	0.210364583	0.250133333	1
5	0.546752813	0.203220521	0.250026667	1
6	0.55071938	0.199275286	0.250005333	1
7	0.552896726	0.197102208	0.250001067	1
8	0.554093413	0.195906374	0.250000213	1
9	0.55475142	0.195248538	0.250000043	1
10	0.555113289	0.194886702	0.250000009	1
11	0.555312311	0.194687687	0.250000002	1
12	0.555421771	0.194578228	0.25	1
13	0.555481974	0.194518026	0.25	1
14	0.555515086	0.194484914	0.25	1



Remark: Given an initial state what's the limit point?

$$\vec{x}^* = \lim_{n\to\infty} \vec{x}(n) = \lim_{n\to\infty} A^n \overline{x_0}$$

Modeling infectious diseases

Epidemics:

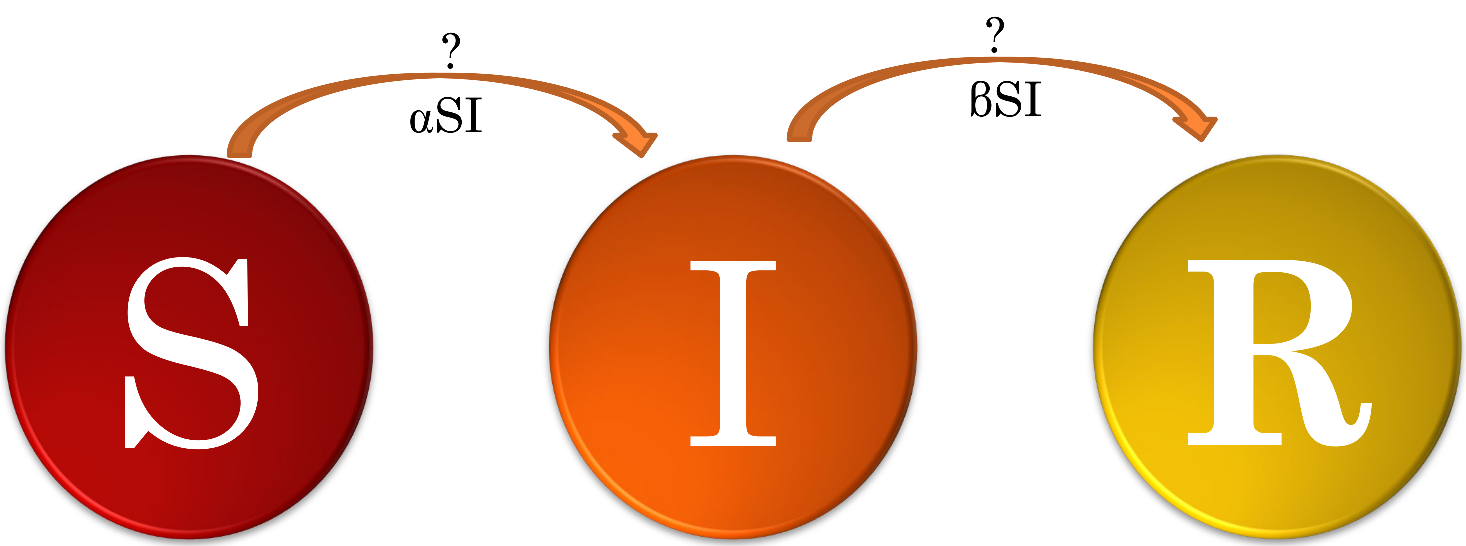
Daniel Bernoulli was the first to derive a math model for the spread of infectious disease.

One of his models is called the SIR model

S~ people who are susceptible to the disease

I~ people who are infected with the disease

R~ people who recovered and no longer susceptible



Assume the population carries the birth rate of β and the death rate δ

$$S(n+1)-S(n)= \text{Gains} - \text{Losses}$$
$$= \beta\, S(n) - \delta\, S(n) - \alpha\, S(n)\, I(n)$$
$$I(n+1) - I(n)= \lambda\, S(n)\, I(n) - g\, I(n) - \delta\, I(n)$$

$$R(n+1) - R(n)= g\, I(n) - \delta\, R(n)$$

Homework §6.1 #1-4