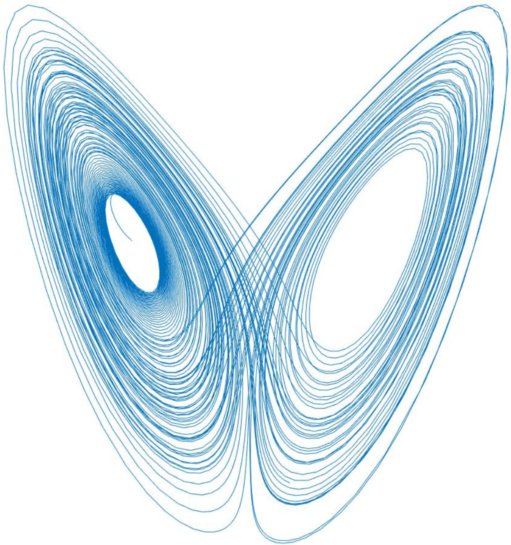


We will consider differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

We will also use the notations

$$\frac{dy}{dt} = y' = \dot{y}$$

Usually we will be solving Initial Value Problems (IVP) where we are given y and some value of t:

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Example

Consdier the equation

$$y' = r\,y, \qquad y(0) = y_0$$

This has the solution

$$y(t) = y_0 e^{rt}$$

Example

Consider the equation

$$y'' + y = 0, \qquad y(0) = y_0,\ y'(0) = v_0$$

This has the solution

$$y(t) = y_0 \cos(t) + v_0 \sin(t)$$

We can rewrite this as a system of first order equations. Let

$$y_1 = y, \quad y_2 = y'$$

Then

$$y_1' = y' = y_2, \quad y_2' = y'' = -y_1$$

So

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}$$

Or, in matrix-vector form,

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Existence and Uniqueness

Does $y' = f(t, y)$ always have a unique solution? Alas no.

Example

Show that

$$y' = y^{1/3}, \qquad y(0) = 0$$

has more than one solution.

Example

Show that

$$y' = \begin{cases} -1 & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases} \qquad y(0) = 0$$

has no solution.

Theorem

Let f be continuously differentiable near (t_0, y_0). Then there is an $\alpha > 0$ such that the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

has a unique solution for

$$t_0 \le t < t_0 + \alpha$$

Integrating Differential Equations

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0$$

$$y_n \approx y(t_n), \qquad n = 0, 1, 2, \ldots$$

$$h_n = t_{n+1} - t_n$$

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))ds$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s)ds$$

Systems of Equations

$$\ddot{x}(t) = -x(t)$$

$$y(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

$$\dot{y}(t) = \begin{pmatrix} \dot{x}(t) \\ -x(t) \end{pmatrix}$$

$$= \begin{pmatrix} y_2(t) \\ -y_1(t) \end{pmatrix}$$

$$\ddot{u}(t) = -u(t)/r(t)^3$$
$$\ddot{v}(t) = -v(t)/r(t)^3$$

$$r(t) = \sqrt{u(t)^2 + v(t)^2}$$

$$y(t) = \begin{pmatrix} u(t) \\ v(t) \\ \dot{u}(t) \\ \dot{v}(t) \end{pmatrix}$$

$$\dot{y} = \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \\ -u(t)/f(t)^3 \\ -v(t)/r(t)^3 \end{pmatrix} = \begin{pmatrix} y_3(t) \\ y_4(t) \\ -y_1(t)/r(t)^3 \\ -y_2(t)/r(t)^3 \end{pmatrix}, \qquad r(t) = \sqrt{y_1(t)^2 + y_2(t)^2}$$

Linearized Differential Equations

$$f(t, y) = f(t_c, y_c) + \alpha(t - t_c) + J(y - y_c) + \cdots$$

$$\alpha = \frac{\partial f}{\partial t}(t_c, y_c)$$

$$J = \frac{\partial f}{\partial y}(t_c, y_c)$$

Near (t_c, y_c),

$$f(t, y) \approx f(t_c, y_c) + \alpha(t - t_c) + J(y - y_c)$$

If $f(t_c, y_c) = 0$

$$\dot{y} = f(t, y) \quad \text{is approximated by} \quad \dot{y} = \alpha(t - c) + J(y - y_c)$$

$$\dot{y} = f(t, y)$$

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, \dots, y_n) \\ f_2(t, y_1, \dots, y_n) \\ \vdots \\ f_n(t, y_1, \dots, y_n) \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \dots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \dots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \dots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

$$J = \left(\frac{\partial f_i}{\partial y_j}\right)$$

Linear Systems

$$\dot{y} = Ay$$

Suppose A can be diagonalized

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$A = X\Lambda X^{-1}, \qquad X = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Set

$$y = Xz$$

Then

$$\dot{y} = X\dot{z} = X\Lambda X^{-1} y = X\Lambda X^{-1} Xz = X\Lambda z$$

So

$$\dot{z} = \Lambda z$$

$$z = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \qquad \boxed{y = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t}}$$

$$\ddot{x} + x = 0$$

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of J are $\pm i$ and the solutions are purely oscillatory linear combinations of

$$e^{it} \quad \text{and} \quad e^{-it}$$

$$\dot{y} = \begin{pmatrix} y_3(t) \\ y_4(t) \\ -y_1(t)/r(t)^3 \\ -y_2(t)/r(t)^3 \end{pmatrix}, \qquad r(t) = \sqrt{y_1(t)^2 + y_2(t)^2}$$

$$J = \frac{1}{r^5} \begin{pmatrix} 0 & 0 & r^5 & 0 \\ 0 & 0 & 0 & r^5 \\ 2y_1^2 - y_2^2 & 3y_1y_2 & 0 & 0 \\ 3y_1y_2 & 2y_2^2 - y_1^2 & 0 & 0 \end{pmatrix}$$

$$\lambda = \frac{1}{r^{3/2}} \begin{pmatrix} \sqrt{2} \\ i \\ -\sqrt{2} \\ -i \end{pmatrix}$$

Single Step Methods

Given an approximation at time t, we want to advance the solution to time $t + h$.

Euler's Method

$$\dot{y}(t) \approx \frac{y(t+h) - y(t)}{h}$$

$$\dot{y} = f(t.y)$$

$$y(t+h) \approx y(t) + h\,f(t, y(t))$$

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h\,f(t_n, y_n) \\ t_{n+1} &= t_n + h \end{aligned}}$$

Methods based on quadrature rules

$$y(t+h) = y(t) + \int_{t}^{t+h} f(s, y(s))ds$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s)ds$$

Midpoint rule:

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}s_1\right) \\
y_{n+1} &= y_n + hs_2 \\
t_{n+1} &= t_n + h
\end{aligned}
$$

Trapezoid rule: (Heun method)

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f\left(t_n + h, y_n + hs_1\right) \\
y_{n+1} &= y_n + \frac{h}{2}\frac{s_1 + s_2}{2} \\
t_{n+1} &= t_n + h
\end{aligned}
$$

Simpson's rule: (Classical Runge-Kutta)

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}s_1\right) \\
s_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}s_2\right) \\
s_4 &= f(t_n + h, y_n + hs_3) \\
y_{n+1} &= y_n + \frac{h}{6}\left(s_1 + 2s_2 + 2s_3 + s_4\right) \\
t_{n+1} &= t_n + h
\end{aligned}
$$

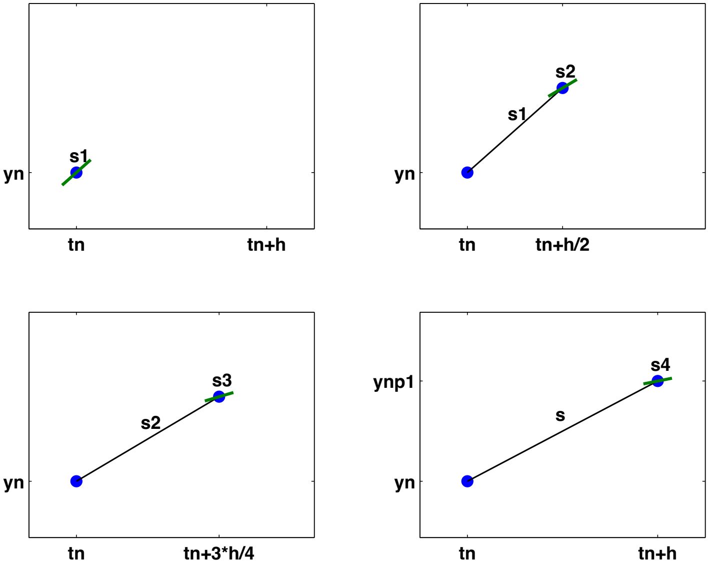
$$s_i = f\left(t_n + \alpha_i h, y_n + h\sum_{j=1}^{i-1}\beta_{ij}s_j\right), \quad i = 1, \dots, k$$

$$y_{n+1} = y_n + h\sum_{i=1}^{k}\gamma_i s_i$$

$$e_{n+1} = h\sum_{i=1}^{k}\delta_i s_i$$

The BS23 Algorithm

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} s_1\right) \\
s_3 &= f\left(t_n + \frac{3}{4} h, y_n + \frac{3}{4} h s_2\right) \\
t_{n+1} &= t_n + h \\
y_{n+1} &= y_n + \frac{h}{9}\left(2s_1 + 3s_2 + 4s_4\right) \\
s_4 &= f(t_{n+1}, y_{n+1}) \\
e_{n+1} &= \frac{h}{72}(-5s_1 + 6s_2 + 8s_3 - 9s_4)
\end{aligned}
$$


s1
yn
tn
tn+h
s2
s1
yn
tn
tn+h/2
s3
s2
yn
tn
tn+3*h/4
s4
ynp1
s
yn
tn
tn+h

Lorenz Attractor

$$\dot{y} = Ay, \qquad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$A = \begin{pmatrix} -\beta & 0 & y_2 \\ 0 & -\sigma & \sigma \\ -y_2 & \rho & -1 \end{pmatrix}$$

There is a fixed point at

$$\begin{pmatrix} \rho - 1 \\ \eta \\ \eta \end{pmatrix} \qquad \eta = \pm\sqrt{\beta(\rho - 1)}$$

Stiffness

A problem is stiff if the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results.

Example

$$\dot{y} = y^2 - y^3, \qquad 0 \le t \le \frac{2}{\eta}$$

$$y(0) = \eta$$