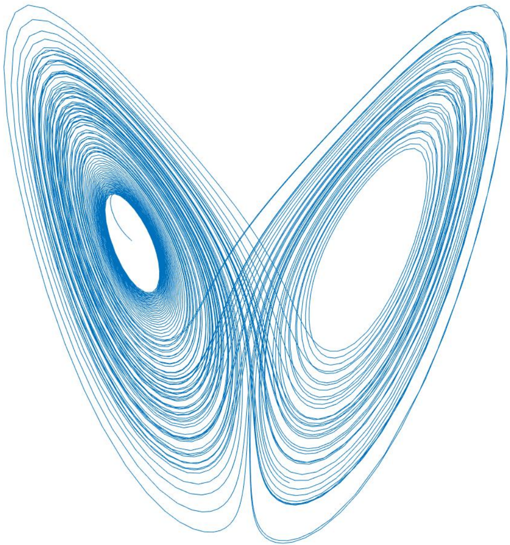

We will consider differential equations of the form

$$\frac{dy}{dt} = f(t, y)$$

We will also use the notations

$$\frac{dy}{dt} = y' = \dot{y}$$

Usually we will be solving Initial Value Problems (IVP) where we are given y and some value of t:

$$y' = f(t, y), \qquad y(t_0) = y_0$$

Example

Consdier the equation

$$y' = r\, y, \qquad y(0) = y_0$$

This has the solution

$$y(t) = y_0 e^{rt}$$

Example

Consider the equation

$$y'' + y = 0, \qquad y(0) = y_0,\ y'(0) = v_0$$

This has the solution

$$y(t) = y_0 \cos(t) + v_0 \sin(t)$$

We can rewrite this as a system of first order equations. Let

$$y_1 = y, \quad y_2 = y'$$

Then

$$y_1' = y' = y_2, \quad y_2' = y'' = -y_1$$

So

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 \end{aligned}$$

Or, in matrix-vector form,

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Existence and Uniqueness

Does $y' = f(t, y)$ always have a unique solution? Alas no.

Example

Show that

$$y' = y^{1/3}, \qquad y(0) = 0$$

has more than one solution.

Example

Show that

$$y' = \begin{cases} -1 & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases} \qquad y(0) = 0$$

has no solution.

Theorem

Let f be continuously differentiable near (t_0, y_0). Then there is an $\alpha > 0$ such that the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

has a unique solution for

$$t_0 \leq t < t_0 + \alpha$$

Integrating Differential Equations

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_0) = y_0$$

$$y_n \approx y(t_n), \qquad n = 0, 1, 2, \ldots$$

$$h_n = t_{n+1} - t_n$$

$$y(t+h) = y(t) + \int_{t}^{t+h} f(s, y(s))ds$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s)ds$$

Systems of Equations

$$\ddot{x}(t) = -x(t)$$

$$y(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

$$\dot{y}(t) = \begin{pmatrix} \dot{x}(t) \\ -x(t) \end{pmatrix}$$

$$= \begin{pmatrix} y_2(t) \\ -y_1(t) \end{pmatrix}$$

$$\ddot{u}(t) = -u(t)/r(t)^3$$
$$\ddot{v}(t) = -v(t)/r(t)^3$$

$$r(t) = \sqrt{u(t)^2 + v(t)^2}$$

$$y(t) = \begin{pmatrix} u(t) \\ v(t) \\ \dot{u}(t) \\ \dot{v}(t) \end{pmatrix}$$

$$\dot{y} = \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \\ -u(t)/f(t)^3 \\ -v(t)/r(t)^3 \end{pmatrix} = \begin{pmatrix} y_3(t) \\ y_4(t) \\ -y_1(t)/r(t)^3 \\ -y_2(t)/r(t)^3 \end{pmatrix}, \qquad r(t) = \sqrt{y_1(t)^2 + y_2(t)^2}$$

Linearized Differential Equations

$$f(t, y) = f(t_c, y_c) + \alpha(t - t_c) + J(y - y_c) + \cdots$$

$$\alpha = \frac{\partial f}{\partial t}(t_c, y_c)$$

$$J = \frac{\partial f}{\partial y}(t_c, y_c)$$

Near (t_c, y_c),

$$f(t, y) \approx f(t_c, y_c) + \alpha(t - t_c) + J(y - y_c)$$

If $f(t_c, y_c) = 0$

$$\dot{y} = f(t, y) \quad \text{is approximated by} \quad \dot{y} = \alpha(t - c) + J(y - y_c)$$

$$\dot{y} = f(t, y)$$

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, \ldots, y_n) \\ f_2(t, y_1, \ldots, y_n) \\ \vdots \\ f_n(t, y_1, \ldots, y_n) \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \ldots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \ldots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \ldots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

$$J = \left(\frac{\partial f_i}{\partial y_j}\right)$$

Linear Systems

$$\dot{y} = Ay$$

Suppose A can be diagonalized

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$A = X\Lambda X^{-1}, \qquad X = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Set

$$y = Xz$$

Then

$$\dot{y} = X\dot{z} = X\Lambda X^{-1} y = X\Lambda X^{-1} X z = X\Lambda z$$

So

$$\dot{z} = \Lambda z$$

$$z = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \qquad \boxed{y = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t}}$$

$$\ddot{x} + x = 0$$

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of J are $\pm i$ and the solutions are purely oscillatory linear combinations of

$$e^{it} \quad \text{and} \quad e^{-it}$$

$$\dot{y} = \begin{pmatrix} y_3(t) \\ y_4(t) \\ -y_1(t)/r(t)^3 \\ -y_2(t)/r(t)^3 \end{pmatrix}, \qquad r(t) = \sqrt{y_1(t)^2 + y_2(t)^2}$$

$$J = \frac{1}{r^t} \begin{pmatrix} 0 & 0 & r^5 & 0 \\ 0 & 0 & 0 & r^5 \\ 2y_1^2 - y_2^2 & 3y_1y_2 & 0 & 0 \\ 3y_1y_2 & 2y_2^2 - y_1^2 & 0 & 0 \end{pmatrix}$$

$$\lambda = \frac{1}{r^{3/2}} \begin{pmatrix} \sqrt{2} \\ i \\ -\sqrt{2} \\ -i \end{pmatrix}$$

Single Step Methods

Given an approximation at time t, we want to advance the solution to time $t + h$.

Euler's Method

$$\dot{y}(t) \approx \frac{y(t+h) - y(t)}{h}$$

$$\dot{y} = f(t.y)$$

$$y(t+h) \approx y(t) + h\,f(t, y(t))$$

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h\,f(t_n, y_n) \\ t_{n+1} &= t_n + h \end{aligned}}$$