The written portion of the second midterm will consist of problems taken from the following list.

1. Consider the points
$$(0,20),(1,7),(2,0)$$
   (a) Find the least squares line through the points.
   (b) Find the best exponential function of the form
$$y = C + D \cdot 2^{-x}$$
   through the points.
   (c) Plot the points, the line and the curve on the same graph. Which gives a better fit, the line or the exponential?

2. (a) Find the QR factorization and the pseudoinverse of the matrix
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$
   (b) Find the minimal least squares solution of $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   (c) Plot the line of least squares solutions and mark the minimal least squares solution.

3. Points $P = (s, s, s)$ and $Q = (t, 3t, -1)$ lie on lines that never meet. Find s and t to minimize the distance $\|P - Q\|$ between the points.

4. Let $p_3(x)$ be a degree 3 polynomial, and let $p_2(x)$ be its interpolating polynomial at the three points $x = -h, 0, h$. Prove that
$$\int_{-h}^{h} p_3(x)\,dx = \int_{-h}^{h} p_2(x)\,dx$$
   What does this fact say about Simpson's Rule?

5. (a) How accurate is the following quadrature rule?
$$\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
   (b) Now use this rule to devise a rule that can be used on any interval $[a,b]$. Make the change of variables $x = \dfrac{b-a}{2}u + \dfrac{a+b}{2}$ to get
$$\int_{a}^{b} f(x)\,dx \approx \ ?$$
   How accurate is it?

6. (a) One way to estimate the logarithm on the interval $[1, 2]$ is by approximating the integral

$$\log x = \int_1^x \frac{1}{t}\, dt$$

What is the approximation using the trapezoidal rule? What if we used Simpson's rule?

   (b) We could also approximate $\log x$ by solving the differential equation

$$\frac{dy}{dx} = \frac{1}{x}, \quad y(1) = 0$$

What is the approximation using Euler's method for solving the differential equation? What if we used the Heun (Runga-Kutta order 2) method?

7. Consider the ODE

$$\dot{y} = f(t)$$

(Notice that the RHS does not depend on y.) Show that $s_2 = s_3$ in the fourth-order Runge-Kutta scheme and that RK4 is equivalent to Simpson's Rule for the integral

$$\int_{t_n}^{t_n+h} f(s)\, ds$$

8. Recall that the implicit trapezoidal method advances the solution of $\dot{y} = f(t, y)$ from time step n to $n + 1$ by solving the equation

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

   (a) Derive the Euler and implicit trapezoidal method for the scalar equation

$$\dot{y} = \lambda y, \quad y(0) = 1$$

Suppose $\lambda = -10$. How small does the step size have to be for the Euler method to accurately reflect the exact solution? The implicit trapezoidal?

   (b) Derive the Euler and implicit trapezoidal methods for solving the linear system of equations

$$\dot{y} = Ay$$

where A is an $n \times n$ matrix.

   (c) Write out the Euler and implicit trapezoidal schemes for solving the harmonic oscillator

$$\ddot{y} + y = 0, \quad y(0) = 0, \ \dot{y}(0) = 1$$

With $h = 0.1$, how well do these schemes approximate the exact solution $y(t) = \sin(t)$?