NAME: ____

Problem	Points
A1	/10
A2	/10
A3	/10
A4	/10
B1	/10
B2	/10
B3	/10
B4	/10
Total	/60

INSTRUCTIONS:

1. Complete your choice of any 3 of the problems in part A, and your choice of any 3 of the problems in part B. You should, therefore, complete 6 problems total.
2. Include any code you have written for each problem, and any figures you have produced. Attach your solutions in the order of the problems, with this page on top.
3. Show all details of your work.
4. The exam is due Wed, Dec 13 at 11:59 PM. Exams must be submitted as a single PDF file through Canvas.
5. If you have any questions on the exam, you may email me (tmcmillen@fullerton.edu).
6. All work on this exam must be yours and yours alone. **No collaboration is allowed.** You may use your course texts, class notes and the MATLAB help pages.

By signing my name below, I certify that the attached work is mine alone.

Signed: ______________________________

Part A:

1. Suppose you had a computer called the Marc-5 that used floating point arithmetic in binary with 5 bits for the fraction. So, for example, the decimal number 0.3 would be represented in the Marc-5 by

$$1.00110 \cdot 2^{-2}$$

Explain what would happen if you entered the following commands in MATLAB on the Marc-5.

```
t = 0.1
n = 1:10
e = n/10 - n*t
```

2. Consider the equation $x - 9^{-x} = 0$.
   (a) Prove that there is a solution in the interval $(0, 1)$.
   (b) Find the interpolating polynomial through the points at $x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$ for the function on the left hand side of the equation. Set the interpolating polynomial to zero and solve to find an approximate solution to the equation.
   (c) Perform one iteration of the secant method using the starting guesses $x_0,\ x_1$. How close do you get to the exact zero (`0.408004` to six digits) in one step?

3. The polynomial $p(x) = (x-1)^2(x-2)^2(x-3)$ has three zeros: $1, 2,$ and 3. If a_0 and b_0 are any two real numbers such that $a_0 < 1$ and $b_0 > 3$, then $f(a_0)f(b_0) < 0$. Thus, on the inteval $[a_0, b_0]$ the bisection method will converge to one of the three zeros. If $a_0 < 1$ and $b_0 > 3$ are selected such that $c_n = (a_n + b_n)/2$ is never equal to $1, 2,$ or 3 for any $n \geq 1$, then the bisection method will *always* converge to which zero? Why?

4. Find the *plane* that gives the best fit to the 4 values $z = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $z = C + Dx + Ey$ at those 4 points are $Ax = z$ with three unknowns (C, D, E). What is A? At the center of the square, show that $C + Dx + Ey =$ average of the z's.

Part B:

1. The following table shows the volume of arctic sea ice in million sq. km. by year. [1]

2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
5.26	4.87	4.56	3.57	5.21	5.22	4.62	4.53	4.82	4.79	4.36	3.92

   (a) Find the least squares line through the points.
   (b) The point at 2012 seems like an outlier. Find the least squares line through the points with the data at year 2012 removed.
   (c) Plot the points and lines on the same graph.
   (d) Based on these fits, estimate when the volume of arctic sea ice will reach zero.

2. **The case of the missing large mammalian predators.**

   The Lotka-Volterra predator-prey equations are

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{R}\right) - \alpha xy\\ \frac{dy}{dt} &= -ky + \beta xy\end{aligned}$$

   Here x is the prey population (e.g. rabbits) and y is the predator population (e.g. foxes). R is the carrying capacity of the prey. If the ecosystem cannot support enough prey (i.e. the carrying capacity is too low) there is not enough for the predators to eat. This explains why there are no large mammalian predators in Australia. Suppose $\alpha = \beta = .01$, $r = 2$ and $k = 1$. Start with $x(0) = 300$ prey and $y(0) = 150$ predators. Make plots of

   - the number of predators and number of prey versus time, and
   - the number of predators vs. the number of prey

   over 50 units of time. For the carrying capacity take:

   (a) $R = 300$
   (b) $R = 70$

   Describe what happens in each case. Find the minimal carrying capacity R that allows the predators to survive.

3. **Interplanetary long jumps.**

   Bob Beamon jumped 8.90 m in the 1968 Olympic games in Mexico City, shattering the old record. The equations governing the jumper's motion are

$$\dot{x} = v\cos\theta, \quad \dot{y} = v\sin\theta, \quad \dot{\theta} = -\frac{g}{v}\cos\theta, \quad \dot{v} = -\frac{D}{m} - g\sin\theta,$$

   where

$$D = \frac{c\rho s}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

   The parameters in Mexico City are $g = 9.81$ m/s^2, m = 80 kg, $c = 0.72$, $s = 0.50$ m^2, and the takeoff angle was $\theta_0 = \pi/8$. The air density is $\rho = 0.94\,\mathrm{kg/m^3}$.

   (a) Find Beamon's initial velocity for the jump.
   (b) Suppose Beamon were to repeat his jump on the moon, with the same takeoff velocity and angle, where $g = 1.62519$ and $\rho = 0$. How far would he go?
   (c) On Venus, $g = 8.87$ m/s and the air density is $\rho = 65$ kg/m^3. What would the takeoff velocity have to be on Venus to beat Beamon's mark? What is more important for distance traveled, air density or gravitational acceleration?

[1] `https://nsidc.org/cryosphere/sotc/sea_ice.html`

4. **Strange satellites.**

The general n-body problem involves n mutually attracting masses $m_1, \ldots, m_n$ at position vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, satisfying the $3n$-dimensional second order differential equation

$$\ddot{\mathbf{x}}_i = -\sum_{j \neq i} \frac{g m_j (\mathbf{x}_i - \mathbf{x}_j)}{\|\mathbf{x}_i - \mathbf{x}_j\|^3}, \quad i = 1, 2, \ldots, n$$

(a) The center of mass of the system is $\mathbf{x}_c = \dfrac{1}{M} \displaystyle\sum_{i=1}^{n} m_i \mathbf{x}_i$, where $M = m_1 + m_2 + \cdots + m_n$. Show that $\ddot{\mathbf{x}}_c = 0$, so that $\mathbf{x}_c = \mathbf{a}t + \mathbf{b}$, i.e. the center of mass moves in a straight line. Thus one may change coordinates to the frame of reference in which the center of mass is fixed at the origin.

In the *restricted three body problem*, it is assumed that two larger bodies affect but are not affected by a third, smaller body. This could describe, for example, the motion of an Earth-Moon satellite. To simplify, the masses of the larger bodies are scaled to $1 - \mu$ and μ and their positions relative to the center of mass are $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$. Write y_1, y_2, y_3 as the coordinates of the smaller mass. Under these assumptions the equations of motion are

$$\begin{aligned}
\ddot{y}_1 &= 2y_5 + y_1 - \frac{\mu(y_1 + \mu - 1)}{\left(y_2^2 + y_3^2 + (y_1 + \mu - 1)^2\right)^{3/2}} - \frac{(1-\mu)(y_1 + \mu)}{\left(y_2^2 + y_3^2 + (y_1 + \mu)^2\right)^{3/2}} \\
\ddot{y}_2 &= -2y_4 + y_2 - \frac{\mu y_2}{\left(y_2^2 + y_3^2 + (y_1 + \mu - 1)^2\right)^{3/2}} - \frac{(1-\mu) y_2}{\left(y_2^2 + y_3^2 + (y_1 + \mu)^2\right)^{3/2}} \\
\ddot{y}_3 &= -\frac{\mu y_3}{\left(y_2^2 + y_3^2 + (y_1 + \mu - 1)^2\right)^{3/2}} - \frac{(1-\mu) y_3}{\left(y_2^2 + y_3^2 + (y_1 + \mu)^2\right)^{3/2}}
\end{aligned}$$

(b) Periodic planar orbits exist. For $\mu = 1/81.45$, corresponding to the Earth-Moon system, a planar periodic orbit with $y_3 = 0$ has initial conditions

$$(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3) = (0.994, 0, 0, 0, -2.0015851063790825224, 0).$$

The period of this orbit is 17.06521656. Write a code to compute this orbit and plot the trajectory in the $y_1 y_2$-plane. Use `'events'` to find the period.

(c) Another periodic planar orbit has initial conditions $(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3) = (0.87978, 0, 0, 0, -0.3797, 0)$. Compute this orbit and find its period. Plot the trajectory of one period of the orbit.