We consider the problem of solving nonlinear problems

$$\boxed{f(x) = 0}$$

The most common algorithms use a combination of methods, which we consider separately.

Bisection

Let's compute $\sqrt{2}$ by finding a zero of

$$f(x) = x^2 - 2$$

Since $f(2) = 2$ and $f(1) = -1$, we know there is a zero between 1 and 2. Then we can consider the average of these, $3/2$. Since

$$f(3/2) = \frac{1}{4}$$

we know there is a zero between 1 and $3/2$. We could then continue by taking the average of these.

Generally,

```
Find a and b such that f(a)*f(b)<0

while b-a > tol
    x = (a+b)/2
    If f(a)*f(x) < 0
        b=x
    else
        a=x
    end
end
```

Bisection pros: Always works, robust, can calculate zero to any tolerance

Bisection cons: Slow

Notice that the error is halved upon each iteration. Thus, if

$$e_n = x_n - x_*$$

where x_* is the zero we are looking for,

$$e_{n+1} \approx \frac{1}{2} e_n$$

This linear behavior is much too slow. Let's consider a faster method.

Newton's Method

Newton's method is based on making an initial guess x_0 and tracking the tangent line at x_0 back to the $x-$axis for the next guess. We get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem

Let f be twice continuously differentiable on an interval (a, b). Suppose that $f(x_*) = 0$ for some $a < x_* < b$ and $f'(x_*) \neq 0$. Then there exists $\varepsilon > 0$ such that if $x_0 \in (x_* - \varepsilon, x_* + \varepsilon)$, and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

then $x_n \to x_*$. Moreover, for some $M > 0$, we have

$$|x_{n+1} - x_*| \leq M|x_n - x_*|^2$$

In other words, in terms of the error e_n,

$$e_{n+1} \leq M\, e_n^2$$

In this sense, the convergence in Newton's method is *quadratic*.

Proof.

By Taylor's Theorem, we have

$$0 = f(x_*) = f(x_n) + f'(x_n)(x_* - x_n) + \frac{f''(\xi)}{2}(x_* - x_n)^2$$

for some ξ between x_n and x_*. Thus

$$\begin{aligned} x_{n+1} - x_* &= x_n - x_* - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - x_* - \frac{1}{f'(x_n)}\left[-f'(x_n)(x_* - x_n) - \frac{f''(\xi)}{2}(x_* - x_n)^2\right] \\ &= \frac{f''(\xi)}{2f'(x_n)}(x_n - x_*)^2 \end{aligned}$$

Choose ε such that $f'(x)$ is bounded below by some $\delta > 0$ for $x \in (x_* - \varepsilon, x_* + \varepsilon)$. Then since $f''(x)$ is continuous, we have

$$M = \sup \frac{f''(\xi)}{2f'(x)}$$

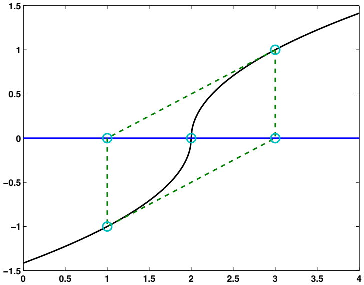
and

$$|x_{n+1} - x_n| \leq M|x_n - x_*|^2$$

as long as $x_0 \in (x_* - \varepsilon, x_* + \varepsilon)$, as required. $\square$

Unfortunately, it doesn't always work.

A "perverse" example.



$$f(x) = \text{sign}(x-2)\sqrt{|x-2|}$$

With $x_0 = 3$, we have

$$x_1 = 1,\ x_2 = 0,\ x_3 = -1,\ x_4 = 3,\ \ldots$$

Newton's Method:

Pros: Fast, efficient

Cons: Need good initial guess, formula for $f'(x)$

Secant Method

A related method that doesn't need a formula for $f'(x)$ is the secant method. In place of $f'(x_n)$, we use

$$s_n = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Notice that $s_n \approx f'(x_n)$.
Thus we get the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{s_n}$$

With secant method we do not need to calculate $f'(x)$, and we get similar convergence properties.

It can be shown

$$e_{n+1} = \mathcal{O}(e_n e_{n-1})$$

and

$$e_{n+1} = \mathcal{O}(e_n^{\phi})$$

where $\phi = (1+\sqrt{5})/2 \approx 1.62$ is the golden ratio. Thus secant method is superlinear but not quadratic.

Inverse Quadratic Interpolation

Whereas the secant method uses two previous iterates, Inverse Quadratic Interpolation (IQI) uses three. Why not?

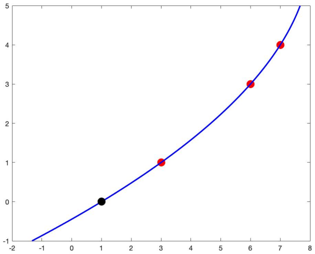
The idea is to fit a quadratic $P(y)$ through the three previous iterates, but here

$$x = P(y)$$

(A “sideways” parabola.)

Then the next iterate is where P crosses the $x-$axis. This is then just

$$x_{n+1} = P(0)$$



If we have `a, b, c` as three previous iterates, we can implement IQI as follows:

```
while abs(c-b) > tol
    x = polyinterp([f(a), f(b), f(c)], [a, b, c], 0);
    a = b;
    b = c;
    c = x;
end
```

Zeroin

The `zeroin` algorithm (implemented in Matlab as `fzero`) combines the reliability of bisection with the speed of the secant method and IQI.

- Start with a and b such that $f(a)f(b) < 0$.
- Use a secant step to give c between a and b.
- Repeat the following steps until $|b - a|$ is within a given tolerance or $f(b) = 0$.
- Arrange a, b, c so that
  - $f(a)f(b) < 0$
  - $|f(b)| \leq |f(a)|$.
  - c is the previous value of b
- If $c \neq a$ consider an IQI step.
- If $c = a$, consider a secant step.
- If the IQI or secant step is in the interval $[a, b]$, take it.
- If the step is not in the interval, use bisection.