12.1- Modeling of Systems of Differential Equations

DEFINITION

- The system of first-order differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$\frac{dx_1}{dt} = f_n(x_1, x_2, \ldots, x_n)$$

is called an **autonomous** system if $f_i(x_1,x_2,\ldots,x_n)$ does not depend on t for $i=1,2,\ldots,n$.

REMARKS

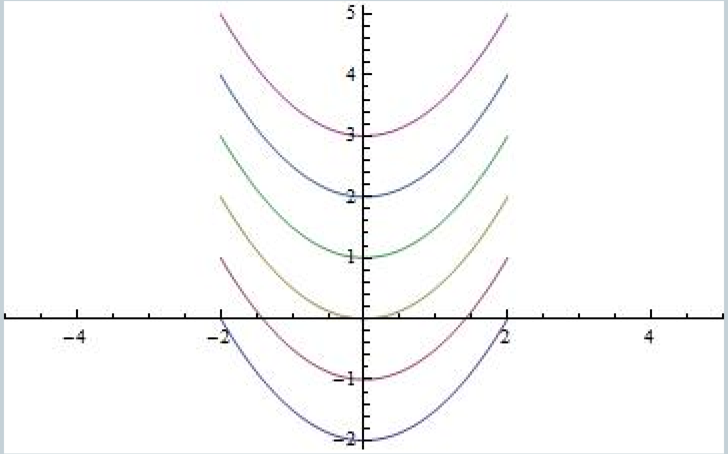
- Such systems are time independent.
- The solutions
$$\vec{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$
are parameterized curves in $\mathbb{R}^n$-space.

EXAMPLE:

$$\frac{dx_2}{dx_1}=x_1$$

or

$$\left.\begin{aligned} \frac{dx_1}{dt}&=x_1 \\ \frac{dx_2}{dt}&={x_1}^2 \end{aligned}\right\} \frac{dx_2}{dx_1}=\frac{dx_2/dt}{dx_1/dt}=x_1$$



REMARKS (continued)

- These solution curves are known as solution **trajectories**, **paths**, or **orbits** in ⍰^{n}.
- The ⍰^{n} –space is known as the **phase space**.
- $\overrightarrow{x^*} = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix}$ is an equilibrium point if $f_i(x_i{}^*)=0$ for

  i=1,2,…,n.
- $\overrightarrow{x^*}$ is stable if for every ε>0, there exists a δ>0 such that if $\left|\vec{x}(t_0) - \vec{x}^*\right| < \varepsilon$, then $\left|\vec{x}(t) - \vec{x}^*\right| < \delta$ for all $t>t_0$.

REMARKS (continued)

- $\overrightarrow{x^*}$ is asymptotically stable if $\overrightarrow{x^*}$ is stable <u>and</u> there is an ε>0 such that if $\left|\vec{x}(t_0)-\overrightarrow{x^*}\right|<\varepsilon$ then $\lim_{t\to\infty}\left|\vec{x}(t)-\overrightarrow{x^*}\right|=0.$
- $\overrightarrow{x^*}$ is unstable if it is not stable.

EXAMPLES

EXAMPLE 1:

Consider a first-order autonomous system:

$$\frac{dx_1}{dt} = -x_1 + x_2$$
$$\frac{dx_2}{dt} = -x_1 - x_2$$

Where are the equilibrium points?

$$\left.\begin{array}{l} -x_1 + x_2 \overset{set}{=} 0 \\ -x_1 - x_2 \overset{set}{=} 0 \end{array}\right\} \left.\begin{array}{l} x_1^* \\ x_2^* \end{array}\right\} x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLES (continued)

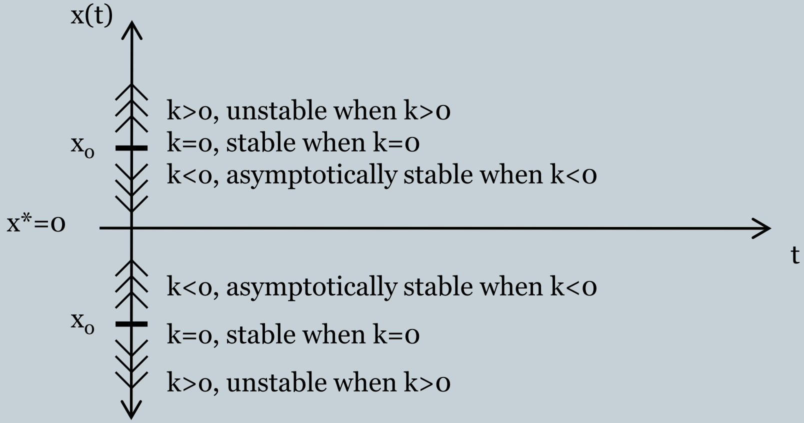
EXAMPLE 2:

We next study the stability of equilibrium points.

$\frac{dx}{dt} = dx$, where k is a constant (k plays an important role in the stability of x^*)

$x(t) = x_0 e^{kt}$

x*=0 is an equilibrium point.



REMARKS

- The 2x2 system can be converted into

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ where } A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

- Similar to the 1x1 system, the eigenvalues of the linear autonomous system will dictate the stability of the equilibrium points.
- $\overrightarrow{x}^*$ is asymptotically stable if the real part of the eigenvalues are <0, stable if the real parts are= 0, and unstable if a real part is >0.

EXAMPLE

- For $$\frac{dx_1}{dt} = -x_1 + 5$$ $$\frac{dx_2}{dt} = 3x_2$$

  we have $$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}, \text{ where } A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

  The equilibrium point $\overrightarrow{x^*} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ satisfies $A\overrightarrow{x^*} + \vec{b} = \vec{0}$.

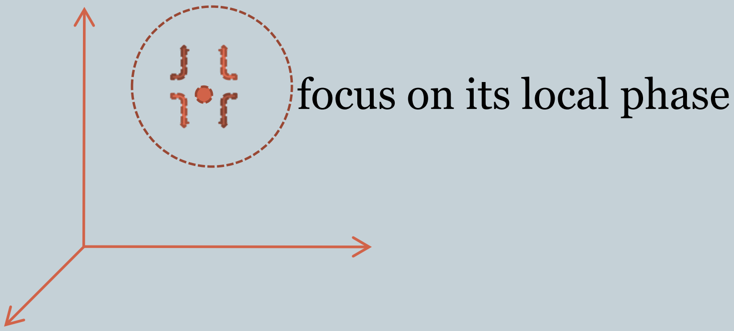
  The eigenvalues of A are -1 and 3. Since one of the eigenvalues is greater than zero, then the equilibrium point is unstable.

REMARKS

- The eigenvalues of a linear autonomous system can determine the stability of the equilibrium points.
- However, many interesting systems are non-linear.
- In these instances, one must rely on the phase space near the equilibrium point to determine its stability

EXAMPLES

EXAMPLE 1:



EXAMPLE 2:

Consider a nonlinear autonomous system:

$$\left.\begin{aligned}\frac{dx}{dt} &= 2y^2 \\ \frac{dy}{dt} &= y\end{aligned}\right\} = \text{Ax+b, where } A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2y^2 \\ 0 \end{bmatrix}.$$

The eigenvalues of A are 0 and 1. Therefore the equilibrium point is unstable.

REMARK

- When one eigenvalue is positive the equilibrium point of the nonlinear system is unstable.
- However, even when the eigenvalues are non-positive, the equilibrium point may still be unstable due to the nonlinear term.

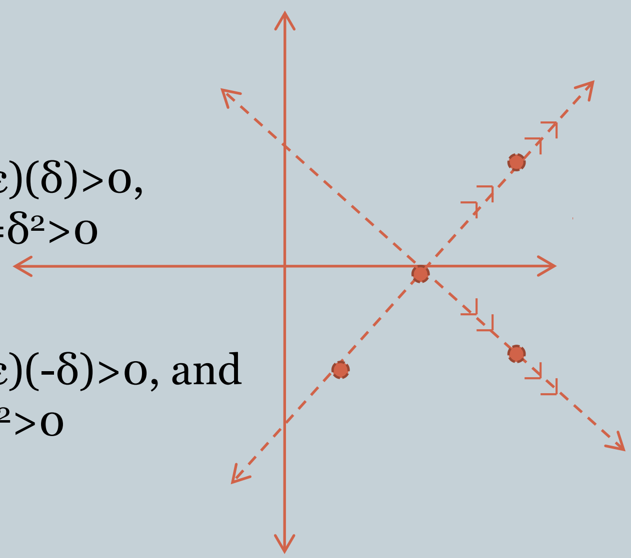
DRAWING PHASES

- EXAMPLE

  Consider the system dx/dt=y^2; dy/dt=xy. The equilibrium points are (a,), where a is arbitrary.

at $(a-\varepsilon,\delta)$,
$dy/dt= (a-\varepsilon)(\delta)>0$,
and $dx/dt=\delta^2>0$

at $(a+\varepsilon,\delta)$,
$dy/dt= (a+\varepsilon)\delta>0$, and
$dx/dt=\delta^2>0$



at $(a-\varepsilon,-\delta)$,
$dy/dt= (a-\varepsilon)(-\delta)>0$, and
$dx/dt=(-\delta)^2>0$

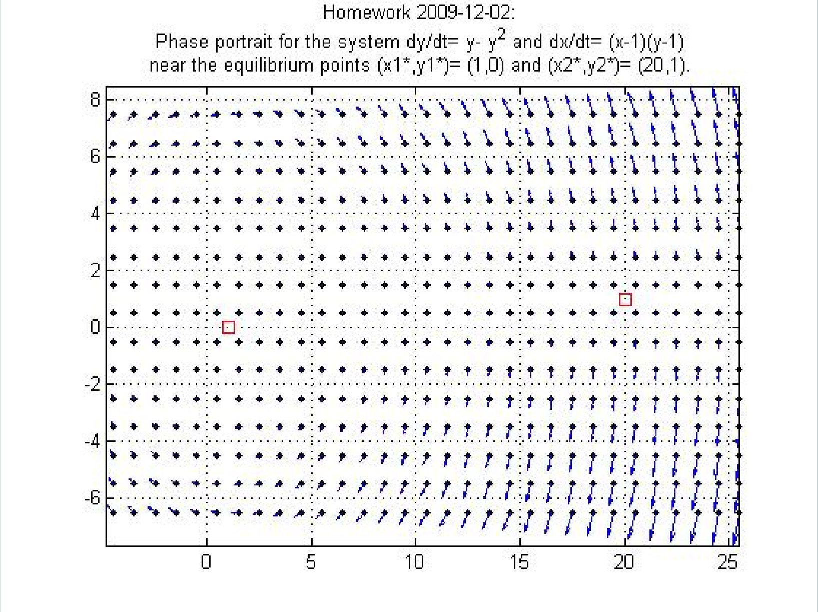
at $(a+\varepsilon,-\delta)$,
$dy/dt= (a+\varepsilon)(-\delta)>0$,
and $dx/dt=(-\delta)^2>0$

MATLAB CODE

- Consider the system,

$$\frac{dx}{dt} = y - y^2$$
$$\frac{dy}{dt} = (x-1)(x+1)$$

with the equilibrium points (1,0) and (a,1), where a is arbitrary. It's phase portrait looks like this:



MATLAB CODE

```
xstar1= 1; ystar1= 0;
dx1=1; dy1=1;
Nx1=15; Ny1=7;
%[X1,Y1 ]= meshgrid([-Nx1:Nx1]*dx1+ xstar1, [-Ny1:Ny1]*dy1+ ystar1);
a=20;
xstar2= a; ystar2= 1;
%this plot contains both equilibrium points
[X3,Y3]= meshgrid([-Nx1:Nx1]*dx1+ (xstar1+ xstar2)/2, [-Ny1:Ny1]*dy1+ (ystar1+ ystar2)/2);
DX3= Y3- Y3.*Y3;          %since dx/dt= y-y^2
DY3= (X3-1).*(Y3-1);      %since dy/dt= y
%subplot(3,1,3);
plot(X3,Y3,'.k'); hold on;
plot(xstar1,ystar1,'rs');
plot(xstar2,ystar2,'rs');
quiver(X3,Y3,DX3,DY3,'b');
axis tight;
grid on;
title({'Homework 2009-12-02:';['Phase portrait for the system dy/dt= y- y^2 and dx/dt= (x-1)(y-1)']; ['near the equilibrium points (x1*,y1*)= (' num2str(xstar1) ',' num2str(ystar1) ') and (x2*,y2*)= (' num2str(xstar2) ',' num2str(ystar2) ').']})
```

HOMEWORK

- **Draw the phase portraits for the previous two examples**
- **#8,9 Section 11.1**