Section 1.3
Solution to Dynamical Systems

Difference Equation 1: Solution

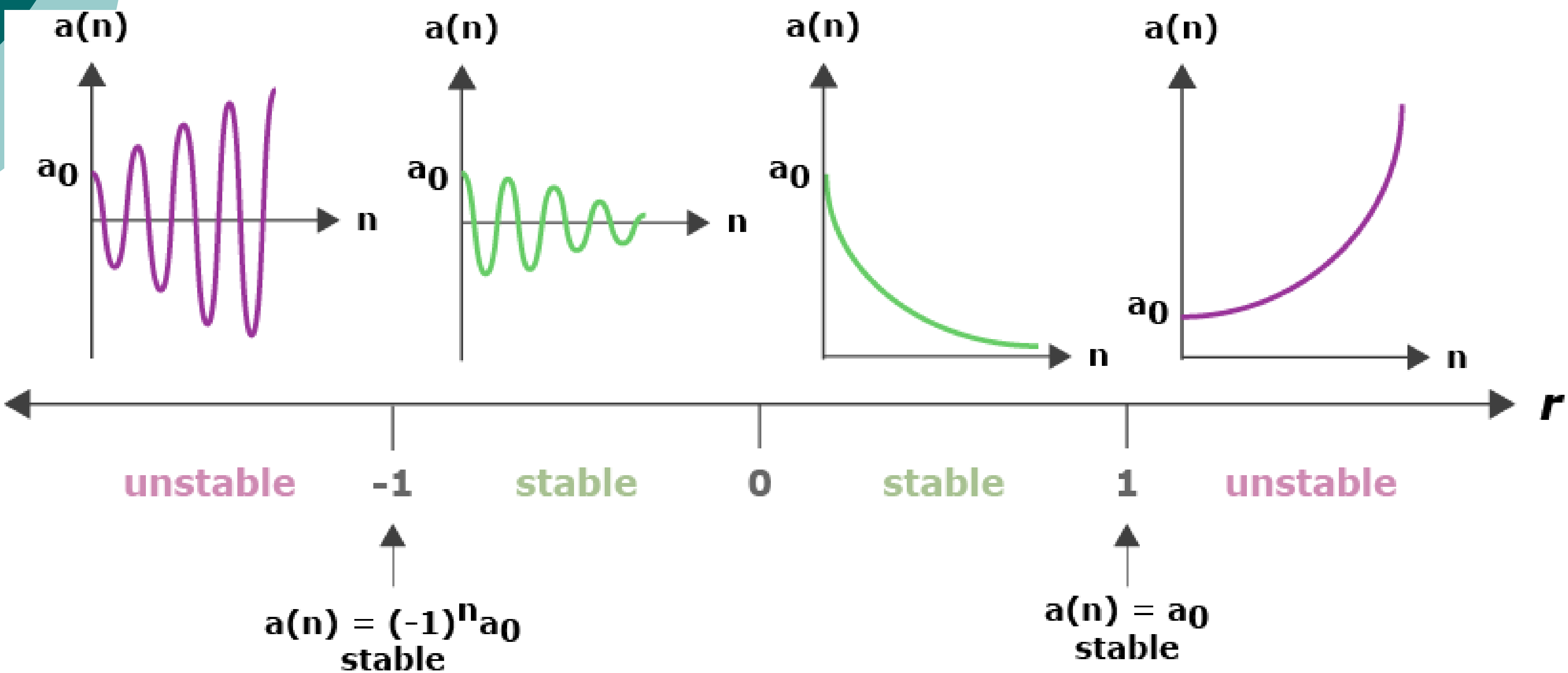
Difference Equation

$$a(n+1) - a(n) = ka(n)$$
$$\Rightarrow a(n+1) = ra(n); r = k+1$$

With a(0) = a_0

$$\begin{aligned} a(n+1) &= ra(n) \\ &= r(ra(n-1)) = r^2(ra(n-2)) = \dots \\ &= r^{n+1}(a(0)) \\ &= \boxed{r^{n+1}a_0} \end{aligned}$$

Difference Equation 1: Behaviors



Difference Equation 2: Solution

Difference Equation

$$a(n+1) - a(n) = ka(n) + b$$
$$\Rightarrow a(n+1) = ra(n) + b; r = k+1$$

With $a(0) = a_0$

$$\begin{aligned} a(n+1) &= r(a(n)) + b \\ &= r(ra(n-1) + b) + b = r^2 a(n-1) + b(1+r) \\ &= r^2(ra(n-2) + b) + b(1+r) = r^3 a(n-2) + b(1+r+r^2) \\ &= \boxed{r^{n+1} a_0 + b(1 + r + \ldots + r^n)} \end{aligned}$$

Difference Equation 2: Continued

Note:

$$(1-r)(1+r+\ldots+r^{n}) = 1-r^{n+1} \leftarrow \text{telescopic series, r} \neq 1$$

So:

$$a(n+1) = r^{n+1}a_0 + b\left[\frac{(1-r^{n+1})}{1-r}\right]$$

$$= \boxed{\begin{array}{l} r^{n+1}\left(a_0 - \dfrac{b}{1-r}\right) + \dfrac{b}{1-r} \quad \text{when } r \neq 1 \\ a_0 + (n)b \quad \text{when r} = 1 \end{array}}$$

Difference Equation 2: Behaviors

The behaviors of the solutions to DE2 are almost identical to those of DE1, except when r=1, which grows linearly.

In the event |r|<1, where does a(n) converge?

$$a(n+1) = r^{n+1}\left(a_0 - \frac{b}{1-r}\right) + \frac{b}{1-r} \quad \text{as n} \to \infty$$

$$a(n) \to \frac{b}{1-r}$$

This is called a limit point.

Difference Equation 2: Remarks

a* is an equilibrium point of DE2 when

$$a^* = ra^* + b \Rightarrow a^* = \frac{b}{1-r}$$

i.e. once a(n) = a* it is there forever

Applications: Loan Model Revisited

B(n) ~ balance of loan at nth month
r ~ monthly interest rate (r = APR/12)
M ~ monthly payment

Recall:

$$B(n+1) = (1+r)^{n+1}\left(B_0 - \frac{M}{1-(1+r)}\right) + \frac{M}{1-(1+r)}$$

$$\Leftrightarrow$$

$$B(n+1) = (1+r)^{n+1}\left(B_0 + \frac{M}{r}\right) - \frac{M}{r}$$

Loan Model: How long to pay off?

When the loan is paid off, B(n+1) = 0

$$(1+r)^{n+1}\left(B_0 + \frac{M}{r}\right) = \frac{M}{r}$$

$$(n+1)\log(1+r) = \log\left(\frac{\frac{M}{r}}{B_0 + \frac{M}{r}}\right)$$

$$\boxed{(n+1) = \frac{1}{\log(1+r)}\log\left(\frac{\frac{M}{r}}{B_0 + \frac{M}{r}}\right)}$$

How long it will take to pay off the loan

Loan Model: What is monthly payment?

$$\frac{M}{r}\left(1-(1+r)^{n+1}\right) = B_0(1+r)^{n+1}$$

$$M = r\left(\frac{B_0(1+r)^{n+1}}{1-(1+r)^{n+1}}\right)$$

$$\boxed{M = \left(\frac{rB_0}{(1+r)^{-(n+1)}-1}\right)}$$

Monthly payment for the loan

Loan Model: How much Principle?

Suppose we have the rate (r=rate/12) and the monthly payment (M), how much can we afford to borrow?

$$B_0(1+r)^{n+1} = \frac{M}{r}(1-(1+r)^{n+1})$$

$$B_0 = \frac{M}{r} \bullet \frac{1-(1+r)^{n+1}}{(1+r)^{n+1}}$$

$$\boxed{B_0 = \frac{M}{r} \bullet \left((1+r)^{-(n+1)} - 1\right)}$$

Principle that can be borrowed

Homework –Section 1.3

- # (1-4)C, 5, 8, 9, 14