11.4 Solutions to Autonomous Differential Equations

Math 370

Definition

A first-order differential equation

$$\frac{dy}{dt} = g(x, y)$$

is called autonomous differential equation

if g(x,y) = g(y) * function of y only, no x involved

Ex:

$$\frac{dy}{dx} = \frac{1}{2}y$$

$$\frac{du}{dt} = 2u^2$$

$$\frac{dw}{dt} = 4$$

Note:

- To solve any first-order differential equation we need an initial solution

$$y(x_0) = y_0$$

Otherwise, we get indefinitely many solutions.

ex:

$$\frac{dy}{dx} = \frac{1}{2}y$$

$$\frac{dy}{y} = \frac{1}{2}dx$$

$$\ln y = \frac{1}{2}x + c$$

$$y(x) = e^c \cdot e^{\frac{1}{2}x} = ke^{\frac{1}{2}x}$$ for any constant k

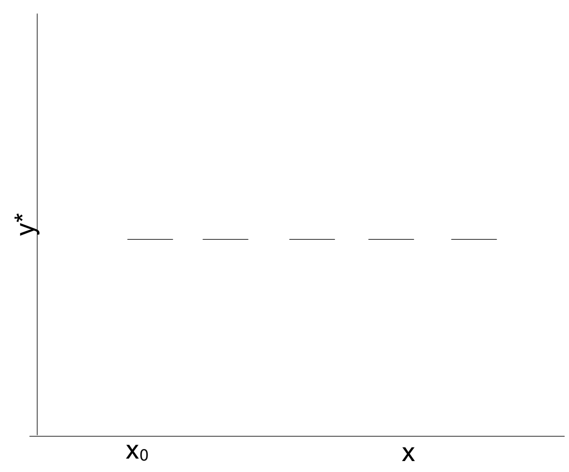
Definition

- Given a 1st – order Autonomous Differential Equation

$$\frac{dy}{dx} = g(y)$$

the value y*, such that g(y*)=0 is called an equilibrium value (fixed point)

@y* => $$\frac{dy^*}{dx} = g(x^*) = 0$$ y(x) = y*



- Ex: $$\frac{dy}{dx} = y - 3$$ and $$y(x_0) = 3$$

@ y =3; $\frac{dy}{dx} = 0$ => y(x) = k

but $y(x_0) = 3$

=> y(x) = 3

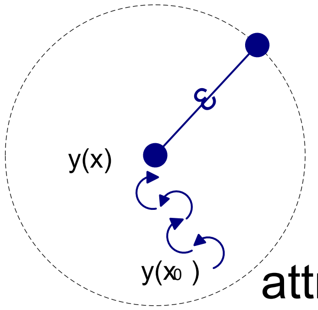
Definition

- An equilibrium value y_0 of a 1st order $\dfrac{dy}{dx} = g(y)$

  Autonomous Differential Equation

  is called stable if there is a $\varepsilon > 0$

  such that $|\, y(x_0) - y^* \,| < \varepsilon$



attracts everything nearby

y(x) -> y^* eventually

Definition

- Given a 1st – order Autonomous Differential Equation $\frac{dy}{dx} = g(y)$

the phase line is a plot along the y-axis which includes

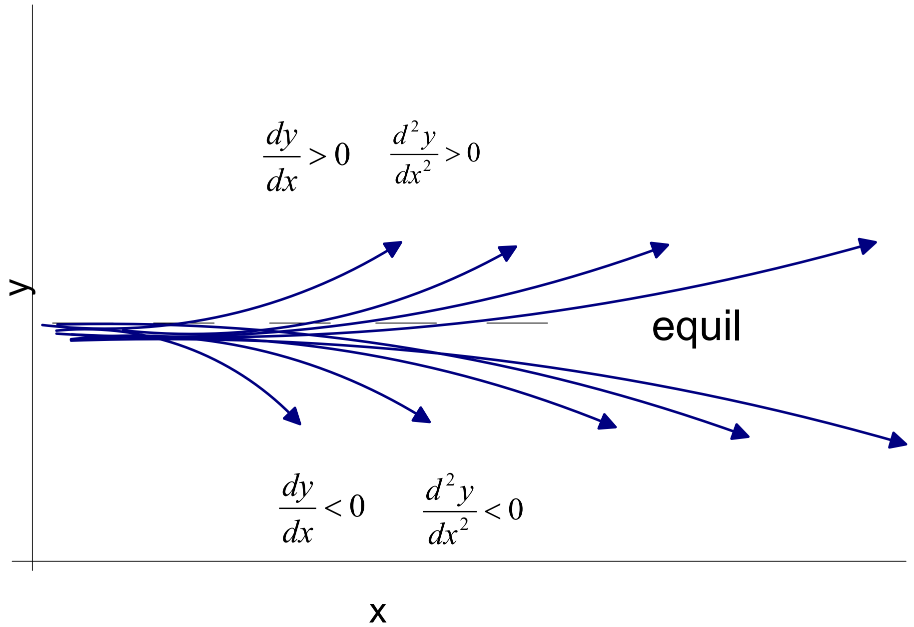
1) the equilibrium value y^*

2) positive / negative values of slope $\frac{dy}{dx}$

3) positive / negative values of concavity $\frac{d^2y}{dx^2}$

- Continue with the previous example, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y-3) = \frac{dy}{dx}$$

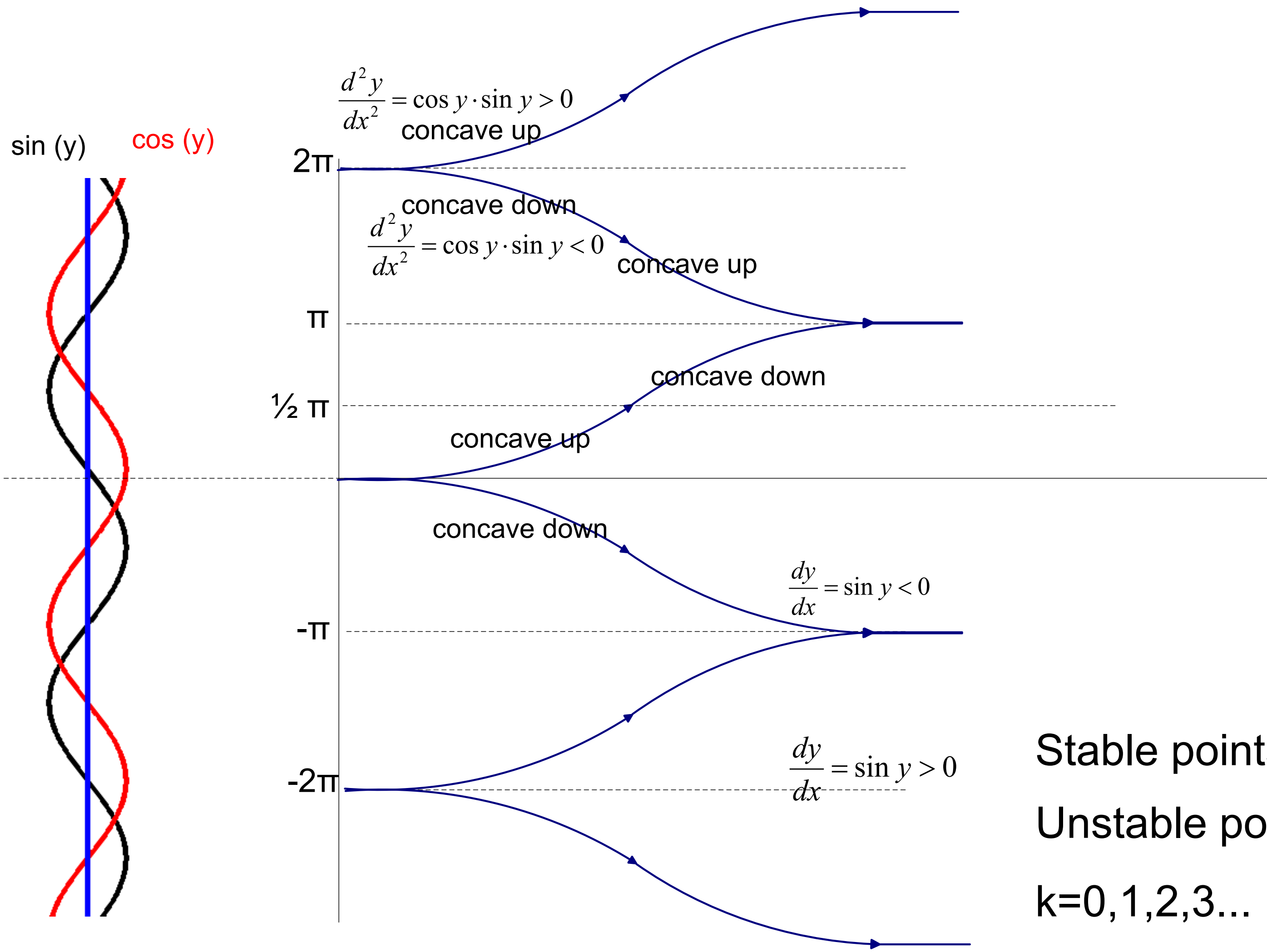


Ex: Given a 1st order Autonomous Differential Equation

$$\frac{dy}{dx} = \sin y = 0$$

$$y^* = \pm \pi k; k = 0,1,2,3\ldots$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx} = \cos y \sin y$$



Stable points=±(2k+1) π

Unstable points=±(2k+1) π

k=0,1,2,3...

Homework:

- Section 11.4:

1a,d,2,5,6,8