1. Evaluate the multiple integral

$$\iint_D \frac{y}{1+x^2}\,dA,$$

where $D$ is bounded by $y=\sqrt{x},\ y=0,\ x=1$.

*Solution.* (rev #17) The region is $0 \le y \le \sqrt{x},\ 0 \le x \le 1$, so we can write the integral as an iterated integral:

$$\begin{aligned}
\iint_D \frac{y}{1+x^2}\,dA &= \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2} dy\,dx \\
&= \int_0^1 \left.\frac{y^2/2}{1+x^2}\right|_{y=0}^{y=\sqrt{x}} dx \\
&= \frac{1}{2}\int_0^1 \frac{x}{1+x^2} dx \qquad (u-\text{sub}:\ u=1+x^2,\ du=2x\,dx) \\
&= \frac{1}{4}\int_1^2 \frac{du}{u} \\
&= \left.\frac{1}{4}\ln u\right|_1^2 \\
&= \boxed{\frac{1}{4}\ln 2}
\end{aligned}$$

□

2. Evaluate the double integral by reversing the order of integration

$$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$$

*Solution.* (rev #13) The region of integration is $x \le y \le 1,\ 0 \le x \le 1$. In reversed order, this is $0 \le x \le y,\ 0 \le y \le 1$, so the integral is

$$\begin{aligned}
\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx &= \int_0^1 \int_0^y \cos(y^2)\,dx\,dy \\
&= \int_0^1 \cos(y^2) y\,dy \qquad \big(u-\text{sub } u=y^2,\ du=2ydy\big) \\
&= \int_0^1 \cos(u)\frac{1}{2}du \\
&= \left.\frac{1}{2}\sin(u)\right|_0^1 \\
&= \boxed{\frac{\sin 1}{2}}
\end{aligned}$$

□

3. Evaluate the integral by converting to polar coordinates

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$$

*Solution.* (12.3 #23) The region of integration is $0 \le y \le \sqrt{9-x^2}$, $-3 \le x \le 3$. In polar coordinates, this is $0 \le \theta \le \pi$, $0 \le r \le 3$. So, in polar coordinates the integral is

$$\begin{aligned}
\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx &= \int_{0}^{\pi}\int_{0}^{3} \sin(r^2)r\,dr\,d\theta \\
&= \pi\int_{0}^{3} \sin(r^2)r\,dr \qquad (u-\text{sub} : u = r^2,\ du = 2r\,dr) \\
&= \frac{\pi}{2}\int_{0}^{9} \sin(u)du \\
&= \frac{\pi}{2}(-\cos u)\Big|_{0}^{9} \\
&= \boxed{\frac{\pi}{2}(1-\cos 9)}
\end{aligned}$$

□

4. (a) Write the integral in the order $dx\,dy\,dz$.

$$\int_{0}^{1}\int_{y}^{1}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy$$

*Solution.* (12.5 #33) The region of integration is $0 \le z \le y$, $y \le x \le 1$, $0 \le y \le 1$. This is the region under the plane $z = y$ and above the triangle in the $xy-$plane bounded by $y = x, y = 0, x = 1$. So $x$ is bounded by the planes $x = y$ and $x = 1$, and $y$ is bounded by $z$ and $y = 1$. Therefore, the integral is

$$\boxed{\int_{0}^{1}\int_{y}^{1}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy = \int_{0}^{1}\int_{z}^{1}\int_{y}^{1} f(x,y,z)\,dx\,dy\,dz}$$

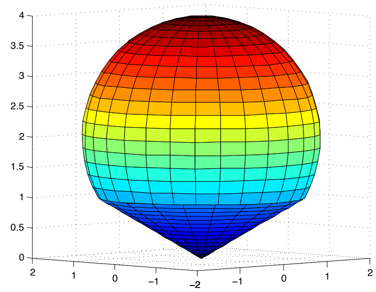
□

(b) (BONUS 5 points) Write the integral in the order for which the lower limits of the three integrals are all zero.

*Solution.* $$\boxed{\int_{0}^{1}\int_{y}^{1}\int_{0}^{y} f(x,y,z)\,dz\,dx\,dy = \int_{0}^{1}\int_{0}^{x}\int_{0}^{y} f(x,y,z)\,dz\,dy\,dx}$$ □

5. (a) Sketch the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 4\cos\phi$.

*Solution.* The sphere $\rho = 4\cos\phi$ is the sphere of radius 2 centered at $(0, 0, 2)$. So above the cone $\phi = \pi/3$ and below the sphere looks like the figure below:




$\square$

(b) Find the volume of the solid in part (a).

*Solution.* (12.7 #27) The solid is described in spherical coordinates by $0 \le \rho \le 4\cos\phi$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/3$. The volume is the integral of 1, so

$$\begin{aligned}
V(E) &= \iiint_E dV = \int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\phi} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi \\
&= 2\pi \int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi = 2\pi \int_0^{\pi/3} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=4\cos\phi} d\phi \\
&= \frac{2\pi}{3} 4^3 \int_0^{\pi/3} \cos^3(\phi)\sin(\phi)\, d\phi \qquad (u-\text{sub}:\ u = \cos(\phi),\ du = -\sin(\phi)d\phi) \\
&= \frac{2\pi}{3} 4^3 \int_{1/2}^{1} u^3 du = \frac{2\pi}{3} 4^3 \frac{1}{4} u^4 \Big|_{1/2}^{1} \\
&= \boxed{10\pi}
\end{aligned}$$

$\square$

(c) (BONUS 5 points) Find the centroid of the solid in part (a).

*Solution.* By symmetry, the $x$ and $y$ coordinates of the centroid are both zero, so we just need to find $\bar{z}$. We use the fact that in spherical coordinates $z = \rho\cos\phi$, so

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\phi} \rho\cos(\phi)\rho^2\sin(\phi)\, d\rho\, d\theta\, d\phi \\
&= 2\pi \int_0^{\pi/3} \frac{1}{4}\rho^4 \Big|_0^{4\cos\phi} \cos(\phi)\sin(\phi) d\phi \\
&= 2\pi 4^3 \int_0^{\pi/3} \cos^5(\phi)\sin(\phi)\, d\phi \qquad (u-\text{sub}:\ u = \cos\phi,\ du = -\sin(\phi)\, d\phi) \\
&= 2\pi 4^3 \int_{1/2}^{1} u^5 du \\
&= \boxed{21\pi}
\end{aligned}$$

Therefore, the $z$-coordinate of the centroid is $\bar{z} = \frac{1}{V(E)}\iiint_E z dV = 21\pi/10\pi = 2.1$. So the centroid is $\boxed{(0, 0, 2.1)}$.

$\square$

6. (a) Find the area of the region bounded by the curves $y = x/3$, $y = 3x$, $y = 3/x$ and $y = 1/x$.

*Solution.* If we make the transformation $u = xy$, $v = y/x$, then in the $uv-$plane, the region is $1/3 \leq v \leq 3$, $1 \leq u \leq 3$. Solving for $x$ and $y$ in this transformation gives us $x = \sqrt{u/v}$, $y = \sqrt{uv}$.

To find the area we need the Jacobian of the transformation $x = \sqrt{u/v}$, $y = \sqrt{uv}$, which is

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{bmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{u}v^{-3/2} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{bmatrix} \\
&= \frac{1}{4v} + \frac{1}{4v} \\
&= \boxed{\frac{1}{2v}}
\end{aligned}$$

The area of the region is the integral of 1.

$$\begin{aligned}
A(D) &= \iint_D dA \\
&= \int_{1/3}^{3} \int_{1}^{3} \frac{1}{2v} du\, dv \\
&= \int_{1/3}^{3} \frac{dv}{v} \\
&= \boxed{\ln 9}
\end{aligned}$$

NOTE: You could also do this by making the transformation $x = u/v$, $y = v$. In this case the Jacobian is $1/v$, and the transformed region in the $uv-$plane is $1 \leq u \leq 3$, $\sqrt{u/3} \leq v \leq \sqrt{3u}$. You could then do the integral as a type 1 region. □

(b) (BONUS 5 points) Find the centroid of the region in part (a).

*Solution.* By symmetry the $x$ and $y$ coordinates of the centroid are equal, so we just need to find

$$\begin{aligned}
\iint_D y\, dA &= \int_{1/3}^{3} \int_{1}^{3} \frac{\sqrt{uv}}{2v} du\, dv \\
&= \frac{1}{2} \int_{1/3}^{3} v^{-1/2} dv \int_{1}^{3} \sqrt{u}\, du \\
&= \frac{4}{9}\left(9 - \sqrt{3}\right)
\end{aligned}$$

Therefore, the centroid is $\boxed{(\bar{x}, \bar{y}) = \left(\frac{4(9-\sqrt{3})}{9\ln 9}, \frac{4(9-\sqrt{3})}{9\ln 9}\right)}$ (FYI, this is approximately $(1.47, 1.47)$.)

□