1. (a) Show that $\mathbf{F}$ is conservative. (b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\mathbf{F}(x,y) = (1+xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}, \quad C : \mathbf{r}(t) = (t+\sin \pi t)\,\mathbf{i} + (2t+\cos \pi t)\,\mathbf{j},\ 0 \le t \le 1.$$

Solution. (a) We look for a potential function f such that $\mathbf{F} = \nabla f$. We need

$$\frac{\partial f}{\partial x} = (1+xy)e^{xy} \quad \Rightarrow \qquad f(x,y) = xe^{xy} + C(y)$$

$$\frac{\partial f}{\partial y} = e^y + x^2 e^{xy} \quad \Rightarrow \qquad f(x,y) = e^y + xe^{xy} + D(x)$$

These are consistent if $C(y) = e^y,\ D(x) = 0$. So $\mathbf{F}$ has the potential function

$$\boxed{f(x,y) = e^y + xe^{xy}}$$

and therefore $\mathbf{F}$ is conservative.

(b) By the fundamental theorem for line integrals

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1,1) - f(0,1) = 2e - e = \boxed{e}$$

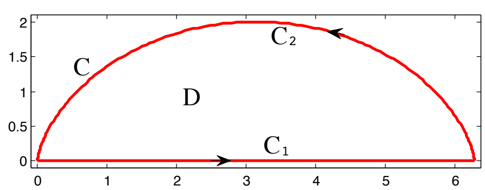
□

2. Use Green's Theorem to find the area bounded by one arc of the cycloid

$$x = a(t-\sin t),\ y = a(1-\cos t),\ a > 0,\ 0 \le t \le 2\pi,$$

and the $x-$axis. (Hint: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.)

Solution. By Green's Theorem, $A(D) = -\int_C y\,dx$, where C is the positively oriented curve bounding the region. The curve C in this case is made up of two pieces: C_1 the portion of the $x-$axis, and C_2 the cycloid, as shown in the figure below (for $a = 1$).



The integral on C_1, $\int_{C_1} y\,dx$ is zero since $y = 0$ on this part of the curve. The part C_2 has the parametrization as given above, but oriented in the opposite direction. Therefore,

$$\begin{aligned}
A(D) &= -\int_C y\,dx \\
&= -\int_{C_1} y\,dx - \int_{C_2} y\,dx \\
&= \int_{-C_2} y\,dx \qquad \left(\text{since } \textstyle\int_{C_1} y\,dx = 0 \text{ and } -\int_{C_2} y\,dx = \int_{-C_2} y\,dx\right) \\
&= \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt \\
&= a^2 \int_0^{2\pi} \left[1 - 2\cos t + \frac{1}{2}(1+\cos 2t)\right] dt \qquad (\text{using half angle identity}) \\
&= \boxed{3\pi a^2}
\end{aligned}$$

$\square$

3. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

Solution. This surface is the graph of a function over the unit disk, so we apply the formula

$$\begin{aligned}
A(S) &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint_D \sqrt{1 + y^2 + x^2}\, dA \qquad \text{(switch to polar coordinates)} \\
&= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r\, dr\, d\theta \\
&= 2\pi \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r\, dr \qquad \text{(make } u- \text{ sub } u = 1 + r^2\text{)} \\
&= 2\pi \int_1^2 \sqrt{u}\, \frac{du}{2} \\
&= \pi \frac{2}{3} u^{3/2} \Big|_1^2 \\
&= \boxed{\frac{2\pi}{3}\left(2^{3/2} - 1\right)}
\end{aligned}$$

$\square$

4. (a) A uniform fluid that flows vertically downward (heavy rain) is described by the vector field $\mathbf{F}(x, y, z) = \langle 0, 0, -1 \rangle$. Find the total flux through the cone $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 \le 1$.

   (b) The rain is driven sideways by a strong wind so that it falls at a 45° angle, and it is described by $\mathbf{F}(x, y, z) = -\frac{1}{\sqrt{2}}\langle 1, 0, 1 \rangle$. Now what is the flux through the cone?

Solution. (a) The surface is the graph of a function over the unit disk D, so we can use the formula

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle dA \\
&= -\iint_D dA \qquad \text{(since the } x- \text{ and } y-\text{components of } \mathbf{F} \text{ are zero)} \\
&= -A(D) \\
&= \boxed{-\pi}
\end{aligned}$$

(b) In this case we will need to calculate $\partial z/\partial x$, since the $x-$component of $\mathbf{F}$ is non-zero. This is

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$$

Thus,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle dA \\
&= \iint_D -\frac{1}{\sqrt{2}}\langle 1,0,1\rangle\cdot\left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{\partial z}{\partial y}, 1\right\rangle dA \\
&= -\frac{1}{\sqrt{2}}\iint_D \left(-\frac{x}{\sqrt{x^2+y^2}}+1\right) dA \qquad \text{(switch to polar coordinates)} \\
&= -\frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^1 (-\cos\theta+1)\, r\, dr\, d\theta \quad \left(\text{since } \tfrac{x}{\sqrt{x^2+y^2}} = \tfrac{r\cos\theta}{r} = \cos\theta\right) \\
&= -\frac{1}{\sqrt{2}}\int_0^{2\pi} \frac{1}{2}(-\cos\theta+1)\, d\theta \\
&= -\frac{1}{2\sqrt{2}}(-\sin\theta+\theta)\Big|_0^{2\pi} \\
&= \boxed{-\frac{\pi}{\sqrt{2}}}
\end{aligned}$$

NOTES: (1) We are using the convention that the upward direction is positive. The flux in both cases is negative because the fluid is flowing down.

(2) Notice that, although in both cases the speed of the fluid is the same ($=1$), the flux is much less in the second case since the normal component of the velocity is smaller, on average, than in the first case. □

5. Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the unit sphere $x^2+y^2+z^2=1$, oriented outward from the origin.

Solution. The divergence of $\mathbf{F}$ is $\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$. The Divergence Theorem implies

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \nabla\cdot\mathbf{F}\, dV \\
&= \iiint_E 3\, dV \\
&= 3\iiint_E dV \\
&= 3\,V(E) \quad \left(V(E) = \frac{4\pi}{3} \text{ is the volume of the sphere of radius 1}\right) \\
&= \boxed{4\pi}
\end{aligned}$$

□

BONUS Suppose the components of a vector field $\mathbf{F}(x, y, z)$ have continuous second derivatives. Let S be the boundary surface of a simple solid region in $\mathbb{R}^3$. Show that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$$

Solution. We recall a fact about the curl, namely that the divergence of the curl is zero: $\nabla \cdot \operatorname{curl} \mathbf{F} = 0$. Let E be the region bounded by S. Then, by the Divergence Theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \operatorname{curl} \mathbf{F} \, dV = \iiint_E 0 \, dV = 0.$$

$\square$