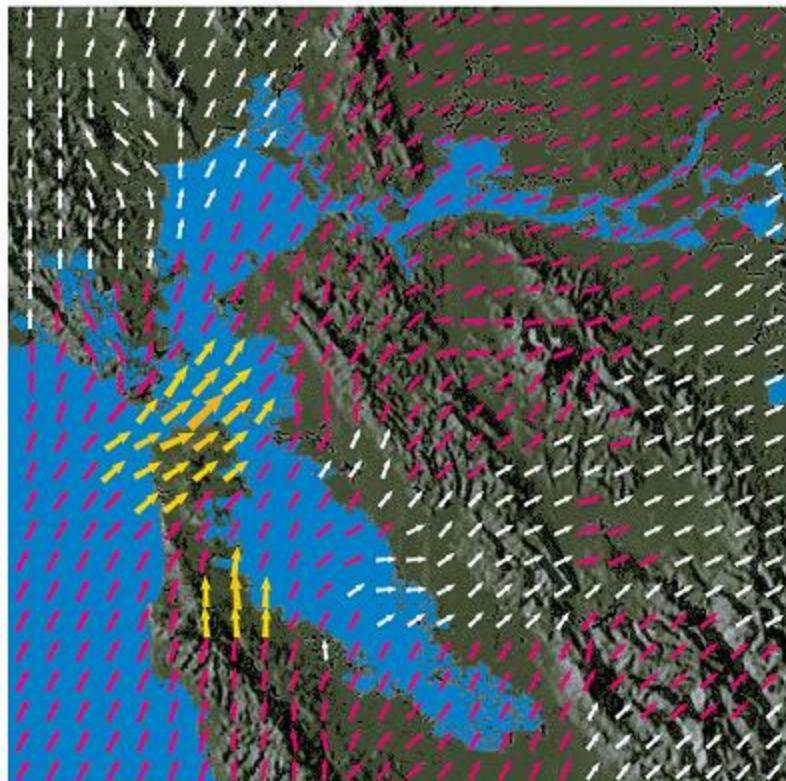


13.1 Vector Fields

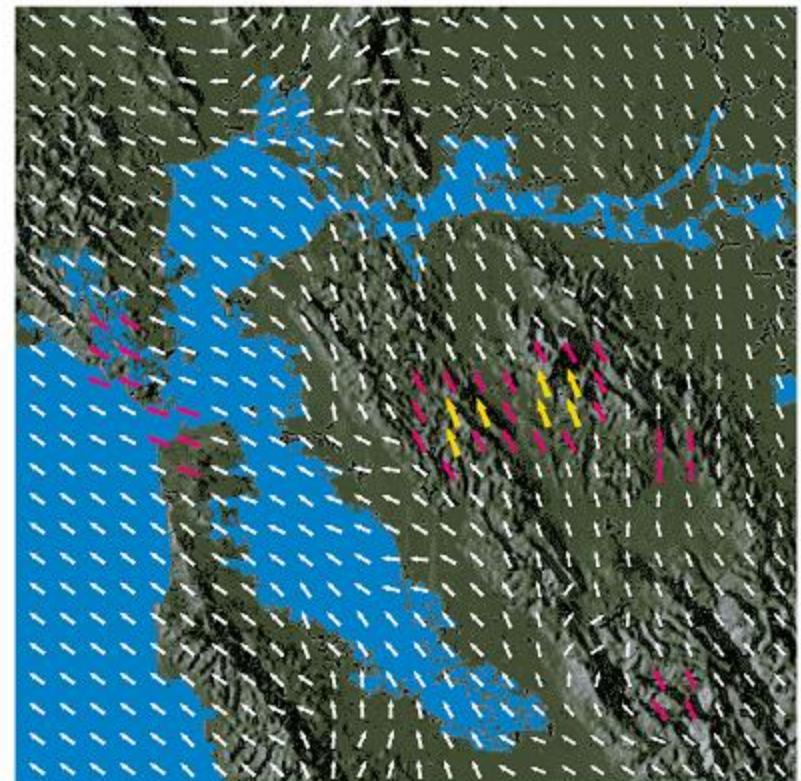
The vectors in Figure 1 are air velocity vectors that indicate the wind speed and direction at points 10 m above the surface elevation in the San Francisco Bay area.

We see at a glance from the largest arrows in part (a) that the greatest wind speeds at that time occurred as the winds entered the bay across the Golden Gate Bridge. Part (b) shows the very different wind pattern 12 hours earlier.

Vector Fields



(a) 6:00 PM, March 1, 2010



(b) 6:00 AM, March 1, 2010

Velocity vector fields showing San Francisco Bay wind patterns

Figure 1

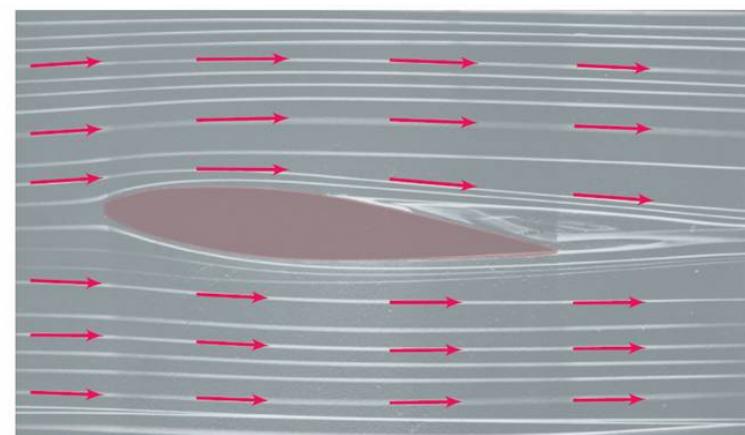
Vector Fields

Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field*.

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia



(b) Airflow past an inclined airfoil

Velocity vector fields
Figure 2

Vector Fields

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field.

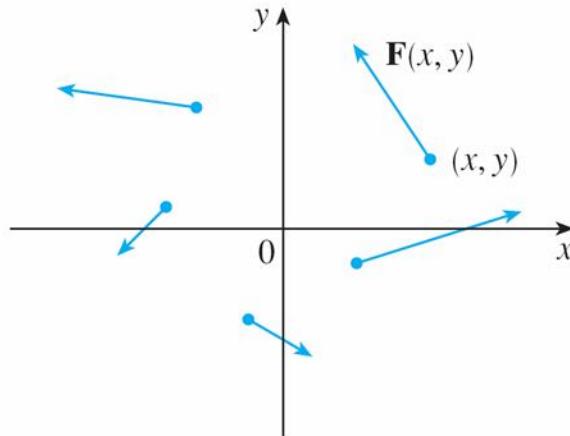
In general, a vector field is a function whose domain is a set of points in \mathbb{R}^2 (or \mathbb{R}^3) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in \mathbb{R}^2 (a plane region). A **vector field on \mathbb{R}^2** is a function \mathbf{F} that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Vector Fields

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y) .

Of course, it's impossible to do this for all points (x, y) , but we can gain a reasonable impression of \mathbf{F} by doing it for a few representative points in D as in Figure 3.



Vector field on \mathbb{R}^2

Figure 3

Vector Fields

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

or, for short,

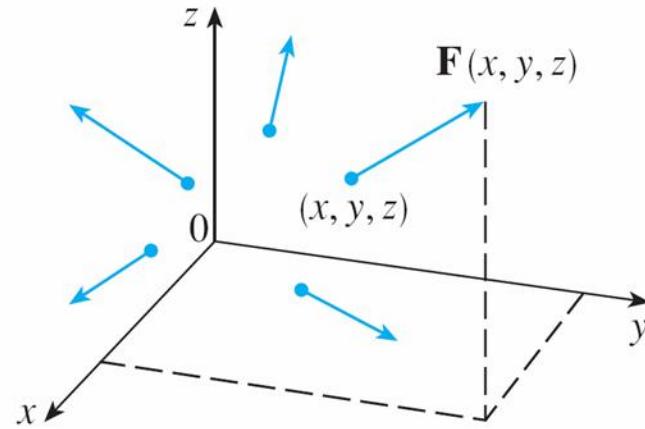
$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of \mathbb{R}^3 . A **vector field on \mathbb{R}^3** is a function \mathbf{F} that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Vector Fields

A vector field \mathbf{F} on \mathbb{R}^3 is pictured in Figure 4.



Vector field on \mathbb{R}^3

Figure 4

We can express it in terms of its component functions P , Q , and R as

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}$$

Vector Fields

As with the vector functions, we can define continuity of vector fields and show that \mathbf{F} is continuous if and only if its component functions P , Q , and R are continuous.

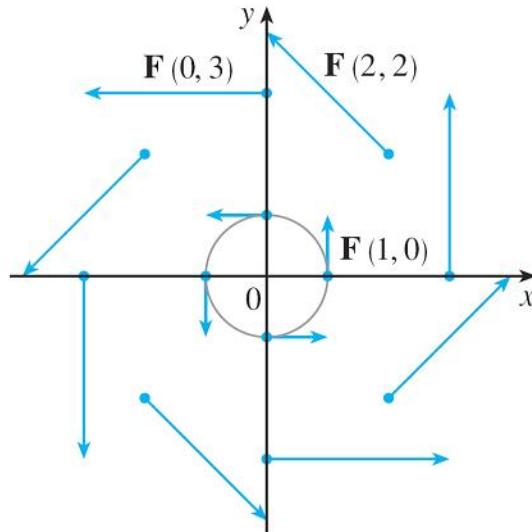
We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

Then \mathbf{F} becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector \mathbf{x} .

Example 1 – Solution

A vector field on \mathbb{R}^2 is defined by $\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$.

Since $\mathbf{F}(1, 0) = \mathbf{j} = \langle 0, 1 \rangle$: start at the point $(1, 0)$ in Figure 5.



$$\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$$

Figure 5

Since $\mathbf{F}(0, 1) = -\mathbf{i} = \langle -1, 0 \rangle$

Example 1 – Solution

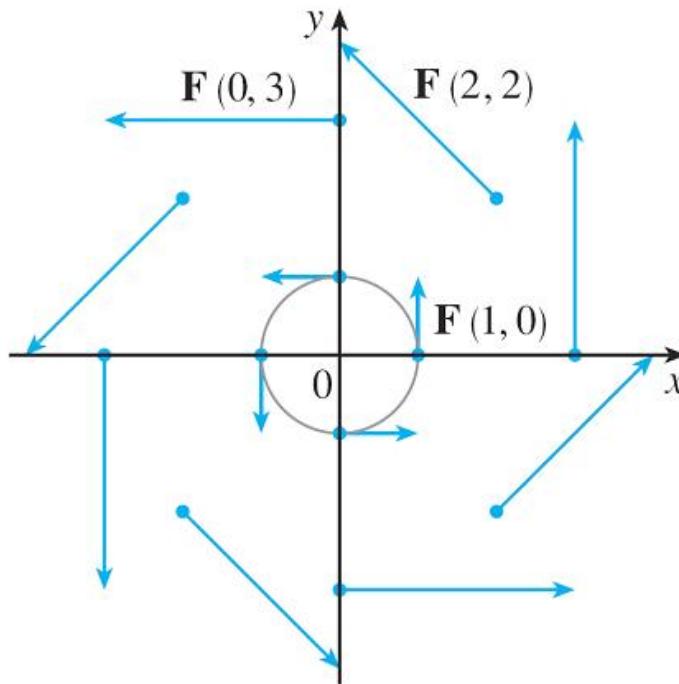
cont'd

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
(1, 0)	$\langle 0, 1 \rangle$	(-1, 0)	$\langle 0, -1 \rangle$
(2, 2)	$\langle -2, 2 \rangle$	(-2, -2)	$\langle 2, -2 \rangle$
(3, 0)	$\langle 0, 3 \rangle$	(-3, 0)	$\langle 0, -3 \rangle$
(0, 1)	$\langle -1, 0 \rangle$	(0, -1)	$\langle 1, 0 \rangle$
(-2, 2)	$\langle -2, -2 \rangle$	(2, -2)	$\langle 2, 2 \rangle$
(0, 3)	$\langle -3, 0 \rangle$	(0, -3)	$\langle 3, 0 \rangle$

Example 1 – Solution

cont'd

It appears from Figure 5 that each arrow is tangent to a circle with center the origin.



$$\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}$$

Figure 5

Example 1 – Solution

cont'd

To confirm this, we take the dot product of the position vector $\mathbf{x} = x \mathbf{i} + y \mathbf{j}$ with the vector $\mathbf{F}(\mathbf{x}) = \mathbf{F}(x, y)$:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) &= (x \mathbf{i} + y \mathbf{j}) \cdot (-y \mathbf{i} + x \mathbf{j}) \\ &= -xy + yx = 0\end{aligned}$$

This shows that $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle x, y \rangle$ and is therefore tangent to a circle with center the origin and radius $|\mathbf{x}| = \sqrt{x^2 + y^2}$.

Notice also that

$$|\mathbf{F}(x, y)| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = |\mathbf{x}|$$

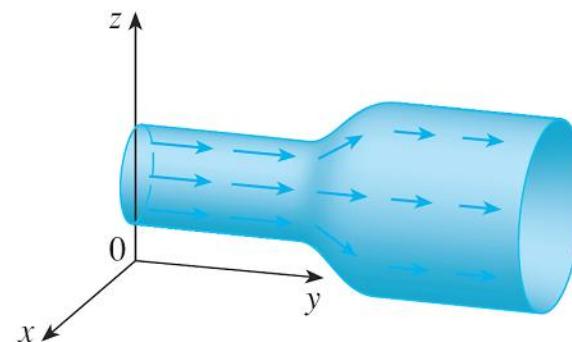
so the magnitude of the vector $\mathbf{F}(x, y)$ is equal to the radius of the circle.

Example 3

Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point (x, y, z) .

Then \mathbf{V} assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so \mathbf{V} is a vector field on \mathbb{R}^3 called a **velocity field**.

A possible velocity field is illustrated in Figure 13.



Velocity field in fluid flow

Figure 13

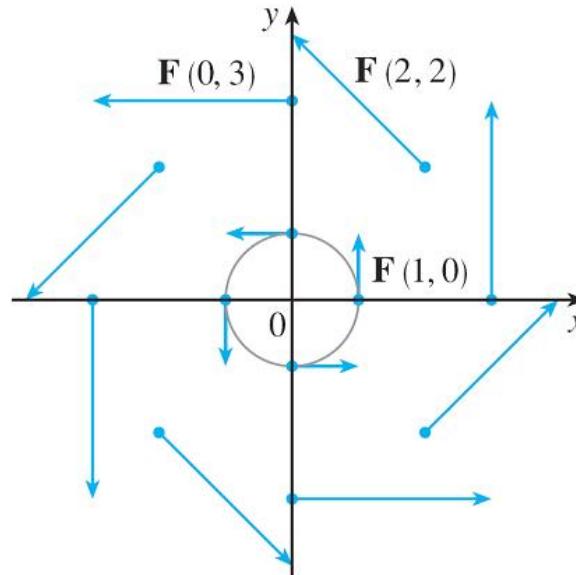
Example 3

cont'd

The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics.

For instance, the vector field in Example 1 could be used as the velocity field describing the counterclockwise rotation of a wheel.



Example 4

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant.

Let's assume that the object with mass M is located at the origin in \mathbb{R}^3 (For instance, M could be the mass of the earth and the origin would be at its center.)

Example 4

cont'd

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

3 $|\mathbf{F}| = \frac{mMG}{r^2}$

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

[Physicists often use the notation \mathbf{r} instead of \mathbf{x} for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}.$]

Example 4

cont'd

The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force $\mathbf{F}(\mathbf{x})$] with every point \mathbf{x} in space.

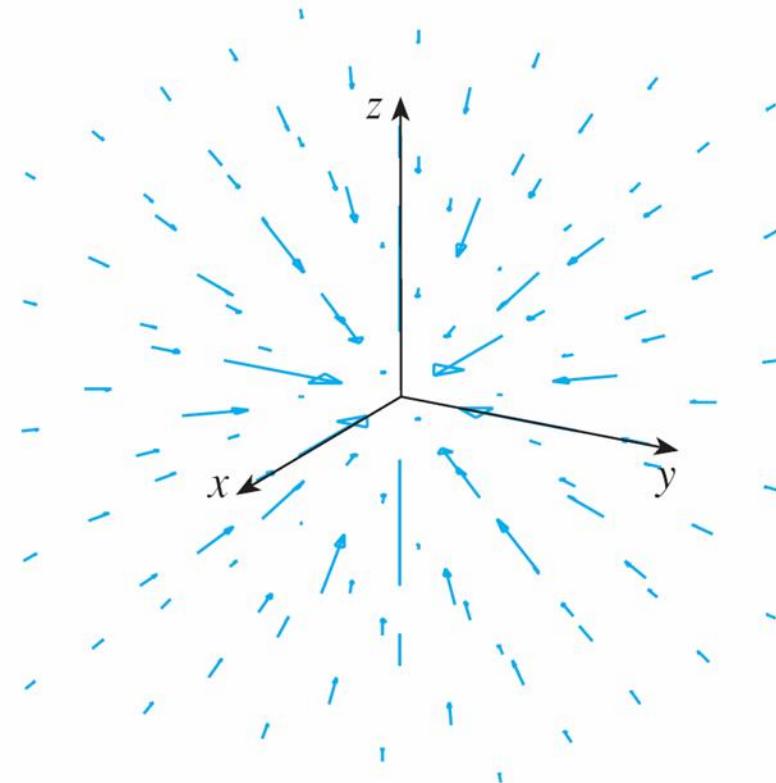
Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that $\mathbf{x} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mM Gy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k}$$

Example 4

cont'd

The gravitational field \mathbf{F} is pictured in Figure 14.



Gravitational force field

Figure 14

Example 5

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

4

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3} \mathbf{x}$$

where ε is a constant (that depends on the units used).

For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive.

Example 5

cont'd

Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force \mathbf{F} , physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q} \mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x}$$

Then \mathbf{E} is a vector field on \mathbb{R}^3 called the **electric field** of Q .

Gradient Fields

If f is a scalar function of two variables, recall that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}$$

Therefore ∇f is really a vector field on \mathbb{R}^2 and is called a **gradient vector field**.

Likewise, if f is a scalar function of three variables, its gradient is a vector field on \mathbb{R}^3 given by

$$\nabla f(x, y, z) = f_x(x, y, z) \mathbf{i} + f_y(x, y, z) \mathbf{j} + f_z(x, y, z) \mathbf{k}$$

Example 6

Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f . How are they related?

Solution:

The gradient vector field is given by

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \\ &= 2xy \mathbf{i} + (x^2 - 3y^2) \mathbf{j}\end{aligned}$$

Example 6 – Solution

cont'd

Figure 15 shows a contour map of f with the gradient vector field.

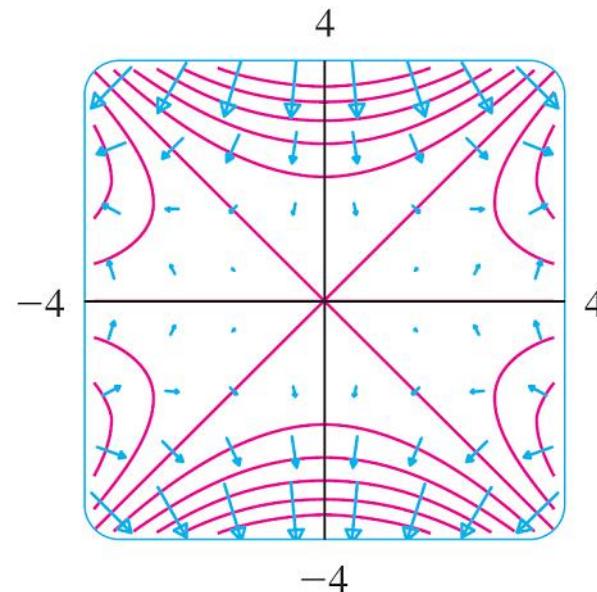


Figure 15

Notice that the gradient vectors are perpendicular to the level curves.

Example 6 – Solution

cont'd

Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart.

That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph.

Gradient Fields

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$.

In this situation f is called a **potential function** for \mathbf{F} .

Not all vector fields are conservative, but such fields do arise frequently in physics.

Gradient Fields

For example, the gravitational field \mathbf{F} in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{i} + \frac{-mM Gy}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{k}$$

$$= \mathbf{F}(x, y, z)$$

13.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C .

Such integrals are called *line integrals*, although “curve integrals” would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals

We start with a plane curve C given by the parametric equations

1

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}$, and we assume that C is a smooth curve. [This means that \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.]

Line Integrals

If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$. (See Figure 1.)

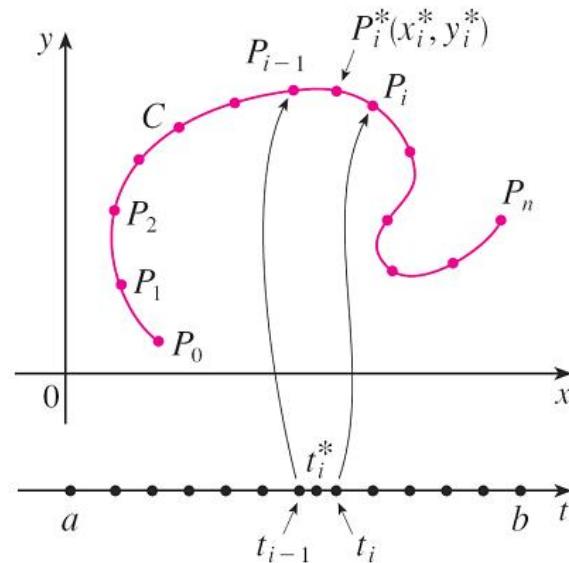


Figure 1

Line Integrals

We choose any point $P_i^*(x_i^*, y_i^*)$ in the i th subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.)

Now if f is any function of two variables whose domain includes the curve C , we evaluate f at the point (x_i^*, y_i^*) , multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum.

Line Integrals

Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y) \, ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

if this limit exists.

We have found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Line Integrals

A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

3

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b .

Line Integrals

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t : Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Line Integrals

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leq x \leq b$.

Formula 3 then becomes

$$\int_C f(x, y) \, ds = \int_a^b f(x, 0) \, dx$$

and so the line integral reduces to an ordinary single integral in this case.

Line Integrals

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area.

In fact, if $f(x, y) \geq 0$, $\int_C f(x, y) ds$ represents the area of one side of the “fence” or “curtain” in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

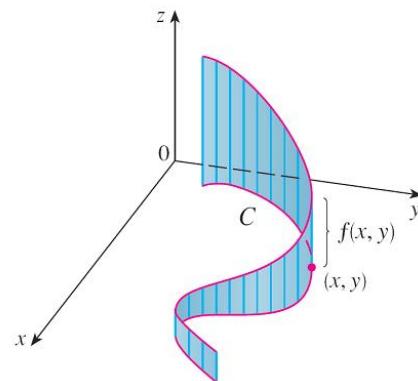


Figure 2

Example 1

Evaluate $\int_C (2 + x^2y) \, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution:

In order to use Formula 3, we first need parametric equations to represent C .

Recall that the unit circle can be parametrized by means of the equations

$$x = \cos t \quad y = \sin t$$

and the upper half of the circle is described by the parameter interval $0 \leq t \leq \pi$. (See Figure 3.)

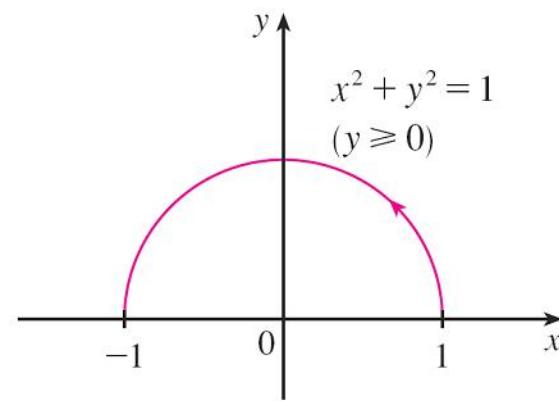


Figure 3

Example 1 – Solution

cont'd

Therefore Formula 3 gives

$$\int_C (2 + x^2y) \, ds = \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} \, dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t) \, dt$$

$$= \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi$$

$$= 2\pi + \frac{2}{3}$$

Line Integrals

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves C_1, C_2, \dots, C_n , where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i .

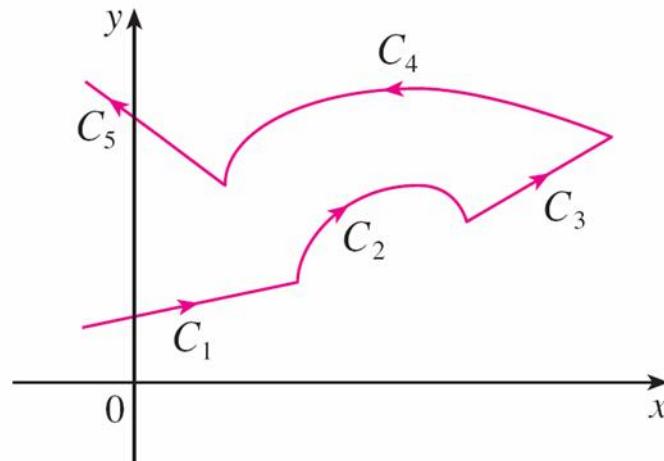


Figure 4

A piecewise-smooth curve

Line Integrals

Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C :

$$\int_C f(x, y) \, ds = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \cdots + \int_{C_n} f(x, y) \, ds$$

Line Integrals

Any physical interpretation of a line integral $\int_C f(x, y) ds$ depends on the physical interpretation of the function f .

Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C .

Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*) \Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*) \Delta s_i$.

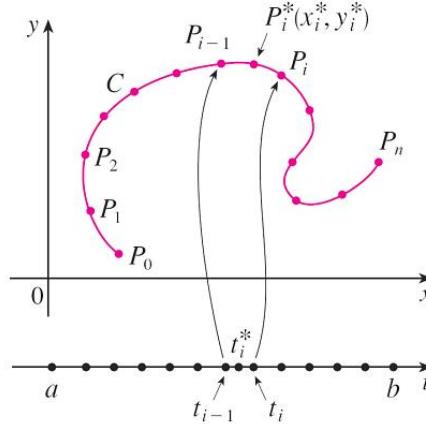


Figure 1

Line Integrals

By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*) \Delta s_i = \int_C \rho(x, y) ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.]

Line Integrals

The **center of mass** of the wire with density function ρ is located at the point (\bar{x}, \bar{y}) , where

4

$$\bar{x} = \frac{1}{m} \int_C x \rho(x, y) \, ds \quad \bar{y} = \frac{1}{m} \int_C y \rho(x, y) \, ds$$

Line Integrals

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2.

They are called the **line integrals of f along C with respect to x and y** :

5

$$\int_C f(x, y) \, dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i$$

6

$$\int_C f(x, y) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

Line Integrals

When we want to distinguish the original line integral $\int_C f(x, y) ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t : $x = x(t)$, $y = y(t)$, $dx = x'(t) dt$, $dy = y'(t) dt$.

7

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

Line Integrals

It frequently happens that line integrals with respect to x and y occur together.

When this happens, it's customary to abbreviate by writing
 $\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.

Line Integrals

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at \mathbf{r}_0 and ends at \mathbf{r}_1 is given by

8

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

Line Integrals

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C , with the positive direction corresponding to increasing values of the parameter t . (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

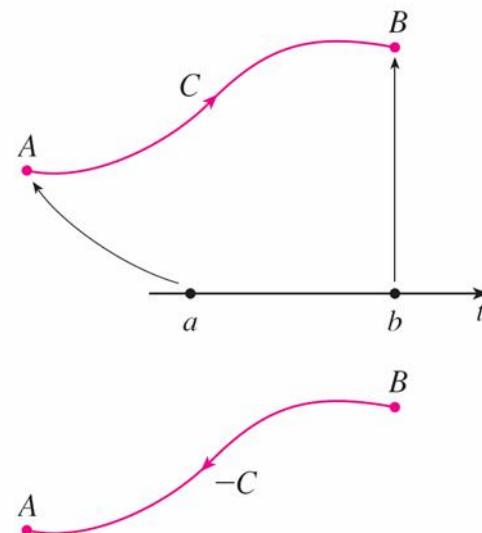


Figure 8

Line Integrals

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y) dx = - \int_C f(x, y) dx \quad \int_{-C} f(x, y) dy = - \int_C f(x, y) dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y) ds = \int_C f(x, y) ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C .

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation $\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}$.

If f is a function of three variables that is continuous on some region containing C , then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z) \, ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

Line Integrals in Space

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \quad \int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)| \, dt = L$$

where L is the length of the curve C .

Line Integrals in Space

Line integrals along C with respect to x , y , and z can also be defined. For example,

$$\begin{aligned}\int_C f(x, y, z) \, dz &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta z_i \\ &= \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

10 $\int_C P(x, y, z) \, dx + Q(x, y, z) \, dy + R(x, y, z) \, dz$

by expressing everything (x , y , z , dx , dy , dz) in terms of the parameter t .

Example 5

Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$.
(See Figure 9.)

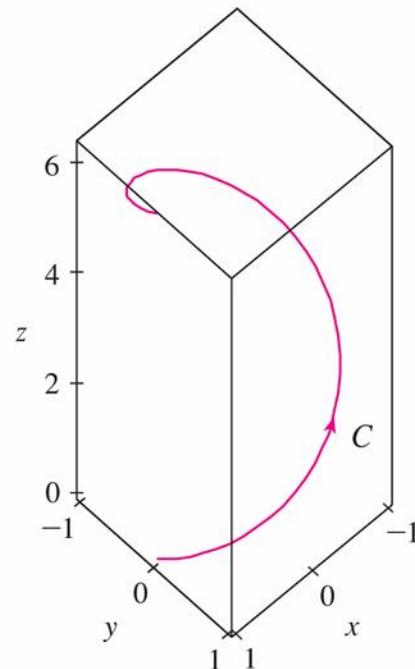


Figure 9

Example 5 – Solution

Formula 9 gives

$$\begin{aligned}\int_C y \sin z \, ds &= \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\&= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt \\&= \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t) \, dt \\&= \frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_0^{2\pi} \\&= \sqrt{2} \pi\end{aligned}$$

Line Integrals of Vector Fields

Recall that the work done by a variable force $f(x)$ in moving a particle from a to b along the x -axis is $W = \int_a^b f(x) dx$.

Then we have found that the work done by a constant force \mathbf{F} in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a continuous force field on \mathbb{R}^3 . (A force field on \mathbb{R}^2 could be regarded as a special case where $R = 0$ and P and Q depend only on x and y .)

We wish to compute the work done by this force in moving a particle along a smooth curve C .

Line Integrals of Vector Fields

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.)

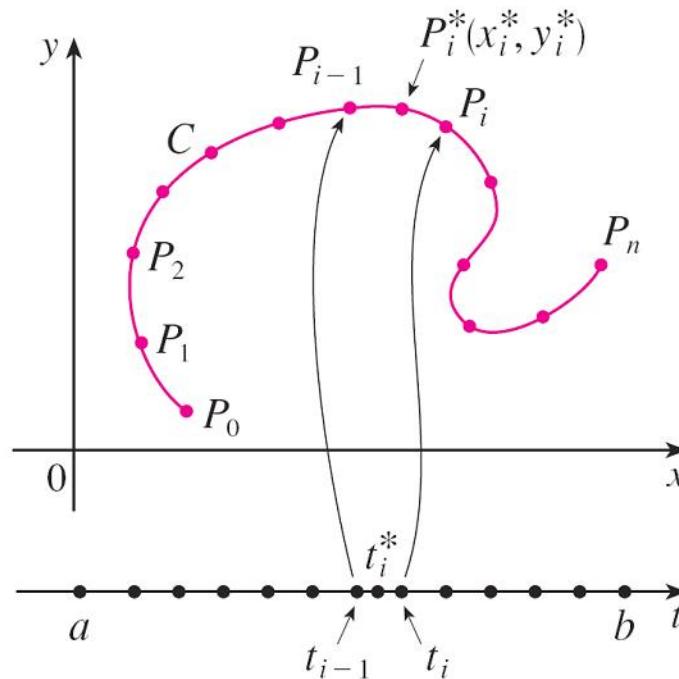


Figure 1

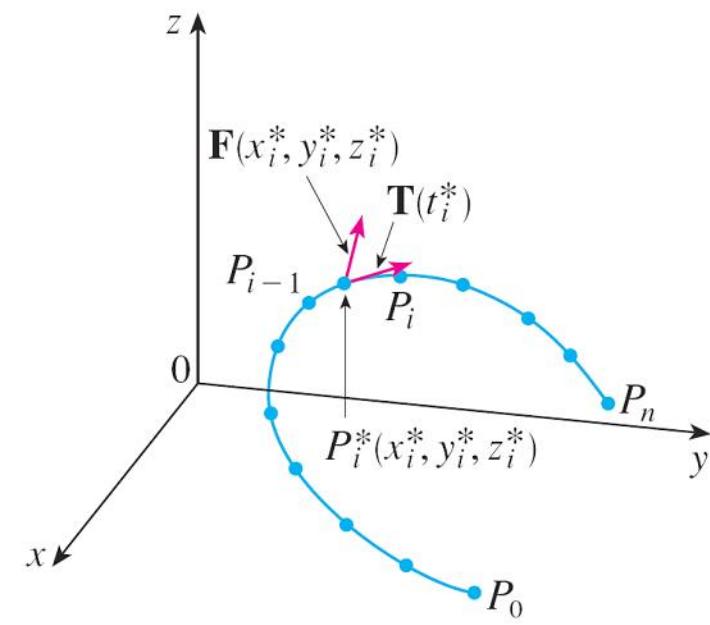


Figure 11

Line Integrals of Vector Fields

Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i th subarc corresponding to the parameter value t_i^* .

If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^* .

Line Integrals of Vector Fields

Thus the work done by the force \mathbf{F} in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along C is approximately

11

$$\sum_{i=1}^n [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C .

Line Integrals of Vector Fields

Intuitively, we see that these approximations ought to become better as n becomes larger.

Therefore we define the **work W** done by the force field \mathbf{F} as the limit of the Riemann sums in [11], namely,

[12]

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z) \, ds = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

Line Integrals of Vector Fields

If the curve C is given by the vector equation

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k},$$

then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well.

Line Integrals of Vector Fields

Therefore we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of \mathbf{F} along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

When using Definition 13, remember that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$.

Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t) dt$.

Example 7

Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} - xy \mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}, 0 \leq t \leq \pi/2.$

Solution:

Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t \mathbf{i} - \cos t \sin t \mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Example 7 – Solution

cont'd

Therefore the work done is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{\pi/2} (-2 \cos^2 t \sin t) \, dt \\ &= 2 \left. \frac{\cos^3 t}{3} \right|_0^{\pi/2} \\ &= -\frac{2}{3}\end{aligned}$$

Line Integrals of Vector Fields

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field \mathbf{F} on \mathbb{R}^3 is given in component form by the equation $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$.

We use Definition 13 to compute its line integral along C :

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b (P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}) \cdot (x'(t) \mathbf{i} + y'(t) \mathbf{j} + z'(t) \mathbf{k}) dt \\ &= \int_a^b [P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) \\ &\quad + R(x(t), y(t), z(t)) z'(t)] dt\end{aligned}$$

Line Integrals of Vector Fields

But this last integral is precisely the line integral in [10].
Therefore we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy + R \, dz \quad \text{where } \mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

For example, the integral $\int_C y \, dx + z \, dy + x \, dz$ could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$$

13.3 The Fundamental Theorem for Line Integrals

Recall that Part 2 of the Fundamental Theorem of Calculus can be written as

1

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$.

We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

The Fundamental Theorem for Line Integrals

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f , then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Example 1

Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3} \mathbf{x}$$

in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C .

Solution:

We know that \mathbf{F} is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Example 1 – Solution

cont'd

Therefore, by Theorem 2, the work done is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(2, 2, 0) - f(3, 4, 12)$$

$$= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$= mMG \left(\frac{1}{2\sqrt{2}} - \frac{1}{13} \right)$$

Independence of Paths

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .

Since we know that the line integral of any conservative vector field \mathbf{F} is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Independence of Paths

The physical interpretation is that the work done by a conservative force field as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves.

4 Theorem Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Independence of Paths

We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D . (So D doesn't contain any of its boundary points.)

In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D .

The question remains: How is it possible to determine whether or not a vector field \mathbf{F} is conservative? Suppose it is known that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is, $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

Independence of Paths

Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

5 Theorem If $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem 5 is true only for a special type of region.

Independence of Paths

To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

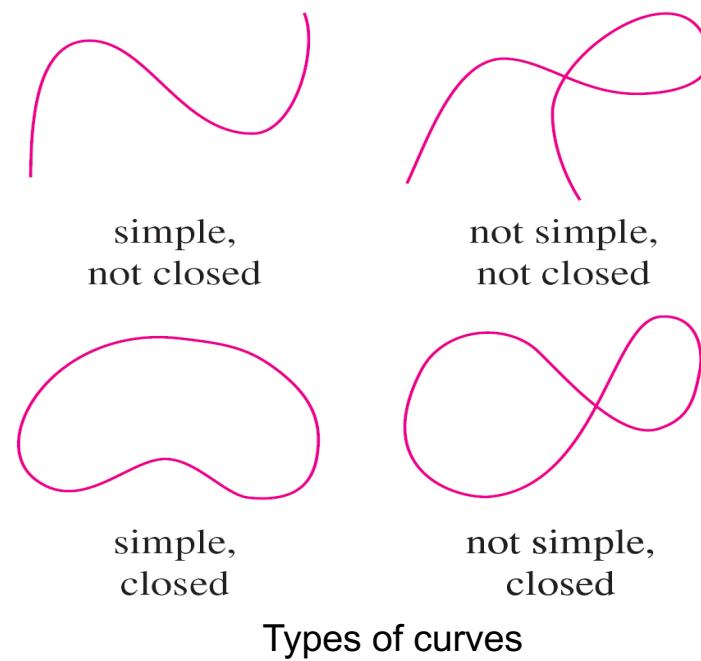


Figure 6

Independence of Paths

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition.

A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D .

Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

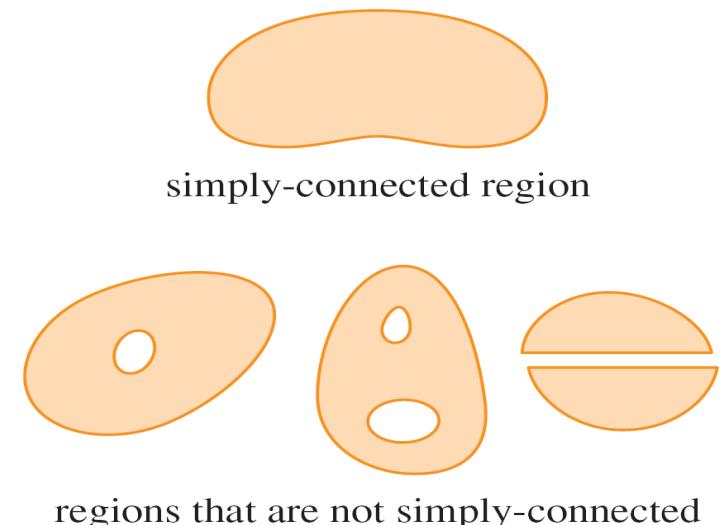


Figure 7

Independence of Paths

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on \mathbb{R}^2 is conservative.

6 Theorem Let $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then \mathbf{F} is conservative.

Example 2

Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y) \mathbf{i} + (x - 2) \mathbf{j}$$

is conservative.

Solution:

Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$.

Then $\frac{\partial P}{\partial y} = -1$ $\frac{\partial Q}{\partial x} = 1$

Since $\partial P / \partial y \neq \partial Q / \partial x$, \mathbf{F} is not conservative by Theorem 5.

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field \mathbf{F} that moves an object along a path C given by $\mathbf{r}(t)$, $a \leq t \leq b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C . According to Newton's Second Law of Motion, the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t) dt$$

Conservation of Energy

$$\begin{aligned} &= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] dt \quad (\text{By formula } \frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] \\ &\quad = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)) \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 dt = \frac{m}{2} [|\mathbf{r}'(t)|^2]_a^b \quad (\text{Fundamental Theorem of Calculus}) \\ &= \frac{m}{2} (|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) \end{aligned}$$

Therefore

15

$$W = \frac{1}{2}m |\mathbf{v}(b)|^2 - \frac{1}{2}m |\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

Conservation of Energy

The quantity $\frac{1}{2}m |\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \quad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that \mathbf{F} is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$.

Conservation of Energy

In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$.

Then by Theorem 2 we have

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= - \int_C \nabla P \cdot d\mathbf{r} \\ &= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] \\ &= P(A) - P(B) \end{aligned}$$

Conservation of Energy

Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant.

This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

13.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C . (See Figure 1. We assume that D consists of all points inside C as well as all points on C .)

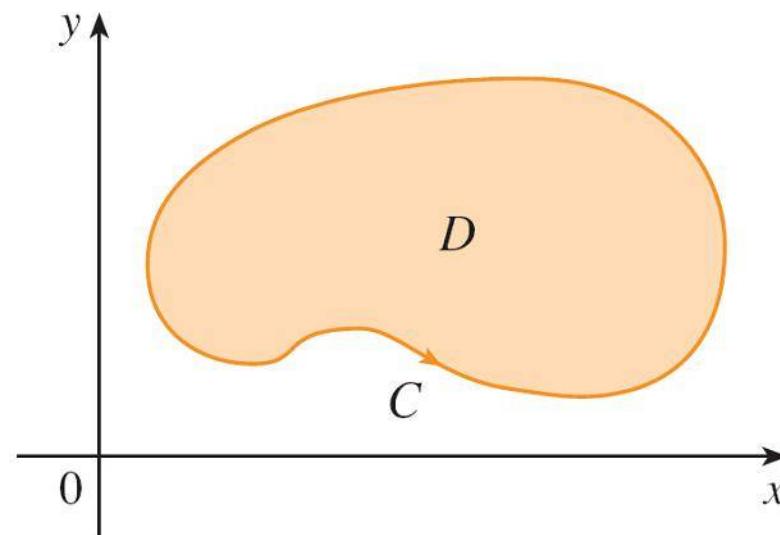
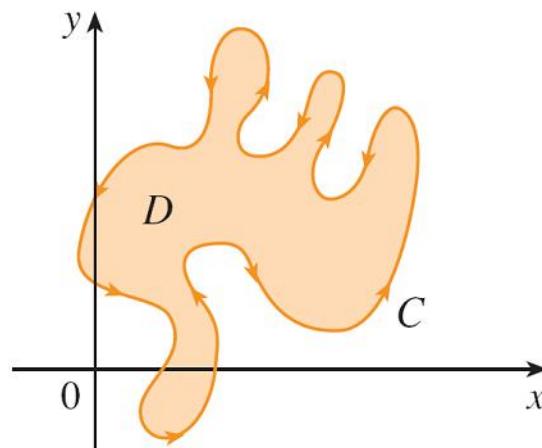


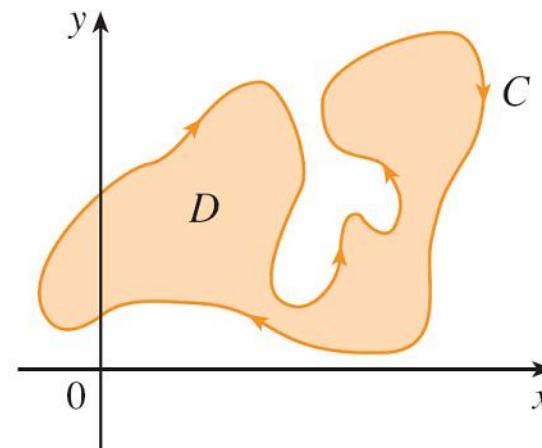
Figure 1

Green's Theorem

In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C . Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C .
(See Figure 2.)



(a) Positive orientation



(b) Negative orientation

Figure 2

Green's Theorem

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example 1

Evaluate $\int_C x^4 dx + xy dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

Solution:

Although the given line integral could be evaluated as usual by the methods, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead.

Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4).

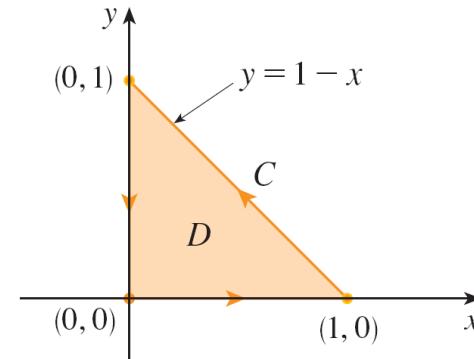


Figure 4

Example 1 – Solution

cont'd

If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4 dx + xy dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

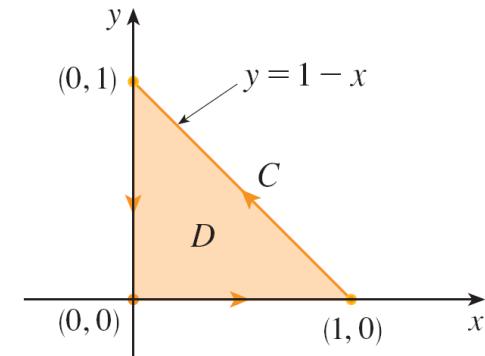
$$= \int_0^1 \int_0^{1-x} (y - 0) dy dx$$

$$= \int_0^1 \left[\frac{1}{2}y^2 \right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{2} \int_0^1 (1 - x)^2 dx$$

$$= -\frac{1}{6}(1 - x)^3 \Big|_0^1$$

$$= \frac{1}{6}$$



Green's Theorem

In Example 1 we found that the double integral was easier to evaluate than the line integral.

But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction.

For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C , then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P \, dx + Q \, dy = 0$$

no matter what values P and Q assume in the region D .

Green's Theorem

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1 \, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0$$

$$Q(x, y) = x$$

$$P(x, y) = -y$$

$$Q(x, y) = 0$$

$$P(x, y) = -\frac{1}{2}y$$

$$Q(x, y) = \frac{1}{2}x$$

Green's Theorem

Then Green's Theorem gives the following formulas for the area of D :

5

$$A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx$$

Green's Theorem

Formula 5 can be used to explain how planimeters work.

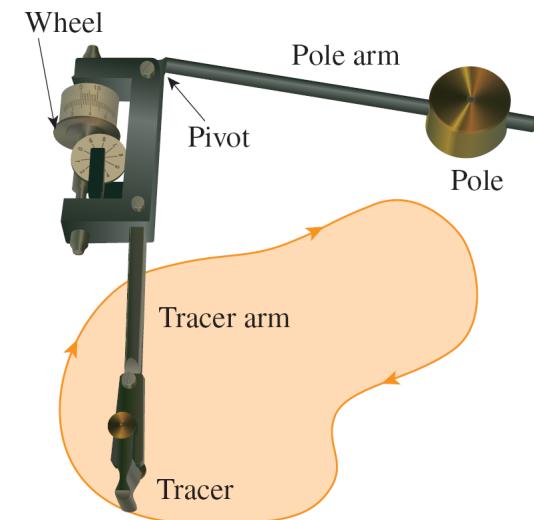
A **planimeter** is a mechanical instrument used for measuring the area of a region by tracing its boundary curve.

These devices are useful in all the sciences: in biology for measuring the area of leaves or wings, in medicine for measuring the size of cross-sections of organs or tumors, in forestry for estimating the size of forested regions from photographs.

Green's Theorem

Figure 5 shows the operation of a polar planimeter: The pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm.

The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region.



A Keuffel and Esser polar planimeter

Figure 5

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions.

For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple.

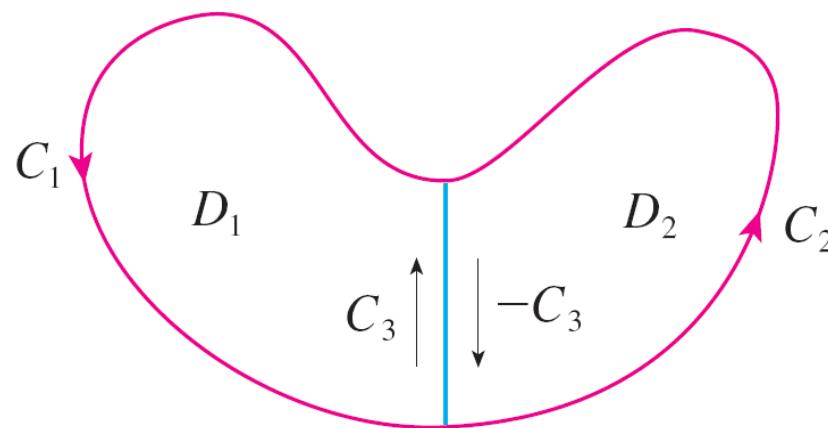


Figure 6

Extended Versions of Green's Theorem

The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P \, dx + Q \, dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P \, dx + Q \, dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Extended Versions of Green's Theorem

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

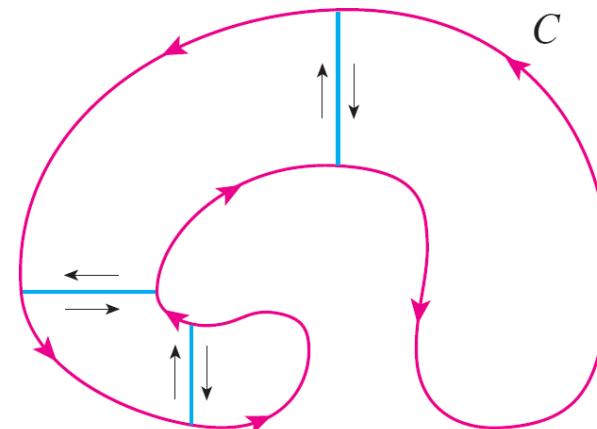


Figure 7

Example 4

Evaluate $\oint_C y^2 dx + 3xy dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

Notice that although D is not simple, the y -axis divides it into two simple regions (see Figure 8).

In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

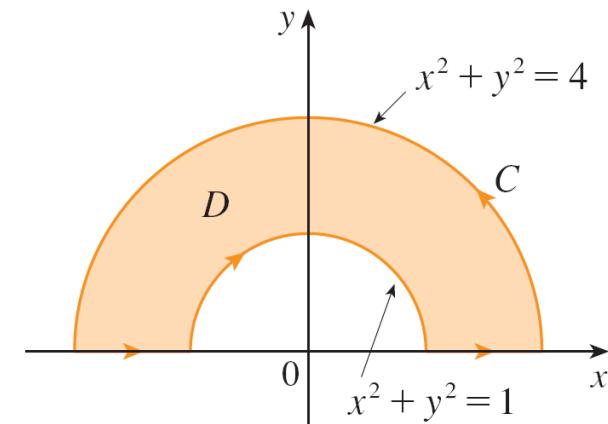


Figure 8

Example 4 – Solution

cont'd

Therefore Green's Theorem gives

$$\begin{aligned}\oint_C y^2 dx + 3xy dy &= \iint_D \left[\frac{\partial}{\partial x} (3xy) - \frac{\partial}{\partial y} (y^2) \right] dA \\&= \iint_D y dA \\&= \int_0^\pi \int_1^2 (r \sin \theta) r dr d\theta \\&= \int_0^\pi \sin \theta d\theta \int_1^2 r^2 dr \\&= [-\cos \theta]_0^\pi \left[\frac{1}{3} r^3 \right]_1^2 = \frac{14}{3}\end{aligned}$$

Extended Versions of Green's Theorem

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.

Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2 .

We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2 .

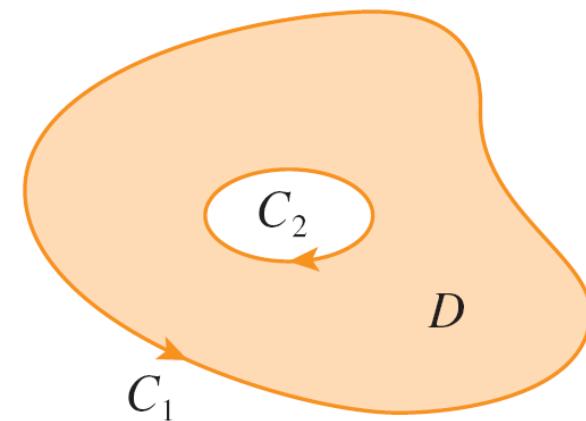


Figure 9

Extended Versions of Green's Theorem

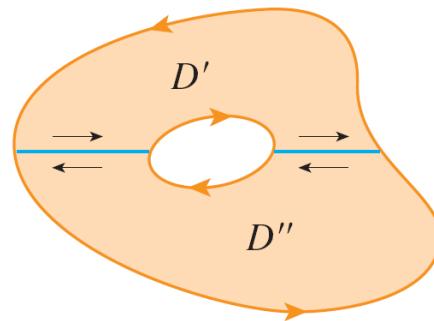


Figure 10

If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'' , we get

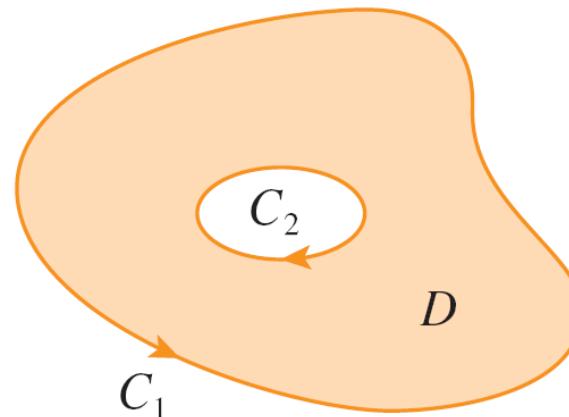
$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_{\partial D'} P \, dx + Q \, dy + \int_{\partial D''} P \, dx + Q \, dy\end{aligned}$$

Extended Versions of Green's Theorem

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P \, dx + Q \, dy + \int_{C_2} P \, dx + Q \, dy = \int_C P \, dx + Q \, dy$$

which is Green's Theorem for the region D .



13.5 Curl and Divergence

Curl

If $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a vector field on \mathbb{R}^3 and the partial derivatives of P , Q , and R all exist, then the **curl** of \mathbf{F} is the vector field on \mathbb{R}^3 defined by

1

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

Curl

It has meaning when it operates on a scalar function to produce the gradient of f :

$$\nabla f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field \mathbf{F} as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

$$\begin{aligned} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \operatorname{curl} \mathbf{F} \end{aligned}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

2

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$

Example 1

If $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$, find $\text{curl } \mathbf{F}$.

Solution:

Using Equation 2, we have

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

Example 1 – Solution

cont'd

$$\begin{aligned} &= \left[\frac{\partial}{\partial y} (-y^2) - \frac{\partial}{\partial z} (xyz) \right] \mathbf{i} - \left[\frac{\partial}{\partial x} (-y^2) - \frac{\partial}{\partial z} (xz) \right] \mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x} (xyz) - \frac{\partial}{\partial y} (xz) \right] \mathbf{k} \\ &= (-2y - xy) \mathbf{i} - (0 - x) \mathbf{j} + (yz - 0) \mathbf{k} \\ &= -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k} \end{aligned}$$

Curl

Recall that the gradient of a function f of three variables is a vector field on \mathbb{R}^3 and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Curl

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If \mathbf{F} is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

Curl

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if \mathbf{F} is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”)

4 Theorem If \mathbf{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then \mathbf{F} is a conservative vector field.

Curl

The reason for the name *curl* is that the curl vector is associated with rotations.

Another occurs when \mathbf{F} represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $\text{curl } \mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

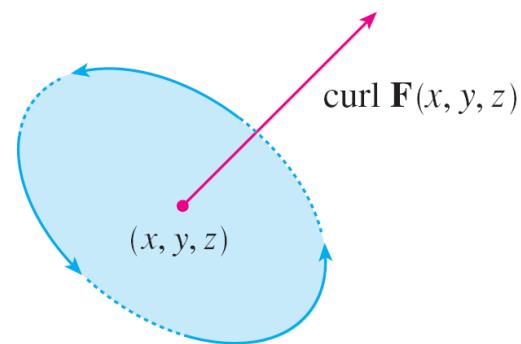


Figure 1

Curl

If $\operatorname{curl} \mathbf{F} = \mathbf{0}$ at a point P , then the fluid is free from rotations at P and \mathbf{F} is called **irrotational** at P .

In other words, there is no whirlpool or eddy at P .

If $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

Divergence

If $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a vector field on \mathbb{R}^3 and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of \mathbf{F}** is

9

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\operatorname{curl} \mathbf{F}$ is a vector field but $\operatorname{div} \mathbf{F}$ is a scalar field.

In terms of the gradient operator

$\nabla = (\partial/\partial x) \mathbf{i} + (\partial/\partial y) \mathbf{j} + (\partial/\partial z) \mathbf{k}$, the divergence of \mathbf{F} can be written

10

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

Example 4

If $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} + y^2 \mathbf{k}$, find $\operatorname{div} \mathbf{F}$.

Solution:

By the definition of divergence (Equation 9 or 10) we have

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \frac{\partial}{\partial x} (xz) + \frac{\partial}{\partial y} (xyz) + \frac{\partial}{\partial z} (-y^2)$$

$$= z + xz$$

Divergence

If \mathbf{F} is a vector field on \mathbb{R}^3 , then $\text{curl } \mathbf{F}$ is also a vector field on \mathbb{R}^3 . As such, we can compute its divergence.

The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a vector field on \mathbb{R}^3 and P , Q , and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

Divergence

If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then $\operatorname{div} \mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, $\operatorname{div} \mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z) .

If $\operatorname{div} \mathbf{F} = 0$, then \mathbf{F} is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f .

Divergence

If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Divergence

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P \mathbf{i} + \nabla^2 Q \mathbf{j} + \nabla^2 R \mathbf{k}$$

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work.

We suppose that the plane region D , its boundary curve C , and the functions P and Q satisfy the hypotheses of Green's Theorem.

Then we consider the vector field $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$.

Vector Forms of Green's Theorem

Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P \, dx + Q \, dy$$

and, regarding \mathbf{F} as a vector field on \mathbb{R}^3 with third component 0, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Vector Forms of Green's Theorem

Therefore

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

Vector Forms of Green's Theorem

Equation 12 expresses the line integral of the tangential component of \mathbf{F} along C as the double integral of the vertical component of $\operatorname{curl} \mathbf{F}$ over the region D enclosed by C . We now derive a similar formula involving the *normal* component of \mathbf{F} .

If C is given by the vector equation

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \quad a \leq t \leq b$$

then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|} \mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|} \mathbf{j}$$

Vector Forms of Green's Theorem

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|} \mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|} \mathbf{j}$$

(See Figure 2.)

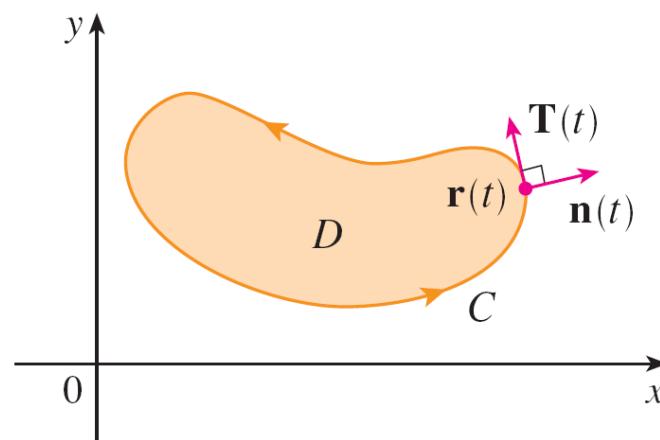


Figure 2

Vector Forms of Green's Theorem

Then, from equation

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

we have

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n} \, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) |\mathbf{r}'(t)| \, dt \\ &= \int_a^b \left[\frac{P(x(t), y(t)) y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t)) x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| \, dt \\ &= \int_a^b P(x(t), y(t)) y'(t) \, dt - Q(x(t), y(t)) x'(t) \, dt \\ &= \int_C P \, dy - Q \, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \end{aligned}$$

by Green's Theorem.

Vector Forms of Green's Theorem

But the integrand in this double integral is just the divergence of \mathbf{F} . So we have a second vector form of Green's Theorem.

13

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

This version says that the line integral of the normal component of \mathbf{F} along C is equal to the double integral of the divergence of \mathbf{F} over the region D enclosed by C .

13. 6 Parametric Surfaces and Their Areas

Parametric Surfaces

We describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t , and a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v . We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv -plane.

Parametric Surfaces

So x , y , and, z , the component functions of \mathbf{r} , are functions of the two variables u and v with domain D .

The set of all points (x, y, z) in \mathbb{R}^3 such that

2

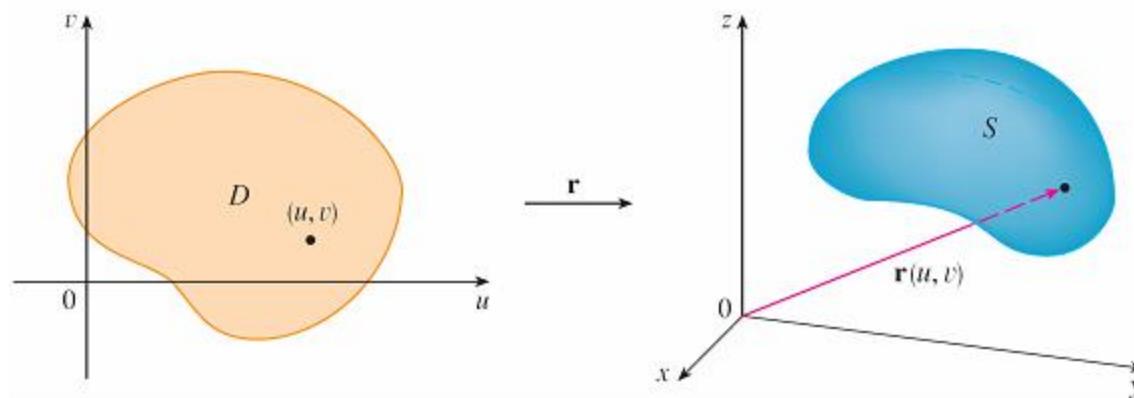
$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

and (u, v) varies throughout D , is called a **parametric surface** S and Equations 2 are called **parametric equations** of S .

Parametric Surfaces

Each choice of u and v gives a point on S ; by making all choices, we get all of S .

In other words, the surface is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D . (See Figure 1.)



A parametric surface

Figure 1

Example 1

Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v \mathbf{j} + 2 \sin u \mathbf{k}$$

Solution:

The parametric equations for this surface are

$$x = 2 \cos u \quad y = v \quad z = 2 \sin u$$

Example 1 – Solution

cont'd

So for any point (x, y, z) on the surface, we have

$$\begin{aligned}x^2 + z^2 &= 4 \cos^2 u + 4 \sin^2 u \\&= 4\end{aligned}$$

This means that vertical cross-sections parallel to the xz -plane (that is, with y constant) are all circles with radius 2.

Example 1 – Solution

cont'd

Since $y = v$ and no restriction is placed on v , the surface is a circular cylinder with radius 2 whose axis is the y -axis (see Figure 2).

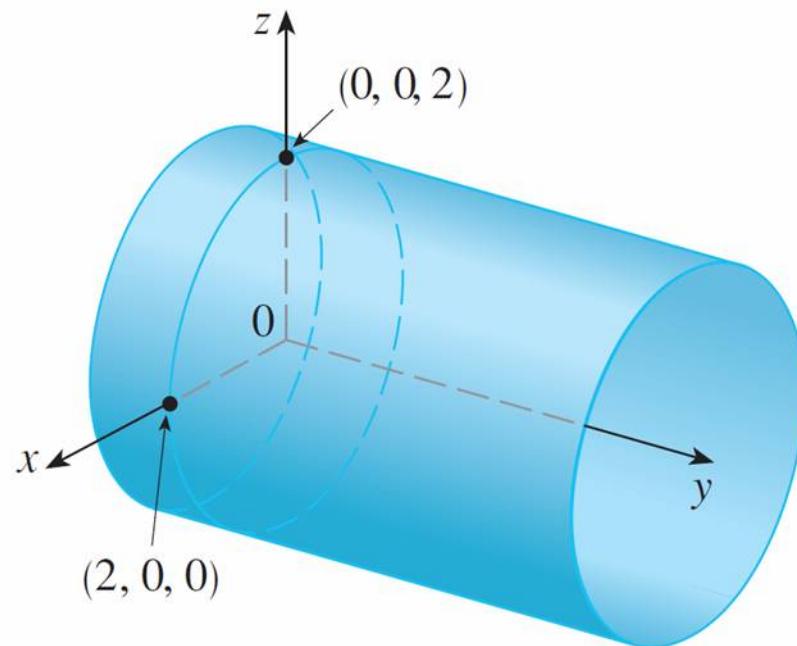


Figure 2

Parametric Surfaces

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S , one family with u constant and the other with v constant.

These families correspond to vertical and horizontal lines in the uv -plane.

Parametric Surfaces

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S . (See Figure 4.)

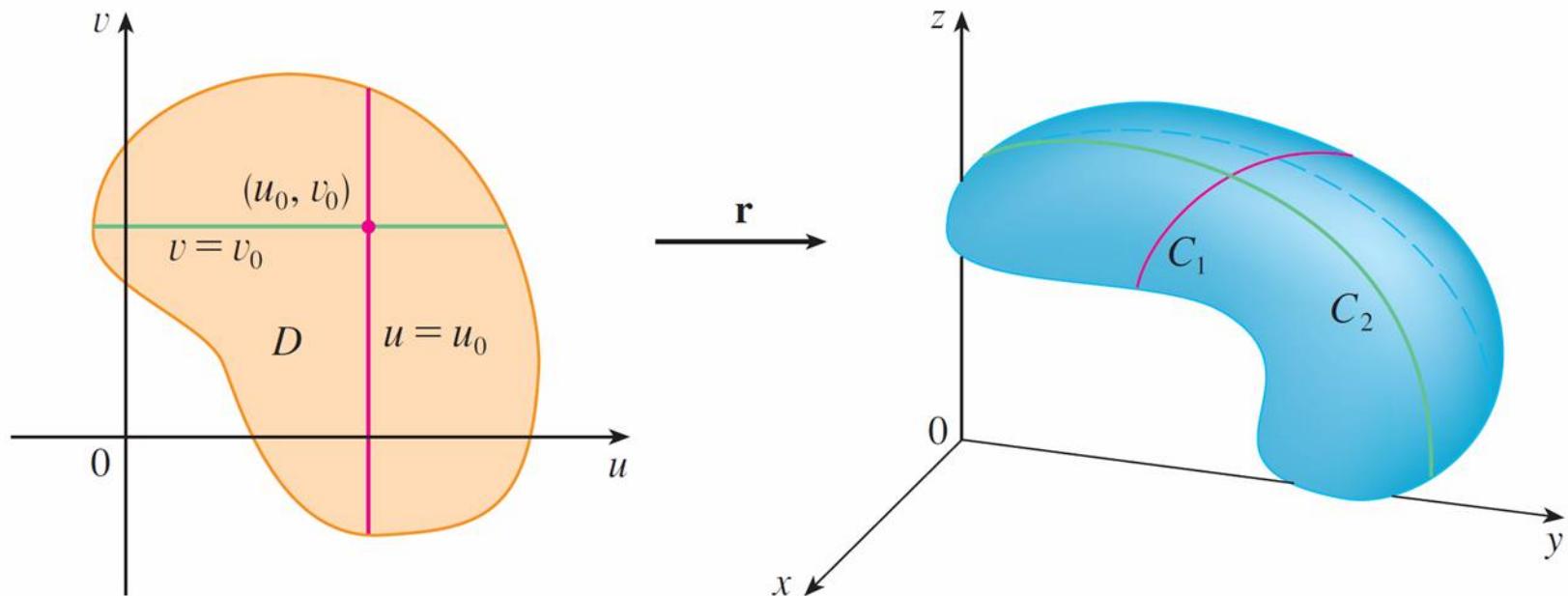


Figure 4

Parametric Surfaces

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S .

We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.)

In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves, as we see in the next example.

Example 4

Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

Solution:

The sphere has a simple representation $\rho = a$ in spherical coordinates, so let's choose the angles ϕ and θ in spherical coordinates as the parameters.

Example 4 – Solution

cont'd

Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates, we obtain

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

as the parametric equations of the sphere.

The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

Example 4 – Solution

cont'd

We have $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$, so the parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$.

The grid curves with ϕ constant are the circles of constant latitude (including the equator).

The grid curves with θ constant are the meridians (semi-circles), which connect the north and south poles (see Figure 7).

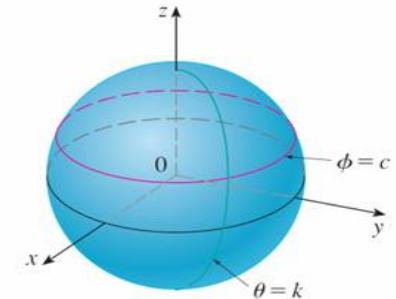
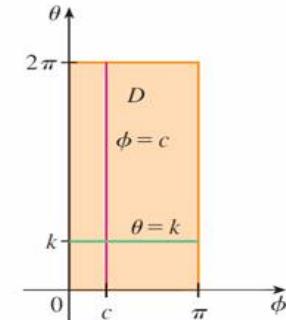


Figure 7

Parametric Surfaces

Note:

We saw in Example 4 that the grid curves for a sphere are curves of constant latitude and longitude.

For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude.

Describing a point on a parametric surface (like the one in Figure 5) by giving specific values of u and v is like giving the latitude and longitude of a point.

Surfaces of Revolution

Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x -axis, where $f(x) \geq 0$.

Let θ be the angle of rotation as shown in Figure 10.

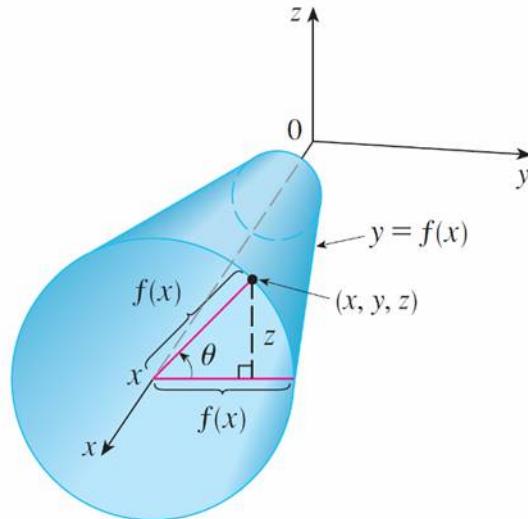


Figure 10

Surfaces of Revolution

If (x, y, z) is a point on S , then

3

$$x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta$$

Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S .

The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

Example 8

Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leq x \leq 2\pi$, about the x -axis. Use these equations to graph the surface of revolution.

Solution:

From Equations 3, the parametric equations are

$$x = x \quad y = \sin x \cos \theta \quad z = \sin x \sin \theta$$

and the parameter domain is $0 \leq x \leq 2\pi$, $0 \leq \theta \leq 2\pi$.

Example 8 – Solution

cont'd

Using a computer to plot these equations and rotate the image, we obtain the graph in Figure 11.

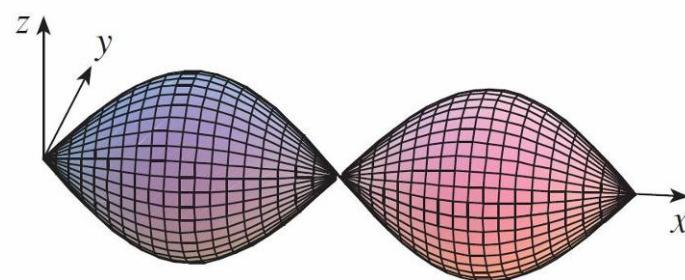


Figure 11

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

Tangent Planes

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S . (See Figure 12.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of \mathbf{r} with respect to v :

$$4 \quad \mathbf{r}_v = \frac{\partial x}{\partial v} (u_0, v_0) \mathbf{i} + \frac{\partial y}{\partial v} (u_0, v_0) \mathbf{j} + \frac{\partial z}{\partial v} (u_0, v_0) \mathbf{k}$$

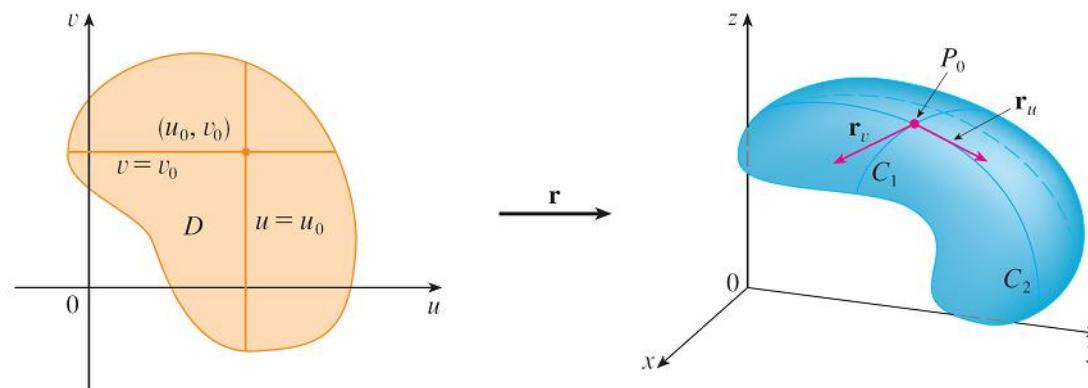


Figure 12

Tangent Planes

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S , and its tangent vector at P_0 is

$$5 \quad \mathbf{r}_u = \frac{\partial x}{\partial u} (u_0, v_0) \mathbf{i} + \frac{\partial y}{\partial u} (u_0, v_0) \mathbf{j} + \frac{\partial z}{\partial u} (u_0, v_0) \mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth** (it has no “corners”).

For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors \mathbf{r}_u and \mathbf{r}_v , and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example 9

Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

Solution:

We first compute the tangent vectors:

$$\begin{aligned}\mathbf{r}_u &= \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \\ &= 2u \mathbf{i} + \mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_v &= \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k} \\ &= 2v \mathbf{j} + 2 \mathbf{k}\end{aligned}$$

Example 9 – Solution

cont'd

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix}$$
$$= -2v \mathbf{i} - 4u \mathbf{j} + 4uv \mathbf{k}$$

Notice that the point $(1, 1, 3)$ corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2 \mathbf{i} - 4 \mathbf{j} + 4 \mathbf{k}$$

Example 9 – Solution

cont'd

Therefore an equation of the tangent plane at $(1, 1, 3)$ is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

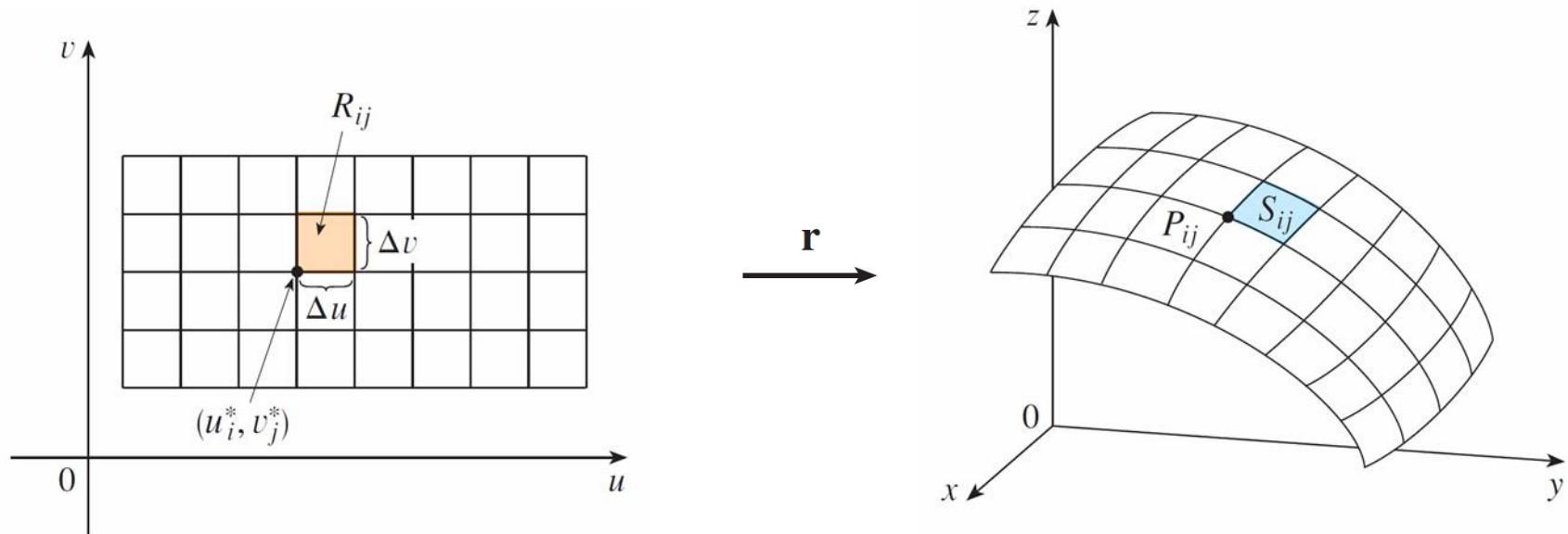
or

$$x + 2y - 2z + 3 = 0$$

Surface Area

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij} .

Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij} .



The image of the subrectangle R_{ij} is the patch S_{ij} .

Surface Area

The part of S_{ij} the surface that corresponds to R_{ii} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners.

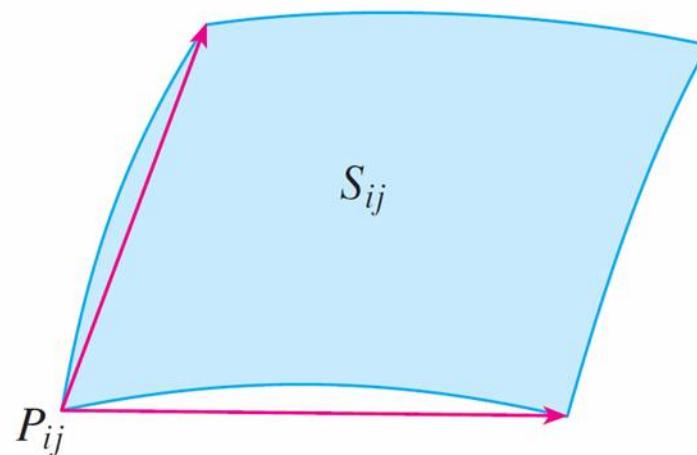
Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} .

Surface Area

Figure 15(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients.



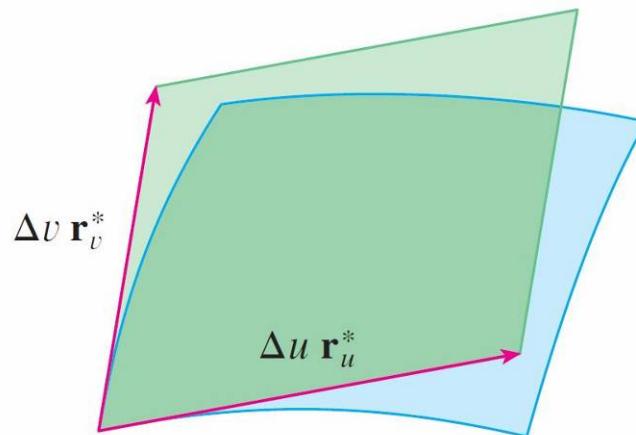
Approximating a patch by a parallelogram.

Figure 15(a)

Surface Area

So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$.

This parallelogram is shown in Figure 15(b) and lies in the tangent plane to S at P_{ij} .



Approximating a patch by a parallelogram.

Figure 15(b)

Surface Area

The area of this parallelogram is

$$|(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

and so an approximation to the area of S is

$$\sum_{i=1}^m \sum_{j=1}^n |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

This approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv.$$

Surface Area

This motivates the following definition.

6

Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D , then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}$ $\mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$

Example 10

Find the surface area of a sphere of radius a .

Solution:

In Example 4 we found the parametric representation

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

Example 10 – Solution

cont'd

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix}$$

Example 10 – Solution

cont'd

$$= a^2 \sin^2\phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}$$

Thus

$$\begin{aligned} |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4\phi \cos^2\theta + a^4 \sin^4\phi \sin^2\theta + a^4 \sin^2\phi \cos^2\phi} \\ &= \sqrt{a^4 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi} \\ &= a^2 \sqrt{\sin^2\phi} \\ &= a^2 \sin \phi \end{aligned}$$

since $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$.

Example 10 – Solution

cont'd

Therefore, by Definition 6, the area of the sphere is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA \\ &= \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi \, d\theta \\ &= a^2 \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi \\ &= a^2(2\pi)2 \\ &= 4\pi a^2 \end{aligned}$$

Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters.

The parametric equations are

$$x = x \quad y = y \quad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x} \right) \mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y} \right) \mathbf{k}$$

and

7

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}$$

Surface Area of the Graph of a Function

Thus we have

$$8 \quad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area formula in Definition 6 becomes

$$9 \quad A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

Example 11

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution:

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3.
(See Figure 16.)

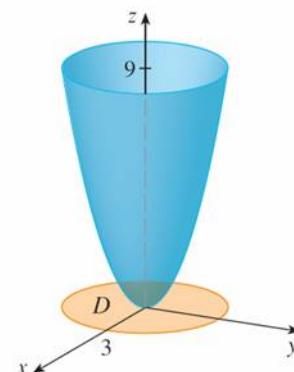


Figure 16

Example 11 – Solution

cont'd

Using Formula 9, we have

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)} dA \end{aligned}$$

Example 11 – Solution

cont'd

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r \sqrt{1 + 4r^2} dr$$

$$= 2\pi \left(\frac{1}{8} \right) \frac{2}{3} (1 + 4r^2)^{3/2} \Big|_0^3$$

$$= \frac{\pi}{6} (37\sqrt{37} - 1)$$

13.7 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

Suppose f is a function of three variables whose domain includes a surface S .

We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S .

We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Suppose that a surface has a vector equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv .

Then the surface S is divided into corresponding patches S_{ij} as in Figure 1.

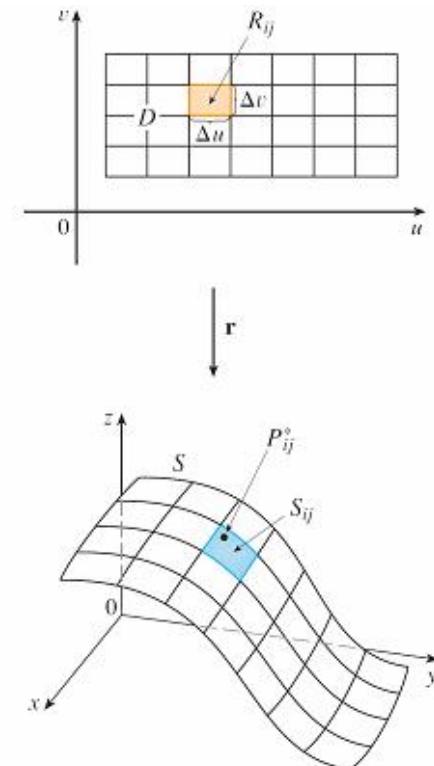


Figure 1

Parametric Surfaces

We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

1
$$\iint_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Notice the analogy with the definition of a line integral and also the analogy with the definition of a double integral.

Parametric Surfaces

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane.

In our discussion of surface area we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

are the tangent vectors at a corner of S_{ij} .

Parametric Surfaces

If the components are continuous and \mathbf{r}_u and \mathbf{r}_v are nonzero and nonparallel in the interior of D , it can be shown from Definition 1, even when D is not a rectangle, that

2

$$\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z) \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt$$

Parametric Surfaces

Observe also that

$$\iint_S 1 \, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D .

When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

Example 1

Compute the surface integral $\iint_S x^2 dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

We use the parametric representation

$$\begin{aligned}x &= \sin \phi \cos \theta & y &= \sin \phi \sin \theta & z &= \cos \phi & 0 \leq \phi \leq \pi \\0 \leq \theta &\leq 2\pi\end{aligned}$$

That is,

$$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$$

We can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin \phi$$

Example 1 – Solution

cont'd

Therefore, by Formula 2,

$$\begin{aligned}\iint_S x^2 dS &= \iint_D (\sin \phi \cos \theta)^2 |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA \\&= \int_0^{2\pi} \int_0^\pi \sin^2 \phi \cos^2 \theta \sin \phi d\phi d\theta \\&= \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi \\&= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta \int_0^\pi (\sin \phi - \sin \phi \cos^2 \phi) d\phi \\&= \frac{1}{2} [\theta + \frac{1}{2} \sin 2\theta]_0^{2\pi} [-\cos \phi + \frac{1}{3} \cos^3 \phi]_0^\pi \\&= \frac{4\pi}{3}\end{aligned}$$

Parametric Surfaces

If a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z) \, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_S x \rho(x, y, z) \, dS \quad \bar{y} = \frac{1}{m} \iint_S y \rho(x, y, z) \, dS \quad \bar{z} = \frac{1}{m} \iint_S z \rho(x, y, z) \, dS$$

Graphs

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \quad y = y \quad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x} \right) \mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y} \right) \mathbf{k}$$

Thus

3

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 + 1}$$

Graphs

Therefore, in this case, Formula 2 becomes

$$4 \quad \iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

Similar formulas apply when it is more convenient to project S onto the yz -plane or xz -plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection onto the xz -plane, then

$$\iint_S f(x, y, z) \, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1} \, dA$$

Example 2

Evaluate $\iint_S y \, dS$, where S is the surface $z = x + y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$. (See Figure 2.)

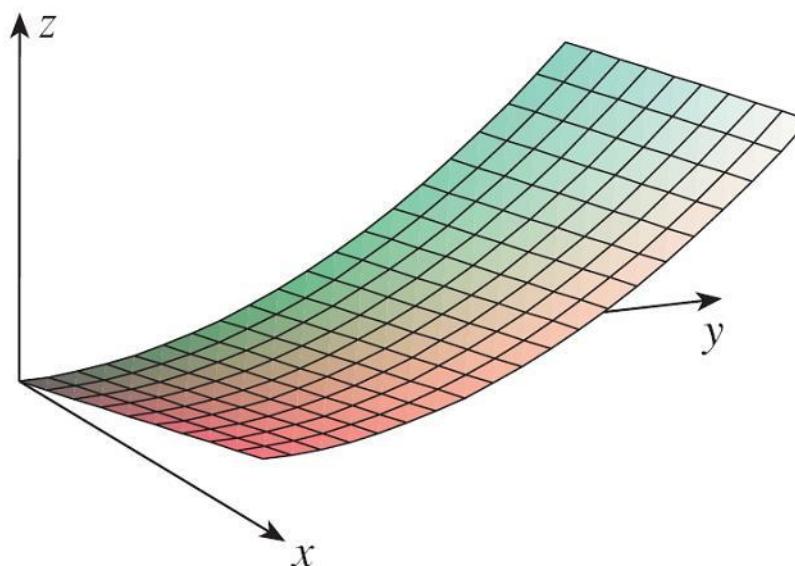


Figure 2

Example 2 – Solution

Since

$$\frac{\partial z}{\partial x} = 1 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned}\iint_S y \, dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\ &= \int_0^1 \int_0^2 y \sqrt{1 + 1 + 4y^2} \, dy \, dx \\ &= \int_0^1 dx \sqrt{2} \int_0^2 y \sqrt{1 + 2y^2} \, dy \\ &= \sqrt{2} \left(\frac{1}{4}\right)^{\frac{2}{3}} (1 + 2y^2)^{3/2} \Big|_0^2 = \frac{13\sqrt{2}}{3}\end{aligned}$$

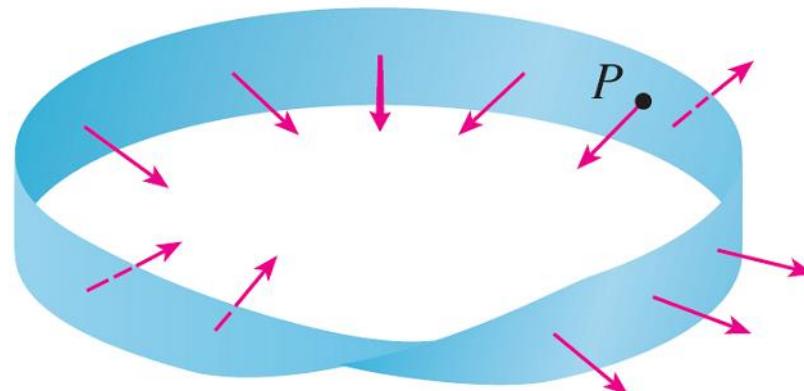
Graphs

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces S_1, S_2, \dots, S_n that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z) \, dS = \iint_{S_1} f(x, y, z) \, dS + \cdots + \iint_{S_n} f(x, y, z) \, dS$$

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).]

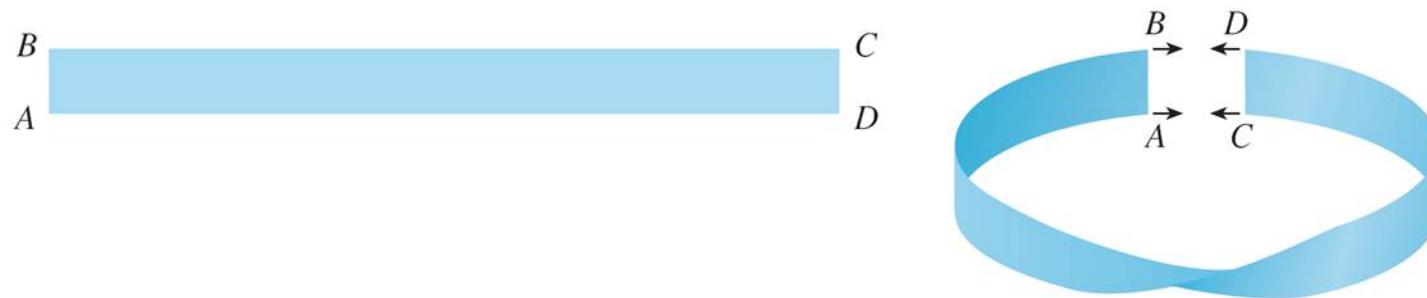


A Möbius strip

Figure 4

Oriented Surfaces

You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5.



Constructing a Möbius strip

Figure 5

Oriented Surfaces

If an ant were to crawl along the Möbius strip starting at a point P , it would end up on the “other side” of the strip (that is, with its upper side pointing in the opposite direction).

Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.)

Therefore a Möbius strip really has only one side.

From now on we consider only orientable (two-sided) surfaces.

Oriented Surfaces

We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).

There are two unit normal vectors \mathbf{n}_1 and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z) . (See Figure 6.)

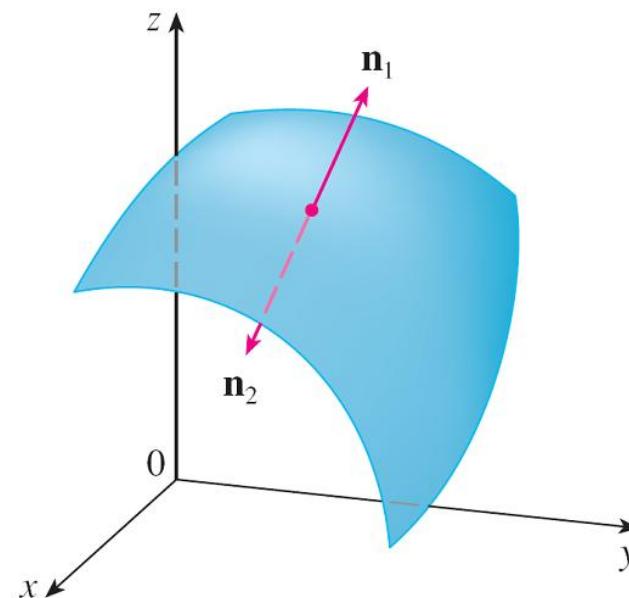
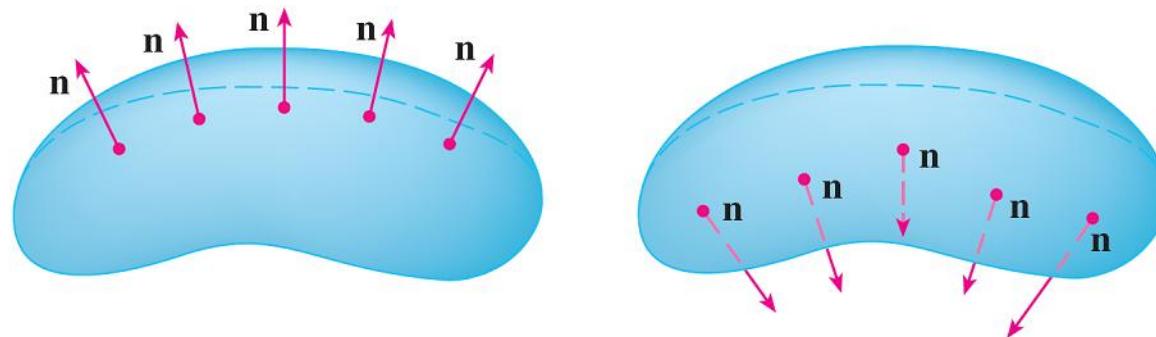


Figure 6

Oriented Surfaces

If it is possible to choose a unit normal vector \mathbf{n} at every such point (x, y, z) so that \mathbf{n} varies continuously over S , then S is called an **oriented surface** and the given choice of \mathbf{n} provides S with an **orientation**.

There are two possible orientations for any orientable surface (see Figure 7).



The two orientations of an orientable surface

Figure 7

Oriented Surfaces

For a surface $z = g(x, y)$ given as the graph of g , we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \quad \mathbf{n} = \frac{-\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

Oriented Surfaces

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

6

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$.

For instance, the parametric representation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$.

Oriented Surfaces

We know that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} = \frac{1}{a} \mathbf{r}(\phi, \theta)$$

Observe that \mathbf{n} points in the same direction as the position vector, that is, outward from the sphere (see Figure 8).

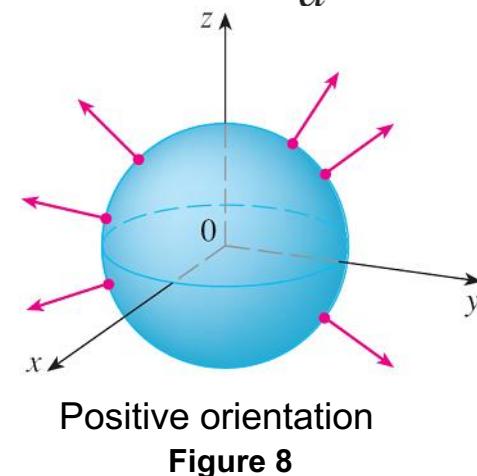
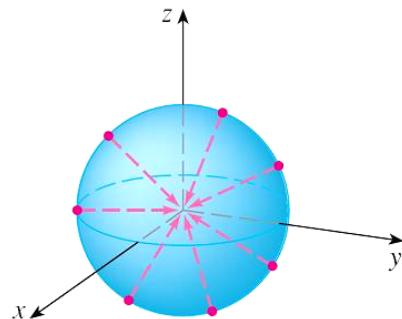


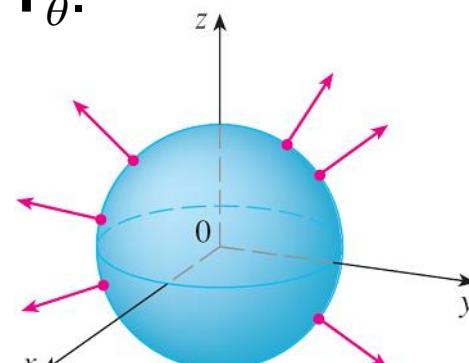
Figure 8

Oriented Surfaces

The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.



Negative orientation
Figure 9



Positive orientation
Figure 8

For a **closed surface**, that is, a surface that is the boundary of a solid region E , the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E , and inward-pointing normals give the negative orientation (see Figures 8 and 9).

Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector \mathbf{n} , and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing *through* S . (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.)

Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

Surface Integrals of Vector Fields

We divide S into small patches S_{ij} , as in Figure 10 (compare with Figure 1).

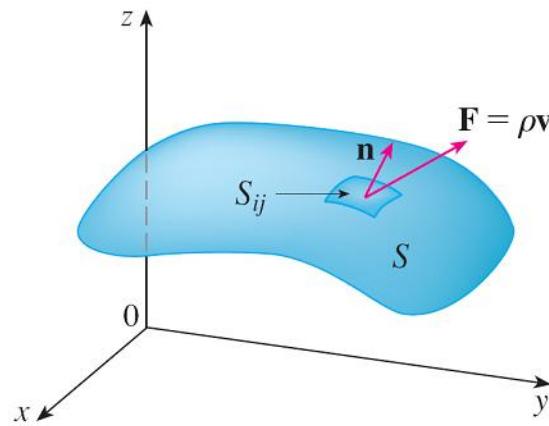


Figure 10

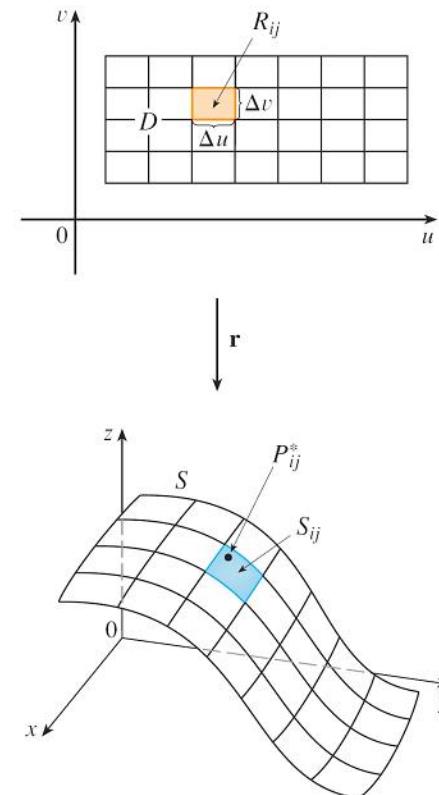


Figure 1

Surface Integrals of Vector Fields

Then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal \mathbf{n} by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where ρ , \mathbf{v} , and \mathbf{n} are evaluated at some point on S_{ij} .
(Recall that the component of the vector $\rho \mathbf{v}$ in the direction of the unit vector \mathbf{n} is $\rho \mathbf{v} \cdot \mathbf{n}$.)

Surface Integrals of Vector Fields

By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho \mathbf{v} \cdot \mathbf{n}$ over S :

7

$$\iint_S \rho \mathbf{v} \cdot \mathbf{n} \, dS = \iint_S \rho(x, y, z) \mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z) \, dS$$

and this is interpreted physically as the rate of flow through S .

If we write $\mathbf{F} = \rho \mathbf{v}$, then \mathbf{F} is also a vector field on \mathbb{R}^3 and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

Surface Integrals of Vector Fields

A surface integral of this form occurs frequently in physics, even when \mathbf{F} is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of \mathbf{F} over S .

8 Definition If \mathbf{F} is a continuous vector field defined on an oriented surface S with unit normal vector \mathbf{n} , then the **surface integral of \mathbf{F} over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This integral is also called the **flux** of \mathbf{F} across S .

In words, Definition 8 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S .

Surface Integrals of Vector Fields

If S is given by a vector function $\mathbf{r}(u, v)$, then \mathbf{n} is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| dA\end{aligned}$$

where D is the parameter domain. Thus we have

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$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

Example 4

Find the flux of the vector field $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$$

$$0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

Then

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos \phi \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \sin \phi \cos \theta \mathbf{k}$$

and,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k}$$

Example 4 – Solution

cont'd

Therefore $\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta$ and, by Formula 9, the flux is

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) dA \\ &= \int_0^{2\pi} \int_0^\pi (2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta) d\phi d\theta \\ &= 2 \int_0^\pi \sin^2 \phi \cos \phi d\phi \int_0^{2\pi} \cos \theta d\theta + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \\ &= 0 + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \quad \left(\text{since } \int_0^{2\pi} \cos \theta d\theta = 0 \right) \\ &= \frac{4\pi}{3}\end{aligned}$$

by the same calculation as in Example 1.

Surface Integrals of Vector Fields

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right)$$

Surface Integrals of Vector Fields

Thus Formula 9 becomes

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$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of S ; for a downward orientation we multiply by -1 .

Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$.

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.

Surface Integrals of Vector Fields

For instance, if \mathbf{E} is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of \mathbf{E} through the surface S . One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

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$$Q = \epsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ϵ_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\epsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$.)

Surface Integrals of Vector Fields

Therefore, if the vector field \mathbf{F} in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3} \pi \varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance.

Surface Integrals of Vector Fields

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

13.8 Stokes' Theorem

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve).

Figure 1 shows an oriented surface with unit normal vector \mathbf{n} .

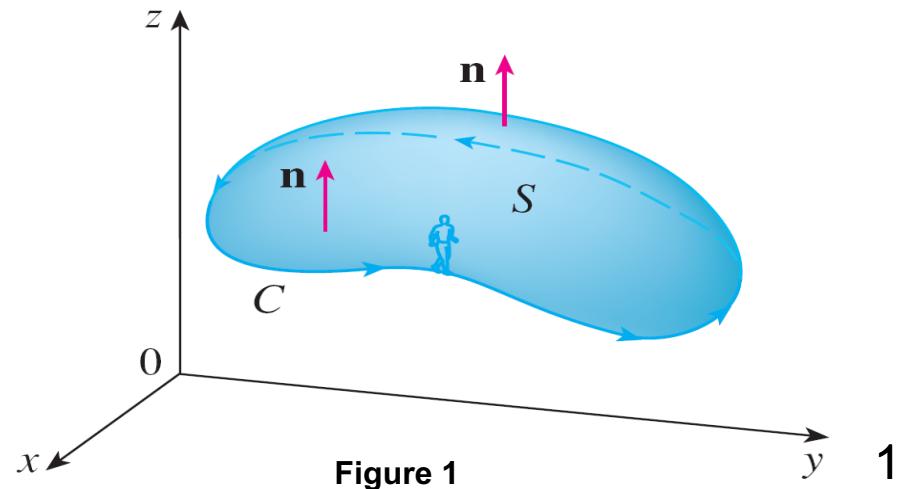


Figure 1

Stokes' Theorem

The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure.

This means that if you walk in the positive direction around C with your head pointing in the direction of \mathbf{n} , then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds \quad \text{and} \quad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} \, dS$$

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of \mathbf{F} is equal to the surface integral over S of the normal component of the curl of \mathbf{F} .

The positively oriented boundary curve of the oriented surface S is often written as ∂S , so Stokes' Theorem can be expressed as

1

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Stokes' Theorem

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus.

As before, there is an integral involving derivatives on the left side of Equation 1 (recall that $\text{curl } \mathbf{F}$ is a sort of derivative of \mathbf{F}) and the right side involves the values of \mathbf{F} only on the *boundary* of S .

Stokes' Theorem

In fact, in the special case where the surface S is flat and lies in the xy -plane with upward orientation, the unit normal is \mathbf{k} , the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$$

This is precisely the vector form of Green's Theorem.

Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Example 1

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

Solution:

The curve C (an ellipse) is shown in Figure 3.

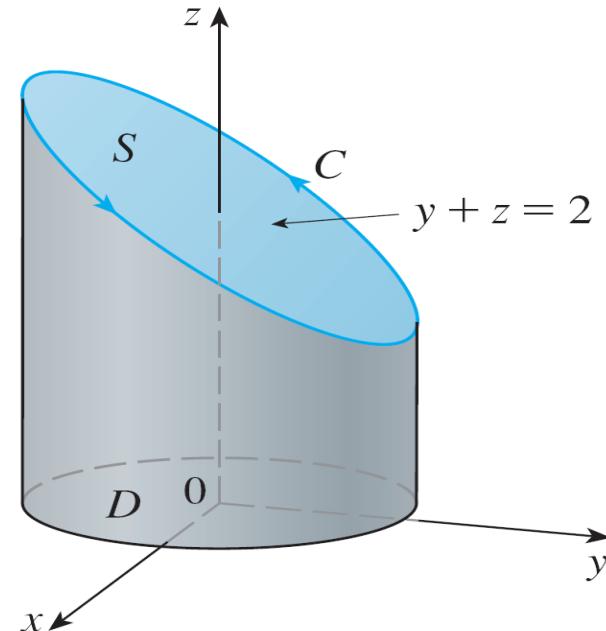


Figure 3

Example 1 – Solution

cont'd

Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem.

We first compute

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y) \mathbf{k}$$

Although there are many surfaces with boundary C , the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C .

Example 1 – Solution

cont'd

If we orient S upward, then C has the induced positive orientation.

The projection D of S onto the xy -plane is the disk $x^2 + y^2 \leq 1$ and so using equation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y) dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r dr d\theta$$

Example 1 – Solution

cont'd

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2 \frac{r^3}{3} \sin \theta \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta$$

$$= \frac{1}{2}(2\pi) + 0$$

$$= \pi$$

Stokes' Theorem

We know that \mathbf{F} is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C . Given C , suppose we can find an orientable surface S whose boundary is C .

Then Stokes' Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0.

Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C .

13.9 The Divergence Theorem

We write Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

where C is the positively oriented boundary curve of the plane region D .

If we were seeking to extend this theorem to vector fields on \mathbb{R}^3 , we might make the guess that

1

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z) \, dV$$

where S is the boundary surface of the solid region E .

The Divergence Theorem

It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem.

Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ($\operatorname{div} \mathbf{F}$ in this case) over a region to the integral of the original function \mathbf{F} over the boundary of the region.

We state the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.)

The Divergence Theorem

The boundary of E is a closed surface, and we use the convention, that the positive orientation is outward; that is, the unit normal vector \mathbf{n} is directed outward from E .

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E , given with positive (outward) orientation. Let \mathbf{F} be a vector field whose component functions have continuous partial derivatives on an open region that contains E . Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of \mathbf{F} across the boundary surface of E is equal to the triple integral of the divergence of \mathbf{F} over E .

Example 1

Find the flux of the vector field $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

First we compute the divergence of \mathbf{F} :

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$.

Example 1 – Solution

cont'd

Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \operatorname{div} \mathbf{F} \, dV$$

$$= \iiint_B 1 \, dV$$

$$= V(B)$$

$$= \frac{4}{3} \pi (1)^3$$

$$= \frac{4\pi}{3}$$

The Divergence Theorem

Let's consider the region E that lies between the closed surfaces S_1 and S_2 , where S_1 lies inside S_2 . Let \mathbf{n}_1 and \mathbf{n}_2 be outward normals of S_1 and S_2 .

Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal \mathbf{n} is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2 .
(See Figure 3.)

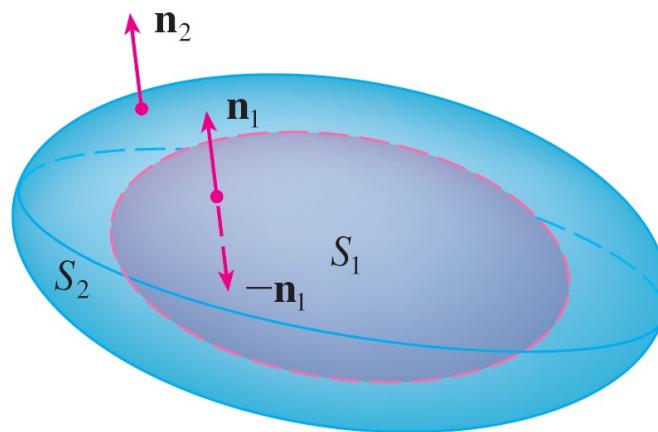


Figure 3

The Divergence Theorem

Applying the Divergence Theorem to S , we get

$$\begin{aligned} 7 \quad \iiint_E \operatorname{div} \mathbf{F} dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_S \mathbf{F} \cdot \mathbf{n} dS \\ &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 dS \\ &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \end{aligned}$$

Example 3

We considered the electric field:

$$\mathbf{E}(\mathbf{x}) = \frac{\epsilon Q}{|\mathbf{x}|^3} \mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector.

Use the Divergence Theorem to show that the electric flux of \mathbf{E} through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\epsilon Q$$

Example 3 – Solution

The difficulty is that we don't have an explicit equation for S_2 because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so we let S_1 be a small sphere with radius a and center the origin. You can verify that $\operatorname{div} \mathbf{E} = 0$.

Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E} dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} dS$$

Example 3 – Solution

cont'd

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at \mathbf{x} is $\mathbf{x}/|\mathbf{x}|$.

Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) = \frac{\varepsilon Q}{|\mathbf{x}|^4} \mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$.

Example 3 – Solution

cont'd

Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS = \frac{\varepsilon Q}{a^2} \iint_{S_1} dS = \frac{\varepsilon Q}{a^2} A(S_1) = \frac{\varepsilon Q}{a^2} 4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of \mathbf{E} is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss's Law for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.]

The Divergence Theorem

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ . Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area.

The Divergence Theorem

If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a , then $\operatorname{div} \mathbf{F}(P) \approx \operatorname{div} \mathbf{F}(P_0)$ for all points in B_a since $\operatorname{div} \mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F} dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0) dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \rightarrow 0$ and suggests that

8

$$\operatorname{div} \mathbf{F}(P_0) = \lim_{a \rightarrow 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

The Divergence Theorem

Equation 8 says that $\operatorname{div} \mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0 . (This is the reason for the name *divergence*.)

If $\operatorname{div} \mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**.

If $\operatorname{div} \mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

The Divergence Theorem

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_2 .

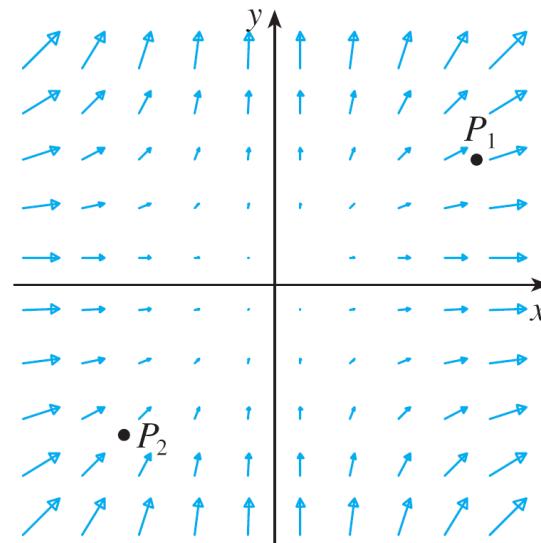


Figure 4

The vector field $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j}$

The Divergence Theorem

Thus the net flow is outward near P_1 , so $\operatorname{div} \mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2 , on the other hand, the incoming arrows are longer than the outgoing arrows.

Here the net flow is inward, so $\operatorname{div} \mathbf{F}(P_2) < 0$ and P_2 is a sink.

We can use the formula for \mathbf{F} to confirm this impression. Since $\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j}$, we have $\operatorname{div} \mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

Summary

We collect all higher-dimensional versions of the Fundamental Theorem of Calculus together here (without hypotheses) so that you can see more easily their essential similarity.

Notice that in each case we have an integral of a “derivative” over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

Fundamental Theorem of Calculus

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

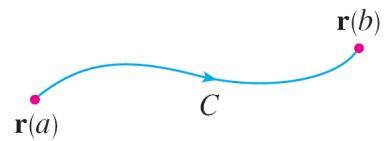


Summary

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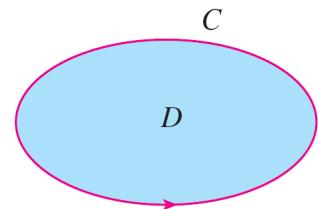
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P \, dx + Q \, dy$$

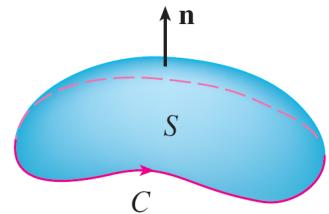


Summary

cont'd

Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

