1. Find the area of the parallelogram with vertices $A(-2,1)$, $B(0,4)$, $C(4,2)$, and $D(2,-1)$.

   Solution. (10.4 #27) We find vectors forming the sides of the parallelogram. We can let $\mathbf{v} = \overrightarrow{AB} = \langle 2,3\rangle$ and $\mathbf{u} = \overrightarrow{AD} = \langle 4,-2\rangle$. Then the area of the parallelogram is $|\mathbf{v}\times\mathbf{u}|$, which is the determinant of the matrix with $\mathbf{v}$ and $\mathbf{u}$ as rows. That is,

$$A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} = 4(3) - 2(-2) = \boxed{14}$$

$\square$

2. Find an equation of the plane through the point $(1,-1,-1)$ and parallel to the plane $5x-y-z=6$.

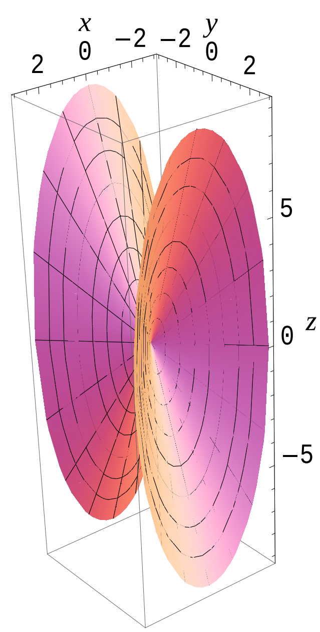
   Solution. (10.5 #23) We need a point and the normal vector. In this case we have the point, and the normal vector can be read off the equation for the parallel plane. The plane $5x-y-z=6$ has normal vector $\mathbf{n} = \langle 5,-1,-1\rangle$. The parallel plane has normal in the same direction, so we can take the same normal. The equation for the plane is therefore,

$$5(x-1)-(y+1)-(z+1)=0 \quad \text{or} \quad \boxed{5x-y-z=7}$$

$\square$

3. Classify and sketch the surface $y^2 = x^2 + \frac{1}{9}z^2$.

   (10.6 #21) This is an elliptical cone. The cone opens in the $y-$direction. The cross-sections parallel to the $xz-$plane are ellipses that are wider in the $z-$direction.



4. For the curve

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\,\sin 2t\,\mathbf{k}$$

(a) Find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = 0$.

Solution. (10.7 #45) We first find the tangent vector: $\mathbf{r}'(t) = \langle -\sin t, 3, 4\cos 2t\rangle$. The tangent vector at $t = 0$ is $\mathbf{r}'(0) = \langle 0, 3, 4\rangle$. To find the unit tangent, we find the magnitude: $|\mathbf{r}'(0)| = \sqrt{0^2 + 3^2 + 4^2} = 5$. Then the unit tangent is

$$\boxed{\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle}$$

□

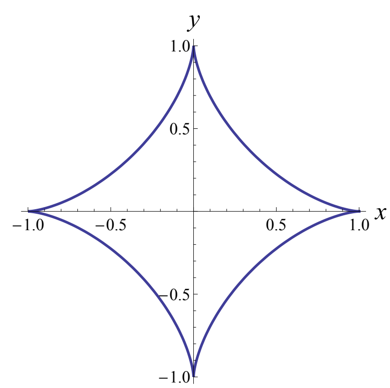
(b) Find the parametric equations for the tangent line to the curve at $t = 0$.

Solution. The tangent line is $\mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is a vector to a point on the curve and $\mathbf{v}$ is a tangent vector at the point. We can use $\mathbf{v} = \mathbf{r}'(0)$ for the tangent vector. For the point, we have $\mathbf{r}(0) = \langle 1, 0, 0\rangle$. Then the line is $\langle 1, 0, 0\rangle + t\langle 0, 3, 4\rangle = \langle 1, 3t, 4t\rangle$. The parametric equations are

$$\boxed{x = 1,\ y = 3t,\ z = 4t}$$

□

5. The hypocycloid (shown below) is the curve parametrized by $x = \cos^3 t$, $y = \sin^3 t$. Find the length of the curve.



Solution. First, we note that the curve is traversed once as t goes from 0 to 2π. The length of the four parts of the curve in each quadrant are equal, so we can find the total length by calculating any of these and multiplying by four. So we calculate, first of all, $|\mathbf{r}'(t)|$:

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -3\cos^2 t\,\sin t, 3\sin^2 t\,\cos t\rangle \\
|\mathbf{r}'(t)| &= \sqrt{(-3\cos^2 t\,\sin t)^2 + (3\sin^2 t\,\cos t)^2} \\
&= 3\sqrt{\cos^4 t\,\sin^2 t + \sin^4 t\,\cos^2 t} \\
&= 3\sqrt{\cos^2 t\,\sin^2 t(\cos^2 t + \sin^2 t)} \\
&= 3|\cos t\,\sin t|
\end{aligned}$$

To find the total length we will find the length of the piece in the first quadrant and multiply by four:

$$
\begin{aligned}
L &= 4\int_0^{\pi/2} |\mathbf{r}'(t)|\,dt \\
&= 4\int_0^{\pi/2} 3\cos t\,\sin t\,dt \quad u-\text{sub } u=\sin t \\
&= 4\int_0^{1} 3u\,du \\
&= 12\frac{1}{2}u^2|_0^1 \\
&= \boxed{6}
\end{aligned}
$$

$\square$

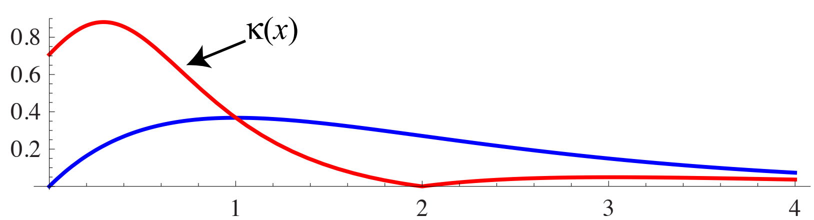
6. (a) Compute the curvature of the plane curve $y = x\,e^{-x}$ for $x \geq 0$.

Solution. The curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1+(y'(x))^2]^{3/2}}$, so we need to calculate the derivatives.

$$y'(x) = e^{-x}(1-x) \qquad \text{and} \qquad y''(x) = e^{-x}(x-2)$$

So, $\boxed{\kappa(x) = \dfrac{e^{-x}|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}}$. $\square$

(b) The curve $y = x\,e^{-x}$ and the curvature of this curve are shown on the graph below. Identify which is which.



(c) At what point does the curve $y = x\,e^{-x}$ have minimal curvature?

Solution. The smallest the curvature can be is zero. For this curve, this minimum is achieved at $\boxed{x=2}$. $\square$

BONUS Given nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, do the equations $\mathbf{a}\times\mathbf{c} = \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{c} = |\mathbf{a}|$ uniquely determine the vector $\mathbf{c}$?

Solution. YES. The reason is as follows. Let θ be the angle between $\mathbf{a}$ and $\mathbf{c}$. The first equation implies $|\mathbf{a}|\cdot|\mathbf{c}|\sin\theta = |\mathbf{b}|$. The second equation implies $|\mathbf{a}|\cdot|\mathbf{c}|\cos\theta = |\mathbf{a}|$. Dividing these equations gives $\tan\theta = |\mathbf{b}|/|\mathbf{a}|$, and the second equation implies $\cos\theta \geq 0$. Thus, the angle between $\mathbf{a}$ and $\mathbf{c}$ is uniquely determined. The two equations also determine the length of $\mathbf{c}$. Since the length and direction of $\mathbf{c}$ are determined, the vector is determined. $\square$