1. Find the equation of (a) the tangent plane and (b) the normal line (normal to the tangent plane) to the surface at the point

$$x + y + z = e^{xyz}, \qquad (0, 0, 1)$$

Solution. (a) Write the equation as

$$f(x, y, z) = x + y + z - e^{xyz} = 0$$

The normal to the surface is the gradient, so we calculate

$$\nabla f(x, y, z) = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle$$

The normal at the point $(0, 0, 1)$ is

$$\mathbf{n} = \nabla f(0, 0, 1) = \langle 1, 1, 1 \rangle$$

Therefore, the equation for the tangent plane is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$, or

$$\boxed{x + y + z - 1 = 0}$$

(b) The normal line is $\mathbf{r}_0 + t\mathbf{n}$, or

$$\langle 0, 0, 1 \rangle + t\langle 1, 1, 1 \rangle = \langle t, t, 1 + t \rangle$$

The parametric equations for the line are $\boxed{x = t,\ y = t,\ z = 1 + t}$. □

2. (a) Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}$$

Solution. We can show that f approaches different values along different paths to the origin, and therefore the limit does not exist. Along the line $x = 0$,

$$f(0, y) = \frac{-4y^2}{2y^2} = -2$$

Along the line $y = 0$,

$$f(x, 0) = \frac{x^4}{x^2} = x^2 \to 0$$

as $(x, y) \to (0, 0)$. So the limit along $x = 0$ is 2, but the limit along $y = 0$ is 0. Therefore, <u>the limit does not exist</u>. □

(b) Determine the set of points at which the following function is continuous.

$$f(x, y) = \begin{cases} \dfrac{x^4 - 4y^2}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution. Since the limit does not exist at $(0, 0)$, the function is not continuous there. This is the only point where a discontinuity could exist, so the set where the function is continuous is $\boxed{\{(x, y) : (x, y) \neq (0, 0)\}}$. □

3. Find the maximum rate of change of

$$f(x,y) = x^2y + \sqrt{y}$$

at the point $(2,1)$. In which direction does it occur?

Solution. The maximal rate of change is the norm of the gradient, and occurs in the direction of the gradient, so we calculate

$$\nabla f(x,y) = \left\langle 2xy, x^2 + \frac{1}{2\sqrt{y}} \right\rangle$$

So, the maximal rate of change at $(2,1)$ occurs in the direction $\boxed{\nabla f(2,1) = \left\langle 4, 4\frac{1}{2} \right\rangle}$, and the maximal rate of change is $\boxed{|\nabla f(2,1)| = \frac{\sqrt{145}}{2}}$. □

4. Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 + 4x - 4y$ on the disk $x^2 + y^2 \leq 9$.

Solution. We need to find the critical points inside the boundary, and on the boundary. It is probably easiest to complete the squares and write

$$f(x,y) = (x+2)^2 + (y-2)^2 - 8$$

This is a paraboloid with vertex at $(-2,2,-8)$, so there is a single critical point in the interior of D, namely $(-2,2)$.

To find the critical points on the boundary, we can use Lagrange multipliers. We write the boundary as $g(x,y) = x^2 + y^2 = 9$. Then the extrema of f constrained to $g = 9$ are solutions of $\nabla f = \lambda \nabla f$. This gives us the three equations

$$\begin{aligned} f_x &= \lambda g_x \qquad & 2(x+2) &= \lambda 2x \qquad (1)\\ f_y &= \lambda g_x \qquad & 2(y-2) &= \lambda 2y \qquad (2)\\ g(x,y) &= 9 \qquad & x^2 + y^2 &= 9 \qquad (3) \end{aligned}$$

Equations (1) and (2) can be solved for λ and equated:

$$\lambda = \frac{x+2}{x} = \frac{y-2}{y}$$

Solve for y to get $y = -x$. Now we substitute into equation (3):

$$x^2 + (-x)^2 = 9 \quad \Rightarrow \quad x^2 = \frac{9}{2} \quad \Rightarrow \quad x = \pm\frac{3}{\sqrt{2}}$$

So there are three critical points: $(-2,2), (3/\sqrt{2}, -3/\sqrt{2}), (-3/\sqrt{2}, 3/\sqrt{2})$. We evaluate f at each of these points:

$$f(-2,2) = -8,\ f(3/\sqrt{2}, -3/\sqrt{2}) = 9 + 12\sqrt{2},\ f(-3/\sqrt{2}, 3/\sqrt{2}) = 9 - 12\sqrt{2}$$

Therefore, the maximum is $\boxed{9 + 12\sqrt{2}}$, and the minimum is $\boxed{-8}$.

NOTE: You can also do this by parametrizing the boundary as $x = 3\cos t, y = 3\sin t$, and setting the derivative of $f(3\cos t, 3\sin t)$ equal to zero. □

5. Show that $u = \ln\sqrt{x^2+y^2}$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

Solution. To make things easier, use properties of logarithms to write $u = \frac{1}{2}\ln(x^2+y^2)$. We take the derivatives:

$$\begin{aligned}
u_x &= \frac{1}{2}\frac{1}{x^2+y^2}2x \\
&= \frac{x}{x^2+y^2} \\
u_{xx} &= \frac{(x^2+y^2) - x2x}{(x^2+y^2)^2} \\
&= \frac{y^2-x^2}{(x^2+y^2)^2} \\
u_y &= \frac{1}{2}\frac{1}{x^2+y^2}2y \\
&= \frac{y}{x^2+y^2} \\
u_{yy} &= \frac{(x^2+y^2) - y2y}{(x^2+y^2)^2} \\
&= \frac{x^2-y^2}{(x^2+y^2)^2}
\end{aligned}$$

Therefore,

$$u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0$$

□