MATH 250B - Linear Algebra & Differential Equations

Additional Practice Problems – Summer 2023

1. (a) Solve the following initial value problems:

$$a) y' + y = 1 + t^2 \quad y(2) = 3 \;\; b)\; y' + y\cos t = 0 \;\;,\;\; c)\; y' + y\sin t = 0, \;\; c)\; y' + \frac{2t}{1+t^2}y = \frac{1}{1+t^2}.$$

   (b) Find a continuous solution of the IVP

$$y' + y = g(t), \qquad y(0) = 0$$

   where $g(t) = \begin{cases} 2 & \text{if} \quad 0 \le t \le 1 \\ 0 & \text{if} \quad t > 1 \end{cases}$

   $y' + y = g(t), y(0) = 0,$

$$g(t) = \begin{cases} 2, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } t > 1. \end{cases}$$

   The integrating factor is $\mu(t) = e^{\int dt} = e^t \implies \frac{d}{dt}(e^t y) = e^t g(t) \implies [e^t y]_0^t = \int_0^t e^t f(t)dt \implies e^t y - y(0) = \int_0^t e^t g(t)dt \implies e^t y - 0 = \int_0^t e^t g(t)dt \implies y(t) = e^{-t}\int_0^t e^t g(t)dt.$
   If $0 \le t \le 1$, $\int_0^t e^t g(t)dt = 2\int_0^t e^t dt = 2(e^t - 1) \implies y(t) = 2e^{-t}(e^t - 1).$
   If $t > 1$, $\int_0^t e^t g(t)dt = \int_0^1 e^t g(t)dt + \int_1^t e^t g(t)dt = \int_0^1 e^t dt = e - 1 \implies y(t) = e^{-t}(e - 1).$

2. Let $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues of the matrix A.

   (b) Find all of vectors $v \neq 0$ such that $Av = \lambda v$ where λ is an eigenvalue of A. Such vector is called an eigenvector of A corresponding to the eigenvalue λ.

3. Find all values of k for which the matrix $\begin{pmatrix} 2 & k-3 & k^2 \\ 2 & 1 & 4 \\ 1 & k & 0 \end{pmatrix}$ is invertible.

4. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix}$. Find $\det(B)$ in terms of $\det(A)$.

5. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} a-5d & b-5e & c-5f \\ 3g & 3h & 3i \\ -d+3g & -e+3h & -f+3i \end{pmatrix}$. Find $\det(B)$ in terms of $\det(A)$.
   To obtain the given matrix from A, we add 5 times the middle row to the top row, we multiply the last row by 3, we multiply the middle row by -1, we add a multiple of the last row to the middle row, and we perform a row permutation. The combined effect of these operations is to multiply the determinant of A by (1) · (3) · (-1) · (-1) = 3. Hence, the given matrix has determinant $3\det(A)$.

6. Let $A = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 1 & 5 \\ 0 & -1 & 2 \end{pmatrix}$. Show that A is invertible and find its inverse, using the method of cofactors.

7. Consider the following system

$$\begin{cases} x_1 + 2x_2 - x_3 = 5, \\ 3x_1 + x_2 - 2x_3 = 9, \\ -x_1 + 4x_2 + 2x_3 = 0. \end{cases} \qquad (\star).$$

(a) Show that solving $(\star)$ is equivalent to solving $A\mathbf{x} = \mathbf{b}$, where A, $\mathbf{x}$ and $\mathbf{b}$ aree to be determined.

(b) Find the augmented matrix $A^\sharp$ associated with $(\star)$.

(c) Perform the Gaussian elimination on the augmented matrix $A^\sharp$.

(d) Find the solution of the linear system $(\star)$.

8. (a) Determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

i) $y' + 2y = 0$.

ii) $y'' + y' - 6y = 0$.

(b) Determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

i) $t^2y'' + 4ty' + 2y = 0$.

ii) $t^2y'' - 4ty' + 4y = 0$.

9. Verify that the given vector function x defines a solution to $x' = Ax + b$ for the given A and b.

1) $x = \begin{pmatrix} e^{4t} \\ -2e^{4t} \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2) $x = \begin{pmatrix} 2te^t + e^t \\ 2te^t - e^t \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 4e^t \end{pmatrix}$.

10. Write the vector formulation for the given system of differential equations:

1) $x_1' = -4x_1 + 3x_2 + 4t$, $x_2' = 6x_1 - 4x_2 + t^2$.

2) $x_1' = e^{2t}x_2$, $x_2' + \sin tx_1 = 1$.

3) $x_1' = t^2x_1 - tx_2$, $x_2' = -\sin tx_1 + x_2$.

11. Verify that for all values of s and t, $(s, s - 2t, 2s + 3t, t)$ is a solution to the linear system:

$$\begin{cases} x_1 + x_2 - x_3 + 5x_4 = 0, \\ 2x_2 - x_3 + 7x_4 = 0, \\ 4x_1 + 2x_2 - 3x_3 + 13x_4 = 0 \end{cases}$$

12. Find A^2, A^3 and A^4 if $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{pmatrix}$.

13. If $A = \begin{pmatrix} 2 & -5 \\ 6 & -6 \end{pmatrix}$, calculate A^2 and verify that $A^2 - 4A + 18I_2 = 0_2$.

14. Find numbers x, y, and z such that the matrix $A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ satisfies

$$A^2 + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = I_3.$$

15. Use the properties of the transpose to prove that
   (a) AA^T is a symmetric matrix.
   (b) $(ABC)^T = C^T B^T A^T$.

16. Verify that for all values of t, $(1-t, 2+3t, 3-2t)$ is a solution of the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 6, \\ x_1 - x_2 - 2x_3 = -7, \\ 5x_1 + x_2 - x_3 = 4. \end{cases}$$

17. Determine all values of the constants a and b for which the following system has (a) no solution, (b) an infinite number of solutions, and (c) a unique solution.

$$\begin{cases} x_1 - ax_2 = 3, \\ 2x_1 + x_2 = 6, \\ -3x_1 + (a+b)x_2 = 1 \end{cases}$$

18. Execute Gaussian elimination on $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & -6\alpha \\ 2\alpha & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ \beta \end{pmatrix}$.

Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left(\begin{array}{cc|c} 2 & -6\alpha & 3 \\ 2\alpha & -1 & \beta \end{array}\right) \overset{1}{\sim} \left(\begin{array}{cc|c} 1 & -3\alpha & 3/2 \\ 2\alpha & -1 & \beta \end{array}\right) \overset{2}{\sim} \left(\begin{array}{cc|c} 1 & -3\alpha & 3/2 \\ 0 & -1+6\alpha & \beta - 3\alpha \end{array}\right).$$

$$\boxed{\textbf{1. } M_1(1/2) \quad \textbf{2. } A_{12}(-2\alpha)}$$

• If $\alpha \neq \frac{1}{6}$, the system has a unique solution, obtained by back substitution. We obtain $x_2 = \frac{3\alpha-\beta}{6\alpha-1}$ and $x_1 = 3\left(\frac{1}{2} + \alpha\frac{3\alpha-\beta}{6\alpha-1}\right)$.

• If $\alpha = \frac{1}{6}$ and $\beta \neq \frac{1}{2}$, there is an inconsistency. The system has no solution.

• If $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, the last row leads to $0 = 0$. And the system has infinitely many solutions. Indeed, any couple (x_1, x_2) such that $x_1 = \frac{3}{2} + 3\alpha x_2$ is a solution of our system. The set of solution is therefore:

$$\mathcal{S} = \left\{ \left(\frac{3}{2} + 3\alpha t, t\right) : t \in \mathbb{R} \right\}.$$

19. Let $A = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{pmatrix}$. Show that $A\mathbf{x} = \mathbf{b}$ is not always solvable for all $\mathbf{b} \in \mathbb{R}^3$, and describe the set of $\mathbf{b} \in \mathbb{R}^3$ (as a plane or a line) such that the system is solvable.

20. Find all the values of h such that the following vector equation has solution(s),

$$x_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} h \\ -3 \\ -5 \end{pmatrix}.$$

21. Let $A = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{pmatrix}$. Show that $A\mathbf{x} = \mathbf{b}$ is not always solvable for all $\mathbf{b} \in \mathbb{R}^3$, and describe the set of $\mathbf{b} \in \mathbb{R}^3$ (as a plane or a line) such that the system is solvable.

22. Solve the linear system with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{array}\right).$$

After a row-reduction and using back substitution, we obtain $x_3 = 2,\ x_2 = -1,\ x_1 = 2$.

23. Find all the values h so that the linear system with the following augmented matrix is consistent:Applying row reduction to:

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right)$$

Using Gaussian elimination we obtain the following equivalent matrices:

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2)}$$

So, the system is inconsistent when $h = -4$; it has a unique solution if $h \neq -4$. The solution is (x_1, x_2), with $x_1 = -\frac{32}{h+4}$ and $x_2 = -\frac{8}{h+4}$.

24. Row reduce the following augmented matrix. Determine the leading ones and pivot columns.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 3 \end{array}\right).$$

Using Gaussian elimination we obtain the following equivalent matrices:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 3 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -17 \end{array}\right) \overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -17 \\ 0 & 0 & -3 & -6 \end{array}\right) \overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & \frac{17}{3} \\ 0 & 0 & 1 & 2 \end{array}\right).$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2) \text{ and } \mathrm{A}_{13}(-4) \ \textbf{2. } \mathrm{P}_{23}(-2) \ \textbf{3. } \mathrm{M}_2(-\tfrac{1}{3}) \text{ and } \mathrm{M}_3(-\tfrac{1}{3})}$$

So, Columns 1, 2, 3 are pivot columns, and $(1,1)$, $(2,2)$, $(3,3)$ entries are pivot positions. Using a backward substitution, we obtain $x_3 = 2$, $x_2 = \frac{17}{3} - 4x_3 = -\frac{7}{3}$ and $x_1 = 5 - 2x_2 - 4x_3 = \frac{5}{3}$.

25. Find the solutions of the system with augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{array}\right).$$

Using Gaussian elimination we obtain the following equivalent matrices:

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{array}\right) \overset{1}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{array}\right) \overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(2) \quad \textbf{2. } \ M_2(-\tfrac{1}{7})}$$

So, using the corresponding system, we obtain $x_3 = -2$ and $x_1 - 2x_2 = 4 + x_3 = 2$. Thus, $x_1 = 2x_2 + 2$. Hence there are indefinitely many solutions, given by the set

$$\mathcal{S} = \{(2t+2, t, -2) : t \in \mathbb{R}\}.$$

26. Find the solutions of the system with augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The augmented matrix is already row-echelon. Using back substitution, we obtain $x_4 = -7,\ x_3 = t,\ x_2 = 1 - 3t,\ x_1 = 2 + 2t$ for all $t \in \mathbb{R}$.

27. Determine the conditions on (h, k) so that the following system has no solution, one solution, and infinitely many solutions.

$$x_1 - 3x_2 = 1, \quad 2x_1 + hx_2 = k.$$

Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & h & k \end{array}\right) \overset{1}{\sim} \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & h+6 & k-2 \end{array}\right).$$

$$\boxed{\mathbf{1.}\ \mathrm{A}_{12}(-2)}$$

So, the system is inconsistent when $h = -6$ and $k \neq 2$; it has infinitely many solution if $h = -6$ and $k = 2$; it has a unique solution if $h \neq -6$.

28. Find the rank of

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 2 & 4 & 0 & 0 & 5 \\ 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & -1 & 4 \end{pmatrix}.$$

29. Find the reduced row-echelon and the rank of

$$\begin{pmatrix} 3 & 0 & 1 & 2 \\ 6 & -2 & 1 & 0 \\ 3 & 6 & 0 & -6 \end{pmatrix}.$$

30. Solve the systems

$$\begin{cases} 3x_1 - x_2 - x_3 = 0, \\ x_1 + x_2 + x_3 = 0, \\ 2x_1 + x_3 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} 2x_2 + x_3 = 1, \\ 4x_1 - 5x_2 - 7x_3 = 2, \\ x_1 - x_2 + 2x_3 = 3 \\ x_1 + x_3 = 1. \end{cases}$$