Munkres Chapter 2.17 exercises

Cherie Li

06/05/2018

Exercise 2.17.4: Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Solution: Let X be a set and let U open in X and A closed in. X. Notice that $A - U = A \cap U^c$ and that $U - A = U \cap A^c$. By definition of open, U^c is closed. Similarly, by definition of closed, A^c is open. Since the intersection of open sets is also open, $U - A$ closed and since the intersection of closed sets is also closed, $A - U$ closed.

Exercise 2.17.11: Show that the product of two Hausdorff spaces is Hausdorff.

Solution: Let T_1 and T_2 be Haursdorff spaces and let $T_1 \times T_2$ be the product topology. Let $(x_1, y_1) \in T_1 \times T_2$ be arbitrary. Similarly, let $(x_2, y_2) \in T_1 \times T_2$ be arbitrary, where $x_1 \neq x_2$ and $y_1 \neq y_2$. Then there exist open neighborhoods $U_1, U_2 \in T_1$ and $V_1, V_2 \in T_2$ such that $U_1 \cap U_2 = \varnothing$ and $V_1 \cap V_2 = \varnothing$. Then $U_1 \times V_1 \in T_1 \times T_2$ and $U_2 \times V_2 \in T_1 \times T_2$ and are disjoint. By definition, $T_1 \times T_2$ is Hausdorff.

Exercise 2.17.12: Show that a subspace of a Hausdorff space is Hausdorff.

Solution: Let X be a set with topology T. Let subspace topology $T_y = \{Y \cap U | U \in T\}$. Let $x_1, x_2 \in T_y$ be arbitrary. Then, there exists disjoint open sets U_1, U_2 such that $x_1 \in Y \cap U_1$ and $x_2 \in Y \cap U_2$. Then $Y \cap U_1$ and $Y \cap U_2$ also disjoint. Therefore, since x_1, x_2 were arbitrary, the subspace is Hausdorff.

Exercise 2.17.13: Show that X is Hausdorff if and only if the diagonal $\triangle = \{x \times x | x \in X\}$ is closed in $X \times X$.

Solution: First let X be a Hausdorff space. We want to show $\triangle$ is closed in $X \times X$, which is the same thing as showing the complement is open. Let $\triangle^C = \{x \times y | x, y \in X, x \neq y\}$. Since $x \neq y$ and X is Hausdorff, then there exist disjoint open sets U_1, U_2 in X such that $x \in U_1$, $y \in U_2$. By definition of product topology, $U_1 \times U_2$ open in $X \times X$. Also, $U_1 \times U_2 \in \triangle^C$ since they are disjoint. That is, for all $a \times b \in U_1 \times U_2$, $a \in U_1, b \in U_2$ so $a \neq b$. Then by definition of open set, where every point has an open interval containing it contained in the set, $\triangle^C$ is open in $X \times X$.