Munkres Chapter 2.13 exercises

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Exercise 2.13.4:

(a) If $\{T_\alpha\}$ is a family of topologies on X show that $\cap T_\alpha$ is a topology on X. Is $\cup T_\alpha$ a topology on X?

(b) Let $\{T_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections T_α, and a unique largest topology contained in all T_α.

(c) If $X = \{a, b, c\}$, let $T_1 = \{\varnothing, X, \{a\}, \{a, b\}\}$ and $T_2 = \{a\}, \{b, c\}\}$ Find the smallest topology containing T_1 and T_2 and the largest topology contained in T_1 and T_2.

Solution:

(a) To show that $\cap T_\alpha$ is a topology on X we need to show that the three properties of topology hold. Note that $X, \varnothing \in \cap T_\alpha$ since each T_α is a topology, and by property of topology, $X, \varnothing \in T_\alpha$. Now we want to show that $\cap T_\alpha$ is closed under arbitrary unions. Notice that for all T_α, $\cap T_\alpha \subseteq T_\alpha$. Since each T_α is a topology, it is closed under arbitrary unions, so $\cap T_\alpha$ is closed under arbitrary unions. Next, we show that $\cap T_\alpha$ is closed under finite intersections. Notice again that $\cap T_\alpha \subseteq T_\alpha$ for each T_α. Since each T_α is a topology, it also contains the finite intersection of any elements. Thus, finite intersections of $\cap T_\alpha$ are contained in all T_α, so are contained in $\cap T_\alpha$.

(b) Let T be the topology generated by the subbasis $\cup T_\alpha$ where $\{T_\alpha\}$ are all the topologies of X. By definition of subbasis, T is the collection of all unions of finite intersections of elements of $\cup T_\alpha$, so $T_\alpha \subseteq T$ for all T_α. is an element Suppose that there is another topology T_1 that contains all topologies on X. Then $T \subseteq T_1$, since by definition of topology, all unions of elements and finite intersections must be open sets as well. So T is the unique smallest topology on X.
For the unique largest topology on X contained in all T_α, consider the intersection of all topologies $\cap T_\alpha$. Suppose that there is another topology T_2 that is contained in all of T_α. Then $T_2 \in \cap T_\alpha$, so this is the unique largest topology.

(c) Taking $T_1 \cup T_2$ as our subbasis should give us the largest topology. The topology generated by unions of finite intersections of $T_1 \cup T_2$ is $T_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. The intersection $T_1 \cap T_2$ should give the smallest topology contained in them, which is $T_4 = \{\varnothing, X, \{a\}\}$

Exercise 2.13.5: Show that if A is a basis for a topology on X, then the topology generated by A equals the intersection of all topologies on X that contain A. Prove the same if A is a subbasis.

Solution: Let X be a set together with a topology T. Let A be a basis for T. We want to show that $T = \cap T_\alpha$, where $\{T_\alpha\} := \{T_\alpha | A \subseteq T_\alpha\}$. We show by double containment. First, we show that $T \subseteq \cap T_\alpha$. By Lemma 13.1, T is the collection of all unions of elements of A. Since $A \in \cap T_\alpha$, and by our previous results $\cap T_\alpha$ is a topology on X, it is closed under arbitrary unions, so $T \in \cap T_\alpha$. Now, we show that $\cap T_\alpha \in T$. Notice that $A \subseteq T$, so $T = T\alpha$ for some α. Therefore, $\cap T_\alpha \in T$.

Exercise 2.13.6: Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable.

Solution: It suffices to show that neither is $\mathbb{R}_l$ is not finer than $\mathbb{R}_k$ nor is $\mathbb{R}_k$ finer than $\mathbb{R}_l$. By definition of finer, we want to show that $\mathbb{R}_k \not\subset \mathbb{R}_l$ and $\mathbb{R}_l \not\subset \mathbb{R}_k$ respectively. First we show that $\mathbb{R}_k \not\subset \mathbb{R}_l$. Let B_l be a basis for $\mathbb{R}_l$. Let the set $B \in B_l$ be a basis element. Consider the basis element $[a, b) \in B$. Notice that there is no open interval of the form (c, d) or $(c, d) - K$ such that $(c, d) \subseteq [a, b)$ and $a \in (c, d)$. Note that if $(c, d) \subseteq [a, b)$, then $a < c$. If $a \in (c, d)$, then $a > c$. By Lemma 13.3, $\mathbb{R}_l$ is not finer than $\mathbb{R}_k$. Now we want to show that $\mathbb{R}_l \not\subset \mathbb{R}_k$. Let B_k be a basis for $\mathbb{R}_k$. Let $B' \subset B_k$. Consider the basis element $(-1, 1) - K \in B'$. Notice that there is no half open interval $[a, b)$ such that $0 \in (a, b)$ and $(a, b) \subset (-1, 1) - K$. If $0 \in (a, b)$, then $b > 0$, and there exists $n \in \mathbb{Z}^+$ such that $0 < \frac{1}{n} < b$. So $(a, b) \not\subset (-1, 1) - K$. If $[a, b) \subset (-1, 1) - K$, then there does not exist $\frac{1}{n}$ for $n \in \mathbb{Z}^+$ such that $\frac{1}{n} \in [a, b)$. Then $0 \notin [a, b)$.