Munkres Chapter 2.18 exercises

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Exercise 2.18.5: Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Solution: Let (a, b) be a subspace of $\mathbb{R}$. Consider the function $f(x) = \frac{x-a}{b-a}$, where $x \in (a, b)$. Notice that $0 < f(x) < 1$ and that it is bijective. Since $b > a$, $b - a > 0$, so the function is continuous, since it is just a linear function $x - a$ over some nonzero constant. Also, we have that the inverse is also continuous: $f(y) = y(b - a) + a$, as this is just the equation of a line with slope $(b - a)$ and intercept a as well as bijective.
Now consider the same function for $0 \leq f(x) \leq 1$. Notice when $x = a$, the function evaluates to 0 which is in the interval $[0, 1]$. Also when $x = b$, we have that the function evaluates to 1. Also the function is continuous and bijective still.
Since function and its inverse are both continuous, then f is a homeomorphism and $(0, 1)$ homeomorphic to (a, b) and $[0, 1]$ homeomorphic to $[a, b]$.

Exercise 2.18.10: Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x|f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function $h(x) = min\{f(x), g(x)\}$. Show that h is continuous. [$Hint$: Use the pasting lemma.]

Solution:

(a) Let Y be an ordered set in the order topology and let $f, g : X \to Y$ be continuous. To show that the set $A := \{x|f(x) \leq g(x)\}$ is closed it suffices to show that $A^c := \{x|f(x) > g(x)\}$ is open in X. Assume A^c is nonempty and let $a \in A^c$ be arbitrary. Let $y_1, y_2, y_3 \in Y$ such that $y_1 \leq g(a) < y_2 < f(a) \leq y_3$. Notice that (y_1, y_2) and (y_2, y_3) open in Y, since if $f(a)$ is the maximal element or $g(a)$ is the minimal element, we have $[g(a), y_2), (y_2, f(a)]$ instead. These are open intervals in order topology, so their intersection $(y_1, y_2) \cap (y_2, y_1)$ is also open. Since f is continuous, by definition we have that $f^{-1}\big((y_1, y_2) \cap (y_2, y_1)\big)$ is also open in X where $a \in f^{-1}\big((y_1, y_2) \cap (y_2, y_1)\big) \subset A^c$. So A^c is open in X and A is closed.

(b) Notice from the previous part that the set $A := \{x|f(x) \leq g(x)\}$ and symmetrically $B := \{x|g(x) \leq f(x)\}$ are both closed in X. Also, notice that $X = A \cup B$ since all x must have $g(x) > f(x)$, $g(x) = f(x)$ or $g(x) < f(x)$ and that their intersection is when $g(x) = f(x)$. By Theorem 18.3 (the pasting lemma), f and g give the continuous function $h : X \to Y$, where $h(x) = f(x)$ if $x \in A$. So if $x \in A$, then $f(x) < g(x)$. Similarly, if $x \in B$, then $g(x) > f(x)$. This is the same as the function min, defined in the statement.