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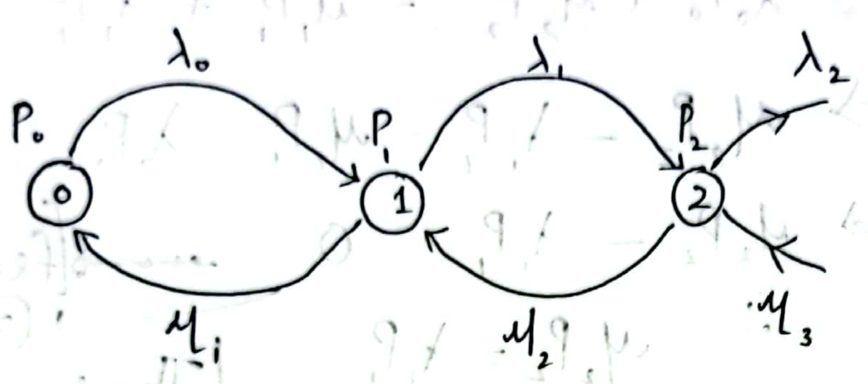
Prof little's formulae



P_0 is the probability that the system is in state 0
P_1 " " " " " " " state 1
P_2 " " " " " " " state 2

State	Rate in = Rate out
0	$\mu_1 P_1 = \lambda_0 P_0$
1	$\lambda_0 P_0 + \mu_2 P_2 = \mu_1 P_1 + \lambda_1 P_1$
2	$\lambda_1 P_1 + \mu_3 P_3 = \mu_2 P_2 + \lambda_2 P_2$
3	
⋮	
n	

λ = arrival rate

μ = service rate

$\boxed{\text{state } 0}$

$$\mu_1 P_1 = \lambda_0 P_0$$

$$\Rightarrow \mu_1 P_1 - \lambda_0 P_0 = 0 \longrightarrow (1)$$

Thus,

$$P_1 = \frac{\lambda_0 P_0}{\mu_1}$$

$\boxed{\text{state } 1}$

$$\lambda_0 P_0 + \mu_2 P_2 = \mu_1 P_1 + \lambda_1 P_1$$

$$\Rightarrow \mu_2 P_2 - \lambda_1 P_1 = \mu_1 P_1 - \lambda_0 P_0$$

$$\Rightarrow \mu_2 P_2 - \lambda_1 P_1 = 0 \longrightarrow \text{[from (1)]}$$

$$\therefore \mu_2 P_2 = \lambda_1 P_1 \longrightarrow (2)$$

Thus,

$$P_2 = \frac{\lambda_1 P_1}{\mu_2}$$

$$= \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \cdot P_0$$

$\boxed{\text{state } 3}$

$$\lambda_1 P_1 + \mu_3 P_3 = \mu_2 P_2 + \lambda_2 P_2$$

$$\Rightarrow \mu_3 P_3 - \lambda_2 P_2 = \mu_2 P_2 - \lambda_1 P_1$$

$$\Rightarrow \mu_3 P_3 - \lambda_2 P_2 = 0 \longrightarrow (3) \text{ [from (2)]}$$

$$\therefore \mu_3 P_3 = \lambda_2 P_2$$

Thus,

$$P_3 = \frac{\lambda_2 . P_2}{\mu_3}$$

$$= \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_2} \cdot P_0$$

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Now, if we observe these values that we are getting for the different state probabilities, we will get:

$$P_n = \left(\frac{\lambda_{n-1} \cdot \lambda_{n-2} \cdot \lambda_{n-3} \cdots \lambda_3 \lambda_2 \lambda_1 \lambda_0}{\mu_n \cdot \mu_{n-1} \cdot \mu_{n-2} \cdots \mu_4 \cdot \mu_3 \cdot \mu_2 \cdot \mu_1}\right) \cdot P_0$$

In general, we can write

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0}$$

IMP

$$\boxed{P_n = \rho^n \cdot P_0}$$, where $\rho = \frac{\lambda}{\mu}$

rho

We know the sum of all probabilities = 1.

$$\therefore \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \rho^n \cdot P_0 = 1$$

$$\Rightarrow P_0 = \left\{\sum_{n=0}^{\infty} \rho^n\right\}^{-1}$$

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μ should always be greater that λ, because the service rate should be higher than the arrival rate of the customer, or the queue will be crowded.

Sum of infinite numbers

$$\sum_{n=0}^{\infty} \rho^n = \rho^0 + \rho^1 + \rho^2 + \dots$$
$$= 1 + \rho^1 + \rho^2 + \dots$$

This is a geometric progression

$$= \frac{1}{1-\rho}, |\rho| < 1$$

the common term

$$\therefore P_0 = \left\{ \sum_{n=0}^{\infty} \rho^n \right\}^{-1}$$

$$= \left\{ \frac{1}{1-\rho} \right\}^{-1}$$

$$= 1 - \rho$$

$$\boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

This is the derived value of P_0

Derivation of expected number of customers in the ~~Queuing~~ system (L_s)

[Including the one infront of the queue getting service]

$$\boxed{L_s = \sum_{n=0}^{\infty} n \cdot P_n}$$ → This is the expected number of customers in the queuing system

$$= \sum_{n=0}^{\infty} n \cdot \rho^n \cdot P_0$$

$$= (1-\rho) \cdot \rho \sum_{n=0}^{\infty} n \cdot \rho^{n-1}$$

$$\left[\because P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho\right]$$

$$= (1-\rho)\rho \cdot \frac{d}{d\rho}\left[\sum_{n=0}^{\infty} \rho^n\right]$$

$$= (1-\rho)\rho \cdot \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

[As we have seen before, the sum of infinite terms in a geometric progression $= \frac{1}{1 - \text{common term}}$]

differentiating the above equation, we get,

$$\Rightarrow \cancel{(1-\rho)}\cdot\rho\cdot\frac{1}{(1-\rho)^{2}}$$

$$= \frac{\rho}{(1-\rho)}$$

$$= \frac{\lambda/\mu}{1-\lambda/\mu}$$

$$= \frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}}$$

$$= \frac{\lambda}{\mu-\lambda}$$

$$\therefore \boxed{L_s = \frac{\lambda}{\mu-\lambda}}$$

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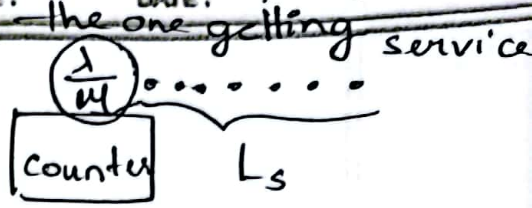

Expected no. of customers in the queuing system

$$L_q = L_s - \frac{\lambda}{\mu}$$

(L_q is the value excluding the customer getting service)

$$= \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu}$$

$$= \frac{\mu\lambda - \mu\lambda + \lambda^2}{\mu(\mu-\lambda)}$$

$$\boxed{L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}}$$

Similarly, expected ~~no. of customers~~ time spent:

(i) by a customer in the system

$$W_s = \frac{L_s}{\lambda}$$

$$= \frac{\lambda/\mu-\lambda}{\lambda}$$

$$\therefore \boxed{W_s = \frac{1}{\mu-\lambda}}$$

(ii) by a customer in the queuing system

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{1}{\lambda}$$

$$\boxed{W_q = \frac{\lambda}{\mu(\mu-\lambda)}}$$