DISCRETE TIME MARKOV CHAIN

A Discrete Time Markov Chain can be used to describe the behavior of a system that jumps from one state to another state with a certain probability, and this probability of transition to the next state **depends only on what state the system is in currently**, i.e. it **does not depend on** which states the system was in **prior** to the **current state.**

Discrete-time Markov chains {Xn,n = 0, 1, 2,...} make transitions only at integer times:

$$\mathbf{P(X_{n+1} = j \mid X_n = i)}$$

In other words, the chain can only stay in each state for an integer amount of time before making the next transition.
So changes to the system can only happen at one of those discrete time values.

An example is a board game like Chutes and "Snakes and Ladders" in which pieces move around on the board according to a die roll.
If you are looking at the board at the beginning of someone's turn and wondering what the board will look like at the beginning of the next person's turn, it doesn't matter how the pieces arrived at their current positions (the past history of the system). All that matters is that the pieces are where they currently are (the current system state) and the upcoming die roll (the probabilistic aspect). This is discrete because changes to the system state can only happen on someone's turn.

Example 11.4

Consider the Markov chain shown in Figure 11.7.

a. Find $P(X_4 = 3|X_3 = 2)$.
b. Find $P(X_3 = 1|X_2 = 1)$.
c. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2)$.
d. If we know $P(X_0 = 1) = \frac{1}{3}$, find $P(X_0 = 1, X_1 = 2, X_2 = 3)$.

a. By definition

$$P(X_4 = 3|X_3 = 2) = p_{23} = \frac{2}{3}.$$

b. By definition

$$P(X_3 = 1|X_2 = 1) = p_{11} = \frac{1}{4}.$$

c. We can write

$$\begin{aligned} P(X_0 = 1, X_1 = 2) &= P(X_0 = 1)P(X_1 = 2|X_0 = 1) \\ &= \frac{1}{3} \cdot p_{12} \\ &= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}. \end{aligned}$$

d. We can write

$$\begin{aligned} &P(X_0 = 1, X_1 = 2, X_2 = 3) \\ &\quad = P(X_0 = 1)P(X_1 = 2|X_0 = 1)P(X_2 = 3|X_1 = 2, X_0 = 1) \\ &\quad = P(X_0 = 1)P(X_1 = 2|X_0 = 1)P(X_2 = 3|X_1 = 2) \quad \text{(by Markov property)} \\ &\quad = \frac{1}{3} \cdot p_{12} \cdot p_{23} \\ &\quad = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \\ &\quad = \frac{1}{9}. \end{aligned}$$

Link: https://www.probabilitycourse.com/chapter11/11_2_1_introduction.php
Try solving these problems:
https://www.probabilitycourse.com/chapter11/11_2_7_solved_probs.php

CONTINUOUS-TIME MARKOV CHAIN (CTMC)

A continuous-time Markov chain is one in which changes to the system can happen at any time along a continuous interval.

we will consider a random process $\{X_{(t)}, t \in [0,\infty)\}$
Again, we assume that we have a countable state space $S \subset \{0,1,2,\cdots\}$

- If $\mathbf{X_{(0)}=i}$ then $\mathbf{X_{(t)}}$ stays in **state i** for a **random amount of time, say T1** where **T1 is a continuous random variable**.

- **At time T1,** the process jumps to a **new state j** and will spend a **random amount of time T2** in that state, and so on.

- As it will be clear shortly, **the random variables T1, T2,... have exponential distribution**. The probability of going from state i to state j is shown by $\mathbf{p_{ij}}$

- Thus, the time that a continuous-time Markov chain spends in state i (called the **holding time**) will have **Exponential(λi) distribution**, where **λi is a nonnegative real number.**

- We further assume that the **λi's are bounded,** i.e., there exists a **real number M<∞ such that λi<M, for all i∈S**

Thus, a continuous Markov chain has two components.
-> First, we have a **discrete-time Markov chain,** called the **jump chain or** the **embedded Markov chain,** that gives us the **transition probabilities $\mathbf{p_{ij}}$.**
-> Second, for each state we have a **holding time parameter $\boldsymbol{\lambda_i}$** that controls the amount of time spent in each state.

An example is the number of cars that have visited a drive-through at a local fast-food restaurant during the day.
A car can arrive **at any time t** rather than at discrete time intervals. Since **arrivals are basically independent**, if you know the number of cars that have gone through by 10:00 a.m., what happened before 10:00 a.m. doesn't give you any additional information that would be useful in predicting the number of cars that **will have visited** the drive-through **by noon**.