Данные материалы были использованы для подготовки команды Казахстанского филиала МГУ на IMC-2018, где ребята выступили очень успешно: Бекмаганбетов Бекарыс (ММ-2) взял золото (6 задач из 10), Аскергалиев Ануар (ВМК-2) взял серебро (4 задачи из 10), а Журавская Александра (ВМК-4) получила бронзовую награду (2 задачи из 10).

В брошюре собраны 219 задач из различных источников: задачи IMC, задачи Putnam, Putnam And Beyond (Andreescu), Задачи студенческих олимпиад (Садовничий, Григорьян, Конягин) и другие.

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1 Telescope

1.1 Problems

25.06.2018

1. (putnam-and-beyond-365) Find the sum $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

2. (putnam-and-beyond-365) Let $a_0 = 1$, $a_1 = 3$, $a_{n+1} = \frac{a_n^2+1}{2}$, for $n \geqslant 1$. Prove that
$$\frac{1}{a_0+1} + \frac{1}{a_1+1} + \cdots + \frac{1}{a_n+1} + \frac{1}{a_{n+1}-1} = 1,$$
for $n \geqslant 1$.

3. (putnam-and-beyond-368) Let
$$a_n = \frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}-\sqrt{2n-1}}$$
for $n \geqslant 1$. Prove that $a_1 + a_2 + \cdots + a_{40}$ is a positive integer.

4. (putnam-and-beyond-366) Let λ be a root of unity. Prove that
$$\lambda^{-1} = \sum_{n=0}^{\infty} \lambda^n (1-\lambda)(1-\lambda^2)(1-\lambda^3)\cdots(1-\lambda^n)$$
with the convention that the 0th term of the series is 1.

5. (putnam-2016-B1) Let x_0, x_1, x_2, ... be the sequence such that $x_0 = 1$ and for $n \geqslant 0$,
$$x_{n+1} = \ln(e^{x_n} - x_n).$$
Show that the infinite series $x_0 + x_1 + x_2 + \ldots$ converges and find its sum.

6. (putnam-2014-A3) Let $a_0 = \frac{5}{2}$ and $a_k = a_{k-1}^2 - 2$ for $k \geqslant 1$. Compute
$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$
in closed form.

1.2 Hints

1. (putnam-and-beyond-365)
$$\frac{1}{\sqrt{n}+\sqrt{n+1}}=\sqrt{n+1}-\sqrt{n}$$

2. (putnam-and-beyond-365)
$$2(a_{k+1}-1)=(a_k-1)(a_k+1)$$
.

3. (putnam-and-beyond-368) $a=\sqrt{2n-1}$, $b=\sqrt{2n+1}$.

4. (putnam-and-beyond-369) Sum $a_{n+1}+a_n$.

5. (putnam-2016-B1) $e^x \geqslant x+1$.

6. (putnam-2014-A3) $a_k+1=(a_{k-1}-1)(a_{k-1}+1)$, $a_k^2-4=a_{k-1}^2(a_{k-1}^2-4)$.

2 Recurrent

2.1 Problems

26.06.2018

1. (putnam-and-beyond-299) Define the sequence $(a_n)_{n\geqslant 0}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n,$$

for $n \geqslant 0$. Prove that n divides a_n for all $n \geqslant 1$.

2. (putnam-and-beyond-302) A sequence u_n is defined by $u_0 = 2$, $u_1 = \frac{5}{2}$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, for $n \geqslant 1$. Prove that for all positive integers n,

$$[u_n] = 2^{\frac{2^n - (-1)^n}{3}}.$$

3. (putnam-and-beyond-305) Find the general term of the sequence given by $x_0 = 3$, $x_1 = 4$, and

$$(n+1)(n+2)x_n = 4(n+1)(n+3)x_{n-1} - 4(n+2)(n+3)x_{n-2},$$

$n \geqslant 2$.

4. (putnam-and-beyond-307) Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1 + 4a_n + \sqrt{1 + 24a_n}}{16}$$

for all $n \geqslant 1$. Find an explicit formula for a_n in terms of n.

5. (putnam-2015-A2) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_n - 1 - a_{n-2}$ for $n \geqslant 2$. Find an odd prime factor of a_{2015}.

6. (putnam-and-beyond-306) Let (x_n), $n \geqslant 0$ be defined by the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$, with $x_0 = 0$.

   - Show that the expression
   $$x_n^2 - x_{n-1}x_{n+1}$$
   depends only on b and x_1, but not on a.
   - Show that the expression
   $$f_n^2 - f_{n-1}f_{n+1}$$
   where $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$.

7. (putnam-2017-A2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and $Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$ for all $n \geqslant 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

2.2 Hints

1. (putnam-and-beyond-299) $a_n = n * f_n$.
2. (putnam-and-beyond-302) $u_n = 2^{x_n} + 2^{-x_n}$.
3. (putnam-and-beyond-305) $y_n = \frac{x_n}{n+3}$.
4. (putnam-and-beyond-307) $b_n = \sqrt{1 + 24a_n}$.
5. (putnam-2015-A2) $(x^5 + y^5)|(x^{2015} + y^{2015})$.
6. (putnam-and-beyond-306) Induction.
7. (putnam-2017-A2) See putnam-and-beyond-306. Note that $Q_n(x)$ has linear recurrent realation by $Q_{n-1}(x)$ and $Q_{n-2}(x)$.

3 Linear algebra

3.1 Problems

27.06.2018

1. (putnam-and-beyond-215) Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

2. (putnam-2015-A6) Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.

3. (putnam-and-beyond-216) Let X, Y be $n \times n$ matrices. Prove that

$$\det(I_n - XY) = \det(I_n - YX).$$

4. (putnam-and-beyond) Let A be $n \times n$ matrices. Prove that

$$\det(I_n + A^2) \geqslant 0$$

5. (putnam-and-beyond-217) Let A and B be $n \times n$ matrices that commute. Prove that if $det(A + B) \geqslant 0$, then $det(A^k + B^k) \geqslant 0$ for all $k \geqslant 1$.

6. (putnam-and-beyond-226) Let A, B, C be $n \times n$ matrices, $n \geqslant 1$, satisfying

$$ABC + AB + BC + AC + A + B + C = O_n.$$

7. (putnam-and-beyond-230) Let A and B be $n \times n$ matrices, $n \geqslant 1$, satisfying $AB - B^2A^2 = I_n$ and $A^3 + B^3 = O_n$. Prove that $BA - A^2B^2 = I_n$.

8. A is orthogonal matrix and $I + A$ is non singular. Prove that $(I + A)^{-1}(I - A)$ is skew symmetric. (кососимметрическая)

9. A is orthogonal matrix, where n is even. Suppose that $\det(A) = -1$. Show that $\det(I - A) = 0$.

10. Let A, B is two invertable $n \times n$ real matrices. Assume that $A + B$ is invertable. Then prove that $A^{-1} + B^{-1}$ is also invertable.

11. Let X be any arbitrary $n \times n$ square matrix, n is even. Prove that

$$tr(X^n) \geqslant ndet(X).$$

12. Matrix A, B, C, $D \in R^{n \times n}$ such that

$$AC - BD = I_n, AD + BC = O_n$$

   - Prove that $CA - DB = I_n$, $DA + CB = O_n$
   - $\det(AC) \geqslant 0$, $(-1)^n \det(BD) \geqslant 0$.

3.2 Hints

1. (putnam-and-beyond-215) If A is invertable, then use matrix row transformation. If A is not invertable, $A_t = A + tI_n$ is invertable for all $t \neq -\lambda_k$. Checking equation is polynomial equation.

2. (putnam-2015-A6) If A and B is invertable, then checked it. If A or B is not invertable, $A_t = A + tI_n$, $B_t = B + tI_n$ is invertable for all $t \neq -\lambda_k$.

3. (putnam-and-beyond-216) Use problem 215.

4. (putnam-and-beyond) $I_n + A^2 = (I_n + iA)(I_n - iA)$, where $i = \sqrt{-1}$.

5. (putnam-and-beyond-217) Complex roots of $x^k = 1$ divides into complex conjugate pairs.

6. (putnam-and-beyond-226) If $XY = I$ then X, Y commute.

7. (putnam-and-beyond-230) $(A + iB^2)(B + iA^2) = I_n$.

8. B is skew symmetric $\Leftrightarrow B^T = -B$.

9. $\det(AB) = \det(A)\det(B) = \det(BA)$.

10. $\det(A^{-1} + B^{-1}) \neq 0$.

11. $det(X) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$, $tr(X^n) = \lambda_1^n + \lambda_2^n + \cdots + \lambda_n^n$.

12. $(A + \mathrm{i}B)(C + \mathrm{i}D) = I_n$.

4 Polynomial

4.1 Problems

28.06.2018

1. Про многочлен $f(x) = x^{10} + a_9x^9 + ... + a_0$ известно, что $f(1) = f(-1)$, $f(2) = f(-2)$, $f(3) = f(-3)$, $f(4) = f(-4)$, $f(5) = f(-5)$. Докажите, что $f(x) = f(-x)$ для любого действительного x.

2. Известно, что $P(x)$ – многочлен степени n такой, что для всех $t \in \{1, 2, 2^2, ..., 2^n\}$ верно соотношение $P(t) = \frac{1}{t}$. Найдите $P(0)$.

3. Известно, что для многочлена степени n верно, что $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $\dots$, $f(n) = 2^n$. Найдите $f(n+1)$.

4. Пусть x, y – натуральные числа такие, что:
$$A = \frac{x^2 + y^2}{xy + 1}$$
– целое. Найдите все возможные значения A.

5. Пусть x, y – натуральные числа такие, что выражение $A = \frac{x^2+y^2+1}{xy}$ – целое. Найдите все возможные значения A.

6. Произведение квадратных трехчленов $x^2 + a_1x + b_1$, $x^2 + a_2x + b_2$, $\dots$, $x^2 + a_nx + b_n$ равно многочлену $P(x) = x^{2n} + c_1x^{2n-1} + c_2x^{2n-2} + \dots + c_{2n-1}x + c_{2n}$, где коэффициенты c_1, c_2, $\dots$, c_{2n} положительны. Докажите, что для некоторого k ($1 \leqslant k \leqslant n$) коэффициенты a_k и b_k положительны.

7. (putnam-and-beyond-169) Let $P(x)$ be a polynomial with all roots real and distinct and such that none of its zeros is equal to 0. Prove that the polynomial $x^2P''(x) + 3xP'(x) + P(x)$ also has all roots real and distinct.

8. (putnam-and-beyond-172) Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n$ are real. Prove that
$$P'\left(\frac{x_1 + x_2}{2}\right) P'\left(\frac{x_{n-1} + x_n}{2}\right) \neq 0.$$

9. (putnam-and-beyond-173) Let $a_1, a_2, \dots, a_n$ be positive real numbers. Prove that the polynomial $P_(x) = x_n - a_1x^{n-1} - a_2x^{n-2} - \dots - a_n$ has a unique positive zero.

10. (putnam-and-beyond-174) Prove that the zeros of the polynomial
$$P(z) = z^7 + 7z^4 + 4z + 1$$
lie inside the disk of radius 2 centered at the origin.

11. (putnam-and-beyond-175) For $a \neq 0$ a real number and $n > 2$ an integer, prove that every nonreal root z of the polynomial equation $x^n + ax + 1 = 0$ satisfies the inequality
$$|z| \geqslant \sqrt[n]{\frac{1}{n-1}}.$$

12. Prove that for any distinct integers $a_1, a_2, \dots, a_n$ the polynomial $P(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1$ cannot be written as a product of two nonconstant polynomials with integer coefficients.

4.2 Hints

1. Многочлен $g(x) = f(x) - f(-x)$.
2. Многочлен $F(t) = P(t) * t - 1$.
3. Многочлен $f(x) = 1 + \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + \cdots + \frac{x(x-1)\cdots(x-n+1)}{1 \cdot 2 \cdot 3 \cdots n}$.
4. Идея: прыжки Виета. Если $x = y$, то легко решить явно. Если существует решение (x, y), где $x > y$, то существует решение (y, z), где $y > z$ (теорема Виета для квадратного уравнения).
5. Идея: прыжки Виета.
6. Теорема Виета для c_{2n} и c_1.
7. (putnam-and-beyond-169) Total differential. Rolle's theorem:

   If a real-valued function f is continuous on a proper closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.
8. (putnam-and-beyond-172) $P'(x) = P(x)\left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + \cdots + \frac{1}{x-x_n}\right)$.
9. (putnam-and-beyond-173) $\frac{P(x)}{x^n}$.
10. (putnam-and-beyond-174) $\left|\frac{P(z)}{z^7}\right|$. Triangle inequlaity.
11. (putnam-and-beyond-175) $z = r(\cos t + i \sin t)$.
12. (putnam-and-beyond-185) $P(x) = Q(x)R(x)$. What about $Q(x) + R(x)$?

5 Calculus

5.1 Problems

29.06.2018

1. (PUTNAM-1987-B1) Evaluate
$$\int_0^1 \frac{\sqrt{\ln(9-x)}dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$$

2. (PUTNAM-1989-A2) Evaluate
$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dydx,$$
where a and b are positive.

3. (PUTNAM-1990-B1) Find all real-valued continuously differentiable functions f on the real line such that for all x,
$$f(x)^2 = \int_0^x \left(f(t)^2 + f'(t)^2\right) dt + 1990.$$

4. (PUTNAM-1998-B1) Find the minimum value of
$$\frac{(x+\frac{1}{x})^6 - x^6 - \frac{1}{x^6} - 2}{(x+\frac{1}{x})^3 + (x^3 + \frac{1}{x^3})}$$
for $x > 0$.

5. (PUTNAM-1998-A3) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a)f'(a)f''(a)f'''(a) \geqslant 0.$$

6. (PUTNAM-2006-B5) For each continuous function $f : [0,1] \to \mathbb{R}$, let
$$I(f) = \int_0^1 x^2 f(x)dx$$
$$J(f) = \int_0^1 x(f(x))^2 dx.$$
Find the maximum value of $I(f) - J(f)$ over all such functions f.

7. (PUTNAM-2005-B3) Find all differentiable functions $f : \mathbb{R}^+ \to \mathbb{R}^=$ for which there is a positive real number a such that
$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$
for all $x > 0$.

8. Let function $f(x)$ is continuous on $\mathbb{R}$ and $f^3(x) + f(x) = x$. Calculate
$$I = \int_0^2 f(x)dx.$$

9. If $f(x)$ is continuous and differentiable on $[0;1]$ with $f(0) = 0$, $f(1) = 1$. Evaluate minimum value of
$$\int_0^1 \left(f'(x)\right)^2 dx.$$

5.2 Hints

1. (PUTNAM-1987-B1) Symmetric variable substitution $9 - x = t + 3$.
2. (PUTNAM-1987-B1) From double integral to repeated integral.
3. (PUTNAM-1990-B1) Differentiate the equation.
4. (PUTNAM-1998-B1) Simplify. $x + \frac{1}{x} \geqslant 2$ for $x > 0$.
5. (PUTNAM-1998-A3) Without lose of generalization $f(x) > 0$. Without lose of generalization $f'(x) > 0$. It $f(x) > 0$, $f'(x) > 0$ for all x, then $f''(x) > 0$.
6. (PUTNAM-2006-B5) Cauchy-Schwarz inequality for $(I(f))^2$.
7. (PUTNAM-2005-B3) Variable substitution $t = \frac{a}{x}$. Differentiate the equation.
8. Variable substitution $f(x) = y$.
9. Cauchy-Schwarz inequality.

6 Series

6.1 Problems

02.07.2018

1. (putnam-and-beyond-352) Show that the series
$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}} + \cdots$$
converges when $|x| > 1$, and in this case find its sum.

2. (putnam-and-beyond-355) Let $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{x_n} < 80$$

3. (putnam-and-beyond-357) Does the series
$$\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2+1})$$
converges?

4. (putnam-and-beyond-369) Prove that
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1^2 - 2^2 + 3^2 - \cdots + (-1)^{k+1}k^2} = \frac{2n}{n+1}$$

5. (putnam-and-beyond-370) Prove that
$$\sum_{n=1}^{9999} \frac{1}{(\sqrt{n} + \sqrt{n+1})(\sqrt[4]{n} + \sqrt[4]{n+1})} = 9$$

6. (putnam-and-beyond-371) Let $a_n = \sqrt{1 + \left(1 + \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 - \frac{1}{n}\right)^2}$, $n \geqslant 1$. Prove that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{20}}$$
is a positive integer.

7. (putnam-and-beyond-373) Let $a_n = 3n + \sqrt{n^2-1}$ and $b_n = 2(\sqrt{n^2-n} + \sqrt{n^2+n})$, $n \geqslant 1$. Show that
$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \cdots + \sqrt{a_{49} - b_{49}} = A + B\sqrt{2}$$
for some integer A and B.

8. Compute the product
$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{F_n^2}\right)$$
where F_n is the n-th Fibonacci number.

9. (putnam-and-beyond-378) Let x be a positive number less than 1. Compute the product
$$\prod_{n=1}^{\infty} \left(1 + x^{2^n}\right)$$

10. (ВШЭ-2012) Сходится ли ряд
$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}+1}}?$$

11. (Садовничий-7-1) Последовательность задается: $x_1 = a > 1$, $x_{n+1} = x_n^2 - x_n + 1$. Найдите
$$\sum_{n=1}^{\infty} \frac{1}{x_n}$$

12. (Садовничий-7-4) Вычислить:
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)2^n}$$

13. (Садовничий-7-7) Вычислить:
$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{4\cdot 5\cdot 6} + \frac{1}{7\cdot 8\cdot 9} + \cdots$$

14. (putnam-2016-B6) Evaluate
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

15. (putnam-2015-A3) Compute
$$\log_2\left(\prod_{a=1}^{2015}\prod_{b=1}^{2015}(1+e^{2\pi iab/2015})\right)$$
Here i is the imaginary unit (that is, $i^2=-1$).

16. (putnam-2014-A3) Let $a_0 = \frac{5}{2}$ and $a_k = a_{k-1}^2 - 2$ for $k \geqslant 1$. Compute
$$\prod_{k=0}^{\infty}\left(1 - \frac{1}{a_k}\right)$$
in closed form.

17. (putnam-2011-A2) Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that
$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \ldots a_n}$$
converges, and evaluate S.

18. (putnam-2001-B3) For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate
$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n}.$$

19. (putnam-1999-A4) Sum the series
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}.$$

20. (Алфутова-устинов-11-65) Вычислите суммы:
   - $C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n$.
   - $C_n^1 + 2^2C_n^2 + 3^2C_n^3 + \cdots + n^2C_n^n$.

6.2 Hints

1. (putnam-and-beyond-352) Add $\frac{1}{1-x}$.
2. (putnam-and-beyond-355) Double series $\sum_{x_j<10^n} \frac{1}{x_j} = \sum_{1\leqslant i\leqslant n} \sum_{10^{i-1}\leqslant j<10^i} \frac{1}{x_j}$
3. (putnam-and-beyond-357) $\sin(\pi\sqrt{n^2+1}) = (-1)^n \sin(\frac{\pi}{\sqrt{n^2+1}+n})$
4. (putnam-and-beyond-369) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k+1}k^2 = (-1)^{k+1}\frac{k(k+1)}{2}$
5. (putnam-and-beyond-370) $a_n = \sqrt[4]{n+1} - \sqrt[4]{n}$
6. (putnam-and-beyond-371) $a_n = \frac{1}{4}(b_{n+1} - b_n)$
7. (putnam-and-beyond-373) $a_k - b_k = \frac{1}{2}\left((\sqrt{k} - \sqrt{k+1}) - (\sqrt{k-1} - \sqrt{k})\right)$
8. Let (x_n), $n \geqslant 0$ be defined by the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$, with $x_0 = 0$.
   - Show that the expression
     $$x_n^2 - x_{n-1}x_{n+1}$$
     depends only on b and x_1, but not on a.
   - Evaluate
     $$f_n^2 - f_{n-1}f_{n+1}$$
     where $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$.
9. (putnam-and-beyond-378) Multiply by $(1+x)$.
10. (ВШЭ-2012) $n^{n+1} < 2n$
11. (Садовничий-7-1) $\frac{1}{x_n} = \frac{1}{x_{n+1}-1} - \frac{1}{x_n-1}$
12. (Садовничий-7-4) Интегрируем дважды степенной ряд $\sum_{n=1}^{\infty} z^n$.
13. (Садовничий-7-7)
14. (putnam-2016-B6) $\frac{1}{k+1} = \int_0^1 x^k dx$
15. (putnam-2015-A3) $z^n - 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$. $(1+\omega^b) = \frac{1-\omega^{2b}}{1-\omega^b}$.
16. (putnam-2014-A3) $a_k + 1 = (a_{k-1} - 1)(a_{k-1} + 1)$, $a_k^2 - 4 = a_{k-1}^2(a_{k-1}^2 - 4)$.
17. (putnam-2011-A2) $S_m - S_{m-1} = \frac{1}{a_1a_2\cdots a_m}$, where $S_m = \frac{b_1b_2\cdots b_n}{(b_1+2)(b_2+2)\cdots(b_n+2)}$
18. (putnam-2001-B3) Since $(k-1/2)^2 = k^2 - k + 1/4$ and $(k+1/2)^2 = k^2 + k + 1/4$, we have that $\langle n\rangle = k$ if and only if $k^2 - k + 1 \le n \le k^2 + k$. Hence
    $$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n} = \sum_{k=1}^{\infty} \sum_{n,\langle n\rangle=k} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n} = \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{2^k + 2^{-k}}{2^n} = \sum_{k=1}^{\infty}(2^k+2^{-k})(2^{-k^2+k}-2^{-k^2-k}) = 3.$$
19. (putnam-1999-A4) Denote the series by S, and let $a_n = 3^n/n$:
    $$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_m(a_m+a_n)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{a_n(a_m+a_n)},$$
    Thus
    $$2S = \left(\sum_{n=1}^{\infty} \frac{n}{3^n}\right)^2.$$
    $S = 9/32$.
20. (Алфутова-устинов-11-65) $f(x) = (1+x)^n$, $f'(x)$, $f''(x)$.

7 Linear algebra

7.1 Problems

03.07.2018

1. (putnam-and-beyond-199) Let M be an $n \times n$ complex matrix. Prove that there exist Hermitian matrices A and B such that $M = A + iB$. (A matrix X is called Hermitian if $X^T = \overline{X}$).

2. (putnam-and-beyond-200) Do there exist $n \times n$ matrices A and B such that $AB - BA = I_n$?

3. (putnam-and-beyond-201) Let A and B be 2×2 matrices with real entries satisfying $(AB - BA)^n = I_2$ for some positive integer n. Prove that n is even and $(AB - BA)^4 = I_2$.

4. (putnam-and-beyond-205) Let A and B be $n \times n$ matrices with real entries satisfying
$$tr(AA^T + BB^T) = tr(AB + A^T B^T).$$
Prove that $A = B^T$.

5. (putnam-and-beyond-206) Let $x_1, x_2, \dots, x_n$ be arbitrary numbers ($n \geqslant 1$). Compute the determinant
$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

6. (putnam-and-beyond-213) Let A and B be 3×3 matrices with real elements such that $\det A = \det B = \det(A + B) = \det(A - B) = 0$. Prove that $\det(xA + yB) = 0$ for any real numbers x and y.

7. (putnam-and-beyond-222) Let A and B be 2×2 matrices with integer entries such that A, $A+B$, $A+2B$, $A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries.

8. (putnam-and-beyond-243) Let A be the $n \times n$ matrix whose i, j entry is $i + j$ for all $i, j = 1, 2, \dots, n$. What is the rank of A?

9. (putnam-and-beyond-244) For integers $n \geqslant 2$ and $0 \leqslant k \leqslant n - 2$, compute the determinant
$$\begin{vmatrix} 1^k & 2^k & 3^k & \dots & n^k \\ 2^k & 3^k & 4^k & \dots & (n+1)^k \\ 3^k & 4^k & 5^k & \dots & (n+2)^k \\ \dots & \dots & \dots & \dots & \dots \\ n^k & (n+1)^k & (n+2)^k & \dots & (2n-1)^k \end{vmatrix}$$

10. (putnam-and-beyond-248) Let $A : V \to W$ and $B : W \to V$ be linear maps between finite-dimensional vector spaces. Prove that the linear maps AB and BA have the same set of nonzero eigenvalues, counted with multiplicities.

11. (putnam-and-beyond-249) Let A, B be 2×2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$. Prove that if $\det(A^3 + B^3) = 1$, then $A^2 = O_2$.

12. (putnam-and-beyond-258) Let A and B be 2×2 matrices with determinant equal to 1. Prove that
$$tr(AB) - (trA)(trB) + tr(AB^{-1}) = 0.$$

13. (putnam-and-beyond-259) Find the 2×2 matrices with real entries that satisfy the equation

$$X^3 - 3X^2 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$$

14. (putnam-and-beyond-261) Let A and B be 3×3 matrices. Prove that

$$3\det(AB - BA) = tr((AB - BA)^3)$$

15. (putnam-2015-B3) Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

16. (putnam-2014-A2) Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

17. (ncumc-2017-5) Find the maximal set of points in $\mathbb{C}$ such that there are no complex Hermitian positively definite matrices of identical sizes A, B for which the point is an eigenvalue of matrix $(A + B)^{-1}(I + AB)$.

18. (Садовничий-2-1-11) Вычислить определитель

$$\begin{vmatrix} P(x) & P(x+1) & \dots & P(x+n) \\ P'(x) & P'(x+1) & \dots & P'(x+n) \\ \dots & \dots & \dots & \dots \\ P^{(n)}(x) & P^{(n)}(x+1) & \dots & P^{(n)}(x+n) \end{vmatrix}$$

где $P(x) = x(x+1)(x+2) \cdot (x+n)$.

19. (Садовничий-2-1-20) Пусть A, B — квадратные матрицы порядка 2017. Доказать, что если $AB = 0$, то хотя бы одна из матриц $A + A^T$ или $B + B^T$ вырождена.

20. (Садовничий-2-1-21) Пусть A, B, C — квадратные матрицы порядка $n \times n$. Доказать, что

$$rg(AB) + rg(BC) \leqslant rg(B) + rg(ABC)$$

21. (Садовничий-2-1-27) Пусть A, B — квадратные матрицы порядка $n \times n$. Верно ли, что матрицы AB и BA имеют одинаковый характеристический многочлен?

7.2 Hints

1. (putnam-and-beyond-199) $\overline{M^T} = \overline{A^T} - i\overline{B^T} = A - iB$.
2. (putnam-and-beyond-200) $tr(AB) = tr(BA)$.
3. (putnam-and-beyond-201) $(AB - B)^2 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = kI_2$, where k is a root of unity.
4. (putnam-and-beyond-205) $tr(XX^T)$ is the sum of the squares of the entries of X.
$$tr[(A - B^T)(A - B^T)^T] = 0$$
5. (putnam-and-beyond-206) The Vandermonde determinant is equal to $\prod_{i>j}(x_i - x_j)$.
6. (putnam-and-beyond-213) $\det(xA+yB) = a_0(x)y^3+a_1(x)y^2+a_2(x)y+a_3(x)$. Idea: $\det(xA+xB)$, $det(xA - xB)$.
7. (putnam-and-beyond-222) $\det C = \pm 1$, $P(x) = \det(A + xB)$.
8. (putnam-and-beyond-243) $n \geqslant 2$, $rank = 2$
9. (putnam-and-beyond-244) The polynomials $P_j(x) = (x + j)^k$, $j = 0, 1, \ldots, n - 1$, lie in the $(k + 1)$-dimensional real vector space of polynomials of degree at most k.
10. (putnam-and-beyond-248) Let A, B, C, D be $n \times n$ matrices such that $AC = CA$. Prove that
$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = det(AD - CB).$$
11. (putnam-and-beyond-249)
$$A^3 + B^3 = (C^3 + I)B^3.$$
$$C^3 + I = (C + I)(C + \varepsilon I)(C + \varepsilon^2 I).$$
$$P(-1)P(-\varepsilon)P(-\varepsilon^2) = 1$$
where $P(x) = x^2 - mx + n$.
12. (putnam-and-beyond-258) $P_A(\lambda) = \det(\lambda I_n - A)$, $P_A(A) = 0_n$.
13. (putnam-and-beyond-259) Get det. If $\det X = 0$, then $X^2 = (tr(X))X$ and $tr(X) = 2$ or $tr(X) = -1$.

    If $\det(X - 3I) = 0$, then 3 is eigenvalue. $\det(X^3 - 3X + 4I_2) = 0$.
14. (putnam-and-beyond-261) $(AB-BA)^3 - c_1(AB-BA)^2 + c2(AB-BA) - c_3I_3 = O_3$ Take trace.
15. (putnam-2015-B3) If a, b, c, d are in arithmetic progression, then we may write
$$a = r - 3s, b = r - s, c = r + s, d = r + 3s$$
for some r, s. If $s = 0$, then clearly all powers of M are in S. Also, if $r = 0$, then one easily checks that M^3 is in S.

    We now assume $rs \neq 0$, and show that in that case M cannot be in S. First, note that the characteristic polynomial of M is $x^2 - 2rx - 8s^2$, and since M is nonsingular (as $s \neq 0$), this is also the minimal polynomial of M by the Cayley-Hamilton theorem. By repeatedly using the relation $M^2 = 2rM + 8s^2 I$, we see that for each positive integer, we have $M^k = t_k M + u_k I$ for unique real constants t_k, u_k (uniqueness follows from the independence of M and I). Since M is in S, we see that M^k lies in S only if $u_k = 0$.

On the other hand, we claim that if $k > 1$, then $rt_k > 0$ and $u_k > 0$ if k is even, and $t_k > 0$ and $ru_k > 0$ if k is odd (in particular, u_k can never be zero). The claim is true for $k = 2$ by the relation $M^2 = 2rM + 8s^2I$. Assuming the claim for k, and multiplying both sides of the relation $M^k = t_kM + u_kI$ by M, yields

$$M^{k+1} = t_k(2rM + 8s^2I) + u_kM = (2rt_k + u_k)M + 8s^2t_kI,$$

implying the claim for $k + 1$.

16. (putnam-2014-A2) Sub n from $(n - 1)$ row.

17. (ncumc-2017-5) Let c be an eigenvalue of the operator in question, i.e. $(A+B)^{-1}(I+AB)x = cx$ for some non-zero vector x and some complex number c. Then, $x + ABx = c(Ax + Bx)$.

    Mark $Bx = y$. Hence, $(y, x) > 0$, and it is the only condition for x, y. Equation rewritten in the form $A(y - cx) = cy - x$. Moreover, $(cy - x, y - cx) > 0$ due to the fact that A be positively definite. It is also possible that $y = cx$, $cy = x$. this takes place for $x = y$, $c = 1$. Let us introduce a notation $c = a + bi$. Then,

    $$a((x, x) + (y, y)) + bß((y, y) - (x, x)) - (1 + a^2 + b^2)(x, y) > 0.$$

    Consequently, $a > 0$, and for any $a > 0$ and any b, one can find matrices and almost orthogonal vectors x, y of identical lengths such that the inequality takes place.

18. (Садовничий-2-1-11) Умножим k-ю строку на $\frac{(x+n)^{k-1}}{(k-1)!}$. Ряд Тейлора.

19. (Садовничий-2-1-20)

20. (Садовничий-2-1-21) $rg(A + B) \leqslant rg(A) + rg(B)$, $rg(A + B) \geqslant rg(AB) + n$.

21. (Садовничий-2-1-27)

8 Calculus

8.1 Problems

04.07.2018

1. (imc-1994-1-2) Let $f \in C^1(a,b)$, $\lim_{x\to a+} f(x) = +\infty$, $\lim_{x\to b-} f(x) = -\infty$ and
$$f'(x) + f^2(x) \geqslant -1$$
for $x \in (a,b)$. Prove that $b - a \geqslant \pi$ and give an example where $b - a = \pi$.

2. (imc-1994-2-1) Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that $|f'(x)| \leqslant \lambda |f(x)|$ for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

3. (imc-1995-1-2) Let f be a continuous function on $[0,1]$ such that for every $x \in [0,1]$ we have $\int_x^1 f(t)dt \geqslant \frac{1-x^2}{2}$. Show that $\int_0^1 f^2(t)dt \geqslant \frac{1}{3}$.

4. (imc-1996-1-2) Evaluate the definite integral
$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx$$
where n is a natural number.

5. (imc-1998-1-3) Let $f(x) = 2x(1-x)$, $x \in \mathbb{R}$. Define $f_n(x) = f(f(...f(x)...))$ (n times).

   a Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

   b Compute $\int_0^1 f_n(x)dx$ for $n = 1, 2, \ldots$

6. (imc-1998-1-4) The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2$, $f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0,1)$ for which
$$f(\xi)f'(\xi) + f''(\xi) = 0.$$

7. (imc-1998-1-5) Let P be an algebraic polynomial of degree n having only real zeros and real coefficients. Prove that for every real x the following inequality holds:
$$(n-1)(P'(x))^2 \geqslant nP(x)P''(x).$$

8. (imc-1998-1-6) Let $f : [0,1] \to \mathbb{R}$ be a continuous function with the property that for any x and y in the interval, $xf(y) + yf(x) \leqslant 1$. Show that
$$\int_0^1 f(x)dx \leqslant \frac{\pi}{4}.$$

9. (imc-1999-1-3) Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality
$$\left| \sum_{k=1}^{n} 3^k (f(x+ky) - f(x-ky)) \right| \leqslant 1.$$
for every positive integer n and for all x, $y \in \mathbb{R}$. Prove that f is a constant function.

10. (imc-2000-1-1) Is it true that if $f : [0,1] \to [0,1]$ is

    a monotone increasing;

    a monotone decreasing

    then there exists an $x \in [0,1]$ for which $f(x) = x$?

11. (imc-2002-1-2) Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

12. (imc-2003-1-2) Evaluate the limit
$$\lim_{x\to+0}\int_x^{2x}\frac{\sin^m(t)}{t^n}dt$$
$m, n \in \mathbb{N}$

13. (imc-2004-2-2) Let $f, g : [a, b] \to [0, \infty)$ be continuous and nondecreasing functions such that for each$x \in [a, b]$ we have
$$\int_a^x \sqrt{f(t)}dt \leqslant \int_a^x \sqrt{g(t)}dt$$
$$\int_a^b \sqrt{f(t)}dt \leqslant \int_a^b \sqrt{g(t)}dt$$
Prove that
$$\int_a^b \sqrt{1+f(t)}dt \leqslant \int_a^b \sqrt{1+g(t)}dt$$

14. (imc-2005-1-3) Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that
$$\left|\int_0^1 f^3(x)dx - f^2(0)\int_0^1 f(x)dx\right| \leqslant \max_{0\leqslant x\leqslant 1}|f'(x)|\left(\int_0^1 f(x)dx\right)^2$$

15. (imc-2005-2-4) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that
$$\frac{f'''(\xi)}{6} = \frac{f(1)-f(-1)}{2} - f'(0).$$

16. (imc-2006-2-3) Compare $tan(sinx)$ and $sin(tanx)$ for all $x \in (0, \frac{\pi}{2})$.

17. (imc-2007-2-3) Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

18. (imc-2009-2-2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$, and for all $x \in [0, \infty)$,
$$f''(x) - 5f'(x) + 6f(x) \geqslant 0.$$
Prove that for all $x \in [0, \infty)$,
$$f(x) \geqslant 3e^{2x} - 2e^{3x}.$$

19. (imc-2010-1-1) Let $0 < a < b$. Prove that
$$\int_a^b (x^2+1)e^{-x^2}dx \geqslant e^{-a^2} - e^{-b^2}$$

20. (imc-2013-1-2) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose $f(0) = 0$. Prove that there exists $\xi \in (-\frac{-\pi}{2}, \frac{\pi}{2})$ such that
$$f''(\xi) = f(\xi)(1 + 2\tan^2(\xi)).$$

21. (imc-2014-2-3) Let $f(x) = \frac{sinx}{x}$, for $x > 0$, and let n be a positive integer. Prove that
$$\left|f^{(n)}(x)\right| < \frac{1}{n+1},$$
where $f(n)$ denotes the n-th derivative of f.

22. (imc-2016-2-2) Today, Ivan the Confessor prefers continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying
$$f(x) + f(y) \geqslant |x - y|$$
for all pairs x, $y \in [0, 1]$. Find the minimum of $\int_0^1 f(x)dx$over all preferred functions.

8.2 Hints

1. (imc-1994-1-2) $arctg(f(x))$.
2. (imc-1994-2-1) $g(x) = lnf(x) - \lambda x$ is not increasing. $f(x) \geqslant e^{\lambda(x-y)} f(y)$.
3. (imc-1995-1-2) $\int_0^1 (f(t) - t)^2 dt \geqslant 0$.
4. (imc-1996-1-2) $I_n = I_{n-2}$.
5. (imc-1998-1-3) $x_n = f_n(x_0)$ has 0 in limit. $f_n(x) = \frac{1}{2} - 2^{2^n - 1} \left(x - \frac{1}{2}\right)^{2^n}$.
6. (imc-1998-1-4) $g(x) = \frac{1}{2} f(x)^2 + f'(x)$. $h(x) = \frac{x}{2} - \frac{1}{f(x)}$.
7. (imc-1998-1-5) Find $P'(x)/P(x)$ and $P''(x)/P(x)$.
8. (imc-1998-1-6) $I + \int_0^{\frac{\pi}{2}} f(sin\theta) cos\theta d\theta = \int_0^{\frac{\pi}{2}} f(cos\theta) sin\theta d\theta$.
9. (imc-1999-1-3) $3^n (f(x+ny) - f(x-ny) \leqslant 2$.
10. (imc-2000-1-1) yes, no.
11. (imc-2002-1-2) $f(x) < f(0) + x \cdot f(0) = (1+x) f(0)$
12. (imc-2003-1-2) $f(x)/x$ is decreasing.
13. (imc-2004-2-2) The length ot the graph of F is $\geqslant$ the length of the graph of G. This is clear since both functions are convex.
14. (imc-2005-1-3) Integrate $-Mf(x) \leqslant f(x) f'(x) \leqslant Mf(x)$
15. (imc-2005-2-4) $g(x) = -\frac{f(-1)}{2} x^2 (x-1) - f(0)(x^2 - 1) + \frac{f(1)}{2} x^2 (x+1) - f'(0) x (x-1)(x+1)$
16. (imc-2006-2-3) $f(x) = tan(sinx) - sin(tanx)$ increase.
17. (imc-2007-2-3) Suppose $f(x) \neq x$. Let $[a, b]$ be the smallest closed interval that contains C.
18. (imc-2009-2-2) $g(x) = f'(x) - 2f(x)$, $f'(x) - 2f(x) \geqslant -2e^{3x}$
19. (imc-2010-1-1) $f(x) = \int_0^x (t^2 + 1) e^{-t^2} dt$, $g(x) = -e^{-x^2}$.
20. (imc-2013-1-2) $g(x) = f(x) cosx$
21. (imc-2014-2-3) $g(x) = x^{n+1} \left(f^n(x) - \frac{1}{n+1}\right)$
22. (imc-2016-2-2) Triangle ineqality.

9 Sequences and series

9.1 Problems

05.07.2018

1. (imc-1995-2-2) Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that $b_0 = 1$,
$$b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}.$$
Calculate
$$\sum_{n=1}^{\infty} b_n 2^n.$$

2. (imc-1996-2-5) Prove that
$$\lim_{x\to+\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

3. (imc-1997-1-1) Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers, such that $lim_{n\to\infty}\varepsilon_n = 0$. Find
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon_n\right).$$

4. Дискретное преобразование Абеля
$$\sum_{k=m}^{n} a_k b_k = a_n B_n - a_m B_{m-1} - \sum_{k=m}^{n-1} (a_{k+1} - a_k) B_k$$

5. (imc-1999-1-2) Does there exist a bijective map $\pi : N \to N$ such that
$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

6. (imc-1999-2-3) Assume that $x_1, \ldots, x_n \geqslant -1$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove that $\sum_{i=1}^{n} x_i \leqslant \frac{n}{3}$.

7. (imc-2001-1-3) Find
$$\lim_{t\to 1-0} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}.$$

8. (imc-2002-1-3) Let n be a positive integer and let $a_k = \frac{1}{C_n^k}$, $b_k = 2^{k-n}$ for $k = 1, \ldots, n$. Show that
$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} = 0.$$

9. (imc-2003-1-1) Let a_1, a_2 , ... be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2} a_n$ for all n. Prove that the sequence
$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$
has a finite limit or tends to infinity.

10. (imc-2003-2-6) Let $(a_n)_{n\in N}$ be the sequence defined by $a_0 = 1$,
$$a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}$$
Find the limit
$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k}$$

11. (imc-2008-2-3) Let n be a positive integer. Prove that 2^{n-1} divides

$$\sum_{0\leqslant k<n/2} C_n^{2k+1}5^k$$

12. (imc-2010-1-2) Compute the sum of the series

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots$$

13. (imc-2010-1-3) Define the sequencex_1, x_2, ... inductively by $x_1=\sqrt{5}$, $x_{n+1}=x_n^2-2$ for $n\geqslant 1$. Compute

$$\lim_{n\to\infty}\frac{x_1\cdot x_2\cdot x_3\cdots x_n}{x_{n+1}}$$

14. (imc-2010-2-1) A sequence x_1, x_2, ... of real numbers satisfies $x_{n+1}=x_n\cos x_n$ for all $n\geqslant 1$. Does it follow that this sequence converges for all initial values x_1?

    A sequence y_1, y_2, ... of real numbers satisfies $y_{n+1}=y_n\sin y_n$ for all $n\geqslant 1$. Does it follow that this sequence converges for all initial values y_1?

15. (imc-2010-2-2) Let a_0, a_1 , ..., a_n be positive real numbers such that $a_{k+1}-a_k\geqslant 1$ for all $k=0,1,\ldots,n-1$. Prove that

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_n-a_0}\right)\leqslant\left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_n}\right)$$

16. (imc-2011-2-1) Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with $\frac{1}{2}<a_n<1$ for all $n\geqslant 0$. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0=a_0$ and

$$x_{n+1}=\frac{a_{n+1}+x_n}{1+a_{n+1}x_n}$$

    What are the possible values of $\lim_{n\to\infty}x_n$? Can such a sequence diverge?

17. (imc-2012-2-2) Define the sequence a_0, a_1, ...inductively by $a_0=1$, $a_1=\frac{1}{2}$,

$$a_{n+1}=\frac{na_n^2}{1+(n+1)a_n}$$

    for $n\geqslant 1$. Show that the series

$$\sum_{k=1}^{\infty}\frac{a_{k+1}}{a_k}$$

    converges and determine its value.

18. (imc-2013-1-4) Let $n\geqslant 3$ and let x_1, ..., x_n be nonnegative real numbers. Define $A=\sum_{i=1}^n x_i$, $B=\sum_{i=1}^n x_i^2$, $C=\sum_{i=1}^n x_i^3$. Prove that

$$(n+1)A^2B+(n-2)B^2\geqslant A^4+(2n-2)AC$$

19. (imc-2014-1-2) Consider the following sequen

$$(a_n)_{n=1}^{\infty}=(1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,1,\ldots)$$

    Find all pairs (α,β) of positive real numbers such that

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{\infty}a_k}{n^{\alpha}}=\beta$$

20. (imc-2015-1-2) For a positive integer n , let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \leqslant \frac{n^2}{4}$$

When does equality hold?

21. (imc-2015-1-3) Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and $F(n) = \frac{2}{5}F(n-1) - F(n-2)$ for $n \geqslant 2$. Determine whether or not

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$$

is a rational number.

22. (imc-2016-1-3) Let n be a positive integer. Also let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leqslant \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n}(a_i + b_i)}$$

23. (imc-2016-2-1) Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$$

Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leqslant 2$$

9.2 Hints

1. $a_n = 1 + \sqrt{b_n}$, telescope.
2. (imc-1996-2-5) Riemann sum for $f(t) = \frac{t}{(1+t^2)^2}$, where $h = \frac{1}{\sqrt{x}}$.
3. (imc-1997-1-1) Riemann sum for $f(x) = \ln(x)$.
4. Проверить
5. (imc-1999-1-2) Дискретное преобрзование Абеля. $\pi(1) + ... + \pi(n) \geqslant 1 + ... + n$.
6. (imc-1999-2-3) $(x+1)(x-1/2)^2 \geqslant 0$.
7. (imc-2001-1-3) $\int_0^\infty \frac{dx}{1+e^x}$.
8. (imc-2002-1-3)
$$\frac{2^n}{n}\left[\sum_{k=0}^{n-1}\frac{1}{C_{n-1}^k}\right] = \sum_{k=1}^{n}\frac{2^n}{n}$$
9. (imc-2003-1-1) $b_{n+1} > b_n$.
10. (imc-2003-2-6) $f(x) = \sum_{n=0}^\infty a_n x^n$. $f'(x) = ?$. $\ln(f(x)) = \int_0^x \frac{f'(t)}{f(t)} dt$
11. (imc-2008-2-3) Bine formula for Fibonacci.
12. (imc-2010-1-2) $\frac{x^{4k}}{4k}$
13. (imc-2010-1-3) $x_{n+1}^2 - 4 = x_n^2(x_n^2 - 4)$
14. (imc-2010-2-1) No, $x_n = (-1)^{n-1}\pi$. Yes, Since the function $t \to t\sin t$ is continuous, $y_{n+1} = |y_n|\sin|y_n| \to |a|\sin|a| = a$
15. (imc-2010-2-2) Apply induction on n.
16. (imc-2011-2-1) $u_n = arctanha_n$ (hyperbolic).
17. (imc-2012-2-2) $ka_k = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1}$
18. (imc-2013-1-4) $p(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$. The $(n-3)$-th derivative of p has three nonnegative real roots $0 \leqslant u \leqslant v \leqslant w$.
19. (imc-2014-1-2) Let see first $n(n+1)/2$ addend.
20. (imc-2015-1-2) $2^{s-1} - 1 \leq n \leqslant 2^s - 1$.
21. (imc-2015-1-3) $F(n) = 2^n - 2^{-n}$
22. (imc-2016-1-3) $\frac{XY-Y^2}{X+Y} = Y - \frac{2Y^2}{X+Y}$
23. (imc-2016-2-1) interchang the sum, $\sum_{k=m}^\infty \frac{1}{k^2} < \frac{1}{n-\frac{1}{2}}$.

10 Linear algebra

10.1 Problems

09.07.2018

1. (imc-1994-1-1) Let A be a $n \times n$, $n \geqslant 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leqslant n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.

2. (imc-1995-1-5) Let A and B be real $n \times n$ matrices. Assume that there exist $n+1$ different real numbers t_1, t_2, $\dots$, t_{n+1} such that the matrices $C_i = A + t_i B$, $i = 1, 2, \dots, n+1$, are nilpotent (i.e. $C_i^n = 0$). Show that both A and B are nilpotent.

3. (imc-1995-2-1) Let A be 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u \in \mathbb{R}^3$. Prove that:
   - $A^T = -A$;
   - there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$.

4. (imc-1996-1-1) Let for $j = 0, \dots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 & a_1 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{vmatrix}$$

   Calculate $det(A)$.

5. (imc-1996-1-3) The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

6. (imc-1997-1-3) Let A and B be real $n \times n$ matrices such that $A^2 + B^2 = AB$. Prove that if $BA - AB$ is an invertible matrix then n is divisible by 3.

7. (imc-1997-2-2) Let M be an invertible matrix of dimension $2n \times 2n$, represented in block form as

$$M = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

$$M^{-1} = \begin{vmatrix} E & F \\ G & H \end{vmatrix}$$

   Show that $\det M \cdot \det H = \det A$.

8. (imc-1999-1-1)
   - Show that for any $m \in N$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix.
   - Show that $\det A > 0$ for every real $m \times m$ matrix satisfying $A^3 = A + I$.

9. (imc-2000-1-3) A and B are square complex matrices of the same size and $rank(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$.

10. (imc-2002-2-1) Compute the determinant of the $n \times n$ matrix $A = a_{ij}$:

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & i \neq j \\ 2, & i = j \end{cases}$$

11. (imc-2003-1-3) Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.)

12. (imc-2003-2-1) Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

13. (imc-2004-2-1) Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

Find BA.

14. (imc-2005-1-1) Let A be the $n \times n$ matrix, whose (i, j)-th entry is $i + j$ for all $i, j = 1, 2, \ldots, n$. What is the rank of A?

15. (imc-2007-2-4) Let $n > 1$ be an odd positive integer and $A = (a_{ij})$ be an $n \times n$ matrix with:

$$a_{ij} = \begin{cases} 2, & i = j \\ 1, & i - j = \pm 2 \pmod n \\ 0, & \text{otherwise} \end{cases}$$

Find $\det(A)$.

16. (imc-2008-2-5) Let $n > 1$ be an odd positive integer and $A = (a_{ij})$ be an $n \times n$ matrix with:

$$a_{ij} = \begin{cases} 1, & i + j \text{ is prime} \\ 0, & \text{otherwise} \end{cases}$$

Prove that $\det A = k^2$ for some integer k.

17. (imc-2008-2-2) Let A, B and C be real square matrices of the same size, and suppose that A is invertible. Prove that if $(A - B)C = BA^{-1}$, then $C(A - B) = A^{-1}B$.

18. (imc-2009-2-3) Let A, $B \in M_n(C)$ be two $n \times n$ matrices such that $A^2B + BA^2 = 2ABA$. Prove that there exists a positive integer k such that $(AB - BA)^k = 0$.

19. (imc-2010-1-5) Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geqslant a^2 + b^2 + c^2$$

Prove that

$$1 + 2(abc)^n \geqslant a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.

20. (imc-2013-1-1) Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

21. (imc-2014-1-1) Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $trace(M) = a$ and $det(M) = b$.

22. (imc-2014-2-2) Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix with real entries, and let λ_1, λ_2, $\ldots$, λ_n denote its eigenvalues. Show that

$$\sum_{1 \leqslant i < j \leqslant n} a_{ii}a_{jj} \geqslant \sum_{1 \leqslant i < j \leqslant n} \lambda_i\lambda_j$$

and determine all matrices for which equality holds.

23. (imc-2015-1-1) For any integer $n \geqslant 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that $det(A) = det(B)$. Does the same conclusion follow for matrices with complex entries?

24. (imc-2017-1-1) Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

10.2 Hints

1. (imc-1994-1-1) we have at least two non-zero elements in every column of A^{-1}.
2. (imc-1995-1-5) n^2 polynom for each matrix element
3. (imc-1995-2-1) Set $v_1 = -a_{23}$, $v_2 = a_{13}$, $v_3 = -a_{12}$.
4. (imc-1996-1-1) $(-1)^n a_0 + a_n 2^{n-1} d^n$
5. (imc-1996-1-3) The minimal polynomial of A is a divisor of $x^2 - 1$. Eigenvalue equal to ± 1 and its multiplicity is less or equal to 1. Matrix is diagonalizable.
6. (imc-1997-1-3) $S = A + e^{i\frac{2\pi}{3}} B$. Then $S\overline{S} = e^{i\frac{2\pi}{3}}(BA - AB)$.
7. (imc-1997-2-2) The definition. If A or H invertable. $\det(XY) = \det(YX)$.
8. (imc-1999-1-1) The minimal polynomial of A is a divisor of $p(x) = x^3 - x - 1$ has a positive real root and two conjugated complex roots. Matrix is diagonalizable. $\det A$ equal to product of eigenvalues.
9. (imc-2000-1-3) Jordan form.
10. (imc-2002-2-1) $det(A) = n + 1$
11. (imc-2003-1-3) The minimal polynomial of A is a divisor of $3x^3 - x^2 - x - 1$. This polynomial has three different roots. This implies that A is diagonalizable. The eigenvalues of the matrices A and D are all roots of polynomial. One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of D^k, which are the k-th powers of the eigenvalues, tend to either 0 or 1 and the limit $M = \lim D^k$ is idempotent. Then $\lim A^k = C^{-1}MC$ is idempotent as well.

    Минимальный многочлен делит характеристический многочлен матрицы. Любой аннулирующий многочлен делится на минимальный[1]. Минимальный многочлен единственен. Множество корней минимального многочлена совпадает с множеством корней характеристического многочлена матрицы.
12. (imc-2003-2-1) $(A + I)(B + I) = I = (B + I)(A + I)$
13. (imc-2004-2-1) $A = (A_1 A_2)^T$, $B = (B_1 B_2)$.
14. (imc-2005-1-1) $rank = 2$
15. (imc-2007-2-4) $A = B^2$.
16. (imc-2008-2-5) $\left|C00C\right| = k^2$ Now let X' be the matrix obtained from A by replacing the first row by $(100\dots0$, and let Y be the matrix obtained from A by replacing the entry a_{11} by 0. By multilinearity of the determinant, $det(A) = det(X') + det(Y)$. Note that X' can be written as $\left|10vX\right|$ for some $(n-1)\times(n-1)$-matrix X and some column vector v. Then $det(A) = det(X) + det(Y)$. Now consider two cases. If n is odd, then X is of type(C), and Y is of type (B). Therefore, $|det(A)| = |det(X)|$ is a perfect square. If n is even, then X is of type (B), and Y is of type (C); hence $|det(A)| = |det(Y)|$ is a perfect square. The set of primes can be replaced by any subset of $\{2\} \cup \{3, 5, 7, 9, 11, \dots\}$.
17. (imc-2008-2-2) $XY = I = YX$.
18. (imc-2009-2-3) $\delta^2 B = 0$. $\delta^k(B^k) = k!(\delta B)^k$. Cayley-Hamilton Theorem.

    Set $X = AB - BA$. The matrix X commutes with A. $X^{m+1} = AX^mB - BX^mA$. Take trace.
19. (imc-2010-1-5) $\begin{vmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{vmatrix}$

20. (imc-2013-1-1) The transforms given by A and B strictly increase the length of every nonzero vector, this can be seen easily in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product AB also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than -1.

21. (imc-2014-1-1) Note that $trace(M) = a$ and $det(M) = b$ if and only if the two eigenvalues of M are solutions of $x^2 - ax + b = 0$.

22. (imc-2014-2-2) Eigenvalues of a real symmetric matrix are real, hence the inequality makes sense. Similarly, for Hermitian matrices diagonal entries as well as eigenvalues have to be real.

$$\sum_i a_{ii}^2 \geqslant \sum_{i,j} a_{ij}^2 = \sum_i \lambda_i^2$$

23. (imc-2015-1-1) $X = AB^{-1}$, $X^3 = I$. In case of complex matrices the statement is false.

24. (imc-2017-1-1)$A^4 - A = 0$. The matrix represents a rotation by $2\pi/3$.

11 Number Theory

11.1 Problems

10.07.2018

1. Докажите, что существует бесконечно много чисел, не представимых в виде суммы 3 точных квадратов.
2. Доказать, что существует бесконечно много чисел, не представимых в виде суммы трёх кубов.
3. Докажите, что уравнение не имеет решений в целых числах: $x^3 + 21y^2 + 5 = 0$.
4. Докажите, что $x^2 + xy - y^2 = 3$ не имеет целых решений.
5. Существует ли такое натуральные n, что $n^2 + n + 1$ делится на 2015?
6. Решите в натуральных числах $x! - 1 = y^2$.
7. Докажите, что если $a^2 + b^2 + ab$ делится на 11, то a и b делятся на 11.
8. Найдите все пятерки простых чисел, для которых число $q_1^4 + q_2^4 + q_3^4 + q_4^4 + q_5^4$ равно произведению двух последовательных четных натуральных чисел.
9. Докажите, что уравнение $x^3 + y^3 = 4(x^2y + xy^2 + 1)$ не имеет решений в целых числах.
10. Найдите все такие пары простых чисел p и q, что $p^3 - q^5 = (p+q)^2$.
11. Дана последовательность $a_n = 1 + 2^n + 3^n + 4^n + 5^n + 6^n + 7^n$. Существуют ли a_n, делящиеся на 2016?
12. Докажите, что число $60^{2017} + 2017^{60}$ – составное.
13. Пусть p – простое число, $p > 2$. Докажите, что любой простой делитель числа $2^p - 1$ имеет вид $2kp + 1$.
14. Докажите, что ни при каком целом k число $k^2 + k + 1$ не делится на 101.
15. Пусть $(a, b) = 1$.
    - Пусть $a|bc$. Докажите, что $a|c$.
    - Докажите, что $(a, bc) = (a, c)$.
16. Докажите, что дробь несократима при всех натуральных n:
    а) $\frac{21n+4}{14n+3}$; б) $\frac{7n+16}{3n+7}$;
17. Докажите, что если a делится на b, то $(a, b) = b$. При каких целых n следующая дробь будет целой:
    а) $\frac{n^2+1}{n+1}$; б) $\frac{n^3+n+1}{n^2-n+1}$; в) $\frac{n^4+1}{n^2+n+1}$.
18. Докажите, что при $m \neq n$ выполняется равенство: $(a^m - 1, a^n - 1) = a^{(m,n)} - 1 (a > 1)$.
19. Пусть $f_k = 2^{2^k} + 1$ – числа Ферма.
    - Составьте рекуррентную формулу f_k и f_{k-1}.
    - Докажите, что $(f_n, f_m) = 1$, где $n \neq m$.
20. Пусть F_n – числа Фибоначчи ($F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.
    - Докажите, что $F_n = F_k \cdot F_{n-k+1} + F_{k-1} \cdot F_{n-k}$.

- Докажите, что $(F_n, F_m) = F_{(n,m)}$.

21. Пусть $b \neq d$ и $\frac{a}{b}$, $\frac{c}{d}$ – несократимые дроби. Докажите, что $\frac{a}{b} + \frac{c}{d}$ – не целое число.

22. Докажите, что $\frac{[a,b,c]^2}{[a,b]*[b,c]*[a,c]} = \frac{(a,b,c)^2}{(a,b)*(b,c)*(c,a)}$.

23. При каком наибольшем n число $n^3 + 100$ делится на $n + 10$.

24. Найдите все такие натуральные n, что $80^n - 1$ делится на $8^n - 1$ без остатка.

25. Найти $(5^m + 7^m, 5^n + 7^n)$.

26. Докажите, что число $(36a + b)(36b + a)$ не может быть точной степенью двойки.

27. Известно, что $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$. Пусть $h = (x, y, z)$. Докажите, что $hxyz$ – полный квадрат.

12 Train

12.1 Problems

15-20.07.2018

1. Доказать, что если матрицы A и B действительные положительно определенные симметричные порядка n, то
$$\det(A+B) \geqslant \det(A) + \det(B)$$

2. Пусть функция $f(x)$ интегрируема по Риману на отрезке $[0;1]$, причем $|f(x)| \leqslant 1$. Доказать неравенство
$$\int_0^1 \sqrt{1-f(x)^2}dx \leqslant \sqrt{1-\left(\int_0^1 f(x)dx\right)^2}$$

3. Функция $f(x)$ дважды непрерывно дифференцируема на отрезке $[a,b]$ и имеет на $[a,b]$ не менее трех различных нулей. Доказать, что существует точка $\xi \in [a,b]$ такая, что
$$f(\xi) + f''(\xi) = 2f'(\xi)$$

4. Let $a_1 = a_2 = 97$ and
$$a_{n+1} = a_n a_{n-1} + \sqrt{(a_n^2-1)(a_{n-1}^2-1)}$$
Prove that
   - $2+2a_n$ is a perfect square;
   - $2+\sqrt{2+2a_n}$ is a perfect square.

5. Evaluate
$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k(n-k)!+(k+1)}{(k+1)!(n-k)!}$$

6. Determine whether or not there exist integers $a, b, c > 2018$ satisfying the equation
$$a^3 + 2b^3 + 4c^3 = 6abc + 1.$$

7. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that
$$\int_0^1 (f'(x))^2 dx = 1$$
Prove that $|f(1) - f(0)| < 1$.

8. Let $A \in \mathbb{R}^{m,n}$, $B \in \mathbb{R}^{n,p}$, $C \in \mathbb{R}^{p,q}$, $D \in \mathbb{R}^{q,m}$ be such that
$$A^T = BCD, B^T = CDA, C^T = DAB, D^T = ABC$$
Prove that $(ABCD)^2 = ABCD$.

9. Let $P(x)$ be a polynomial with real coefficients of degree at least 2. Prove that if there is a real number a such that
$$P(a)P''(a) > (P'(a))^2,$$
then P has at least tow nonreal zeros.

10. Let $A, B \in \mathbb{R}^{2018,2018}$ such that $AB = BA$, $A^{2018} = I$, $B^{2018} = I$ and $tr(AB) = 2018$. Prove that $tr(A) = tr(B)$.

11. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(z)f(\mathrm{i}z) = z^2$ for all $z \in \mathbb{C}$. Prove that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

12. a) k is even positive integer and A is a real symmetric $n \times n$ matrix such that

$$(tr(A^k))^{k+1} = (tr(A^{k+1}))^k.$$

Prove that $A^n = tr(A)A^{n-1}$.

b) Does the assertion from a) also hold for odd positive integers k.

12.2 Hints

1. (Садовничий, стр. 229, №23) Докажем при $A = I$. Докажем при $A = C^2$, где C — симметрическая положительно определенная.

2. (Садовничий, стр. 120, №23) Неравенство Коши-Шварца.

3. (Садовничий, стр. 101, №5) Теорема Ролля для функции $g(x) = e^{-x}f(x)$.

4. (Titu 360, p.80, №55) The expression $a_1^2 - 1 = x^2$ and $2a_1 + 2 = y^2$ ask for substitution: $a_1 = \frac{1}{2}(b^2 + \frac{1}{b^2})$, hence $b + \frac{1}{b} = 14$. Setting $b = c^2$, and $c = 2 + sqrt3$.
   We will prove that $a_n = \frac{1}{2}\left(c^{4F_n} + c^{-4F_n}\right)$.

5. (Titu 360, p.195, №13) Is equal to $1 - \frac{1}{(n+1)!} + \frac{2^n}{n!}$.

6. (AOPS, USA December TST for IMO 2012, Problem 3)
   $$x^3 + y^3 + z^3 - 3xyz = \det\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}$$
   Let $w^3 = 2$. Then
   $$x^3 + 2y^3 + 4z^3 - 6xyz = \det\begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix} \cdot \det\begin{pmatrix} 1 & w & w^2 \\ w^2 & 1 & w \\ w & w^2 & 1 \end{pmatrix}$$

7. (Titu 360, p.218, №56)
   $$\int_0^1 (f'(x) - 1)^2 dx \geqslant 0$$

8. (AOPS) $X = ABCD$.
   $$A = D^T C^T B^T = ABCDABCDA = X^2 A \Leftrightarrow X = X^3$$
   Минимальный аннулирующий многочлен матрицы X делит многочлен $t^3 - t$, то есть имеет корни среди $\{-1, 0, 1\}$, причем все они имеют степень не выше 1. Значит, матрица диагонализируема.
   $$AA^T = ABCD = X \Rightarrow (Xe, e) = (A^T e, A^T e) \geqslant 0 \Rightarrow$$
   $$X \geqslant 0 \Rightarrow \lambda(X) \geqslant 0$$
   На диагонали стоят только 0 или 1, то есть $X^2 = X$.

9. (Titu 360, p.45, №57) Differentiate the equation:
   $$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}$$

10. (AOPS) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = tr(AB)$.

11. (Titu 360, p.36, №39) Substitute z with zi.

12. (AOPS)
    $$(\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k)^{\frac{1}{k}} = (\lambda_1^{k+1} + \lambda_2^{k+1} + \cdots + \lambda_n^{k+1})^{\frac{1}{k+1}}$$
    .