Problem Set 1

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Problem 1

If a and b are two even integers, then a + b is even.
Solution:
Direct proof: Since a and b are two even integers, there exist two integers x and y such that a = 2x and b = 2y.
Then, a + b = 2x + 2y = 2(x + y),
Since (x + y) is an integer, a + b must be an even integer.

Problem 2

Prove that $\sqrt{2}$ is irrational.
Solution:
Prove by contradiction: Assume $\sqrt{2}$ is rational, there exists $\frac{p}{q} = \sqrt{2}$ such that p and q are integers, p and q have no common divisor.
Calculation steps:

$$\frac{p}{q} = \sqrt{2}$$

$$\Leftrightarrow \frac{p^2}{q^2} = 2$$

$$\Leftrightarrow p^2 = 2q^2$$

Since $p^2 = 2q^2$, p^2 must be even, so p must be even as well, there exist an integer k such that p = 2k. So, $p^2 = (2k)^2$

$$\Leftrightarrow (2k)^2 = 2q^2$$

$$\Leftrightarrow 2k^2 = q^2$$

(This implies that both p^2 and q^2 contain a common divisor 2, leads to a contradiction)
Therefore, $\sqrt{2}$ must be irrational.

Problem 3

Prove that if n^2 is even, then n is even.
Solution:
Prove by contrapositve: Rewrite the predicate as "if n is odd, then n^2 is odd."
Since n is odd, there exists an integer k such that n = 2k + 1.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since $2k^2 + 2k$ is an integer, n^2 must be odd, proving the predicate. Therefore, the original predicate is proven.

Problem 4

Show that 1 + 2 + 3 + ... + n = $\frac{n(n+1)}{2}$
Solution: Prove by mathematical induction.
1. Base case, where n = 1, $\frac{1(1+1)}{2} = 1$ (P(n)).
2. Induction Hypothesis:
Assume P(n) is true for an arbitrary natural number k, that:
1 + 2 + 3 + ... + k = $\frac{k(k+1)}{2}$ = P(k).
3. Incduction steps:
Prove that if P(k) is true, then P(k + 1) must also be true.
1 + 2 + 3 + ... + k + (k + 1) = $\frac{k(k+1)}{2} + k + 1$
$\frac{k(k+1)}{2} + k + 1 = (\frac{k}{2} + 1)(k+1) = \frac{k+2}{2}(k+1) = \frac{k+2(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2}$
Since P(k + 1) is true, the original predicate is proved.