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7. Derive the solution of Put option from Black-Schole PDE.

Ans European style put option

$$rF = F_t + \tfrac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S \quad \text{— (1)}$$

$$F(S,T) = (K-S)^+ \quad \text{— (I)}$$

$$F(0,t) = 0 \quad \text{— (II)}$$

$$F(S,t) \to Ke^{-r(T-t)} - S \quad \text{— (IV)}$$

Side Conditions

Convert (1) to a well-known PDE, the heat equation. Suppose F, S, and t are defined in terms of the new variables v, x, and τ as in the following equations:

$$S = Ke^x \longleftrightarrow x = \ln\frac{S}{K} = \ln S - \ln K \quad \text{and} \quad e^{-x} = K/S$$

$$t = T - \frac{2\tau}{\sigma^2} \longleftrightarrow \tau = \frac{\sigma^2}{2}(T-t)$$

$$F(S,t) = Kv(x,\tau) \quad \text{— (2)}$$

The multivariable form of the chain rule can be used to determine new expressions for the derivatives present in (1):

จะได้ $F_S = (Kv(x,\tau))_S = K(v_x x_S + v_\tau \overset{0}{\tau_S}) = K\left(\frac{v_x}{S} + 0\right) = e^{-x}v_x$ — (3)

$$F_t = (Kv(x,\tau))_t = K(v_x \overset{0}{x_t} + v_\tau \tau_t) = K\left(0 + v_\tau\left(\frac{-\sigma^2}{2}\right)\right) = \frac{-K\sigma^2}{2}v_\tau \quad \text{— (4)}$$

$$F_{SS} = (e^{-x}v_x)_S$$

$$= e^{-x}\frac{\partial v_x}{\partial S} + v_x\frac{\partial}{\partial S}e^{-x}$$

$$= v_x\frac{\partial}{\partial S}e^{-x} + e^{-x}\frac{\partial v_x}{\partial S}$$

$(v_x(x,\tau))_S = v_{xx}x_S + v_{x\tau}\tau_S$

$$= v_x\frac{\partial}{\partial x}\frac{\partial x}{\partial S}e^{-x} + e^{-x}\left(v_{xx}\overset{1/S}{\frac{dx}{dS}} + v_{x\tau}\overset{0}{\frac{d\tau}{dS}}\right)$$

$$= -e^{-x}v_x\underset{1/S}{\frac{dx}{dS}} + \frac{e^{-x}}{S}v_{xx}$$

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$$= \frac{-e^{-x}}{S} V_x + \frac{e^{-x}}{S} V_{xx}$$

$$= \frac{e^{-x}}{S}(V_{xx} - V_x) \quad ; \text{จาก } S = Ke^x$$

$$F_{SS} = \frac{e^{-2x}}{K}(V_{xx} - V_x) \quad \text{—— (5)}$$

แทน (2), (3), (4), (5) ใน (1) จะได้ :

$$r(Kv(x,\tau)) = -\frac{K\sigma^2}{2} v_\tau + \frac{1}{2}\sigma^2 S^2 \left(\frac{e^{-2x}}{K}(V_{xx} - V_x)\right) + rS(e^{-x} V_x)$$

แทน $K = 2r/\sigma^2$ จะได้ $rV = -\frac{\sigma^2}{2} v_\tau + \frac{1}{2}\sigma^2 (V_{xx} - V_x) + rV_x$

$$\frac{\sigma^2}{2} v_\tau = -rv + \frac{1}{2}\sigma^2 (v_{xx} - v_x) + rv_x$$

$$v_\tau = v_{xx} + (K-1) v_x - Kv \quad \text{—— (6)}$$

The Final condition for F is converted by this change of variables into an initial condition since when $t = T$, $\tau = 0$. The initial condition $v(x,0)$ is then found to be

$$\begin{aligned} Kv(x,0) &= F(S,T) \\ &= (K - S)^+ \\ &= K(1 - e^x)^+ \\ v(x,0) &= (1 - e^x)^+ \end{aligned}$$

Since $\lim_{S \to 0^+} F(S,t) = \lim_{x \to -\infty} Kv(x,\tau) \Rightarrow \lim_{x \to -\infty} v(x,\tau) = 0$

Likewise since as $\lim_{S \to \infty} x = \infty$,

$$\lim_{S \to \infty} F(S,t) = Ke^{-r(T-t)} - S = \lim_{x \to \infty} Kv(x,\tau)$$

$$e^{-K\tau} - e^x = \lim_{x \to \infty} v(x,\tau)$$

Hence, we have derived a pair of boundary conditions for the partial differential equation in (6). So, the original Black-Scholes initial, boundary value problem can be recast in the form of the following.

$$v_\tau = v_{xx} + (K-1) v_x - Kv \quad \text{—— (*)}$$

$$v(x,0) = (1 - e^x)^+ \quad \text{—— (**)}$$

$$v(x,\tau) \to 0 \quad \text{—— (x)}$$

$$v(x,\tau) \to e^{-K\tau} - e^x \quad \text{—— (y)}$$

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Now another change of variables is needed. If α and β are constants then we can introduce the new dependent variable u.

$$v(x,\tau) = e^{\alpha x+\beta\tau}u(x,\tau) \quad (7)$$
$$v_x = e^{\alpha x+\beta\tau}(\alpha u(x,\tau)+u_x) \quad (8)$$
$$v_{xx} = e^{\alpha x+\beta\tau}(\alpha^2 u(x,\tau)+2\alpha u_x+u_{xx}) \quad (9)$$
$$v_\tau = e^{\alpha x+\beta\tau}(\beta u(x,\tau)+u_\tau) \quad (10)$$

Substituting the expressions found in (7)-(10) into (*) :

$$u_\tau = (\alpha^2+(k-1)\alpha-k-\beta)u+(2\alpha+k-1)u_x+u_{xx} \quad (11)$$

Since α and β are arbitrary constants, they can now be chosen appropriately to simplify (11). Ideally, the coefficients of u_x and u would be zero. Solving the two equations:

$$0 = \alpha^2+(k-1)\alpha-k-\beta$$
$$0 = 2\alpha+k-1$$

yields $\alpha=(1-k)/2$ and $\beta=-(k+1)^2/4$. The initial condition for u can be derived from the initial condition given in (**)

$$v(x,0) = (1-e^x)^+$$
$$u(x,0) = e^{(k-1)x/2}(1-e^x)^+$$
$$= e^{(k-1)x/2}\begin{cases}0 & \text{if } x\geq 0\\ 1-e^x & \text{if } x<0\end{cases}$$
$$= \begin{cases}0 & \text{if } x\geq 0\\ e^{(k-1)x/2}-e^{(k+1)x/2} & \text{if } x<0\end{cases}$$
$$= \left(e^{(k-1)x/2}-e^{(k+1)x/2}\right)^-$$

Likewise, we can derive boundary conditions at $x=\pm\infty$ for u from (x) and (y). To summarize, the original Black-Scholes partial differential equation, initial, and boundary conditions have been converted to the following system of equations.

$$u_\tau = u_{xx} \quad (12)$$
$$u(x,0) = \left(e^{(k-1)x/2}-e^{(k+1)x/2}\right)^- \quad (13)$$
$$u(x,\tau)\to 0 \quad (14)$$
$$u(x,\tau)\to e^{\frac{(k-1)}{2}[x+(k-1)\tau/2]}-e^{\frac{(k+1)}{2}[x+(k+1)\tau/2]} \quad (15)$$

The PDE of (12) is the well-known heat equation of mathematical physics. The Fourier Transform technique introduced earlier will now be used to solve this initial boundary value problem.

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• Solving the Black-Scholes Equation:

taking the Fourier transform of both sides of the heat equation in (12)

$$F\{u_\tau\} = F\{u_{xx}\}$$
$$\int_{-\infty}^{\infty} u_\tau e^{-iwx}\,dx = \int_{-\infty}^{\infty} u_{xx} e^{-iwx}\,dx$$
$$\frac{d}{d\tau}\int_{-\infty}^{\infty} u(x,\tau)e^{-iwx}\,dx = (iw)^2\int_{-\infty}^{\infty} u(x,\tau)e^{-iwx}\,dx$$
$$\frac{d\hat{u}}{d\tau} = -w^2\hat{u}$$

where $\hat{u}$ is the Fourier transform of $u(x,\tau)$. The last equation is an ordinary differential equation of the type used to model exponential decay. Seperating variables and solving this equation produces a solution of the form: $\hat{u}(w,\tau) = De^{-w^2\tau}$

The expression represented by D is any quantity which is constant with respect to τ. Even though this equation was solved using ordinary differential equations techniques, the quantity $\hat{u}$ is a function of both w and τ. To evaluate D, we can set $\tau = 0$ and determine that $\hat{u}(w,0) = D$. Thus, D is merely the Fourier Transform of the initial conditions in (13). For simplicity of notation, we will write $D = \hat{f}(w)$. Thus, the Fourier transformed solution to the heat equation is $\hat{u}(w,\tau) = \hat{f}(w)e^{-w^2\tau}$

Now, this solution must be inverse Fourier transformed and then have its variables changed back to the original variables of the Black-Scholes equation.

$$F^{-1}\{\hat{u}(w,\tau)\} = F^{-1}\{\hat{f}(w)e^{-w^2\tau}\}$$

โดย $F^{-1}\{e^{-w^2\tau}\} = \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/(4\tau)}$

$$u(x,\tau) = (e^{(k-1)x/2} - e^{(k+1)x/2})^{+} * \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/(4\tau)}$$
$$= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}(e^{(k-1)z/2} - e^{(k+1)z/2})^{+}e^{-\frac{(x-z)^2}{4\tau}}\,dz$$

แทนค่า z ด้วย $x + \sqrt{2\tau}\,y$ จะได้

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{(k-1)(x+\sqrt{2\tau}y)/2} - e^{(k+1)(x+\sqrt{2\tau}y)/2}\right)^{+}\cdot e^{-y^2/2}\,dy$$

Thus far, the solution to the initial boundary value problem (12) $\longrightarrow$ (15) is given by:

$$u(x,\tau) = e^{(k-1)(x/2)+(k-1)^2(\tau/4)}\,\Phi\left(\frac{-x}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}\right) - e^{(k+1)(x/2)+(k+1)^2(\tau/4)}\,\Phi\left(\frac{-x}{\sqrt{2\tau}} - \frac{1}{2}(k+1)\sqrt{2\tau}\right)$$

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Now, beginning the task of reconverting variables to those of the Black-Scholes initial boundary value problem as stated in (I) to (IV). Using the change of variables in (7) this solution can be re-written in terms of the function $v(x,\tau)$ where

$-(\cancel{k^2}+2k\cancel{+1})+(\cancel{k^2}-2k\cancel{+1}) = -4k$

จะได้ $(-\cancel{4}k)\dfrac{\tau}{\cancel{4}}$

$= -k\tau$

$$v(x,\tau) = e^{-(k-1)(x/2)-(k+1)^2\tau/4}\cdot u(x,\tau)$$

$$= e^{-\cancel{(k-1)(x/2)}-(k+1)^2(\tau/4)+\cancel{(k-1)(x/2)}+(k-1)^2(\tau/4)}\,\Phi\left(\frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k-1)\sqrt{2\tau}\right)$$

$$-e^{-(k-1)(x/2)-\cancel{(k+1)^2(\tau/4)}+(k+1)(x/2)+\cancel{(k+1)^2(\tau/4)}}\,\Phi\left(\frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}\right)$$

$$= e^{-k\tau}\,\Phi\left(\overbrace{\frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k-1)\sqrt{2\tau}}^{-w'}\right)-e^{x}\,\Phi\left(\underbrace{\frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}_{-w}\right) \longrightarrow \text{Z}$$

Now, using the change of variables,

$$w = \frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k+1)\sqrt{2\tau} = \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \longrightarrow -w = \frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}$$

$$w' = \frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k-1)\sqrt{2\tau} = w-\sigma\sqrt{T-t} \longrightarrow -w' = \sigma\sqrt{T-t}-w$$

$\boxed{S = Ke^{x}}$

แทน $-w$ และ $-w'$ ใน Z จะได้ $v(x,\tau) = e^{-r(T-t)}\Phi(\sigma\sqrt{T-t}-w)-\underbrace{e^{x}}_{S/K}\Phi(-w)$

$\times K$ ทั้งสองข้างของสมการ

$$P(S,t) = Ke^{-r(T-t)}\Phi(\sigma\sqrt{T-t}-w)-S\Phi(-w)$$

$$\boxed{P(S,t) = Ke^{-r(T-t)}\Phi(\sigma\sqrt{T-t}-w)-S\Phi(-w)}$$

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