LAPLACE TRANSFORMS

① $\mathcal{L}\{H(t)\} = \frac{1}{s}$

② $\mathcal{L}\{\delta(t)\} = 1$

③.a $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

③ $\mathcal{L}\{\sin(\omega t)H(t)\} = \frac{\omega}{s^2+\omega^2}$

④ $\mathcal{L}\{\cos(\omega t)H(t)\} = \frac{s}{s^2+\omega^2}$

⑤ $\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}F(s)$

⑥ $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

⑦ $\mathcal{L}\{f(t-\tau)\} = e^{-\tau s}F(s)$

⑧ $\mathcal{L}\{\frac{d}{dt}f\} = sF(s) - f(0^-)$

$$\mathcal{L}\left\{\frac{d}{dt}\cdot\left(\frac{d}{dt}f\right)\right\} = s(sF(s) - f(0^-)) - \dot{f}(0)$$
$$= s^2F(s) - sf(0) - \dot{f}(0)$$

⑨ IVT: If poles of $F(s)$ have -ve real parts $\Rightarrow f(0^+) = \lim_{s\to\infty} sF(s)$

⑩ FVT: " $\rightarrow f(\infty) = \lim_{s\to 0} sF(s)$

⑪
$$\mathcal{L}\left\{\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n t\sqrt{1-\zeta^2}\right)\right\} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

STATE SPACE REPRESENTATION

① Model

$$\begin{cases}\dot{x} = Ax + Bu & \Rightarrow \text{dynamics}\\ y = Cx + Du & \Rightarrow \text{output}\end{cases}$$

② Transfer func.:

$$G(s) = C\underbrace{(sI-A)^{-1}}_{}B + D$$

$$\frac{\operatorname{adj}(sI-A)}{\det(sI-A)} \longrightarrow \begin{bmatrix} a & b \\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

③ Linearization:

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{\bar{x},\bar{u}}, \quad D = \left.\frac{\partial h}{\partial u}\right|_{\bar{x},\bar{u}}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \cdot & - & - \\ & & \\ - & - & - \end{bmatrix} \rightarrow \text{Row } i, \text{ col } j \quad \frac{\partial f_i \leftarrow i^{th}\ f \text{ comp.}}{\partial x_j \leftarrow j^{th}\ x \text{ comp.}}$$

LTI SYSTEM

① Complete response

$$y(t) = \underbrace{e^{A(t-t_0)} x(t_0)}_{\text{Zero input response}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{Zero state response}}$$

$$\begin{cases} \dot{x} = Ax \\ x(t_0) \end{cases} \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \quad x(t_0) = 0$$

② Matrix exponential / state transition matrix

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = 1 + At + \frac{At}{2!} + \dots = \underbrace{\begin{bmatrix} e^{\lambda_1} & & - - \\ 0 & \ddots & \\ \vdots & & e^{\lambda_n} \end{bmatrix}}_{A \text{ is diagonalizable}}$$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & \vdots \\ \vdots & 0 & \lambda_n \end{bmatrix}$$

FIRST ORDER SYSTEM

① Form:

$$G(s) = \frac{\mu}{1 + s\tau}$$

② Performance metrics:

Steady state gain : μ

Rise time : 2τ (time b/w $0.9 y_{ss}$ & $0.1 y_{ss}$)

5% settling time : 3τ

2% " : 4τ

1% " : 5τ

SECOND ORDER SYSTEM

① Form

$$G(s) = \frac{\mu \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

② Performance metrics:

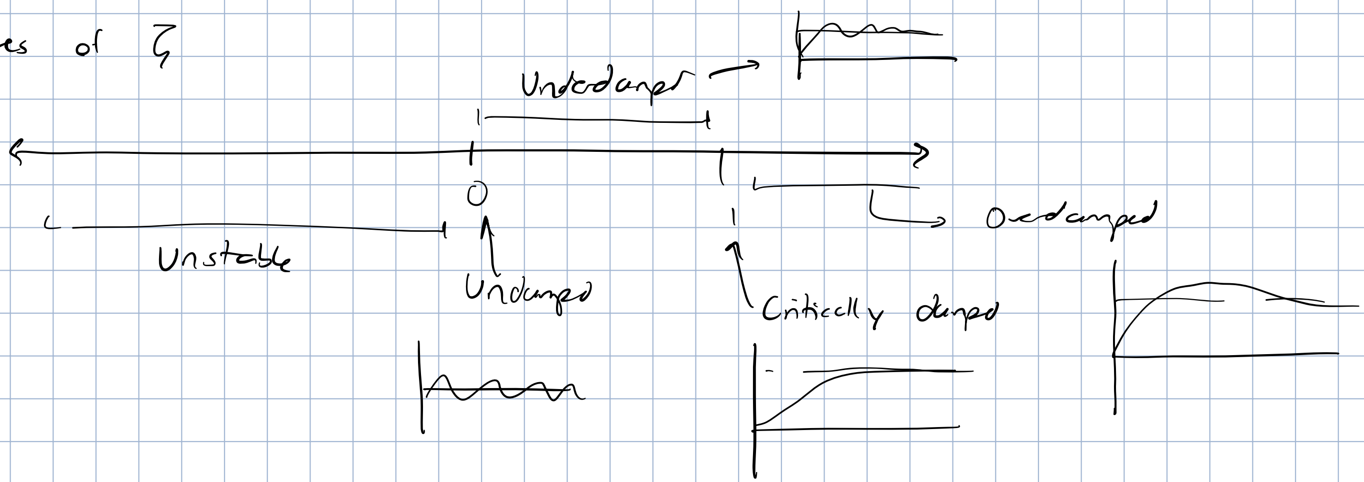
Steady state gain : μ

% Overshoot $= 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$

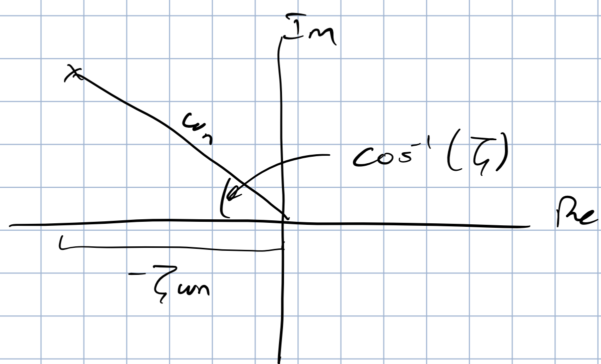
Settling time at $\varepsilon\%$ $= -\frac{1}{\zeta\omega_n}\ln(0.01\,\varepsilon)$

↳ $2\% = \frac{4}{\zeta\omega_n}$

④ Values of ζ



⑥ Poles :



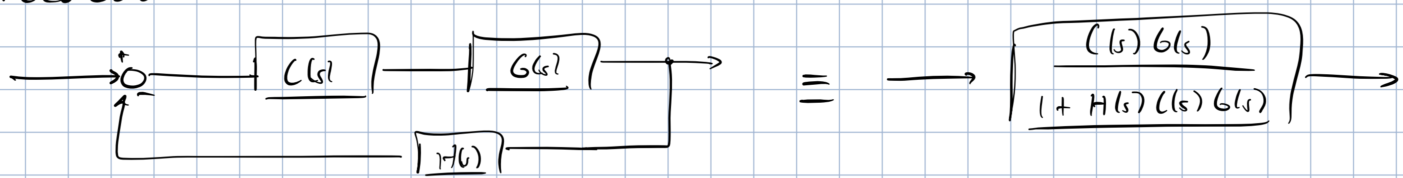
SYSTEM IDENTIFICATION

① $Y = D'\theta$

↳ $\hat{\theta} = \underset{\theta}{\arg\min} \| Y - D^T\theta \|^2 = (D^T D)^{-1} D^T Y$

SYSTEM ANALYSIS

① Feedback



Routh Hurwitz

① Method

1. Characteristic polynomial
2. 1^{st} 2 rows of table

s^n	a_n	a_{n-2}	$\cdot$ — —	0
s^{n-1}	a_{n-1}	a_{n-3}	- - -	0

3. Calculate rest

$$\begin{array}{c|cc} s^n & a_n & a_{n-2} \\ s^{n-1} & a_{n-1} & a_{n-3} \\ & A & \end{array} \qquad A = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

② Theorem

$\Pi(s)$ is Hurwitz (no +ve real part roots) $\Leftrightarrow$ Routh 1st col. is non-zero & all same sign

③ Unstable poles:

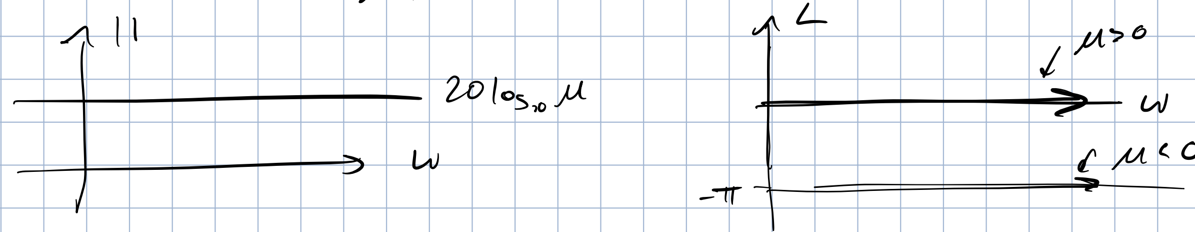
\# of unstable poles = # of sign changes

Freq. Response

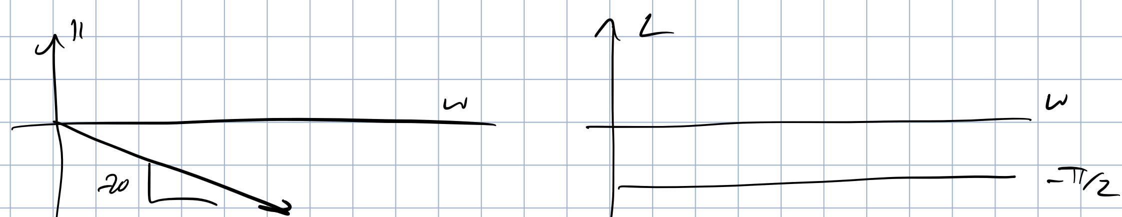
$$a\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow |G(j\omega)|\, a \sin(\omega t + \angle G(j\omega))$$

Bode Plots

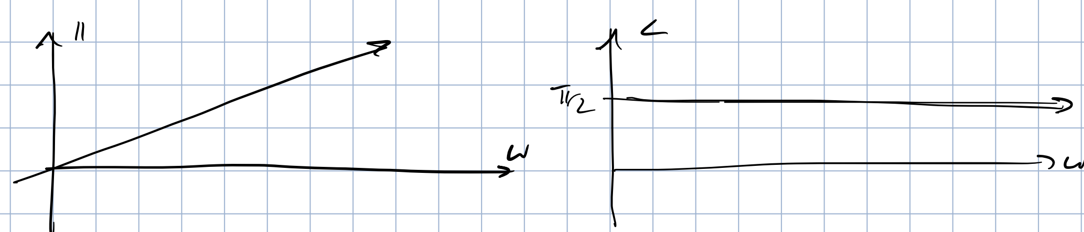
① Constant $(G(s) = \mu)$



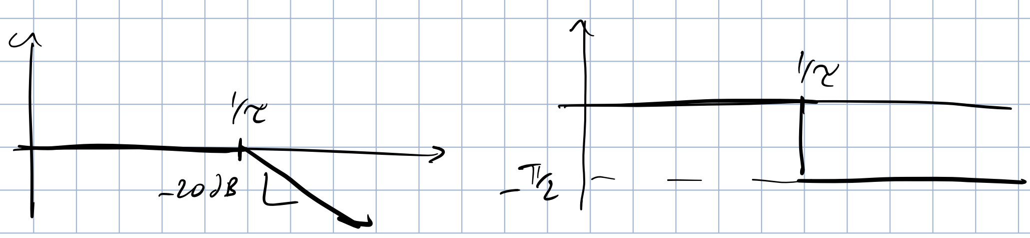
② $G(s) = 1/\tau s$



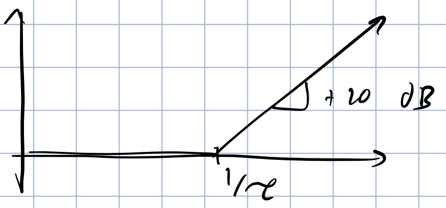
③ $G(s) = s$

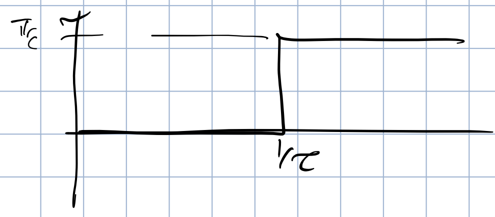


④ $G(s) = \dfrac{1}{1+\tau s}$

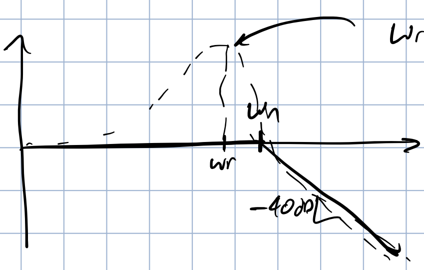


⑤ $G(s) = 1 + \tau s$





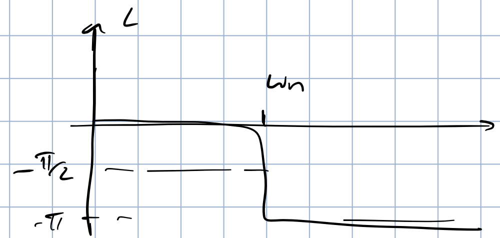
⑥ $G(s) = \dfrac{1}{1 + \frac{2\xi}{\omega_n} s + \frac{s^2}{\omega_n^2}}$



$\omega_r = \omega_n \sqrt{1 - 2\xi^2}$

$\} \ \xi < 1/\sqrt{2}$

$\} \ \xi > 1/\sqrt{2}$



⑦ Low pass, high-pass, band-pass

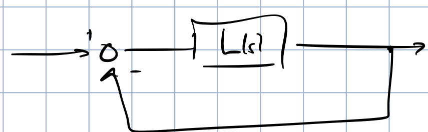
Low pass: $\dfrac{\mu}{1 + \tau s}$ High pass: $\dfrac{s}{1 + \tau s}$

Band-pass: Concat. low pass & high pass

Nyquist

① Theorem:

P = # of poles w/ +ve real roots. N = # of loops around ±1



is A.S. $\Leftrightarrow$ $N = P$

② Corollaries

A: $\forall \omega$, $|G(j\omega)| < 1 \Rightarrow$ closed loop stable

B: $\forall \omega$, $\angle G(j\omega) \in [0, \pi) \Rightarrow$ "

③ Gain margin:

$k_{m_{dB}} = 0 - |G(j\omega_\pi)| \Rightarrow > 0_{dB} \Rightarrow$ stable

④ Phase margin

$\varphi_m = \angle G(j\omega_c) - (-\pi) \Rightarrow > 0° \Rightarrow$ stable

Controller Synthesis

System metrics

① Stability: K_m & $\varphi_m > 0$

② Robust stability: K_m & φ_m high

③ Static perf: high μ or $p > 0$ $\Rightarrow$ low steady state error

$$F(s) = \frac{\mu}{s^{p} + \mu} \xrightarrow{\text{steady state}} \begin{cases} \frac{\mu}{1+\mu}, & p = 0 \\ 1, & p > 0 \\ 0, & p < 0 \end{cases}$$

④ Dynamic perf.: high ω_c $\rightarrow$ faster response

⑤ Disturbance rejection: $\omega_c >$ minimum

⑥ Noise attenuation: $\omega_c <$ maximum

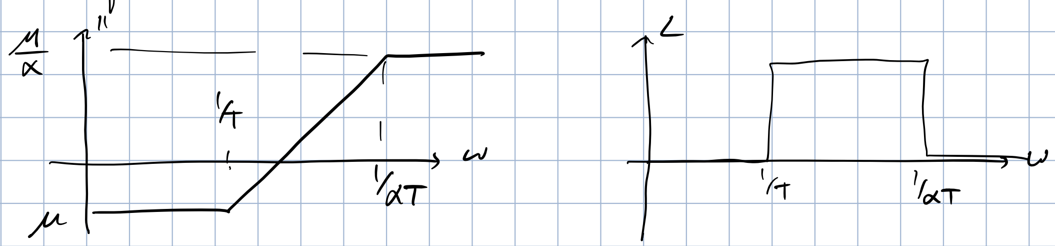
⑦ Realizability: $C(s)$ is proper

Lead compensator

① Form

$$C(s) = \mu \frac{1 + Ts}{1 + \alpha T s}, \quad T > 0, \ \mu > 0, \ 0 < \alpha < 1$$

② Bode plot:



③ Effect:

increase phase at ω_c $\rightarrow$ $\varphi_m > 0$

ω_c increases $\rightarrow$ higher freq. response

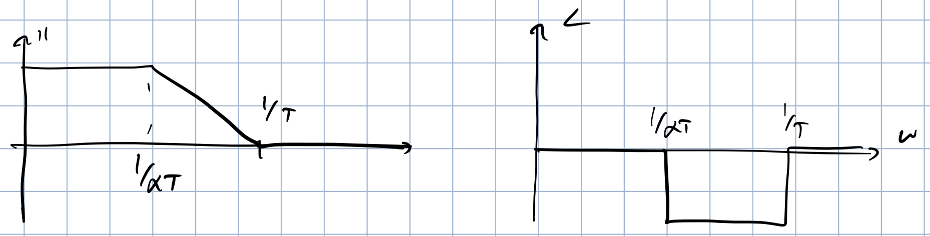
$\alpha \rightarrow 0$ $\longrightarrow$ PD controller

Lag compensator

① Form

$$C(s) = \mu \frac{1 + Ts}{1 + \alpha T s}, \quad \mu, T > 0, \quad \alpha > 1$$

② Bode:



③ Effect:

Increase gain @ low freq. → low steady state error

Could put $\varphi_m < 0$ if too close to wc

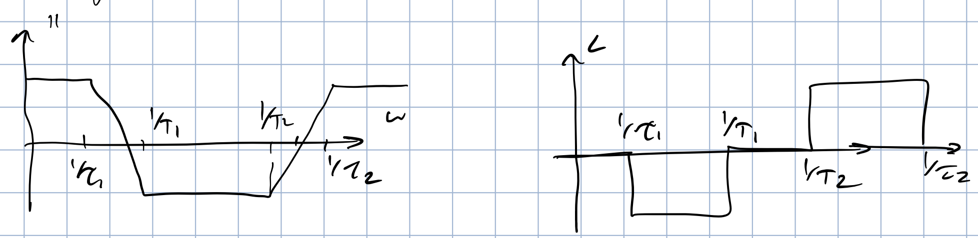
$\alpha \to \infty$, PI controller

Lead-lag compensator

① Form

$$C(s) = \mu \frac{(1+T_1 s)\ (1+T_2 s)}{(1+\tau_1 s)\ (1+\tau_2 s)} \qquad \left|\frac{1}{\tau_1}\right| < \left|\frac{1}{T_1}\right| < \left|\frac{1}{T_2}\right| < \left|\frac{1}{\tau_2}\right|$$

② Bode plot



③ Effect:

Realizable PID

PID

Actual PID: $C(s) = K_p + K_d s + \frac{K_I}{s}$

Real PID: $C(s) = K_p + \frac{K_d s}{1 + \frac{K_d s}{K_p N}} + \frac{K_I}{s}$

Root locus

① Rules:

A: n loci, n = # of zeros & poles

B: As μ increase, poles go from poles of open loop to zeros of open loop

C: Loci are symmetric about real

D: Loci cannot crossover

E: Portion of Re axis left to odd # of poles & zeros included in loci

F: extra loci (no pole-zero pair) go to ∞

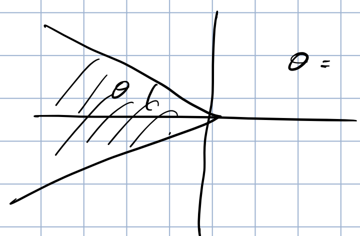
G: Asymptotes:

$$x_n = \frac{1}{n-m}\left(\sum p_i - \sum z_i\right)$$

$$\varphi_{a_k} = \frac{2k+1}{n-m}\pi \qquad (k \in [0, n-m+1])$$

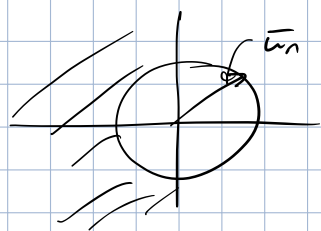
① Pole placement

A: Damping ratio: $\xi \geq \bar{\xi}$



$\theta = \cos^{-1}(\bar{\xi})$

B: $\omega_n \geq \bar{\omega}_n$



C: $\sigma = \xi\omega_n \geq \bar{\sigma}$

Pole placement

① Cannonical form:

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \implies \left\{ \begin{array}{l} A = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \\ -a_0 & \cdots & & \cdots & -a_{n-1} \end{bmatrix}}_{n} \\ B = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{n} \\ C = [b_0 \; b_1 \; \cdots \; b_{n-1}] \end{array} \right.$$

② Eigen value for stability

$$|\lambda I - (A - BK)| = 0$$

③ Choose $\lambda_1^* \ldots \lambda_n^*$ & match w/ $\lambda_1, \ldots, \lambda_n$

Remember:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1 \ \ldots \ y_m] = \begin{bmatrix} x_1 y_1 & \ldots & x_1 y_m \\ \vdots & & \\ x_n y_1 & \ldots & x_n y_m \end{bmatrix}$$