INTRODUCTION

Diff Eq:

Let $x(t) = f(t)$

$$x(t) = f\left(x, \alpha, \underbrace{\frac{d}{dt}x, \frac{d^2}{d^2t}x^2}_{\text{Diff.}} \ldots\right)$$

"Solve" a DE?

↳ Take DE ⇒ find original definition of dep. var. in terms of indep. variables.

Signals:

Carries info

System:

$$\text{Signal} \longrightarrow \boxed{f(\ldots)} \longrightarrow \text{Signal}\ldots$$

Algebraic:

Ex://
$$\frac{dx}{dt} = \alpha x(t)$$

① Separate:

$$\frac{dx}{x} = \alpha\, dt$$

② Integrate:

$$\int \frac{1}{x}\, dx = \int \alpha\, dt$$

$$\ln x + c_1 = \alpha t + \boxed{c_2}$$

Constants matter!

$$\ln x = \alpha t + c_3 \qquad c_3 = c_2 - c_1$$

③ Solve for x:

$$x = e^{\alpha t + c_3}$$

$$x = c e^{\alpha t} \Rightarrow c = e^{c_3}$$

Numerically: out-of-scope.

Important: constants b/c they encaps. initial cond.

↳ Initial cond: $x(t_0) = x_0$ (t_0 doesn't have to be 0)

or $x'(t_0)$, or $x''(t_0)$

↳ Boundary cond: $y(x = x_1)$, $y''(x = x_2)$, ...

Ex://
$$\frac{dx}{dt} = ax - bx^2 \qquad (a, b > 0)$$

↳ $a < 0 \Rightarrow$ exp. decay

① Separation:

$$\int \frac{dx}{ax - bx^2} = \int dt$$

② PFD:

$$\int \frac{1/a}{x} + \int \frac{b/a}{a-bx} = \int dt$$

③ Integral:

$$\frac{1}{a}\ln x + \frac{b}{a}\left(-\frac{1}{b}\right)\ln(a-bx) = t + c$$

$$\frac{1}{a}\left(\ln x - \ln(a-bx)\right) = t+c$$

$$\ln\left(\frac{x}{a-bx}\right) = a(t+c)$$

$$\frac{x}{a-bx} = e^{a(t+c)}$$

④ Solve for x:

$$x = e^{a(t+c)}(a-bx)$$

$$= ae^{a(t+c)} - bxe^{a(t+c)}$$

$$x + bxe^{a(t+c)} = ae^{a(t+c)}$$

$$x = \frac{ae^{a(t+c)}}{1+be^{a(t+c)}}$$

$$= \vdots$$

$$= \frac{a}{b} \cdot \frac{1}{1+e^{-a(t-t_0)}}$$

You can do simple analysis: $\boxed{\lim_{t\to\infty} x = \frac{a}{b}}$ → Steady state.

Easier way to get S.S.:

$$\frac{df}{dx} = 0$$

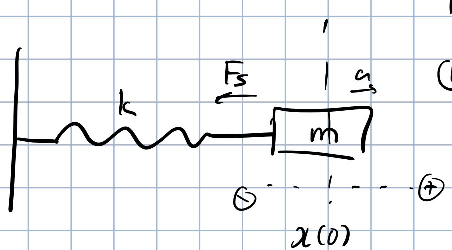
$$\frac{dx}{dt} = ax - bx^2 = 0$$

$$x(a-bx) = 0$$

$$\therefore x = 0,\ a/b \Rightarrow 2 \text{ s.s. soln.}$$

Ex:// Spring

Model via D.E.



① Equations using defn.

$$F_s = ma$$

$$-kx = m \cdot \frac{d^2x}{dt^2}$$

Taylor Expansion

Problem: function are hard to integrate

Soln: approx. func via polynomials (via Taylor method)

Taylor Theorem

Say that approx. func $f(x)$ at $x = x_0$.

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

i^{th} order: $\dfrac{f^{(i)}(x_0)}{i!}(x - x_0)^i$

Ex:// Prove $\sin\theta = \theta$ for small θ.

① Function + center

$f(\theta) = \sin\theta$

Center: small $\theta \sim 0$

② Taylor exp.

$$f(\theta) = \sin(0) + \frac{f'(0)}{1!}\theta + \frac{f''(0)}{2!}\theta^2 + \dots$$

$$= 0 + \theta + 0 + \dots$$

$$\sim \theta$$

Q: n^{th} order Taylor polynomial

↳ $f(x) \longrightarrow$ T.E $\longrightarrow$ only go up to n^{th} derivative $(x - x_0)^n$

Useful expansions:

1. $\sin\theta \sim \theta$
2. $\cos\theta \sim 1 - \theta^2/2$
3. $e^x \sim 1 + x$
4. $\sqrt{1+x} \sim 1 + x/2$
5. $\dfrac{1}{1 \pm x} \sim 1 \mp x$
6. $\dfrac{1}{\sqrt{1+x}} \sim 1 - x/2$

Use in compound T. E.

Ex:// T. E. of $\dfrac{1}{\sqrt{R^2 + D^2}}$

① Component func:

Looks like $\dfrac{1}{\sqrt{1+x}} \Rightarrow$ variable sub

$$\frac{1}{\sqrt{R^2 + D^2}} = \frac{1}{R\sqrt{1 + D^2/R^2}} = \frac{1}{R} \cdot \frac{1}{\sqrt{1 + D^2/R^2}}$$

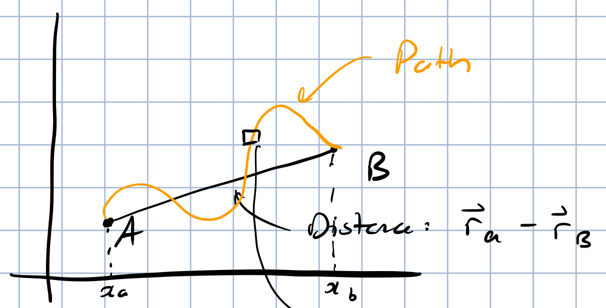
Let $x = D^2/R^2$

② T.F. of comp. of fn:

$$\frac{1}{R}\cdot\frac{1}{\sqrt{1+D^2/R^2}} = \frac{1}{R}\left(1-\frac{D^2}{2R^2}\right)$$

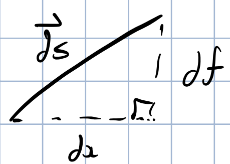
Important note: make sure dep. variable in expansion is <u>unitless</u>

Distance vs Path Length



Q: Path length?

If s is a function of x (f(x))



$$\therefore\ ds^2 = dx^2 + df^2$$

$$ds = \sqrt{dx^2 + df^2}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{df}{dx}\right)^2}$$

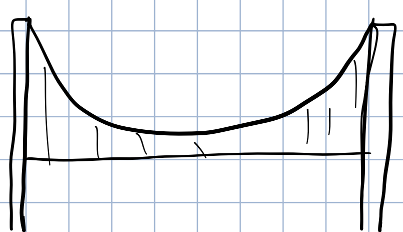
$$S = \int ds$$

$$= \int_{x_a}^{x_b} \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx$$

$$= \boxed{\int_{x_a}^{x_b} \sqrt{1+(f'(x))^2}\, dx}$$

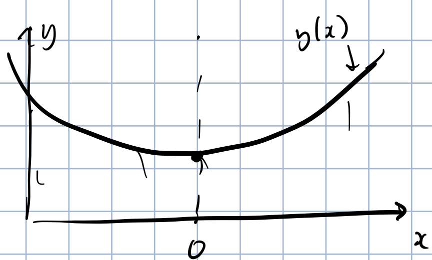
Hanging cable problem

Problem:



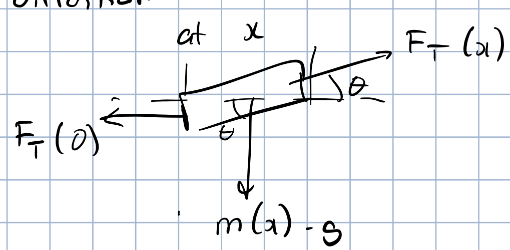
What is shape of cable?

① Model problem via physics:



L_s

a. Consider differential:



b. Use physics

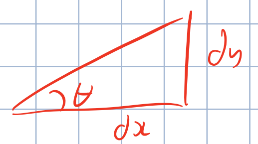
$\hat{x}$: $F_T(x)\cos\theta = F_T(0)$ — ①

$\hat{y}$: $F_T(x)\sin\theta = m(x)\cdot g$ — ②

② ÷ ①: $\tan\theta = \dfrac{m(x)\cdot g}{F_T(0)}$

② Introduce derivative:

Trick:

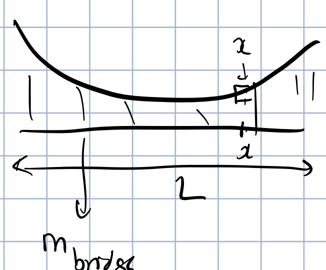


$\Rightarrow \tan\theta = \dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{m(x)\cdot g}{F_T(0)}$$

③ Classify $m(x)$

1) Case #1: Cable weight is negligible



Assuming linear density

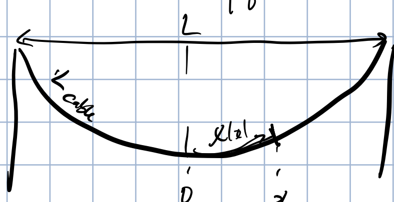
$$m(x) = \frac{m_{bridge}}{L}\cdot x$$

$$\therefore \frac{dy}{dx} = \underbrace{\left[\frac{m_{bridge}\cdot g}{F_T(0)\cdot L}\right]}_{c_1}\cdot x$$

$$\frac{dy}{dx} = c_1 x$$

$$y = \frac{c_1}{2}\cdot x^2 \sim \underline{\text{Parabola}}$$

2) Case #2: Cable supports own weight



$\Rightarrow$ Weight at pt. x?

Assume linear density:

$$m(x) = \frac{m_{cable}}{L_{cable}}\cdot \ell(x)$$

$$\therefore \frac{dy}{dx} = \underbrace{\left[\frac{m_{cable}\cdot g}{L_{cable}\cdot F_T(0)}\right]}_{c_1}\cdot \ell(x)$$

$$dy = c_1 \ell(x) \cdot dx$$

① Take 2nd derivative

2nd order diff eq. $\Longrightarrow$

$$\frac{d^2y}{dx^2} = c_1 \cdot \frac{d\ell(x)}{dx} \longrightarrow$$ Path differential!

$$= c_1 \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

② Variable sub:

$$s = \frac{dy}{dx}$$

$$\therefore \frac{ds}{dx} = c_1 \cdot \sqrt{1 + s^2}$$

③ Integrate:

$$\frac{dy}{dx} = s = \frac{e^{c_1 x} - e^{-c_1 x}}{2}$$

$$y = c_1 \cdot \frac{e^{c_1 x} + e^{-c_1 x}}{2} + c_3$$

$$\boxed{y = c_1 \cdot \cosh(c_1 x) + c_3}$$

- Partial diff eqns:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \cdot \frac{\partial^2 y}{\partial^2 x} \Rightarrow y = f(t, x)$$

$$\vdots$$

$$y = A \sin k\left(x - \sqrt{\frac{T}{\rho}}\, t\right)$$