CSE6706:

Advanced Digital Image Processing

Dr. Md. Monirul Islam

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Spectral/Frequency Domain Analysis of Images

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1D Fourier Series and its Inverse

Review

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\frac{2\pi n}{T}t}$$

- The inverse Fourier series is

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t}$$

1D Fourier Transform and its Inverse

Review

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt$$

- The inverse Fourier transform is

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ut} du$$

Continuous Impulse Function

Review

$$\delta(t) = \begin{cases} \infty & \text{if } \; t = 0 \\ 0 & \text{if } \; t \neq 0 \end{cases}$$

with the constraint,

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Sifting Property of Continuous Impulse Function

Review

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

- Evaluates the function at the location of the impulse

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Sifting Property of Continuous Impulse Function

Review

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt = f(t_0)$$

- Evaluates the function at the location of the impulse

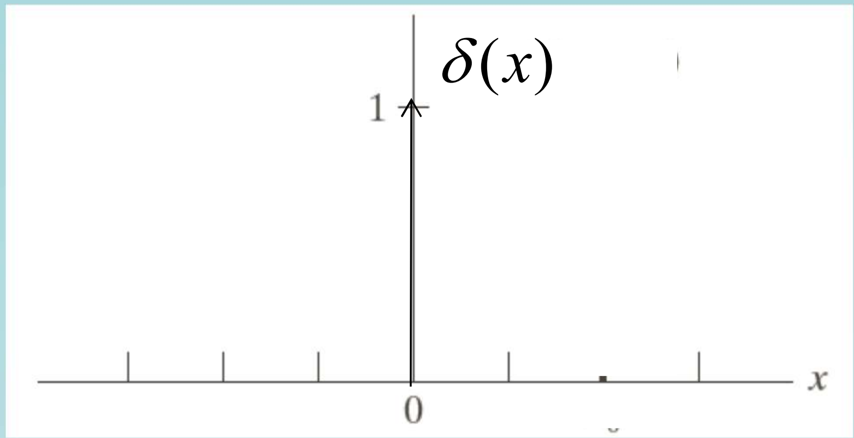
Discrete Unit Impulse Function

Review

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

with the constraint,

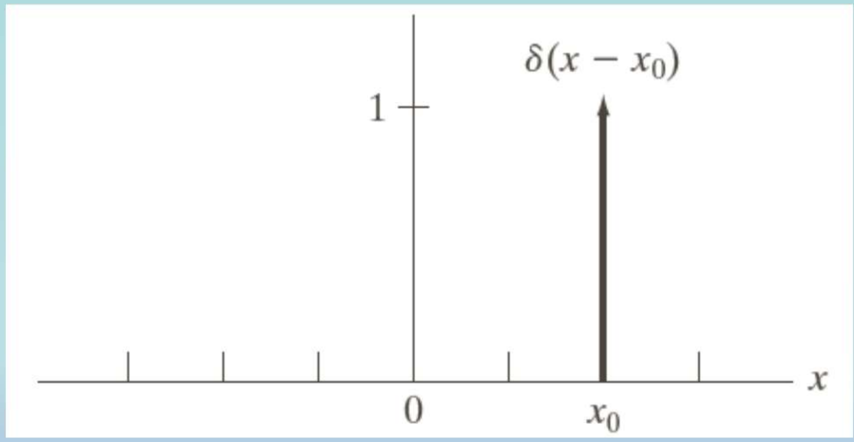
$$\sum_{x=-\infty}^{x=\infty} \delta(x) = 1$$



Sifting Property of Discrete Unit Impulse Function

Review

$$\sum_{x=-\infty}^{x=\infty} \delta(x-x_0) f(x) = f(x_0)$$



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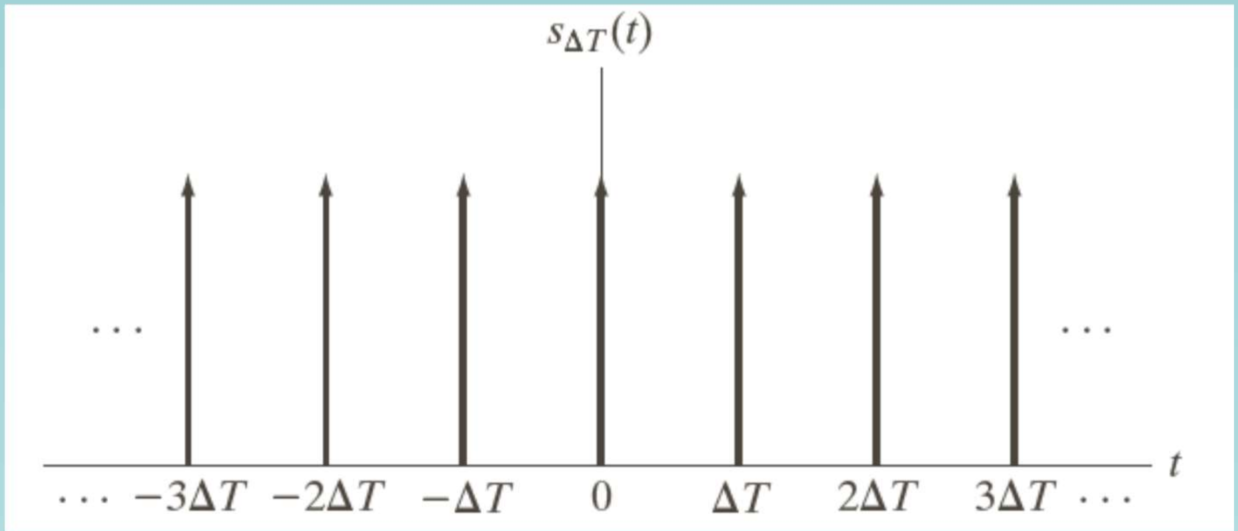
Impulse Train: $s_{\Delta T}(t)$

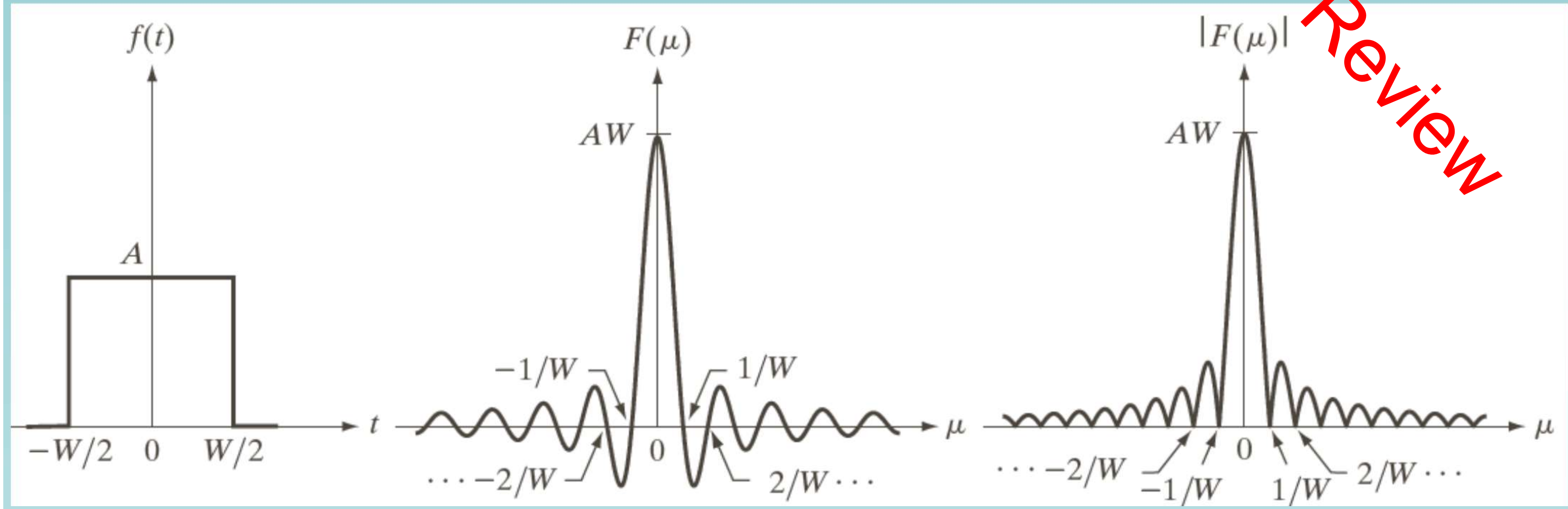
Review



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

1D Fourier Transform and its Inverse

Review



$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt = \int_{-W/2}^{W/2} A e^{-j2\pi ut} dt = \cdots$$

$$= \frac{A}{j2\pi u}\left[e^{j\pi uW} - e^{-j\pi uW}\right] = AW \frac{\sin(\pi uW)}{\pi uW} = AW \mathrm{sinc}(\pi uW)$$

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1D FT of Impulse Function: $\delta(t)$

$$F(u) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi ut} \delta(t) dt$$

$$= \int_{-\infty}^{\infty} f(t) \delta(t) dt$$

$$= f(0)$$

$$= e^{-j2\pi u0} = e^{0} = 1$$

1D FT of Impulse Function: δ(t-t_0)

$$F(u) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi ut} \delta(t - t_0) dt$$

$$= \int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt$$

$$= f(t_0)$$

$$= e^{-j2\pi ut_0}$$

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Recall: 1D FT and its Inverse

Forward Transform:

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt$$

Inverse Transform:

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ut} du$$

- FT of $f(t)$ is $F(u)$

- Prove that FT of $F(t)$ is $f(-u)$: *Exercise!*

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1D FT of Impulse Functions

Fourier transorm of, $f(t) = \delta(t - t_0)$

is

$F(u) = e^{-j2\pi u t_0}$

Then, Fourier Transform of $F(t)$ = ?

1D FT of Impulse Functions

Fourier transorm of, $f(t) = \delta(t - t_0)$

is

$F(u) = e^{-j2\pi u t_0}$

Fourier transform of, $F(t) = e^{-j2\pi t t_0}$

is

$f(-u) = \delta(-u - t_0)$

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1D FT of Impulse Functions

$$\text{Fourier transform of, } F(t) = e^{-j2\pi t t_0}$$

is

$$f(-u) = \delta(-u - t_0)$$

$$\text{Let, } t_0 = -a,$$

$$\text{Fourier transform of, } F(t) = e^{j2\pi t a}$$

is

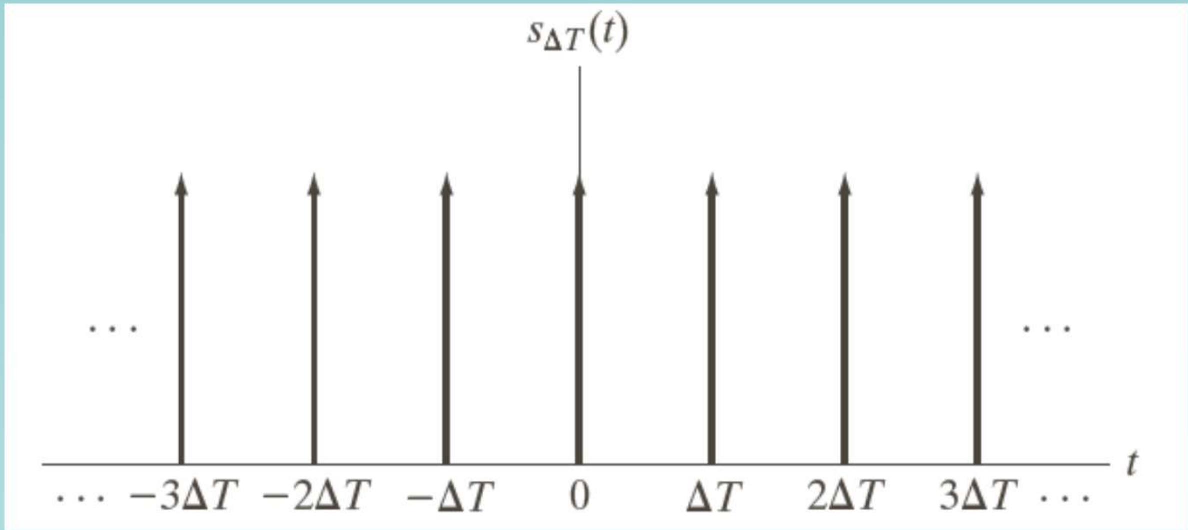
$$f(-u) = \delta(-u + a) = \delta(a - u) = \delta(u - a)$$

1D FT of Impulse Functions

That is,

$$\Im\left\{e^{j2\pi ta}\right\} = \delta(u-a)$$

1D FT of Impulse Trains



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

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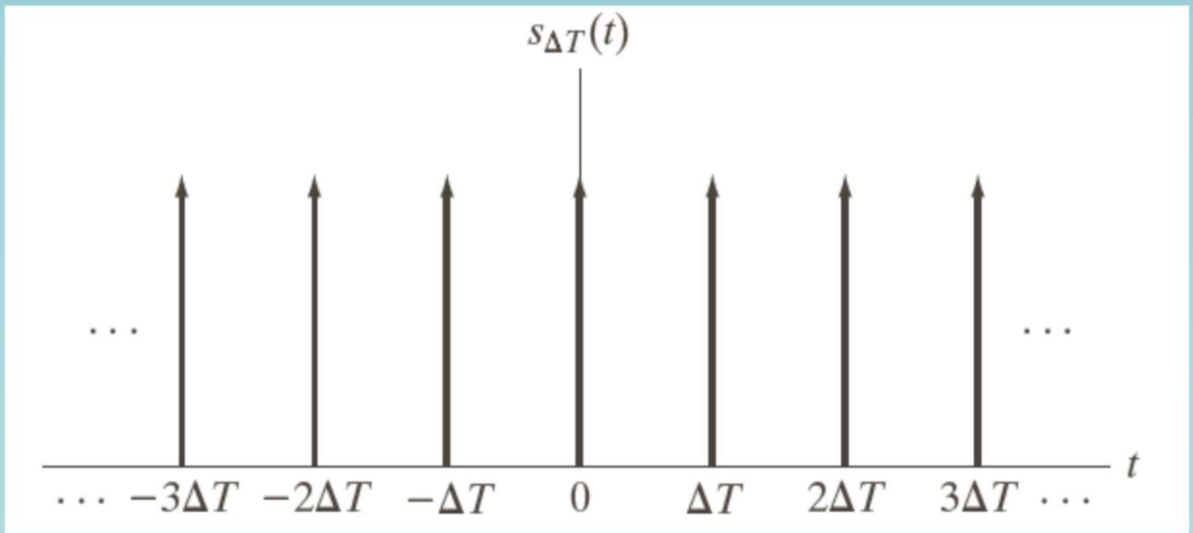
1D FT of Impulse Trains



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

- $s_{\Delta T}(t)$ is periodic
- So can be represented as a Fourier series

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\frac{2\pi n}{\nabla T}t}$$

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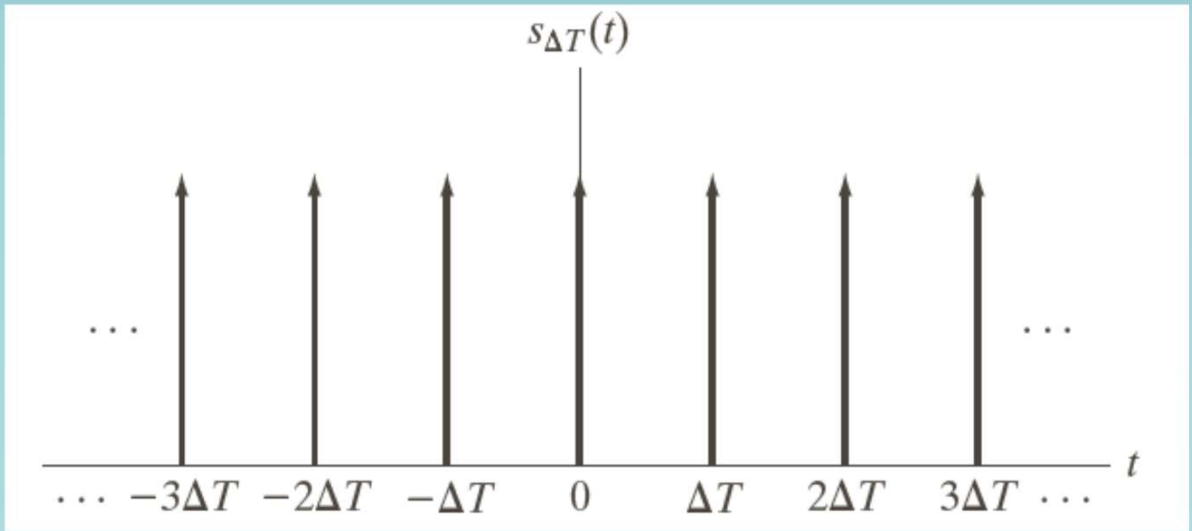
1D FT of Impulse Trains



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

- $s_{\Delta T}(t)$ is periodic
- So can be represented as a Fourier series

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\frac{2\pi n}{\nabla T}t}$$

$$c_n = \frac{1}{\nabla T}\int_{-\nabla T/2}^{\nabla T/2} s_{\nabla T}(t) e^{-j\frac{2\pi n}{T}t} dt$$

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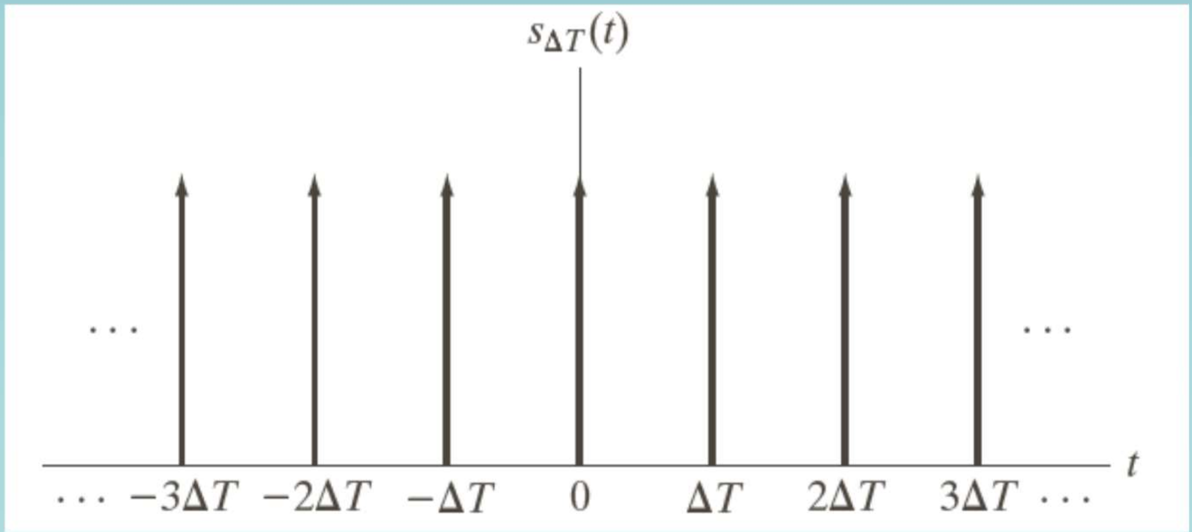
1D FT of Impulse Trains



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\frac{2\pi n}{\nabla T}t}$$

$$c_n = \frac{1}{\nabla T}\int_{-\nabla T/2}^{\nabla T/2} s_{\nabla T}(t) e^{-j\frac{2\pi n}{T}t} dt = \frac{1}{\nabla T}\int_{-\nabla T/2}^{\nabla T/2} \delta(t) e^{-j\frac{2\pi n}{T}t} dt$$

$$= \frac{1}{\nabla T} e^{-j\frac{2\pi n}{T}0} = \frac{1}{\nabla T}$$

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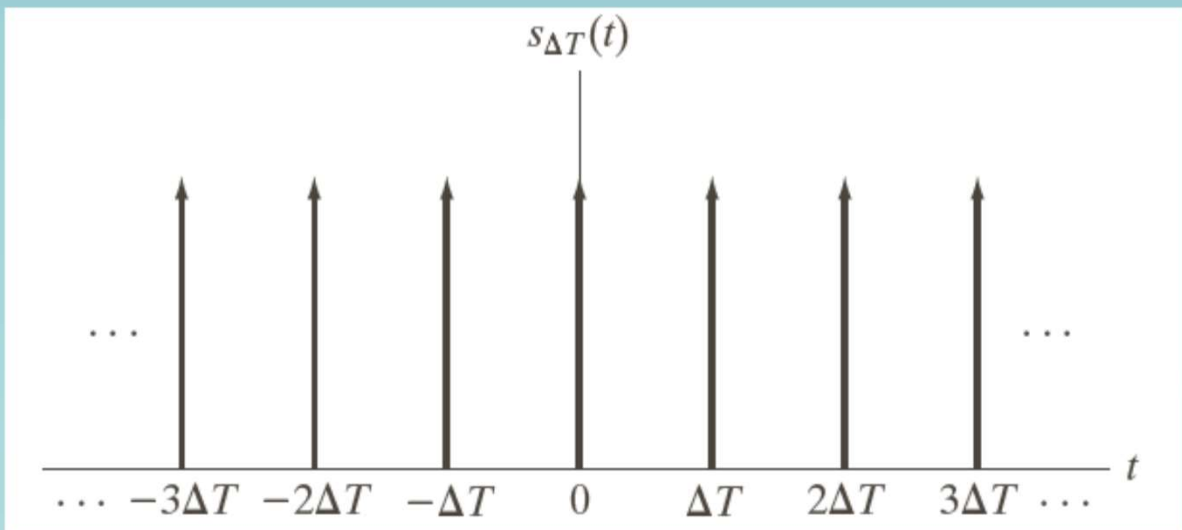
1D FT of Impulse Trains



$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{j\frac{2\pi n}{\nabla T}t}$$

$$s_{\nabla T}(t) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} e^{j\frac{2\pi n}{\nabla T}t}$$

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1D FT of Impulse Trains

$$s_{\nabla T}(t) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} e^{j\frac{2\pi n}{\nabla T}t}$$

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1D FT of Impulse Trains

$$s_{\nabla T}(t) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} e^{j\frac{2\pi n}{\nabla T}t}$$

$$S(u) = \Im\{s_{\nabla T}(t)\} = \Im\left\{ \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} e^{j\frac{2\pi n}{\nabla T}t} \right\}$$

$$= \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \Im\left\{ e^{j\frac{2\pi n}{\nabla T}t} \right\}$$

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1D FT of Impulse Trains

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \Im\left\{ e^{j\frac{2\pi n}{\nabla T}t} \right\}$$

But we know, $\Im\left\{ e^{j2\pi ta} \right\} = \delta(u-a)$

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1D FT of Impulse Trains

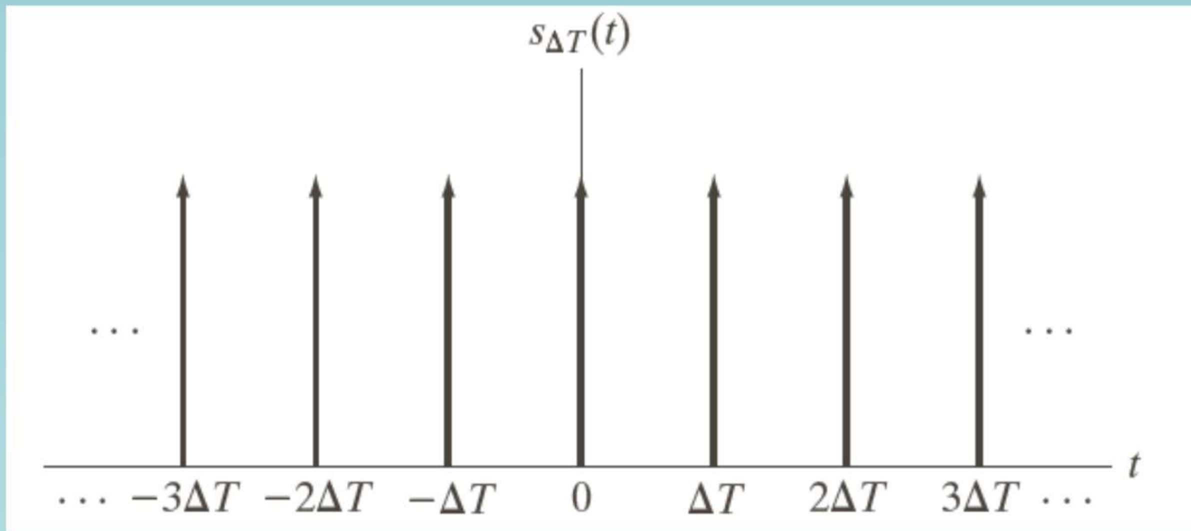
But we know, $\Im\left\{e^{j2\pi ta}\right\} = \delta(u-a)$

Therefore,

$$S(u) = \Im\left\{s_{\nabla T}(t)\right\} = \frac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty}\Im\left\{e^{j\frac{2\pi n}{\nabla T}t}\right\}$$

$$= \frac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty}\delta(u-\frac{n}{\nabla T})$$

1D FT of Impulse Trains



Impulse Train:

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

After Transform:

$$S(u) = \Im\{s_{\nabla T}(t)\} = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

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1D FT of Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau$$

1D FT of Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right]e^{-j2\pi ut}dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi ut}dt\right]d\tau$$

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1D FT of Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

$$\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ut} dt \right] d\tau$$

but, $\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ut} dt = \Im\{h(t-\tau)\}$ which is $H(u) e^{-j2\pi u\tau}$

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1D FT of Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

$$\begin{aligned}
\Im\{f(t) * h(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi ut} dt \\
&= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ut} dt \right] d\tau \\
&= \int_{-\infty}^{\infty} f(\tau) H(u) e^{-j2\pi u\tau} d\tau = H(u) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi u\tau} d\tau
\end{aligned}$$

1D FT of Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

$$\Im\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ut} dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau) H(u) e^{-j2\pi u\tau} d\tau = H(u) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi u\tau} d\tau$$

$$= H(u) F(u)$$

1D FT of Convolution

$$\Im\left\{f(t) * h(t)\right\} = H(u)F(u)$$

$$f(t) * h(t) \Leftrightarrow H(u)F(u)$$

- Fourier transform of the convolution of 2 spatial domain functions is the product of their Fourier transforms

1D FT of Convolution

$$\Im\{f(t)*h(t)\} = H(u)F(u)$$

$$f(t)*h(t) \Leftrightarrow H(u)F(u)$$

$$\Im\{f(t)h(t)\} = H(u)*F(u)$$

$$f(t)h(t) \Leftrightarrow H(u)*F(u)$$

- Similarly, Fourier transform of the product of 2 spatial domain functions is the convolution of their Fourier transforms

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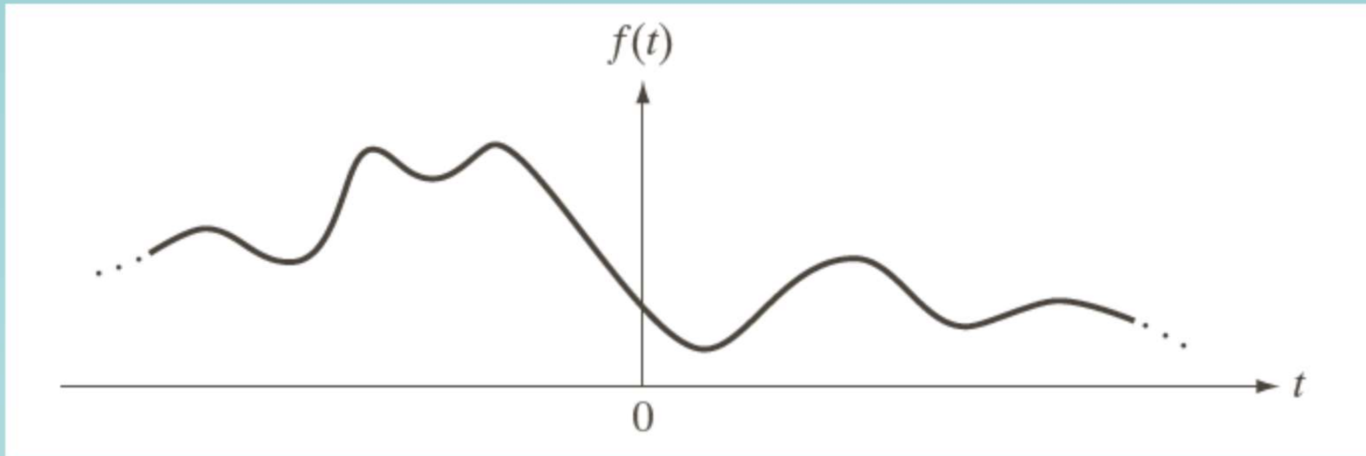
1D FT of Sampled Function



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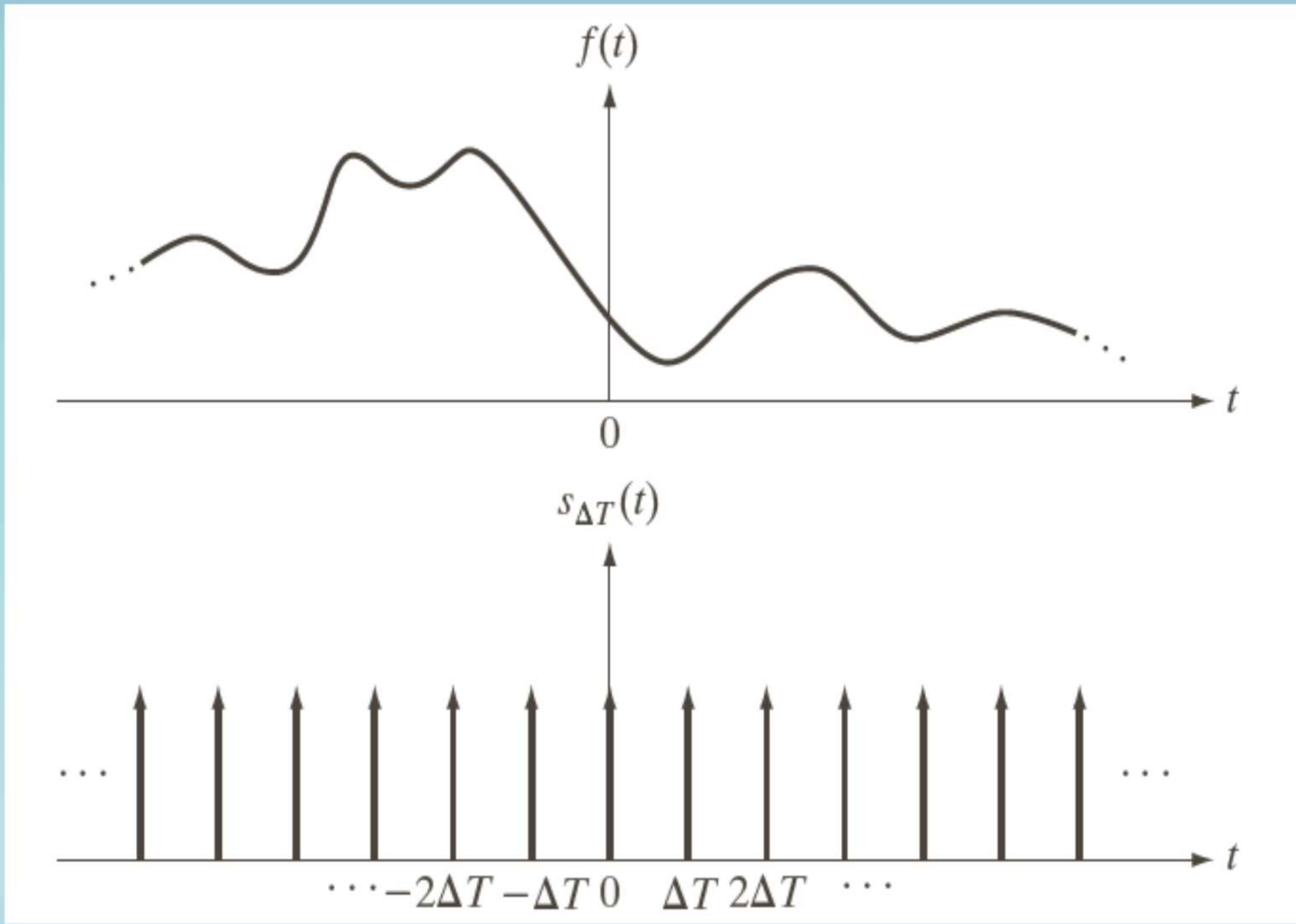
1D FT of Sampled Function



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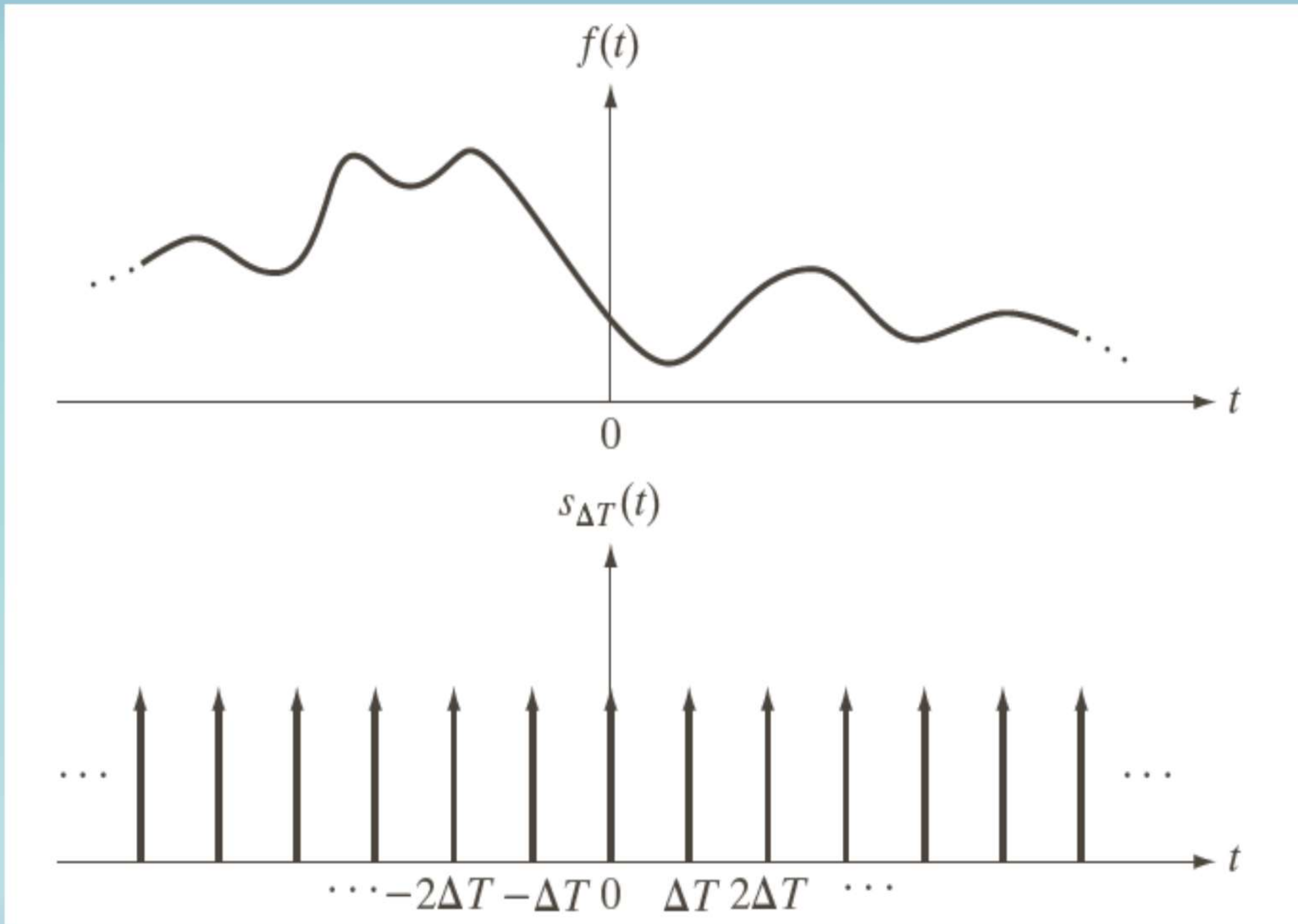
1D FT of Sampled Function



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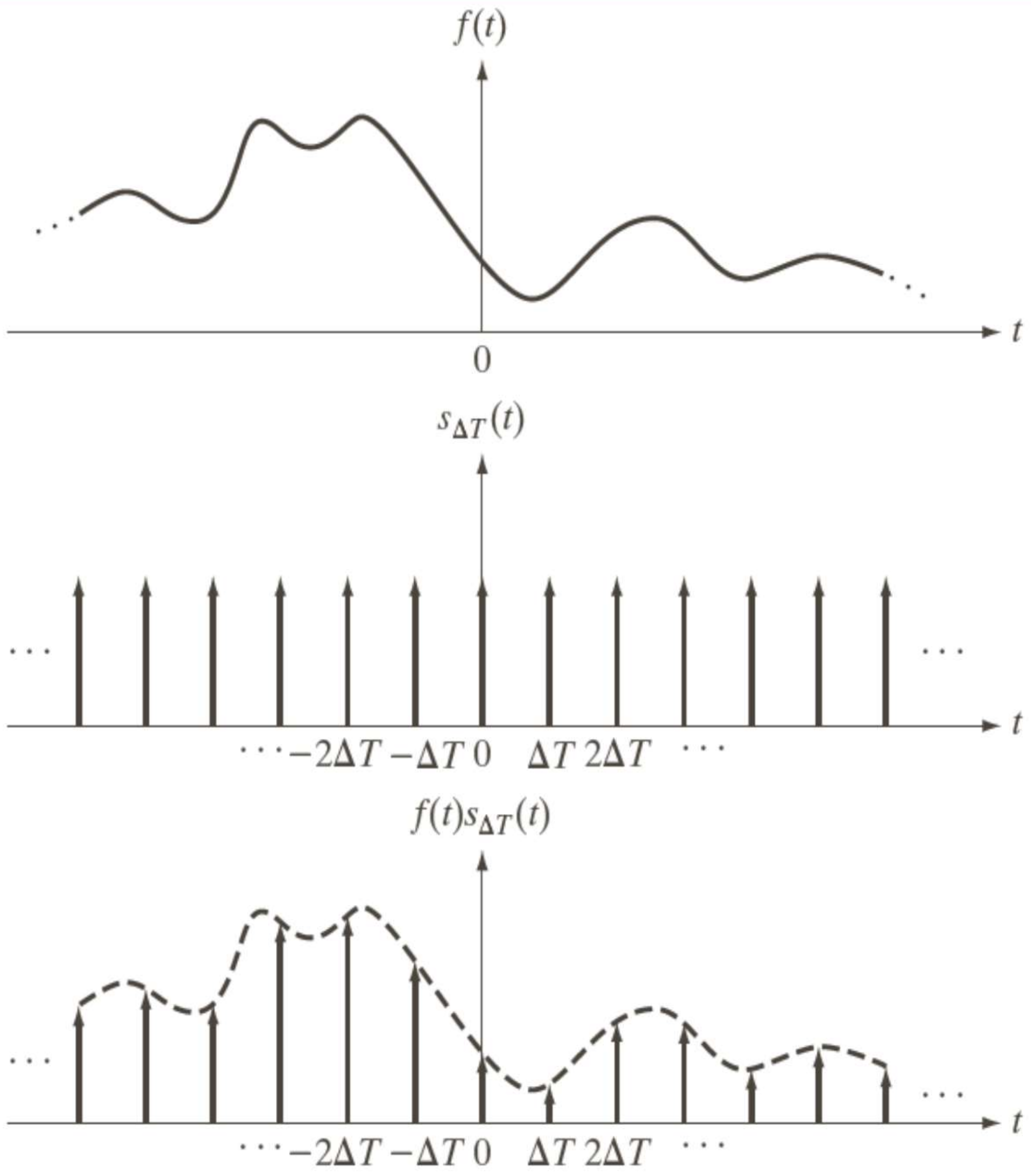
$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

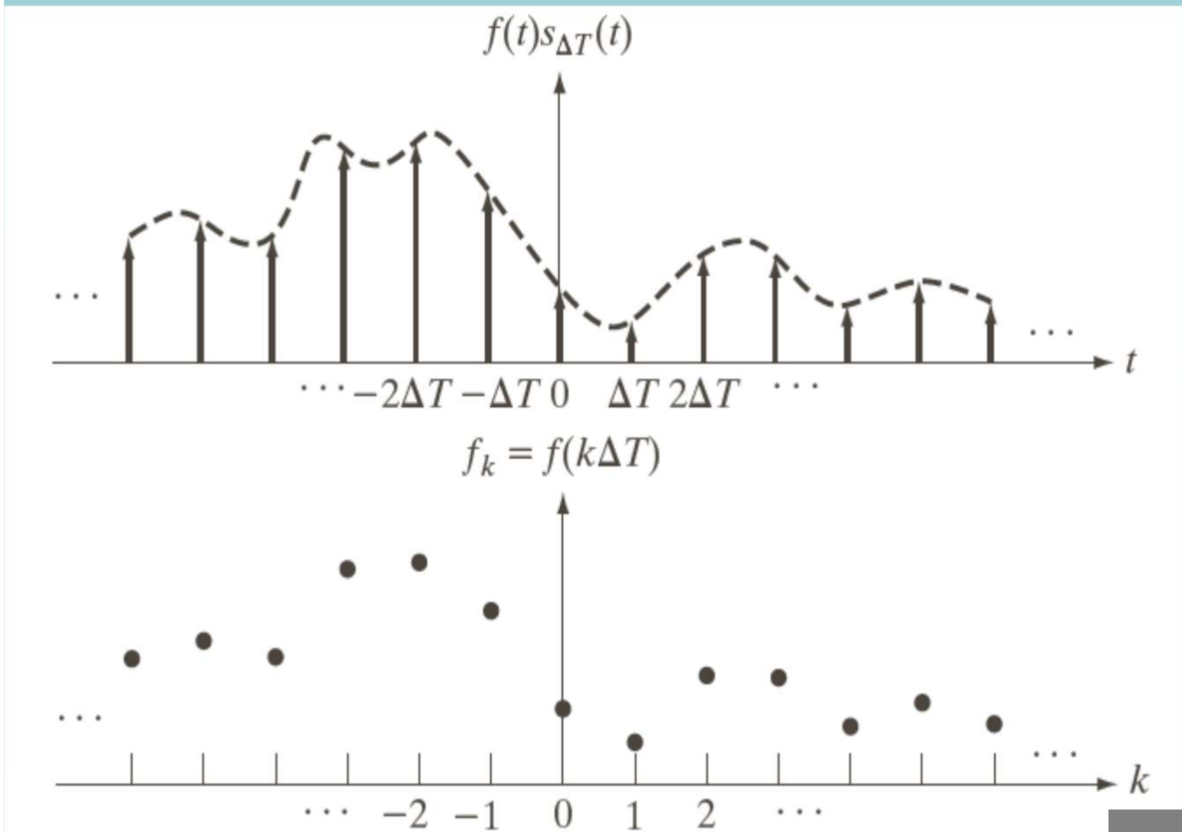
1D FT of Sampled Function



$$\tilde{f}(t) = f(t)s_{\nabla T}(t)$$

$$= \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

1D FT of Sampled Function



$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\nabla T)dt$$

$$= f(k\nabla T)$$

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1D FT of Sampled Function

$$\tilde{f}(t) = f(t)s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

1D FT of Sampled Function

$$\tilde{f}(t) = f(t)s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$\tilde{F}(u) = \Im\left\{\tilde{f}(t)\right\} = \Im\left\{f(t)s_{\nabla T}(t)\right\} = F(u) * S(u)$$

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1D FT of Sampled Function

$$\tilde{f}(t) = f(t)s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$\tilde{F}(u) = \Im\left\{\tilde{f}(t)\right\} = \Im\left\{f(t)s_{\nabla T}(t)\right\} = F(u) * S(u)$$

We know,

$$S(u) = \Im\left\{s_{\nabla T}(t)\right\} = \frac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

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1D FT of Sampled Function

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta\left(u - \frac{n}{\nabla T}\right)$$

$$\widetilde{F}(u) = F(u) * S(u)$$

1D FT of Sampled Function

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

$$\widetilde{F}(u) = F(u) * S(u)$$

$$= \int_{-\infty}^{\infty} F(\tau) S(u - \tau) d\tau$$

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1D FT of Sampled Function

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

$$\widetilde{F}(u) = F(u) * S(u)$$

$$= \int_{-\infty}^{\infty} F(\tau) S(u-\tau) d\tau$$

$$= \frac{1}{\nabla T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{n=\infty} \delta(u - \tau - \frac{n}{\nabla T}) d\tau$$

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1D FT of Sampled Function

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

$$\widetilde{F}(u) = F(u) * S(u)$$

$$= \int_{-\infty}^{\infty} F(\tau) S(u-\tau) d\tau$$

$$= \frac{1}{\nabla T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{n=\infty} \delta(u - \tau - \frac{n}{\nabla T}) d\tau$$

$$= \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} F(\tau) \delta(u - \frac{n}{\nabla T} - \tau) d\tau$$

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1D FT of Sampled Function

$$S(u) = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

$$\widetilde{F}(u) = F(u) * S(u)$$

$$= \int_{-\infty}^{\infty} F(\tau) S(u - \tau) d\tau$$

$$= \frac{1}{\nabla T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{n=\infty} \delta(u - \tau - \frac{n}{\nabla T}) d\tau$$

$$= \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} F(\tau) \delta(u - \frac{n}{\nabla T} - \tau) d\tau$$

$$= \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} F(u - \frac{n}{\nabla T})$$

1D FT of Sampled Function

For original function: $\Im\left\{f(t)\right\} = F(u)$

For sampled function: $\Im\left\{\widetilde{f}(t)\right\} = \widetilde{F}(u) = \dfrac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty} F(u - \dfrac{n}{\nabla T})$

- $\widetilde{F}(u)$ copies $F(u)$ at interval of $\dfrac{1}{\nabla T}$

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1D FT of Sampled Function

For original function: $\Im\left\{f(t)\right\} = F(u)$

For sampled function: $\Im\left\{\tilde{f}(t)\right\} = \tilde{F}(u) = \dfrac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty} F(u - \dfrac{n}{\nabla T})$

- Although $\tilde{f}(t)$ may be finite, its FT, $\tilde{F}(u)$ is
  - infinite
  - continuous
  - periodic

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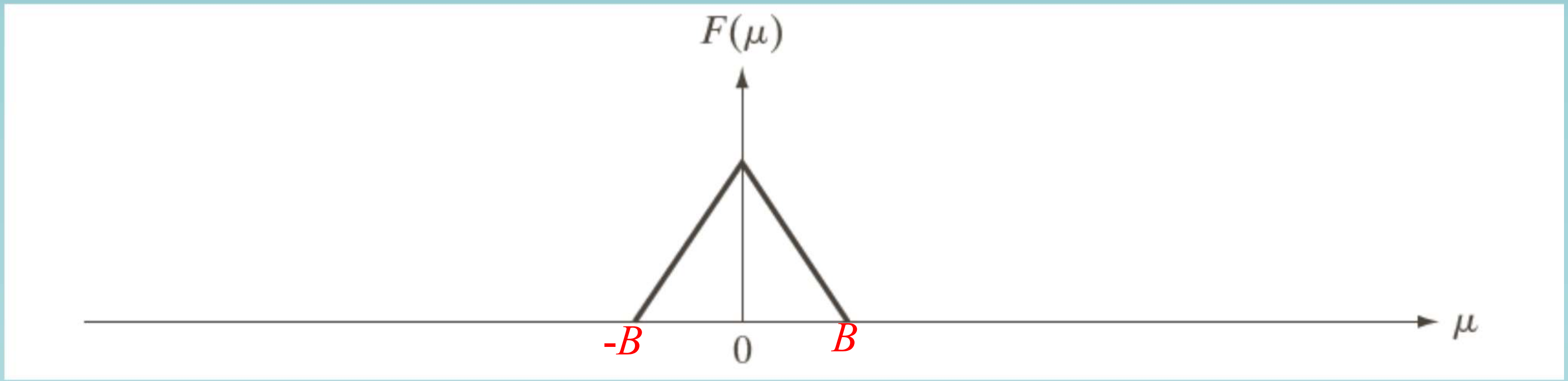
1D FT of Sampled Function



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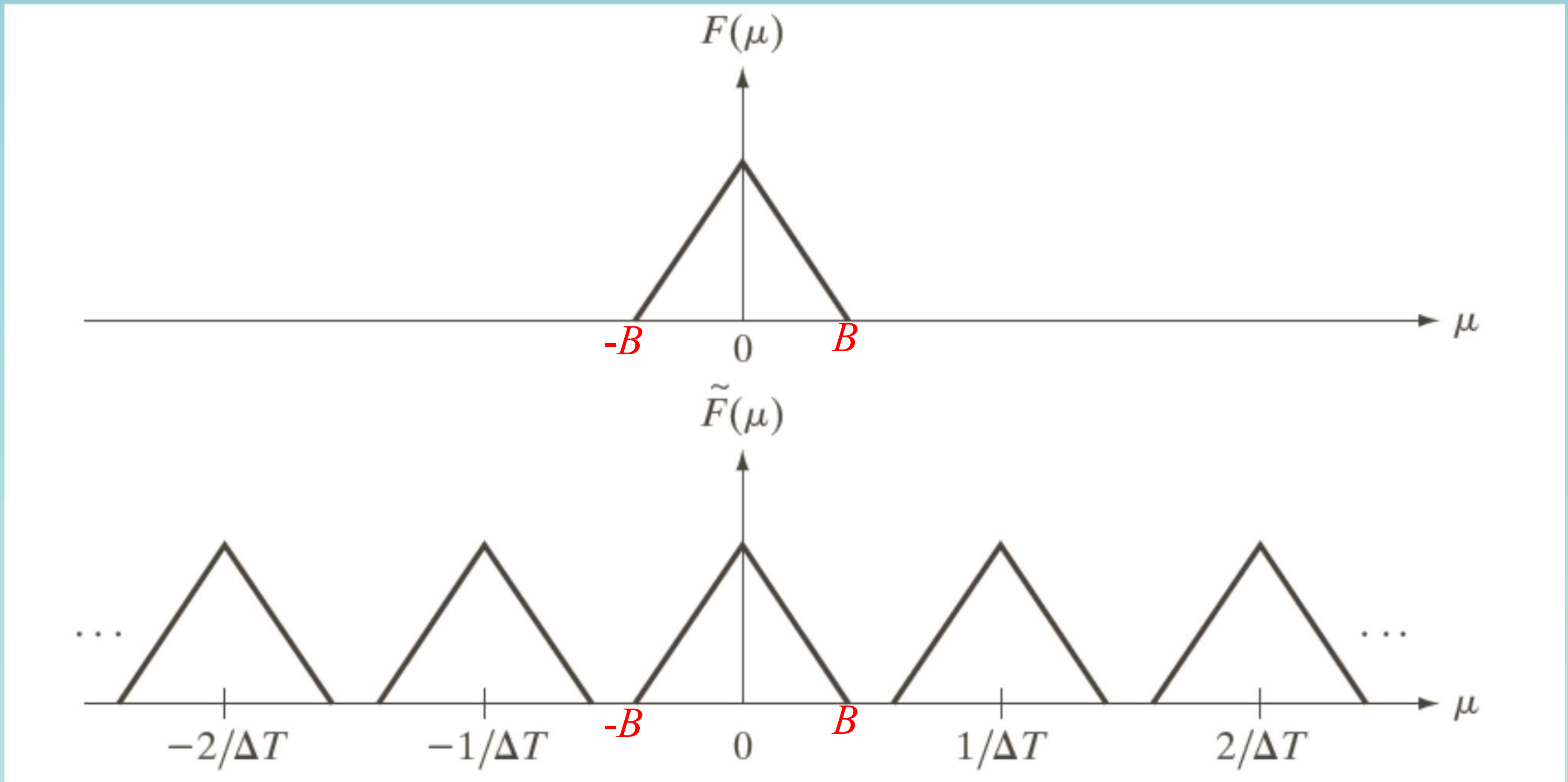
1D FT of Sampled Function



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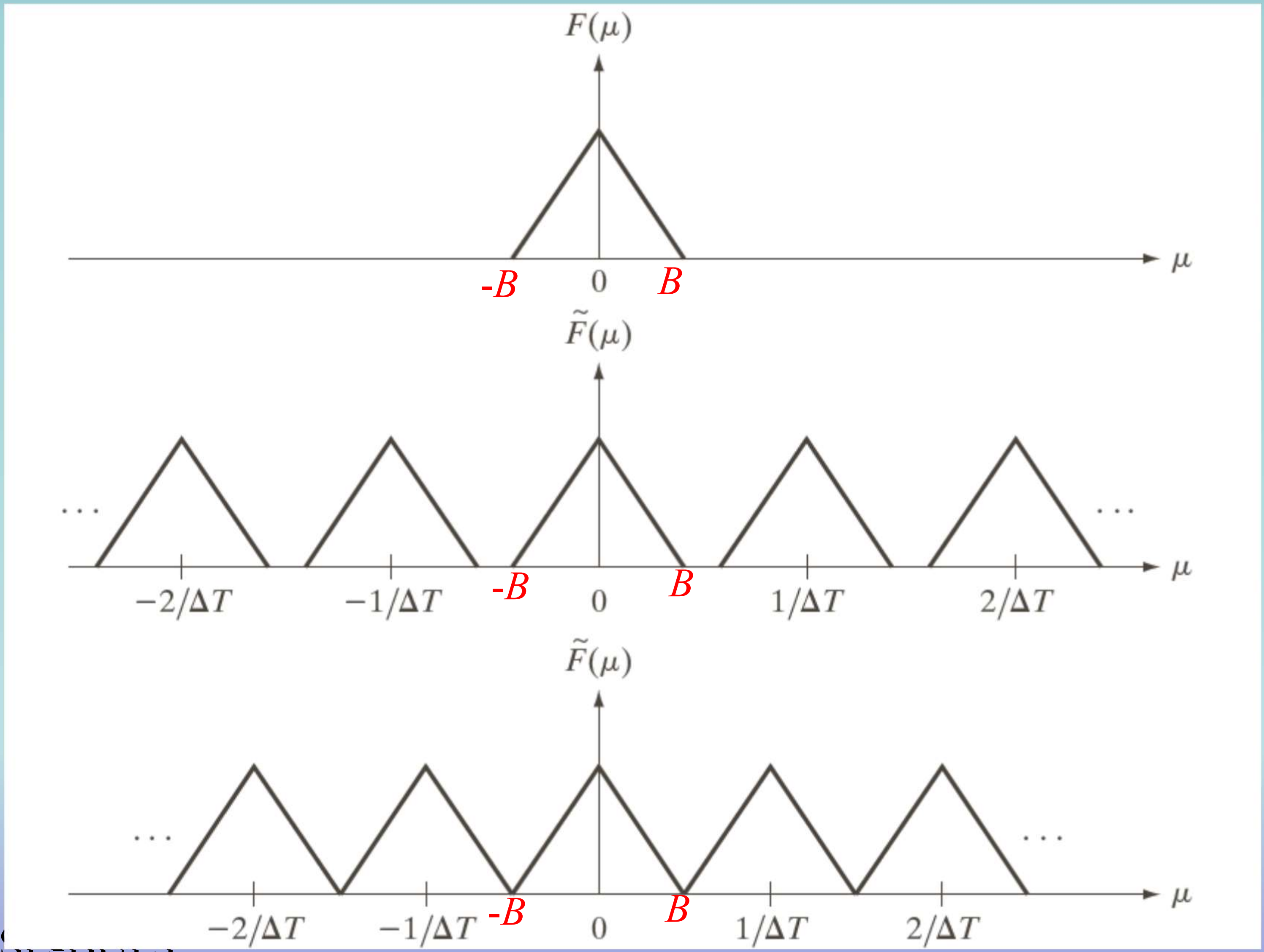
1D FT of Sampled Function



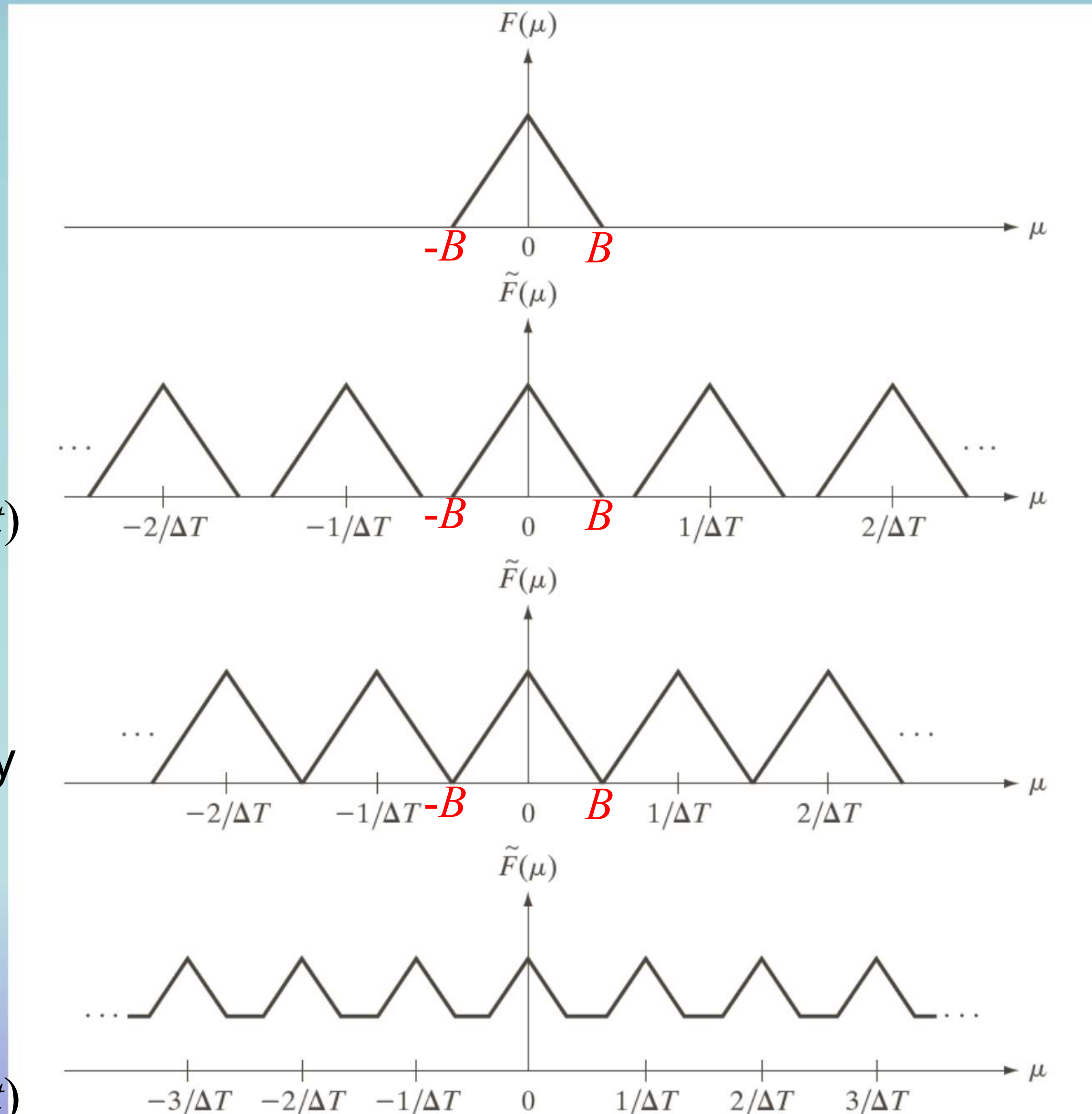
1D FT of Sampled Function

Original FT

FTof over sampled $f(t)$

FTof critically sampled $f(t)$

FTof under sampled $f(t)$



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Discrete Fourier Transform (DFT) of a Sampled Function

$$\Im\left\{f(t)\right\} = F(u)$$

$$\Im\left\{\tilde{f}(t)\right\} = \tilde{F}(u) = \frac{1}{\nabla T}\sum_{n=-\infty}^{n=\infty} F(u - \frac{n}{\nabla T})$$

Discrete Fourier Transform (DFT) of a Sampled Function

$$\widetilde{F}(u)=\int_{-\infty}^{\infty}\widetilde{f}(t)e^{-j2\pi ut}dt\,,\quad \text{where}\ \ \widetilde{f}(t)=\sum_{n=-\infty}^{n=\infty}f(t)\delta(t-n\nabla T)$$

Discrete Fourier Transform (DFT) of a Sampled Function

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt \,, \quad \text{where } \tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T) e^{-j2\pi ut} dt$$

CS

Discrete Fourier Transform (DFT) of a Sampled Function

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt \,, \quad \text{where } \tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T) e^{-j2\pi ut} dt$$

$$= \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} \delta(t - n\nabla T) dt$$

CS

Discrete Fourier Transform (DFT) of a Sampled Function

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt, \quad \text{where } \tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T) e^{-j2\pi ut} dt$$

$$= \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} \delta(t - n\nabla T) dt$$

$$= \sum_{n=-\infty}^{n=\infty} f(n\nabla T) e^{-j2\pi un\nabla T}$$

Discrete Fourier Transform (DFT) of a Sampled Function

$$\tilde{F}(u) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ut} dt \,, \quad \text{where } \tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T) e^{-j2\pi ut} dt$$

$$= \sum_{n=-\infty}^{n=\infty} \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} \delta(t - n\nabla T) dt$$

$$= \sum_{n=-\infty}^{n=\infty} f(n\nabla T) e^{-j2\pi un\nabla T} = \sum_{n=-\infty}^{n=\infty} f_n e^{-j2\pi un\nabla T}$$

Discrete Fourier Transform (DFT) of a Sampled Function

$$\widetilde{F}(u) = \sum_{n=-\infty}^{n=\infty} f_n e^{-j2\pi un\nabla T}$$

- f_n is discrete function
- $\widetilde{F}(u)$ is
  - infinite
  - continuous
  - Periodic with period $\frac{1}{\nabla T}$

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Discrete Fourier Transform (DFT) of a Sampled Function

- Let, we take M samples of $\tilde{F}(u)$ from $u = 0$ to $u = \dfrac{1}{\nabla T}$

$$\nabla u = \frac{1}{M \nabla T}, \text{ where } m = 0,1,2,\cdots M-1$$

$$\text{Then, } F_m = \sum_{n=0}^{n=M-1} f_n e^{-j2\pi (m \nabla u) n \nabla T} = \sum_{n=0}^{n=M-1} f_n e^{-j2\pi mn/M}$$

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Discrete Fourier Transform (DFT) of a Sampled Function

$$F_m = \sum_{n=0}^{n=M-1} f_n e^{-j2\pi mn/M} \quad \text{for } m = 0,1,2,\cdots, M-1$$

- Thus, given $\{f_n\}$, we can get $\{F_m\}$
- Similarly, given $\{F_m\}$, we can get $\{f_n\}$

$$f_n = \frac{1}{M} \sum_{m=0}^{m=M-1} F_m e^{j2\pi mn/M} \quad \text{for } n = 0,1,2,\cdots, M-1$$

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Discrete Fourier Transform (DFT) of a Sampled Function

- Replacing m and n by u and x

$$F(u) = \sum_{x=0}^{x=M-1} f(x) e^{-j2\pi ux/M} \quad \text{for } u = 0,1,2,\cdots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{u=M-1} F(u) e^{j2\pi ux/M} \quad \text{for } x = 0,1,2,\cdots, M-1$$

Discrete Fourier Transform (DFT) of a Sampled Function

$$F(u) = \sum_{x=0}^{x=M-1} f(x) e^{-j2\pi ux/M} \quad \text{for u} = 0,1,2,\cdots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{u=M-1} F(u) e^{j2\pi ux/M} \quad \text{for } x = 0,1,2,\cdots, M-1$$

- M is not period, rather is the No. of samples taken
- ∇T is sampling interval of $f(x)$
- $\frac{1}{\nabla T}$ is
  - sampling frequency of $f(x)$
  - length of one complete period of $F(u)$

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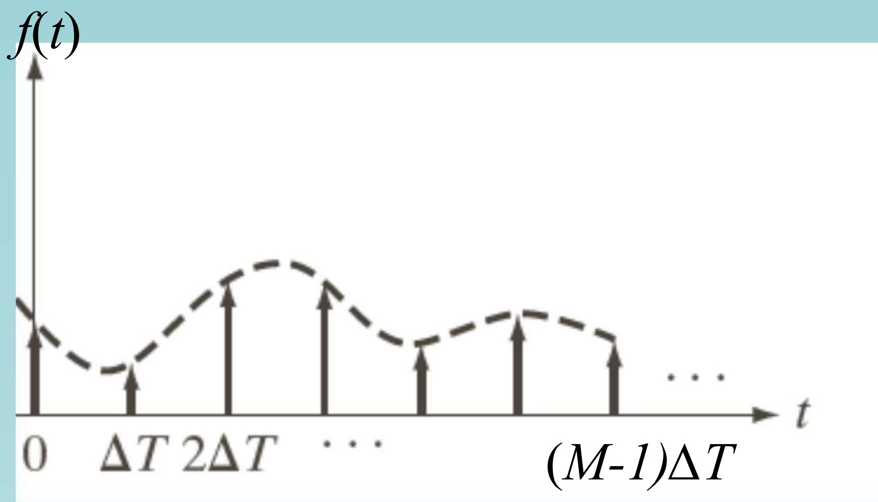
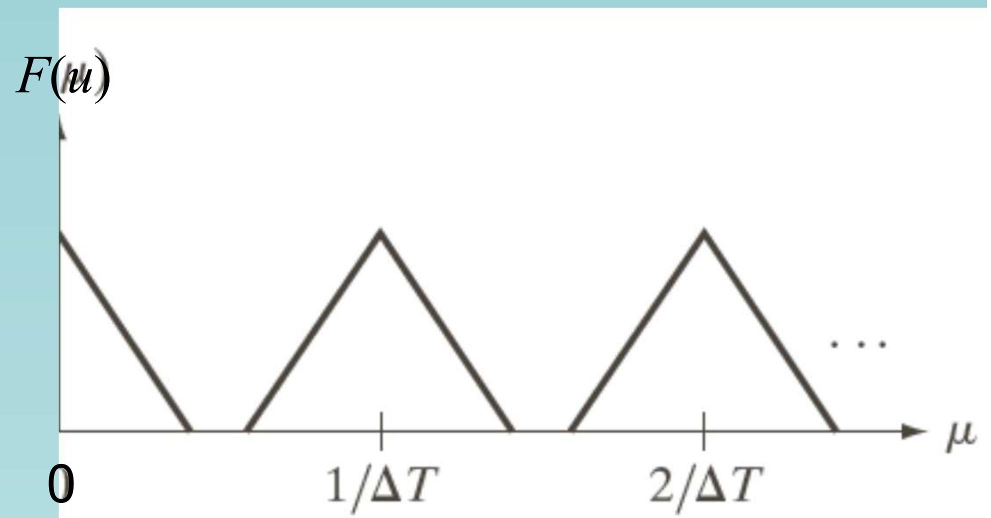
Discrete Fourier Transform (DFT) of a Sampled Function



T



One period in frequency domain

$$T = M\nabla T$$

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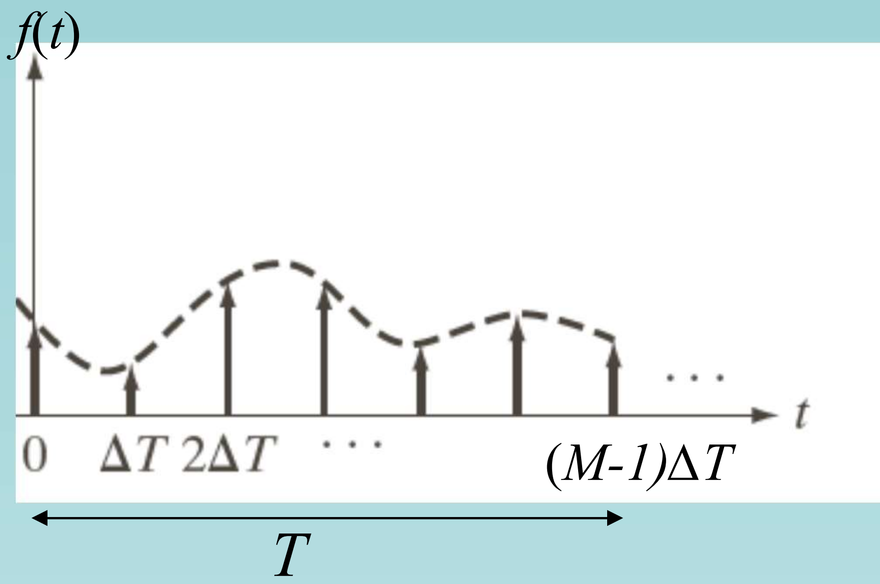
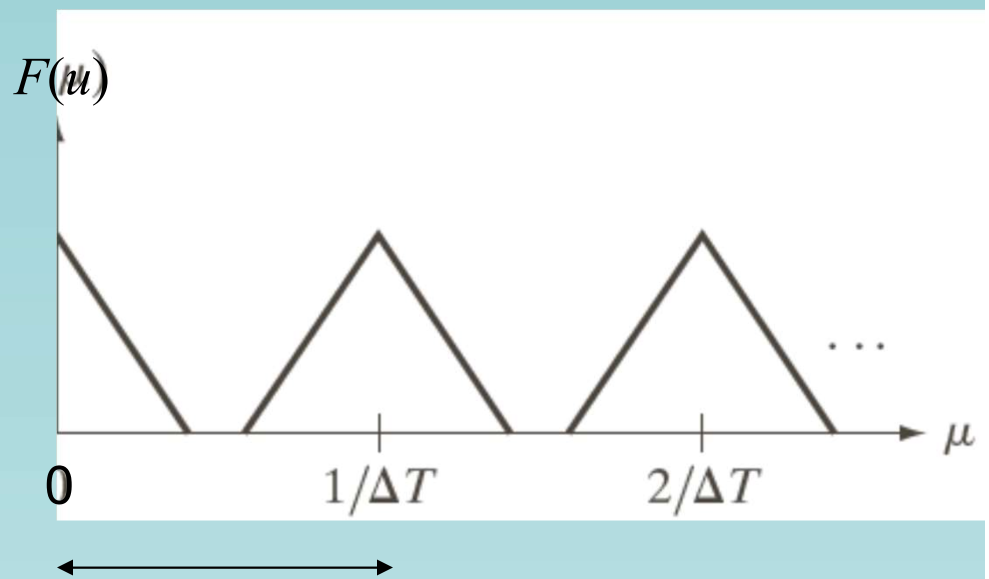
Discrete Fourier Transform (DFT) of a Sampled Function





One period in frequency domain

$$T = M\nabla T, \quad \text{and}$$

$$\nabla u = \frac{1/\nabla T}{M} = \frac{1}{M\nabla T}$$

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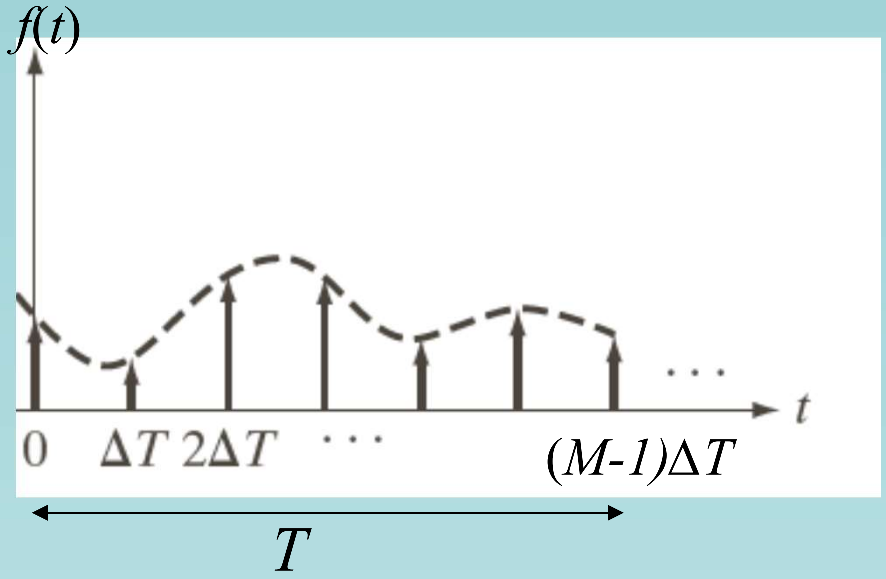
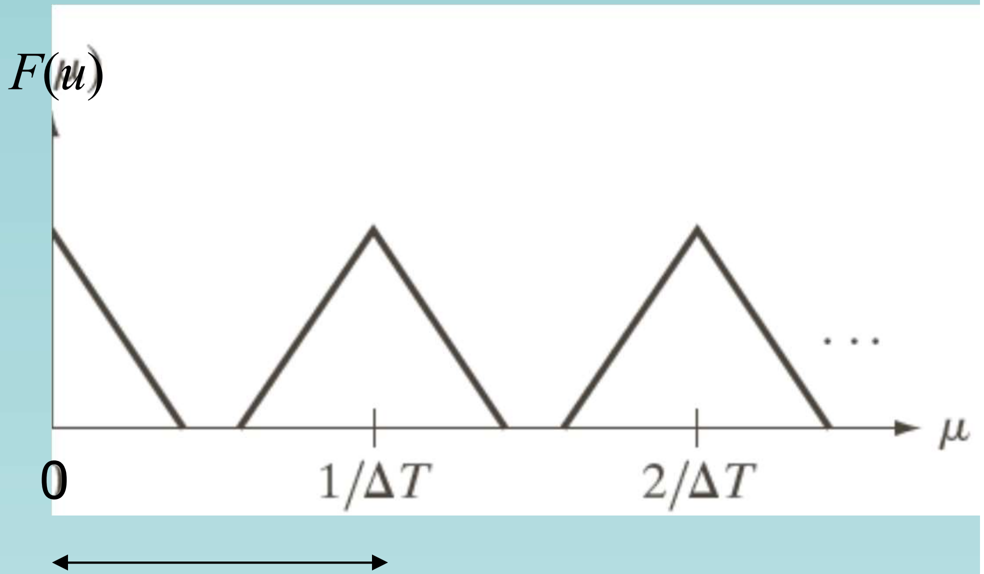
Discrete Fourier Transform (DFT) of a Sampled Function





- $F(u)$ is periodic: $F(u) = F(u+kM)$

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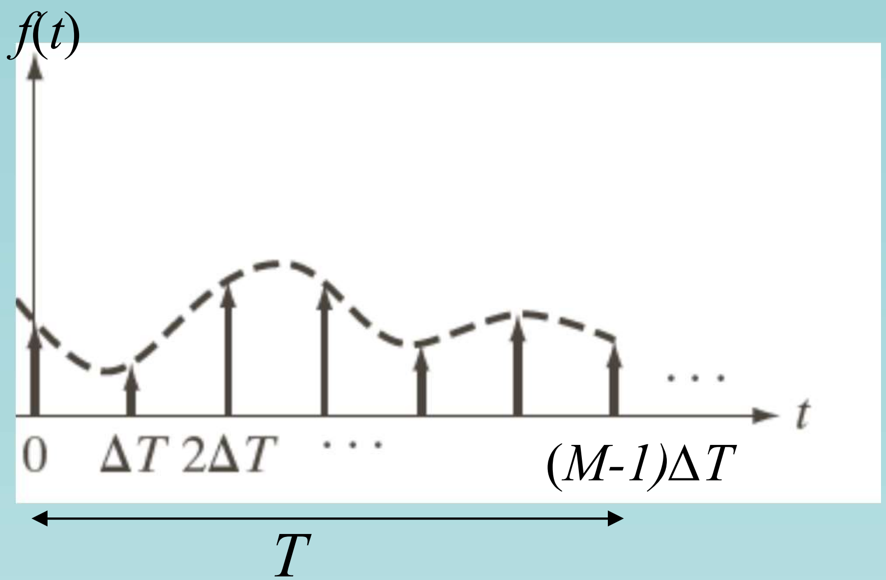
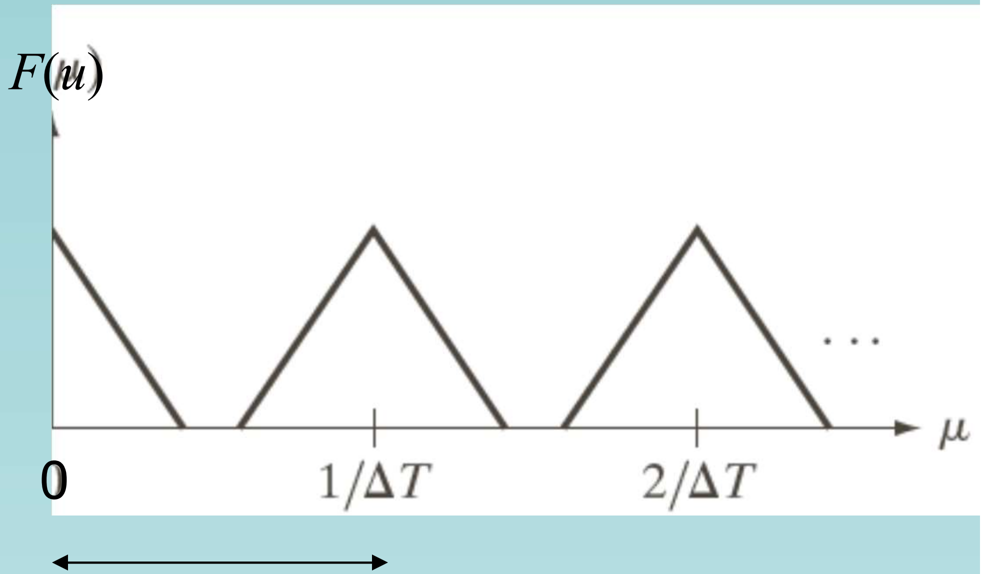
Discrete Fourier Transform (DFT) of a Sampled Function





- If $f(x)$ is derived from the inverse of $F(u)$, $f(x)$ is also periodic: $f(x) = f(x+kM)$

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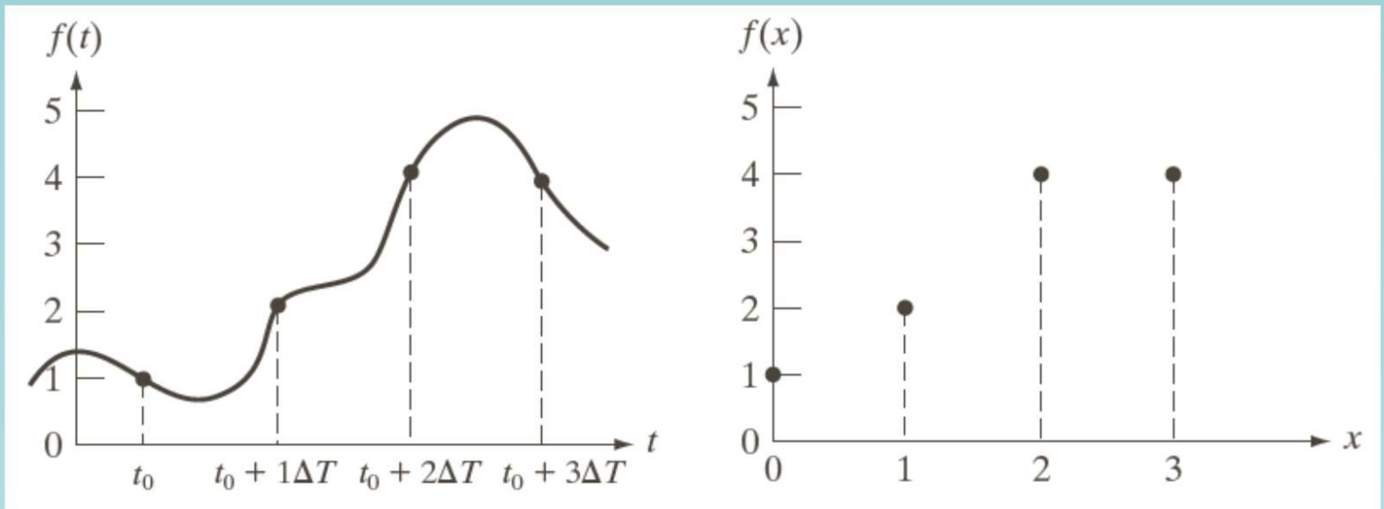
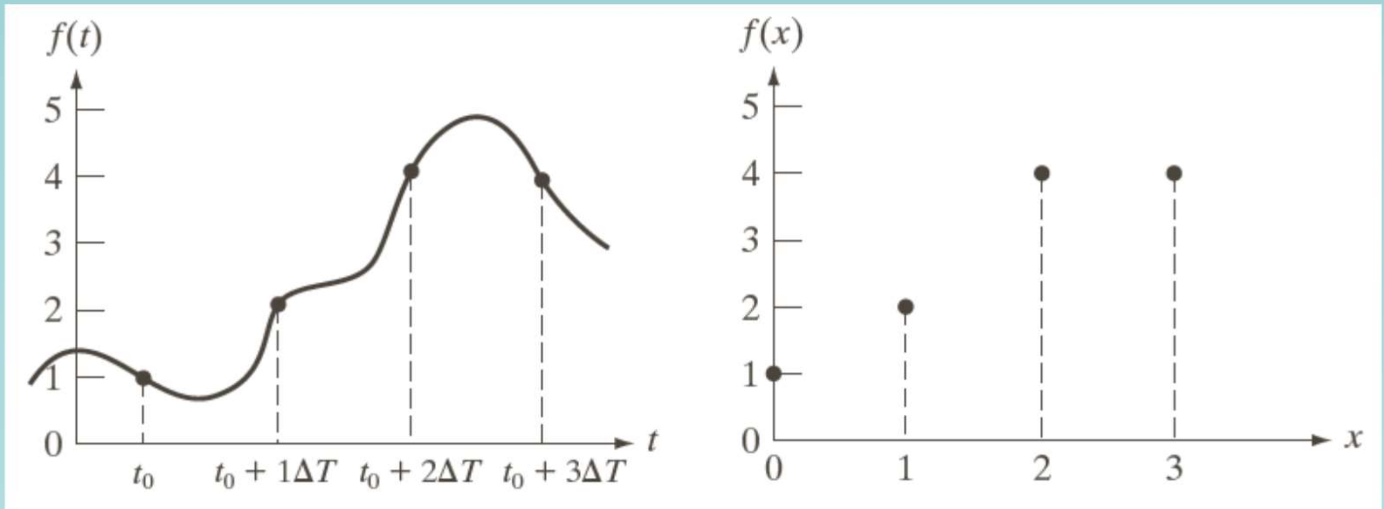
Example of DFT



Example of DFT



$$F(0) = \sum_{x=0}^{x=4} f(x) = f(0) + f(1) + f(2) + f(3)$$

$$= 1 + 2 + 4 + 4 = 11$$

$$F(1) = \sum_{x=0}^{x=4} f(x) e^{-j2\pi(1)x/4}$$

$$= 1e^{0} + 2e^{-j\pi/4} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

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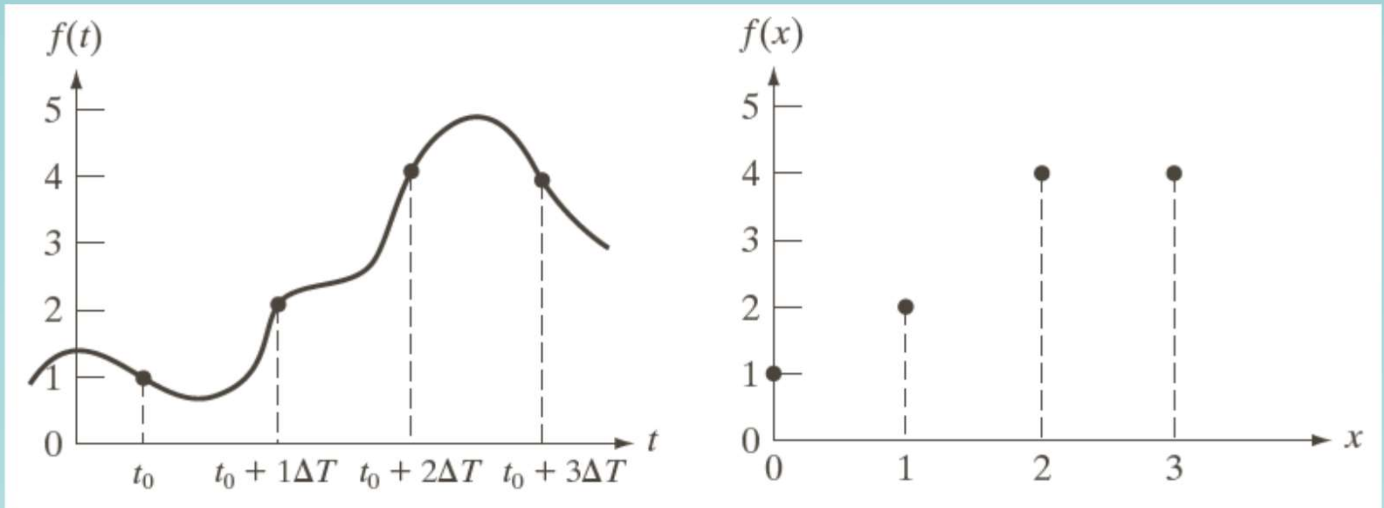
Example of DFT



$$F(0) = 11$$

$$F(1) = -3 + 2j$$

$$F(2) = -1$$

$$F(3) = -3 - 2j$$

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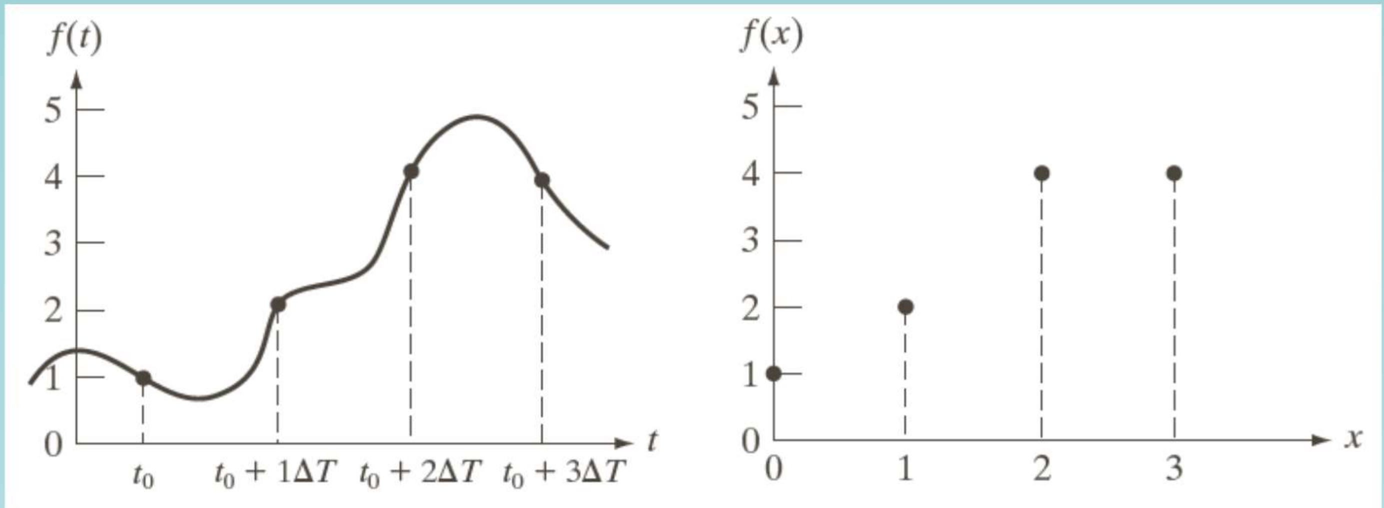
Example of DFT



$$F(0) = 11$$

$$F(1) = -3 + 2j$$

$$F(2) = -1$$

$$F(3) = -3 - 2j$$

$$f(0) = \frac{1}{4}\sum_{u=0}^{u=3} F(u)e^{j2u\pi(0)/4}$$

$$= \frac{1}{4}\sum_{u=0}^{u=3} F(u)$$

$$= \frac{1}{4}\left[11 - 3 + 2j - 1 - 3 - 2j\right] = 1$$

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