CSE6706:

Advanced Digital Image Processing

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Spectral/Frequency Domain Analysis of Images

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Recall: DFT of a Sampled Function for a Single Period

Review

$$F(u) = \sum_{x=0}^{x=M-1} f(x) e^{-j2\pi ux/M} \qquad \text{for } u = 0,1,2,\cdots,M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{u=M-1} F(u) e^{j2\pi ux/M} \qquad \text{for } x = 0,1,2,\cdots,M-1$$

2D DFT and Its Inverse

Review

- Forward transform:

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)},\ \text{for } u = 0, 1, ..., M\text{-}1 \text{ and } v = 0, 1, . . ., N\text{-}1$$

- Inverse transform:

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)},\ \text{for } x = 0, 1, ..., M-1 \text{ and } y = 0, 1, . . ., N\text{-}1$$

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Review: 1D FT and Convolution

$$\Im\left\{f(t)*h(t)\right\}=H(u)F(u)$$

$$f(t)h(t)=H(u)*F(u)$$

$$f(t)*h(t)\Leftrightarrow H(u)F(u)$$

$$f(t)h(t)\Leftrightarrow H(u)*F(u)$$

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2D FT and Convolution

$$\Im\left\{f(x,y)*h(x,y)\right\}=H(u,v)F(u,v)$$

$$f(x,y)*h(x,y)\Leftrightarrow H(u,v)F(u,v)$$

$$\Im\left\{f(x,y)h(x,y)\right\}=H(u,v)*F(u,v)$$

$$f(x,y)h(x,y)\Leftrightarrow H(u,v)*F(u,v)$$

Properties of DFT: Translation

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

Properties of DFT: Rotation

Let,

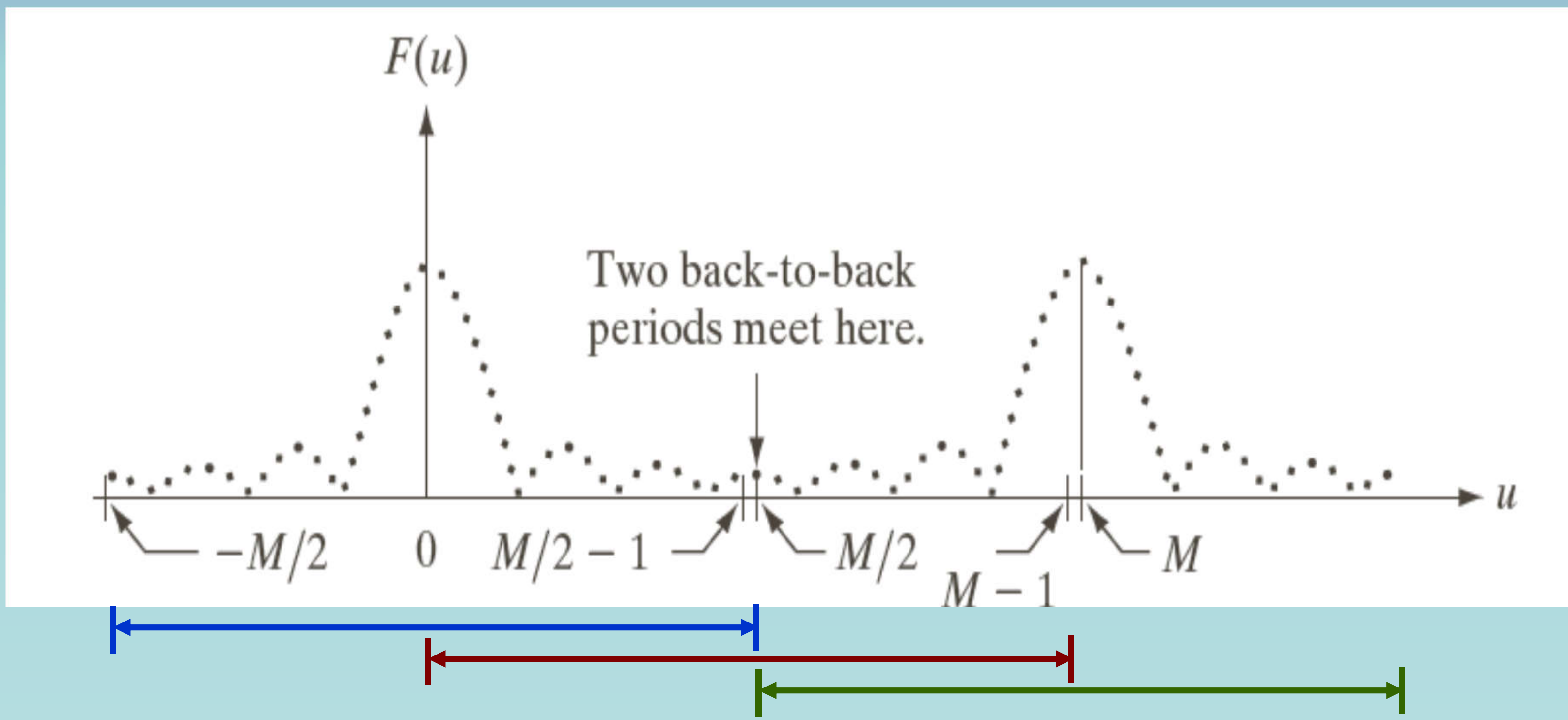
$$x = r\cos\theta, y = r\sin\theta, u = \omega\cos\varphi, v = \omega\sin\varphi$$

Now, if $f(r,\theta) \Leftrightarrow F(\varphi,\omega)$

Then,

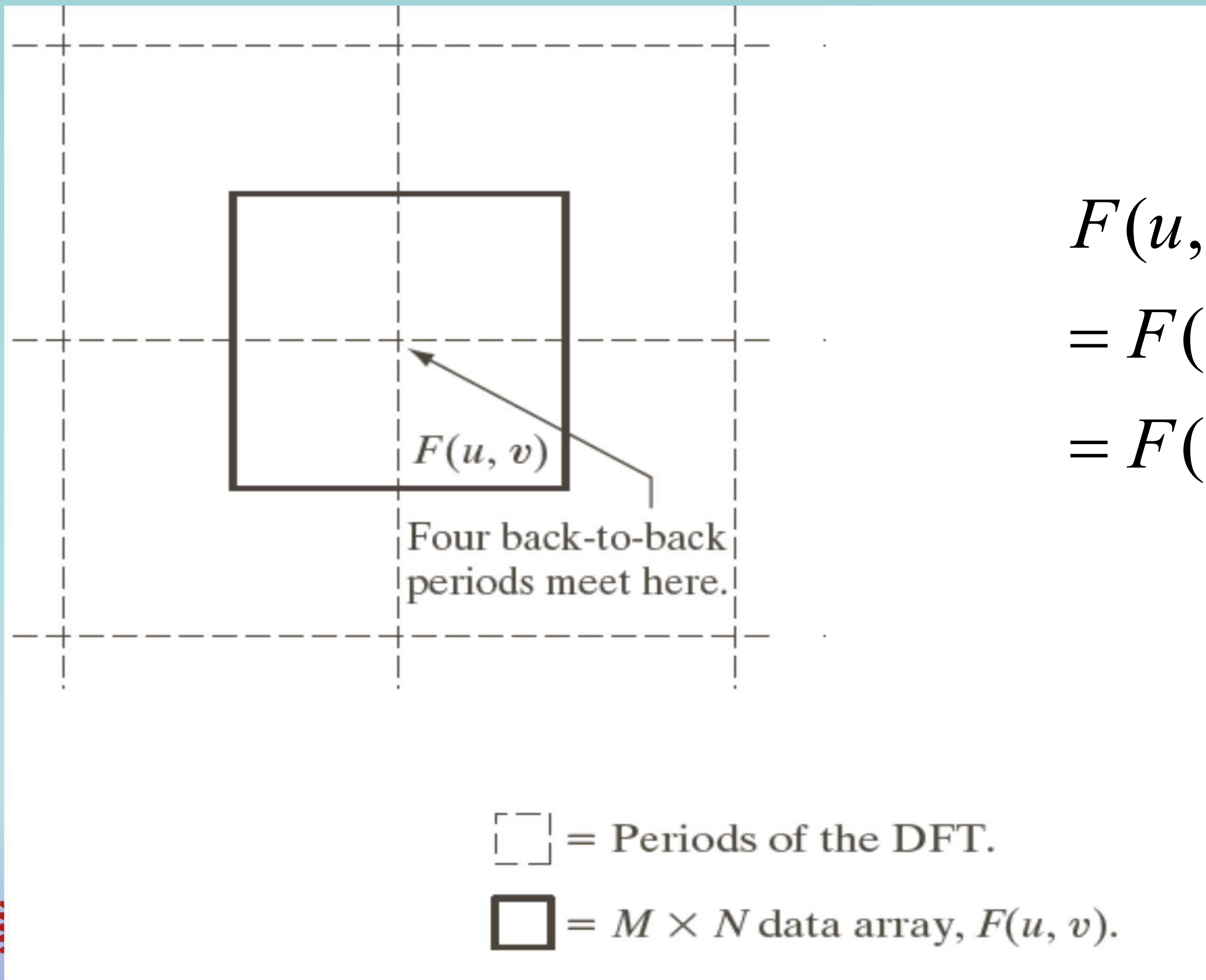
$$f(r,\theta+\theta_0) \Leftrightarrow F(\varphi,\omega+\theta_0)$$

Properties of DFT: Periodicity (1D)



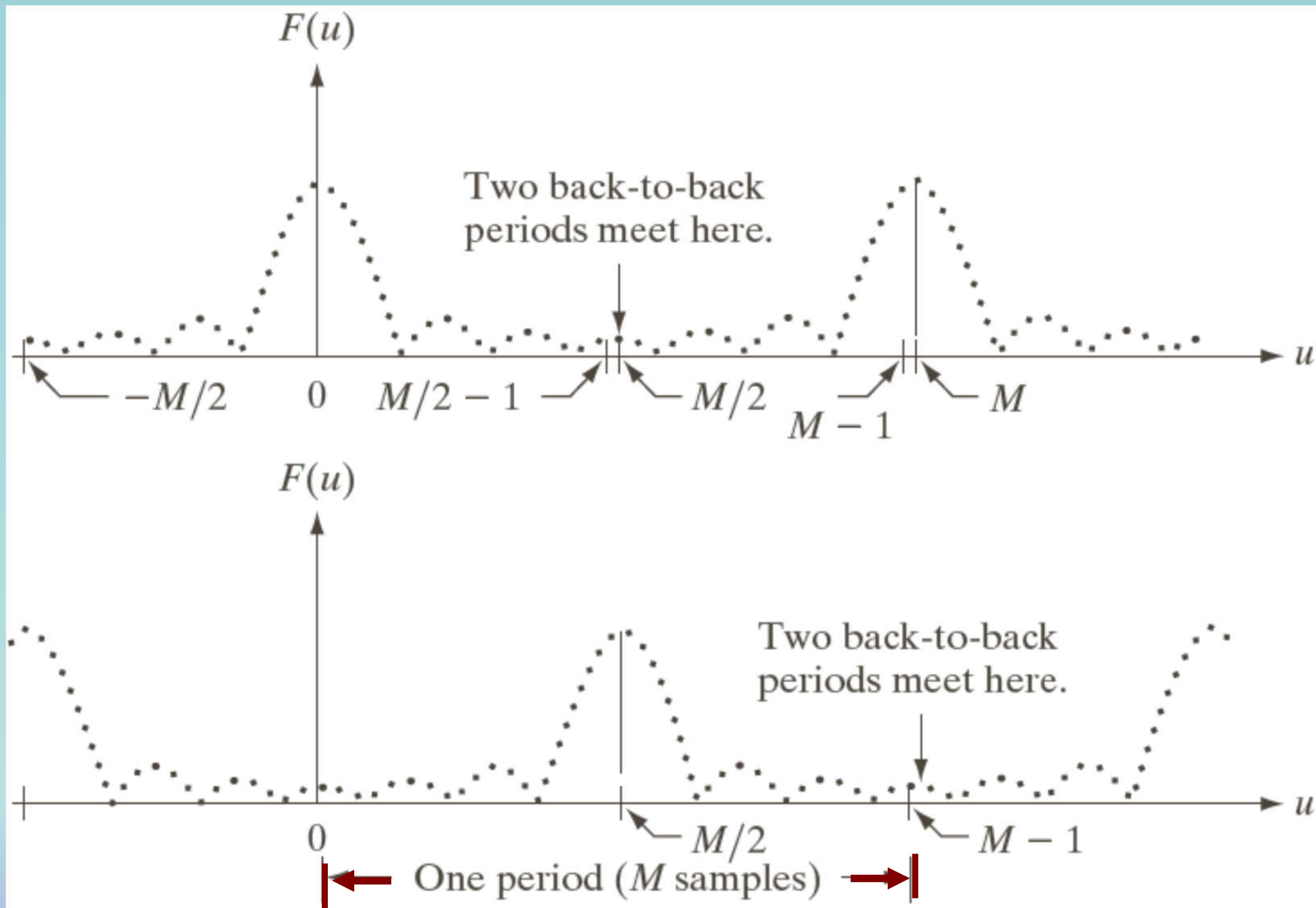
$$F(u) = F(u + kM)$$

Properties of DFT: Periodicity (2D)



$$F(u,v) = F(u + k_1 M, v)$$
$$= F(u, v + k_2 N)$$
$$= F(u + k_1 M, v + k_2 N)$$

Properties of DFT: Periodicity (1D)



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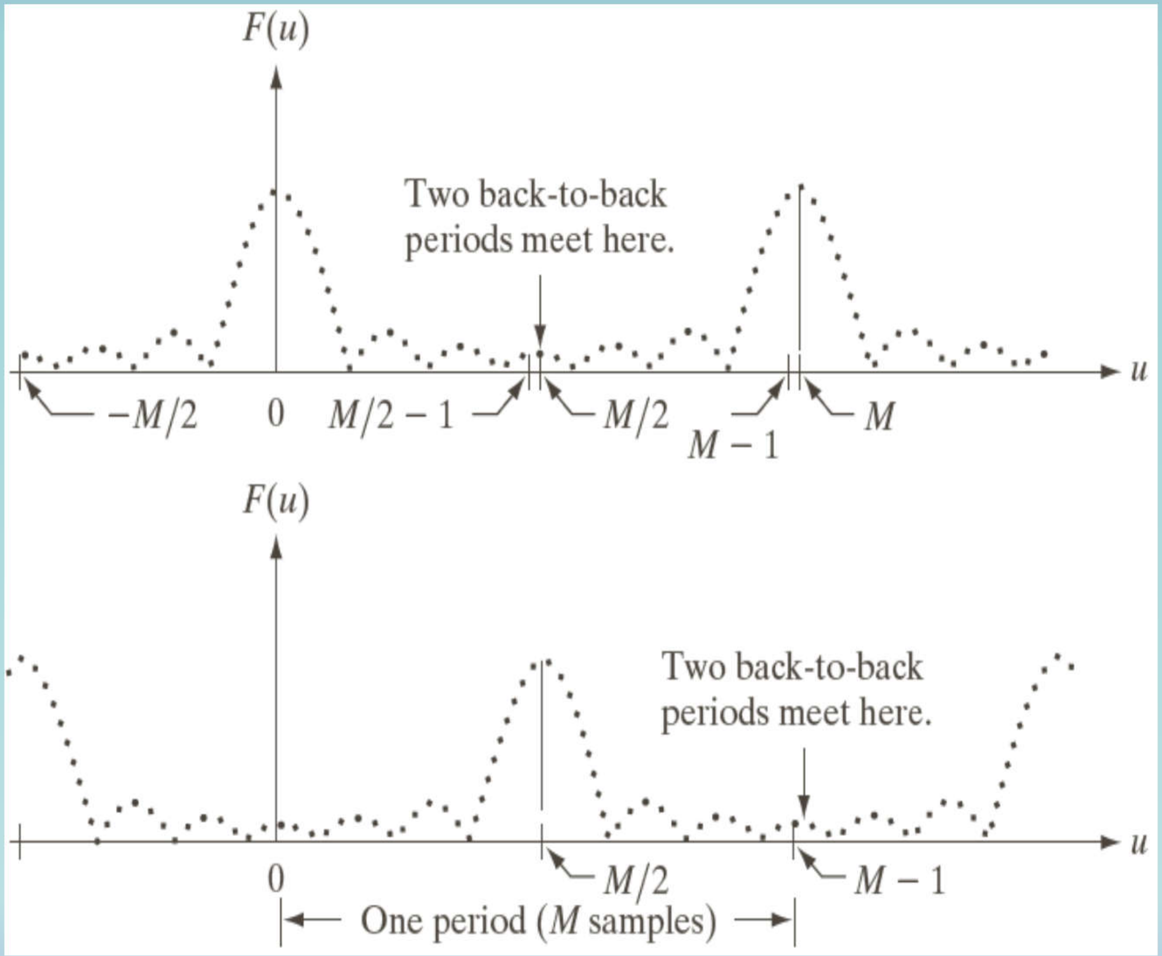
Properties of DFT: Periodicity (1D)



$$f(x)e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u-u_0)$$

$$\text{where, } u_0 = M/2$$

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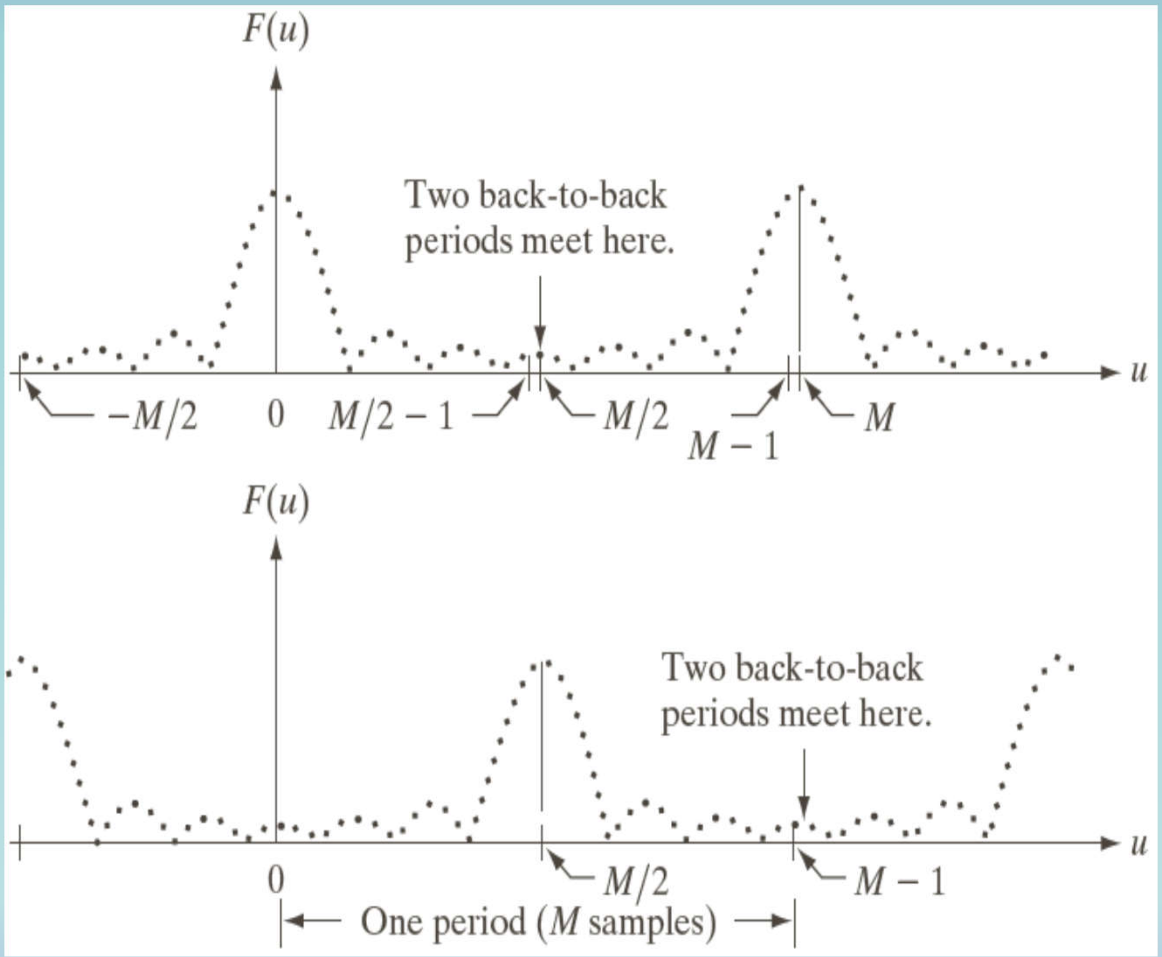
Properties of DFT: Periodicity (1D)



$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u-u_0)$$

$$\begin{aligned} & f(x)e^{j2\pi(u_0x/M)} \\ &= f(x)e^{j2\pi(Mx/2M)} \\ &= f(x)e^{j\pi x} \\ &= f(x)(e^{j\pi})^x = f(x)(-1)^x \end{aligned}$$

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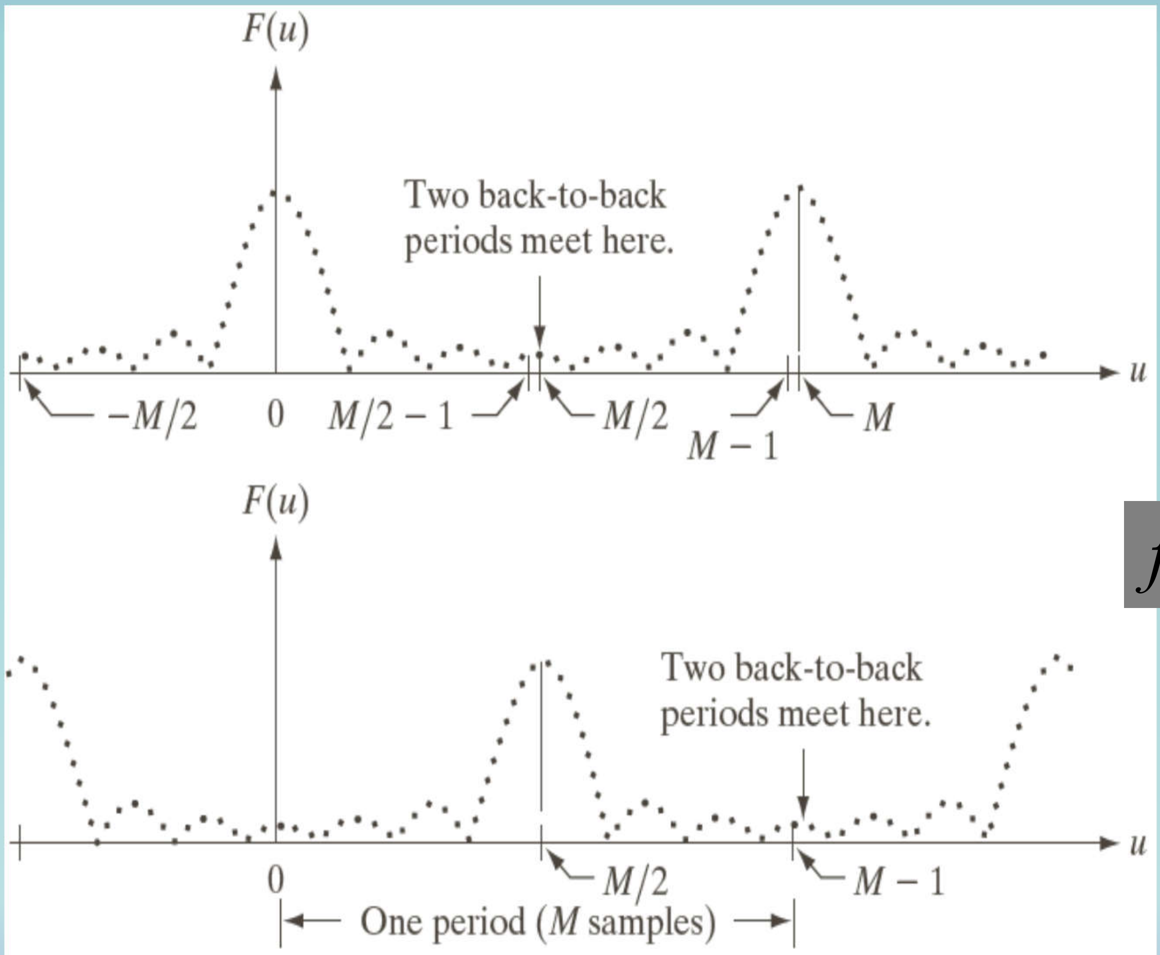
Properties of DFT: Periodicity (1D)



$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

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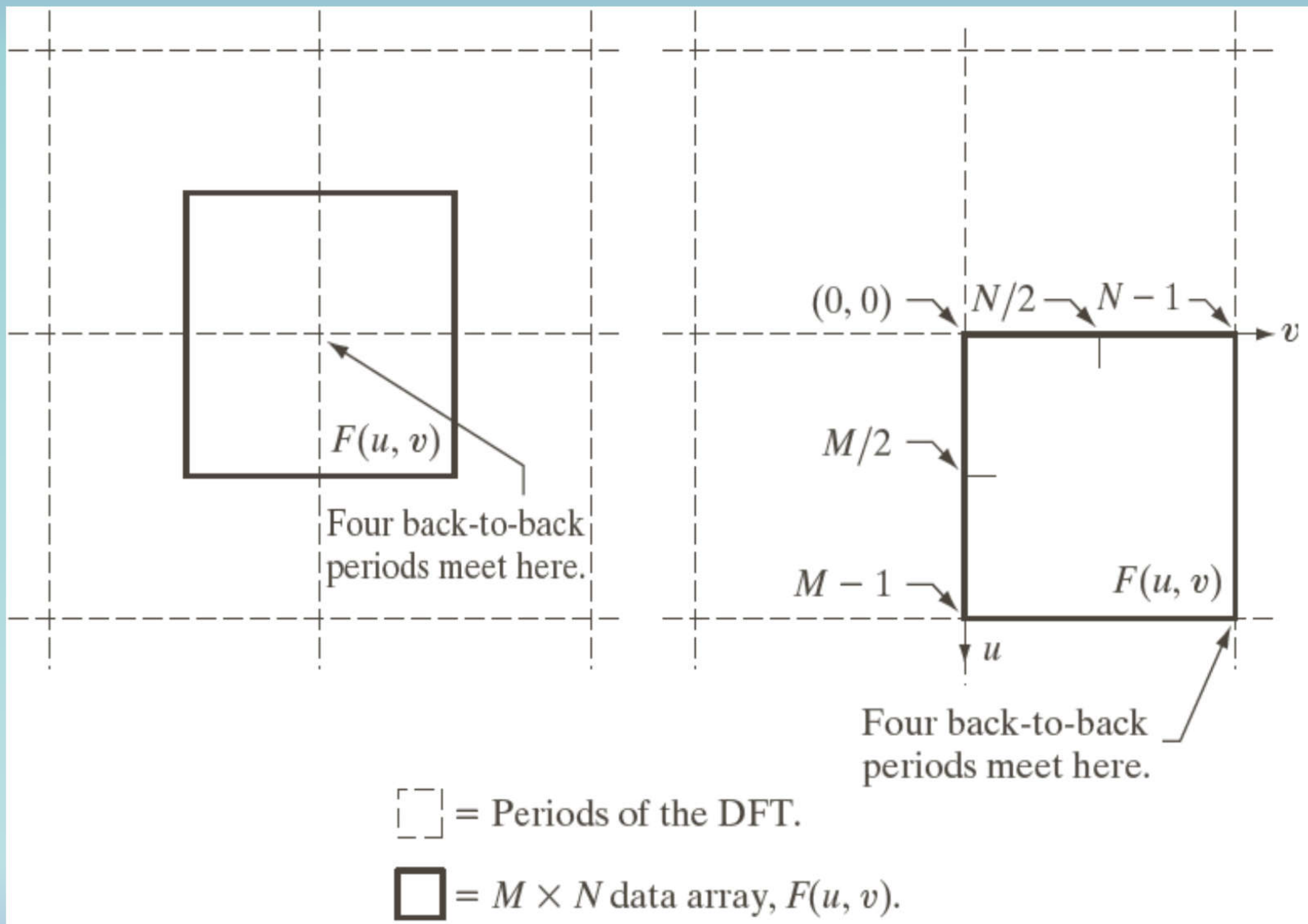
Properties of DFT: Periodicity (2D)



$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$

Useful Terms in 2D DFT

$F(u,v)$ can be written as $\left|F(u,v)\right|e^{-j\phi(u,v)}$

where,

$$\left|F(u,v)\right| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2} \quad \text{and} \quad \phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$$

Fourier spectrum

and $R(u, v)$ and $I(u, v)$ are real and imaginary components of $F(u, v)$

Useful Terms in 2D DFT

Power spectrum:

$$P(u,v) = \left|F(u,v)\right|^2 = R^2(u,v) + I^2(u,v)$$

Coefficients of 2D DFT

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

- The FT coefficient at $u = 0$, $v = 0$:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

Coefficients of 2D DFT

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

- The FT coefficient at other *u*'s and *v*'s ?

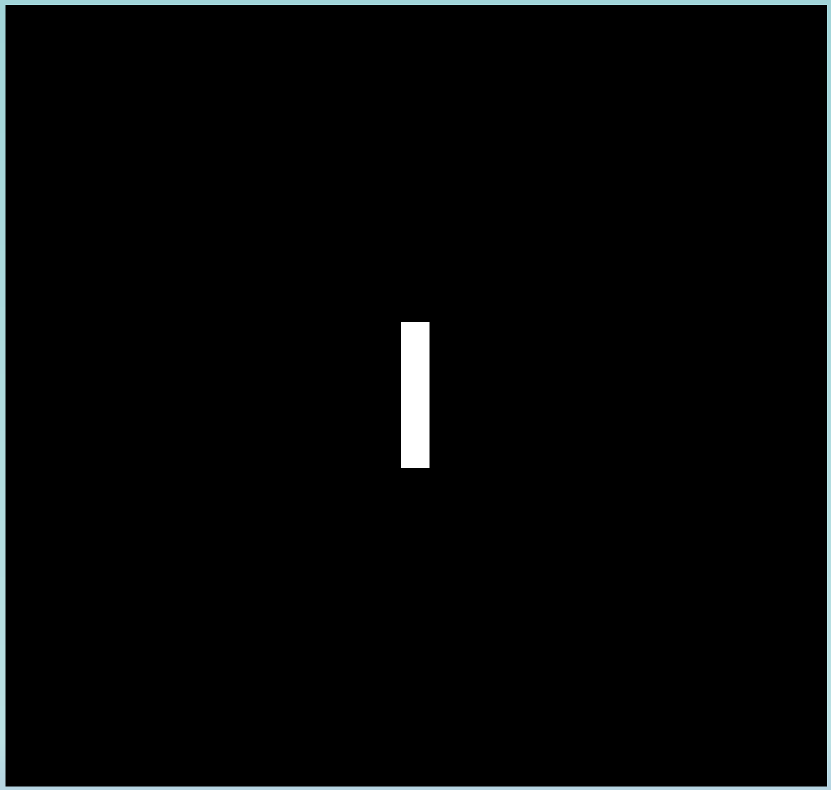
Coefficients of 2D DFT

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

- The FT coefficient at other *u*'s and *v*'s ?

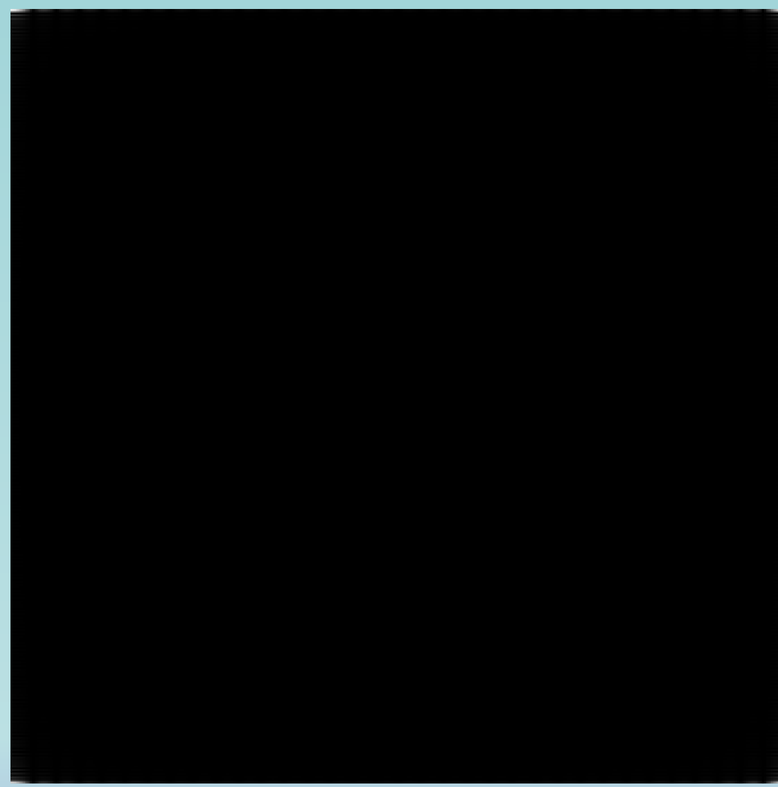
$$F(u,v)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} f(x,y)[\cos 2\pi(ux/M+vy/N)-j\sin 2\pi(ux/M+vy/N)]$$

Examples of 2D DFT (1)



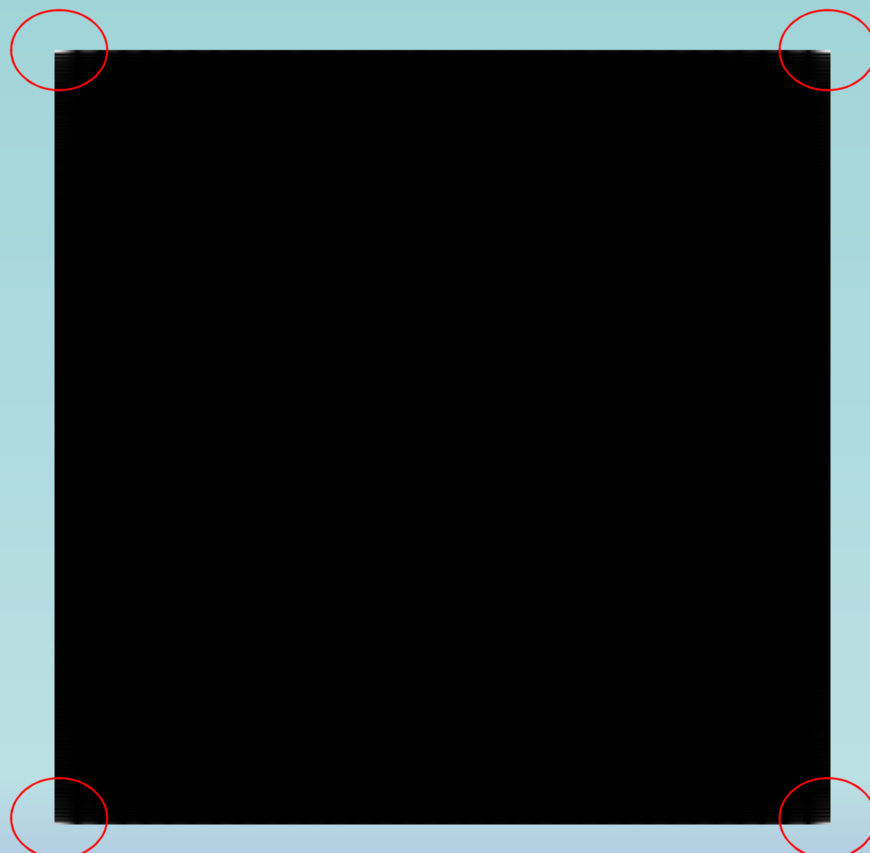
An image with a
rectangular object

Examples of 2D DFT (1)



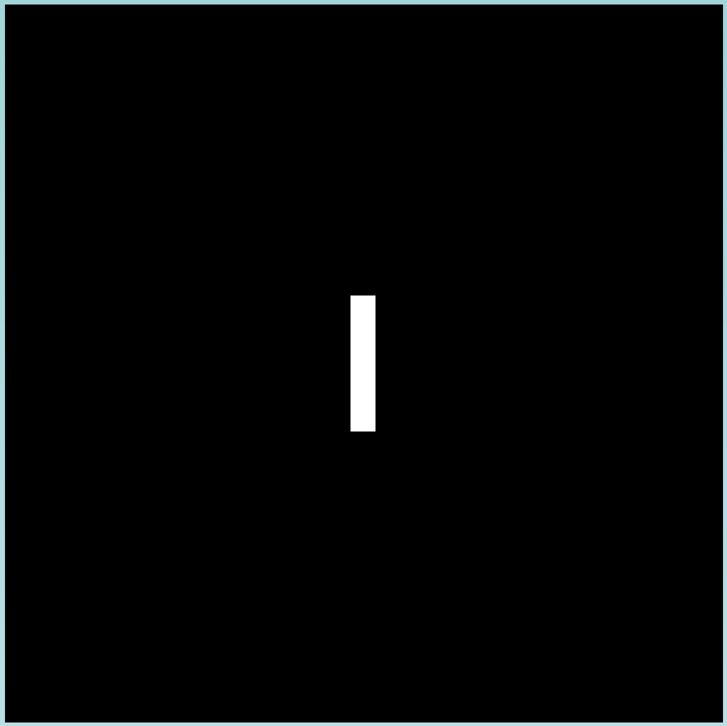
FT of the image with
a rectangular object

Examples of 2D DFT (1)

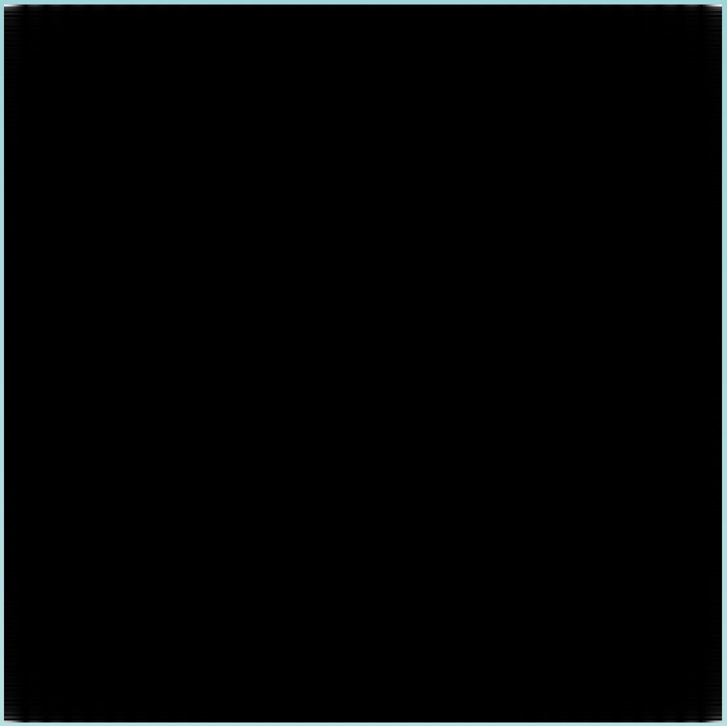


Spectrum is visible
only in corners

Examples of 2D DFT (1)

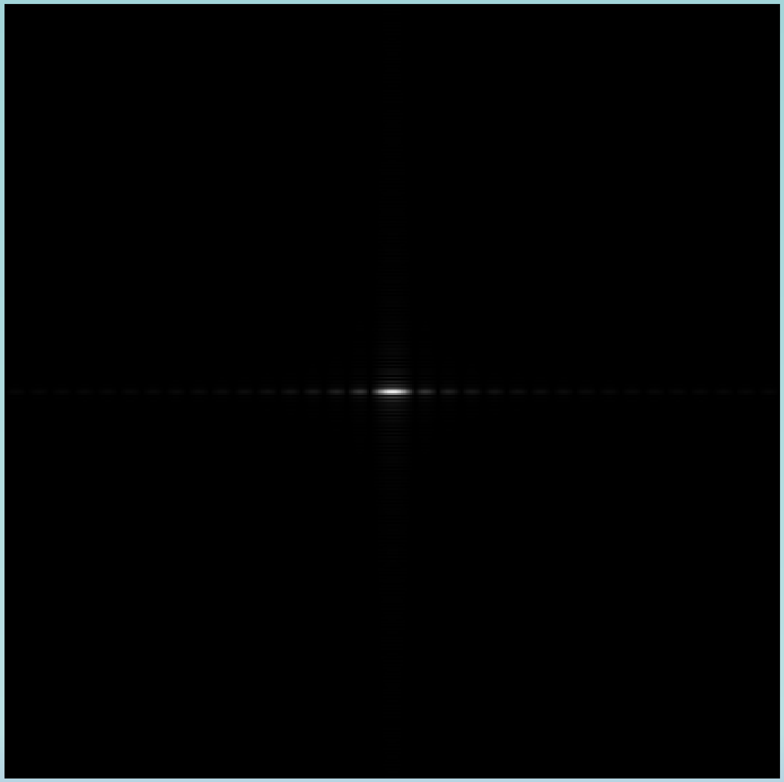


The image



Spectrum

Examples of 2D DFT (1)



Centered

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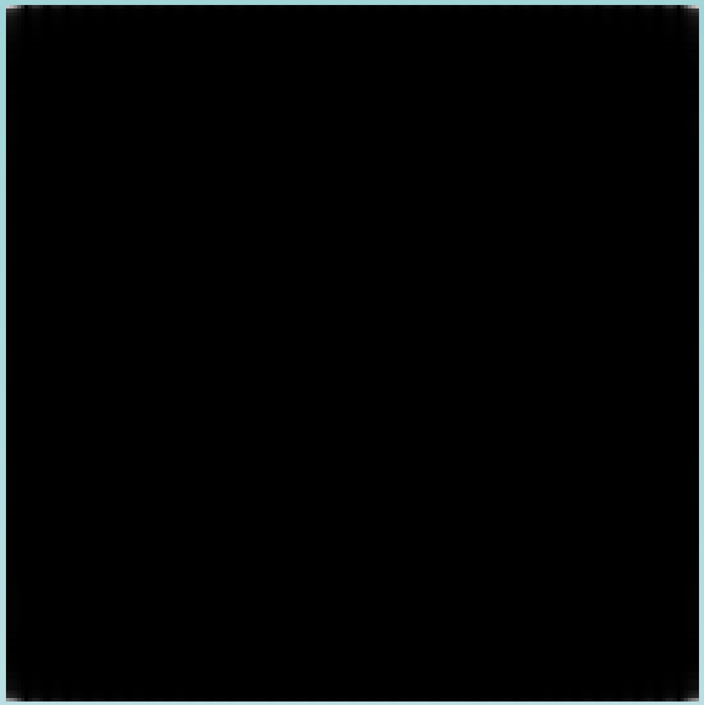
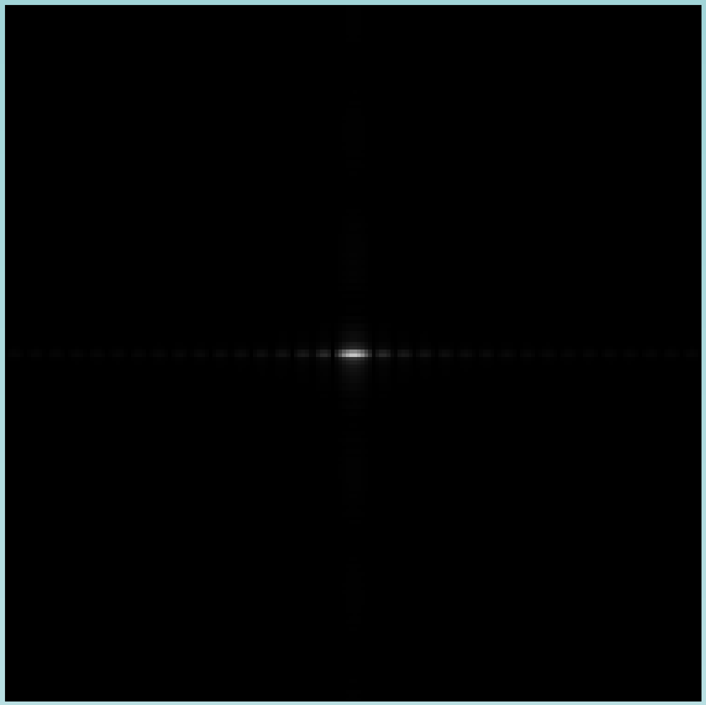
Examples of 2D DFT (1)



Before centering



After centering

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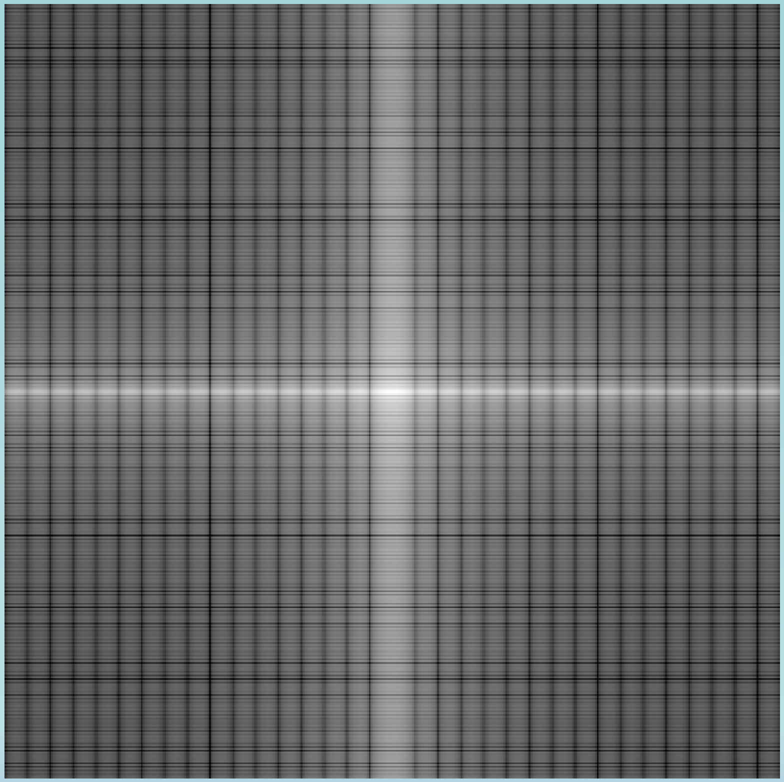
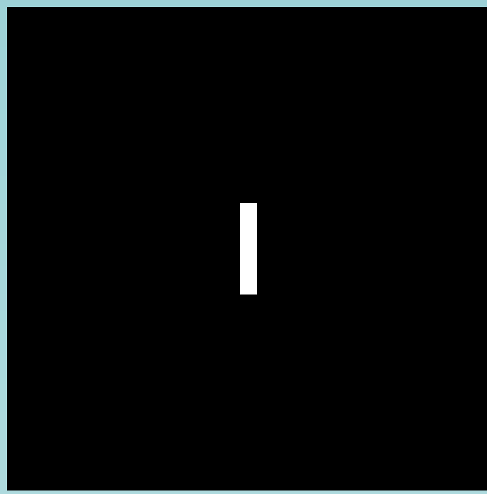
Examples of 2D DFT (1)



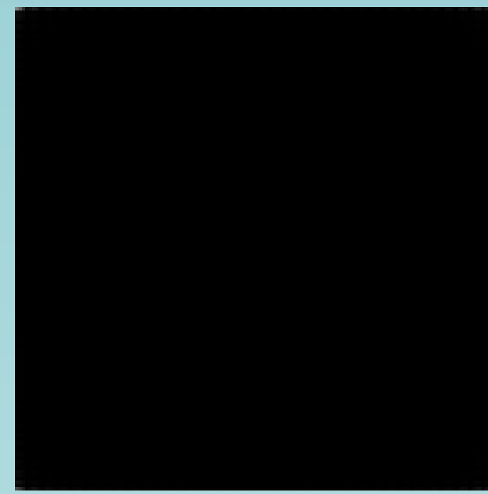
After Log transform

Examples of 2D DFT (1)

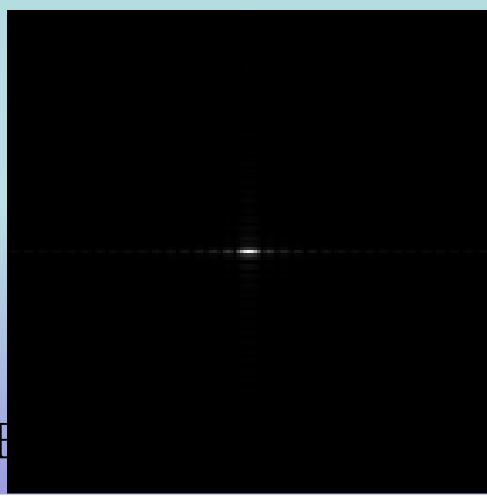
The image



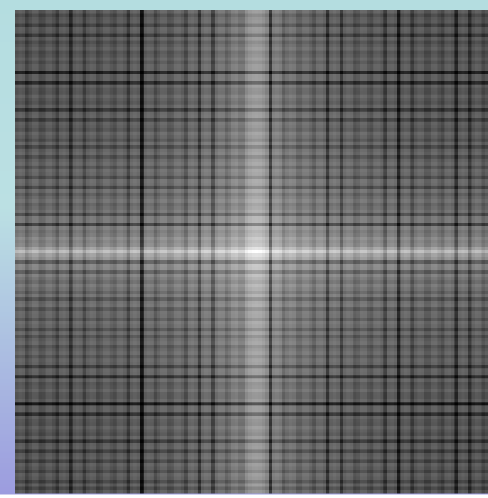
Spectrum before centering



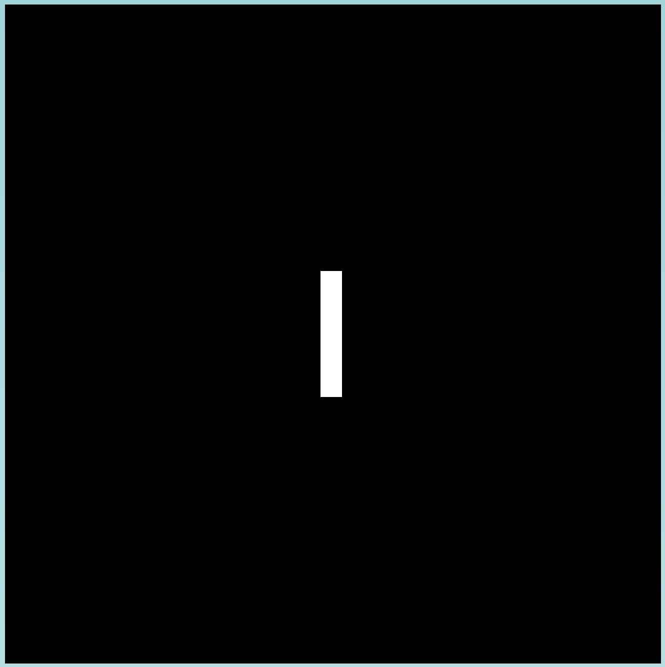
Spectrum after centering

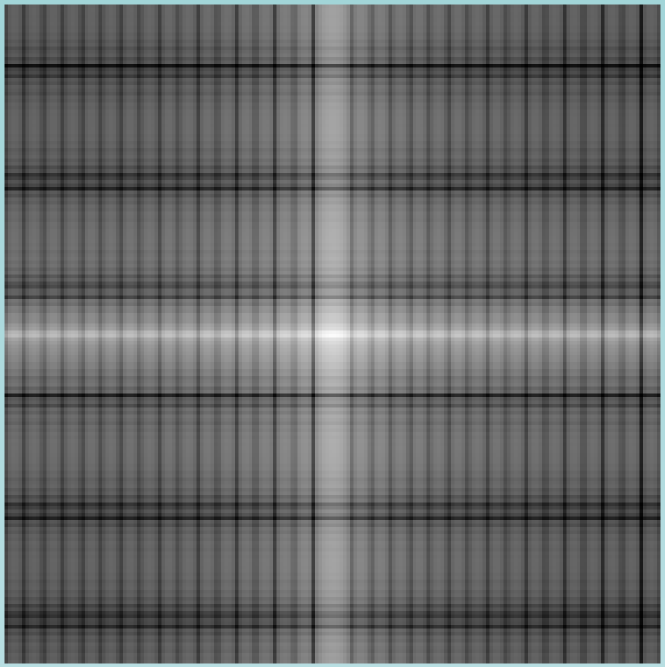


Spectrum after Log Transform

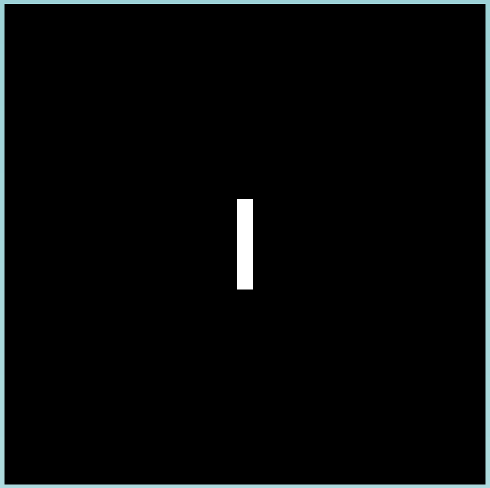


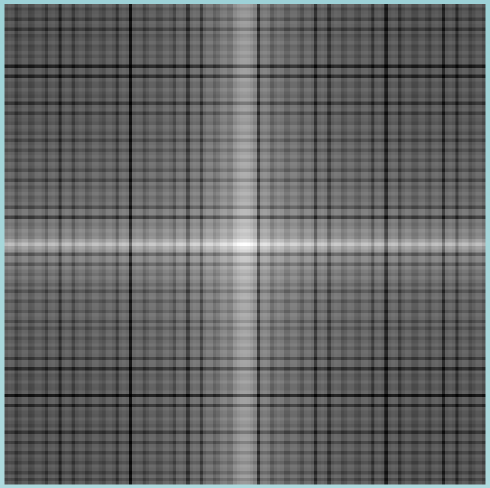
Examples of 2D DFT (1)

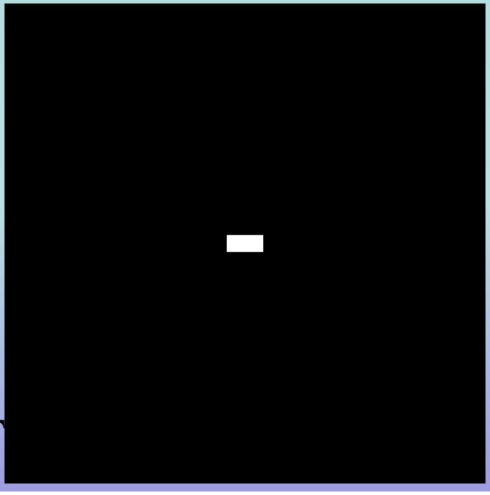


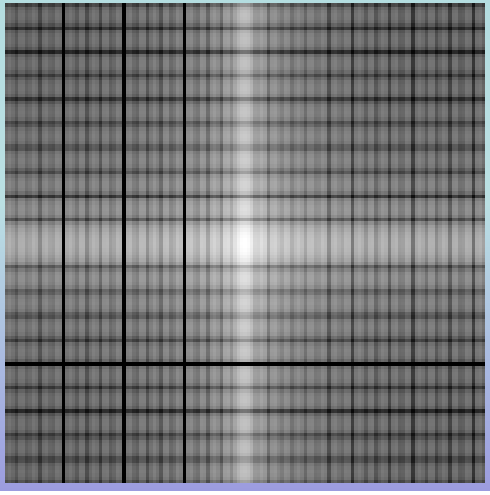


Examples of 2D DFT (2)

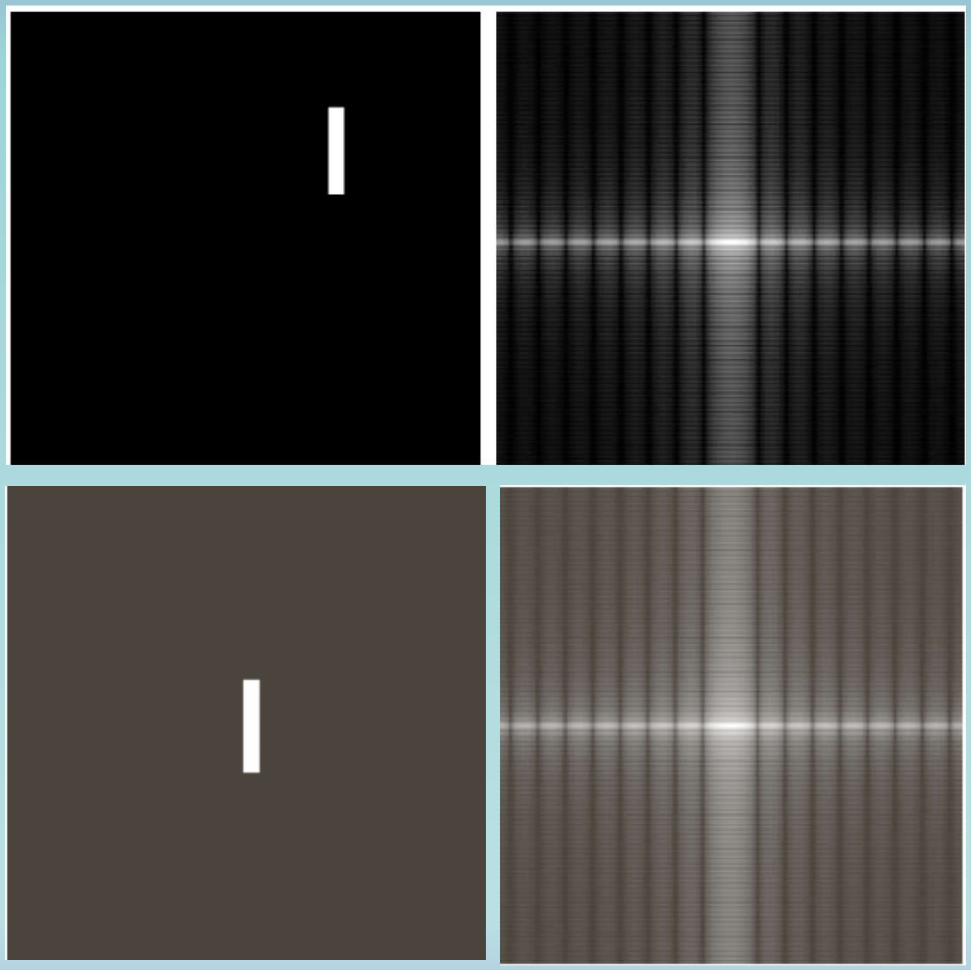








Translation



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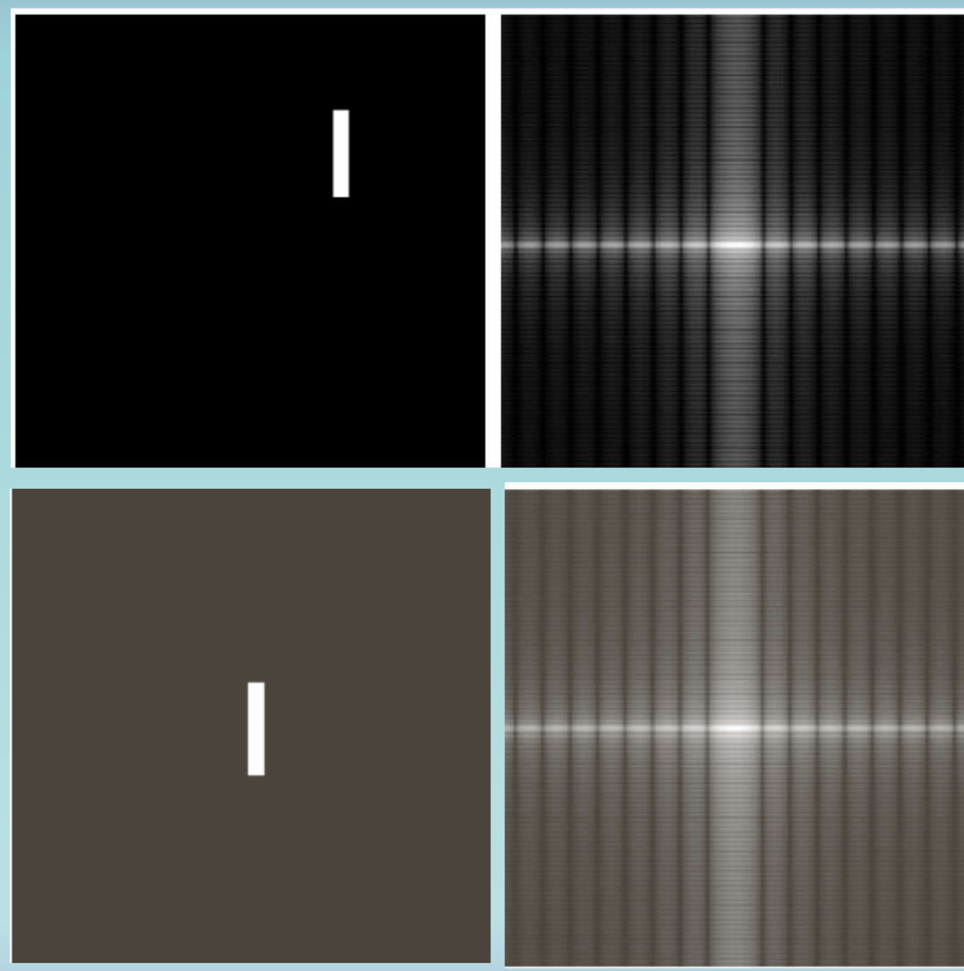
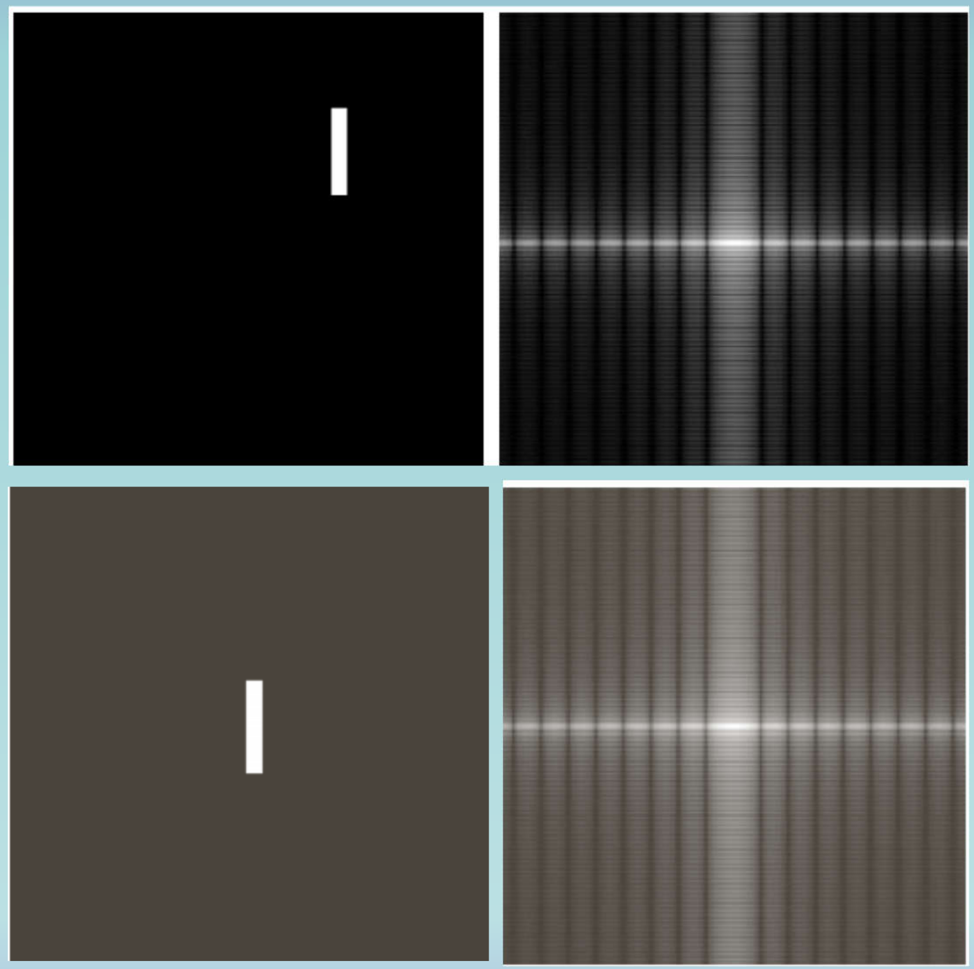
Translation



$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

Translation



$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}$$

$$\text{as } |e^{i\theta}| = 1, \quad \left|F(u,v)e^{-j2\pi(x_0u/M+y_0v/N)}\right| = \left|F(u,v)\right| \times 1 = \left|F(u,v)\right|$$

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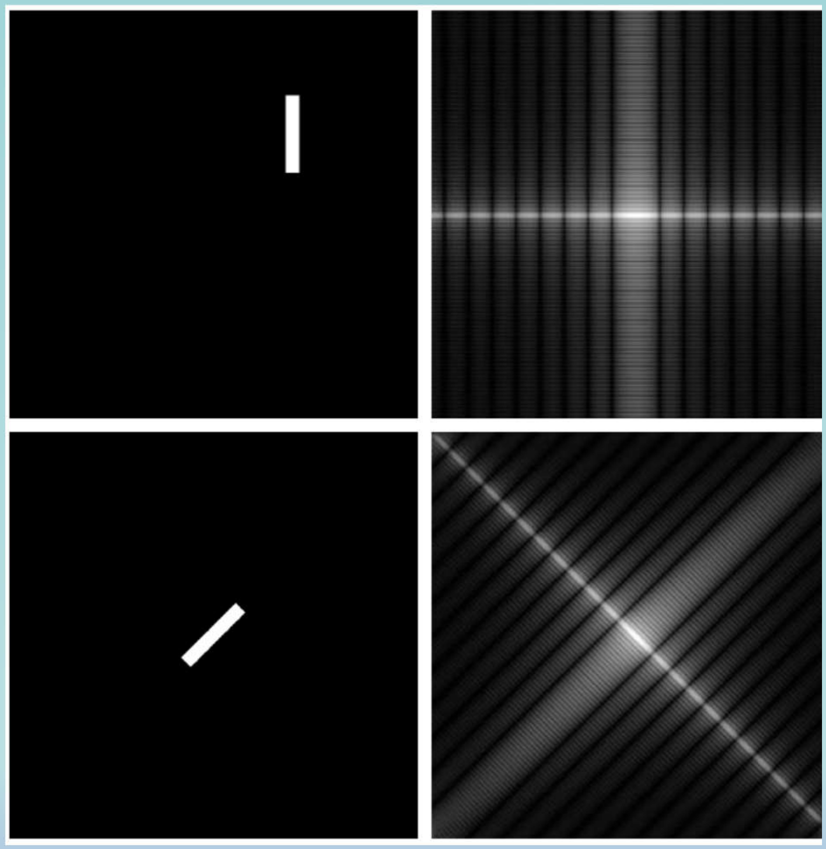
Rotation



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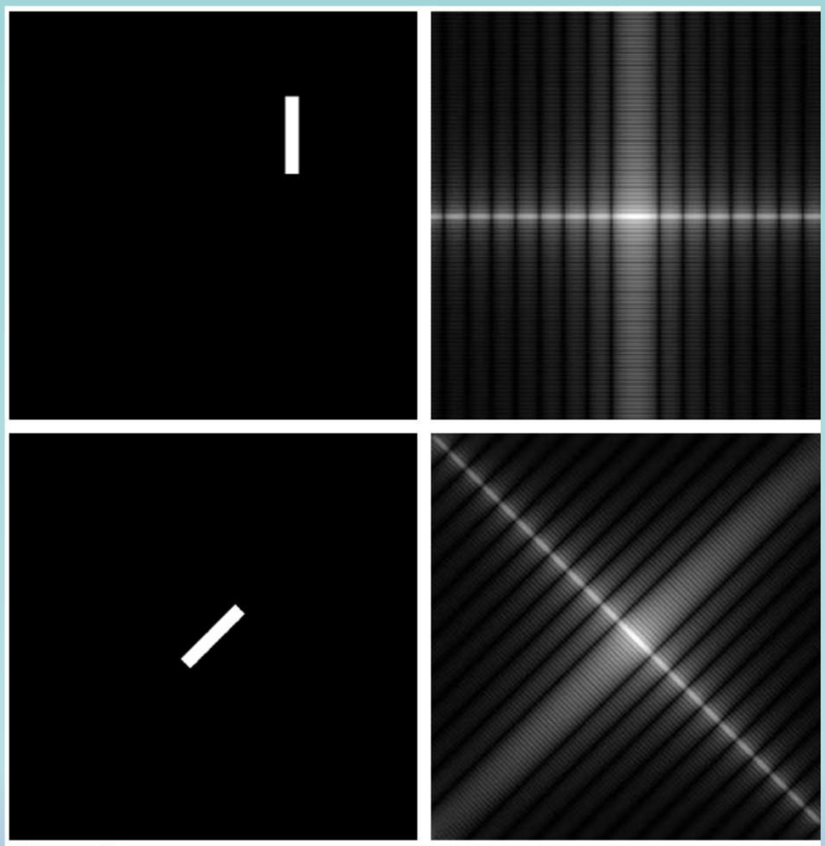
Rotation



$$f(r,\theta+\theta_0) \Leftrightarrow F(\varphi,\omega+\theta_0)$$

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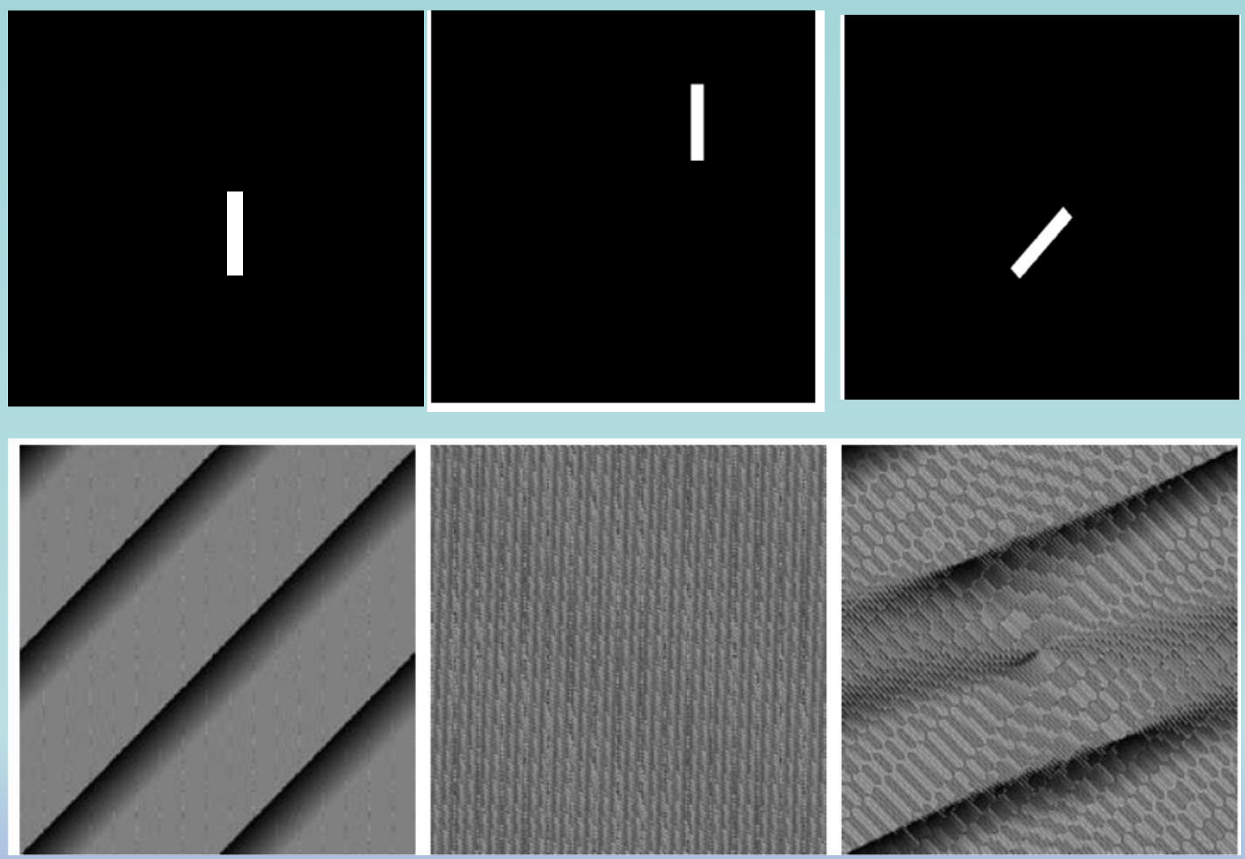
Phase Angle of 2D DFT



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Phase Angle of 2D DFT

Original image　　　　Phase angle

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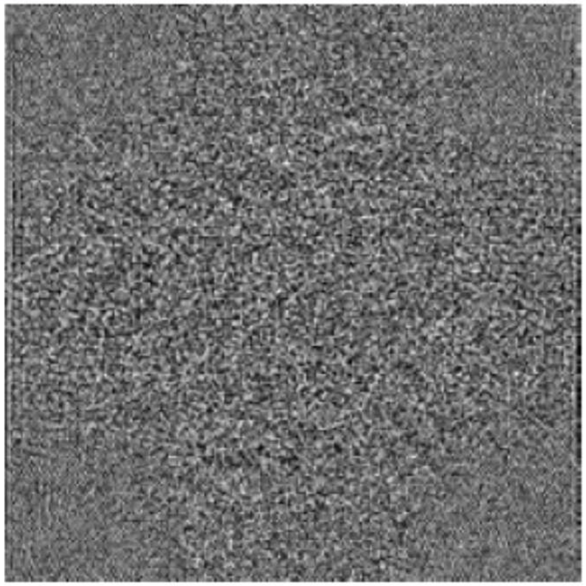
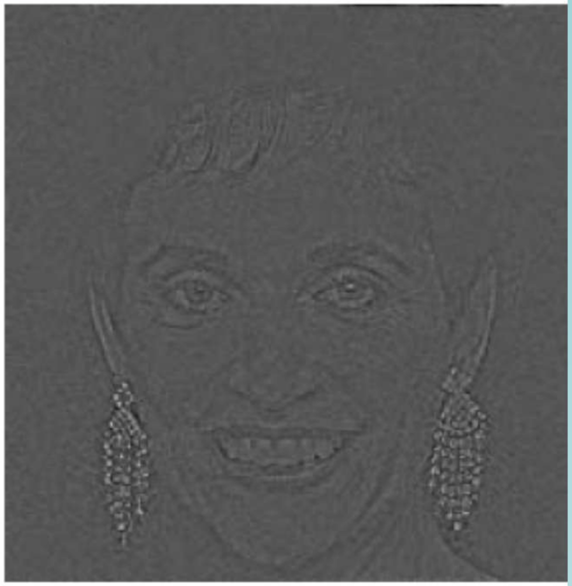
Phase Angle of 2D DFT

Original image



Phase angle, $\varphi(u, v)$



Reconstructed only from Phase angle

$$\text{Let, } |F(u,v)| = 1$$

$$f(x,y) \Leftrightarrow |F(u,v)| e^{j\phi(u,v)} = e^{j\phi(u,v)}$$

$$f(x,y) \Leftrightarrow e^{j\phi(u,v)}$$

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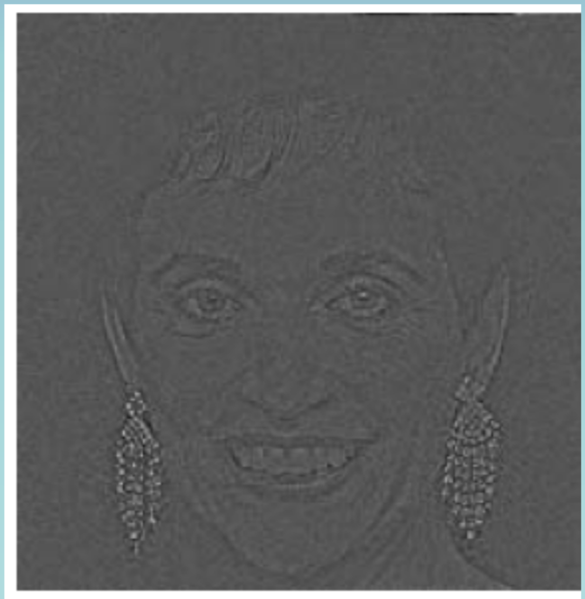
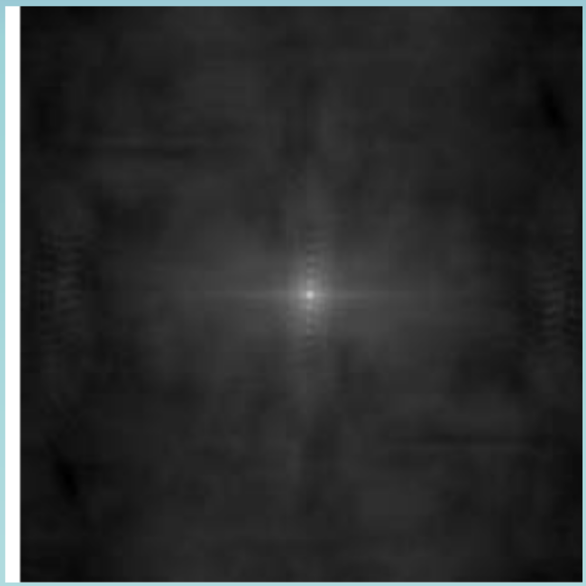
Phase Angle of 2D DFT

Original image



Reconstructed only from Phase angle



Reconstructed only from Spectrum

$$\text{Let, } \phi(u,v) = 0$$

$$f(x,y) \Leftrightarrow |F(u,v)| \times 1 = |F(u,v)|$$

$$f(x,y) \Leftrightarrow |F(u,v)|$$

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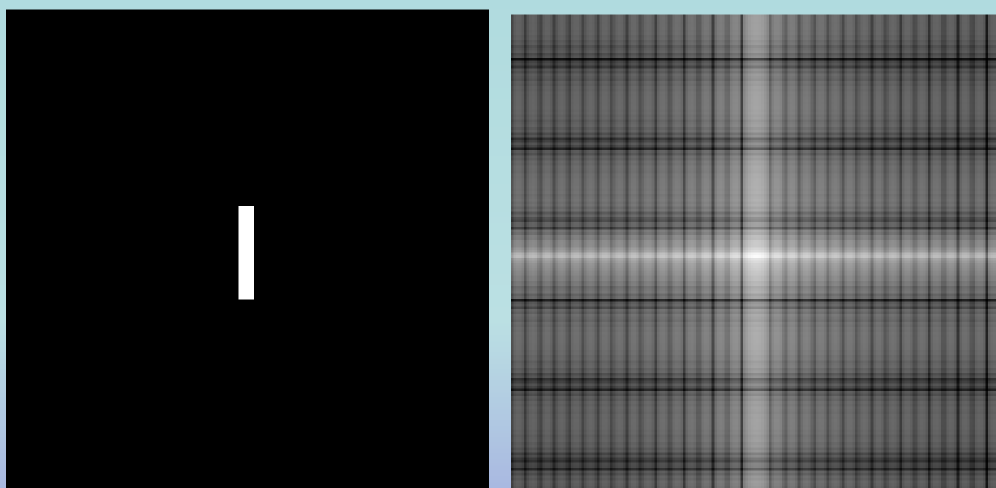
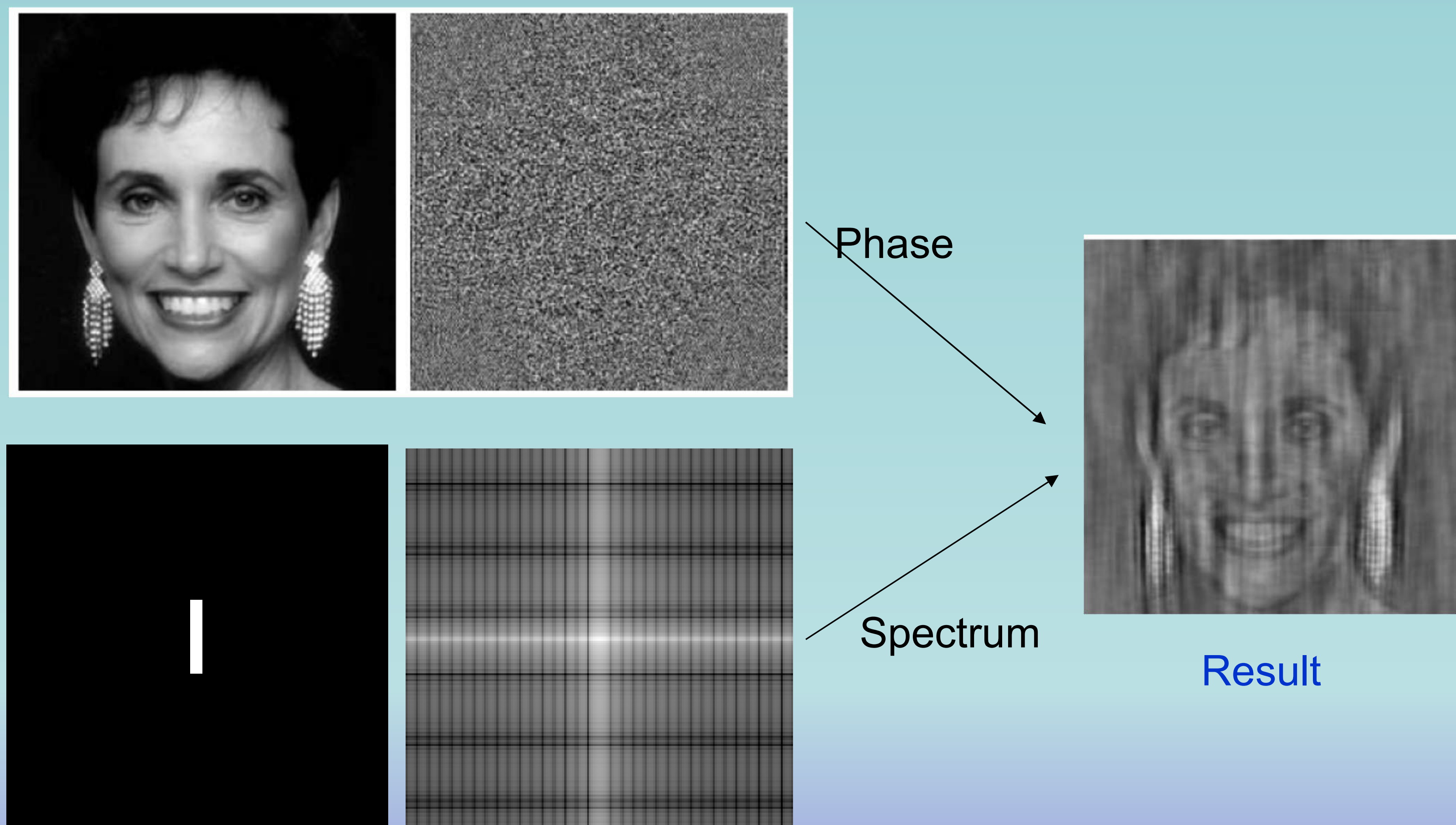
Phase Angle of 2D DFT

Use this
Phase angle



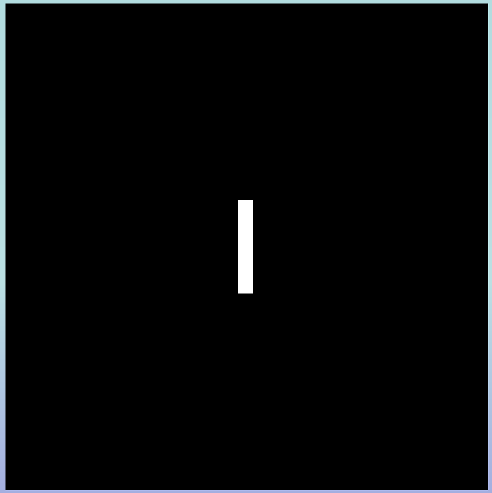
Use this
Spectrum

Phase Angle of 2D DFT

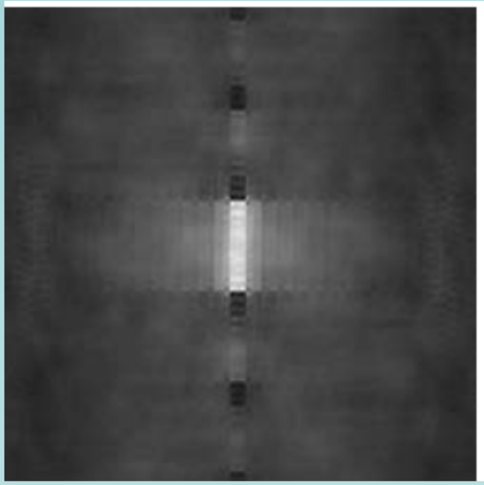


Phase Angle of 2D DFT

Spectrum



Phase



Result

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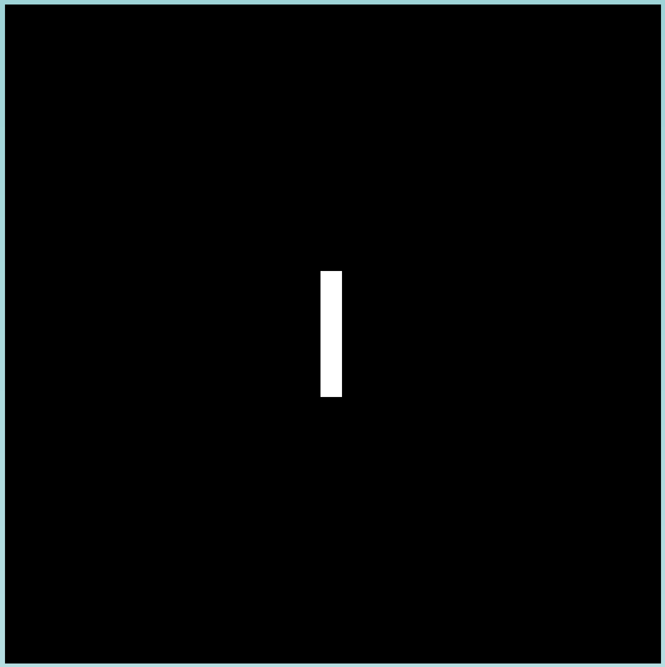
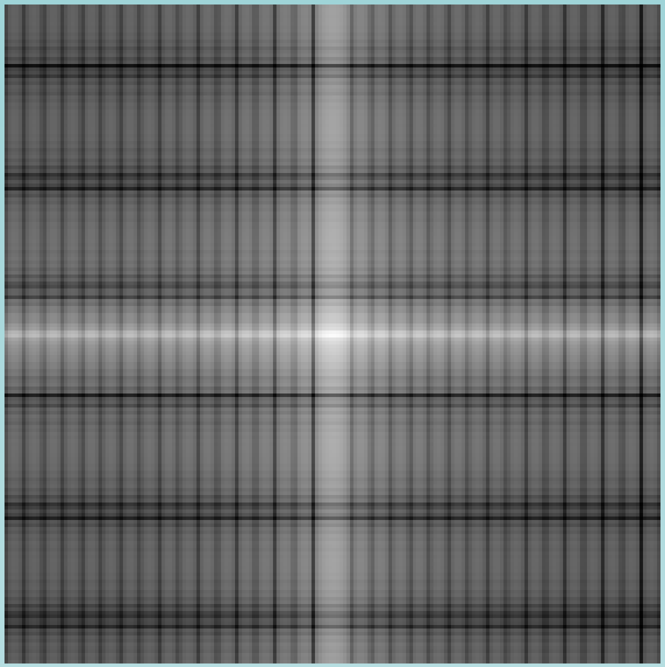
Other Characteristics in Frequency Domain





- spatial and frequency domain terms: difficult to relate directly
- However, general statements can be made, because
  - Frequency is related to spatial rate of change

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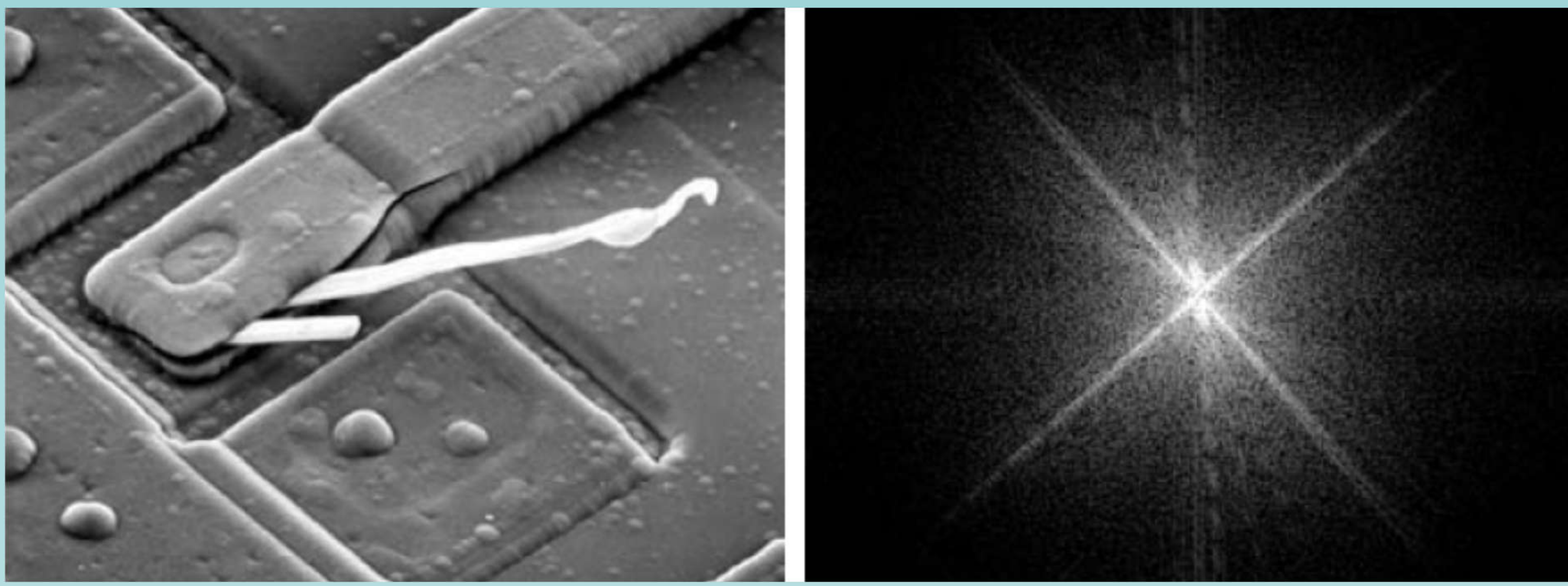
Other Characteristics in Frequency Domain

$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = MN \frac{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)}{MN} = MN\bar{f}(x,y)$$

- Lowest frequency component is proportional to average gray level

Other Characteristics in Frequency Domain



- Moving away from the origin, low frequencies correspond to slowly varying intensities
- Further away, high frequencies correspond to sharp changes in the intensities

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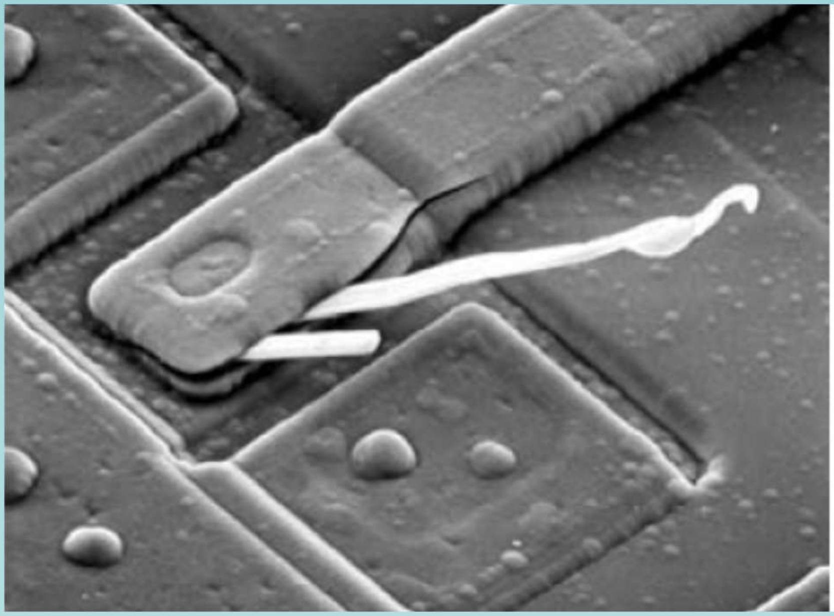
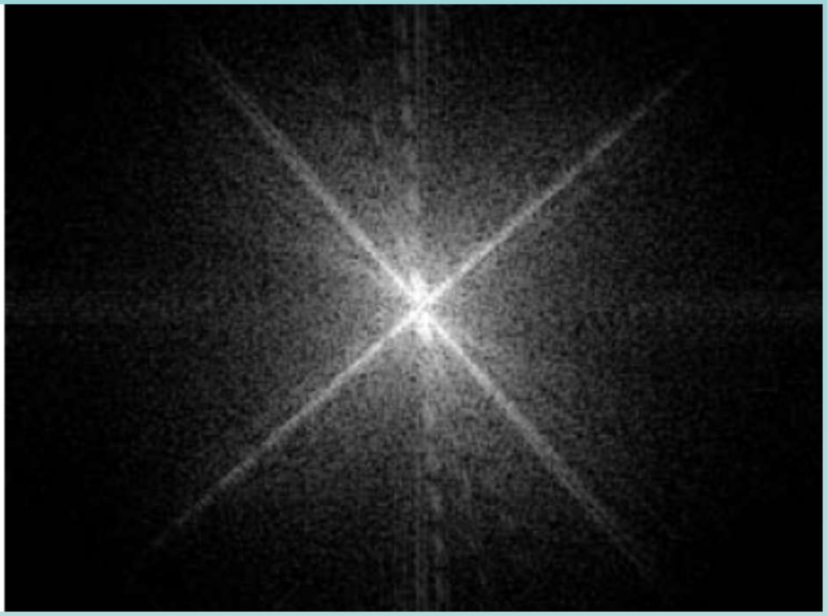
Filtering in Frequency Domain





- Filter
  - Low frequencies
  - High frequencies
  - A particular band of frequencies

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Filtering in Frequency Domain

Filtering is done by convolution

$$\Im\{f(x,y)*h(x,y)\} = H(u,v)F(u,v)$$

$$f(x,y)*h(x,y) = \Im^{-1}\{F(u,v)H(u,v)\}$$

- Filter
  - Low frequencies: set $H(u, v) = 0$ for low u, v
  - High frequencies : set $H(u, v) = 0$ for high u, v
  - A particular band of frequencies : set $H(u, v) = 0$ for other u, v

Filtering in Frequency Domain

Filtering is done by convolution

$$\Im\{f(x,y) * h(x,y)\} = H(u,v)F(u,v)$$

$$f(x,y) * h(x,y) = \Im^{-1}\{F(u,v)H(u,v)\}$$

- $H(u, v)$ is the filter or transfer function
- Both $H(u, v)$ and $F(u, v)$ are centered

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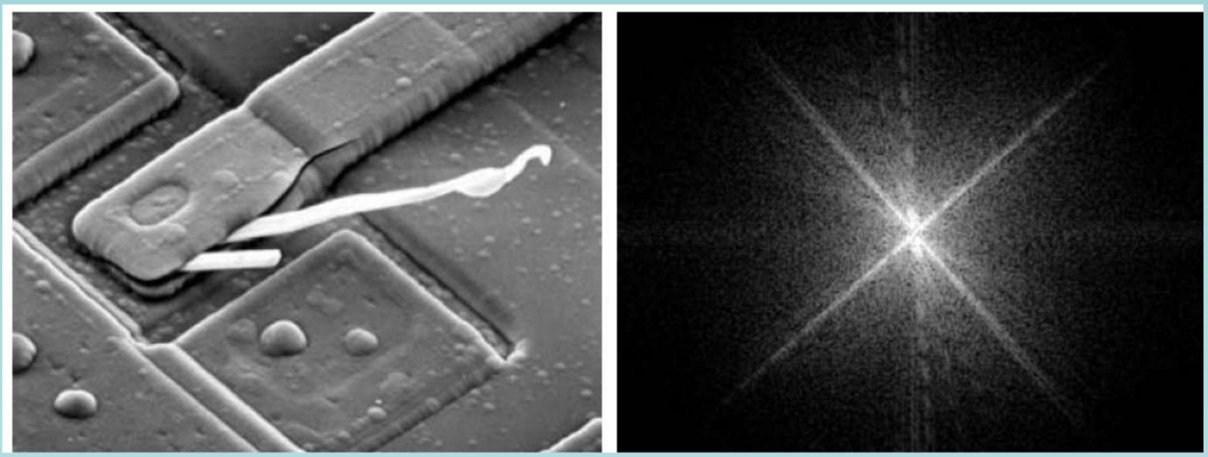
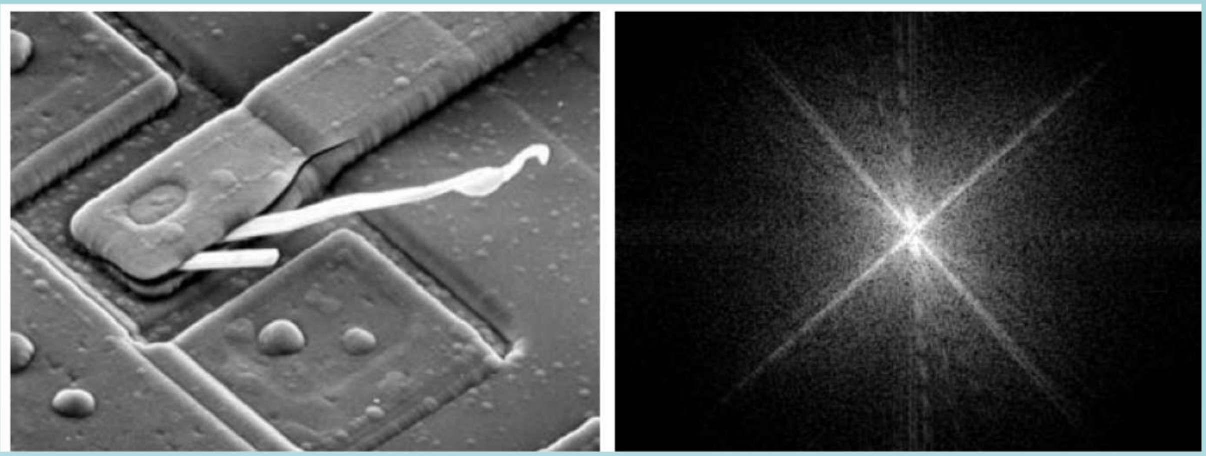
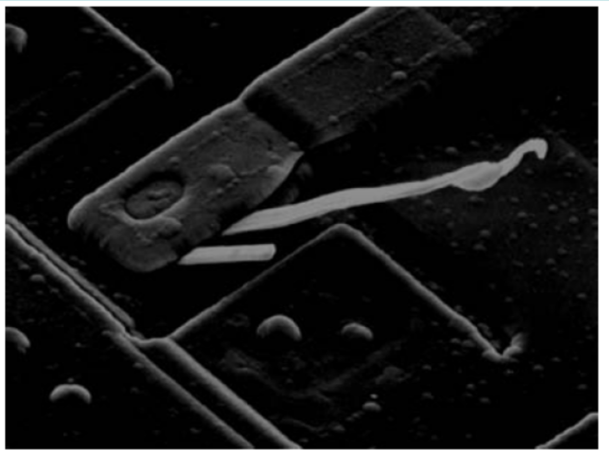
Filtering Example



Filtering Example



Setting, $H(M/2, N/2)=0$, others unchanged, then IDFT



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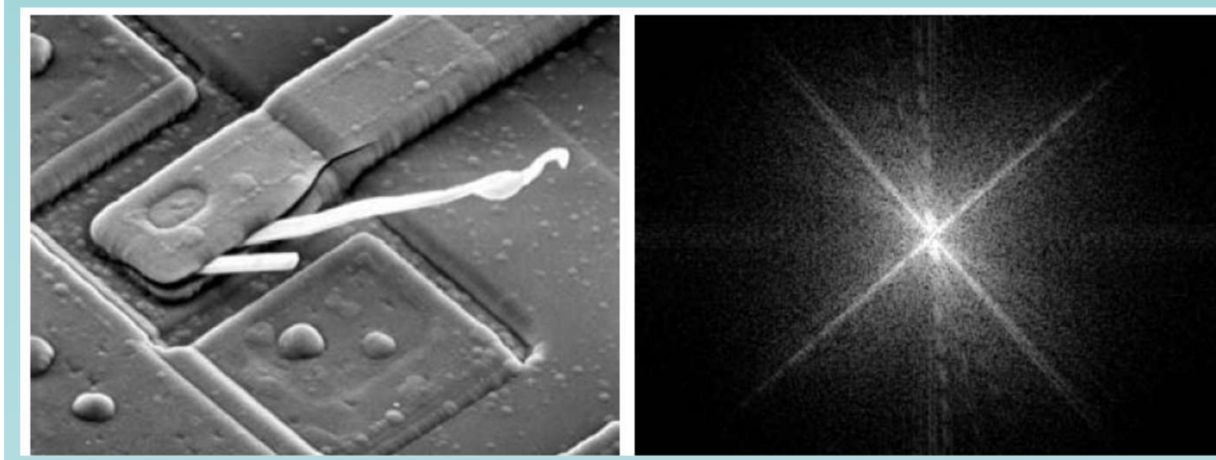
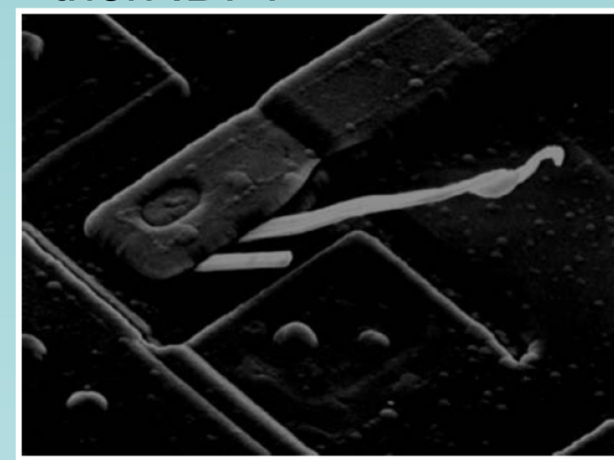
Filtering Example



Setting, $H(M/2, N/2)=0$,
others unchanged,
then IDFT



- Average of the output image should be zero as $H(M/2, N/2)=0$
- That means,
  - there are many negative coefficient in the image

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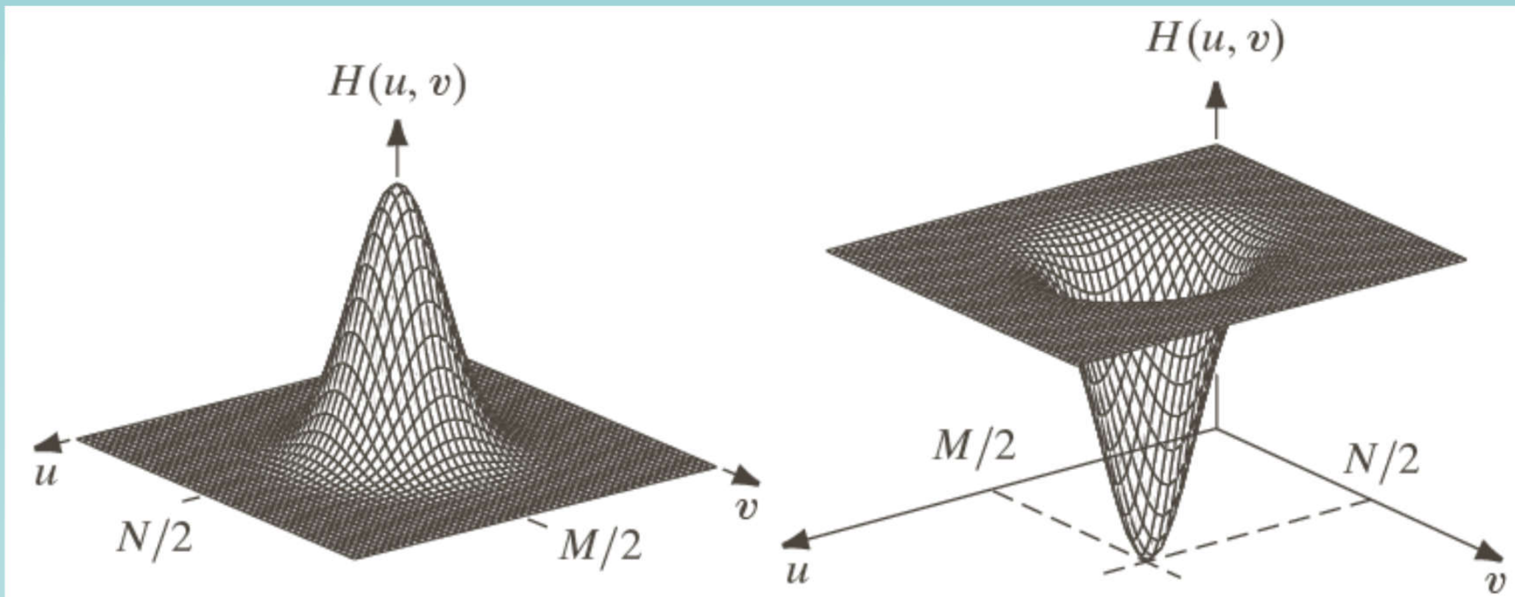
Lowpass and Highpass Filters



- Low pass filter: attenuates high frequencies
  - smoothes an image
- High pass filter: attenuates low frequencies
  - sharpens an image

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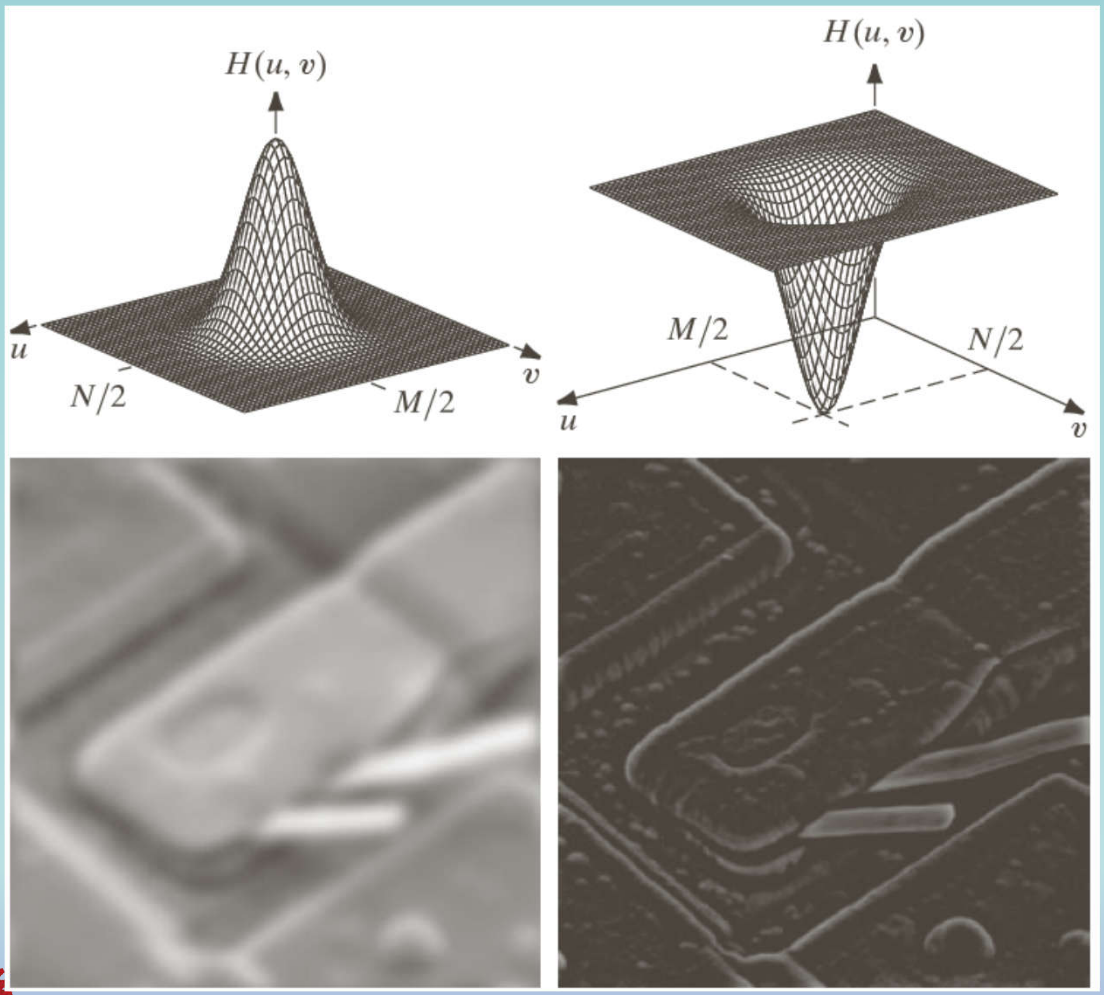
Lowpass and Highpass Filters



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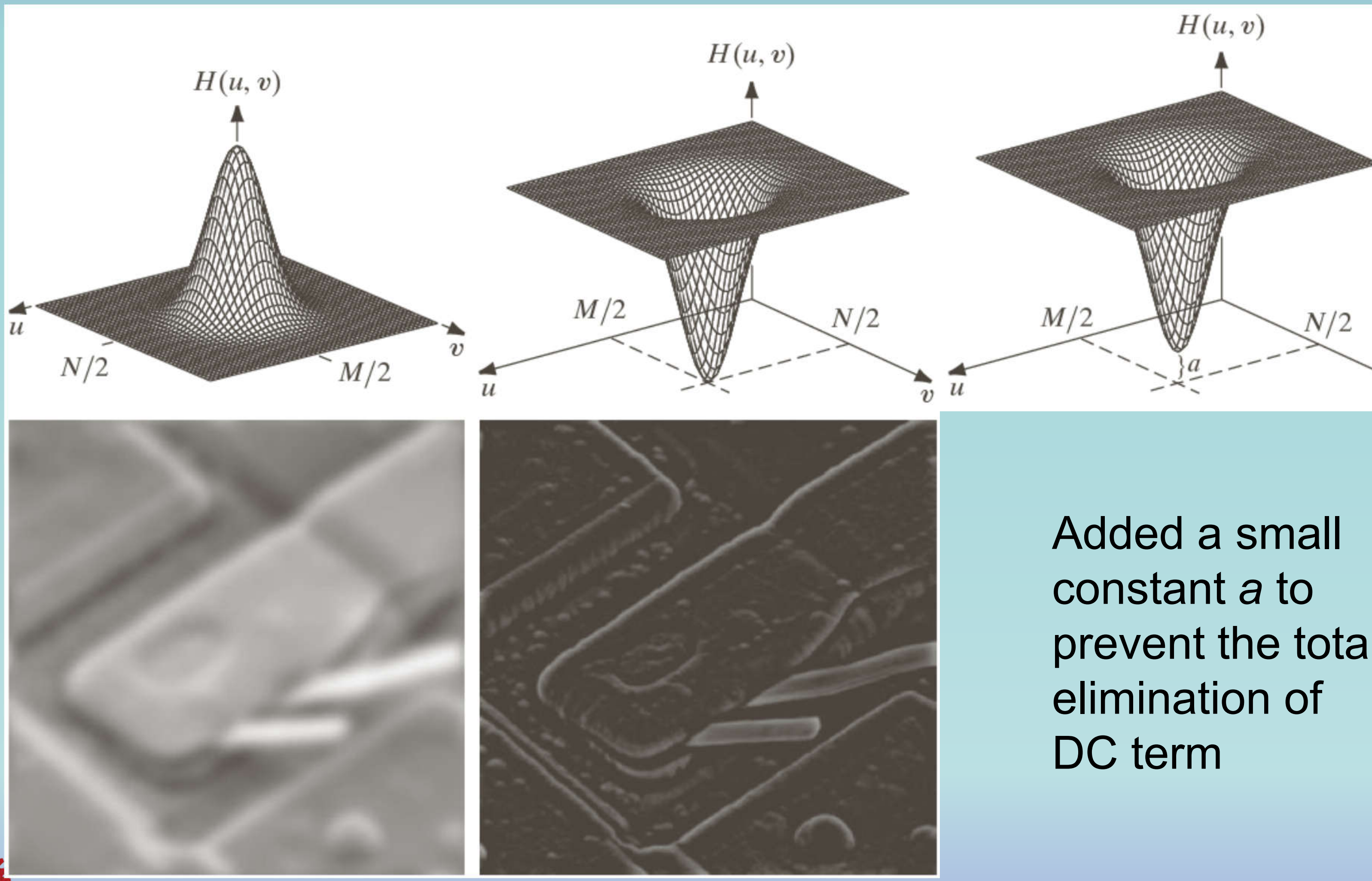
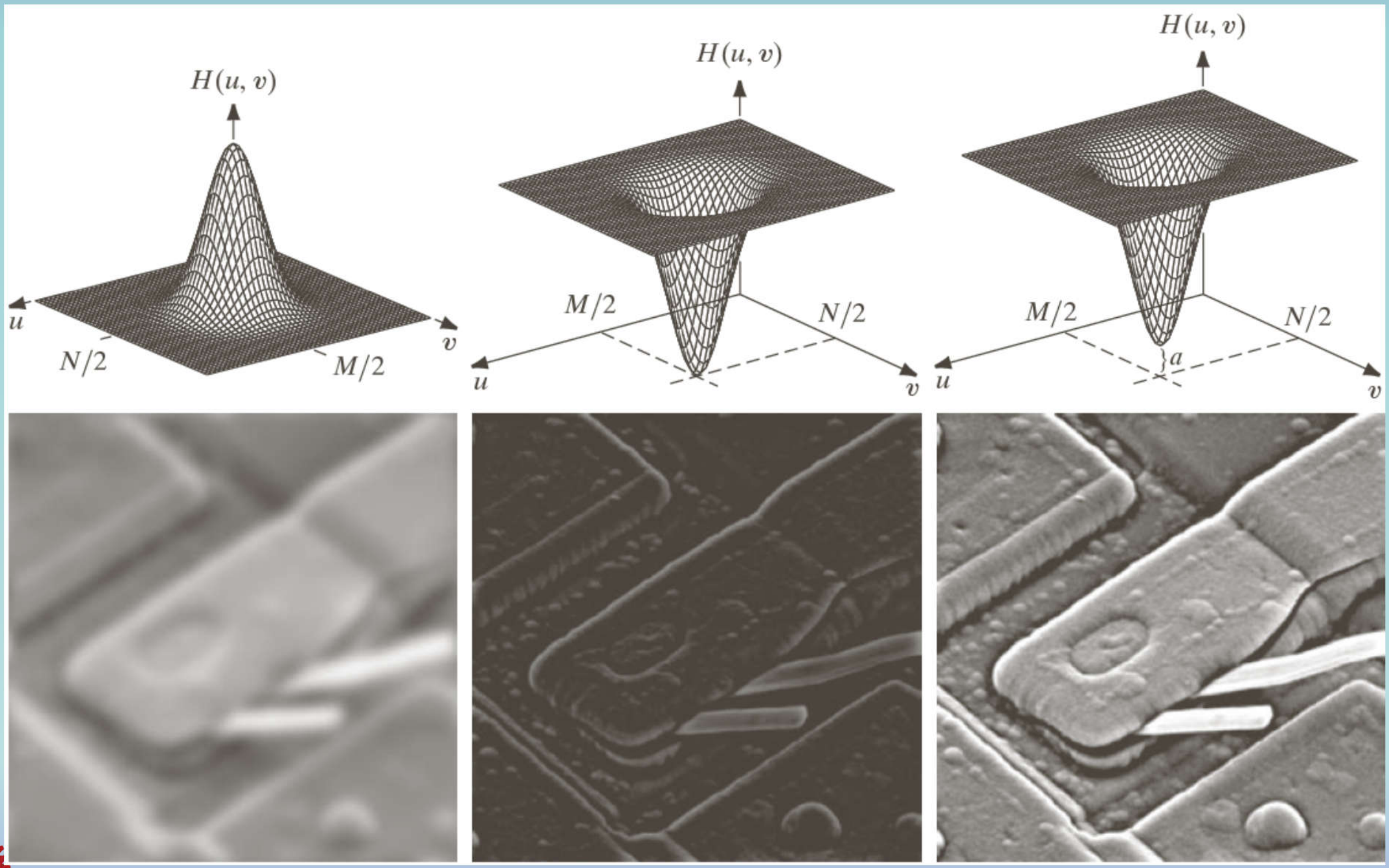
Lowpass and Highpass Filters



Added a small constant *a* to prevent the total elimination of DC term

Lowpass and Highpass Filters



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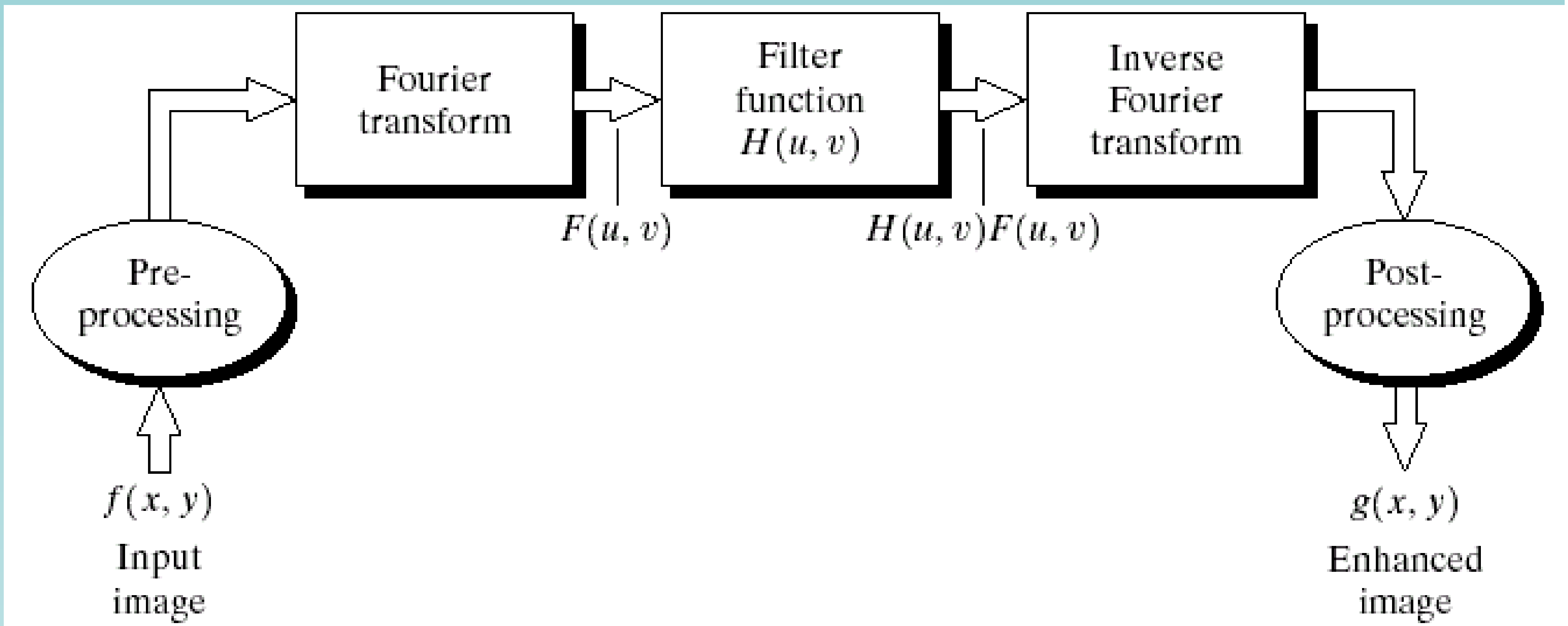
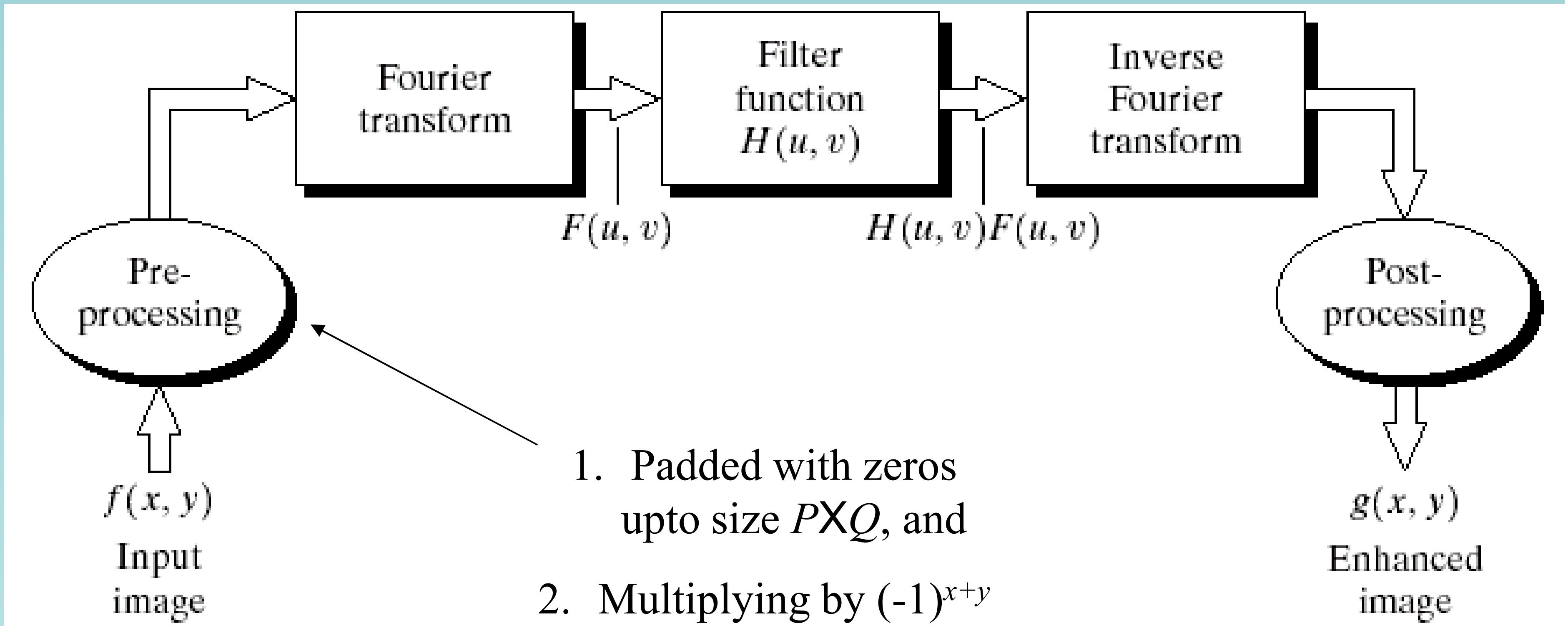
Filtering Steps in Frequency Domain



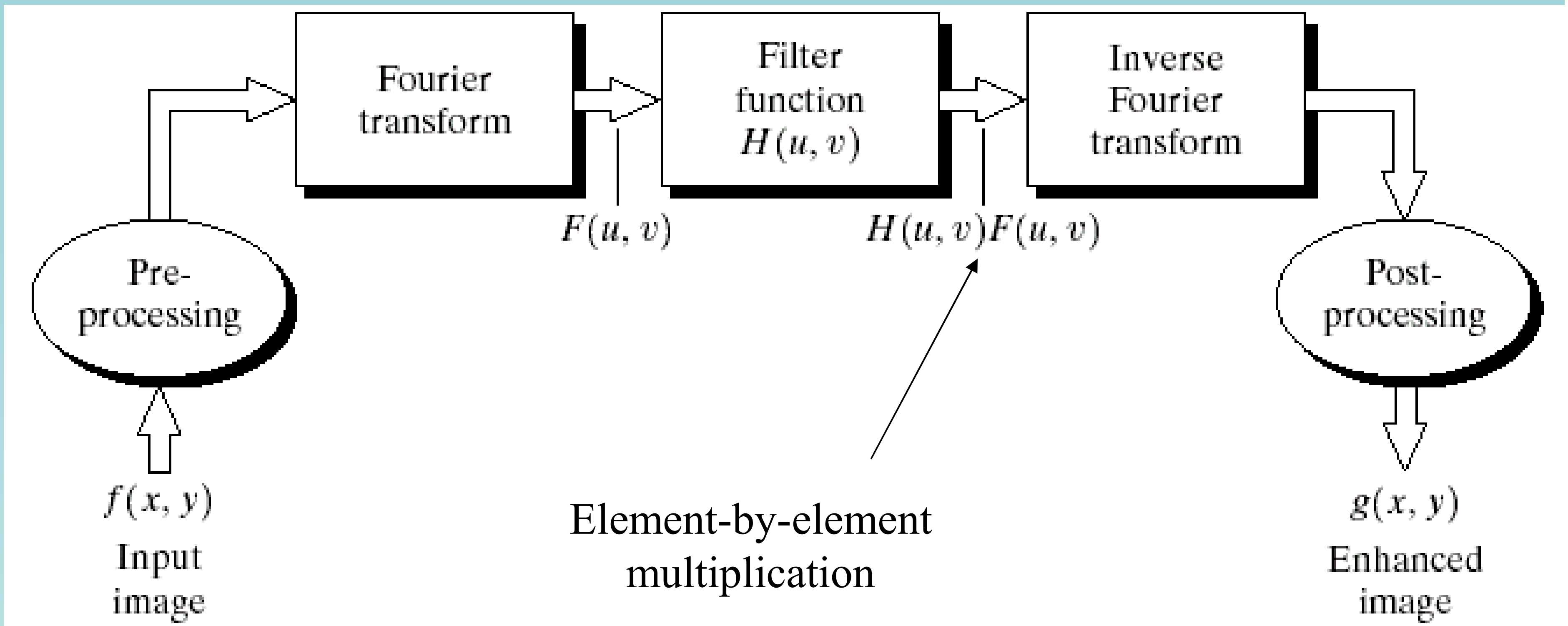
Filtering Steps in Frequency Domain



1. Padded with zeros upto size PXQ, and
2. Multiplying by $(-1)^{x+y}$

where, P>=2M-1, Q>=2N-1,

Filtering Steps in Frequency Domain



The Result is $G(u, v) = F(u, v)H(u, v)$

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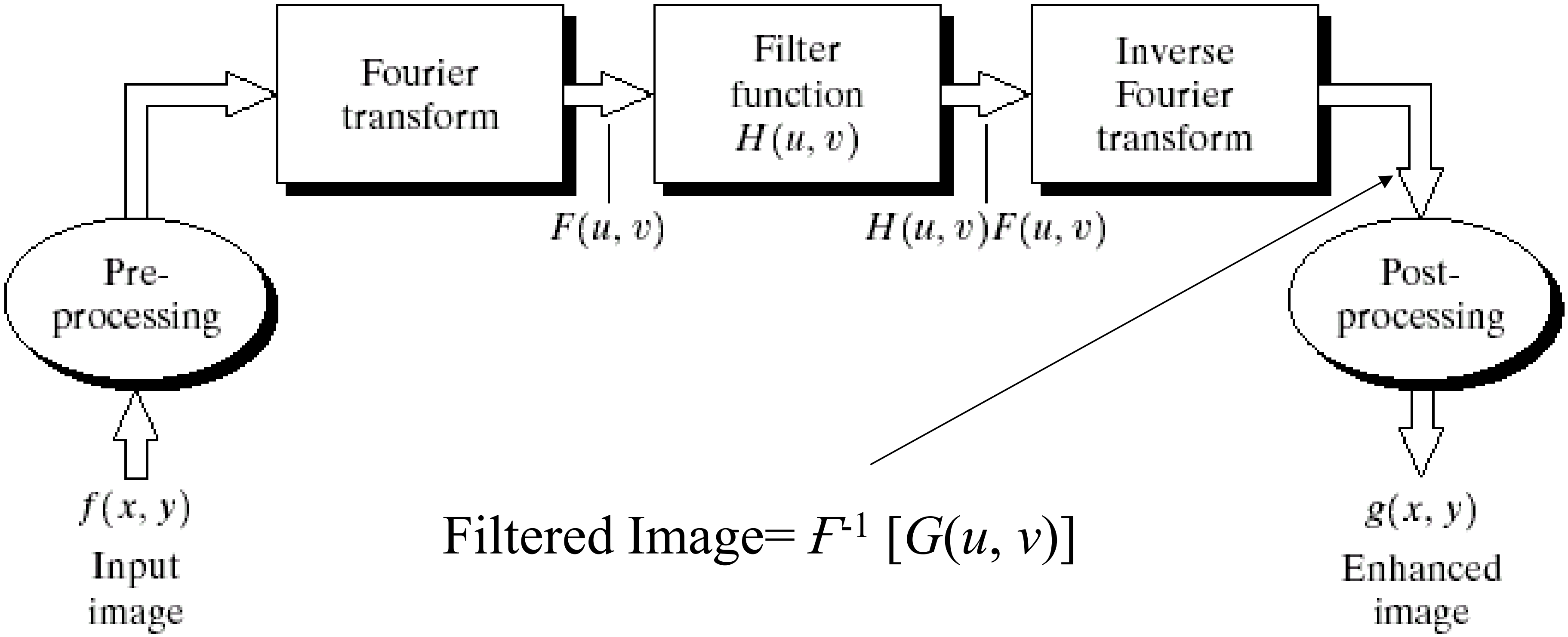
Filtering Steps in Frequency Domain



Filtered Image= F^{-1} $[G(u, v)]$

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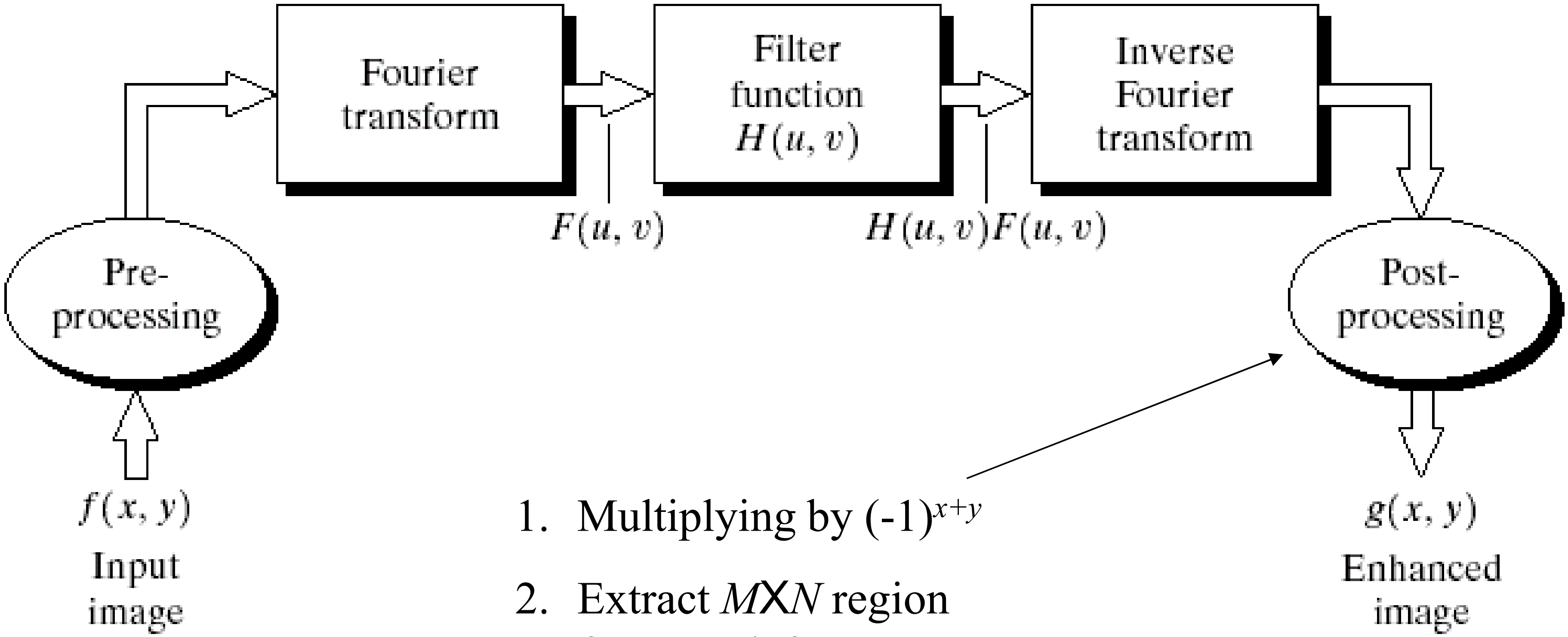
Filtering Steps in Frequency Domain



1. Multiplying by $(-1)^{x+y}$
2. Extract MXN region from top left corner

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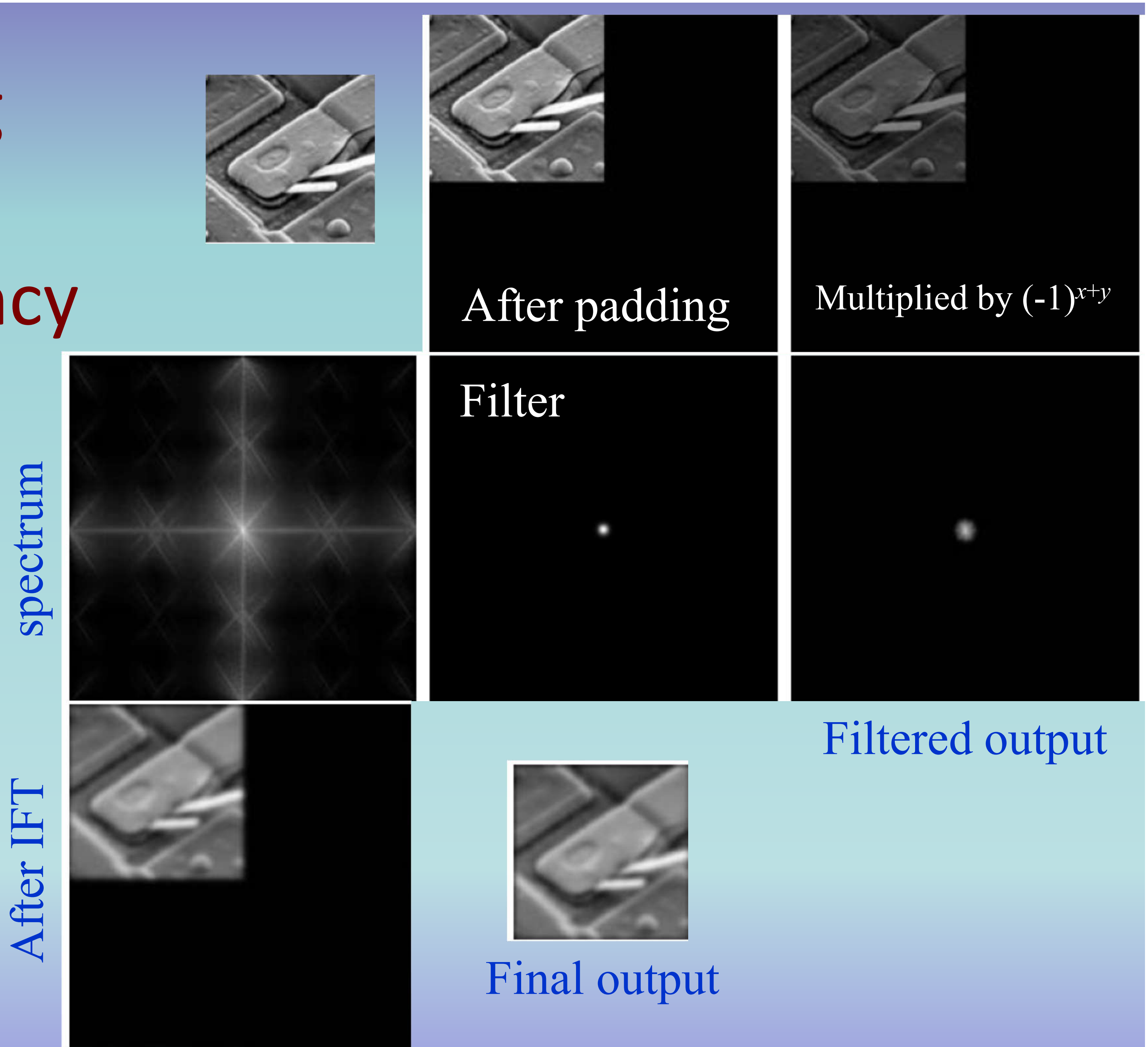
Filtering Steps in Frequency Domain



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Correspondence Between Spatial and Frequency Domain Filters

$$g(x, y) = f(x, y) * h(x, y) = \Im^{-1}\left\{H(u, v)F(u, v)\right\}$$

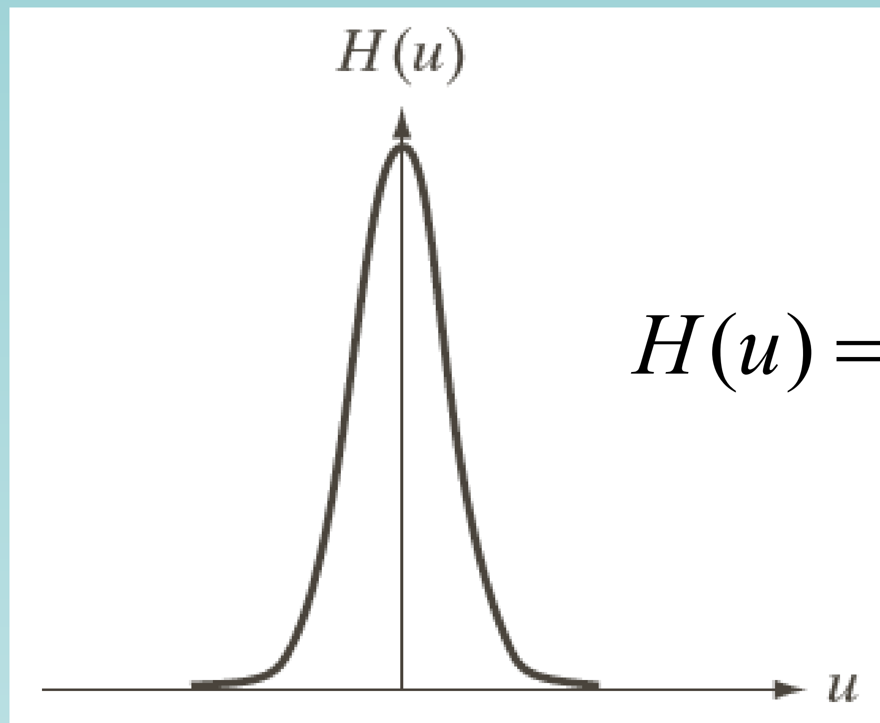
Correspondence Between Spatial and Frequency Domain Filters

$$g(x,y) = f(x,y) * h(x,y) = \Im^{-1}\left\{H(u,v)F(u,v)\right\}$$

$$h(x,y) \Leftrightarrow H(u,v)$$

- We can get the spatial filter than find its frequency domain equivalent
- However, filtering is more intuitive in frequency domain
- So, alternatively, get a filter in frequency domain, find its IDFT

Correspondence Between Spatial and Frequency Domain Filters



$$H(u) = Ae^{-u^2/2\sigma^2}$$

Gaussian 1D filter in Frequency domain

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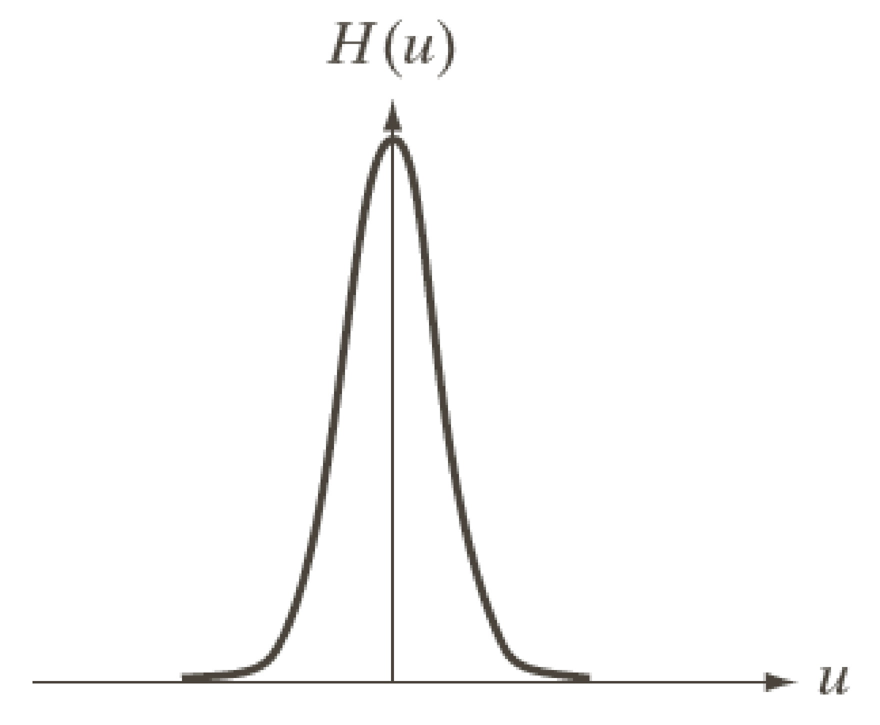
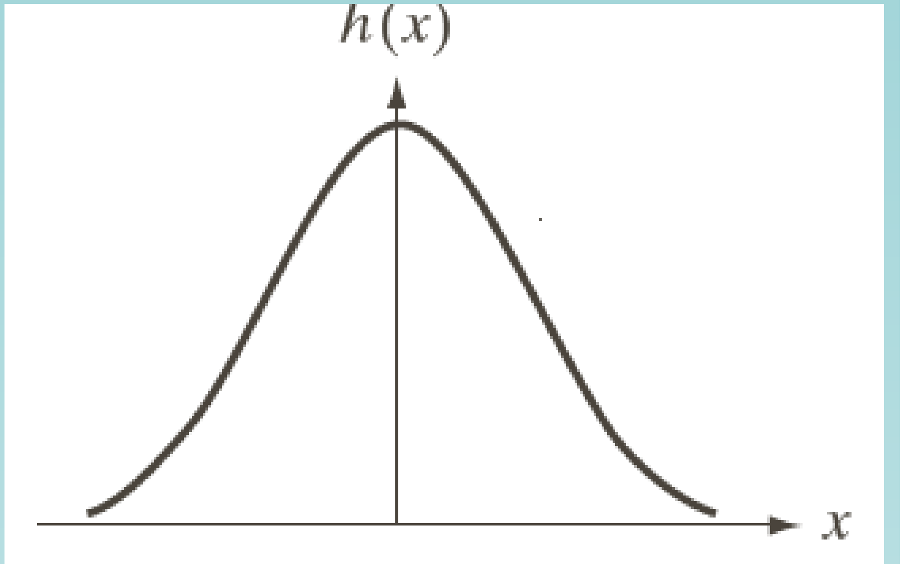
Correspondence Between Spatial and Frequency Domain Filters





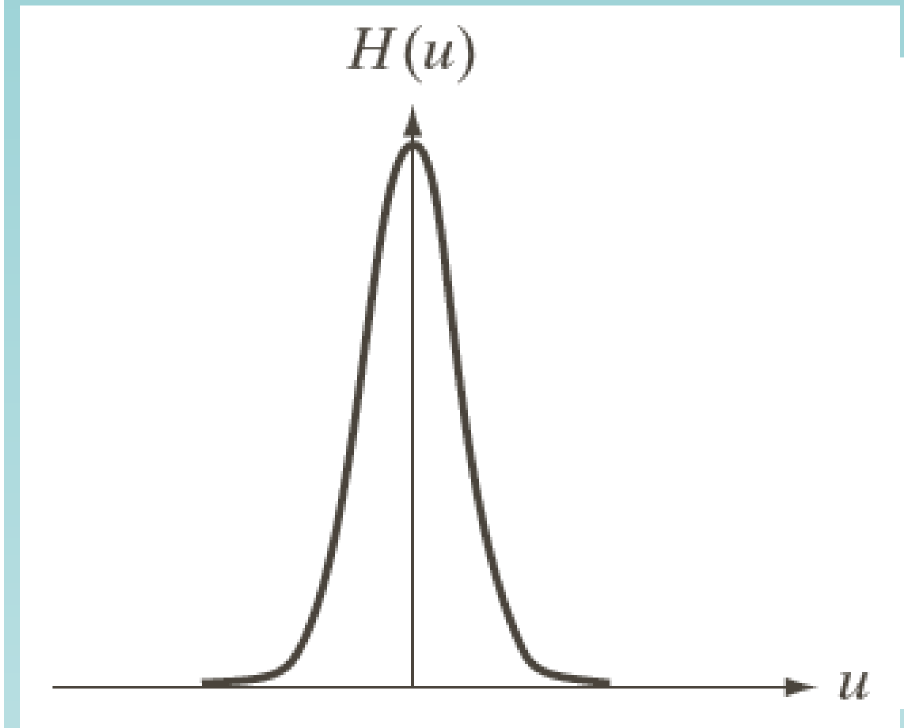
$$H(u) = Ae^{-u^2/2\sigma^2}$$

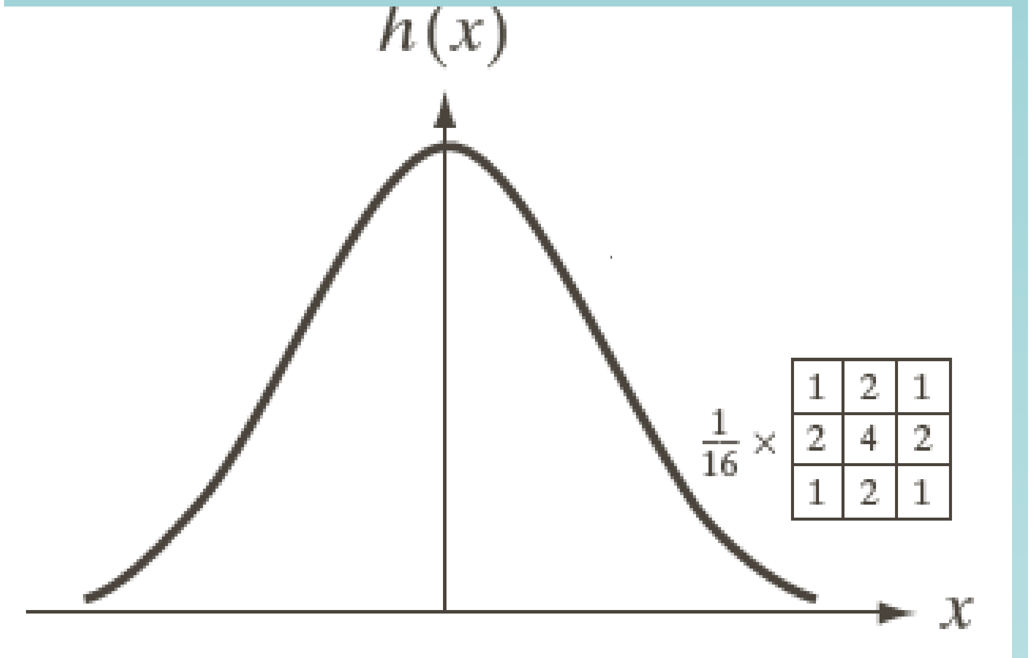
$$h(x) = 2\sqrt{\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

Gaussian 1D filter in Frequency domain

Gaussian 1D filter in Spatial domain

Correspondence Between Spatial and Frequency Domain Filters





$$H(u) = Ae^{-u^2/2\sigma^2}$$

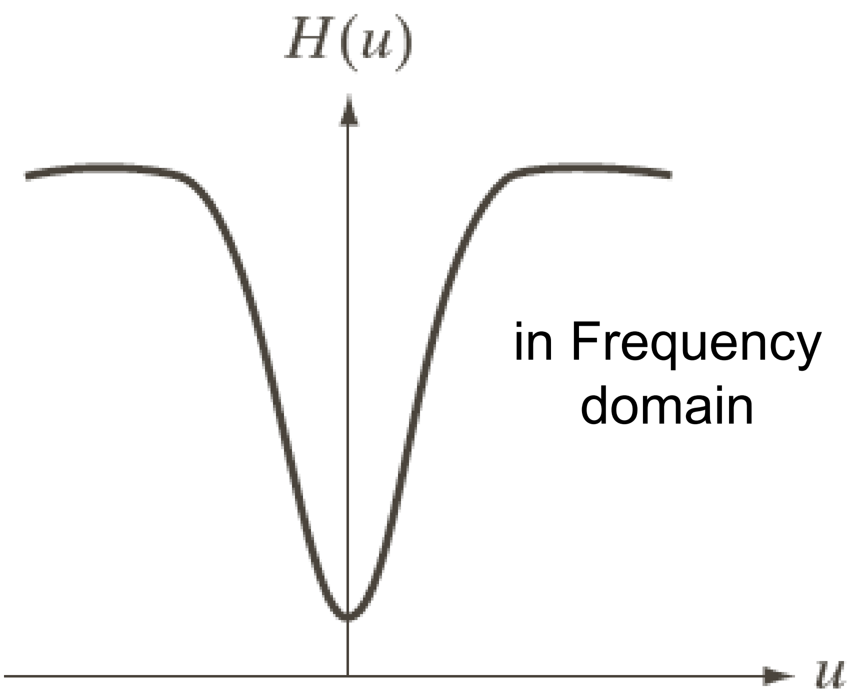
$$h(x) = 2\sqrt{\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

$$\frac{1}{16} \times \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

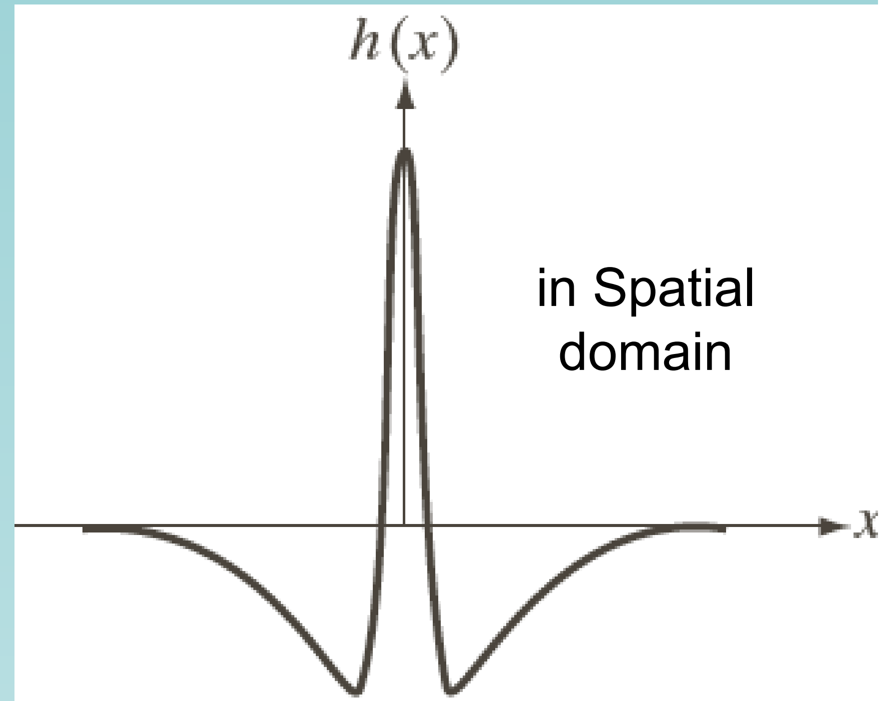
Gaussian 1D filter in Frequency domain

Gaussian 1D filter in Spatial domain

Correspondence Between Spatial and Frequency Domain Filters



in Frequency domain



in Spatial domain

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

$$h(x) = 2\sqrt{\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2 x^2} - 2\sqrt{\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2 x^2}$$

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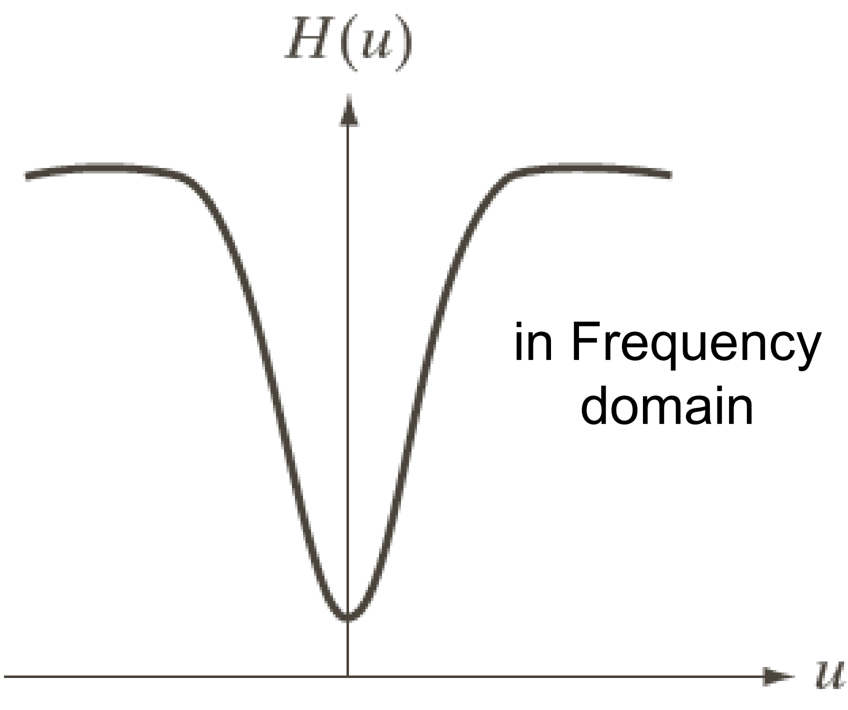
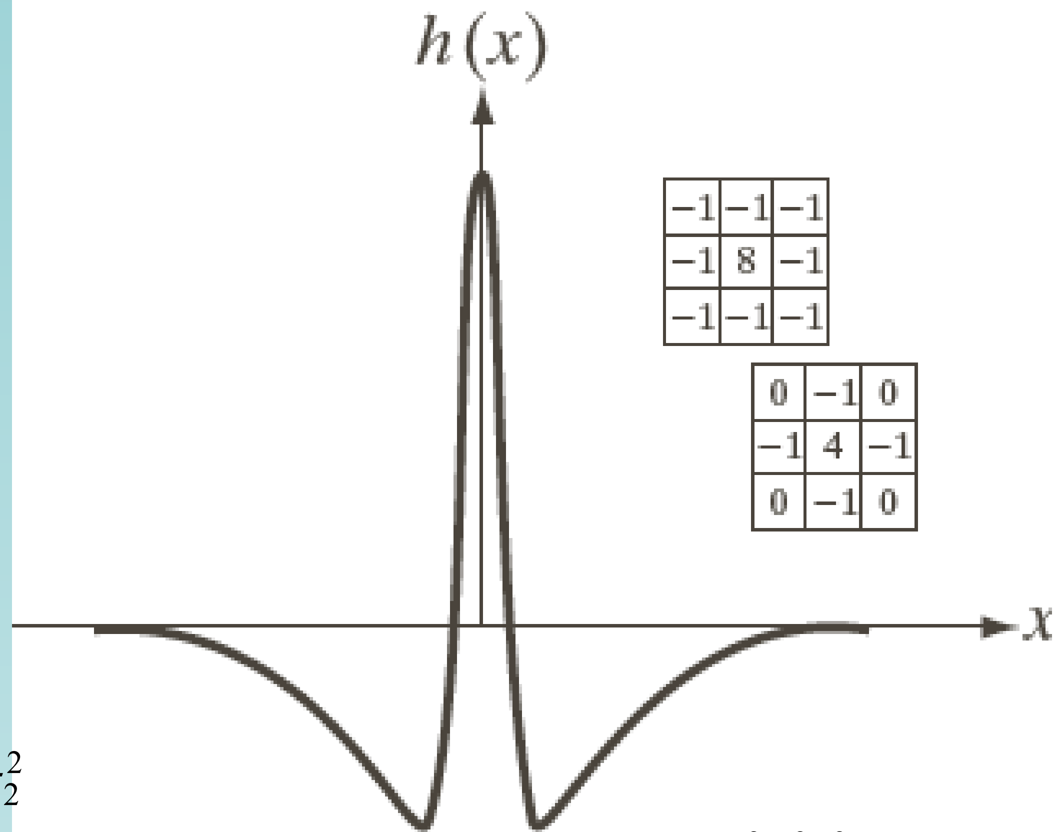
Correspondence Between Spatial and Frequency Domain Filters





−1	−1	−1
−1	8	−1
−1	−1	−1

0	−1	0
−1	4	−1
0	−1	0

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

$$h(x) = 2\sqrt{\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - 2\sqrt{\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

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Correspondence Between Spatial and Frequency Domain Filters

600X600

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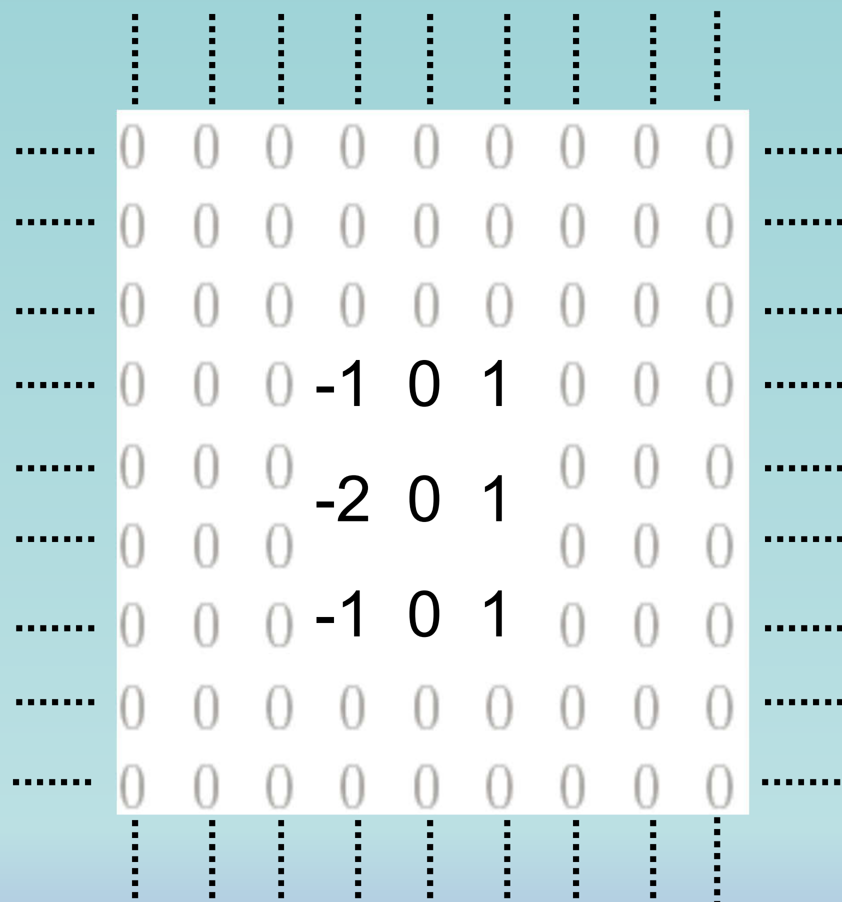
Correspondence Between Spatial and Frequency Domain Filters

−1	0	1
−2	0	2
−1	0	1

Spatial domain Sobel Mask of size 3X3

Correspondence Between Spatial and Frequency Domain Filters



Padded with zero's to be of size 602X602

Padding dim>=600+3-1

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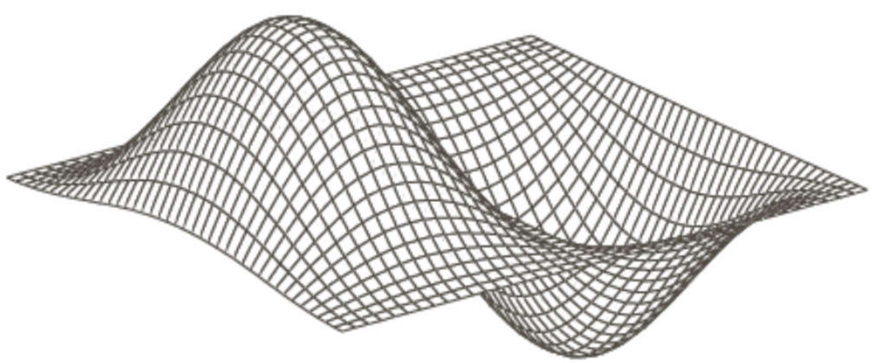
Correspondence Between Spatial and Frequency Domain Filters

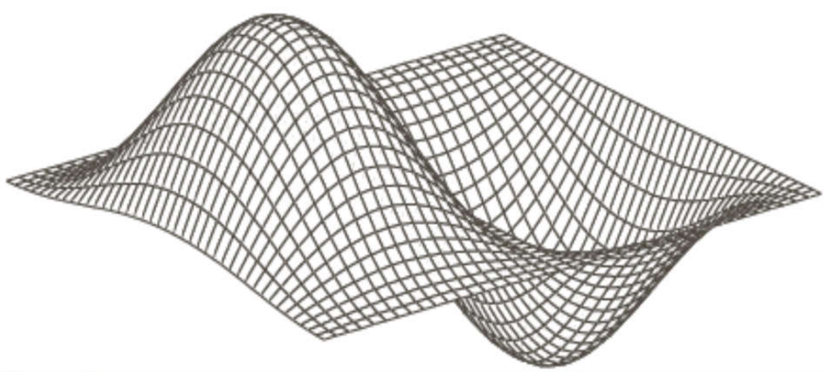
−1	0	1
−2	0	2
−1	0	1



Surface view
after padding

Correspondence Between Spatial and Frequency Domain Filters

−1	0	1
−2	0	2
−1	0	1



Surface view after padding

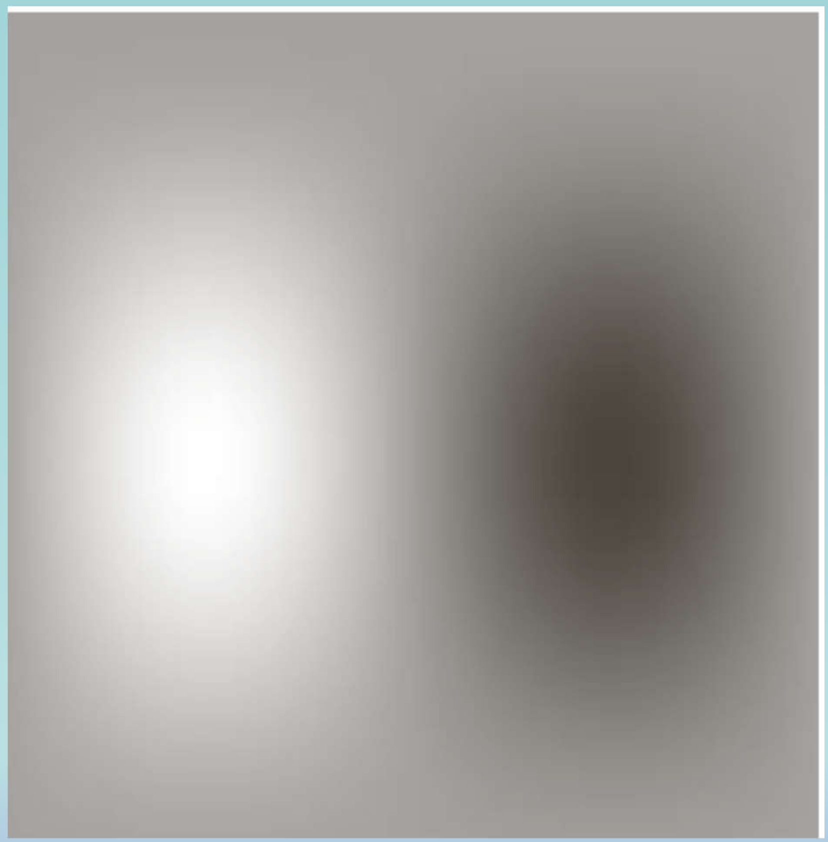


Image View

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Correspondence Between Spatial and Frequency Domain Filters

Steps to find the frequency domain filter:

- Multiply the extended sobel filter by $(-1)^{x+y}$
- Find its FT
- Set its real part to zero [to remove the parasitic components]
- Multiply the Frequency domain filter by $(-1)^{u+v}$

Correspondence Between Spatial and Frequency Domain Filters

Result of applying the frequency domain filter

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Correspondence Between Spatial and Frequency Domain Filters

Result of applying the frequency domain filter

Result of applying the Sobel filter directly

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Image Smoothing in Frequency Domain

3 types of Smoothing filters:

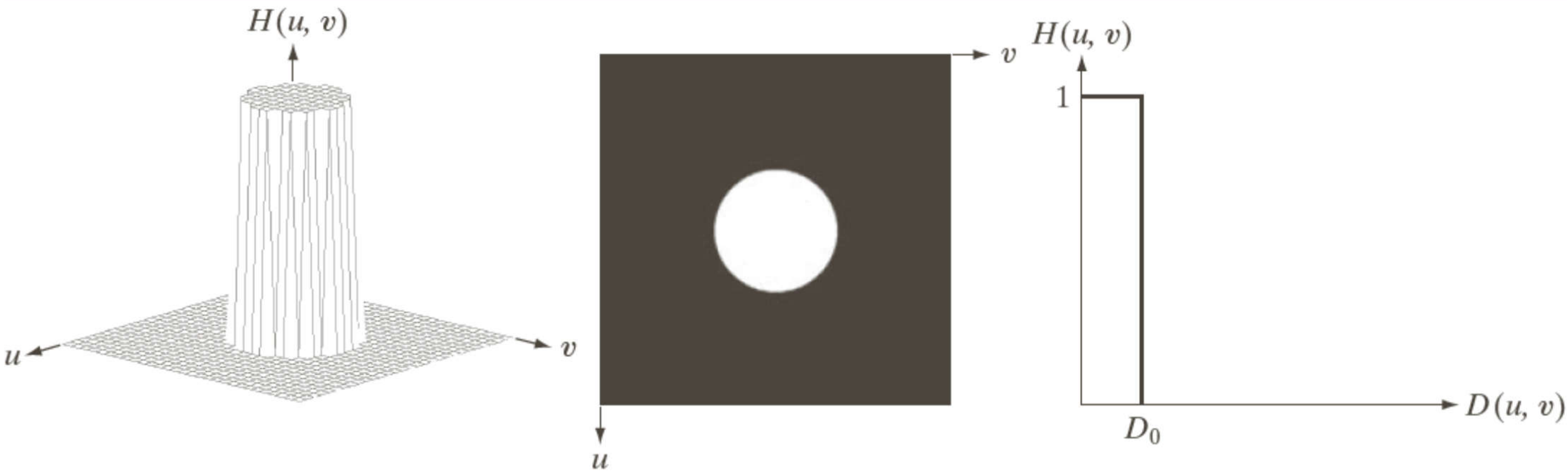
- Ideal Low Pass Filter (ILPF)
- Butterworth Low Pass Filter
- Gaussian Low Pass Filter (GLPF)

Ideal Low Pass Filter

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

$$\text{where,} \quad D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$

Ideal Low Pass Filter



3D Surface View

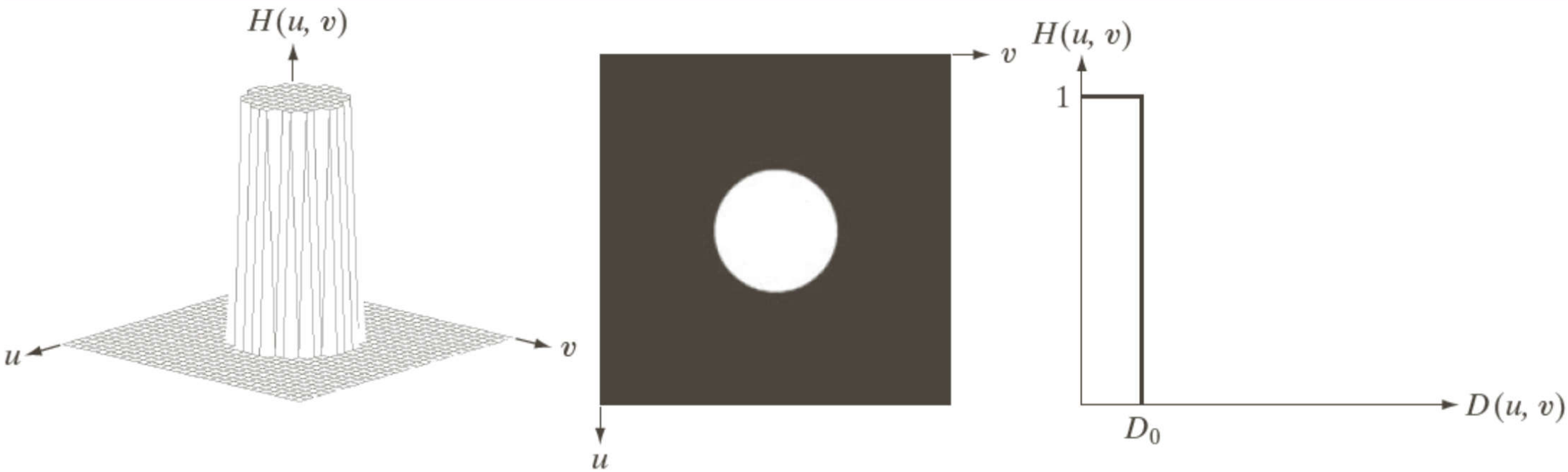
Image View

Radial Cross Section

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

CSE-BUET

Ideal Low Pass Filter



- D_0 is cutoff frequency
- Different filters can be designed by changing D_0
- Changing D_0 means allowing different amount of energy

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Ideal Low Pass Filter

Total Energy:

$$F_T = \sum_{u=0}^{u=M-1} \sum_{v=0}^{v=N-1} \left| F(u,v) \right|^2$$

A filter with radius D_0 capture α% of energy:

$$\alpha = 100 \sum_u \sum_v \left| F(u,v) \right|^2 \Big/ F_T$$

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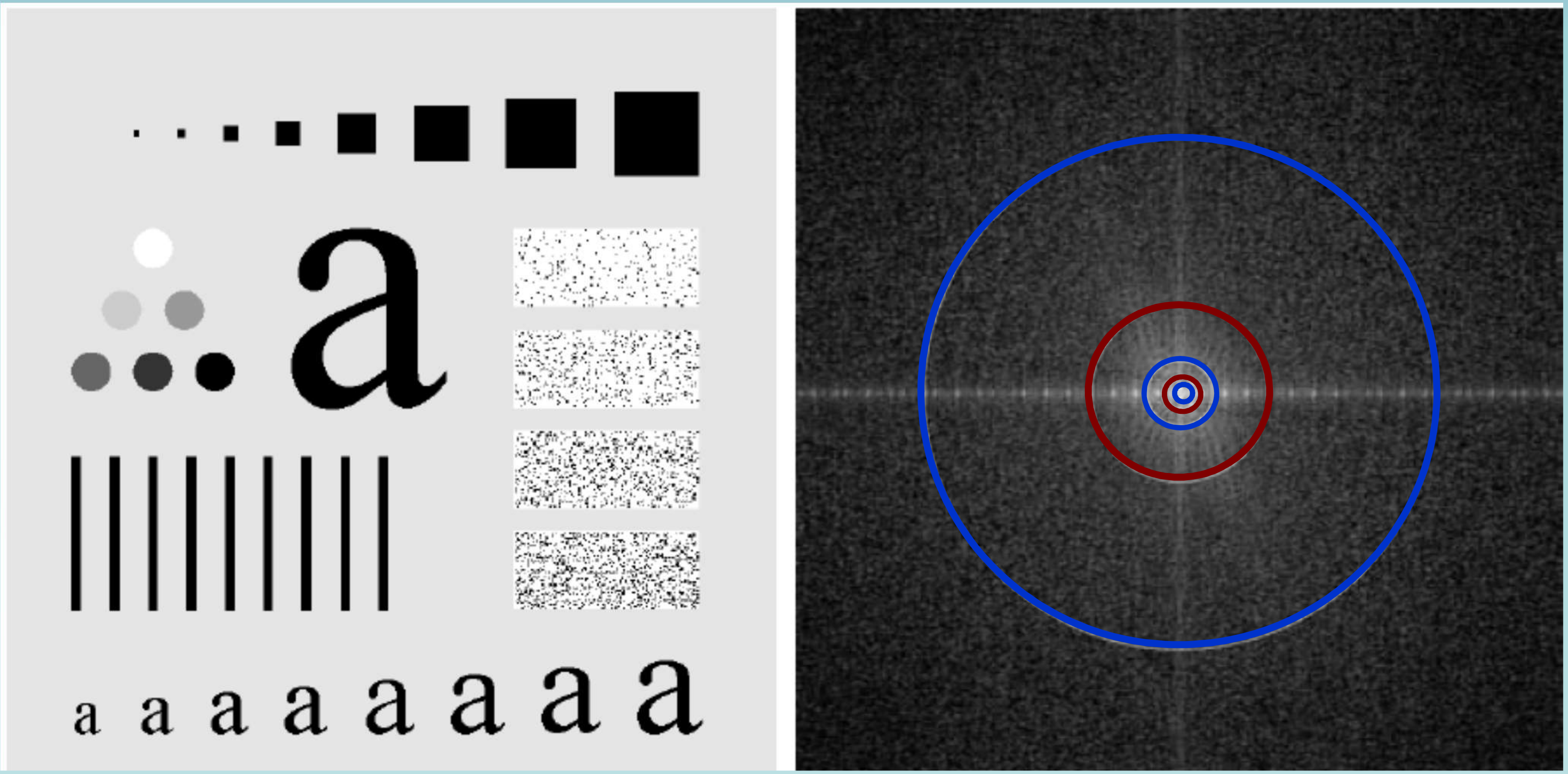
Ideal Low Pass Filter



The superimposed circles have radius 10, 30, 60, 160, 460 and enclose 87.0, 93.1, 95.7, 97.8 and 99.2% energy

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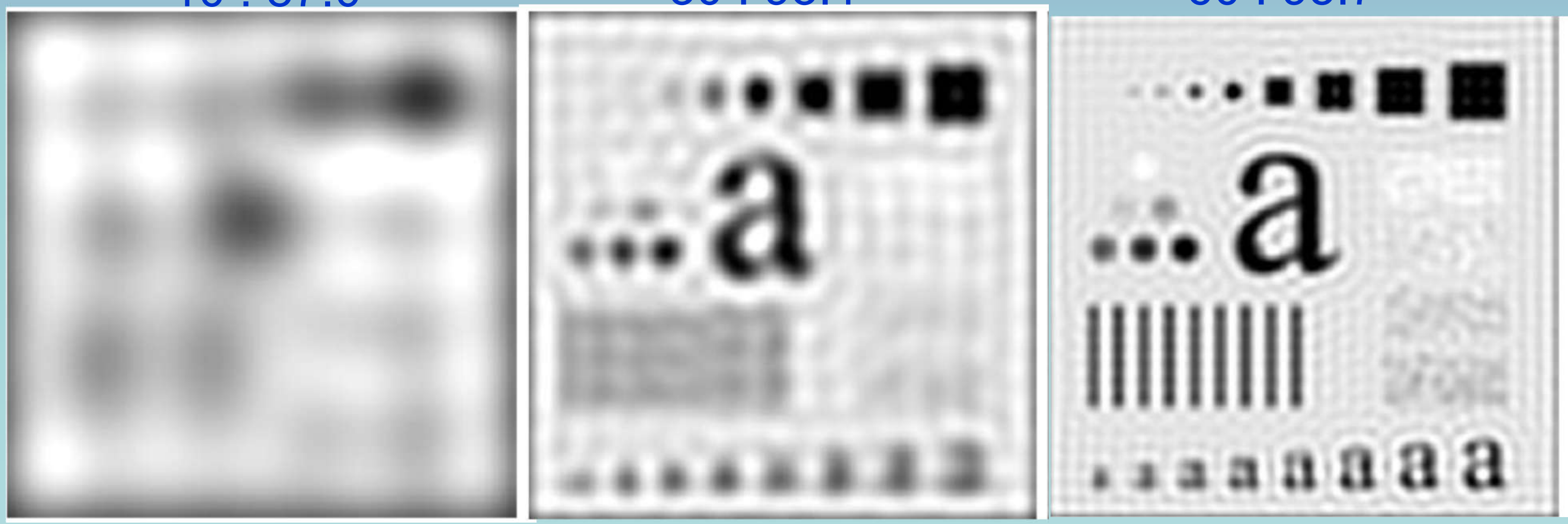
Ideal Low Pass Filter

Original   10 : 87.0   30 : 93.1

60 : 95.7   160 : 97.8   460 : 99.2

Ideal Low Pass Filter

10 : 87.0 | 30 : 93.1 | 60 : 95.7



- Blurring and ringing effect increases as D_0 decreases

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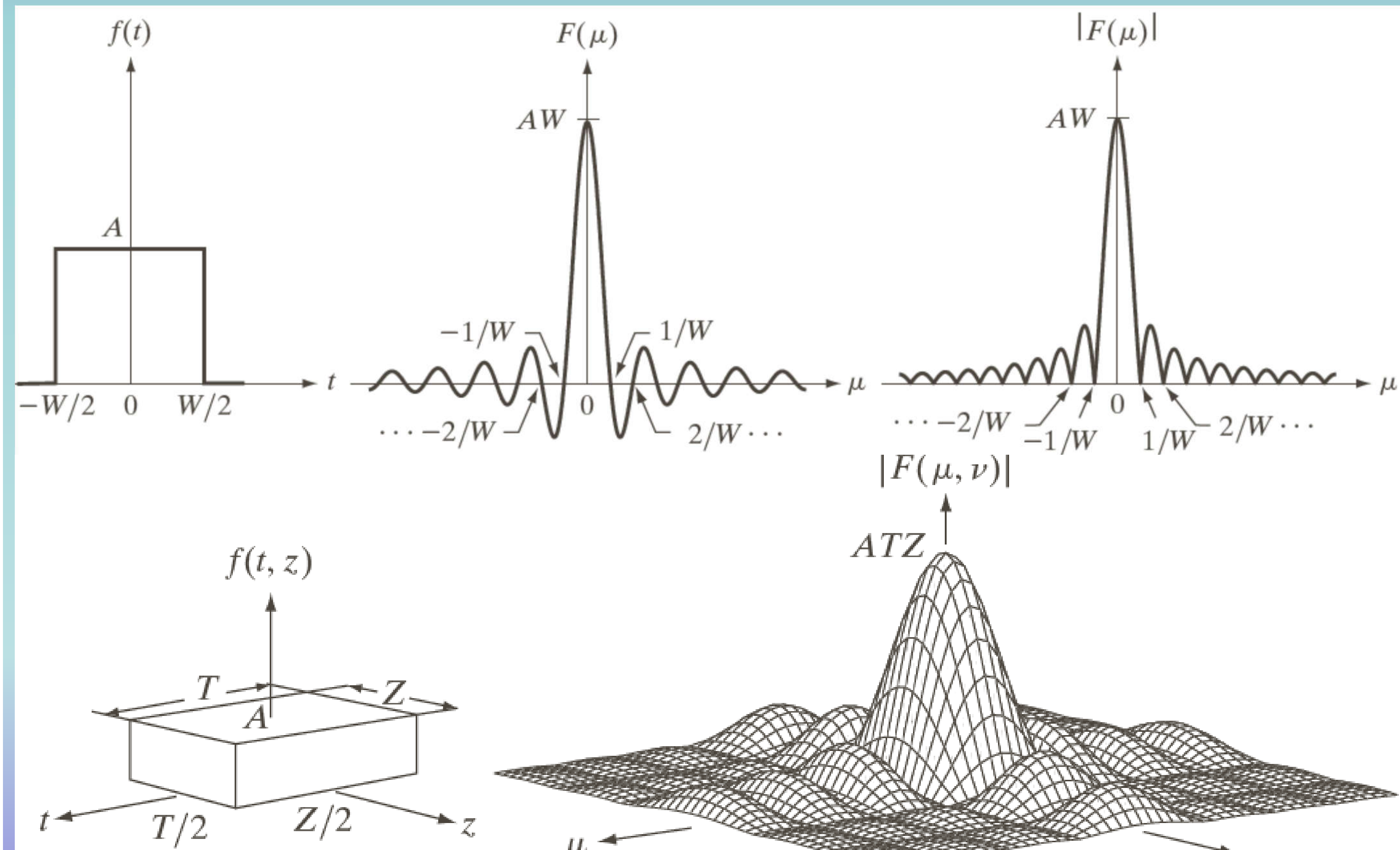
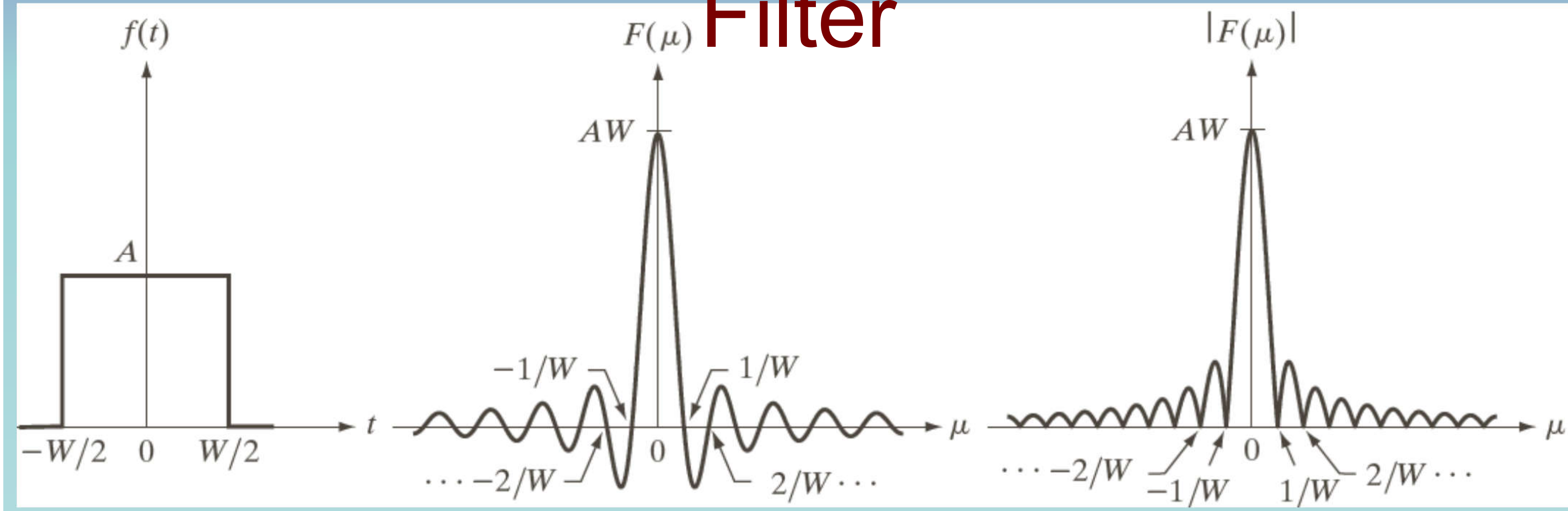
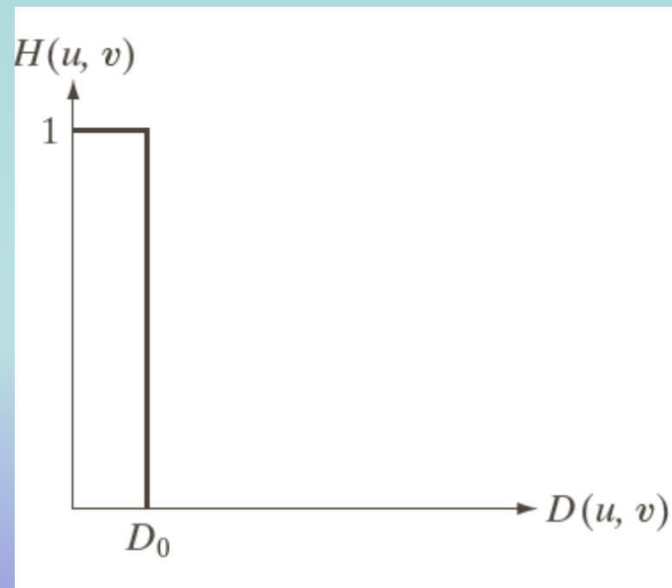
Box and Sinc Function



Box Function and Ideal Low Pass Filter





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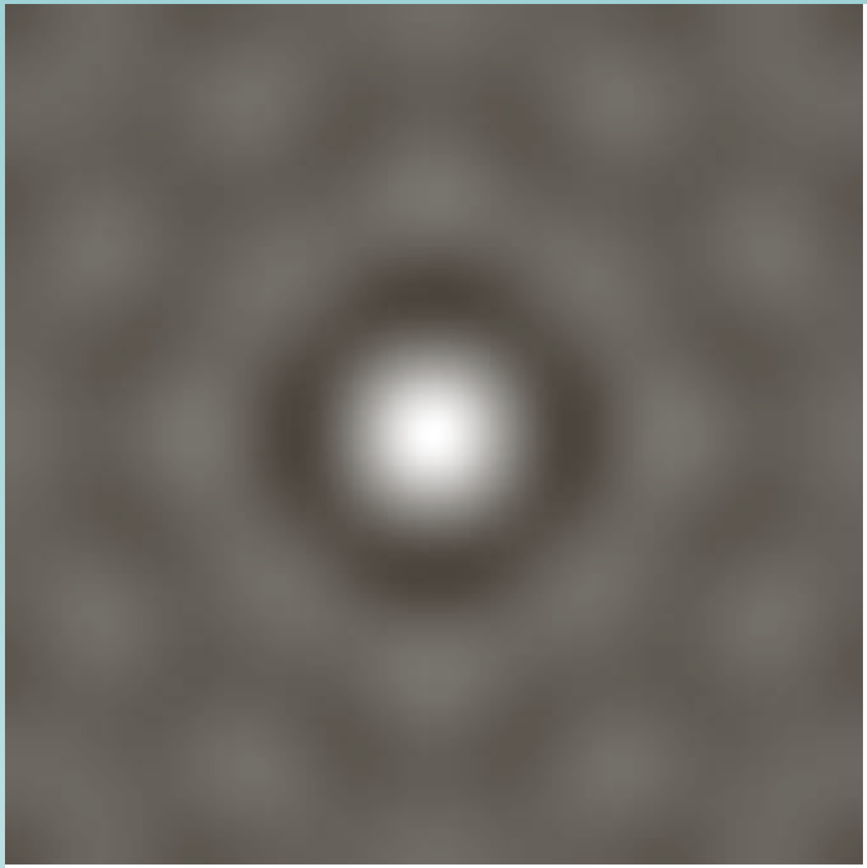
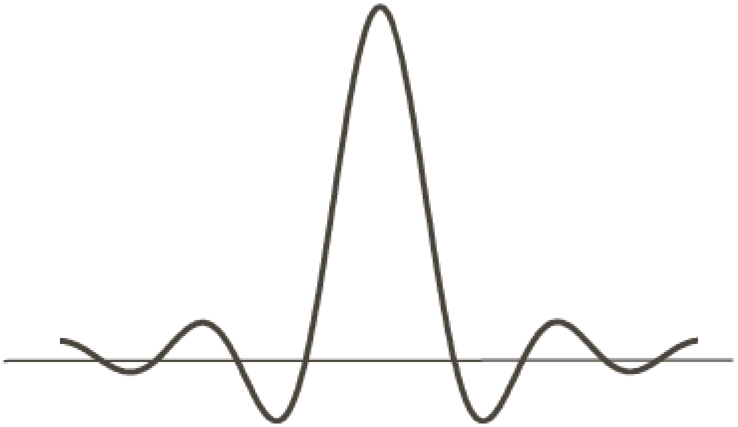
Ideal Low Pass Filter with *Radius* 10

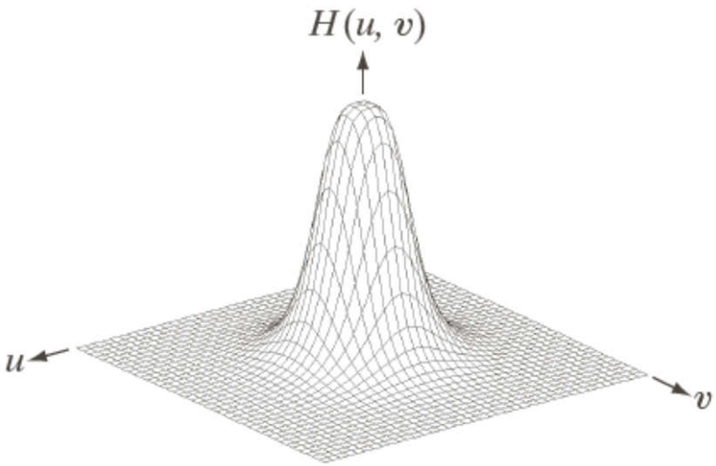




Intensity profile through the centre: $h(x)$

Spatial Domain Representation: $h(x, y)$

Butterworth Low Pass Filter



3D Surface View

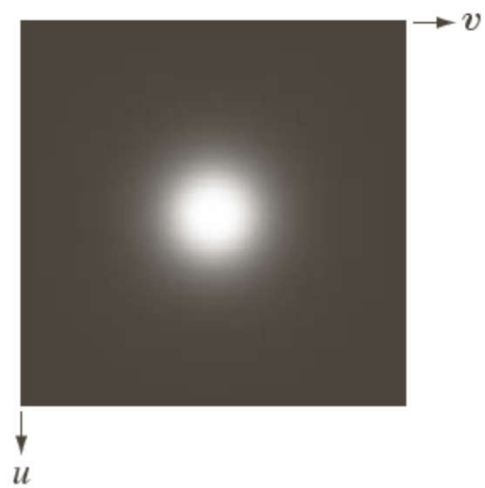
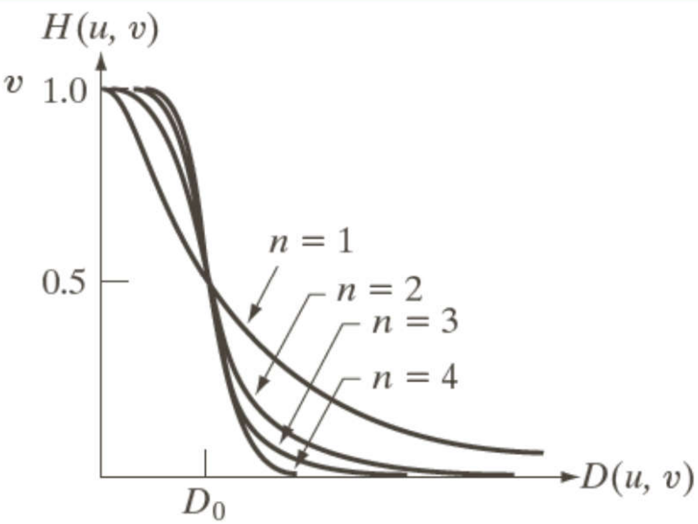


Image View



Radial Cross Section

$$H(u,v) = \frac{1}{1 + \left[D(u,v) / D_0 \right]^{2n}}$$

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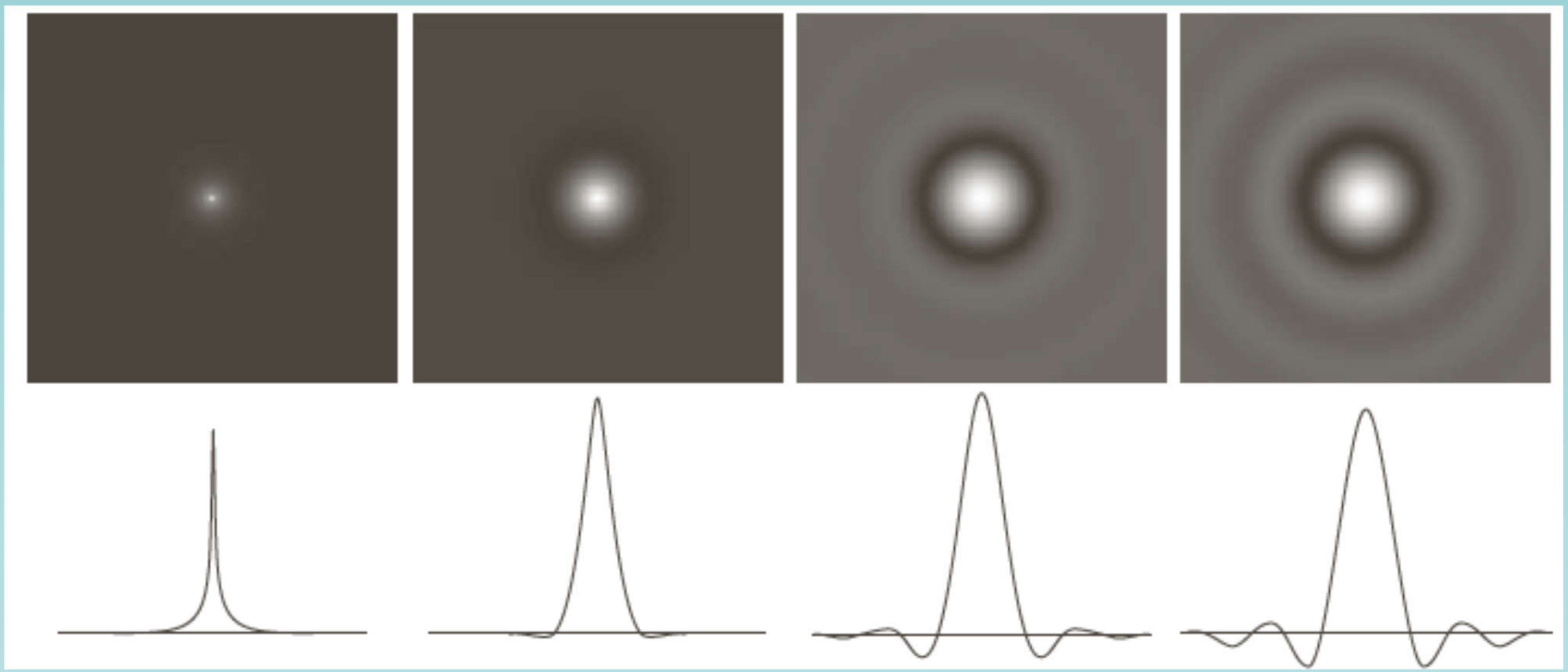
Butterworth Low Pass Filter



$n = 1$ $n = 2$ $n = 5$ $n = 20$

$$H(u,v) = \frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

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Butterworth Low Pass Filter with n = 2

10 : 87.0

30 : 93.1

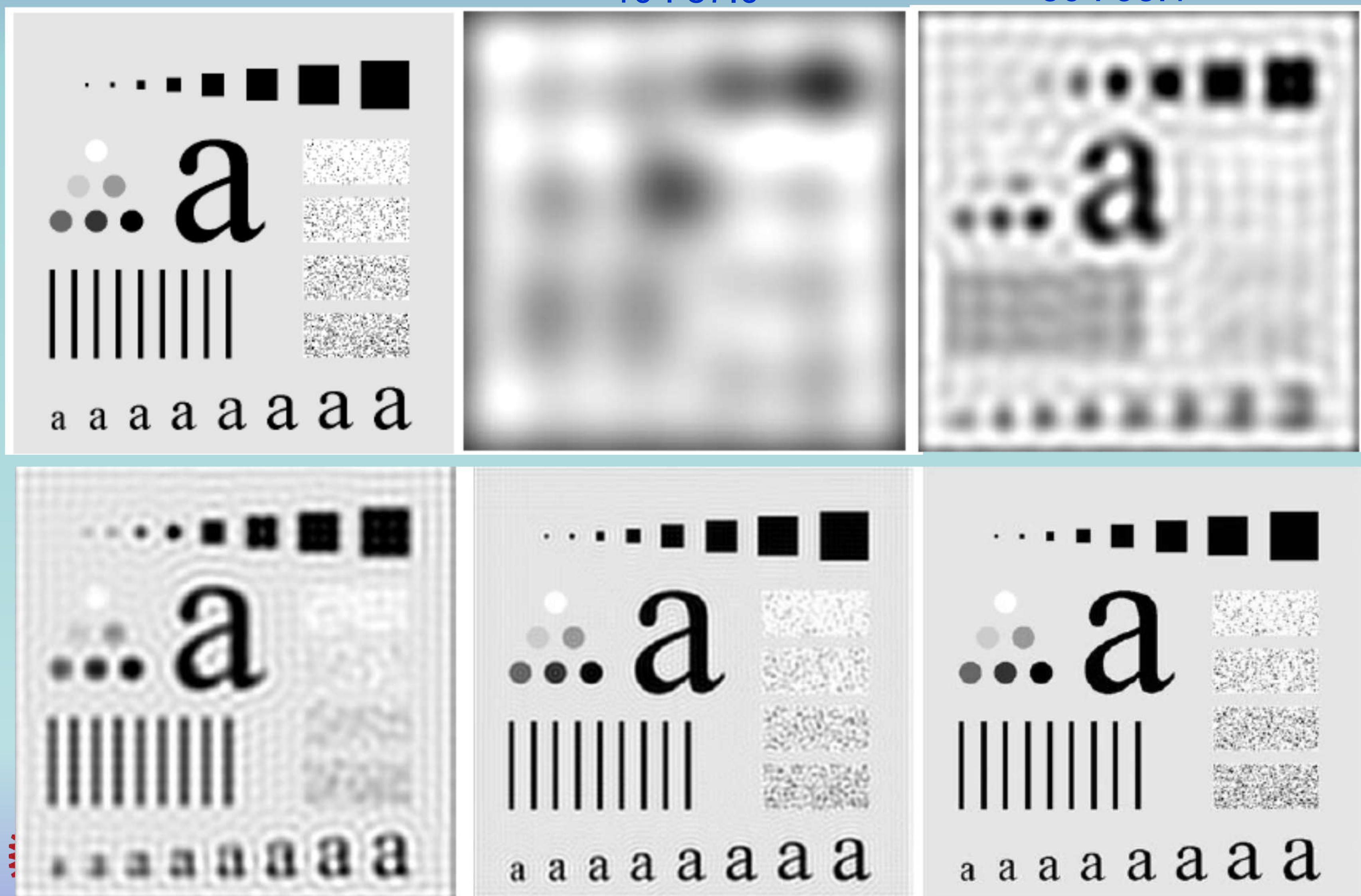
60 : 95.7

160 : 97.8

460 : 99.2

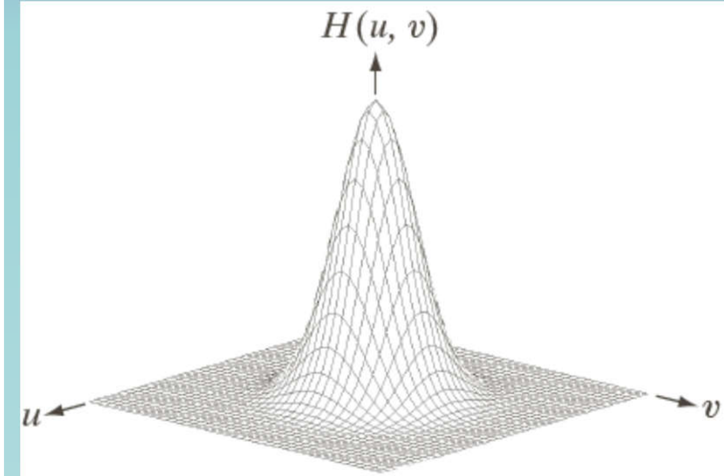
Ideal Low Pass Filter

10 : 87.0 30 : 93.1



60 : 95.7 160 : 97.8 460 : 99.2

Gaussian Low Pass Filter



3D Surface View

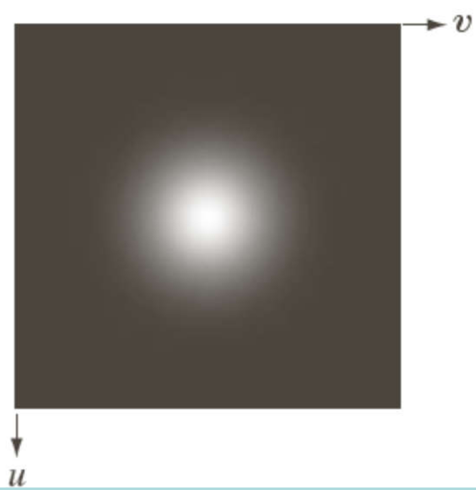
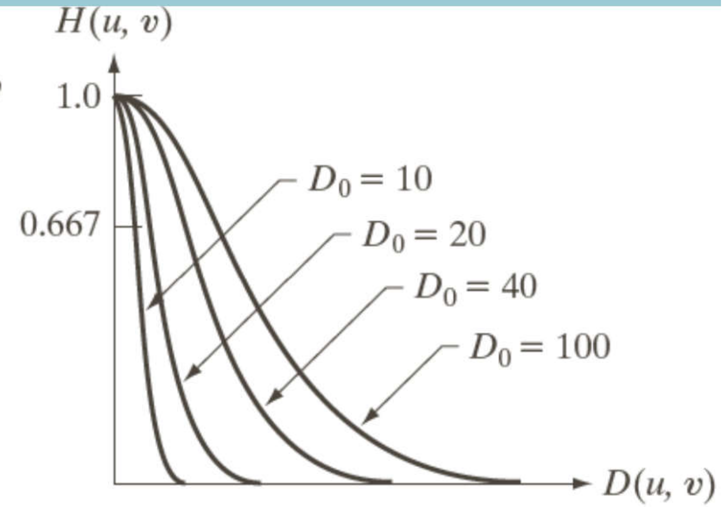


Image View



Radial Cross Section

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

or

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

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Gaussian Low Pass Filter

10 : 87.0

30 : 93.1

60 : 95.7

160 : 97.8

460 : 99.2

Comparison Betn GLPF and BLPF

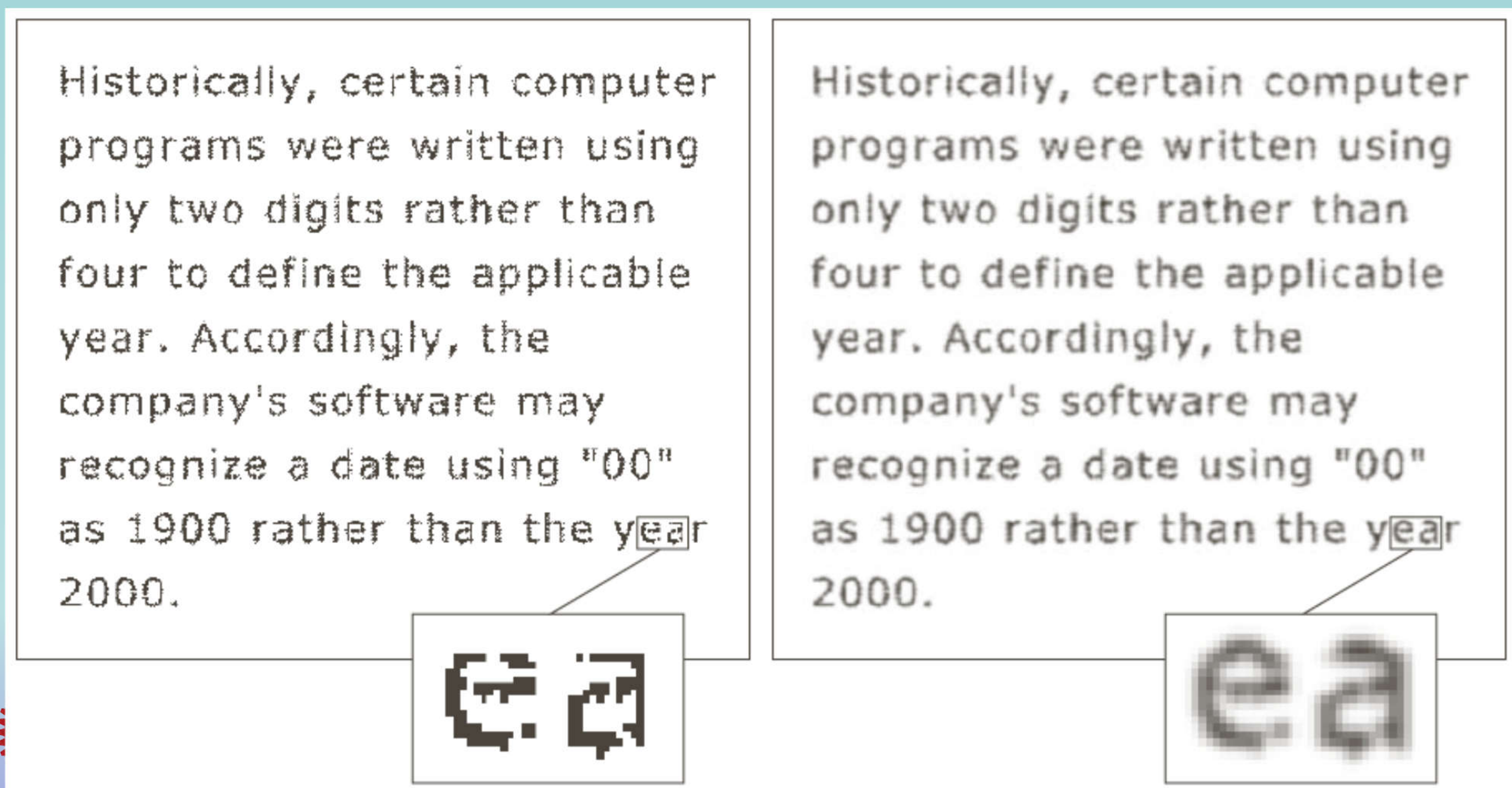
10 : 87.0 | 30 : 93.1 | 60 : 95.7

Gaussian

Butterworth

Application of Low Pass Filtering

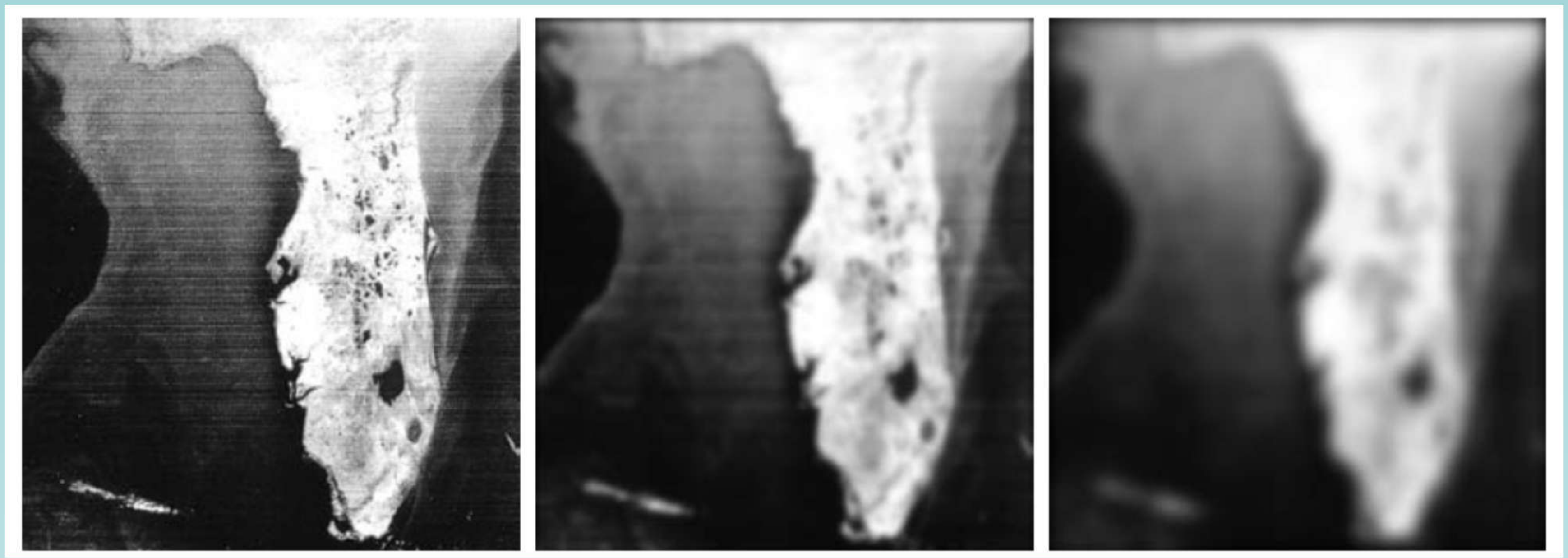
Applying GLPF with D_0 = 80



Application of Low Pass Filtering

In satellite imagery:

GLPF with $D_0 = 50$ GLPF with $D_0 = 20$



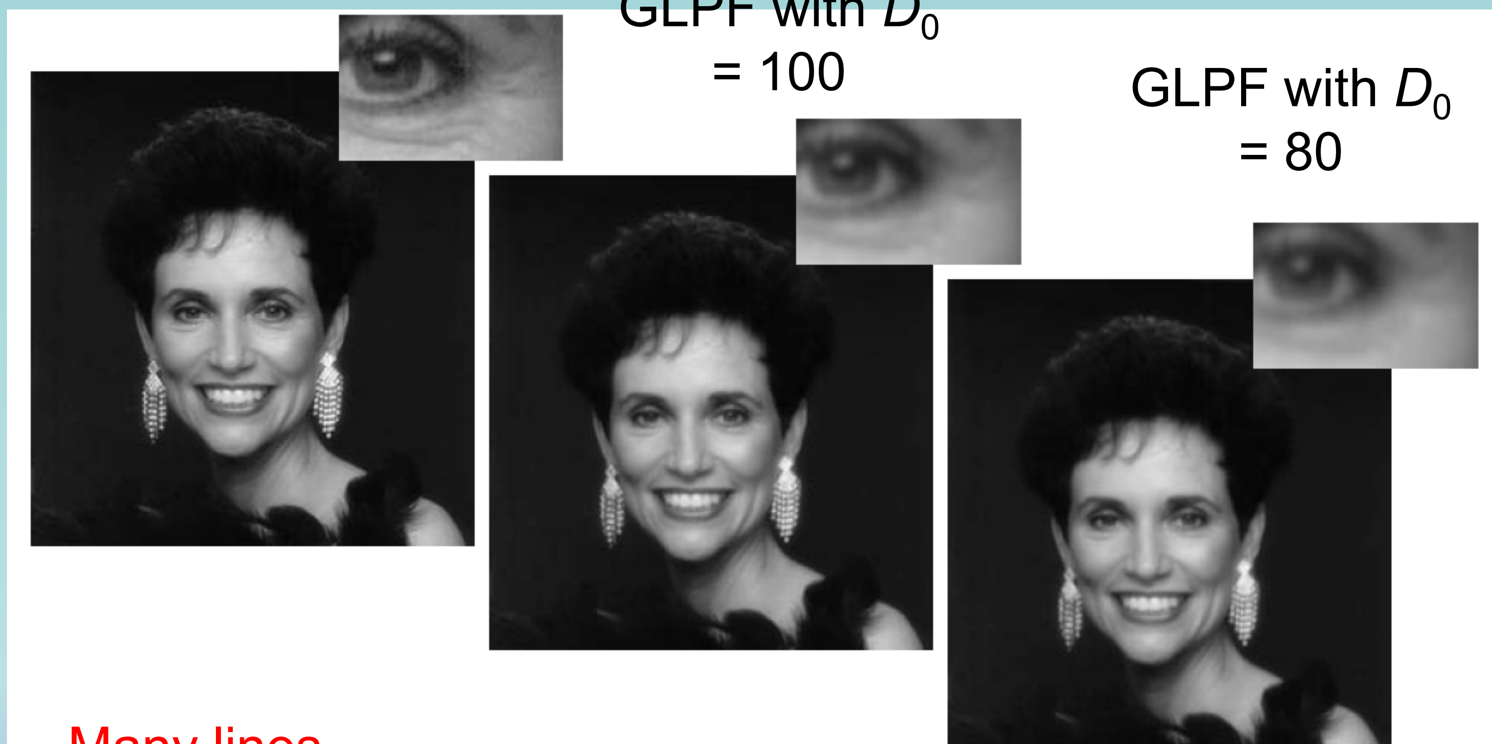
Many horizontal lines produced by the scanning sensor

Reduction in horizontal lines

Removing fine details

Application of Low Pass Filtering

'Cosmetic' processing:



Many lines near the eye

Lines reduced, looking softer