CSE6706:

Advanced Digital Image Processing

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Spectral/Frequency Domain Analysis of Images

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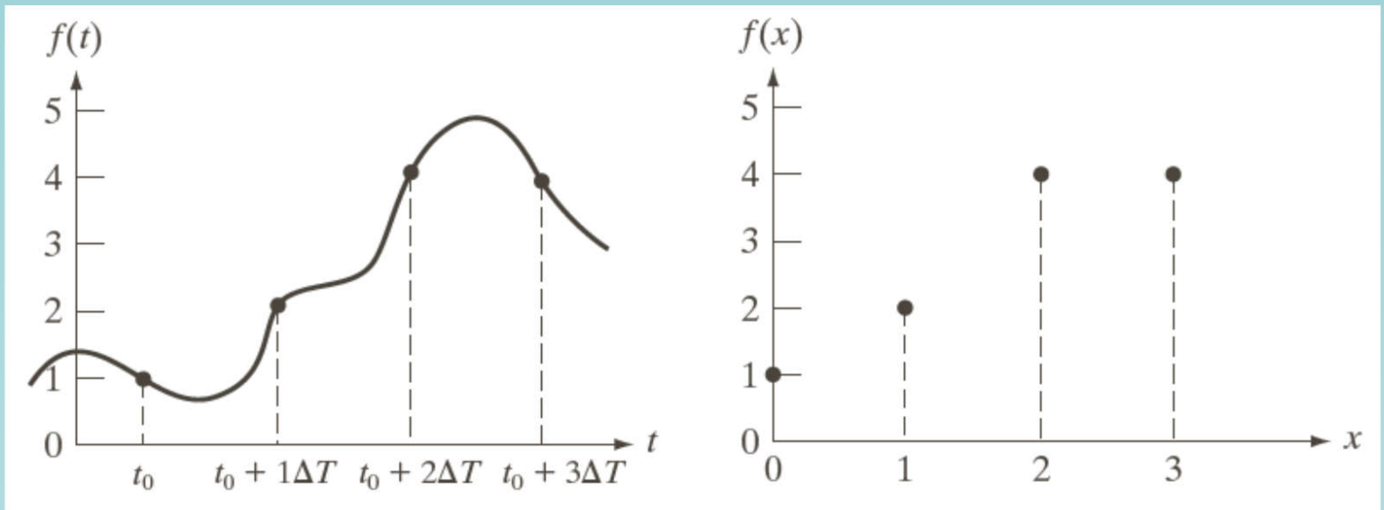
Example of DFT



$$F(0) = 11$$
$$F(1) = -3 + 2j$$
$$F(2) = -1$$
$$F(3) = -3 - 2j$$

$$f(0) = \frac{1}{4}\sum_{u=0}^{u=3} F(u)e^{j2u\pi(0)/4}$$
$$= \frac{1}{4}\sum_{u=0}^{u=3} F(u)$$
$$= \frac{1}{4}\left[11 - 3 + 2j - 1 - 3 - 2j\right] = 1$$

Spectral Technique for Motion Detection

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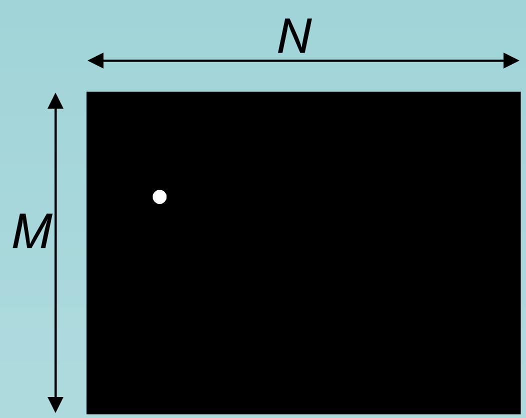
Spectral Technique for Motion Detection



Frame, $f(x, y, 0)$

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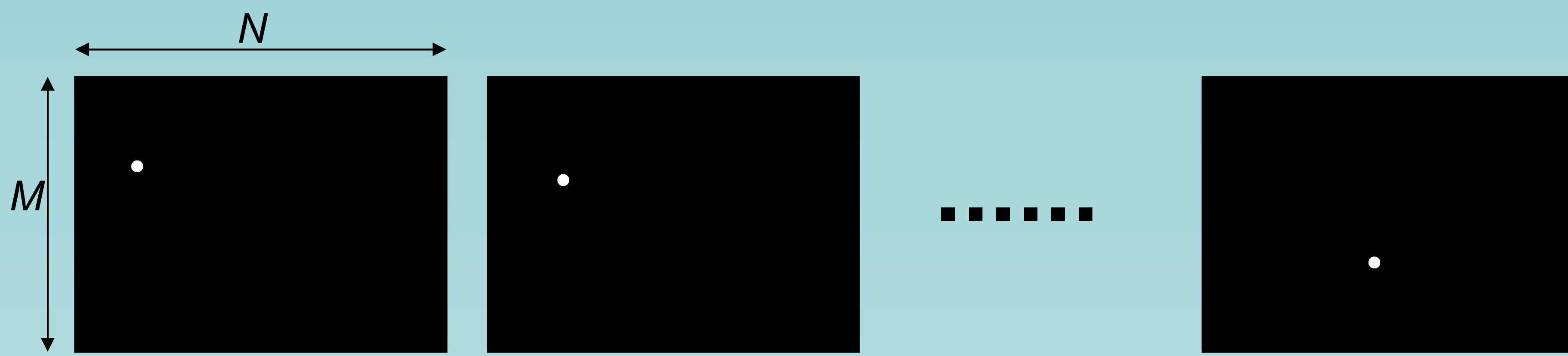
Spectral Technique for Motion Detection



Frame, $f(x, y, 0)$ Frame, $f(x, y, 1)$ Frame, $f(x, y, K\text{-}1)$

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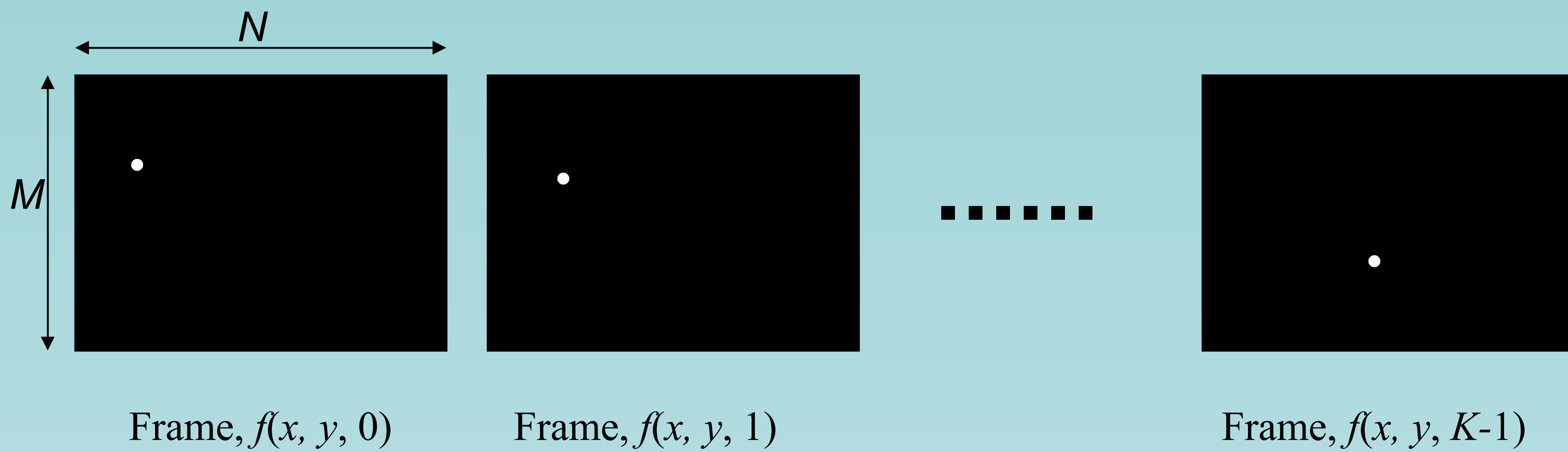
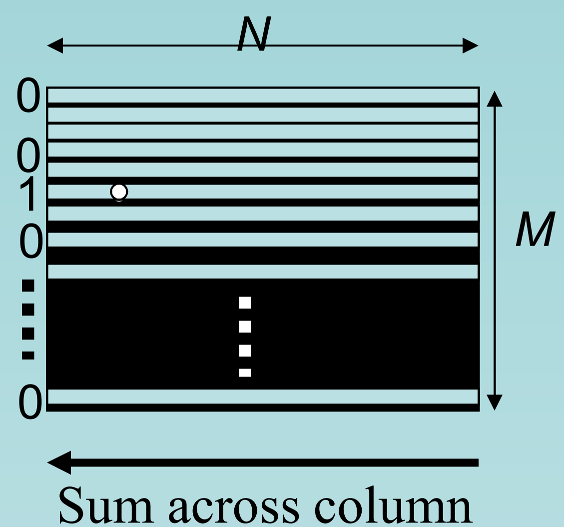
Spectral Technique for Motion Detection



- Objective: determine the motion of the white pixel

Spectral Technique for Motion Detection



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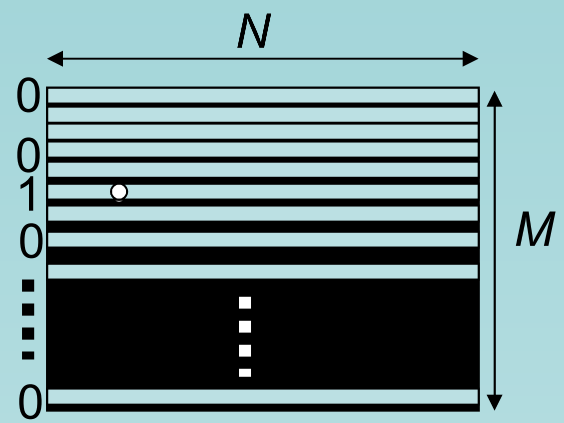
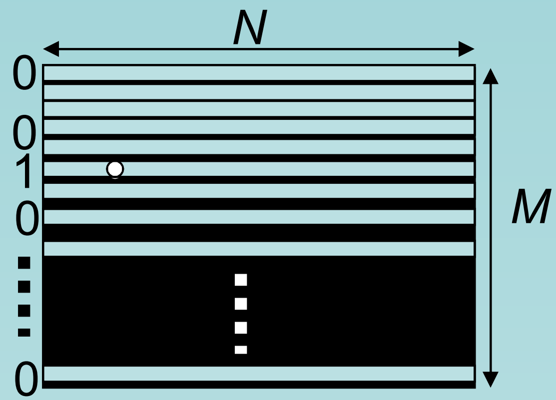
Spectral Technique for Motion Detection

$$e^{j2\pi a_1 x\Delta t} \times$$



Spectral Technique for Motion Detection

$$e^{j2\pi a_1 x\Delta t} \times$$



Sum across row= $e^{j2\pi a_1 x\Delta t}$

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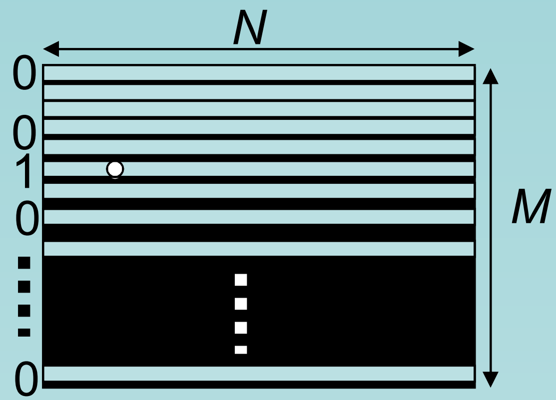
Spectral Technique for Motion Detection

If the object is at coordinate $(x`, y`)$

$$e^{j2\pi a_1 x'\Delta t} \quad \times$$



Sum across row= $e^{j2\pi a_1 x'\Delta t}$

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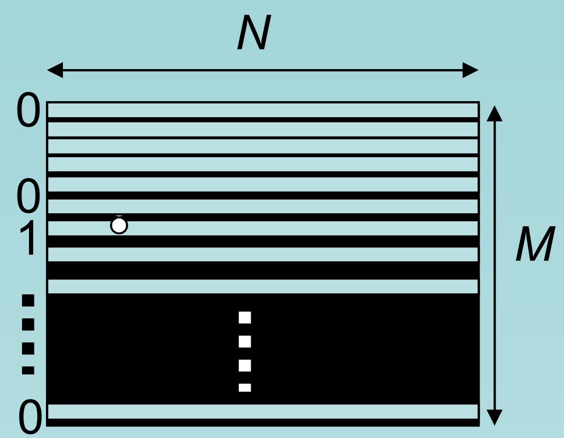
Spectral Technique for Motion Detection

In next frame, the object moves to coordinate $(x`+1, y`)$

$$e^{j2\pi a_1 x'\Delta t} \times$$



Sum across row= $e^{j2\pi a_1 (x'+1)\Delta t}$

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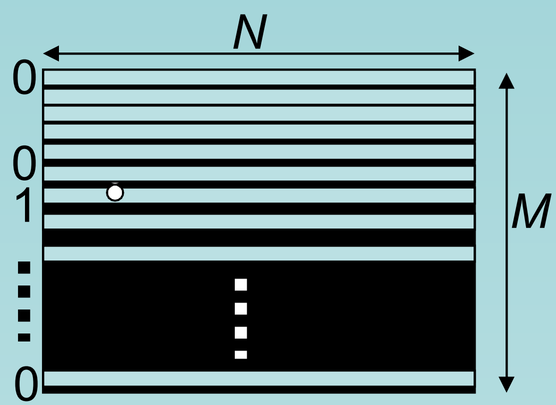
Spectral Technique for Motion Detection

At any instant of time *t*

$$e^{j2\pi a_1 x' \Delta t} \times$$



Sum across row= $e^{j2\pi a_1 (x'+t)\Delta t}$ for $t = 0, 1, \ldots, K\text{-}1$

Spectral Technique for Motion Detection

$$e^{j2\pi a_1(x'+t)\Delta t} = \cos\left[2\pi a_1(x'+t)\Delta t\right] + j\sin\left[2\pi a_1(x'+t)\Delta t\right]$$

- A sinusoidal equation with frequency a_1
- If object moves v_1 pixels per frame, the frequency would be $a_1 v_1$
- The frequency response will have two peaks at $a_1 v_1$ and K- $a_1 v_1$

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Spectral Technique for Motion Detection

$$e^{j2\pi a_2(y'+t)\Delta t} = \cos\left[2\pi a_2(y'+t)\Delta t\right] + j\sin\left[2\pi a_2(y'+t)\Delta t\right]$$

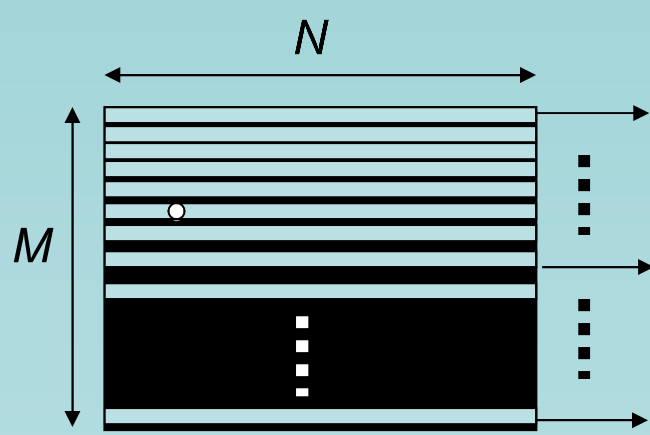
- A similar expression can be derived if the object is projected in y direction
- The corresponding component will be a_2 and v_2

Spectral Technique for Motion Detection

- For a general image with multiple moving objects, the spectrum will have peaks
  - One for stationary background at frequency 0
  - Two for each moving object at location proportional its corresponding velocity

Spectral Technique for Motion Detection

If the background is not zero



$$\sum_{y=0}^{N-1} f(x,y,t)$$

$$\sum_{y=0}^{N-1} f(x,y,t) \quad \times e^{j2\pi a_1 x \Delta t}$$

$$\sum_{y=0}^{N-1} f(x,y,t)$$

Sum across row, $g(t,a_1) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y,t)e^{j2\pi a_1 x \Delta t}$

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Spectral Technique for Motion Detection

$$g(t, a_1) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y, t) e^{j 2\pi a_1 x \Delta t}$$

$$g(t, a_2) = \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f(x, y, t) e^{j 2\pi a_2 y \Delta t}$$

Spectral Technique for Motion Detection

$$g(t,a_1)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y,t)e^{j2\pi a_1 x\Delta t}$$

$$g(t,a_2)=\sum_{y=0}^{N-1}\sum_{x=0}^{M-1} f(x,y,t)e^{j2\pi a_2 y\Delta t}$$

$$G_x(u_1,a_1)=\sum_{t=0}^{K-1} g_x(t,a_1)e^{-j2\pi u_1 t/K}$$

$$G_y(u_2,a_2)=\sum_{t=0}^{K-1} g_y(t,a_2)e^{-j2\pi u_2 t/K}$$

Spectral Technique for Motion Detection

$$G_x(u_1, a_1) = \frac{1}{K}\sum_{t=0}^{K-1} g_x(t, a_1)e^{-j2\pi u_1 t/K}$$

$$G_y(u_2, a_2) = \frac{1}{K}\sum_{t=0}^{K-1} g_y(t, a_2)e^{-j2\pi u_2 t/K}$$

- The relation between frequencies and velocities:

$$u_1 = a_1 v_1$$

$$u_2 = a_2 v_2$$

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Spectral Technique for Motion Detection

- v_1= 10 pixels per K frames (images)
- K = 30 frames
- Frame rate 2 frames /second
- Pixels are 0.5 meter apart

- What is the velocity?

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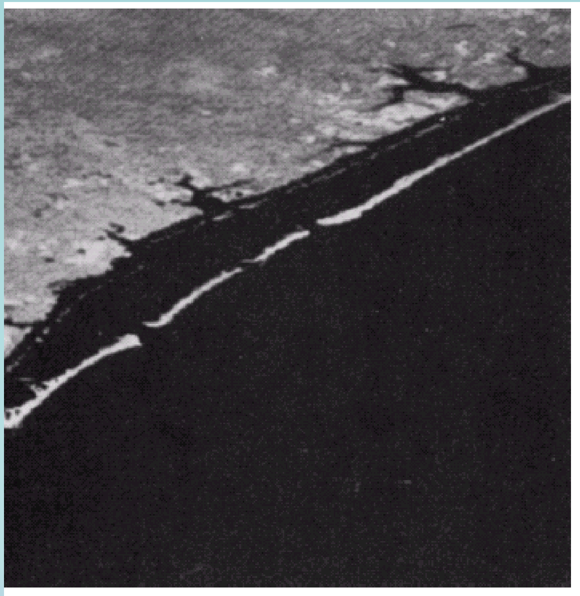
- LandSAT sequence of 32 frames
- Superimposed target moving 0.5 pixel in *x* direction and 1 pixel in *y* direction
- The target is not visible here!



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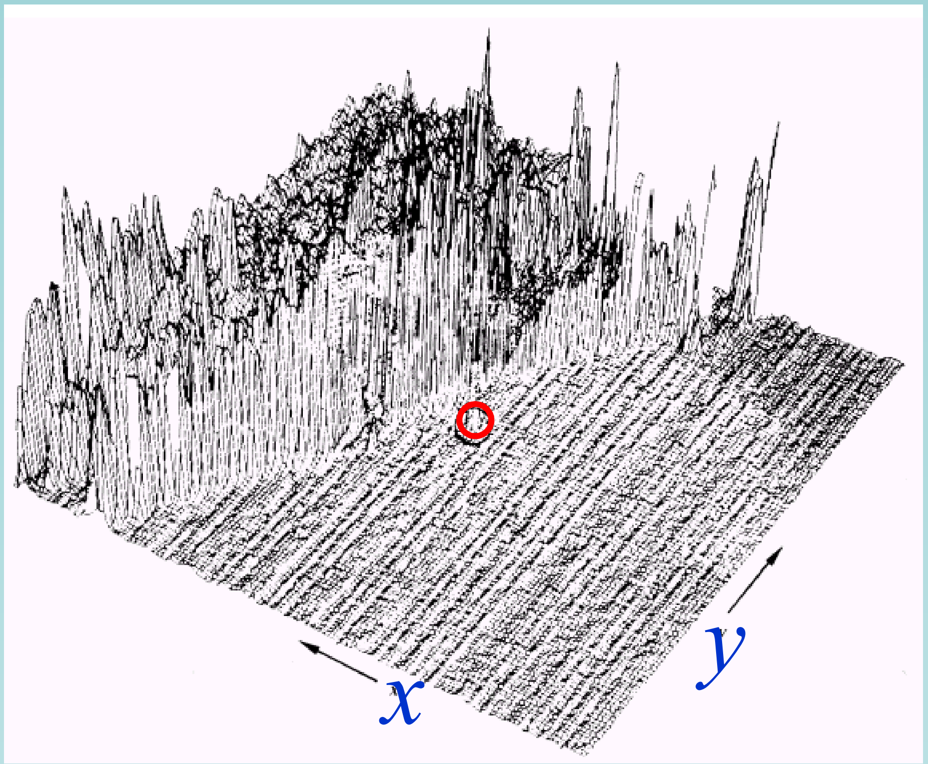
- The target shown in red circle in surface plot of the intensity image



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Spectral Technique for Motion Detection

$$G_x(u_1, a_1) = \frac{1}{K}\sum_{t=0}^{K-1} g_x(t, a_1) e^{-j2\pi u_1 t/K}$$

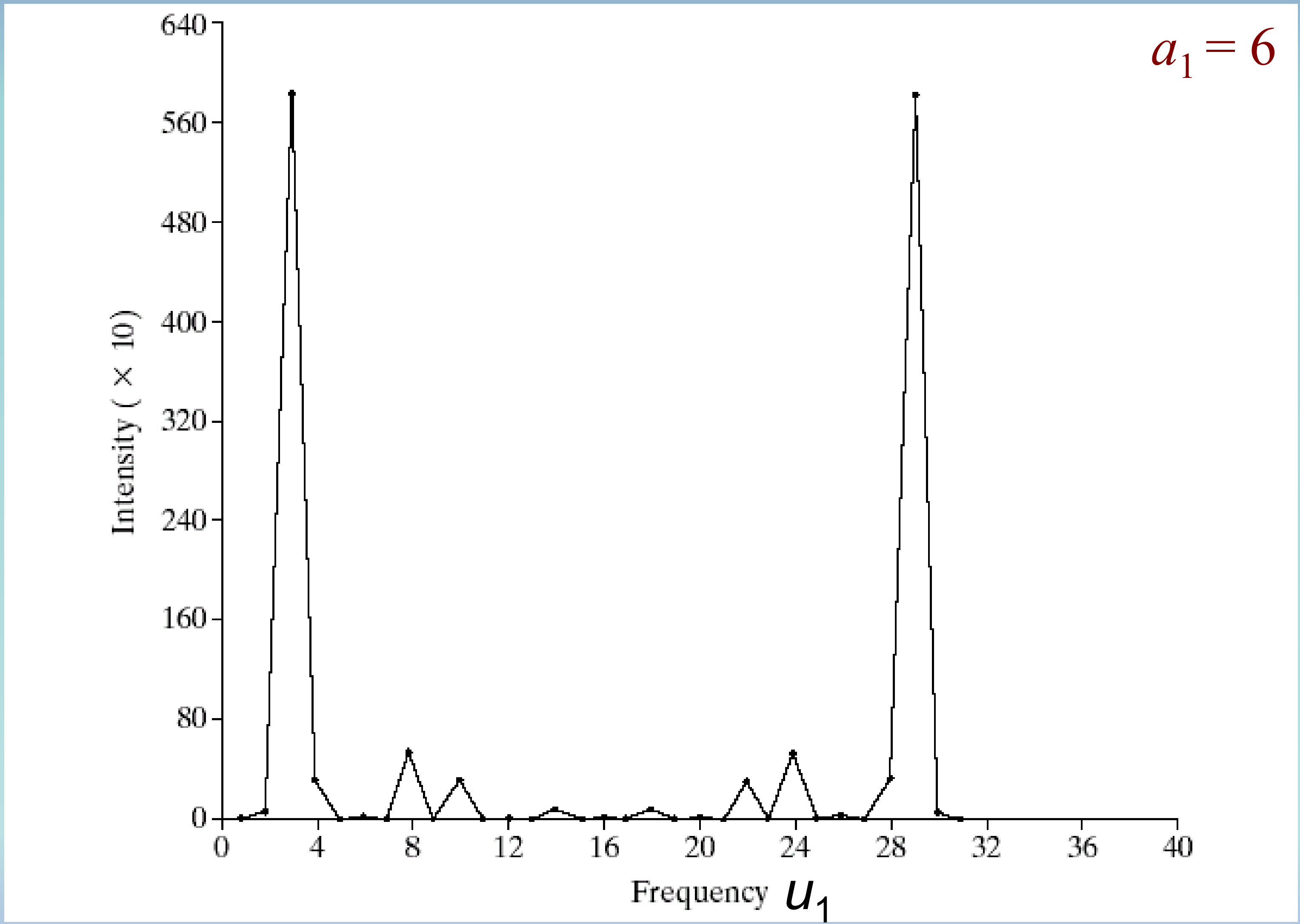
$$G_y(u_2, a_2) = \frac{1}{K}\sum_{t=0}^{K-1} g_y(t, a_2) e^{-j2\pi u_2 t/K}$$

$$u_1 = a_1 v_1$$

$$u_2 = a_2 v_2$$

- $a_1 = 6$ and $a_2 = 4$

$$G_x(u_1, a_1)$$



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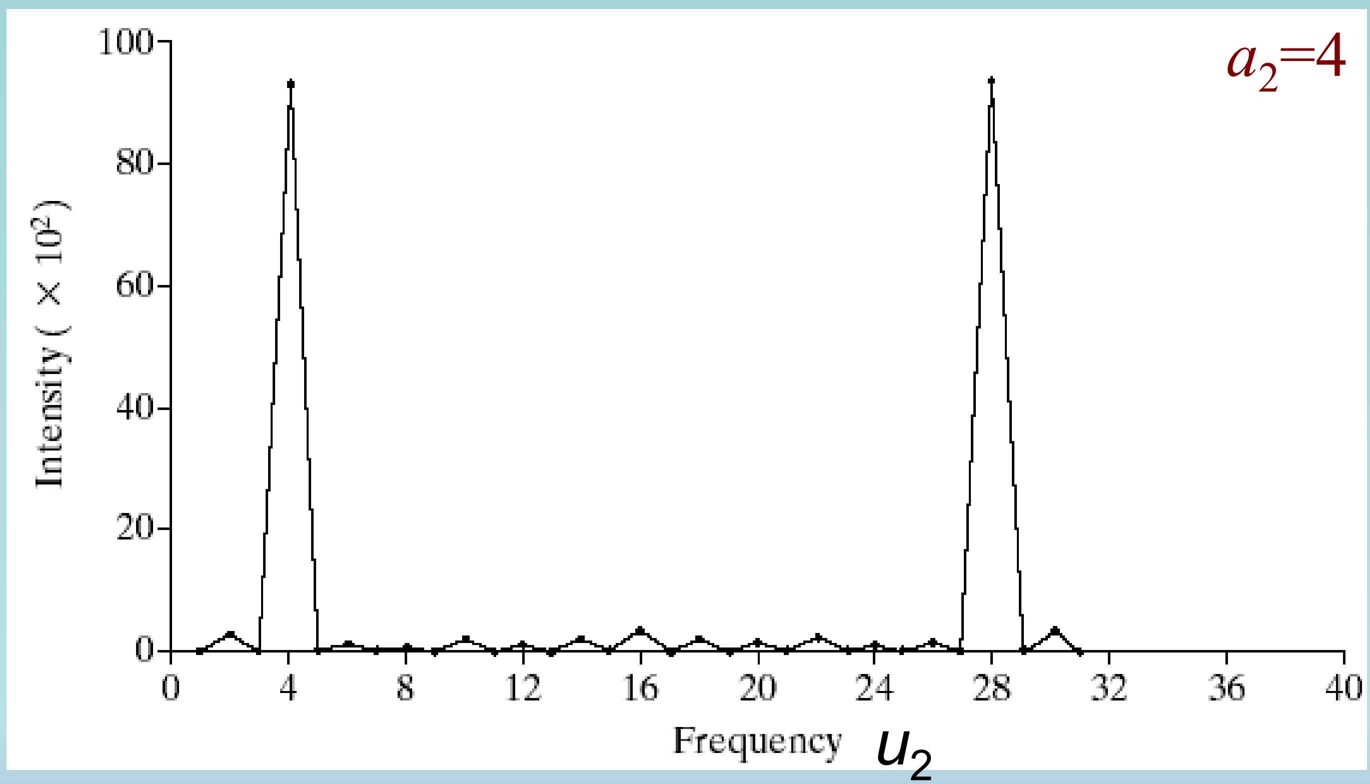
$$G_y(u_2, a_2)$$



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Fourier Transform of 2D Functions

2D Impulse Function

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Continuous 2D Impulse Function

$$\delta(t,z)=\begin{cases}\infty & \text{if } \ t=0, z=0\\ 0 & \text{otherwise}\end{cases}$$

with the constraint,

$$\int_{t=-\infty}^{t=\infty}\int_{z=-\infty}^{z=\infty}\delta(t,z)\,dt\,dz=1$$

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Sifting Property of Continuous 2D Impulse Function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

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Sifting Property of Continuous 2D Impulse Function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

- Evaluates the function at the location of the impulse

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Sifting Property of Continuous 2D Impulse Function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0, z-z_0)\,dt\,dz = f(t_0, z_0)$$

- Evaluates the function at the location of the impulse

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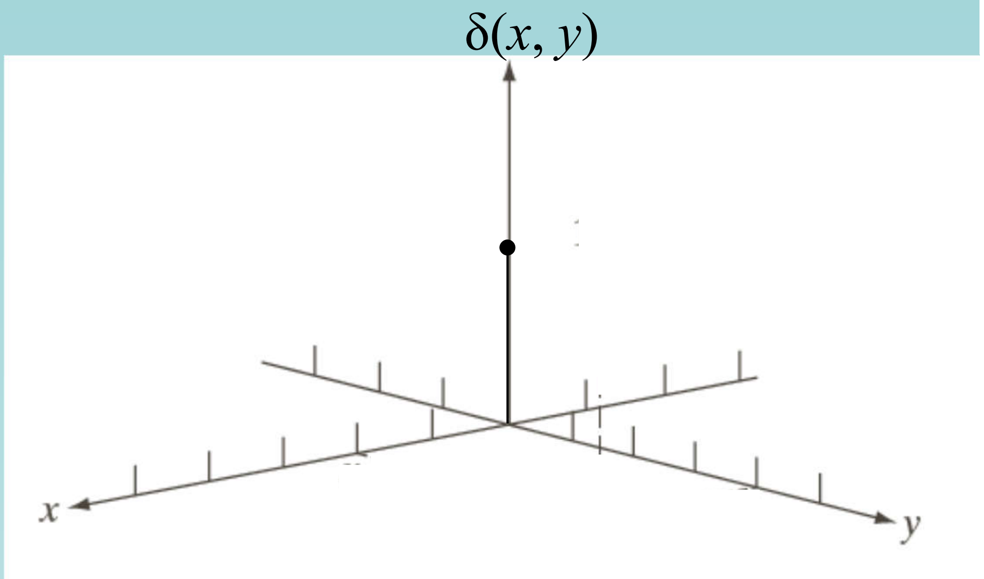
Discrete 2D Impulse Function

$$\delta(x,y) = \begin{cases} 1 & \text{if } \; x = 0, y = 0 \\ 0 & \text{otherwise} \end{cases}$$

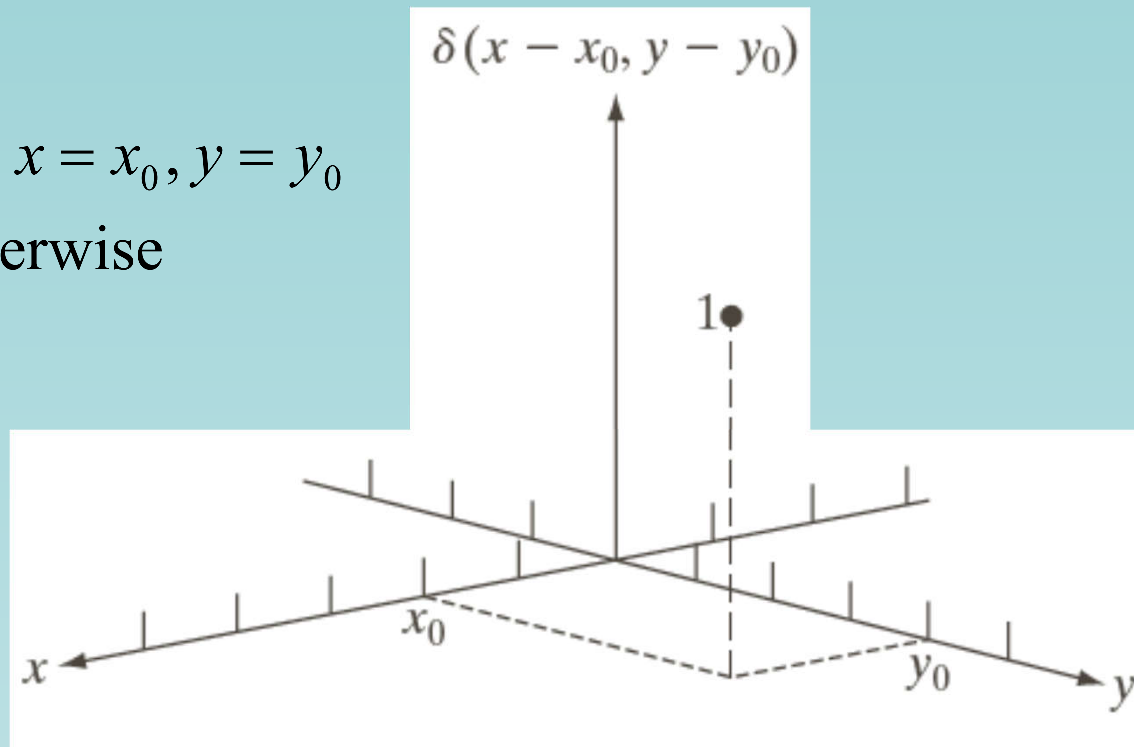


with the constraint,

$$\sum_{x=-\infty}^{x=\infty} \sum_{y=-\infty}^{y=\infty} \delta(x,y) = 1$$

Discrete 2D Impulse Function

$$\delta(x-x_0, y-y_0) = \begin{cases} 1 & \text{if } \; x = x_0, y = y_0 \\ 0 & \text{otherwise} \end{cases}$$



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Sifting Property of Discrete 2D Impulse Function

$$\sum_{x=-\infty}^{x=\infty} \sum_{y=-\infty}^{y=\infty} \delta(x - x_0, x - y_0) f(x, y) = f(x_0, y_0)$$

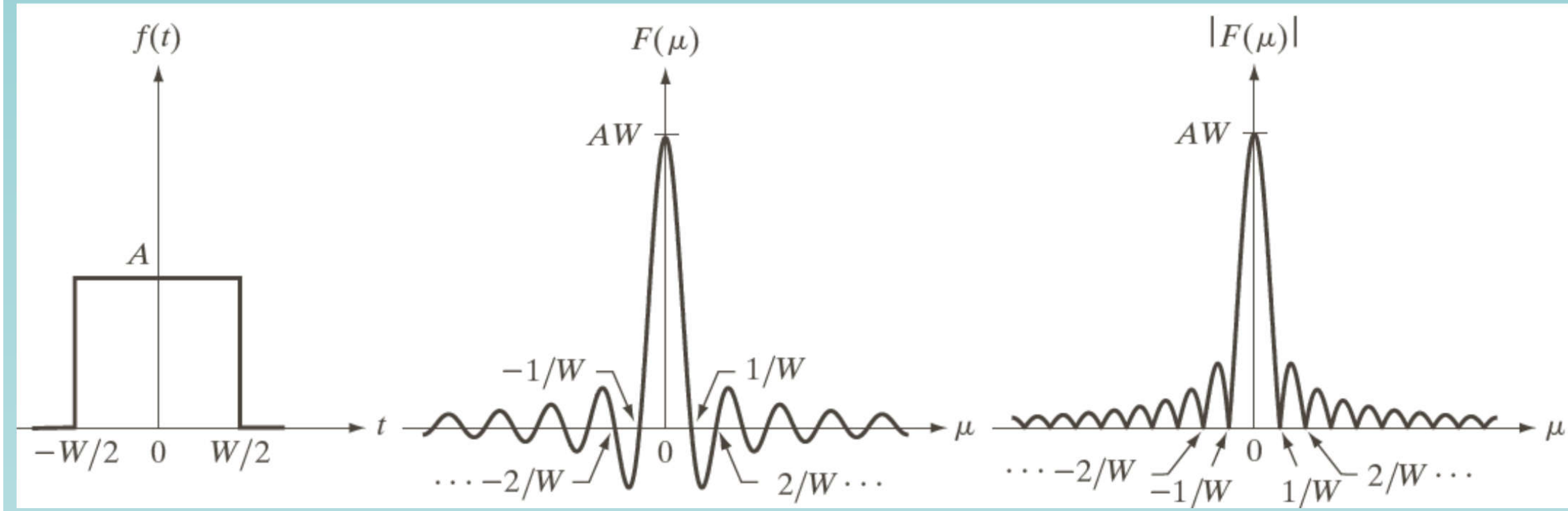
2D Fourier Transform and its Inverse

$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

- The inverse Fourier transform is

$$f(t,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

Recall: 1D Fourier Transform and its Inverse



$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi u t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi u t} dt = \cdots$$

$$= \frac{A}{j2\pi u}\left[e^{j\pi u W} - e^{-j\pi u W}\right] = AW \frac{\sin(\pi u W)}{\pi u W} = AW \operatorname{sinc}(\pi u W)$$

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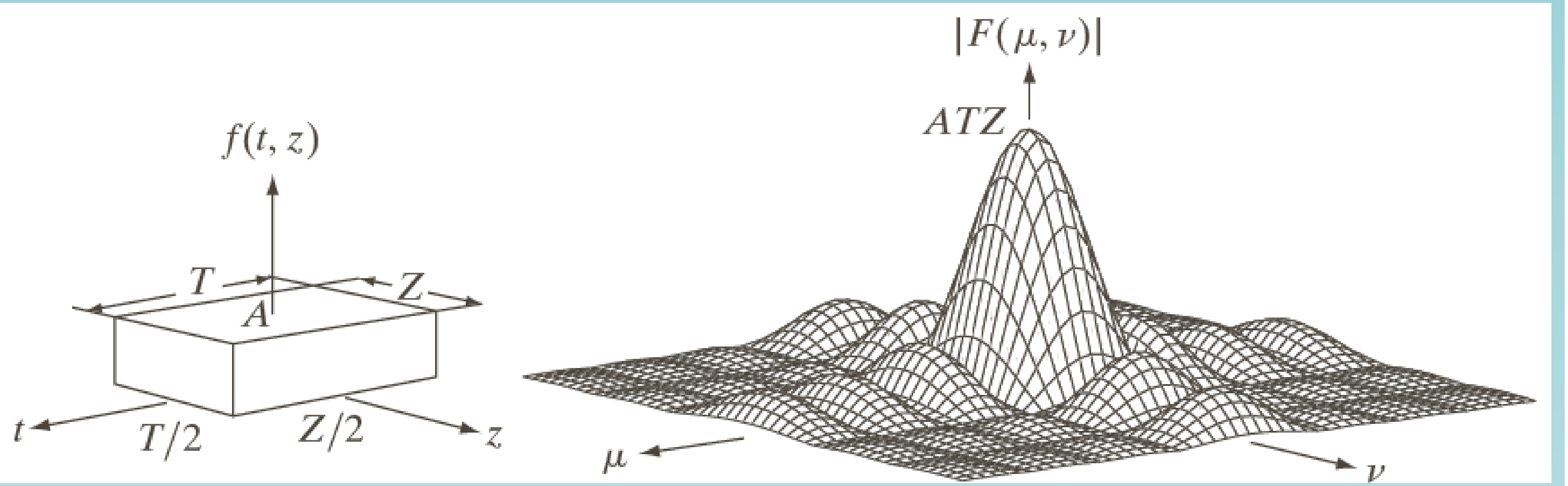
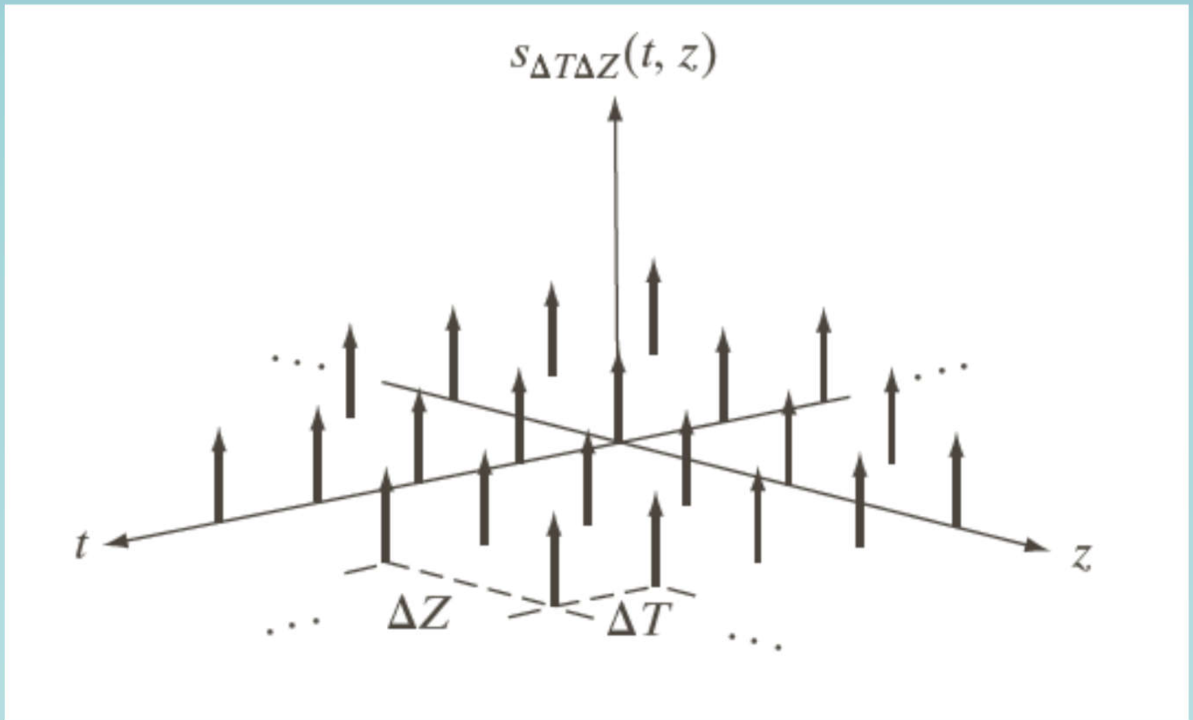
2D Fourier Transform and its Inverse



$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz = \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}dtdz = \cdots$$

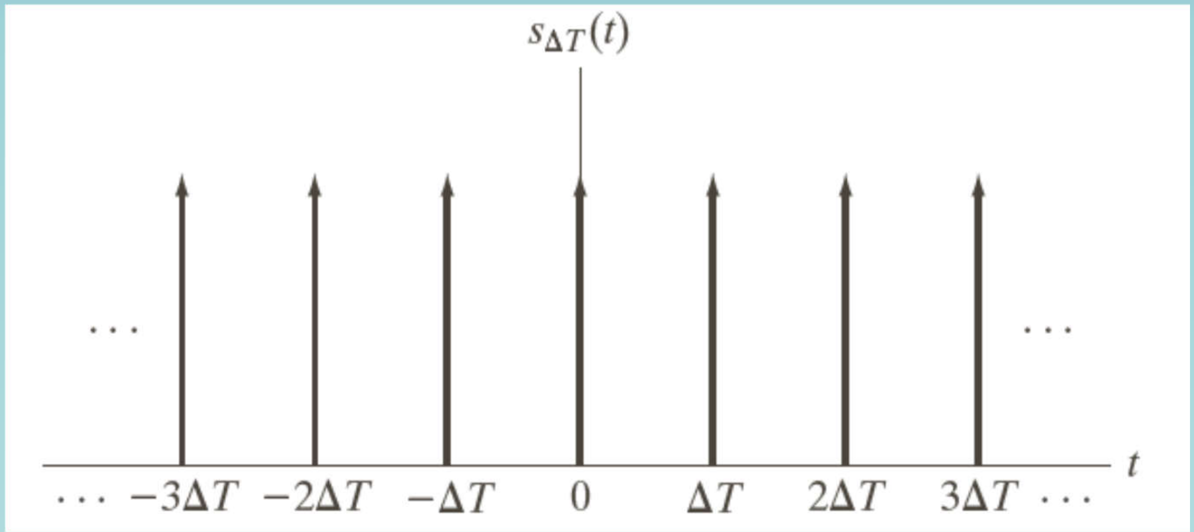
$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right] = ATZ\,\mathrm{sinc}(\pi\mu T)\,\mathrm{sinc}(\pi\nu Z)$$

2D Impulse Train



$$s_{\nabla T, \nabla Z}(t, z) = \sum_{m=-\infty}^{m=\infty} \sum_{n=-\infty}^{n=\infty} \delta(t - m\nabla T, z - n\nabla Z)$$

Recall: FT of 1D Impulse Train



Impulse Train:

$$s_{\nabla T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\nabla T)$$

After Transform:

$$S(u) = \Im\left\{s_{\nabla T}(t)\right\} = \frac{1}{\nabla T} \sum_{n=-\infty}^{n=\infty} \delta(u - \frac{n}{\nabla T})$$

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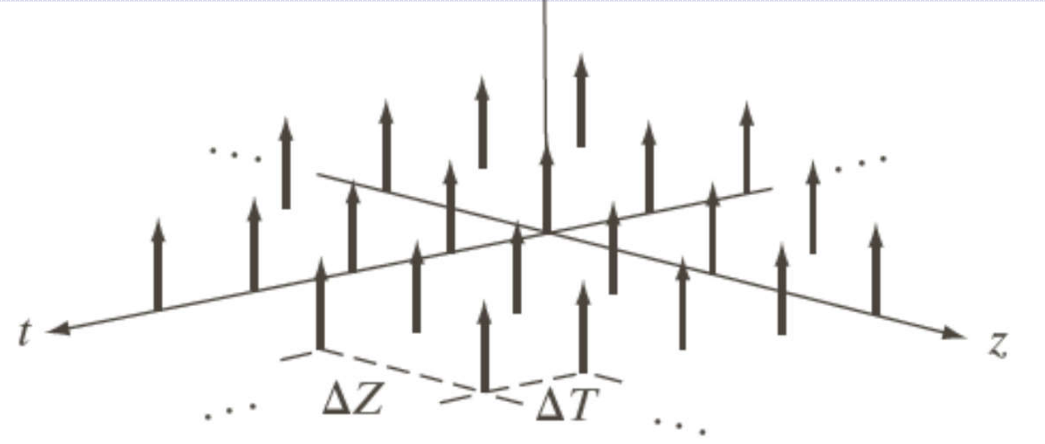
FT of 2D Impulse Train

Impulse Train:

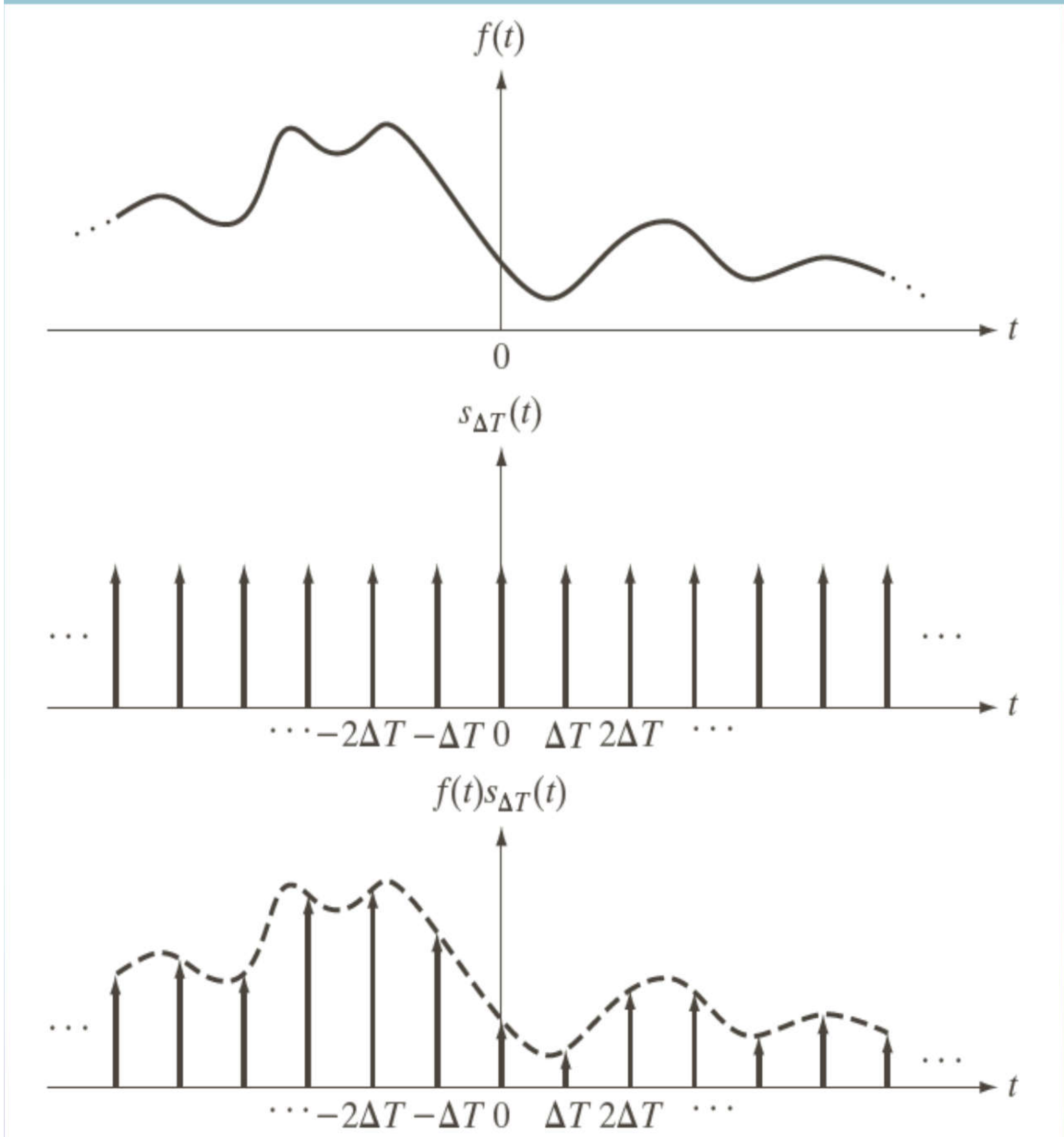
$$s_{\nabla T,\nabla Z}(t,z)=\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}\delta(t-m\nabla T,z-n\nabla Z)$$



After Transform:

$$S(\mu,\nu)=\Im\left\{s_{\nabla T,\nabla Z}(t,z)\right\}=\frac{1}{\nabla T\nabla Z}\sum_{m=-\infty}^{m=\infty}\sum_{n=-\infty}^{n=\infty}\delta\left(\mu-\frac{m}{\nabla T},\nu-\frac{n}{\nabla Z}\right)$$

Recall: 1D Sampled Function



$$\widetilde{f}(t) = f(t)s_{\nabla T}(t)$$

$$= \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\nabla T)$$

2D Sampled Function

$$\tilde{f}(t,z) = f(t,z)s_{\nabla T,\nabla Z}(t,z)$$

$$= \sum_{m=-\infty}^{m=\infty} \sum_{n=-\infty}^{n=\infty} f(t,z)\delta(t - m\nabla T, z - n\nabla Z)$$

Recall: DFT of a Sampled Function for a Single Period

$$F(u) = \sum_{x=0}^{x=M-1} f(x) e^{-j2\pi ux/M} \qquad \text{for } u = 0,1,2,\cdots,M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{u=M-1} F(u) e^{j2\pi ux/M} \qquad \text{for } x = 0,1,2,\cdots,M-1$$

2D DFT and Its Inverse

- Forward transform:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}, \text{ for } u = 0,\ 1,\ ...,\ M\text{-}1 \text{ and } v = 0,\ 1,\ .\ .\ .,\ N\text{-}1$$

- Inverse transform:

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}, \text{ for } x = 0,\ 1,\ ...,\ M-1 \text{ and } y = 0,\ 1,\ .\ .\ .,\ N\text{-}1$$