Propositional Logic

Charles Aunkan Gomes
Lecturer, Dept. of CSE
United International University

What is a Proposition?

A proposition is a declarative sentence that:

- can be **either true or false**
- cannot be **both**
- cannot be **neither**

Let p be the statement “The sun rises in the east”

- p is a proposition as it can be clearly classified as either true or false.
- P is true proposition

Again, let q be the statement “2 plus 2 equals 5”

- q is also a proposition as it can be classified as either true or false.
- q is false proposition

What is not a Proposition?

Statements that cannot be assigned True or False values

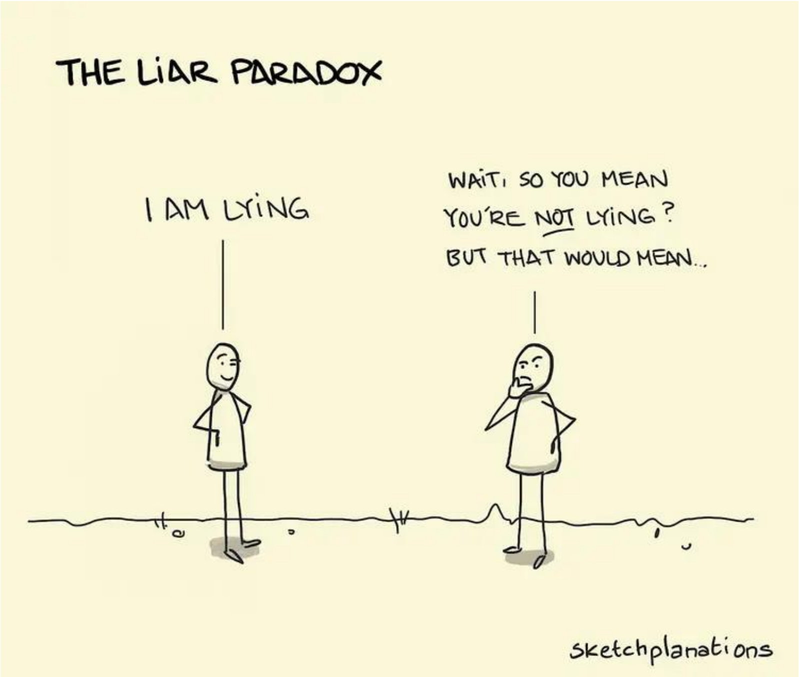
- What is your name?
  - A question. Hence not a declarative sentence.

- Tell me your name.
  - An order. Hence not a declarative sentence.

- $x + 1 = y$
  - An equation. Neither true or false.

Are these propositions?

- x + 1 = y, given that x = 5 and y = 6
  - An equation with values of each variable. **A proposition**.



Are these propositions?

- I am lying.
  - If "I am lying" is true:
    - If you are telling the truth when you say "I am lying," then you must actually be lying. But if you're lying, then the statement "I am lying" cannot be true. BAM!!!
  - If "I am lying" is false:
    - If you're lying when you say "I am lying," then you are not lying (because a liar would be lying about lying). But if you're not lying, then the statement "I am lying" must be true. DOUBLE BAM!!!

Both true and false. Not a proposition.

This is known as "Liar Paradox"

Propositional Variables

We can use variable or symbol to represent a proposition without writing the actual sentence.

For example:

- Consider a propositional variable p
- It can denote the proposition “The earth is round”
- Hence, p = The earth is round
- Easy representation

More examples:

a = Argentina won the world cup

b = Brazil lost to Germany

e = Emi Martinez plays well

Atomic Proposition and Compound Proposition

<u>Atomic propositions</u>: Propositions that cannot be further expressed in terms of simpler propositions.

- I live in Dhaka.
- I am a lecturer of UIU.
- I went to Mars.

<u>Compound propositions</u>: Propositions that are formed from combining two or more atomic propositions.

- I live in Dhaka and I am a lecturer of UIU.
- I live in Dhaka but I went to Mars.
- I am a lecturer of UIU or I live in Dhaka.

Logical Operator

Logical operator: word or phrase used either to **modify** one proposition to make a different proposition or join multiple propositions together to form a compound proposition. It is also known as connectives.

There are several ways in which we commonly modify or combine simple propositions into compound ones. The words/phrases ***and, or, not, if … then…, and …if*** and ***only if …*** can be added to one or more propositions to create a new proposition.

We will study the following logical connectives:

- Negation/ Logical Not
- Conjunction/ Logical And
- Disjunction / Logical Or
- Conditional / Implication
- Bi-conditional

Negation (NOT)

The negation of p, denoted by **¬p.** It means "It is not the case that p."

For example:

- "The earth is flat"
- The negation is: "It is not the case that, the earth is flat."
- Simpler negation : "The earth is not flat".

p	$\neg p$
T	F
F	T

True if the actual proposition is **false**.

Conjunction (AND)

The conjunction of **p** and **q** can be denoted by **p ∧ q.** It means "p and q".

For example:

p = Messi is playing

q = Argentina is winning

p ∧ q = Messi is playing and Argentina is winning

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

True if both propositions are **individually true**.

Disjunction (OR)

The conjunction of **p** and **q** can be denoted by **p ∨ q.** It means "p or q".

For example:

p = 2 + 2 equals 4

q = 4 > 5

p ∨ q = 2 + 2 equals 4 or 4 > 5.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

True if any proposition is **true**.

Exclusive OR

It is a special type of Disjunction. The exclusive or of **p** and **q** can be denoted by **p ⊕ q.**

Example:

p = Bangladesh will win the match.

q = England will win the match.

p ⊕ q = Either Bangladesh or England will win the match.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

True if exactly one proposition is true.

Conditional Statement

If it rains, I will stay home.

WHEN IS THIS SENTENCE TRUE?

- The sentence is true if it is raining, and you are staying home.

What if it doesn't rain?

The sentence does not say anything about what if it does not rain

- So if it does not rain, we can say you have not broken any condition
- So if it does not rain, the sentence is true whether you stay home or not

Conditional Statement

So when is this sentence false?

- The sentence is false if it rains, and you go out anyway

If it rains, I will stay home.

Let P and Q be propositions.

- p : It rains
- q : I will stay home
- p → q : If it rains, I will stay home.

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

- P is called premise/hypothesis
- Q is called conclusion/consequence

$$p \rightarrow q$$

Hypothesis Conclusion

Hypothesis and conclusion

$$\text{If } \underbrace{\text{it rains,}}_{\text{Hypothesis}} \ \underbrace{\text{I will stay home}}_{\text{Conclusion}}$$

The if clause is called the hypothesis / premise, and the remaining clause is called the conclusion/consequence

We assume that the hypothesis is true in order to verify the validity of the conclusion

The conclusion is the outcome of the hypothesis.

Sufficient condition

Generally, the hypothesis is a sufficient condition for the conclusion.

For the proposition $p \rightarrow q$, p is the sufficient condition for q.

For example: If it rains, I will stay home.

While I may stay home for any reason, if it rains, you know that you will find me home.

In other words, knowing that it is raining is sufficient to know that I am home.

"Raining is sufficient for me staying home."

Necessary Condition

You will pass only if you study.

WHEN IS THIS SENTENCE TRUE?

↓

The sentence is true if you study, and you pass

What if you don’t study?

↓

Fail

Necessary Condition

You will pass only if you study.

What if you don't pass?

↓

You may not pass even if you have studied
(probably because you became sick)

- Even though you have studied, you may not pass for a variety of reasons
- So if you do not pass, that does not violate any condition
- So if you do not pass, the sentence is true whether you have studied or not

Note that while you may fail for any possible reason, you must study in order to pass
In another word, in order to pass, it is necessary that you study

Necessary condition and sufficient condition

if p then q

sufficient Necessary

p only if q

sufficient Necessary

Necessary condition and sufficient condition

Generally, the conclusion is a necessary condition for the hypothesis, and the hypothesis is a sufficient condition for the conclusion

- For $p \to q$, q is a necessary condition for p, and p is a sufficient condition for q

In our first example (If it rains, I will stay home), staying home is necessary for you if it rains

- It is raining only if I stay home

In our second example (You will pass only if you study), passing is sufficient to realize that you have studied

- If you have passed, you have studied

DIFFERENT WAYS OF EXPRESSING CONDITIONAL STATEMENT

1. if p, then q
2. if p, q
3. p implies q
4. q if p
5. q when p
6. q whenever p
7. q follows from p
8. q provided that p
9. q unless ¬p
10. p only if q
11. p is sufficient for q
12. a sufficient condition for q is p
13. q is necessary for p
14. a necessary condition for p is q

BICONDITIONAL

You can take the train if and only if you buy ticket.

WHEN IS THIS SENTENCE TRUE?

↓

If you have a ticket, and you board the train
OR
If you don't have a ticket, and the ticket checker does not let you board

When is this sentence false?

↓

If you have a ticket, and the ticket checker still refused to take you
OR
If you don't have a ticket, and you board the train anyway

BICONDITIONAL

You can take the train if and only if you buy ticket.

$p \leftrightarrow q$

$\leftrightarrow$: equivalent

p and q are necessary and sufficient conditions for each other

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

BICONDITIONAL

Different ways to express p $\Leftrightarrow$q :

- p if and only if q
- p iff q
- p is necessary and sufficient for q
- if p then q, and if q, then p

Example:

p: "The number is divisible by 2"

q: "The number is even"

Statement: “The number being divisible by 2 is necessary and sufficient for the number to be even.”


THANK YOU