Nested Quantifiers

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Logical equivalences involving quantifiers

Every CS student learns programming AND discrete mathematics.

P(x): x learns programming

Q(x): x learns DM

$$\forall x\big(P(x) \land Q(x)\big)$$

Every CS student learns programming AND every CS student learns discrete mathematics.

$$\big(\forall x\, P(x)\big) \land \big(\forall x\, Q(x)\big)$$

Logical equivalences involving quantifiers

P(x): x excels in hardware

Q(x): x excels in software

Some CS student excels in hardware OR software.

$$\exists x\left(P(x) \vee Q(x)\right)$$

Some CS student excels in hardware OR some CS student excels in software.

$$\left(\exists x\, P(x)\right) \vee \left(\exists x\, Q(x)\right)$$

Logical equivalences involving quantifiers

$$\forall x\big(P(x) \land Q(x)\big) \equiv \big(\forall x\, P(x)\big) \land \big(\forall x\, Q(x)\big)$$
$$\exists x\big(P(x) \lor Q(x)\big) \equiv \big(\exists x\, P(x)\big) \lor \big(\exists x\, Q(x)\big)$$

De Morgan's Laws for Quantifiers

	$\forall x\, E(x)$	Everyone is evil
Negation	$\neg(\forall x\, E(x))$	Not everyone is evil
	$\equiv \exists x\, \neg E(x)$	Someone is good
	$\exists x\, R(x)$	Rich people exist => Someone is rich
Negation	$\neg(\exists x\, R(x))$	Rich people do not exist => No one is rich
	$\equiv \forall x\, \neg R(x)$	Everyone is poor

De Morgan's Laws for Quantifiers

$$\neg\big(\forall x\ P(x)\big) \equiv \exists x\ \neg P(x)$$
$$\neg\big(\exists x\ P(x)\big) \equiv \forall x\ \neg P(x)$$

It's NOT that some CS student has taken calculus.

$\equiv$ Every CS student has not taken calculus.

It's NOT that every CS student has taken calculus.

$\equiv$ Some CS student has not taken calculus.

De Morgan’s Laws for Quantifiers

1. What are the negations of these statements?

$\forall x(x^2 > x)$

$\exists x(x^2 = 2)$

Domain: All positive integers

2. What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Translating from English to Logical Expressions

$S(x) \equiv x$ is a student in this class,
$P(x) \equiv x$ visited Mexico

Some student in this class has visited Mexico

There is a student in this class who visited Mexico

There is a student x in this class such that x visited Mexico

There is a person x such that x is a student in this class and x visited Mexico (Domain: all people)

$$\exists x\ (S(x) \land P(x))$$

Translating from English to Logical Expressions

Express the statement "Every student in this class has studied calculus"

$C(x)$=x has studied calculus

$S(x)\equiv$x is a student in this class,

"For every person x, if person x is a student in this class then x has studied calculus."

$$\forall x(S(x) \rightarrow C(x))$$

Translating from English to Logical Expressions

Given predicates:

- $L(x) \equiv x$ is a lion
- $F(x) \equiv x$ is a fierce creature
- $C(x) \equiv x$ drinks coffee

The domain consists of **ALL CREATURES**

Express the following statements using the given predicates and quantifiers:

1. All lions are fierce.
2. Some lions do not drink coffee.
3. Some fierce creatures do not drink coffee.

Translating from English to Logical Expressions

Given predicates:

- $P(x) \equiv x$ is a hummingbird
- $Q(x) \equiv x$ is large
- $R(x) \equiv x$ lives on honey
- $S(x) \equiv x$ is richly colored

Express the following statements using the given predicates and quantifiers:

1. All hummingbirds are richly colored.
2. No large birds live on honey.
3. Birds that do not live on honey are dull in color.
4. Hummingbirds are small.

Nested Quantifiers

- Nested quantifiers occur when one quantifier appears within the scope of another in a logical statement.
- Quantifiers are read left-to-right

For all values of x and y, x+y=y+x

$$\forall x \forall y (x + y = y + x)$$

Nested Quantifiers

For every real number x, there is a corresponding real number y such that x+y=0.

How do we represent it using quantifiers?

$$\forall x \exists y (x+y=0)$$

Nested Quantifiers

Express the following statement using predicates and quantifiers:

"If a real number x is negative and another real number y is positive, then their product is negative."

$$\forall x \forall y ((N(x) \land P(y)) \rightarrow N(x.y))$$

N(x) : x is a negative real number.

P(y) : y is a positive real number.

Order of Quantifiers

Commutative law: $\forall x \forall y(x+y=y+x)$

For all real numbers x and for all real numbers y, x+y is equal to y+x

Can we rewrite this as $\forall y \forall x(x+y=y+x)$??

What does it mean?

Is the original meaning changed?

Order of Quantifiers

Some random observation: $\exists x \exists y (xy=6)$

There exists a real number x and there exists a real number y such that xy=6.

Can we rewrite this as $\exists y \exists x (xy=6)$??

What does it mean?

Is the original meaning changed?

Order of Quantifiers

$\forall x \exists y (x+y=0)$

For every real number x, there exists a real number y such that x+y=0.

Can we write it as $\exists y \forall x (x+y=0)$??

No

What does it mean?

There exists a single real number y such that for every real number x, x+y=0.

Is the original meaning changed?

Yes

Order of Quantifiers

- If only universal or only existential quantifiers are nested, their order can be changed

- If both quantifiers are nested, their order cannot be changed, otherwise the meaning will be changed

Find the truth value $(\exists x(x^2 = 2)) \rightarrow (\forall y(y^2 \geq y))$

It means If there exists a real number x such that $x^2 = 2$, then for all real numbers y, $y^2 \geq y$.

The antecedent $\exists x(x^2 = 2)$ is true because $x = \surd 2$ and $x = -\surd 2$ satisfied the $x^2 = 2$.

The consequent $\forall y(y^2 \geq y)$ is false because $y = 0.5$ makes $y^2 = 0.25 < 0.5 = y$

Since the antecedent is true but the consequent is false, the implication as a whole is false

Find out the truth values of the following

$\forall x \exists y (x+y=0)$

True

$\exists y \forall x (x+y=0)$

False

$\forall x \exists y (xy=0)$

True

$\exists y \forall x (xy=0)$

True

$(\exists x (x^2 = 2)) \rightarrow (\forall y (y^2 \geq y))$

False

$\neg \exists x ((-x^2 + 2) = (x^2 + 1))$

False


THANK YOU